

Unfinished Business in A Nonlinear Sigma Model on de Sitter Background

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ABSTRACT

Nonlinear sigma models on de Sitter background possess the same kind of derivative interactions as gravity, and show the same sorts of large spacetime logarithms in correlation functions and solutions to the effective field equations. It was recently demonstrated that these logarithms can be resummed by combining a variant of Starobinsky's stochastic formalism with a variant of the renormalization group. This work considers one of these models and completes two pieces of analysis which were left unfinished: the evolution of the background at two loop order and the one loop beta function.

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1 Introduction

In addition to proposing one of the first theories of primordial inflation [1], Starobinsky was the first to realize that the accelerated expansion of inflation generates a spectrum of gravitational radiation [2]. The numbers are staggering. If $a(\eta)$ is the scale factor at conformal time η , and $\Delta_h^2(k)$ is the tensor power spectrum, then the number of inflationary gravitons with co-moving wave number k and each $\pm$ polarization is,

$$N_{\pm}(\eta, k) = \frac{\pi \Delta_h^2(k)}{64 G k^2} \times a^2(\eta) \longrightarrow \left(\frac{H a}{2k} \right)^2. \quad (1)$$

Notice that the left hand form is valid for a general inflationary background whereas the right hand form is specialized to the de Sitter geometry,

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = a^2 \left[-d\eta^2 + d\vec{x} \cdot d\vec{x} \right] \quad , \quad a(\eta) = -\frac{1}{H\eta}. \quad (2)$$

In the distant past, when $a \simeq 0$, the occupation number (1) is nearly zero. At horizon crossing it becomes $\frac{1}{4}$, and within a single e-foldings it has exceeded unity. Solving the horizon problem requires at least 50 e-foldings of inflation, which causes the occupation number to expand by a factor of $e^{100} \simeq 2.6 \times 10^{43}$. One should bear in mind that this is just the occupation number of a single wave vector $\vec{k}$. Of course the 3-volume is expanding like $a^3(t)$. Integrating over all super-horizon modes gives a constant number density of $\frac{H^3}{32\pi^2}$, or 0.003 distinct super-horizon modes for each Hubble volume.

Loops of these inflationary gravitons change the kinematics and forces exerted by themselves and by other particles. One shows this by quantum correcting the effective field equation of whatever particle is under study using the graviton contribution to its 1PI (one-particle-irreducible) 2-point function. For example, the 1PI 2-point function of photons is a bi-vector density known as the vacuum polarization $i[\mu\Pi^\nu](x; x')$, and was first computed, at 1-graviton loop order, on de Sitter background in 2013 [3] using the simplest graviton gauge [4, 5]. The quantum-corrected Maxwell equation is,

$$\partial_\nu \left[\sqrt{-g} g^{\nu\rho} g^{\mu\sigma} F_{\rho\sigma} \right] + \int d^4 x' \left[{}^\mu\Pi^\nu \right](x; x') A_\nu(x') = J^\mu(x), \quad (3)$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor and J^μ is the current density. When (3) is solved for a static point charge Q , the resulting Coulomb potential is [6],

$$\Phi(\eta, r) = \frac{Q}{4\pi a r} \left\{ 1 + \frac{2G}{3\pi a^2 r^2} + \frac{2GH^2}{\pi} \times \ln(aHr) + O(G^2) \right\}. \quad (4)$$

Solving (3) for a dynamical photon shows a similar 1-loop enhancement of the tree order electric field strength F_0^{0i} [7],

$$F^{0i}(\eta, \vec{x}) = F_0^{0i}(\eta, \vec{x}) \left\{ 1 + \frac{2GH^2}{\pi} \times \ln(a) + O(G^2) \right\}. \quad (5)$$

Many similar calculations have been made in recent years. Single graviton loop contributions on de Sitter background have been evaluated for the 1PI 2-point functions of gravitons [8], fermions [9, 10], massless, minimally coupled scalars [11] and massless, conformally coupled scalars [12, 13, 14]. Large temporal and spatial logarithms analogous to (4-5) have been found for the fermion field strength [15], the massless, minimally coupled scalar exchange potential [16], the graviton mode function [17], and the Newtonian potential of a static point mass [18].

A fascinating feature of these results is the breakdown of perturbation theory that occurs when the large logarithms overwhelm the small loop-counting parameter of $GH^2 \lesssim 10^{-10}$. Graviton loop corrections do not necessarily become large at this point because each higher loop contributes an extra factor of the loop-counting parameter times a large temporal or spatial logarithm, and they all become of order one at the same time. Understanding what happens after the breakdown of perturbation theory requires a way of resumming the series of leading logarithms. Although Starobinsky's stochastic formalism [19, 20] solves the analogous problem for the large logarithms of scalar potential models [21, 22], the derivative interactions of quantum gravity cause it to fail [23]. The renormalization group is another obvious approach but the simplest realization of it fails even to describe scalar potential models [24]. This has remained true despite years of effort [25, 26, 27].

Nonlinear sigma models have the same derivative interactions as quantum gravity but without the indices and the gauge issue. It has long been suspected that they might provide a simplified venue for working out a procedure to sum the leading logarithms [22], and much work has been done with them on de Sitter background [28, 29, 30]. These efforts have recently resulted in a synthesis involving variants of Starobinsky's stochastic technique with the renormalization group [31]. What one does is to derive curvature-dependent effective potentials by integrating derivative interactions out of the field equations in the presence of a constant scalar background. The resulting equation is that of a scalar potential model for which Starobinsky's stochastic procedure applies [19, 20]. This recovers some large logarithms; the remaining ones come by employing the renormalization group on the subset of BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmermann) counterterms which can be regarded as curvature-dependent renormalizations of parameters in the original theory.

The purpose of this paper is to complete two parts of the previous analysis [31]. First, we compute the evolution of the scalar background at two loop order. This permits a nontrivial check of the stochastic prediction, which really is a prediction because the explicit calculation had not been done when it was made. The second thing we do here is to compute the 1-loop beta function, which was not needed to check the 1-loop exchange potentials. Having the beta function allows us to carry out a full RG analysis.

There are five sections in this paper. In section 2 we define the particular nonlinear sigma model to be studied and give the stochastic prediction for its background. The 2-loop expectation value of the background is computed in section 3. In section 4 we work out the 1-loop beta function. Our conclusions comprise section 5.

2 Feynman Rules

The purpose of this section is to explain how to make perturbative computations in the model. We begin by giving the bare Lagrangian, the propagators and the vertices. Then we present those counterterms which are needed for our work. The special counterterms which can be viewed as curvature-dependent renormalizations of the bare theory are distinguished. We next using a curvature-dependent effective potential to derive the stochastic prediction for the 2-loop expectation value which is computed in section 3. The section closes with a review of what has been shown in previous work, and what will be shown in this paper.

2.1 The Bare Theory

The Lagrangian we study consists of two scalar fields $A(x)$ and $B(x)$. Their Lagrangian is,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A \partial_\nu A g^{\mu\nu} \sqrt{-g} - \frac{1}{2}\left(1 + \frac{1}{2}\lambda A\right)^2 \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g}. \quad (6)$$

We work on D -dimensional spacetime in order to facilitate the use of dimensional regularization. Our notation exploits the conformal coordinate system evident in the background geometry (2). Because the metric consists of a scale factor $a(\eta) = -\frac{1}{H\eta}$ times the Minkowski metric, $g_{\mu\nu} = a^2 \eta_{\mu\nu}$, its inverse is $g^{\mu\nu} = \frac{1}{a^2} \eta^{\mu\nu}$, and the measure factor is $\sqrt{-g} = a^D$. The standard partial derivative is denoted as ∂_μ , no matter what tensor it acts upon, and its index is raised and lowered with the Minkowski metric, $\partial^\mu \equiv \eta^{\mu\nu} \partial_\nu$. The Minkowski contraction of the partial derivative with itself is written $\partial^2 \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$, again no matter what tensor it acts upon. When more than one coordinate is present we indicate which one is differentiated by adding a superscript or a subscript, as in $\partial_\mu^y \equiv \frac{\partial}{\partial y^\mu}$ and $\partial_z^\mu \equiv \eta^{\mu\nu} \frac{\partial}{\partial z^\nu}$. The same scheme is sometimes applied to the metric, as in $g_y^{\mu\nu} \equiv g^{\mu\nu}(y)$ and $\sqrt{-g_z} \equiv \sqrt{-g(z)}$. It can be employed as well for the scale factor, as in $a_y \equiv a(y^0)$ and $a_z \equiv a(z^0)$.

The propagators of both fields obey,

$$\partial_\mu \left[\sqrt{-g} g^{\mu\nu} \partial_\nu i\Delta(x; x') \right] = \partial^\mu \left[a^{D-2} \partial_\mu i\Delta(x; x') \right] \equiv \mathcal{D} i\Delta(x; x') = i\delta^D(x - x'). \quad (7)$$

The homogeneous and isotropic solution is [32, 33],

$$i\Delta(x; x') = F(\mathcal{Y}(x; x')) + k \ln(a(\eta)a(\eta')) \quad , \quad k \equiv \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}. \quad (8)$$

Here $\mathcal{Y}(x; x') \equiv a(\eta)a(\eta')H^2\Delta x^2(x; x') \equiv a(\eta)a(\eta')H^2(x - x')^\mu(x - x')^\nu\eta_{\mu\nu}$ is the de Sitter length function and the first derivative of the function $F(\mathcal{Y})$ is,

$$F'(\mathcal{Y}) = -\frac{H^{D-2}}{4(4\pi)^{\frac{D}{2}}} \left\{ \Gamma\left(\frac{D}{2}\right) \left(\frac{4}{\mathcal{Y}}\right)^{\frac{D}{2}} + \Gamma\left(\frac{D}{2}+1\right) \left(\frac{4}{\mathcal{Y}}\right)^{\frac{D}{2}-1} + \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+\frac{D}{2}+2)}{\Gamma(n+3)} \left(\frac{\mathcal{Y}}{4}\right)^{n-\frac{D}{2}+2} - \frac{\Gamma(n+D)}{\Gamma(n+\frac{D}{2}+1)} \left(\frac{\mathcal{Y}}{4}\right)^n \right] \right\}. \quad (9)$$

Using dimensional regularization, the coincidence limits of the propagator and its first two derivatives are seen to be,

$$i\Delta(x; x) = k \left[-\pi \cot\left(\frac{D\pi}{2}\right) + 2 \ln(a) \right] , \quad \partial_\mu i\Delta(x; x') \Big|_{x'=x} = k H a \delta^0_\mu , \quad (10)$$

$$\partial_\mu \partial_\nu i\Delta(x; x') \Big|_{x'=x} = -\left(\frac{D-1}{D}\right) k H^2 g_{\mu\nu} , \quad \partial_\mu i\Delta(x; x) = 2k H a \delta^0_\mu . \quad (11)$$

The bare vertices consist of the $\lambda A \partial B \partial B$ coupling and the $\lambda^2 A^2 \partial B \partial B$ coupling. Figure 1 gives a diagrammatic representation of the bare Feynman rules.

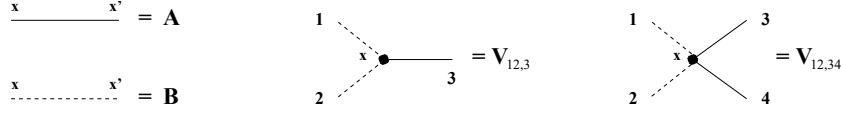


Figure 1: The bare Feynman rules of (6). A lines are solid and B lines are dashed. Both propagators are the same and are described by equations (8-9).

2.2 Counterterms

The Lagrangian (6) is not renormalizable so it requires an infinite number of counterterms. However, this paper only employs the two 1-loop counterterms which are shown in Figure 2.



Figure 2: Diagrammatic representation of the two counterterms we require for this project. The left hand graph represents expression (12) which renormalize the B self-mass and the right hand graph represents (16) which renormalize the 3-point vertex. A lines are solid and B lines are dashed.

The first of the counterterms we require renormalizes the B self-mass at 1-loop order,

$$\Delta \mathcal{L}_{B^2} = -\frac{1}{2} C_{1B^2} \square B \square B \sqrt{-g} - \frac{1}{2} C_{2B^2} R \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} . \quad (12)$$

Absorbing divergences in the B self-mass determines the coefficients C_{1B^2} and C_{2B^2} to be [31],

$$C_{1B^2} = -\frac{\lambda^2 \mu^{D-4}}{16\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)} , \quad (13)$$

$$C_{2B^2} = \frac{\lambda^2 \mu^{D-4}}{4(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \frac{\pi \cot(\frac{D\pi}{2})}{D(D-1)} - \frac{\lambda^2 \mu^{D-4}}{32\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)} \left(\frac{D-2}{D-1}\right) . \quad (14)$$

The term proportional to C_{1B^2} is a higher derivative counterterm which plays no role describing in large inflationary logarithms. In contrast, the term proportional to C_{2B^2} can be

viewed as a curvature-dependent field strength renormalization whose gamma function is,

$$Z_B = 1 + C_{2B^2} \times R + O(\lambda^4) \quad \Longrightarrow \quad \gamma_B \equiv \frac{\partial \ln(Z_B)}{\partial \ln(\mu^2)} = -\frac{\lambda^2 H^2}{32\pi^2} + O(\lambda^4) . \quad (15)$$

The second counterterm renormalizes the 3-point vertex at 1-loop order,

$$\begin{aligned} \Delta \mathcal{L}_{AB^2} = & -\frac{1}{2} C_{1AB^2} \square A \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} - C_{2AB^2} \partial_\mu A \partial_\nu B \square B g^{\mu\nu} \sqrt{-g} \\ & -\frac{1}{2} C_{3AB^2} A \square B \square B \sqrt{-g} - \frac{1}{2} C_{4AB^2} R A \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} . \end{aligned} \quad (16)$$

The terms involving C_{1AB^2} , C_{2AB^2} and C_{3AB^2} contain higher derivatives and have nothing to do with large inflationary logarithms. The term proportional to C_{4AB^2} can be viewed as a curvature-dependent renormalization of the $A\partial B\partial B$ vertex whose beta function we will compute at order λ^3 in section 5.

$$\delta\lambda = C_{4AB^2} \times R + O(\lambda^5) \quad \Longrightarrow \quad \beta \equiv \frac{\partial \delta\lambda}{\partial \ln(\mu)} . \quad (17)$$

2.3 Stochastic Prediction for $\langle A \rangle$

The Heisenberg operator equation for $A(x)$ is,

$$\frac{\delta S}{\delta A} = \partial_\mu \left[\sqrt{-g} g^{\mu\nu} \partial_\nu A \right] - \frac{1}{2} \lambda \left(1 + \frac{1}{2} \lambda A \right) \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} = 0 . \quad (18)$$

A constant A field merely changes the field strength of B . We can therefore capture the effect of undifferentiated A fields to all orders by integrating out the differentiated B fields in equation (18) for constant A ,

$$\partial_\mu B \partial_\nu B \longrightarrow \frac{\partial_\mu \partial'_\nu i \Delta(x; x')|_{x' \rightarrow x}}{(1 + \frac{\lambda}{2} A)^2} = -\frac{(\frac{D-1}{D}) k H^2 g_{\mu\nu}}{(1 + \frac{\lambda}{2} A)^2} \longrightarrow -\frac{\frac{3H^4}{32\pi^2} g_{\mu\nu}}{(1 + \frac{\lambda}{2} A)^2} . \quad (19)$$

Note that we have used expression (11) and the $D = 4$ limit of k from (8).

Substituting (19) in the A field equation (18) gives a scalar potential model which Starobinsky's formalism [19, 20] allows us to describe, at leading logarithm order, by a stochastic field $\mathcal{A}$ obeying the Langevin equation [31],

$$\dot{\mathcal{A}} - \dot{\mathcal{A}}_0 = \frac{\frac{\lambda H^3}{16\pi^2}}{1 + \frac{\lambda}{2} \mathcal{A}} . \quad (20)$$

Note that we have converted from conformal time η to the co-moving time $t = \ln(a)/H$, and dot means derivative with respect to co-moving time.. The stochastic jitter $\dot{\mathcal{A}}_0$ is provided by the time derivative of the infrared truncated, free field mode sum,

$$\mathcal{A}_0(t, \vec{x}) \equiv \int \frac{d^3 k}{(2\pi)^3} \frac{\theta(k-H)\theta(aH-k)H}{\sqrt{2k^3}} \left\{ \alpha_{\vec{k}} e^{i\vec{k} \cdot \vec{x}} + \alpha_{\vec{k}}^\dagger e^{-i\vec{k} \cdot \vec{x}} \right\} . \quad (21)$$

Here $\alpha_{\vec{k}}^\dagger$ and $\alpha_{\vec{k}}$ are the creation and annihilation operators of the field A ,

$$\left[\alpha_{\vec{k}}, \alpha_{\vec{p}}^\dagger\right] = (2\pi)^3 \delta^3(\vec{k}-\vec{p}) \quad , \quad \alpha_{\vec{k}}|\Omega\rangle = 0 \quad \implies \quad \langle\Omega|\mathcal{A}_0^2|\Omega\rangle = \frac{H^2 \ln(a)}{4\pi^2} . \quad (22)$$

Even if the stochastic jitter were absent, $\mathcal{A}(t, \vec{x})$ would roll down the curvature-dependent effective potential,

$$V_{\text{eff}}(A) = -\frac{3H^4}{8\pi^2} \ln\left|1 + \frac{\lambda}{2}A\right| . \quad (23)$$

If one starts from rest at $A = 0$ the exact solution for this behavior is,

$$A_{\text{class}} = \frac{2}{\lambda} \left[\sqrt{1 + \frac{\lambda^2 H^2 \ln(a)}{16\pi^2}} - 1 \right] . \quad (24)$$

The solution of the Langevin equation (20) consists of (24) plus a series in powers of $\mathcal{A}_0$,

$$\mathcal{A} = A_{\text{class}} + \mathcal{A}_0 - \frac{\lambda^2 H^3}{32\pi^2} \int_0^t dt' \mathcal{A}_0 + \frac{\lambda^3 H^3}{64\pi^2} \int_0^t dt' \mathcal{A}_0^2 + O(\lambda^4) . \quad (25)$$

Taking the expectation value gives,

$$\langle\Omega|\mathcal{A}|\Omega\rangle = \frac{\lambda H^2 \ln(a)}{2^4 \pi^2} + \frac{\lambda^3 H^4 \ln^2(a)}{2^{10} \pi^4} + O(\lambda^5) . \quad (26)$$

In the next section we will show that the 2-loop expectation value of the full field $A(x)$ agrees with this stochastic prediction at leading logarithm order.

2.4 What Has Been and Will Be Shown

The title of this paper mentions “unfinished business” because it completes an initial investigation of the model (6) [31]. The earlier study computed the 1-loop self-mass for each field and then used them to quantum-correct the effective field equations to solve for 1-loop corrections to the plane wave mode functions and to the response to a static point source. The fields and their squares were also computed at 2-loop order, with the exception of $A(x)$, whose expectation value was only worked out to 1-loop order. The leading logarithm contributions to the various results are summarized in Table 1 below.

Quantity	Leading Logarithms
$u_A(\eta, k)$	$\left\{1 - \frac{\lambda^2 H^2}{32\pi^2} \ln(a) + O(\lambda^4)\right\} \times \frac{H}{\sqrt{2}k^3}$
$u_B(\eta, k)$	$\left\{1 + 0 + O(\lambda^4)\right\} \times \frac{H}{\sqrt{2}k^3}$
$P_A(\eta, r)$	$\left\{1 - \frac{\lambda^2 H^2}{32\pi^2} \ln(a) + \frac{\lambda^2 H^2}{32\pi^2} \ln(Hr) + O(\lambda^4)\right\} \times \frac{KH}{4\pi} \ln(Hr)$
$P_B(\eta, r)$	$\left\{1 - \frac{\lambda^2 H^2}{32\pi^2} \ln(Hr) + O(\lambda^4)\right\} \times \frac{KH}{4\pi} \ln(Hr)$
$\langle \Omega A(x) \Omega \rangle$	$\left\{1 + O(\lambda^2)\right\} \times \frac{\lambda H^2}{16\pi^2} \ln(a)$
$\langle \Omega A^2(x) \Omega \rangle_{\text{ren}}$	$\left\{1 - \frac{\lambda^2 H^2}{64\pi^2} \ln(a) + O(\lambda^4)\right\} \times \frac{H^2}{4\pi^2} \ln(a)$
$\langle \Omega B(x) \Omega \rangle$	0
$\langle \Omega B^2(x) \Omega \rangle_{\text{ren}}$	$\left\{1 + \frac{3\lambda^2 H^2}{32\pi^2} \ln(a) + O(\lambda^4)\right\} \times \frac{H^2}{4\pi^2} \ln(a)$

Table 1: Color-coded explanations of large logarithms found in studying the 1-loop mode functions $u_{A,B}(\eta, k)$, the 1-loop exchange potentials $P_{A,B}(\eta, r)$ and the 2-loop expectation values of the fields and their squares [31]. Red denotes logarithms which have a stochastic explanation and green indicates those explained by the renormalization group.

As Table 1 indicates, the various leading logarithms can all be explained using a variant of Starobinsky’s stochastic formalism — based on curvature-dependent effective potentials as explained in section 2.3 — or using a variant of the renormalization group — based on curvature-dependent renormalizations of the bare theory, as explained in section 2.2. The goal of this paper is to derive the 2-loop (order λ^3) contribution to the expectation value of $A(x)$ — which will be done in section 3 — and to complete the renormalization group analysis by deriving the 1-loop beta function — which will be done in section 4. Under the usual assumption of no significant higher loop contributions, we will be able to solve the Callan-Symanzik equation.

3 The 2-Loop Expectation Value of $A(x)$

The point of this section is to compute the expectation value of $A(x)$ at 2-loop order, and to compare the leading logarithm result for this with the stochastic prediction (26). The various diagrams are given in Figure 3.

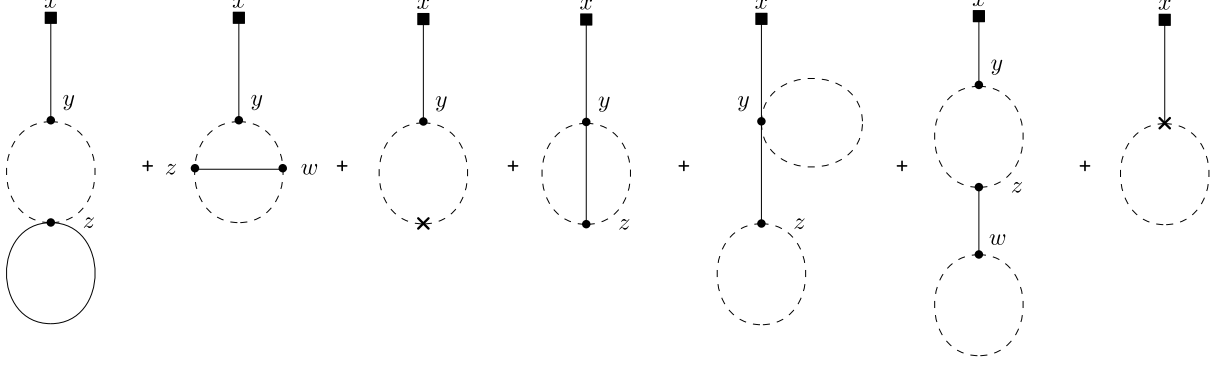


Figure 3: Diagrams contributing to the 2-loop expectation value of $A(x)$. Solid lines represent the A propagator whereas dashed lines stand for the B propagators.

For each diagram we give the initial expression. An explicit partial integration reduction is presented for the first diagram, and the final reductions for each diagram are expressed in terms of $i\Delta(x; x)$, $[i\Delta(x; x)]^2$ and five integrals,

$$I_1 \equiv \int d^D y \sqrt{-g(y)} [i\Delta(x; y)]^2, \quad (27)$$

$$I_2 \equiv \int d^D y \sqrt{-g(y)} i\Delta(x; y) i\Delta(y; y), \quad I_3 \equiv \int d^D y \sqrt{-g(y)} i\Delta(x; y), \quad (28)$$

$$I_4 \equiv \int d^D y \sqrt{-g(y)} i\Delta(x; y) \times I_3(y), \quad I_5 \equiv \int d^D y \sqrt{-g(y)} i\Delta(x; y) \times \dot{I}_3(y). \quad (29)$$

(Note the co-moving time derivative $\partial_t \equiv 1/a \times \partial_0$ of I_3 in I_5 .) The proper treatment of these integrals requires generalizing in-out matrix elements to true expectation values using the Schwinger-Keldysh formalism [34, 35, 36, 37, 38] and is explained in the Appendix. The section closes by expressing the leading logarithm contribution from each diagram as a factor times the 2-loop stochastic result $S \equiv \lambda^3 H^4 \ln^2(a)/2^{10}\pi^4$.

3.1 Computation of the Diagrams

The first 2-loop diagram in Figure 3 has a symmetry factor of $\frac{1}{4}$, and a combination of 3-point and 4-point vertices,

$$A_{2a} \equiv \frac{1}{4}(-i\lambda)(-\frac{i}{2}\lambda^2) \int d^D y \sqrt{-g(y)} g^{\alpha\beta}(y) i\Delta(x; y) \\ \times \int d^D z \sqrt{-g(z)} g^{\mu\nu}(z) \partial_\alpha^y \partial_\mu^z i\Delta(y; z) \partial_\beta^y \partial_\nu^z i\Delta(y; z) i\Delta(z; z). \quad (30)$$

To conserve space in subsequent expressions we will use the Latin letters as superscripts or subscripts to indicate the spacetime arguments of the metric, for example, $\sqrt{g_y} \equiv \sqrt{-g(y)}$ and $g_z^{\mu\nu} \equiv g^{\mu\nu}(z)$. Also, we use $\partial_\mu^y \equiv \frac{\partial}{\partial y^\mu}$ for partial derivatives. We reduce A_{2a} , and all the other contributions, by partially integrating derivatives and then using either the propagator equation (7) or else one of the coincidence limits (10-11). The first step is to

partially integrate the factor of ∂'_α on the last line of (30),

$$A_{2a} = \frac{\lambda^3}{8} \int d^D y i\Delta(x; y) \int d^D z \sqrt{-g_z} g_z^{\mu\nu} \partial_\mu^z i\Delta(y; z) \partial_\nu^z \mathcal{D}_y i\Delta(y; z) i\Delta(z; z) \\ + \frac{\lambda^3}{8} \int d^D y \sqrt{-g_y} g_y^{\alpha\beta} \partial_\alpha^y i\Delta(x; y) \int d^D z \sqrt{-g_z} g_z^{\mu\nu} \partial_\mu^z i\Delta(y; z) \partial_\beta^y \partial_\nu^z i\Delta(y; z) i\Delta(z; z) . \quad (31)$$

Recall that $\mathcal{D}_y \equiv \partial_\alpha^y [\sqrt{-g_y} g_y^{\alpha\beta} \partial_\beta^y]$.

We can use the propagator equation (7) to replace $\mathcal{D}_y i\Delta(y; z)$ on the first line of (31) with $i\delta^D(y - z)$. This permits the integration over y to be performed,

$$A_{2a1} \equiv \frac{\lambda^3}{8} \int d^D y i\Delta(x; y) \int d^D z \sqrt{-g_z} g_z^{\mu\nu} \partial_\mu^z i\Delta(y; z) \partial_\nu^z i\delta^D(y - z) i\Delta(z; z) \\ = \frac{i\lambda^3}{8} \int d^D z \sqrt{-g_z} g_z^{\mu\nu} \partial_\nu^y \left[i\Delta(x; y) \partial_\mu^z i\Delta(y; z) \right]_{y=z} i\Delta(z; z) . \quad (32)$$

The differentiated square bracket can be evaluated using (10-11),

$$\partial_\nu^y \left[i\Delta(x; y) \partial_\mu^z i\Delta(y; z) \right]_{y=z} \\ = \partial_\nu^z i\Delta(x; z) \partial_\mu^z i\Delta(y; z) \Big|_{y=z} + i\Delta(x; z) \partial_\nu^y \partial_\mu^z i\Delta(y; z) \Big|_{y=z} , \quad (33)$$

$$= \partial_\nu^z i\Delta(x; z) \times k H a_z \delta_\mu^0 + i\Delta(x; z) \times -\left(\frac{D-1}{D}\right) k H^2 g_{\mu\nu}^z . \quad (34)$$

Substituting (34) in (32) with $\sqrt{-g_z} g_z^{\mu\nu} = a_z^{D-2} \eta^{\mu\nu}$, partially integrating, and then making use of (11) completes the reduction of A_{2a1} ,

$$A_{2a1} = -\frac{i\lambda^3}{8} k H \int d^D z a_z^{D-1} \partial_0^z i\Delta(x; z) i\Delta(z; z) \\ - \frac{i\lambda^3}{8} (D-1) k H^2 \int d^D z \sqrt{-g_z} i\Delta(x; z) i\Delta(z; z) , \quad (35)$$

$$= \frac{i\lambda^3}{8} k H \int d^D z a_z^{D-1} i\Delta(x; z) \partial_0^z i\Delta(z; z) , \quad (36)$$

$$= \frac{i\lambda^3}{4} k^2 H^2 \int d^D z \sqrt{-g_z} i\Delta(x; z) \longrightarrow \frac{i\lambda^3}{4} k^2 H^2 \times I_3 . \quad (37)$$

The reduction of the term on the last line of (31) begins by exploiting symmetry to write,

$$\sqrt{-g_z} g_z^{\mu\nu} \partial_\mu^z i\Delta(y; z) \partial_\beta^y \partial_\nu^z i\Delta(y; z) = \frac{1}{2} \partial_\beta^y \left[\sqrt{-g_z} g_z^{\mu\nu} \partial_\mu^z i\Delta(y; z) \partial_\nu^z i\Delta(y; z) \right] . \quad (38)$$

We then partially integrate the ∂_β^y and use the propagator equation (7),

$$A_{2a2} \equiv -\frac{\lambda^3}{16} \int d^D y \sqrt{-g_y} \mathcal{D}_y i\Delta(x; y) \int d^D z \sqrt{-g_z} g_z^{\mu\nu} \partial_\mu^z i\Delta(y; z) \partial_\nu^z i\Delta(y; z) i\Delta(z; z) \\ = -\frac{i\lambda^3}{16} \int d^D z \sqrt{-g_z} g_z^{\mu\nu} \partial_\mu^z i\Delta(x; z) \partial_\nu^z i\Delta(x; z) i\Delta(z; z) . \quad (39)$$

Partially integrating the ∂_μ^z and exploiting relations (7) and (11) gives,

$$A_{2a2} = \frac{i\lambda^3}{16} \int d^D z i\Delta(x; z) \mathcal{D}_z i\Delta(x; z) i\Delta(z; z) + \frac{i\lambda^3}{16} \int d^D z \sqrt{-g_z} g_z^{\mu\nu} i\Delta(x; z) \partial_\nu^z i\Delta(x; z) \partial_\mu^z i\Delta(z; z) , \quad (40)$$

$$= -\frac{\lambda^3}{16} \left[i\Delta(x; x) \right]^2 - \frac{i\lambda^3}{8} kH \int d^D z a_z^{D-1} i\Delta(x; z) \partial_0^z i\Delta(x; z) . \quad (41)$$

Now note that,

$$i\Delta(x; z) \partial_0^z i\Delta(x; z) = \frac{1}{2} \partial_0^z \left[i\Delta(x; z) \right]^2 , \quad (42)$$

and partially integrate to reach the form,

$$A_{2a2} = -\frac{\lambda^3}{16} \left[i\Delta(x; x) \right]^2 + \frac{i\lambda^3}{16} (D-1) kH^2 \times I_1 . \quad (43)$$

Combining relations (37) and (43) gives the total for the first diagram,

$$A_{2a} = -\frac{\lambda^3}{16} \left[i\Delta(x; x) \right]^2 + \frac{i\lambda^3}{16} (D-1) kH^2 \times I_1 + \frac{i\lambda^3}{4} k^2 H^2 \times I_3 . \quad (44)$$

For the 2nd diagram of Figure 3 from left, it has a symmetry factor of $\frac{1}{2}$, and three 3-point vertices,

$$A_{2b} \equiv \frac{1}{2} (-i\lambda)^3 \int d^D y \sqrt{-g_y} g_y^{\alpha\beta} i\Delta(x; y) \int d^D z \sqrt{-g_z} g_z^{\mu\nu} \partial_\alpha^y \partial_\mu^y i\Delta(y; z) \times \int d^D w \sqrt{-g_w} \partial_\beta^y \partial_\mu^z i\Delta(y; w) \times \partial_\nu^z \partial_\sigma^w i\Delta(z; z) i\Delta(z; w) . \quad (45)$$

Applying the same sorts of reductions as for A_{2a} we at length reach the form,

$$A_{2b} = \frac{3\lambda^3}{16} \left[i\Delta(x; x) \right]^2 - \frac{i\lambda^3}{8} (D-1) kH^2 \times I_1 - \frac{3i\lambda^3}{4} k^2 H^2 \times I_3 . \quad (46)$$

Now, we have the 3rd diagram of Figure 3 next. It has a symmetry factor of $\frac{1}{2}$ and combines a 3-point vertex with and an insertion of the 2-point counterterm (12),

$$A_{2c} \equiv \frac{1}{2} (-i\lambda) (-iC_{2B^2}) \int d^D y \sqrt{-g_y} g_y^{\alpha\beta} i\Delta(x; y) \times \int d^D z \sqrt{-g_z} g_z^{\mu\nu} \partial_\alpha^y \partial_\mu^y i\Delta(y; z) \partial_\beta^y \partial_\nu^z i\Delta(y; z) R . \quad (47)$$

Note that the counterterm proportional to C_{1B^2} vanishes in dimensional regularization. Note also that the Ricci scalar $R = D(D-1)H^2$ is constant on de Sitter background. After some straightforward reductions we reach the form,

$$A_{2c} = -\frac{\lambda C_{2B^2}}{4} R i\Delta(x; x) . \quad (48)$$

The 4th diagram of Figure 3 has a symmetry factor of $\frac{1}{2}$ and combines 4-point and 3-point vertices,

$$A_{2d} \equiv \frac{1}{2} \left(-\frac{i\lambda^2}{2} \right) (-i\lambda) \int d^D y \sqrt{-g_y} g_y^{\alpha\beta} i\Delta(x; y) \times \int d^D z \sqrt{-g_z} g_z^{\mu\nu} \partial_\alpha^y \partial_\mu^z i\Delta(y; z) \partial_\beta^y \partial_\nu^z i\Delta(y; z) i\Delta(y; z) . \quad (49)$$

After many reductions we find,

$$A_{2d} = -\frac{i\lambda^3}{4} (D-1) k H^2 \times I_2 - \frac{i\lambda^3}{8} k^2 H^2 \times I_3 . \quad (50)$$

The 5th of the diagrams in Figure 3 has a symmetry factor of $\frac{1}{4}$ and also involves 4-point and 3-point vertices,

$$A_{2e} \equiv \frac{1}{4} \left(-\frac{i\lambda^2}{2} \right) (-i\lambda) \int d^D y \sqrt{-g_y} g_y^{\alpha\beta} i\Delta(x; y) \times \partial_\alpha^y \partial_\beta^z i\Delta(y; z) \Big|_{z=y} \times \int d^D w \sqrt{-g_w} g_w^{\mu\nu} i\Delta(y; z) \times \partial_\mu^w \partial_\nu^v i\Delta(w; v) \Big|_{v=w} . \quad (51)$$

Two applications of (11) reduce this diagram to,

$$A_{2e} = -\frac{\lambda^3}{8} (D-1)^2 k^2 H^4 \times I_4 . \quad (52)$$

The penultimate diagram from Figure 3 has a symmetry factor of $\frac{1}{4}$ and involves three 3-point vertices,

$$A_{2f} \equiv \frac{1}{4} (-i\lambda)^3 \int d^D y \sqrt{-g_y} g_y^{\alpha\beta} i\Delta(x; y) \int d^D z \sqrt{-g_z} g_z^{\mu\nu} \partial_\alpha^y \partial_\mu^z i\Delta(y; z) \partial_\beta^y \partial_\nu^z i\Delta(y; z) \times \int d^D w \sqrt{-g_w} g_w^{\rho\sigma} i\Delta(z; w) \times \partial_\rho^w \partial_\sigma^v i\Delta(w; v) \Big|_{v=w} . \quad (53)$$

A long series of reductions yields the form,

$$A_{2f} = \frac{i\lambda^3}{16} (D-1) k H^2 \times I_1 - \frac{i\lambda^3}{8} (D-1) k H^2 \times i\Delta(x; x) \times I_3 - \frac{\lambda^3}{4} (D-1) k^2 H^3 \times I_5 . \quad (54)$$

Lastly, we have the last diagram of Figure 3. It has a symmetry factor of $\frac{1}{2}$ and involves the 3-point counterterm (16),

$$A_{2g} = \frac{1}{2} (-iC_{1AB^2}) \int d^D y \mathcal{D}_y i\Delta(x; y) \times g_y^{\alpha\beta} \partial_\alpha^y \partial_\beta^z i\Delta(y; z) \Big|_{z=y} + 0 + 0 + \frac{1}{2} (-iC_{4AB^2}) \int d^D y \sqrt{-g_y} g_y^{\alpha\beta} i\Delta(x; y) \times R \partial_\alpha^y \partial_\beta^z i\Delta(y; z) \Big|_{z=y} . \quad (55)$$

Application of (7) and (11) gives,

$$A_{2g} = -\frac{C_{1AB^2}}{2} (D-1) k H^2 + \frac{iC_{4AB^2}}{2} (D-1) k H^2 \times I_3 . \quad (56)$$

We now extract the leading logarithm contributions from each diagram (44), (46), (48), (50), (52), (54) and (56). Of course the coincidence limit of the propagator and its square are,

$$i\Delta(x; x) \longrightarrow \frac{H^2}{4\pi^2} \times \ln(a) \quad , \quad \left[i\Delta(x; x) \right]^2 \longrightarrow \frac{H^4}{16\pi^4} \times \ln^2(a) . \quad (57)$$

In the Appendix we show that the leading logarithm contributions from the five integrals (27-29) are,

$$I_1 \longrightarrow -\frac{i}{12\pi^2} \times \ln^2(a) \quad , \quad (58)$$

$$I_2 \longrightarrow -\frac{i}{24\pi^2} \times \ln^2(a) \quad , \quad I_3 \longrightarrow -\frac{i}{3H^2} \times \ln(a) , \quad (59)$$

$$I_4 \longrightarrow -\frac{1}{18H^4} \times \ln^2(a) \quad , \quad I_5 \longrightarrow -\frac{1}{9H^3} \times \ln(a) . \quad (60)$$

Using relations (57) and (58-60) we can express the leading logarithm contribution from each of the seven diagrams in Figure 3 as a number times the stochastic prediction of $S = \lambda^3 H^4 \ln^2(a) / 2^{10} \pi^4$,

$$A_{2a} \longrightarrow -2 \times S \quad , \quad A_{2b} \longrightarrow +8 \times S , \quad (61)$$

$$A_{2c} \longrightarrow +0 \times S \quad , \quad A_{2d} \longrightarrow -4 \times S , \quad (62)$$

$$A_{2e} \longrightarrow +1 \times S \quad , \quad A_{2f} \longrightarrow -2 \times S , \quad (63)$$

$$A_{2g} \longrightarrow +0 \times S \quad . \quad (64)$$

Even if we discount the counterterm diagrams A_{2c} and A_{2g} , which cannot contribute at leading logarithm order, the fact that the remaining five diagrams contrive to add up to $+1 \times S$ represents an impressive confirmation of the stochastic prediction (26).

3.2 Implications

The obvious implication is that the stochastic prediction of section 2.3 is correct to all orders. Because this prediction consists of a “classical” evolution (24), which is accelerated by stochastic jitter, we know that the background rolls down its effective potential (23) for all time, reaching arbitrarily large values. Note that this model provides an explicit contradiction to the view which is sometimes expressed that large inflationary logarithms must always sum up to produce a static, de Sitter invariant result at late times.

4 The 1-Loop Beta Function

The purpose of this section is to compute the 1-loop beta function (17). This requires renormalizing the AB^2 vertex $-iV(x; y; z)$, the diagrams for which are shown in Figure 4.

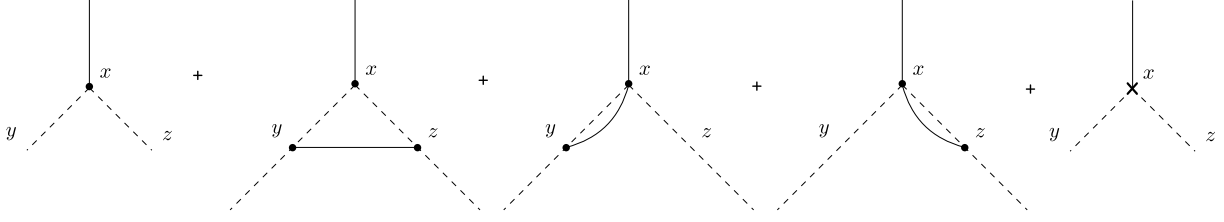


Figure 4: Diagrams contributing to the AB^2 vertex $-iV(x; y; z)$ at tree and 1-loop orders. Recall that A lines are solid whereas B lines are dashed.

We first use partial integration to reduce each diagram as much as possible. Then derivatives are extracted from products of propagators so as to make the result integrable in $D = 4$ dimensions,

$$\int d^D y F(y) \int d^D z G(z) \times -iV(x; y; z) , \quad (65)$$

where $F(y)$ and $G(z)$ are smooth functions. Actually obtaining a finite result requires adding zero in the form,

$$\partial^2 \left[\frac{1}{\Delta x^{D-2}} \right] - \frac{4\pi^{\frac{D}{2}} i \delta^D(\Delta x)}{\Gamma(\frac{D}{2}-1)} = 0 , \quad (66)$$

where Δx^μ is either $(x - y)^\mu$ or $(x - z)^\mu$. This localizes the divergences so that they can be removed by the counterterm (16). We then compute the 1-loop beta function, and employ the renormalization group to solve the Callan-Symanzik equation to all orders.

4.1 Partial Integration Reductions

We start with the first diagrams of Figure 4. It is the tree order vertex,

$$-iV_0(x; y; z) = -i\lambda \sqrt{-g_x} g_x^{\mu\nu} \partial_\mu \delta^D(x-y) \partial_\nu \delta^D(x-z) . \quad (67)$$

The 2nd diagram of Figure 4 gives the most complicated of the 1-loop contributions,

$$\begin{aligned} -iV_{1a}(x; y; z) = & (-i\lambda)^3 \left(-\frac{\partial}{\partial y^\sigma} \right) \left(-\frac{\partial}{\partial z^\beta} \right) \left[\sqrt{-g_x} g_x^{\mu\nu} \sqrt{-g_y} g_y^{\rho\sigma} \partial_\mu^x \partial_\rho^y i\Delta(x; y) \right. \\ & \left. \times \sqrt{-g_z} g_z^{\alpha\beta} \partial_\nu^x \partial_\alpha^z i\Delta(x; z) i\Delta(y; z) \right] . \quad (68) \end{aligned}$$

Note that the derivatives on the external y^σ and z^β legs are partially integrated to act back on the entire diagram. By applying the same partial integration techniques we used in the previous section, and recalling that $\mathcal{D} \equiv \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu]$, one can reduce $-iV_{1a}$ to the form,

$$\begin{aligned} -iV_{1a}(x; y; z) = & \frac{i\lambda^3}{2} \mathcal{D}_x \partial_\sigma^y \partial_\beta^z \left\{ \sqrt{-g_y} g_y^{\rho\sigma} \partial_\rho^y i\Delta(x; y) \sqrt{-g_z} g_z^{\alpha\beta} \partial_\alpha^z i\Delta(x; z) i\Delta(y; z) \right\} \\ & + \frac{\lambda^3}{4} \mathcal{D}_y \mathcal{D}_z \left\{ \left[i\Delta(y; z) \right]^2 \left[\delta^D(x-y) + \delta^D(x-z) \right] \right\} \\ & - \frac{\lambda^3}{2} \partial_\sigma^y \partial_\beta^z \left\{ \sqrt{-g_y} g_y^{\rho\sigma} \partial_\rho^y i\Delta(y; z) \sqrt{-g_z} g_z^{\alpha\beta} \partial_\alpha^z i\Delta(y; z) \left[\delta^D(x-y) + \delta^D(x-z) \right] \right\} . \quad (69) \end{aligned}$$

The 3rd and 4th diagrams of Figure 4 are the next ones that we reduce. The 3rd diagram has a 4-point vertex at x and a 3-point vertex at y , with the external derivative with respect to y^σ partially integrated back on the whole diagram,

$$-iV_{1b}(x; y; z) = -i\lambda \left(-\frac{i\lambda^2}{2}\right) \left(-\frac{\partial}{\partial y^\sigma}\right) \left[\sqrt{-g_x} g_x^{\mu\nu} \sqrt{-g_y} g_y^{\rho\sigma} \partial_\mu^x \partial_\rho^y i\Delta(x; y) \right. \\ \left. \times i\Delta(x; y) \partial_\nu \delta^D(x-z) \right]. \quad (70)$$

Partial integration reduces this to,

$$-iV_{1b}(x; y; z) = \frac{\lambda^3}{4} \mathcal{D}_y \left\{ \sqrt{-g_x} g_x^{\mu\nu} \partial_\mu^x \left[i\Delta(x; y) \right]^2 \partial_\nu \delta^D(x-z) \right\} \\ - \frac{\lambda^3}{2} \partial_\sigma^y \left\{ \sqrt{-g_x} g_x^{\mu\nu} \partial_\mu^x i\Delta(x; y) \sqrt{-g_y} g_y^{\rho\sigma} \partial_\rho^y i\Delta(x; y) \partial_\nu \delta^D(x-z) \right\}. \quad (71)$$

The 4th diagram has a 4-point vertex at x and a 3-point vertex at z , with the external derivative with respect to z^β acted back on everything,

$$-iV_{1c}(x; y; z) = -i\lambda \left(-\frac{i\lambda^2}{2}\right) \left(-\frac{\partial}{\partial z^\beta}\right) \left[\sqrt{-g_x} g_x^{\mu\nu} \sqrt{-g_z} g_z^{\alpha\beta} \partial_\mu^x \partial_\alpha^z i\Delta(x; z) \right. \\ \left. \times i\Delta(x; z) \partial_\nu \delta^D(x-y) \right]. \quad (72)$$

The same partial integrations give,

$$-iV_{1c}(x; y; z) = \frac{\lambda^3}{4} \mathcal{D}_z \left\{ \sqrt{-g_x} g_x^{\mu\nu} \partial_\mu^x \left[i\Delta(x; z) \right]^2 \partial_\nu \delta^D(x-y) \right\} \\ - \frac{\lambda^3}{2} \partial_\beta^z \left\{ \sqrt{-g_x} g_x^{\mu\nu} \partial_\mu^x i\Delta(x; z) \sqrt{-g_z} g_z^{\alpha\beta} \partial_\alpha^z i\Delta(x; z) \partial_\nu \delta^D(x-y) \right\}. \quad (73)$$

It turns out that the last lines of expressions (69), (71) and (73) cancel. To see this, use the delta function on the last line of (73) to flip the x derivative to a y derivative, then extract the derivative and use the delta function to convert x to y ,

$$- \frac{\lambda^3}{2} \partial_\beta^z \left\{ \sqrt{-g_x} g_x^{\mu\nu} \partial_\mu^x i\Delta(x; z) \sqrt{-g_z} g_z^{\alpha\beta} \partial_\alpha^z i\Delta(x; z) \partial_\nu \delta^D(x-y) \right\} \\ = \frac{\lambda^3}{2} \partial_\nu^y \partial_\beta^z \left\{ \sqrt{-g_x} g_x^{\mu\nu} \partial_\mu^x i\Delta(x; z) \sqrt{-g_z} g_z^{\alpha\beta} \partial_\alpha^z i\Delta(x; z) \delta^D(x-y) \right\}, \quad (74)$$

$$= \frac{\lambda^3}{2} \partial_\sigma^y \partial_\beta^z \left\{ \sqrt{-g_y} g_y^{\rho\sigma} \partial_\rho^y i\Delta(y; z) \sqrt{-g_z} g_z^{\alpha\beta} \partial_\alpha^z i\Delta(y; z) \delta^D(x-y) \right\}. \quad (75)$$

Similar manipulations result in partial cancellations between the first lines of (71) and (73) and the second line of (69). The resulting sum of all 1-loop primitive contributions to the vertex is,

$$-iV_{1\text{prim}} = \frac{i\lambda^3}{2} \mathcal{D}_x \partial_y^\rho \partial_z^\alpha \left\{ (a_y a_z)^{D-2} i\Delta(y; z) \partial_\rho^y i\Delta(x; y) \partial_\alpha^z i\Delta(x; z) \right\} \\ + \frac{\lambda^3}{4} \mathcal{D}_y \partial_z^\alpha \left\{ a_z^{D-2} \left[i\Delta(y; z) \right]^2 \partial_\alpha^z \delta^D(x-z) \right\} + \frac{\lambda^3}{4} \mathcal{D}_z \partial_y^\rho \left\{ a_y^{D-2} \left[i\Delta(y; z) \right]^2 \partial_\rho^y \delta^D(x-y) \right\}. \quad (76)$$

4.2 Extracting Divergences

The three curly bracketed expressions of the primitive contribution (76) are each logarithmically divergent. This means that divergences derive entirely from the first term in the expansion (8-9) of the propagator,

$$i\Delta(y; z) = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}} \frac{1}{[a_y a_z (y-z)^2]^{\frac{D}{2}-1}} + \dots \quad (77)$$

For example, the second and third terms of (76) involve the propagator squared,

$$\left[i\Delta(y; z) \right]^2 = \frac{\Gamma^2(\frac{D}{2}-1)}{16\pi^D} \frac{1}{[a_y a_z (y-z)^2]^{D-2}} + \text{UV Finite}, \quad (78)$$

$$\longrightarrow \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}} \frac{\mu^{D-4}}{2(D-3)(D-4)} \frac{i\delta^D(y-z)}{(a_y a_z)^{D-2}} - \frac{\partial^2}{64\pi^4 (a_y a_z)^2} \left[\frac{\ln[\mu^2(y-z)^2]}{(y-z)^2} \right] + \text{UV Finite}. \quad (79)$$

The key to extracting the divergence of the first term in (76) is noting that it must be proportional to the Minkowski metric,

$$(a_y a_z)^{D-2} i\Delta(y; z) \partial_\rho^y i\Delta(x; y) \partial_\alpha^z i\Delta(x; z) \\ = \frac{\Gamma^3(\frac{D}{2}-1)}{64\pi^{\frac{3D}{2}}} \frac{1}{a_x^{D-2} (y-z)^{D-2}} \frac{\partial}{\partial y^\rho} \left[\frac{1}{(x-y)^{D-2}} \right] \frac{\partial}{\partial z^\alpha} \left[\frac{1}{(x-z)^{D-2}} \right] + \text{UV Finite}, \quad (80)$$

$$= \frac{\Gamma^3(\frac{D}{2}-1)}{64\pi^{\frac{3D}{2}}} \frac{\frac{1}{D} \eta_{\rho\alpha} \eta^{\mu\nu}}{a_x^{D-2} (y-z)^{D-2}} \frac{\partial}{\partial x^\mu} \left[\frac{1}{(x-y)^{D-2}} \right] \frac{\partial}{\partial x^\nu} \left[\frac{1}{(x-z)^{D-2}} \right] + \text{UV Finite}. \quad (81)$$

It is simple to extract a d'Alembertian from the contracted derivatives,

$$\partial^\mu \left[\frac{1}{(x-y)^{D-2}} \right] \partial_\mu \left[\frac{1}{(x-z)^{D-2}} \right] = \frac{\partial^2}{2} \left[\frac{1}{(x-y)^{D-2}} \frac{1}{(x-z)^{D-2}} \right] \\ - \frac{\partial^2}{2} \left[\frac{1}{(x-y)^{D-2}} \right] \frac{1}{(x-z)^{D-2}} - \frac{1}{(x-y)^{D-2}} \frac{\partial^2}{2} \left[\frac{1}{(x-z)^{D-2}} \right], \quad (82)$$

$$= \frac{\partial^2}{2} \left[\frac{1}{(x-y)^{D-2}} \frac{1}{(x-z)^{D-2}} \right] - \frac{2\pi^{\frac{D}{2}} i\delta^D(x-y)}{\Gamma(\frac{D}{2}-1)(x-z)^{D-2}} - \frac{2\pi^{\frac{D}{2}} i\delta^D(x-z)}{\Gamma(\frac{D}{2}-1)(x-y)^{D-2}}. \quad (83)$$

Substituting (83) into (81) allows us to isolate and localize the divergence,

$$(a_y a_z)^{D-2} i\Delta(y; z) \partial_\rho^y i\Delta(x; y) \partial_\alpha^z i\Delta(x; z) = -\frac{\Gamma^2(\frac{D}{2}-1)}{32\pi^D} \frac{\frac{1}{D}\eta_{\rho\alpha}}{a_x^{D-2}} \left\{ \frac{i\delta^D(x-y)}{(x-z)^{2D-4}} + \frac{i\delta^D(x-z)}{(x-y)^{2D-4}} \right\} + \text{UV Finite} , \quad (84)$$

$$= \frac{\Gamma(\frac{D}{2}-1)}{8\pi^{\frac{D}{2}}} \frac{\mu^{D-4}\eta_{\rho\alpha}}{D(D-3)(D-4)} \frac{\delta^D(x-y)\delta^D(x-z)}{a_x^{D-2}} + \text{UV Finite} . \quad (85)$$

The total divergent part of the primitive contribution to $-iV_1(x; y; z)$ comes from substituting expressions (79) and (85) in (76),

$$\begin{aligned} -iV_{1\text{div}} = & \frac{i\lambda^3}{2} \mathcal{D}_x \partial_y^\rho \partial_z^\alpha \left\{ \frac{\Gamma(\frac{D}{2}-1)}{8\pi^{\frac{D}{2}}} \frac{\mu^{D-4}\eta_{\rho\alpha}}{D(D-3)(D-4)} \frac{\delta^D(x-y)\delta^D(x-z)}{a_x^{D-2}} \right\} \\ & + \frac{\lambda^3}{4} \mathcal{D}_y \partial_z^\alpha \left\{ a_z^{D-2} \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}} \frac{\mu^{D-4}}{2(D-3)(D-4)} \frac{i\delta^D(y-z)}{(a_y a_z)^{D-2}} \partial_\alpha^z \delta^D(x-z) \right\} \\ & + \frac{\lambda^3}{4} \mathcal{D}_z \partial_y^\rho \left\{ a_y^{D-2} \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}} \frac{\mu^{D-4}}{2(D-3)(D-4)} \frac{i\delta^D(y-z)}{(a_y a_z)^{D-2}} \partial_\rho^y \delta^D(x-y) \right\} . \quad (86) \end{aligned}$$

This expression can be simplified considerably by using the delta functions to reflect derivatives and to evaluate the free scale factors at x . The final result is,

$$\begin{aligned} -iV_{1\text{div}} = & \frac{i\lambda^3 \mu^{D-4}}{16\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)} \left\{ \frac{2}{D} \mathcal{D}_x \left[\frac{\partial^\mu \delta^D(x-y) \partial_\mu \delta^D(x-z)}{a_x^{D-2}} \right] \right. \\ & \left. + \mathcal{D}_y \partial_x^\mu \left[\frac{\delta^D(x-y) \partial_\mu \delta^D(x-z)}{a_x^{D-2}} \right] + \mathcal{D}_z \partial_x^\mu \left[\frac{\partial_\mu \delta^D(x-y) \delta^D(x-z)}{a_x^{D-2}} \right] \right\} . \quad (87) \end{aligned}$$

4.3 The Beta Function

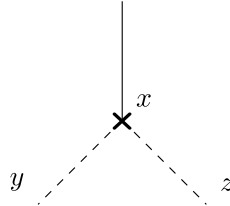


Figure 5: The final diagram on Figure 4 which represents an insertion of the counterterm (16).

The primitive divergence (87) is canceled using the general 1-loop counterterm (16). Varying the counter-action gives,

$$\begin{aligned} \frac{i\delta^3 S_{AB^2}[A, B]}{\delta A(x)\delta B(y)\delta B(z)} = & -iC_{1AB^2}\mathcal{D}_x\left[\frac{\partial^\mu\delta^D(x-y)\partial_\mu\delta^D(x-z)}{a_x^2}\right] \\ & +iC_{2AB^2}\mathcal{D}_y\partial_x^\mu\left[\frac{\delta^D(x-y)\partial_\mu\delta^D(x-z)}{a_x^2}\right] +iC_{2AB^2}\mathcal{D}_z\partial_x^\mu\left[\frac{\partial_\mu\delta^D(x-y)\delta^D(x-z)}{a_x^2}\right] \\ & -iC_{3AB^2}\mathcal{D}_y\mathcal{D}_z\left[\frac{\delta^D(x-y)\delta^D(x-z)}{a_x^D}\right] -iC_{4AB^2}Ra_x^{D-2}\partial^\mu\delta^D(x-y)\partial_\mu\delta^D(x-z) . \end{aligned} \quad (88)$$

Comparison of expressions (87) and (88) shows that the four coefficients are,

$$C_{1AB^2} = \frac{\lambda^3\mu^{D-4}}{16\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}-1)}{D(D-3)(D-4)} , \quad (89)$$

$$C_{2AB^2} = -\frac{\lambda^3\mu^{D-4}}{16\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)} , \quad (90)$$

$$C_{3AB^2} = 0 , \quad (91)$$

$$C_{4AB^2} = 0 . \quad (92)$$

Hence there is no curvature-dependent coupling constant renormalization at 1-loop and the beta function is zero at order λ^3 ,

$$\delta\lambda = C_{4AB^2} \times R + O(\lambda^5) \quad \implies \quad \beta = \frac{\partial\delta\lambda}{\partial\ln(\mu)} = O(\lambda^5) . \quad (93)$$

4.4 Implications

Previous work determined the 1-loop γ function for A [31]. We now have the 1-loop β function, so we can use the Callan-Symanzik equation to study how the n -point Green's functions vary with respect to changes in the renormalization scale μ ,

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right]G_n(x_1; \dots; x_n; \lambda, \mu) = 0 . \quad (94)$$

(The equation for a 1PI n -point function is the same, with the $+n\gamma$ term changed to $-n\gamma$.) One solves (94) by the method of characteristics. First, find a running coupling constant $\bar{\lambda}(\mu)$ such that,

$$\mu\frac{d\bar{\lambda}}{d\mu} = -\beta(\bar{\lambda}(\mu)) , \quad \bar{\lambda}(\mu_0) = \lambda \quad \implies \quad \beta(\lambda)\frac{\partial\bar{\lambda}}{\partial\lambda} = \beta(\bar{\lambda}) . \quad (95)$$

Then the Green's function at scale μ can be expressed in terms of its value at scale μ_0 ,

$$G_n(x_1; \dots; x_n; \lambda; \mu) = G_n(x_1; \dots; x_n; \bar{\lambda}(\mu); \mu_0) \times \exp\left[-n\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma(\bar{\lambda}(\mu'))\right] . \quad (96)$$

Our result (93) for the curvature-dependent beta function is zero at order λ^3 . If we make the usual assumption that 1-loop results dominate, this means that the coupling constant fails to run, that is, $\bar{\lambda}(\mu) = \lambda$. This implies that the perturbative factors of $\ln(\mu)$ exponentiate to give a power,

$$G_n(x_1; \dots; x_n; \lambda; \mu) = G_n(x_1; \dots; x_n; \lambda; \mu_0) \left[\frac{\mu_0}{\mu} \right]^{n\gamma(\lambda)}. \quad (97)$$

Scaling is more complicated in cosmology because the Hubble parameter gives an additional dimensionful parameter, even for massless theories. However, if it is valid for the exchange potential to replace μ by either r or $1/r$, expression (97) provides a highly nontrivial resummation.

5 Conclusions

We have revisited the 2-field model (6) which was analyzed in a recent study of nonlinear sigma models as a paradigm for how to re-sum large inflationary logarithms from fields with derivative interactions like those of quantum gravity [31]. After reviewing the model in section 2, we derived a 2-loop result for the evolution of the background in section 3, and computed the 1-loop beta function in section 4. Our purpose was to check the stochastic prediction (26) for the 2-loop background which was derived in the previous study [31], and to extend the renormalization group analysis of the exchange potentials to all orders.

The seven diagrams of Figure 3 contribute to the 2-loop background, and can be reduced to exact analytic expressions (44), (46), (48), (50), (52), (54) and (56). When the Schwinger-Keldysh formalism is used to extract their leading logarithm contributions (61-64), the total agrees exactly with the stochastic prediction (26). Even if one discounts the counterterm insertions (50) and (56), this still leaves five intricate, 2-loop diagrams which conspire to confirm the stochastic prediction. That must be recognized as a highly nontrivial check of the stochastic formalism.

The implications of our results were discussed at the end of sections 3 and 4. Briefly, these are that we can now sum the leading logarithms to all orders as long as the de Sitter background persists. The scalar background evolves to arbitrarily large values, and this evolution continues to late times. The factors of $\ln(Hr)$ in the exchange potentials sum up to give powers of Hr .

Finally, we again note that it would be interesting to generalize this analysis from de Sitter to a general cosmological background which has undergone primordial inflation. Good analytic approximations for the key correlators (10-11) of the stochastic analysis have recently been derived [39], so it should be possible to work out what happens on a general background. Because these correlators transmit the high scales of primordial inflation to late times [39], it seems as if there may be significant late time effects. It should also be possible to generalize the renormalization group analysis to a general cosmological geometry because the coefficients of counterterms are universal, independent of the background [40].

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6 Appendix: Schwinger-Keldysh Evaluation of (27-29)

Section 3 saw the various 2-loop contributions to the in-out matrix element of $A(x)$ reduced to five integrals (27-29). It turns out that none of these integrals is real, nor do they even converge owing to the vast volume of the infinite future. Both embarrassments are due to the fact that inflationary particle production results in differences between in-out matrix elements and true expectation values. The Schwinger-Keldysh formalism [34, 35, 36, 37, 38] is a diagrammatic technique for giving the true expectation value we want. Fortunately, it is simple to convert in-out matrix elements such as (27-29) to the true expectation values.

There are some excellent review articles on the Schwinger-Keldysh formalism [41, 42, 43]. We confine ourselves here to summarizing the rules and showing how they allow us to convert in-out matrix elements to expectation values [44]. The rules are:

- The diagram topology is identical to that of in-out matrix elements.
- The endpoint of every line (internal and external) carries a $\pm$ polarity, with $+$ standing for a field which evolves forward in time and $-$ corresponding to a field which evolves backwards.
- Vertices (including counterterms) are either all $+$ or all $-$. The $+$ vertices are the same as those of in-out matrix elements, whereas $-$ vertices are complex conjugated.
- Because propagators have two endpoints, each with its own $\pm$ polarity, there are four propagators. They are related to the Feynman propagator $i\Delta(x; y)$ as follows,

$$i\Delta_{++}(x; y) = i\Delta(x; y) \quad , \quad i\Delta_{--}(x; y) = [i\Delta(x; y)]^* \quad , \quad (98)$$

$$i\Delta_{+-}(x; y) = \theta(\eta - \eta_y)[i\Delta(x; y)]^* + \theta(\eta_y - \eta)i\Delta(x; y) \quad , \quad (99)$$

$$i\Delta_{-+}(x; y) = \theta(\eta - \eta_y)i\Delta(x; y) + \theta(\eta_y - \eta)[i\Delta(x; y)]^* \quad , \quad (100)$$

where η_y is the conformal time in y -coordinates.

- If $\mathcal{D}$ is the kinetic operator, the various propagators obey,

$$\mathcal{D}i\Delta_{++}(x; z) = i\delta^D(x-z) \quad , \quad \mathcal{D}i\Delta_{\pm\mp}(x; z) = 0 \quad , \quad \mathcal{D}i\Delta_{--}(x; z) = -i\delta^D(x-z) \quad . \quad (101)$$

Two important consequences of these rules are (1) that the expectation value of a Hermitian operator such as $A(x)$ is real; and (2) that the only net contribution from interaction vertices comes from the past light-cone of external points.

Because the external A line can be viewed as $+$, we see that the Schwinger-Keldysh generalizations of the first three integrals (27-28) are,

$$I_1(t) \longrightarrow \int d^D z \sqrt{-g(z)} \left\{ \left[i\Delta_{++}(x; z) \right]^2 - \left[i\Delta_{+-}(x; z) \right]^2 \right\}, \quad (102)$$

$$I_2(t) \longrightarrow \int d^D z \sqrt{-g(z)} \left\{ i\Delta_{++}(x; z) i\Delta_{++}(z; z) - i\Delta_{+-}(x; z) i\Delta_{--}(z; z) \right\}, \quad (103)$$

$$I_3(t) \longrightarrow \int d^D z \sqrt{-g(z)} \left\{ i\Delta_{++}(x; z) - i\Delta_{+-}(x; z) \right\}. \quad (104)$$

Because $I_3(t)$ is imaginary, the last two integrals become,

$$I_4(t) \longrightarrow \int d^D z \sqrt{-g(z)} \left\{ i\Delta_{++}(x; z) - i\Delta_{+-}(x; z) \right\} \times I_3(t_z), \quad (105)$$

$$I_5(t) \longrightarrow \int d^D z \sqrt{-g(z)} \left\{ i\Delta_{++}(x; z) - i\Delta_{+-}(x; z) \right\} \times \dot{I}_3(t_z), \quad (106)$$

The $++$ and $+-$ propagators in these expressions are obtained from (8-9) by replacing the Minkowski interval Δx^2 in the de Sitter length function $y(x; z) = aa' H^2 \Delta x^2$ with,

$$\Delta x_{++}^2 = \left\| \vec{x} - \vec{z} \right\|^2 - \left(|\eta - \eta_z| - i\epsilon \right)^2, \quad (107)$$

$$\Delta x_{+-}^2 = \left\| \vec{x} - \vec{z} \right\|^2 - \left(\eta - \eta_z + i\epsilon \right)^2. \quad (108)$$

Note that Δx_{+-}^2 and Δx_{++}^2 agree for $\eta_z > \eta$, and are complex conjugates for $\eta_z < \eta$. This guarantees that the integrals over z^μ in expressions (102-106) only contribute when z^μ is in the past light-cone of x^μ , and that $++$ and $+-$ differences are purely imaginary.

The key to evaluating (102-106) is understanding that $I_1(t)$ is logarithmically divergent, and the divergent part of I_2 is a constant times I_3 through the relation,

$$i\Delta_{++}(z; z) = -k\pi \cot\left(\frac{D\pi}{2}\right) + \frac{H^2}{4\pi^2} \ln(a_z) = i\Delta_{--}(z; z). \quad (109)$$

The remaining integrals are finite. This means that we can exploit the $D = 4$ form of the propagator,

$$i\Delta(x; z) = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}} \frac{1}{(aa_z \Delta x^2)^{\frac{D}{2}-1}} - \frac{H^2}{8\pi^2} \ln\left(\frac{1}{4} H^2 \Delta x^2\right) + O(D-4). \quad (110)$$

The square of the leading term has the reduction,

$$\frac{1}{\Delta x_{++}^{2D-4}} = \frac{\mu^{D-4}}{2(D-3)(D-4)} \frac{4\pi^{\frac{D}{2}} i \delta^D(x-z)}{\Gamma(\frac{D}{2}-1)} - \frac{\partial^2}{4} \left[\frac{\ln(\mu^2 \Delta x_{++}^2)}{\Delta x_{++}^2} \right] + O(D-4), \quad (111)$$

$$\frac{1}{\Delta x_{+-}^{2D-4}} = -\frac{\partial^2}{4} \left[\frac{\ln(\mu^2 \Delta x_{+-}^2)}{\Delta x_{+-}^2} \right] + O(D-4). \quad (112)$$

Derivatives can be extracted from the integration to express inverse powers of $\Delta x_{\pm\pm}^2$ in terms of logarithms,

$$\frac{1}{\Delta x_{\pm\pm}^2} = \frac{\partial^2}{4} \left[\ln(\mu^2 \Delta x_{\pm\pm}^2) \right] \quad , \quad \frac{\ln(\mu^2 \Delta x_{\pm\pm}^2)}{\Delta x_{\pm\pm}^2} = \frac{\partial^2}{8} \left[\ln^2(\mu^2 \Delta x_{\pm\pm}^2) - 2 \ln(\mu^2 \Delta x_{\pm\pm}^2) \right] . \quad (113)$$

Differences of these logarithms give,

$$\ln(\mu^2 \Delta x_{++}^2) - \ln(\mu^2 \Delta x_{+-}^2) = 2\pi i \theta(\Delta\eta - \Delta r) , \quad (114)$$

$$\ln^2(\mu^2 \Delta x_{++}^2) - \ln^2(\mu^2 \Delta x_{+-}^2) = 4\pi i \theta(\Delta\eta - \Delta r) \ln[\mu^2(\Delta\eta^2 - \Delta r^2)] , \quad (115)$$

where $\Delta\eta \equiv \eta - \eta_z$ and $\Delta r \equiv \|\vec{x} - \vec{z}\|$.

Exploiting relations (110-115) allows us to evaluate I_1 ,

$$I_1 = \frac{i4H^{D-4}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)} - \frac{1}{3} \ln^2(a) - \frac{2}{3} \ln(a) + \frac{79}{54} + O\left(\frac{1}{a}\right) \right\} . \quad (116)$$

For the remaining integrals it is useful to note,

$$\int d^4z a_z^4 \left[i\Delta_{++}(x; z) - i\Delta_{+-}(x; z) \right] \times f(a_z) = -\frac{i}{3H^2} \int_1^a \frac{da_z}{a_z} \left[1 - \frac{a_z^3}{a^3} \right] \times f(a_z) . \quad (117)$$

Using (117) with different choices of $f(a_z)$ gives the leading logarithms results for the other four integrals,

$$f(a_z) = \frac{H^2 \ln(a_z)}{4\pi^2} \implies I_2 \longrightarrow -\frac{i}{24\pi^2} \times \ln^2(a) , \quad (118)$$

$$f(a_z) = 1 \implies I_3 \longrightarrow -\frac{i}{3H^2} \times \ln(a) , \quad (119)$$

$$f(a_z) = -\frac{i \ln(a')}{3H^2} \implies I_4 \longrightarrow -\frac{1}{18H^4} \times \ln^2(a) , \quad (120)$$

$$f(a_z) = -\frac{i}{3H} \implies I_5 \longrightarrow -\frac{1}{9H^3} \times \ln(a) . \quad (121)$$

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