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A proof of the fermionic theta coinvariant conjecture

Alessandro Iraci [a], Brendon Rhoades [b,*], Marino Romero [c]

[a] Dipartimento di Matematica, Università di Pisa, Pisa, PI, 56127, Italy
[b] Department of Mathematics, University of California, San Diego, La Jolla, CA, 92093-0112, USA
[c] Department of Mathematics, University of Pennsylvania, Philadelphia, PA, 19104-6395, USA

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ABSTRACT

Let $(x_1, \dots, x_n, y_1, \dots, y_n)$ be a list of $2n$ commuting variables, $(\theta_1, \dots, \theta_n, \xi_1, \dots, \xi_n)$ be a list of $2n$ anticommuting variables, and $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] \otimes \wedge\{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n\}$ be the algebra generated by these variables. D'Adderio, Iraci, and Vanden Wyngaerd introduced the *Theta operators* on the ring of symmetric functions and used them to conjecture a formula for the quadruply-graded $\mathfrak{S}_n$-isomorphism type of $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] \otimes \wedge\{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n\}/I$ where I is the ideal generated by $\mathfrak{S}_n$-invariants with vanishing constant term. We prove their conjecture in the 'purely fermionic setting' obtained by setting the commuting variables x_i, y_i equal to zero.

1. Introduction

The *diagonal coinvariant ring* DR_n is obtained from the rank $2n$ polynomial ring $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ by factoring out the ideal generated by $\mathfrak{S}_n$-invariants with vanishing constant term. The ring DR_n is a bigraded $\mathfrak{S}_n$-module; Haiman used algebraic geometry to calculate its isomorphism type [10]. In recent years, researchers in algebraic combinatorics studied variants of DR_n involving mixtures of commuting and anticommuting variables [1–3,9,11–14,16–19]. Drawing terminology from supersymmetry, we will refer to commuting variables as *bosonic* and anticommuting variables as *fermionic*. D'Adderio, Iraci, and Vanden Wyngaerd conjectured [3] a generalization of Haiman's result involving two sets of bosonic and two sets of fermionic variables of which Haiman's result forms the 'purely bosonic case'. We prove the 'purely fermionic case' of their conjecture.

We begin by fixing some notation. Let $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] \otimes \wedge\{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n\}$ be the tensor product

$$\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \otimes \wedge\{\theta_1, \dots, \theta_n, \xi_1, \dots, \xi_n\}$$

of a rank $2n$ symmetric algebra with a rank $2n$ exterior algebra. This ring carries four independent gradings (two bosonic and two fermionic) and the diagonal action of the symmetric group $\mathfrak{S}_n$

$$w \cdot x_i = x_{w(i)} \quad w \cdot y_i = y_{w(i)} \quad w \cdot \theta_i = \theta_{w(i)} \quad w \cdot \xi_i = \xi_{w(i)}$$

preserves this quadrigrading. Writing I_n for the ideal generated by $\mathfrak{S}_n$-invariants with vanishing constant term, the quotient

$$TDR_n := (\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \otimes \wedge\{\theta_1, \dots, \theta_n, \xi_1, \dots, \xi_n\})/I_n \tag{1.1}$$

* Corresponding author.
E-mail addresses: alessandro.iraci@unipi.it (A. Iraci), bprhoades@ucsd.edu (B. Rhoades), mar007@sas.upenn.edu (M. Romero).

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is a quadruply-graded $\mathfrak{S}_n$-module. The notation TDR_n alludes to its status as a 'supersymmetric double' of DR_n with Twice as many generators.

Let Λ denote the ring of symmetric functions in $\mathbf{x} = (x_1, x_2, \dots)$ over the ground field $\mathbb{C}(q,t)$. Given any element $F \in \Lambda$, D'Adderio, Iraci, and Vanden Wyngaerd introduced [3] a *Theta operator* $\Theta_F : \Lambda \to \Lambda$ whose definition is recalled in Section 2. Building on the techniques in [3], D'Adderio and Mellit [4] used Theta operators to prove the rise version of the *Delta Conjecture* of Haglund, Remmel, and Wilson [8]. D'Adderio and Romero [5] rederived and clarified a slew of symmetric function identities using Theta operators, drastically shortening many of their proofs.

In this paper, we will only consider Theta operators indexed by elementary symmetric functions $e_i \in \Lambda$; we abbreviate $\Theta_i := \Theta_{e_i}$. If $G \in \Lambda$ has degree d in the $\mathbf{x}$-variables, then $\Theta_i G$ has degree $d+i$ and will typically involve the parameters q and t (even if G itself does not). D'Adderio, Iraci, and Vanden Wyngaerd conjectured [3] that symmetric functions of the form $\Theta_i \Theta_j \nabla e_{n-i-j}$ (where ∇ is the Bergeron-Garsia nabla operator) determine the quadruply-graded $\mathfrak{S}_n$-structure of TDR_n. Deferring various definitions to Section 2, their conjecture may be stated as follows.

Conjecture 1.1. (D'Adderio-Iraci-Vanden Wyngaerd [3]) *Let $(TDR_n)_{i,j}$ be the piece of TDR_n with homogeneous θ-degree i and ξ-degree j. The space $(TDR_n)_{i,j}$ vanishes whenever $i+j \geq n$. If $i+j<n$ we have*

$$\mathrm{grFrob}((TDR_n)_{i,j}; q, t) = \Theta_i \Theta_j \nabla e_{n-i-j}$$

where q tracks x-degree and t tracks y-degree.

Remark 1.2. D'Adderio et al. define TDR_n in a slightly different way, by considering two sets of fermionic variables in which variables drawn from different sets commute with one another. This does not affect Conjecture 1.1. Indeed, if $V = \mathbb{C}^n$ we have a natural isomorphism of quadruply-graded vector spaces

$$\mathbb{C}[V \oplus V^*] \otimes \wedge(V \oplus V^*) \cong (\mathbb{C}[V] \otimes \wedge V) \otimes \left(\mathbb{C}[V^*] \otimes \wedge V^*\right) \tag{1.2}$$

which commutes with the action of the general linear group $GL(V)$ and its subgroup $\mathfrak{S}_n$ of permutation matrices. In particular, the isomorphism (1.2) restricts to a linear isomorphism between the subspaces of $\mathfrak{S}_n$-invariants with vanishing constant term. Since these subspaces are multihomogeneous, (1.2) also restricts to a linear isomorphism between the ideals which they generate. The isomorphism (1.2) therefore induces an isomorphism between TDR_n as defined in (1.1) and the module defined in D'Adderio et al. in [3].

The case $i = j = 0$ of Conjecture 1.1 amounts to setting the fermionic θ and ξ-variables to zero and is equivalent to Haiman's two-bosonic result $\mathrm{grFrob}(DR_n; q, t) = \nabla e_n$. Setting the y-variables and ξ-variables to zero, Conjecture 1.1 reduces to the one-bosonic, one-fermionic 'superspace coinvariant conjecture' of the Fields Institute combinatorics group (see [18, 19]). If only the ξ-variables are set to zero, Conjecture 1.1 yields a two-bosonic, one-fermionic conjecture of Zabrocki [18] tied to the *Delta operators* on Λ.

In this paper we give additional evidence for Conjecture 1.1 by proving its purely fermionic case. The *fermionic diagonal coinvariant ring*

$$FDR_n := \wedge\{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n\} / \langle \wedge\{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n\}_+^{\mathfrak{S}_n} \rangle \tag{1.3}$$

is obtained from the rank $2n$ exterior algebra $\wedge\{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n\}$ by modding out by the ideal generated by $\mathfrak{S}_n$-invariants with vanishing constant term. Equivalently, the ring FDR_n is obtained from TDR_n by setting the x-variables and y-variables equal to zero. The ring FDR_n is a bigraded $\mathfrak{S}_n$-module. Jongwon Kim and Rhoades introduced FDR_n in [12]; Jesse Kim and Rhoades used FDR_n as a model for resolving a set partition of $\{1, \dots, n\}$ into a linear combination of noncrossing set partitions [11]. Our main result is as follows.

Theorem 1.3. *We have $(FDR_n)_{i,j} = 0$ whenever $i + j \geq n$. When $i + j < n$ we have*

$$\mathrm{Frob}\,(FDR_n)_{i,j} = \Theta_i \Theta_j \nabla e_{n-i-j} \,|_{q=t=0}\,.$$

Remark 1.4. The canonical surjection $DR_n \otimes FDR_n \twoheadrightarrow TDR_n$ applies to show that the bidegree support assertion in Theorem 1.3 implies the corresponding assertion in Conjecture 1.1.

To prove Theorem 1.3, we apply a result of Jongwon Kim and Rhoades [12] which expresses $\mathrm{Frob}\,(FDR_n)_{i,j}$ in terms of Kronecker products of hook-shaped Schur functions. To obtain a recursive structure on the characters $\mathrm{Frob}\,(FDR_n)_{i,j}$, we prove a result (Lemma 3.3) on applying the skewing operators $h_d^\perp$ to Kronecker products $s_\lambda * s_\mu$ which may be of independent interest for studying Kronecker products in general. For the right-hand side $\Theta_i \Theta_j \nabla e_{n-i-j} \,|_{q=t=0}$ of Theorem 1.3, we apply a result of D'Adderio and Romero [5] for applying the operators $h_d^\perp$ to expressions of the form $\Theta_i \Theta_j \widetilde{H}_{(n-i-j)}(\mathbf{x}; q, t)$ and study what happens at the evaluation $q, t \to 0$. We check that the recursions coincide, and Theorem 1.3 is proven.

The rest of the paper is organized as follows. In Section 2 we give the required background on symmetric functions and representation theory. In Section 3 we prove Theorem 1.3.

2. Background

We give background material on symmetric functions and the representation theory of symmetric groups. We use the operation $F[G]$ of *plethysm* throughout. For a more thorough exposition of this material, we refer the reader to [7].

As in the introduction, we write $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ for the graded ring of symmetric functions in the variable set $\mathbf{x} = (x_1, x_2, \dots)$ over the ground field $\mathbb{C}(q,t)$. For $n \geq 0$, we write $e_n = e_n(\mathbf{x}), h_n = h_n(\mathbf{x}) \in \Lambda_n$ for the *elementary* and *complete homogeneous* symmetric functions of degree n.

Bases of Λ_n are indexed by partitions $\lambda \vdash n$. The elementary basis e_λ and homogeneous basis h_λ are defined by setting $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$ and $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots$. We will also use the basis of *Schur functions* $s_\lambda = s_\lambda(\mathbf{x})$ and the basis of *(modified) Macdonald polynomials* $\widetilde{H}_\lambda(\mathbf{x}; q, t)$.

The *Hall inner product* $\langle -, - \rangle$ on Λ is defined by declaring the Schur basis to be orthonormal:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$$

where $\delta_{\lambda,\mu}$ is the Kronecker delta. Given $F \in \Lambda$, write $F^\bullet : \Lambda \to \Lambda$ for the operator $F^\bullet(G) := FG$ of multiplication by F. We write $F^\perp : \Lambda \to \Lambda$ for the adjoint of the operator $F^\bullet$; it is characterized by

$$\langle F^\perp G, H \rangle = \langle G, F^\bullet H \rangle \tag{2.1}$$

for all $G, H \in \Lambda$. The application of $F^\perp$ to a symmetric function is often referred to as *skewing* with respect to F.

Eigenoperators on the Macdonald basis have proven to be remarkable objects in symmetric function theory. We use two such Macdonald eigenoperators in this paper. The first is the *nabla operator* $\nabla : \Lambda \to \Lambda$ defined by

$$\nabla : \widetilde{H}_\mu(\mathbf{x}; q, t) \mapsto \prod_{(i,j) \in \mu} q^{i-1} t^{j-1} \cdot \widetilde{H}_\mu(\mathbf{x}; q, t) \tag{2.2}$$

where the product is over all cells (i, j) in the Young diagram of μ. Similarly, the operator $\Pi : \Lambda \to \Lambda$ is given by

$$\Pi : \widetilde{H}_\mu(\mathbf{x}; q, t) \mapsto \prod_{\substack{(i,j) \in \mu \\ (i,j) \neq (1,1)}} (1 - q^{i-1} t^{j-1}) \cdot \widetilde{H}_\mu(\mathbf{x}; q, t) \tag{2.3}$$

where the product is over all cells $(i, j) \neq (1, 1)$ in the Young diagram of μ. We abbreviate the eigenvalue of $\widetilde{H}_\mu(\mathbf{x}; q, t)$ under the operator Π as

$$\Pi_\mu := \prod_{\substack{(i,j) \in \mu \\ (i,j) \neq (1,1)}} (1 - q^{i-1} t^{j-1}). \tag{2.4}$$

The omission of $(1, 1)$ in this product assures that the operator Π is nonzero and, in fact, invertible.

We are ready to define the Theta operators of [3]. Given $F = F(\mathbf{x}) \in \Lambda$, let $F\left[\frac{\mathbf{x}}{M}\right]$ be the symmetric function obtained by plethystically evaluating F at $\frac{\mathbf{x}}{M}$, where $M = (1-q)(1-t)$. The *Theta operator* $\Theta_F : \Lambda \to \Lambda$ is obtained by conjugating the multiplication operator $F\left[\frac{\mathbf{x}}{M}\right]^\bullet$ by Π. That is, we set

$$\Theta_F := \Pi \circ F\left[\frac{\mathbf{x}}{M}\right]^\bullet \circ \Pi^{-1}. \tag{2.5}$$

Assuming F is homogeneous, the operator Θ_F is homogeneous of degree $\deg(F)$ on the graded ring $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$. As explained in the introduction, we will only use Theta operators indexed by elementary symmetric functions, and so abbreviate $\Theta_d := \Theta_{e_d}$.

We recall some basic ideas from group representation theory. If G is a group and V_1, V_2 are G-modules, we write $V_1 \otimes V_2$ for their *Kronecker product* (or *internal product*). This is the G-module with underlying vector space given by the tensor product of V_1 and V_2 with G-module structure $g \cdot (v_1 \otimes v_2) := (g \cdot v_1) \otimes (g \cdot v_2)$. For us, the group G will either be a symmetric group $\mathfrak{S}_n$ or a parabolic subgroup $\mathfrak{S}_j \times \mathfrak{S}_{n-j}$ thereof.

If G and H are groups, V is a G-module, and W is an H-module, we write $V \boxtimes W$ for the $(G \times H)$-module whose underlying vector space is the tensor product of V and W and whose module structure is determined by $(g, h) \cdot (v \otimes w) := (g \cdot v) \otimes (h \cdot w)$. For us, both G and H will be symmetric groups. We use the distinct notations $\otimes$ and $\boxtimes$ to avoid confusion in the proof of Lemma 3.3 below.

Irreducible representations of the symmetric group $\mathfrak{S}_n$ over $\mathbb{C}$ are in bijective correspondence with partitions of n. Given $\lambda \vdash n$, we write S^λ for the corresponding $\mathfrak{S}_n$-irreducible. If V is any finite-dimensional $\mathfrak{S}_n$-module, there are unique multiplicities $c_\lambda \geq 0$ such that $V \cong \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda$. The *Frobenius image* $\mathrm{Frob}(V) \in \Lambda_n$ is the symmetric function

$$\mathrm{Frob}(V) := \sum_{\lambda \vdash n} c_\lambda \cdot s_\lambda \tag{2.6}$$

obtained by replacing each irreducible factor with the corresponding Schur function. More generally, if $V = \bigoplus_{i \geq 0} V_i$ is a graded $\mathfrak{S}_n$-module with each V_i finite-dimensional, its *graded Frobenius image* is

$$\mathrm{grFrob}(V;q) = \sum_{i \geq 0} \mathrm{Frob}(V_i) \cdot q^i \tag{2.7}$$

and if $V = \bigoplus_{i,j \geq 0} V_{i,j}$ is a bigraded $\mathfrak{S}_n$ module we have the *bigraded Frobenius image*

$$\mathrm{grFrob}(V;q,t) = \sum_{i,j \geq 0} \mathrm{Frob}(V_{i,j}) \cdot q^i t^j. \tag{2.8}$$

Operations on symmetric functions correspond to operations on symmetric group modules via the Frobenius map. For example, if V is an $\mathfrak{S}_n$-module and W is an $\mathfrak{S}_m$-module, the *induction product* of V and W is $V \circ W := \mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} (V \boxtimes W)$ where the embedding $\mathfrak{S}_n \times \mathfrak{S}_m \subset \mathfrak{S}_{n+m}$ is obtained by letting $\mathfrak{S}_n$ permute the first n letters and $\mathfrak{S}_m$ permute the last m letters. We have

$$\mathrm{Frob}(V \circ W) = \mathrm{Frob}(V) \cdot \mathrm{Frob}(W). \tag{2.9}$$

Defined for partitions $\lambda \vdash n$, $\mu \vdash m$, and $\nu \vdash n+m$, the *Littlewood-Richardson coefficients* $c^{\nu}_{\lambda,\mu}$ are the structure coefficients for this product in the Schur basis. They are characterized by either of the formulas

$$S^\lambda \circ S^\mu \cong \bigoplus_{\nu \vdash m+n} c^\nu_{\lambda,\mu} S^\nu \qquad \text{or} \qquad s_\lambda \cdot s_\mu = \sum_{\nu \vdash m+n} c^\nu_{\lambda,\mu} \cdot s_\nu. \tag{2.10}$$

The *Littlewood-Richardson rule* gives a combinatorial interpretation of the nonnegative integers $c^\nu_{\lambda,\mu}$.

As another example, the *Kronecker product* on the space of degree n symmetric functions Λ_n is the bilinear operation $*$ characterized by

$$\mathrm{Frob}(S^\lambda \otimes S^\mu) = s_\lambda * s_\mu \tag{2.11}$$

for all $\lambda, \mu \vdash n$. The nonnegative integers $g_{\lambda,\mu,\nu}$ indexed by triples of partitions $\lambda, \mu, \nu \vdash n$ determined by $s_\lambda * s_\mu = \sum_{\nu \vdash n} g_{\lambda,\mu,\nu} \cdot s_\nu$ are the *Kronecker coefficients*. Finding a combinatorial rule for $g_{\lambda,\mu,\nu}$ is a famous open problem.

3. Proof of Theorem 1.3

Theorem 1.3 asserts an equality of symmetric functions. Our strategy for proving this equality is to show that both sides satisfy the following recursion.

Lemma 3.1. *Let $F, G \in \Lambda$ be symmetric functions with vanishing constant terms. Suppose that $h_j^\perp F = h_j^\perp G$ for all $j \geq 1$. Then $F = G$.*

Proof. For any partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ with $\lambda_1 > 0$, we have

$$\langle F, h_\lambda \rangle = \langle h^\perp_{\lambda_1} F, h_{\bar{\lambda}} \rangle = \langle h^\perp_{\lambda_1} G, h_{\bar{\lambda}} \rangle = \langle G, h_\lambda \rangle \tag{3.1}$$

where $\bar{\lambda} = (\lambda_2, \lambda_3, \dots)$ and the result follows since the h_λ form a basis of Λ and $\langle -, - \rangle$ is an inner product. □

We handle the representation theoretic side $\mathrm{Frob}\,(FDR_n)_{i,j}$ of Theorem 1.3 first. Our starting point is the following result of Jongwon Kim and Rhoades [12] which describes this symmetric function in terms of Kronecker products of hook-shaped Schur functions.

Theorem 3.2. (Jongwon Kim-Rhoades [12]) *We have $(FDR_n)_{i,j} = 0$ whenever $i + j \geq n$. When $i + j < n$ we have*

$$\mathrm{Frob}\,(FDR_n)_{i,j} = s_{(n-i,1^i)} * s_{(n-j,1^j)} - s_{(n-i+1,1^{i-1})} * s_{(n-j+1,1^{j-1})}$$

where by convention $s_{(n-i+1,1^{i-1})} * s_{(n-j+1,1^{j-1})} = 0$ when $i = 0$ or $j = 0$.

Theorem 3.2 was proven by showing that the $\mathfrak{S}_n$-invariant element $\theta_1\xi_1 + \cdots + \theta_n\xi_n \in \wedge\{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n\}$ satisfies a kind of 'bigraded Lefschetz property'. While there exist expressions for the Schur expansion of $s_\lambda * s_\mu$ when $\lambda, \mu \vdash n$ are hooks (see e.g. [15]), these formulas are rather complicated. With an eye towards Lemma 3.1, we give a recursive rule for applying $h_j^\perp$ to an arbitrary Kronecker product of Schur functions.

Lemma 3.3. *Let $1 \leq j \leq n$ and let $\lambda^{(1)}, \lambda^{(2)} \vdash n$ be two partitions. We have*

$$h_j^{\perp}(s_{\lambda^{(1)}} * s_{\lambda^{(2)}}) = \sum_{\substack{\mu \vdash j \\ \nu^{(1)}, \nu^{(2)} \vdash n-j}} c^{\lambda^{(1)}}_{\mu, \nu^{(1)}} \cdot c^{\lambda^{(2)}}_{\mu, \nu^{(2)}} (s_{\nu^{(1)}} * s_{\nu^{(2)}}) \tag{3.2}$$

where the $c^{\lambda^{(i)}}_{\mu, \nu^{(i)}}$ are Littlewood-Richardson coefficients.

The proof of Lemma 3.3 requires both of the module operations $\boxtimes$ and $\otimes$ introduced in Section 2.

Proof. It is well-known that, for any $\mathfrak{S}_n$-module V, the degree $n-k$ symmetric function $h_j^{\perp} \mathrm{Frob}\, V$ has the algebraic interpretation

$$h_j^{\perp} \mathrm{Frob}\, V = \mathrm{Frob}\, \mathrm{Hom}_{\mathfrak{S}_j}(\mathbf{1}_{\mathfrak{S}_j}, \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_j \times \mathfrak{S}_{n-j}} V) \tag{3.3}$$

where the $\mathrm{Hom}_{\mathfrak{S}_j}$-space is an $\mathfrak{S}_{n-j}$-module by means of the second factor of $\mathfrak{S}_j \times \mathfrak{S}_{n-j}$. In our situation, this reads

$$h_j^{\perp}(s_{\lambda^{(1)}} * s_{\lambda^{(2)}}) = \mathrm{Frob}\, \mathrm{Hom}_{\mathfrak{S}_j} \left(\mathbf{1}_{\mathfrak{S}_j}, \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_j \times \mathfrak{S}_{n-j}} (S^{\lambda^{(1)}} \otimes S^{\lambda^{(2)}}) \right) \tag{3.4}$$

$$= \mathrm{Frob}\, \mathrm{Hom}_{\mathfrak{S}_j} \left(\mathbf{1}_{\mathfrak{S}_j}, (\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_j \times \mathfrak{S}_{n-j}} S^{\lambda^{(1)}}) \otimes (\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_j \times \mathfrak{S}_{n-j}} S^{\lambda^{(2)}}) \right) \tag{3.5}$$

$$= \mathrm{Frob}\, \mathrm{Hom}_{\mathfrak{S}_j} \left(\mathbf{1}_{\mathfrak{S}_j}, \bigoplus_{\substack{\mu^{(1)}, \mu^{(2)} \vdash j \\ \nu^{(1)}, \nu^{(2)} \vdash n-j}} c^{\lambda^{(1)}}_{\mu^{(1)}, \nu^{(1)}} \cdot c^{\lambda^{(2)}}_{\mu^{(2)}, \nu^{(2)}} \cdot (S^{\mu^{(1)}} \boxtimes S^{\nu^{(1)}}) \otimes (S^{\mu^{(2)}} \boxtimes S^{\nu^{(2)}}) \right) \tag{3.6}$$

$$= \mathrm{Frob}\, \mathrm{Hom}_{\mathfrak{S}_j} \left(\mathbf{1}_{\mathfrak{S}_j}, \bigoplus_{\substack{\mu^{(1)}, \mu^{(2)} \vdash j \\ \nu^{(1)}, \nu^{(2)} \vdash n-j}} c^{\lambda^{(1)}}_{\mu^{(1)}, \nu^{(1)}} \cdot c^{\lambda^{(2)}}_{\mu^{(2)}, \nu^{(2)}} \cdot (S^{\mu^{(1)}} \otimes S^{\mu^{(2)}}) \boxtimes (S^{\nu^{(1)}} \otimes S^{\nu^{(2)}}) \right) \tag{3.7}$$

$$= \sum_{\substack{\mu^{(1)}, \mu^{(2)} \vdash j \\ \nu^{(1)}, \nu^{(2)} \vdash n-j}} c^{\lambda^{(1)}}_{\mu^{(1)}, \nu^{(1)}} \cdot c^{\lambda^{(2)}}_{\mu^{(2)}, \nu^{(2)}} \cdot \dim \mathrm{Hom}_{\mathfrak{S}_j}(\mathbf{1}_{\mathfrak{S}_j}, S^{\mu^{(1)}} \otimes S^{\mu^{(2)}}) \cdot s_{\nu^{(1)}} * s_{\nu^{(2)}} \tag{3.8}$$

where we used the fact that restriction functors commute with Kronecker products and the consequence

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_j \times \mathfrak{S}_{n-j}} S^{\lambda} \cong \bigoplus_{\substack{\mu \vdash j \\ \nu \vdash n-j}} c^{\lambda}_{\mu, \nu} (S^{\mu} \boxtimes S^{\nu}) \qquad (\lambda \vdash n) \tag{3.9}$$

of Frobenius reciprocity.

The multiplicities $g_{\rho, \mu^{(1)}, \mu^{(2)}} = \dim(\mathrm{Hom}(S^{\rho}, S^{\mu^{(1)}} \otimes S^{\mu^{(2)}}))$ of Schur functions s_{ρ} in general Kronecker products $s_{\mu^{(1)}} * s_{\mu^{(2)}}$ are difficult to compute. However, when $\rho = (j)$ and $S^{\rho} = \mathbf{1}_{\mathfrak{S}_j}$ as in our setting, character orthogonality gives

$$\dim(\mathrm{Hom}_{\mathfrak{S}_j}(\mathbf{1}_{\mathfrak{S}_j}, S^{\mu^{(1)}} \otimes S^{\mu^{(2)}})) = \begin{cases} 1 & \mu^{(1)} = \mu^{(2)}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Adding this information to the above string of equalities gives

$$h_j^{\perp}(s_{\lambda^{(1)}} * s_{\lambda^{(2)}}) \tag{3.11}$$

$$= \sum_{\substack{\mu^{(1)}, \mu^{(2)} \vdash j \\ \nu^{(1)}, \nu^{(2)} \vdash n-j}} c^{\lambda^{(1)}}_{\mu^{(1)}, \nu^{(1)}} \cdot c^{\lambda^{(2)}}_{\mu^{(2)}, \nu^{(2)}} \cdot \dim \mathrm{Hom}_{\mathfrak{S}_j}(\mathbf{1}_{\mathfrak{S}_j}, S^{\mu^{(1)}} \otimes S^{\mu^{(2)}}) \cdot s_{\nu^{(1)}} * s_{\nu^{(2)}} \tag{3.12}$$

$$= \sum_{\substack{\mu \vdash j \\ \nu^{(1)}, \nu^{(2)} \vdash n-j}} c^{\lambda^{(1)}}_{\mu, \nu^{(1)}} \cdot c^{\lambda^{(2)}}_{\mu, \nu^{(2)}} \cdot s_{\nu^{(1)}} * s_{\nu^{(2)}} \tag{3.13}$$

and our proof is complete. $\square$

Our ability to put Lemma 3.3 to good use is bounded by our understanding of the products $c^{\lambda^{(1)}}_{\mu,\nu^{(1)}} \cdot c^{\lambda^{(2)}}_{\mu,\nu^{(2)}}$ of Littlewood-Richardson coefficients appearing therein. Thanks to Theorem 3.2, for our purposes both $\lambda^{(1)}$ and $\lambda^{(2)}$ will be hook-shaped partitions so that these coefficients will be fairly simple. The situation becomes more complicated for general $\lambda^{(1)}, \lambda^{(2)} \vdash n$, but Lemma 3.3 could conceivably be useful for studying Kronecker products $s_{\lambda^{(1)}} * s_{\lambda^{(2)}}$ for partitions other than hooks.

Although we will not need it, for completeness we record the companion Kronecker product recursion involving $e_j^\perp$:

$$e_j^\perp(s_{\lambda^{(1)}} * s_{\lambda^{(2)}}) = \sum_{\substack{\mu \vdash j \\ \nu^{(1)},\nu^{(2)} \vdash n-j}} c^{\lambda^{(1)}}_{\mu',\nu^{(1)}} \cdot c^{\lambda^{(2)}}_{\mu,\nu^{(2)}} (s_{\nu^{(1)}} * s_{\nu^{(2)}}). \tag{3.14}$$

Equation (3.14) has the same right-hand side as Equation (3.2), except that in the first Littlewood-Richardson coefficient the partition $\mu \vdash k$ is replaced by its conjugate $\mu' \vdash k$. Equation (3.14) may be proven in the same way as Equation (3.2), except one uses the formula

$$\dim(\mathrm{Hom}_{\mathfrak{S}_j}(\mathrm{sign}_{\mathfrak{S}_j}, S^{\mu^{(1)}} \otimes S^{\mu^{(2)}})) = \begin{cases} 1 & \mu^{(1)} = (\mu^{(2)})', \\ 0 & \text{otherwise} \end{cases} \tag{3.15}$$

for the multiplicity of the sign representation in a Kroenecker product of irreducibles.

Our application of Lemma 3.3 may be stated as follows. In order to motivate the next result and understand its proof, it will be useful to recall that a product $e_a h_b$ of an elementary symmetric function with a homogeneous symmetric function is a sum

$$e_a h_b = s_{(b,1^a)} + s_{(b+1,1^{a-1})} \tag{3.16}$$

of two successive hook-shaped Schur functions.

Lemma 3.4. *For any $j \geq 1$ and any integers k, ℓ, and m with $k+\ell+m=n$ we have*

$$h_j^\perp\left(s_{(k+\ell,1^m)} * s_{(k+m,1^\ell)} - s_{(k+\ell+1,1^{m-1}} * s_{(k+m+1,1^{\ell-1})}\right) \\ = h_{k+\ell-j}e_m * h_{k+m-j}e_\ell - h_{k+\ell}e_{m-j} * h_{k+m}e_{\ell-j}. \tag{3.17}$$

Proof. By telescoping sums, we may (and will) prove the equivalent assertion

$$h_j^\perp(s_{(k+\ell,1^m)} * s_{(k+m,1^\ell)}) = \sum_{r=0}^{j} h_{k+\ell-j+r}e_{m-r} * h_{k+m-j+r}e_{\ell-r}. \tag{3.18}$$

Applying Equation (3.16), the right-hand side of Equation (3.18) reads

$$\sum_{r=0}^{j} h_{k+\ell-j+r}e_{m-r} * h_{k+m-j+r}e_{\ell-r} \\ = \sum_{r=0}^{j} \left(s_{(k+\ell-j+r,1^{m-r})} + s_{(k+\ell-j+r+1,1^{m-r-1})}\right) * \left(s_{(k+m-j+r,1^{\ell-r})} + s_{(k+m-j+r+1,1^{\ell-r-1})}\right). \tag{3.19}$$

As for the left-hand side of Equation (3.18), Lemma 3.3 yields

$$h_j^\perp\left(s_{(k+\ell,1^m)} * s_{(k+m,1^\ell)}\right) = \sum_{\substack{\mu \vdash j \\ \nu^{(1)},\nu^{(2)} \vdash n-j}} c^{(k+\ell,1^m)}_{\mu,\nu^{(1)}} \cdot c^{(k+m,1^\ell)}_{\mu,\nu^{(2)}} (s_{\nu^{(1)}} * s_{\nu^{(2)}}). \tag{3.20}$$

The Littlewood-Richardson coefficient $c^{(k+\ell,1^m)}_{\mu,\nu^{(1)}}$ is nonzero only when both μ and $\nu^{(1)}$ are hooks. In this case, we have $\mu = (j-r, 1^r)$ for some $r \in \{0, 1, \ldots, j-1\}$ and the Littlewood-Richardson Rule implies

$$s_{(j-r,1^r)} s_{(c,1^{n-j-c})} = s_{(j-r+c,1^{r+n-j-c})} + s_{(j-r+c-1,1^{r+n-j-c+1})} + R \tag{3.21}$$

where R is a sum of Schur functions indexed by non-hook partitions. Therefore, the Schur function $s_{(k+\ell,1^m)}$ appears in this product precisely when $j-r+c = k+\ell$ or $j-r+c-1 = k+\ell$ or, in other words,

$$c = k+\ell-j+r \qquad \text{or} \qquad c = k+\ell-j+r+1.$$

Similarly, the Schur function $s_{(k+m,1^\ell)}$ appears in the product $s_{(j-r,1^r)} s_{(c,1^{n-j-c})}$ precisely when

$$c = k + m - j + r \qquad \text{or} \qquad c = k + m - j + r + 1.$$

This means that

$$\sum_{\substack{\mu \vdash j \\ \nu^{(1)}, \nu^{(2)} \vdash n-j}} c^{(k+\ell,1^m)}_{\mu,\nu^{(1)}} \cdot c^{(k+m,1^\ell)}_{\mu,\nu^{(2)}} (s_{\nu^{(1)}} * s_{\nu^{(2)}}) \tag{3.22}$$

$$= \sum_{r=0}^{j-1} \sum_{c,d} c^{(k+\ell,1^m)}_{(j-r,1^r),(c,1^{n-j-c})} \cdot c^{(k+m,1^\ell)}_{(j-r,1^r),(d,1^{n-j-d})} \left(s_{(c,1^{n-j-c})} * s_{(d,1^{n-j-d})}\right) \tag{3.23}$$

$$= \sum_{r=0}^{j} \left(s_{(k+\ell-j+r,1^{m-r})} + s_{(k+\ell-j+r+1,1^{m-r-1})}\right) * \left(s_{(k+m-j+r,1^{\ell-r})} + s_{(k+m-j+r+1,1^{\ell-r-1})}\right), \tag{3.24}$$

and applying Equation (3.19) finishes the proof. □

We turn to the Theta operator side of Theorem 1.3. Our starting point is an identity of D'Adderio and Romero [5, Theorem 8.2]. We use the standard q-analogs

$$[n]_q := 1 + q + \cdots + q^{n-1} \qquad [n]!_q := [n]_q[n-1]_q \cdots [1]_q \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q \cdot [n-k]!_q} \tag{3.25}$$

of numbers, factorials, and binomial coefficients together with the convention $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ whenever $k < 0$ or $k > n$. We will also need the symmetric functions $E_{n,k} = E_{n,k}(\mathbf{x}; q)$ introduced in [6] which may be defined plethystically by

$$E_{n,k} := q^k \sum_{r=0}^{k} q^{\binom{r}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^r e_n \left[\mathbf{x} \frac{1-q^{-r}}{1-q}\right]. \tag{3.26}$$

Here and throughout, we write $\widetilde{H}_k(\mathbf{x}; q, t)$ for $\widetilde{H}_{(k)}(\mathbf{x}; q, t)$.

Theorem 3.5. (D'Adderio-Romero [5]) *For any integers j, m, ℓ, and k we have*

$$h_j^\perp \Theta_m \Theta_\ell \widetilde{H}_k(\mathbf{x}; q, t) = \sum_{r=0}^{j} \begin{bmatrix} k \\ r \end{bmatrix}_q \sum_{a=0}^{k} \sum_{b=1}^{j-r+a} \Theta_{m-j+r} \Theta_{\ell+k-j-a} \nabla E_{j-r+a,b}$$
$$\times \left(q^{\binom{k-r-a}{2}} \begin{bmatrix} b-1 \\ a \end{bmatrix}_q \begin{bmatrix} b+r-a-1 \\ k-a-1 \end{bmatrix}_q + q^{\binom{k-r-a+1}{2}} \begin{bmatrix} b-1 \\ a-1 \end{bmatrix}_q \begin{bmatrix} b+k-r-a \\ k-a \end{bmatrix}_q \right).$$

We will use Theorem 3.5 to obtain a recursion for the Theta operators at $q = t = 0$. In order to do this, we must first replace the left-hand side of Theorem 3.5 with something closer to the expression $\Theta_i \Theta_j \nabla e_n \mid_{q=t=0}$. This is accomplished with the following lemma.

Lemma 3.6. *For any integers m, ℓ, and k we have*

$$\Theta_m \Theta_\ell \nabla e_k \mid_{t=0} = \Theta_m \Theta_\ell \widetilde{H}_k(\mathbf{x}; q, t) \mid_{t=0}. \tag{3.27}$$

Proof. The first step is to note that

$$\widetilde{H}_k(\mathbf{x}; q, t) = \nabla e_k \mid_{t=0}. \tag{3.28}$$

The left side of Equation (3.28) is well-known to be the graded Frobenius image of the coinvariants of the symmetric group with a single set of bosonic variables. The right side is the bigraded Frobenius image of the coinvariants with two sets of bosonic variables, with the x-variables set to zero. Therefore, the two sides are the same.

We claim that for any symmetric function G, we have

$$\left(\Theta_{e_\lambda} G\right) \mid_{t=0} = \left(\Theta_{e_\lambda} (G \mid_{t=0})\right) \mid_{t=0}. \tag{3.29}$$

It is sufficient to verify (3.29) over a basis. If $G = \widetilde{H}_\mu(\mathbf{x}; q, t)$, then $G \mid_{t=0} = \widetilde{H}_\mu(\mathbf{x}; q, 0)$ is a modified Hall-Littlewood symmetric function; these also give a basis for symmetric functions. Let $D_{\mu,\nu}(q, t)$ be the coefficients in the expansion

$$\widetilde{H}_{\mu}(\mathbf{x};q,0)=\sum_{\nu}D_{\mu,\nu}(q,t)\widetilde{H}_{\nu}(\mathbf{x};q,t) \tag{3.30}$$

Setting $t=0$, we find that

$$D_{\mu,\nu}(q,0)=\delta_{\nu,\mu}. \tag{3.31}$$

Now, applying Theta operators to both sides of (3.30), we find

$$\Theta_{e_\lambda}\widetilde{H}_{\mu}(\mathbf{x};q,0)=\sum_{\lambda}D_{\mu,\nu}(q,t)\Theta_{e_\lambda}\widetilde{H}_{\nu}(\mathbf{x};q,t). \tag{3.32}$$

Assuming that $\Theta_{e_\lambda}\widetilde{H}_{\nu}(\mathbf{x};q,t)$ has no poles at $t=0$, then setting $t=0$ on both sides of (3.32) we get

$$\Theta_{e_\lambda}\widetilde{H}_{\mu}(\mathbf{x};q,0)\mid_{t=0}=\sum_{\lambda}D_{\mu,\nu}(q,0)\Theta_{e_\lambda}\widetilde{H}_{\nu}(\mathbf{x};q,t)\mid_{t=0} \tag{3.33}$$

$$=\Theta_{e_\lambda}\widetilde{H}_{\mu}(\mathbf{x};q,t)\mid_{t=0}, \tag{3.34}$$

which proves (3.29). To show that there are no poles at $t=0$, we see that by definition of Theta operators,

$$\Theta_{e_\lambda}\widetilde{H}_{\mu}(\mathbf{x};q,t)=\sum_{\rho}d^{\lambda}_{\mu,\rho}(q,t)\frac{\Pi_\rho}{\Pi_\mu}\widetilde{H}_{\rho}(\mathbf{x};q,t), \tag{3.35}$$

where $d^{\lambda}_{\mu,\rho}(q,t)$ is the coefficient of $\widetilde{H}_{\rho}(\mathbf{x};q,t)$ in $e_\lambda[\mathbf{x}/M]\widetilde{H}_{\mu}(\mathbf{x};q,t)$. At $t=0$, this product gives

$$e_\lambda\left[\frac{\mathbf{x}}{M}\right]\widetilde{H}_{\mu}(\mathbf{x};q,t)\mid_{t=0}=e_\lambda\left[\frac{\mathbf{x}}{1-q}\right]\widetilde{H}_{\mu}(\mathbf{x};q,0) \tag{3.36}$$

$$=\sum_{\rho}C_{\mu,\rho}(q)\widetilde{H}_{\rho}(\mathbf{x};q,0) \tag{3.37}$$

for some coefficients $C_{\mu,\rho}(q)$. Since the modified Hall-Littlewood symmetric functions are a basis, we must then have $d^{\lambda}_{\mu,\rho}(q,0)=C_{\mu,\rho}(q)$. This means $d^{\lambda}_{\mu,\rho}(q,t)$ has no pole at $t=0$, and since Π_ρ/Π_μ also has no pole at $t=0$, we can conclude that (3.29) holds for any symmetric function G, proving the lemma. □

With Lemma 3.6 in hand, we apply Theorem 3.5 to get a recursive expression for the action of $h_j^{\perp}$ on the symmetric functions $\Theta_m\Theta_\ell\nabla e_k\mid_{q=t=0}$ of Theorem 1.3.

Lemma 3.7. *For any integers j,m,ℓ, and k we have*

$$\begin{aligned}h_j^{\perp}\Theta_m\Theta_\ell\nabla e_k\mid_{q=t=0}=&\sum_{r=0}^{j}\Theta_{m-r}\Theta_{\ell-r}\nabla e_{k-j+2r}\mid_{q=t=0}\\&+\sum_{r=0}^{j-1}(\Theta_{m-r-1}\Theta_{\ell-r}+\Theta_{m-r}\Theta_{\ell-r-1})\nabla e_{k-j+2r+1}\mid_{q=t=0}\\&+\sum_{r=0}^{j-2}\Theta_{m-r-1}\Theta_{\ell-r-1}\nabla e_{k-j+2r+2}\mid_{q=t=0}.\end{aligned}$$

Proof. For convenience, we apply the transformation $a\mapsto k-a$ in Theorem 3.5 to obtain

$$\begin{aligned}h_j^{\perp}\Theta_m\Theta_\ell\widetilde{H}_k(\mathbf{x};q,t)=&\sum_{r=0}^{j}\begin{bmatrix}k\\r\end{bmatrix}_q\sum_{a=0}^{k}\sum_{b=1}^{k+j-r-a}\Theta_{m-j+r}\Theta_{\ell-j+a}\nabla E_{k+j-r-a,b}\\&\times\left(q^{\binom{a-r}{2}}\begin{bmatrix}b-1\\k-a\end{bmatrix}_q\begin{bmatrix}b-k+r+a-1\\a-1\end{bmatrix}_q+q^{\binom{r-a}{2}}\begin{bmatrix}b-1\\k-a-1\end{bmatrix}_q\begin{bmatrix}b+k-r-a\\a\end{bmatrix}_q\right)\end{aligned} \tag{3.38}$$

Our goal is to evaluate Equation (3.38) at $q\to 0$. From Equation (3.27), we see the right-hand sides of both Equation (3.38) and the equation in the statement of the Lemma agree.

At $q\to 0$, in order for the expression in the parentheses of (3.38) to be nonzero, we must have $-1\le a-r\le 1$ (for otherwise both summands involve only positive powers of q). Moreover, by examining the q-binomials, we see that $r\le k$

(from the q-binomial on the first line of (3.38)) and $a \leq j$ (since $b \leq k+j-r-a$, or $b-k+r+a \leq j$, the rightmost q-binomial in both summands vanishes unless $a \leq j$). Notice that, when the summands in the parentheses evaluate at $q \to 0$ to something nonzero, once evaluated they do not depend on b anymore, so we can isolate the sum over b and use the identity

$$\sum_{b=1}^{k+j-r-a} \nabla E_{k+j-r-a,b} = \nabla e_{k+j-r-a}. \tag{3.39}$$

We have three separate cases depending on the value of $a-r \in \{-1,0,1\}$.

Case 1. $a-r=-1$.

In this case, the left summand $q^{\binom{a-r}{2}}\begin{bmatrix} b-1 \\ k-a \end{bmatrix}_q \begin{bmatrix} b-k+r+a-1 \\ a-1 \end{bmatrix}_q$ in the parentheses vanishes and the right summand $q^{\binom{r-a}{2}}\begin{bmatrix} b-1 \\ k-a-1 \end{bmatrix}_q \begin{bmatrix} b+k-r-a \\ a \end{bmatrix}_q$ evaluates to 1 at $q \to 0$. Since $a \geq 0$, the sum restricts to $r \geq 1$ and we have a contribution of

$$\sum_{r=1}^{j} \Theta_{m-j+r}\Theta_{\ell-j+r-1}\nabla e_{k+j-2r+1} \mid_{q=t=0};$$

now making the change of variables $r \mapsto j-r$ we get

$$\sum_{r=0}^{j-1} \Theta_{m-r}\Theta_{\ell-r-1}\nabla e_{k-j+2r+1} \mid_{q=t=0}. \tag{3.40}$$

Case 2. $a-r=0$.

In this case, both summands in the parentheses evaluate to 1 at $q \to 0$, except when $a=r=0$ (in which case the left summand evaluates to 0) or when $a=r=j$ (in which case the right summand evaluates to 0). The contribution of this case is

$$\sum_{r=0}^{j} (2-\delta_{r,0}-\delta_{r,j})\Theta_{m-j+r}\Theta_{\ell-j+r}\nabla e_{k+j-2r} \mid_{q=t=0}$$

where $\delta_{r,0}$ and $\delta_{r,j}$ are Kronecker deltas. We can rewrite this as

$$\sum_{r=0}^{j} \Theta_{m-j+r}\Theta_{\ell-j+r}\nabla e_{k+j-2r} \mid_{q=t=0} + \sum_{r=1}^{j-1} \Theta_{m-j+r}\Theta_{\ell-j+r}\nabla e_{k+j-2r} \mid_{q=t=0}$$

and now making the change of variables $r \mapsto j-r$ in the left summand and $r \mapsto j-r-1$ in the right summand we get

$$\sum_{r=0}^{j} \Theta_{m-r}\Theta_{\ell-r}\nabla e_{k-j+2r} \mid_{q=t=0} + \sum_{r=0}^{j-2} \Theta_{m-r-1}\Theta_{\ell-r-1}\nabla e_{k-j+2r+2} \mid_{q=t=0}. \tag{3.41}$$

Case 3. $a-r=1$.

In this case the right summand in the parentheses vanishes at $q \to 0$ and the left summand evaluates to 1. Since $a \leq j$, the sum restricts to $r \leq j-1$. We get a contribution of

$$\sum_{r=0}^{j-1} \Theta_{m-j+r}\Theta_{\ell-j+r+1}\nabla e_{k+j-2r-1} \mid_{q=t=0};$$

now making the change of variables $r \mapsto j-r-1$ we get

$$\sum_{r=0}^{j-1} \Theta_{m-r-1}\Theta_{\ell-r}\nabla e_{k-j+2r+1} \mid_{q=t=0}. \tag{3.42}$$

The lemma follows immediately by combining the contributions (3.40), (3.41), and (3.42). □

We have all the tools we need to prove our main result Theorem 1.3.

Proof. (of Theorem 1.3) By Theorem 3.2, we aim to show

$$\Theta_m\Theta_\ell\nabla e_k \mid_{q=t=0} = s_{(k+\ell,1^m)} * s_{(k+m,1^\ell)} - s_{(k+\ell+1,1^{m-1})} * s_{(k+m+1,1^{\ell-1})} \tag{3.43}$$

for all integers $k, \ell, m \geq 0$. Notice that both sides of Equation (3.43) are symmetric functions of degree $m+\ell+k$. When $m+\ell+k=0$ both sides specialize to 1 so we assume that $m+\ell+k>0$ and both sides have positive total degree.

We prove this result by induction on the total degree $m+\ell+k$. Let $j \geq 1$. Using Lemma 3.38, we apply the operator $h_j^\perp$ to the left-hand side of Equation (3.43) yielding

$$\begin{aligned}
h_j^\perp \Theta_m \Theta_\ell \nabla e_k|_{q=t=0} &= \sum_{r=0}^{j} \Theta_{m-r}\Theta_{\ell-r}\nabla e_{k-j+2r}\big|_{q=t=0} \\
&\quad + \sum_{r=0}^{j-1} \left(\Theta_{m-r-1}\Theta_{\ell-r} + \Theta_{m-r}\Theta_{\ell-r-1}\right)\nabla e_{k-j+2r+1}\big|_{q=t=0} \\
&\quad + \sum_{r=0}^{j-2} \Theta_{m-r-1}\Theta_{\ell-r-1}\nabla e_{k-j+2r+2}\big|_{q=t=0} \\
(\text{ind. hp.}) &= \sum_{r=0}^{j} s_{k+\ell-j+r,1^{m-r}} * s_{k+m-j+r,1^{\ell-r}} - s_{k+\ell-j+r+1,1^{m-r-1}} * s_{k+m-j+r+1,1^{\ell-r-1}} \\
&\quad + \sum_{r=0}^{j-1} s_{k+\ell-j+r+1,1^{m-r-1}} * s_{k+m-j+r,1^{\ell-r}} - s_{k+\ell-j+r+2,1^{m-r-2}} * s_{k+m-j+r+1,1^{\ell-r-1}} \\
&\quad + \sum_{r=0}^{j-1} s_{k+\ell-j+r,1^{m-r}} * s_{k+m-j+r+1,1^{\ell-r-1}} - s_{k+\ell-j+r+1,1^{m-r-1}} * s_{k+m-j+r+2,1^{\ell-r-2}} \\
&\quad + \sum_{r=0}^{j-2} s_{k+\ell-j+r+1,1^{m-r-1}} * s_{k+m-j+r+1,1^{\ell-r-1}} - s_{k+\ell-j+r+2,1^{m-r-2}} * s_{k+m-j+r+2,1^{\ell-r-2}} \\
(1)-(2) \quad &= \sum_{r=0}^{j} s_{k+\ell-j+r,1^{m-r}} * s_{k+m-j+r,1^{\ell-r}} - \sum_{r=0}^{j} s_{k+\ell-j+r+1,1^{m-r-1}} * s_{k+m-j+r+1,1^{\ell-r-1}} \\
(3)-(4) \quad &\quad + \sum_{r=0}^{j-1} s_{k+\ell-j+r+1,1^{m-r-1}} * s_{k+m-j+r,1^{\ell-r}} - \sum_{r=1}^{j} s_{k+\ell-j+r+1,1^{m-r-1}} * s_{k+m-j+r,1^{\ell-r}} \\
(5)-(6) \quad &\quad + \sum_{r=0}^{j-1} s_{k+\ell-j+r,1^{m-r}} * s_{k+m-j+r+1,1^{\ell-r-1}} - \sum_{r=1}^{j} s_{k+\ell-j+r,1^{m-r}} * s_{k+m-j+r+1,1^{\ell-r-1}} \\
(7)-(8) \quad &\quad + \sum_{r=0}^{j-2} s_{k+\ell-j+r+1,1^{m-r-1}} * s_{k+m-j+r+1,1^{\ell-r-1}} - \sum_{r=2}^{j} s_{k+\ell-j+r,1^{m-r}} * s_{k+m-j+r,1^{\ell-r}}
\end{aligned}$$

where the second equality used induction on degree and the numerals $(1), \ldots, (8)$ on the left of the last expression abbreviate the eight sums therein. We rearrange and make cancellations in these sums, obtaining

$$\begin{aligned}
(1)-(8) \quad &= s_{k+\ell-j,1^m} * s_{k+m-j,1^\ell} + s_{k+\ell-j+1,1^{m-1}} * s_{k+m-j+1,1^{\ell-1}} \\
(3)-(4) \quad &\quad + s_{k+\ell-j+1,1^{m-1}} * s_{k+m-j,1^\ell} - s_{k+\ell+1,1^{m-j-1}} * s_{k+m,1^{\ell-j}} \\
(5)-(6) \quad &\quad + s_{k+\ell-j,1^m} * s_{k+m-j+1,1^{\ell-1}} - s_{k+\ell,1^{m-j}} * s_{k+m+1,1^{\ell-j-1}} \\
(7)-(2) \quad &\quad - s_{k+\ell,1^{m-j}} * s_{k+m,1^{\ell-j}} - s_{k+\ell+1,1^{m-j-1}} * s_{k+m+1,1^{\ell-j-1}}.
\end{aligned}$$

Reindexing these eight summands with $(a), \ldots, (h)$ and applying Equation (3.16) gives

$$\begin{aligned}
(a)+(b) \quad &= s_{k+\ell-j,1^m} * s_{k+m-j,1^\ell} + s_{k+\ell-j+1,1^{m-1}} * s_{k+m-j+1,1^{\ell-1}} \\
(c)-(d) \quad &\quad + s_{k+\ell-j+1,1^{m-1}} * s_{k+m-j,1^\ell} - s_{k+\ell+1,1^{m-j-1}} * s_{k+m,1^{\ell-j}} \\
(e)-(f) \quad &\quad + s_{k+\ell-j,1^m} * s_{k+m-j+1,1^{\ell-1}} - s_{k+\ell,1^{m-j}} * s_{k+m+1,1^{\ell-j-1}} \\
-(g)-(h) \quad &\quad - s_{k+\ell,1^{m-j}} * s_{k+m,1^{\ell-j}} - s_{k+\ell+1,1^{m-j-1}} * s_{k+m+1,1^{\ell-j-1}} \\
(a)+(c) \quad &= h_{k+\ell-j}e_m * s_{k+m-j,1^\ell}
\end{aligned}$$

$$\begin{aligned}
(b)+(e) \qquad & + h_{k+\ell-j}e_m * s_{k+m-j+1,1^{\ell-1}} \\
-(d)-(g) \qquad & - h_{k+\ell}e_{m-j} * s_{k+m,1^{\ell-j}} \\
-(f)-(h) \qquad & - h_{k+\ell}e_{m-j} * s_{k+m+1,1^{\ell-j-1}} \\
& = h_{k+\ell-j}e_m * h_{k+m-j}e_\ell - h_{k+\ell}e_{m-j} * h_{k+m}e_{\ell-j} \\
& = h_j^\perp (s_{k+\ell,1^m} * s_{k+m,1^\ell} - s_{k+\ell+1,1^{m-1}} * s_{k+m+1,1^{\ell-1}})
\end{aligned}$$

where the last step uses Lemma 3.4. In summary, we have

$$h_j^\perp \Theta_m \Theta_\ell \nabla e_k|_{q=t=0} = h_j^\perp (s_{k+\ell,1^m} * s_{k+m,1^\ell} - s_{k+\ell+1,1^{m-1}} * s_{k+m+1,1^{\ell-1}}) \tag{3.44}$$

and since this holds for every $j \geq 1$, by Lemma 3.1 we can deduce that Equation (3.43) holds. This completes the proof of Theorem 1.3. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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