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Haglund's conjecture for multi-t Macdonald polynomials

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ABSTRACT

We provide new approaches to prove identities for the modified Macdonald polynomials via their LLT expansions. As an application, we prove a conjecture of Haglund concerning the multi-t-Macdonald polynomials of two rows.

1. Introduction

In his seminal paper [12], Macdonald introduced the *Macdonald P-polynomials* $P_\mu[X;q,t]$ indexed by partitions μ. The *modified Macdonald polynomials* $\widetilde{H}_\mu[X;q,t]$ are a combinatorial version of the Macdonald P-polynomials and they are characterized as the unique family of symmetric functions satisfying the following *triangularity* and *normalization* axioms (see [6]):

(1) $\widetilde{H}_\mu[X(1-q);q,t]=\sum_{\lambda\geq\mu} a_{\lambda,\mu}(q,t)s_\lambda(X)$,
(2) $\widetilde{H}_\mu[X(1-t);q,t]=\sum_{\lambda\geq\mu'} b_{\lambda,\mu}(q,t)s_\lambda(X)$, and
(3) $\langle\widetilde{H}_\mu, s_{(n)}\rangle=1$,

for suitable coefficients $a_{\lambda,\mu}, b_{\lambda,\mu}\in\mathbb{Q}(q,t)$, where μ' denotes the conjugate partition of μ and $s_\mu(X)$ is the Schur function. The partial order $\leq$ is the *dominance order* on partitions defined by

$$\lambda\leq\mu \text{ if } \lambda_1+\cdots+\lambda_k\leq\mu_1+\cdots+\mu_k \text{ for all } k,$$

$[-]$ denotes the plethystic substitution, and $\langle -,-\rangle$ is the Hall inner product. Haglund, Haiman, and Loehr proved [6] a combinatorial formula for the modified Macdonald polynomials $\widetilde{H}_\mu[X;q,t]$ which generalizes to the *multi-t Macdonald polynomials* $\widetilde{H}_\mu[X;q,t_1,t_2,\dots]$. The polynomial $\widetilde{H}_\mu[X;q,t_1,t_2,\dots]$ specializes to $\widetilde{H}_\mu[X;q,t]$ at $t_1=t_2=\cdots=t$ and depends on an order $c_1,c_2,\dots$ of the cells of μ.

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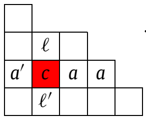


Fig. 1. An example of partition.

LLT polynomials are symmetric functions $\mathrm{LLT}_{\boldsymbol{\nu}}[X;q]$ introduced by Lascoux, Leclerc, and Thibon [11], which depend on a tuple $\boldsymbol{\nu}$ of skew partitions. The LLT polynomial $\mathrm{LLT}_{\boldsymbol{\nu}}[X;q]$ is *unicellular* if every skew partition in $\boldsymbol{\nu}$ is a single cell. Unicellular LLT polynomials are naturally indexed by Dyck paths as well as tuples of skew shapes.

Jim Haglund conjectured [3] a combinatorial formula expanding the multi-t Macdonald polynomials indexed by two-row partitions μ into unicellular LLT polynomials. In this paper, we prove Haglund's conjecture. In the following theorem, we index LLT polynomials with Dyck paths.

Theorem 1.1. *Let $\mu=(n-k,k)$ be a two-row partition and $c_1,\dots,c_k$ be the cells in the upper row. Let $D(h_1,\dots,h_n)$ be the Dyck path of size n whose height of the j-th column is h_j for $1\le j\le n$. Then for $k\le h_1\le\cdots\le h_k$ we have,*

$$\widetilde{H}_\mu[X;q,q^{h_k-k},q^{h_{k-1}-k},\dots,q^{h_1-k}]=\mathrm{LLT}_{D(h_1,\dots,h_k,n,\dots,n)}[X;q],$$

where the left-hand side is the multi-t-Macdonald polynomial $\widetilde{H}_\mu[X;q,t_1,t_2,\dots]$ at $t_i=q^{h_i-k}$ for $1\le i\le k$.

This paper is organized as follows. In Section 2, we provide background on the combinatorics of LLT and modified Macdonald polynomials. Section 3 contains various equivalences of LLT polynomials indexed by different families of skew shapes and a proof of Theorem 1.1 based on these equivalences. In Section 4, we explore 'stretching symmetries' of modified Macdonald polynomials which bear formal similarity to Theorem 1.1. We close in Section 5 with some open problems.

2. Background

2.1. Combinatorics

Let $\mu=(\mu_1,\mu_2,\dots,\mu_\ell)$ be a partition of n. We identify μ with its (French) *Young diagram*

$$\mu=\{(i,j)\in\mathbb{Z}_+\times\mathbb{Z}_+ : j\le\mu_i\}$$

and refer to elements in μ as *cells*. For a cell u in a partition,

- the *content* of $u=(i,j)$ is $c(u):=i-j$,
- the *arm* (resp., *coarm*) of u is the number of cells strictly to the right (resp., left) of u in the same row,
- the *leg* (resp., *coleg*) of u is the number of cells strictly above (resp., below) u in the same column, and
- the *major index* of u is the leg of u plus one.

For example, for a partition $\mu=(5,4,3,2)$, and a cell $c=(2,2)$ as in Fig. 1,

$$\mathrm{arm}(c)=2,\mathrm{coarm}(c)=1,\mathrm{leg}(c)=1,\mathrm{coleg}(c)=2,\text{ and }\mathrm{maj}(c)=2.$$

Let stat be a statistic on cells. For a subset of cells $D\subseteq\mu$, the number $\mathrm{stat}(D)$ is defined by

$$\mathrm{stat}(D):=\sum_{u\in D}\mathrm{stat}(u).$$

For partitions λ and μ with $\mu\subseteq\lambda$, the *skew shape* is the set-theoretic difference $\lambda/\mu:=\lambda-\mu$. A *ribbon* is an edgewise connected skew shape containing no 2×2 block of cells. Note that the contents of the cells of a ribbon are consecutive integers. The *descent set* of a ribbon ν is the set of contents $c(u)$ of those cells $u=(i,j)\in\nu$ such that the cell $v=(i-1,j)$ directly below u also belongs to ν. For an interval $I=[r,r+s]:=\{r,r+1,\dots,r+s\}$, there is a one-to-one correspondence between ribbons of content I and subsets $D\subseteq I\setminus\{r\}$ by considering the descent set of each ribbon (we regard diagonal translations of ribbons as indistinguishable). We denote the ribbon with a content set I and a descent set D by $R_I(D)$. In particular, for any integer a we use $C_a:=R_{\{a\}}(\emptyset)$ to denote a cell of content a. For each subset of cells $D\subseteq\{(i,j)\in\mu : 1<i\}$ where no cell is in the first row, let $D^{(j)}:=\{i:(i,j)\in D\}$. For a partition μ, and a subset D of μ without a cell in the first row, $\boldsymbol{R}_\mu(D)$ is a tuple of ribbons defined by

$$\boldsymbol{R}_\mu(D):=(R_{[1,\mu_1']}(D^{(1)}),R_{[1,\mu_2']}(D^{(2)}),\dots).$$

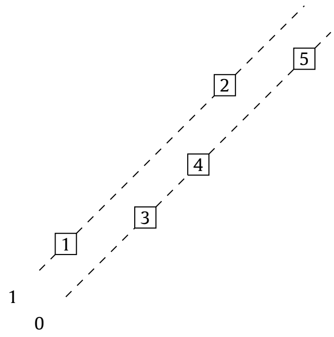


Fig. 2. Labeling of cells.

2.2. LLT polynomials and modified Macdonald polynomials

For a skew partition ν, a *semistandard tableau* of shape ν is a filling of ν with positive integers where each row is weakly increasing from left to right and each column is strictly increasing from bottom to top. For a tuple $\boldsymbol{\nu} = (\nu^{(1)}, \nu^{(2)}, \dots)$ of skew partitions, a semistandard tableau $\boldsymbol{T} = (T^{(1)}, T^{(2)}, \dots)$ of shape $\boldsymbol{\nu}$ is a tuple of semistandard tableaux where each $T^{(i)}$ is a semistandard tableau of shape $\nu^{(i)}$. The set of semistandard tableaux of shape $\boldsymbol{\nu}$ is denoted by $\mathrm{SSYT}(\boldsymbol{\nu})$. For a semistandard tableau $\boldsymbol{T} = (T^{(1)}, T^{(2)}, \dots)$ of shape $\boldsymbol{\nu}$, an *inversion* of $\boldsymbol{T}$ is a pair of cells $u \in \nu^{(i)}$ and $v \in \nu^{(j)}$ such that $T^{(i)}(u) > T^{(j)}(v)$ and either

- $i < j$ and $c(u) = c(v)$, or
- $i > j$ and $c(u) = c(v) + 1$.

Denote by $\mathrm{inv}(\boldsymbol{T})$ the number of inversions in $\boldsymbol{T}$. The *LLT polynomial* $\mathrm{LLT}_{\boldsymbol{\nu}}[X;q]$ is defined by

$$\mathrm{LLT}_{\boldsymbol{\nu}}[X;q] = \sum_{\boldsymbol{T} \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(\boldsymbol{T})} X^{T}.$$

Here, $X^{\boldsymbol{T}} := x_1^{\boldsymbol{T}_1} x_2^{\boldsymbol{T}_2} \cdots$, where $\boldsymbol{T}_i$ is the number of i's in $\boldsymbol{T}$.

If a tuple $\boldsymbol{\nu}$ of skew partitions consists of single cells, the LLT polynomial $\mathrm{LLT}_{\boldsymbol{\nu}}[X;q]$ is called *unicellular*. Dyck paths can also index unicellular LLT polynomials. Given a tuple of n cells, index these cells in order of increasing content, breaking ties by indexing from left to right within a given content. For all i, let h_i be the maximal index j such that the i-th and j-th cell forms an inversion pair, i.e., $i < j$ and either both cells are in the same row or the j-th cell is in the next row and strictly to the left of the i-th cell. Then the correspondence sends the tuple of cells to a Dyck path, where the height of i-th column is given by h_i. For example, a tuple of cells in Fig. 2 corresponds to the Dyck path of height $(2, 4, 5, 5, 5)$.

Haglund–Haiman–Loehr also provided an expansion of the modified Macdonald polynomials into LLT polynomials indexed by tuples of ribbons.

Theorem 2.1. *[6, Section 3] For a partition μ, we have*

$$\widetilde{H}_\mu[X;q,t] = \sum_D q^{-\mathrm{arm}(D)} t^{\mathrm{maj}(D)}\, \mathrm{LLT}_{\boldsymbol{R}_\mu(D)}[X;q],$$

where the sum is over all subsets $D \subseteq \{(i,j) \in \mu : 1 < i\}$.

As a generalization of the above LLT expansion (or a combinatorial formula) of modified Macdonald polynomials, the *multi-t Macdonald polynomial* $\widetilde{H}_\mu[X;q,t_1,t_2\dots]$ is defined as follows: For a partition μ, consider an ordering $c_1, c_2, \dots$ of cells in μ. Then the multi-t-Macdonald polynomial is

$$\widetilde{H}_\mu[X;q,t_1,t_2,\dots] := \sum_D q^{-\mathrm{arm}(D)} \prod_{c_i \in D} t_i^{\mathrm{maj}(c_i)}\, \mathrm{LLT}_{\boldsymbol{R}_\mu(D)}[X;q], \tag{2.1}$$

where the sum is over all subsets $D \subseteq \{(i,j) \in \mu : 1 < i\}$. By specializing each t variable $t_i = t$, the multi-t-Macdonald polynomial is just a usual modified Macdonald polynomial.

3. Proving Haglund's conjecture

3.1. LLT-equivalences

Write $[n]_q := \frac{1-q^n}{1-q}$ for the q-analog of an integer n. Let $\sum_i a_i(q)\boldsymbol{\nu}^{(i)}$ and $\sum_j b_j(q)\boldsymbol{\mu}^{(j)}$ be $\mathbb{N}[q]$-linear combinations of tuples of skew partitions. Following Miller [13], we say these linear combinations are *LLT-equivalent* if for every tuple $\boldsymbol{\lambda}$ of skew partitions, we have

$$\sum_i a_i(q)\,\mathrm{LLT}_{(\boldsymbol{\nu}^{(i)},\boldsymbol{\lambda})}[X;q] = \sum_j b_j(q)\,\mathrm{LLT}_{(\boldsymbol{\mu}^{(j)},\boldsymbol{\lambda})}[X;q].$$

Here $(\boldsymbol{\nu},\boldsymbol{\lambda})$ is the tuple obtained by concatenating $\boldsymbol{\nu}$ and $\boldsymbol{\lambda}$. Abusing notation, we write

$$\sum_i a_i(q)\boldsymbol{\nu}^{(i)} = \sum_j b_j(q)\boldsymbol{\mu}^{(j)},$$

to indicate LLT-equivalence. In this section, we establish a series of LLT-equivalences which are ribbon-analogues of results in [13,10,7].

We prove some of the LLT-equivalences inductively, by showing that both sides satisfy 'linear relations'. To be more precise, we say that a function $f(\alpha)$ of integers α satisfies a *linear relation* in α if we have

$$qf(\alpha)+f(\alpha+2)=[2]_q f(\alpha+1).$$

If a function $f(D)$ is defined for Dyck paths D, we say f satisfies a *row linear relation* if we have

$$qf(D)+f(D'')=[2]_q f(D'),$$

where Dyck paths D, D', and D'' are Dyck paths where they differ at only one row, and the number of boxes in that row below the Dyck path is given by a, $a+1$ and $a+2$, respectively. We define *column linear relation* similarly.

Let R be a ribbon of content $[r-1]$ with a descent set D. We can add a cell of content r to R to obtain a ribbon of content $[r]$ in two ways; add a cell above or to the left of the cell of content $r-1$. In other words, there are two ribbons $R_H^+ := R_{[r]}(D)$ and $R_V^+ := R_{[r]}(D\cup\{r\})$ obtained by adding a cell to R. We used the $+$ sign to mean that one cell is added, and H and V stand for adding a cell horizontally or vertically. Our first LLT equivalence is as follows.

Proposition 3.1. *For a ribbon R of content $[r-1]$, we have the following LLT equivalence*

$$\begin{aligned} R_H^+ + q^\alpha R_V^+ &= [\alpha]_q(R,C_r) - q[\alpha-1]_q(C_r,R)\\ &= \begin{cases} [\alpha]_q(R,C_r)-q[\alpha-1]_q(C_r,R) & \text{if } \alpha\geq 1, \text{ and}\\ -q^\alpha[-\alpha]_q(R,C_r)+q^\alpha[1-\alpha]_q(C_r,R) & \text{if } \alpha<1\end{cases}\end{aligned}$$

for all integers α.

Proof. The formula is easy to see when $\alpha=0,1$. Since each of the three expressions satisfies a linear relation in α, the proposition follows. □

Our second LLT equivalence is a linear relation for LLT polynomials which generalizes the local linear relation of unicellular LLT polynomials given in [10,7].

Proposition 3.2. *For a ribbon R of content $[r-1]$, we have the following LLT equivalence*

$$[k]_q\left(C_r^{k-1},R,C_r\right) = q[k-1]_q\left(C_r^k,R\right)+\left(R,C_r^k\right).$$

Proof. The following linear relation is established in [10]:

$$[2]_q\left(C_r,C_{r-1},C_r\right) = q\left(C_r^2,C_{r-1}\right)+\left(C_{r-1},C_r^2\right).$$

The only difference between this LLT equivalence and ours is that we replaced C_{r-1} with a ribbon R of content $[r-1]$. Since a cell of content r cannot form an inversion pair with a cell of content less than $r-1$, it suffices to care about the cell of content $r-1$ in R (the last cell). The proof of [10, Theorem 3.5] applies to show

$$[2]_q\left(C_r,R,C_r\right) = q\left(C_r^2,R\right)+\left(R,C_r^2\right).$$

Applying this inductively, as in [7, Proof of Theorem 3.4], proves the proposition. □

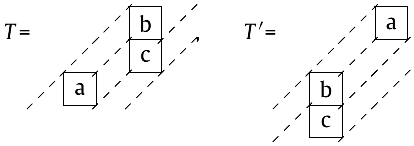
Our third LLT equivalence is a commuting relation between a ribbon and a cell.

Proposition 3.3. *[13] Let R be a ribbon of content $[r]$ with a descent set D. Then we have the following LLT equivalence*

$$(C_r, R) = \begin{cases} q^{-1}(R, C_r) & \text{if } r \in D, \text{ and} \\ (R, C_r) & \text{otherwise.} \end{cases} \tag{3.1}$$

Proof. To prove that two linear combinations of tuples of skew partitions are LLT equivalent, it suffices to show that there is a bijection preserving weight, inversion, and content between semistandard tableaux corresponding to those. Since a similar argument also proves the second case, we only prove the first case.

For semistandard tableaux of shape (C_r, R) and (R, C_r) are the followings tableaux with either of the following four conditions holds: $a > b > c$, $a = b > c$, $b > a > c$, or $b > c \geq a$.



We let the bijection preserve the fillings of the cells of R of content less than $r-1$, so we omit those cells. In Table (3.2), we give the q-weight of each side of (3.1), $\mathrm{inv}(T)$ and $\mathrm{inv}(T')-1$ for each case.

	$a > b > c$	$a = b > c$	$b > a > c$	$b > c \geq a$
T	1	0	0	0
$q^{-1}T'$	0	0	1	0

(3.2)

Let us define a bijection sending T to T' if a, b and c satisfies the second or the fourth case ($a = b > c$ or $b > c \geq a$), and sending T to a tableau obtained from T' by switching a and b otherwise ($a > b > c$ or $b > a > c$). By definition of the bijection, it is obviously weight and content preserving, and (3.2) shows that it is also inversion preserving. □

For the last, we need one more commuting relation between dominoes given in [10].

Proposition 3.4. *[10, Lemma 5.5] Let V and H be vertical and horizontal dominoes of content $[r, r+1]$, respectively. Then we have the following LLT equivalence*

$$(V, H) = (H, V).$$

3.2. Proof of Haglund's conjecture

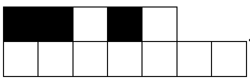
Let $\mu = (n-k, k)$. Note that for a subset $D \subseteq \{(2, j) \in \mu\}$, $\mathrm{arm}(D) = \sum_{(2,j)\in D}(k-j)$. Therefore, the multi-$t$-Macdonald polynomial is given by

$$\widetilde{H}_\mu[X; q, q^{h_k - k}, q^{h_{k-1}-k}, \ldots, q^{h_1 - k}] = \sum_D q^{\sum_{(2,j)\in D} h_{k+1-j} - 2k + j} \mathrm{LLT}_{\boldsymbol{R}_\mu(D)}[X; q],$$

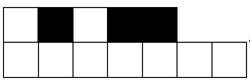
where the sum is over all subsets $D \subseteq \{(2, j) \in \mu\}$. For a subset $D \subseteq \{(2, j) \in \mu\}$, we define

$$D^{rev} = \{(2, k+1-j) : (2, j) \in D\}.$$

For example, let $\mu = (7, 5)$ and

$$D = \{(2,1), (2,2), (2,4)\} =$$  .

Then we have

$$D^{rev} = \{(2,2), (2,4), (2,5)\} =$$  .

The tuple of ribbons corresponding to D and D^{rev} are

$$\boldsymbol{R}_\mu(D) = (V, V, H, V, H, C, C) \quad \text{and} \quad \boldsymbol{R}_\mu(D^{rev}) = (H, V, H, V, V, C, C),$$

where V and H are vertical and horizontal dominoes (of content 1,2) and C is a cell of content 1. Since Proposition 3.4 says that we can swap the vertical and horizontal dominoes (V's and H's),

$$\mathrm{LLT}_{\boldsymbol{R}_\mu(D)}[X;q] = \mathrm{LLT}_{\boldsymbol{R}_\mu(D^{rev})}[X;q].$$

Thus we have

$$\widetilde{H}_\mu[X;q,q^{h_k-k},q^{h_{k-1}-k},\dots,q^{h_1-k}] = \sum_D q^{\sum_{(2,j)\in D} h_j-k-j+1}\,\mathrm{LLT}_{\boldsymbol{R}_\mu(D)}[X;q]. \tag{3.3}$$

Our goal is to show that the right-hand side of (3.3) is the same as $\mathrm{LLT}_{D(h_1,\dots,h_k,n,\dots,n)}[X;q]$.

There are two steps for proving this identity using induction. The first step is to show the identity for "near-staircase" shapes, namely, the case when $h_j = k+j-1+e_j$ for $j=1,\dots,k$, where e_j is either 0 or 1. These partitions are conjugate to partitions containing $(n-k-1, n-k-2,\dots,n-2k)$ and contained in $(n-k,n-k-1,\dots,n-2k+1)$. Proof of the first step is straightforward. When $h_j = k+j-1+e_j$, we have $h_j-k-j+1=e_j$ so by Proposition 3.1, (3.3) becomes

$$\sum_D q^{\sum_{c_j\in D} e_j}\,\mathrm{LLT}_{R_\mu(D)}[X;q] = \mathrm{LLT}_{(P_{e_1},P_{e_2},\dots,P_{e_k},C_1,C_1,\dots,C_1)}[X;q],$$

where $P_1=(C_1,C_2)$ and $P_0=(C_2,C_1)$ and there are $n-2k$ number of C_1's in total.

The second step is to show that (3.3) satisfies both column and row linear relations. Proving the column linear relation is trivial by the formula since a function q_j^h is linear in h_j. Therefore, one only needs to prove the row linear relation. Let $f(h_1,\cdots,h_k)$ be the right-hand side of equation (3.3). Then we need to show that for any positive integer $i\in[k]$ and $a\le n$ satisfying $h_{i-1}\le a$ and $h_{i+2}\ge a+1$, the following holds:

$$q\cdot f(h_1,\dots,h_{i-1},a,a,h_{i+2},\dots,h_k) + f(h_1,\dots,h_{i-1},a+1,a+1,h_{i+2},\dots,h_k) \tag{3.4}$$

$$= [2]_q f(h_1,\dots,h_{i-1},a,a+1,h_{i+2},\dots,h_k) \tag{3.5}$$

When $(2,i)$ and $(2,i+1)$ are both in D, or neither of them is in D, the corresponding sum $\sum_D q^{\sum_{(2,j)\in D} h_j-k-j+1}\,\mathrm{LLT}_{\boldsymbol{R}_\mu(D)}[X;q]$ satisfies (3.4)=(3.5).

When exactly one of $(2,i),(2,i+1)$ is in D, the corresponding function for (3.4) is

$$\begin{aligned}
&\sum_D q^{1+\chi(i,D)(a-k-i+1)+\chi(i+1,D)(a-k-i)} q^{\sum_{(2,j)\in D, j\ne i,i+1} h_j-k-j+1}\,\mathrm{LLT}_{\boldsymbol{R}_\mu(D)}[X;q]\\
&+\sum_D q^{\chi(i,D)(a-k-i+2)+\chi(i+1,D)(a-k-i+1)} q^{\sum_{(2,j)\in D, j\ne i,i+1} h_j-k-j+1}\,\mathrm{LLT}_{\boldsymbol{R}_\mu(D)}[X;q]\\
&\qquad = 2[2]_q q^{a-k-i+1}\sum_{D'} q^{\sum_{(2,j)\in D, j\ne i,i+1} h_j-k-j+1}\,\mathrm{LLT}_{\boldsymbol{R}_\mu(D')}[X;q]
\end{aligned}$$

where $\chi(i,D)$ is 1 if $(2,i)\in D$, 0 otherwise, and the sum in the last line runs over all $D'\subset\{(2,j)\in\mu\}$ satisfying $(2,i)\in D'$ and $(2,i+1)\notin D'$. Here we applied Proposition 3.4.

By a similar argument, one can prove

$$2q^{a-k-i+1}\sum_{D'} q^{\sum_{(2,j)\in D, j\ne i,i+1} h_j-k-j+1}\,\mathrm{LLT}_{\boldsymbol{R}_\mu(D')}[X;q]$$

is the same as (3.5), proving the claim.

We show that proving the two steps is enough to show Haglund's conjecture using induction, where the base case is the conjecture for near-staircase shapes.

By the induction hypothesis, Haglund's conjecture is true for a Dyck path $D(h_1',h_2',\dots,h_k')$ where $h_1'=k$ or $k+1$, and $k+1\le h_i'\le n$ for any $1<i$. Assume that we have any Dyck path $D(h_1,\dots,h_k)$ satisfying $k\le h_i\le n$. There are two cases:

(1) $h_2\ge k+2$. In this case, since we know that Haglund's conjecture holds for $h_1=k$ and $k+1$, it holds for any $k\le h_1\le h_2$ by using the fact that both sides of Haglund's conjecture satisfy the column linear relation.

(2) $h_2=k$ or $k+1$. In this case, consider the largest index i satisfying $h_i=k$. If there is no i or $i=1$, the corresponding two Dyck paths satisfy Haglund's conjecture by an induction. If $i>1$, then we use two previous paths and the row linear relation to show that Haglund's conjecture holds for any i.

4. Stretching symmetries

The starting point for this project was observing a certain 'stretching symmetry' satisfied by the modified Macdonald polynomials $\widetilde{H}_\mu[X;q,t]$. After the authors posted an earlier version of this paper was posted to the arxiv, Mark Haiman [5] informed the authors that this observation follows from the work of Garsia and Tesler [2]. We reproduce this derivation here.

Given a partition $\mu=(\mu_1,\mu_2,\dots)$, let $k\mu$ be the partition $k\mu=(k\mu_1,k\mu_2,\dots)$ obtained by multiplying each part of μ by k. This stretching operation on partitions lifts to the following symmetry of the modified Macdonald polynomials at $t=q^k$.

Theorem 4.1. *[2] Let μ be a partition and k be a positive integer. Then we have*

$$\widetilde{H}_{k\mu}[X;q,q^k]=\widetilde{H}_{k\mu'}[X;q,q^k].$$

Proof. For any partition λ, let $B_\lambda(q,t)$ be the polynomial

$$B_\lambda(q,t):=\sum_{c\in\lambda} q^{\mathrm{coarm}_\lambda(c)}t^{\mathrm{coleg}_\lambda(c)}.$$

For any two partitions λ,ν, Theorem 1.1 of [2] yields the implication

$$B_\lambda(q^r,q^s)=B_\nu(q^r,q^s) \quad\Rightarrow\quad \widetilde{H}_\lambda[X;q^r,q^s]=\widetilde{H}_\nu[X,q^r,q^s] \tag{4.1}$$

for any positive integers r,s. Since $B_{k\mu}(q,q^k)=B_{k\mu'}(q,q^k)$, the result follows. $\square$

Like Theorem 1.1, Theorem 4.1 concerns specializing modified Macdonald polynomials by sending t to a power of q. In a forthcoming paper of the first and second named authors with Donghyun Kim, a different proof [9] of Theorem 4.1 will be given. This proof will show that the equality of B-polynomials $B_\lambda(q,q^k)=B_\nu(q,q^k)$ is equivalent to the equality $\widetilde{H}_\lambda[X;q,q^k]=\widetilde{H}_\mu[X;q,q^k]$ of specialized Macdonald polynomials (so that the converse of the implication (4.1) holds).

For the rest of this section, we study Theorem 4.1, where the partition μ is a single column. To be more precise, we give two new proofs of the following result.

Theorem 4.2. *For nonnegative integers k and ℓ, we have*

$$\widetilde{H}_{(k^\ell)}[X;q,q^k]=\widetilde{H}_{(k\ell)}[X;q,q^k]=\sum_{T\in\mathrm{SYT}(n)} q^{\mathrm{maj}(T)}s_{\lambda(T)}$$

where $\lambda(T)$ is the shape of the standard tableau T.

The final equality in Theorem 4.2 holds because $H_{(n)}[X;q,t]=\sum_{T\in\mathrm{SYT}(n)}q^{\mathrm{maj}(T)}s_{\lambda(T)}$ is the (singly) graded Frobenius image of the coinvariant ring attached to the symmetric group $\mathfrak{S}_n$; see Subsection 4.2 for more details. We prove Theorem 4.2 in two ways: a combinatorial argument using LLT-equivalences and an algebraic argument involving Garsia-Haiman modules.

4.1. Combinatorial proof of Theorem 4.2

Let $\mu=(k^\ell)$. By Theorem 2.1, we have an LLT expansion of the modified Macdonald polynomial of μ at $t=q^k$ by

$$\widetilde{H}_\mu(x;q,q^k)=\sum_{D\subseteq\{(i,j)\in\mu:1<i\}} q^{k\,\mathrm{maj}(D)-\mathrm{arm}(D)}\,\mathrm{LLT}_{\boldsymbol{R}_\mu(D)}[X;q]. \tag{4.2}$$

Note that for the cells in $\{(i,j):1<i\le\ell,1\le j\le k\}$, the q-statistics

$$k\,\mathrm{maj}-\mathrm{arm}$$

in (4.2) are given by $1,2,\dots,k$ for the cells in the top row, $k+1,k+2,\dots,2k$ for the cells in the second top row, and so on.

Choose a subset $E\subseteq\{(i,j)\in\mu:1<i<\ell\}$ consisting of cells not in the first and the last row and take a partial sum of the right-hand-side of (4.2) over subsets $D\subseteq\{(i,j)\in\mu:1<i\}$, where D restricted to $\{(i,j)\in\mu:1<i<\ell\}$ equals to E. This partial sum gives $q^{k\,\mathrm{maj}(E)-\mathrm{arm}(E)}$ times the following sum

$$\sum_D q^{\sum_{(\ell,j)\in D} j}\,\mathrm{LLT}_{\boldsymbol{R}_\mu(D)}[X;q], \tag{4.3}$$

where the sum is over subsets D whose restriction to $\{(i,j)\in\mu : 1<i<\ell\}$ equals to E.

We claim that the summation in (4.3) is LLT equivalent with a *single* tuple of ribbons:

$$\sum_D q^{\sum_{(\ell,j)\in D} j}\boldsymbol{R}_\mu(D)=\left(\boldsymbol{R}_{(k^{\ell-1})}(E),C_\ell^k\right), \tag{4.4}$$

where the sum in the left-hand-side is over subsets D whose restriction to $\{(i,j)\in\mu : 1<i<\ell\}$ equals to E. We prove this claim by induction on k. For the initial case, assume $k=1$. By Proposition 3.1 for $\alpha=1$, we have

$$\boldsymbol{R}_{(1^\ell)}(E)+q\boldsymbol{R}_{(1^\ell)}(E\cup\{\ell\})=\left(R_{[\ell-1]}(E),C_\ell\right),$$

which proves the claim for $k=1$.

Assume $k>1$. Then we have

$$\begin{aligned}
&\sum_D q^{\sum_{j:(\ell,j)\in D} j}\boldsymbol{R}_\mu(D)\\
&=\left(\boldsymbol{R}_{((k-1)^{\ell-1})}(\{(i,j)\in E:1\le i\le k-1\}),C_\ell^{k-1},R_{[\ell]}(E^{(k)})\right)\\
&\qquad\qquad +q^k\left(\boldsymbol{R}_{((k-1)^{\ell-1})}(\{(i,j)\in E:1\le i\le k-1\}),C_\ell^{k-1},R_{[\ell]}(E^{(k)}\cup\{\ell\})\right)\\
&=[k]_q\left(\boldsymbol{R}_{((k-1)^{\ell-1})}(\{(i,j)\in E:1\le i\le k-1\}),C_\ell^{k-1},R_{[\ell-1]}(E^{(k)}),C_\ell\right)\\
&\qquad\qquad -q[k-1]_q\left(\boldsymbol{R}_{((k-1)^{\ell-1})}(\{(i,j)\in E:1\le i\le k-1\}),C_\ell^{k},R_{[\ell-1]}(E^{(k)})\right)\\
&=\left(\boldsymbol{R}_{(k^{\ell-1})}(E),C_\ell^k\right).
\end{aligned}$$

The first equation follows by the induction hypothesis. The second equation follows from Proposition 3.1 and the third equation follows from Proposition 3.2. This proves the claim.

Sliding the cells C_ℓ's to the leftmost part of the diagonal of content $\ell-1$ gives

$$\left(\boldsymbol{R}_{(k^{\ell-1})}(E),C_\ell^k\right)=\left(C_{\ell-1}^k,\boldsymbol{R}_{(k^{\ell-1})}(E)\right). \tag{4.5}$$

Recall that by Proposition 3.3, we can swap a cell of content r and a ribbon R of content $[r]$ where a weight q^{-1} is attached in the case the last cell of R is a descent. Thus,

$$\left(C_{\ell-1}^k,\boldsymbol{R}_{(k^{\ell-1})}(E)\right)=q^{-k|\{j:(\ell-1,j)\in E\}|}\left(\boldsymbol{R}_{(k^{\ell-1})}(E),C_{\ell-1}^k\right), \tag{4.6}$$

Combining (4.4), (4.5) and (4.6), we conclude that

$$\widetilde{H}_\mu[X;q,q^k]=\sum_E q^{k\,\mathrm{maj}(E)-\mathrm{arm}(E)-k|\{j:(\ell-1,j)\in E\}|}\,\mathrm{LLT}_{\left(\boldsymbol{R}_{(k^{\ell-1})}(E),C_{\ell-1}^k\right)}[X;q], \tag{4.7}$$

where the sum is over all subsets $E\subseteq\{(i,j)\in\mu:1<i<\ell\}$. Note that each cell in the top row in E contributes to the q-weight

$$k\,\mathrm{maj}(E)-\mathrm{arm}(E)-k\,|\{j:(\ell-1,j)\in E\}|$$

in (4.7) by $1,2,\dots,k$ as in the initial case. Therefore we can apply the whole procedure again to obtain

$$\widetilde{H}_\mu[X;q,q^k]=\sum_E q^{k\,\mathrm{maj}(E)-\mathrm{arm}(E)-2k|\{j:(\ell-2,j)\in E\}|}\,\mathrm{LLT}_{\left(\boldsymbol{R}_{(k^{\ell-2})}(E),C_{\ell-1}^{2k}\right)}[X;q],$$

where the sum is over all $E\subseteq\{(i,j)\in\mu:1<i<\ell-1\}$. By applying this repeatedly, we prove Theorem 4.2.

4.2. Representation-theoretic proof of Theorem 4.2

For any $\mu\vdash n$, Haiman established [4] the Schur positivity of $\widetilde{H}_\mu[X;q,t]$ by proving

$$\mathrm{grFrob}(V_\mu;q,t)=\widetilde{H}_\mu[X;q,t] \tag{4.8}$$

where V_μ is the *Garsia-Haiman module* [1] attached to μ. The module V_μ is the following subspace of $\mathbb{C}[X_n,Y_n]:=\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]$. Fix a bijective labeling T of the boxes of μ with $1,2,\dots,n$ and define a polynomial $\delta_\mu\in\mathbb{C}[X_n,Y_n]$ by

$$\delta_\mu := \varepsilon_n \cdot \prod_{c \in \mu} x_{T(c)}^{\mathrm{coarm}(c)} y_{T(c)}^{\mathrm{coleg}(c)} \tag{4.9}$$

where $\mathfrak{S}_n$ acts on $\mathbb{C}[X_n, Y_n]$ diagonally and $\varepsilon_n := \sum_{w \in \mathfrak{S}_n} \mathrm{sign}(w) w$ is the antisymmetrizing idempotent. The Garsia-Haiman module V_μ is the smallest linear subspace of $\mathbb{C}[X_n, Y_n]$ containing δ_μ which is closed under the partial differentiation operators $\partial/\partial x_i$ and $\partial/\partial y_i$ for $1 \leq i \leq n$. In particular, when $\mu \vdash n$ is a single row or column, we have an isomorphism of singly-graded $\mathfrak{S}_n$-modules $V_\mu \cong R_n$, where $R_n := \mathbb{C}[X_n]/\langle \mathbb{C}[X_n]_+^{\mathfrak{S}_n} \rangle$ is the type A coinvariant algebra in the x-variables.

Let $\eta : \mathbb{C}[X_n, Y_n] \twoheadrightarrow \mathbb{C}[X_n]$ be the evaluation map that fixes x_i and specializes $y_i \mapsto (x_i)^k$. For any $\mu \vdash n$, we have an $\mathfrak{S}_n$-module homomorphism $\varphi_\mu : V_\mu \to R_n$ given by the composition

$$\varphi_\mu : V_\mu \hookrightarrow \mathbb{C}[X_n, Y_n] \xrightarrow{\eta} \mathbb{C}[X_n] \twoheadrightarrow R_n \tag{4.10}$$

of including V_μ into $\mathbb{C}[X_n, Y_n]$, evaluating along η, and then projecting onto R_n.

Proposition 4.3. *If $n = k\ell$ and $\mu = (k^\ell)$ is a rectangle, then φ_μ is an isomorphism.*

Since the (singly) graded Frobenius image of R_n is

$$\mathrm{grFrob}(R_n; q) = \sum_{\lambda \vdash n} \left(\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} \right) \cdot s_\lambda[X] \tag{4.11}$$

and η evaluates the y-variables to degree k, Proposition 4.3 implies Theorem 4.2. We prove Proposition 4.3 as follows.

Proof. The domain and codomain of φ_μ are both vector spaces of dimension $n!$, so it is enough to show that the image of φ_μ spans R_n. We choose our filling T of the k-by-ℓ rectangle μ so that

$$\delta_\mu = \varepsilon_n \cdot \left(\prod_{\substack{0 \leq i \leq k-1 \\ 0 \leq j \leq \ell-1}} x_{\ell j+i+1}^{i} y_{\ell j+i+1}^{j} \right). \tag{4.12}$$

This corresponds to the 'English reading order' standard filling of μ. The evaluation $\eta(\delta_\mu)$ of the Vandermonde determinant is the image in R_n of the Vandermonde, i.e.,

$$\eta(\delta_\mu) = \varepsilon_n \cdot (x_1^0 x_2^1 \cdots x_n^{n-1}). \tag{4.13}$$

If we endow monomials in $x_1, \dots, x_n$ with the lex term order **with underlying variable order** $x_1 < \cdots < x_n$ the leading monomial of $\eta(\delta_\mu)$ is $x_1^0 x_2^1 \cdots x_n^{n-1}$. We show that any exponent sequence $(a_1, \dots, a_n)$ with $a_i < i$ is the leading monomial of some polynomial in $\eta(V_\mu)$. Since such monomials constitute the standard basis of R_n with respect to the aforementioned term order, this completes the proof.

Suppose we have a componentwise inequality $(a_1, a_2, \dots, a_n) \leq (0, 1, \dots, n-1)$. We apply the Euclidean algorithm to any difference $(i-1) - a_i$ to write

$$(i-1) - a_i = q_i m + r_i,$$

where $q_i \geq 0$ and $0 \leq r_i < m$. We have an element of V_μ given by

$$(\partial/\partial x_1)^{r_1} (\partial/\partial y_1)^{q_1} \cdots (\partial/\partial x_n)^{r_n} (\partial/\partial y_n)^{q_n} (\delta_\mu). \tag{4.14}$$

The image of (4.14) under η has leading monomial $x_1^{a_1} \cdots x_n^{a_n}$, and the argument in the last paragraph completes the proof. $\square$

If $\mu \vdash n$ is not a rectangle, the restriction of η to V_μ is not injective, so the above argument does not go through.

5. Concluding remarks

5.1. Schur positivity

Theorem 4.2 is equivalent to the assertion that

$$\widetilde{H}_{(k\ell)}[X; q, t] - \widetilde{H}_{(k^\ell)}[X; q, t]$$

is divisible by $(q^k - t)$. The proof in Section 3 does not only show the equality of the modified Macdonald polynomials at $t = q^k$ in Theorem 4.2 but also proves the LLT positivity, thus Schur positivity of the quotient

$$\frac{\widetilde{H}_{(k\ell)}[X;q,t] - \widetilde{H}_{(k^\ell)}[X;q,t]}{q^k - t}.$$

For any partition μ, Theorem 4.1 implies that the rational function

$$\frac{\widetilde{H}_{k\mu}[X;q,t] - \widetilde{H}_{k\mu'}[X;q,t]}{q^k - t} \tag{5.1}$$

is a polynomial in q, t, and X. Surprisingly, some (but not all) of the quotients (5.1) are Schur positive. In particular, SAGE computations suggest that if each cell $c = (i, j)$ of μ satisfies either

(1) c is also contained in μ', or
(2) c is under the main diagonal, i.e. $i < j$,

then (5.1) is Schur positive. It is an interesting question to ask for necessary and sufficient conditions for two partitions λ, μ, and $k \geq 0$ so that (5.1) is Schur positive.

5.2. Combinatorial formula for Kostka polynomials

A combinatorial formula for (q,t)-Kostka polynomials is unknown in general, and finding one is one of the most important open problems in algebraic combinatorics.

We recall that a *standard tableau* of a partition $\lambda \vdash n$ is a semistandard tableau consisting of $1, 2, \ldots, n$. We denote the set of standard tableaux of shape λ by $\mathrm{SYT}(\lambda)$. It is well known that (q,t)-Kostka polynomial at $(1,1)$ gives

$$\widetilde{K}_{\lambda,\mu}(1,1) = |\,\mathrm{SYT}(\lambda)|.$$

Therefore, the most desirable form of a combinatorial formula for (q,t)-Kostka polynomials would be given by a generating function for the standard tableaux with two statistics:

$$\widetilde{K}_{\lambda,\mu}(q,t) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{stat}_q(T)} t^{\mathrm{stat}_t(T)}.$$

Theorem 4.2 implies

$$\widetilde{K}_{\lambda,(k^\ell)}(q,q^k) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}.$$

This suggests that the modified Macdonald polynomials (or modified (q,t)-Kostka polynomials) for rectangle μ might have more structure. For example, Theorem 4.2 implies the desirable (q,t)-statistics stat_q and stat_t such that

$$\mathrm{stat}_q + k\mathrm{stat}_t$$

and maj statistics are equidistributed over $\mathrm{SYT}(\lambda)$. We hope that this gives a hint to track the q,t-statistics for (q,t)-Kostka polynomials for rectangles. The (q,t)-Kostka polynomials for partitions of 4 are given in the following table.

$\mu \setminus \lambda$	[4]	[3,1]	[2,2]	[2,1,1]	[1,1,1,1,1]
[4]	1	$q+q^2+q^3$	q^2+q^4	$q^3+q^4+q^5$	q^6
[2,2]	1	$t+qt+q$	t^2+q^2	qt^2+qt+q^2t	q^2t^2
[1,1,1,1]	1	$t+t^2+t^3$	t^2+t^4	$t^3+t^4+t^5$	t^6

5.3. A common generalization of two main theorems

There is an intersection between Theorem 1.1 and Theorem 4.2. More precisely, taking $\mu = (k,k)$ and $h_i = 2k$ for all i's in Theorem 1.1 yields

$$\widetilde{H}_{(n,n)}[X;q,q^n] = \sum_{T \in \mathrm{SYT}(2n)} q^{\mathrm{maj}(T)} X^T. \tag{5.2}$$

Taking $\mu = (2)$ (or $\mu = (1,1)$) in Theorem 4.2 also yields (5.2). Therefore, it is natural to ask the following question.

Question 5.1. *Is there a common generalization of Theorem 4.1 and Theorem 1.1?*

5.4. A new Mahonian statistic

For any $k \geq 1$, we define a statistic maj_k on words $w = w_1 w_2 \dots$ over the positive integers by

$$\mathrm{maj}_k(w) := \sum_{0<j-i<k} \chi((i,j) \in \mathrm{Inv}(w)) + \sum_i i\,\chi((i,i+k) \in \mathrm{Inv}(w))$$

where $\mathrm{Inv}(w)$ is the inversion set of w and for a statement P, we let $\chi(P) = 1$ if P is true and $\chi(P) = 0$ if P is false. We recover the classical major index $\mathrm{maj}_1(w) = \mathrm{maj}(w)$ at $k = 1$. Theorem 4.2 and the formula for modified Macdonald polynomials in [6] imply that for any partition $\mu = (\mu_1, \dots, \mu_k) \vdash n$ and any integer $k \geq 1$, we have

$$\sum_{w \in W(\mu)} q^{\mathrm{maj}_k(w)} = \frac{[n]!_q}{[\mu_1]!_q \cdots [\mu_k]!_q}$$

where $W(\mu)$ is the set of words $w = w_1 \dots w_n$ of content μ and $[n]!_q := [n]_q[n-1]_q \cdots [1]_q$ is the q-analog of $n!$. In particular, the operators maj_k for $k \geq 1$ are equidistributed on $W(\mu)$. Kadell gave [8] a bijective proof of this fact.

Corollary 5.2. *For $n = k\ell$, we have*

$$\widetilde{H}_{(\ell^k)}[X; q^k, q] = \widetilde{H}_{(k\ell)}[X; q, q^k] = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} X^T.$$

Proof. Apply Theorem 4.2 and either the q,t-symmetry $\widetilde{H}_\mu[X; q, t] = \widetilde{H}_{\mu'}[X; t, q]$ or the representation-theoretic argument in Subsection 4.2. □

We define a new statistic maj'_ℓ for words $w = w_1 \dots w_n$ by

$$\mathrm{maj}'_\ell(w) = \sum_i \left\lceil \frac{i}{\ell} \right\rceil \chi((i, i+\ell) \in \mathrm{Inv}(w))$$

Corollary 5.2 and the combinatorial formula for $\widetilde{H}_\mu[X; q, t]$ in [6] implies that the composite statistic

$$k\,\mathrm{maj}_\ell - (n-1)\,\mathrm{maj}'_\ell$$

is equidistributed with the major index on words. That is, for any partition μ of n,

$$\sum_{w \in W(\mu)} q^{k\,\mathrm{maj}_\ell(w) - (n-1)\,\mathrm{maj}'_\ell(w)} = \frac{[n]!_q}{[\mu_1]!_q \cdots [\mu_k]!_q}. \tag{5.3}$$

Question 5.3. *Is there a bijective proof of (5.3)?*

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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