Spanning Configurations and Representation Stability

Brendan Pawlowski

Department of Mathematics
University of Southern California
Los Angeles, CA, 90089, U.S.A.

bpawlows@usc.edu

Eric Ramos*

Department of Mathematics
Bowdoin College
Brunswick, ME, 04011, U.S.A.

e.ramos@bowdoin.edu

Brendon Rhoades†

Department of Mathematics
University of California, San Diego
La Jolla, CA, 92093, U.S.A.

bprhoades@ucsd.edu

Submitted: Mar 19, 2022; Accepted: Dec 24, 2022; Published: Jan 13, 2023

Abstract

Let $V_1, V_2, V_3, \ldots$ be a sequence of $\mathbb{Q}$-vector spaces where V_n carries an action of $\mathfrak{S}_n$. *Representation stability* and *multiplicity stability* are two related notions of when the sequence V_n has a limit. An important source of stability phenomena arises when V_n is the d^{th} homology group (for fixed d) of the configuration space of n distinct points in some fixed topological space X. We replace these configuration spaces with moduli spaces of tuples $(W_1, \ldots, W_n)$ of subspaces of a fixed complex vector space $\mathbb{C}^N$ such that $W_1 + \cdots + W_n = \mathbb{C}^N$. These include the varieties of *spanning line configurations* which are tied to the Delta Conjecture of symmetric function theory.

Mathematics Subject Classifications: 05E10

1 Introduction and Main Result

Let $(V_n)_{n \geqslant 1}$ be a sequence of finite-dimensional $\mathbb{Q}$-vector spaces where each V_n has an action of $\mathfrak{S}_n$. Introduced by Church [1] in a geometric context, *representation stability* gives a notion of the sequence V_n having a limit.

*Supported by NSF grant DMS-1704811.

†Supported by NSF grants DMS-1500838 and DMS-1953781.

Definition 1. *(Church [1])* Let $(V_n)_{n\geqslant 1}$ be a sequence of $\mathfrak{S}_n$ representations, and for each $n \geqslant 1$ let $f_n : V_n \to V_{n+1}$ be a linear map. Then we say that $(V_n)_{n\geqslant 1}$ is *(uniformly) representation stable* with respect to the maps $(f_n)_{n\geqslant 1}$ if for $n \gg 0$

- the map f_n is injective,
- we have $f_n(w \cdot v) = w \cdot f_n(v)$ for all $w \in \mathfrak{S}_n$ and all $v \in V_n$,
- the $\mathfrak{S}_{n+1}$ module generated by the image $f_n(V_n) \subseteq V_{n+1}$ is all of V_{n+1}, and
- the transposition $(n+1, n+2) \in \mathfrak{S}_{n+2}$ acts trivially on the image of the composition $\mathrm{im}(V_n \xrightarrow{f_n} V_{n+1} \xrightarrow{f_{n+1}} V_{n+2}) \subseteq V_{n+2}$.

The isotypic decompositions of a representation stable sequence V_n exhibit limiting properties. A *partition* of n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of positive integers which sum to n. We write $\lambda \vdash n$ to mean that λ is a partition of n and $|\lambda| = n$ for the sum of the parts of λ. The *(English) Ferrers diagram* of λ consists of λ_i left-justified boxes in row i; we identify partitions with their Ferrers diagrams.

Partitions of n are in bijective correspondence with irreducible representations of $\mathfrak{S}_n$; given $\lambda \vdash n$, let S^λ be the corresponding irreducible $\mathfrak{S}_n$-module.

If $\mu = (\mu_1, \mu_2, \dots)$ is a partition and $n \geqslant |\mu| + \mu_1$, the *padded partition* is $\mu[n] \vdash n$ is given by $\mu[n] := (n - |\mu|, \mu_1, \mu_2, \dots)$. Any partition $\lambda \vdash n$ may be expressed uniquely as $\lambda = \mu[n]$ for some partition μ. In fact, if $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ then $\lambda = \mu[n]$ where $\mu = (\lambda_2, \lambda_3, \dots)$.

For any $n \geqslant 1$, the $\mathfrak{S}_n$-module V_n decomposes into a direct sum of irreducibles. There exist unique multiplicities $m_{\mu,n}$ so that $V_n \cong \bigoplus_\mu m_{\mu,n} S^{\mu[n]}$ where the direct sum is over all partitions μ.

Definition 2. The sequence $(V_n)_{n\geqslant 1}$ is *uniformly multiplicity stable* if there exists N such that for any partition μ, we have $m_{\mu,n} = m_{\mu,n'}$ for all $n, n' \geqslant N$.

Church, Ellenberg, and Farb proved that multiplicity and representation stability are essentially equivalent.

Theorem 3. (Church-Ellenberg-Farb [2]) *Let $(V_n)_{n\geqslant 1}$ be a sequence of $\mathfrak{S}_n$-representations. Then $(V_n)_{n\geqslant 1}$ is uniformly multiplicity stable if and only if there exists some collection of linear maps $f_n : V_n \to V_{n+1}$ such that $(V_n)_{n\geqslant 1}$ is representation stable with respect to $(f_n)_{n\geqslant 1}$.*

Theorem 3 notwithstanding, writing down explicit maps $f_n : V_n \to V_{n+1}$ which realize the representation stability of a specific multiplicity stable sequence $(V_n)_{n\geqslant 1}$ can be difficult.

Many geometric instances of representation stability arise from configuration spaces. If X is a topological space and $n \geqslant 1$, the n^{th} *configuration space* of X is the moduli space of n distinct points in X:

$$\mathrm{Conf}_n X := \{(x_1, \dots, x_n) \,:\, x_i \in X \text{ and } x_i \neq x_j \text{ for } i \neq j\}. \tag{1}$$

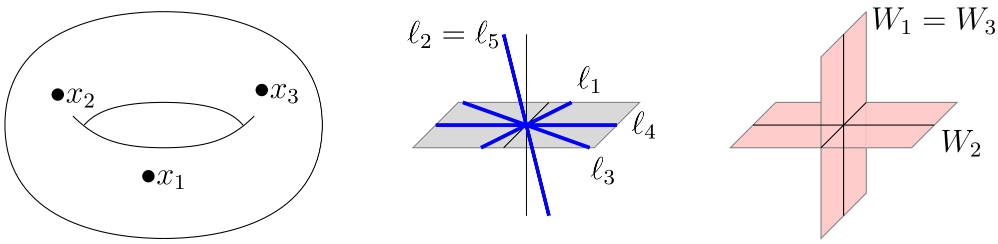

Figure 1: A point configuration, a line configuration, and a 2-plane configuration.

The left of Figure 1 shows a point in $\mathrm{Conf}_3(X)$ where X is the torus. The set $\mathrm{Conf}_n X$ has the subspace topology inherited from the n-fold product $X \times \cdots \times X$.

For $d \geqslant 0$, let $H_d(\mathrm{Conf}_n X)$ be the d^{th} homology group of the n^{th} configuration space of X.[1] The natural action of $\mathfrak{S}_n$ on $\mathrm{Conf}_n(X)$ induces an action on $H_d(\mathrm{Conf}_n X)$. There are many results stating that if the space X is 'nice', the sequence $(H_d(\mathrm{Conf}_n X))_{n \geqslant 1}$ is representation stable [1, 2]. Such stability results can be proven even when the homology of $\mathrm{Conf}_n(X)$ or its $\mathfrak{S}_n$-structure are unknown. In general, it is often easier to prove that a sequence V_n is representation stable than it is to find its $\mathfrak{S}_n$-structure.

We consider a matroidal variation on configuration spaces in which sequences of distinct points are replaced by spanning sequences of m-dimensional subspaces of $\mathbb{C}^k$. Let $\mathrm{Gr}(m,k)$ be the Grassmannian of m-planes in $\mathbb{C}^k$ and let $\mathrm{Gr}(m,k)^n$ be its n-fold self-product. We consider the open subvariety

$$X(m,k,n) := \{(W_1, \ldots, W_n) \in \mathrm{Gr}(m,k)^n \,:\, W_1 + \cdots + W_n = \mathbb{C}^k\} \tag{2}$$

of sequences $(W_1, \ldots, W_n)$ which span $\mathbb{C}^k$. Figure 1 shows a point in $X(1,3,5)$ (middle) and $X(2,3,3)$ (left).

The variety $X(m,k,n)$ is nonempty if and only if $k \leqslant mn$. There is a homotopy equivalence $X(1,n,n) \simeq \mathrm{Fl}(n)$ between $X(1,n,n)$ and the variety of flags in $\mathbb{C}^n$. Pawlowski and Rhoades [6] introduced the variety $X(1,k,n)$ of spanning line configurations $(\ell_1, \ldots, \ell_n)$ in $\mathbb{C}^k$ and presented its cohomology as

$$H^\bullet(X(1,k,n)) = \mathbb{Q}[x_1, \ldots, x_n]/\langle x_1^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1}\rangle \tag{3}$$

where e_d is the degree d elementary symmetric polynomial. The above quotient rings were introduced earlier by Haglund, Rhoades, and Shimozono [4] in the context of Macdonald theory [3].

The symmetric group $\mathfrak{S}_n$ acts naturally on the variety $X(m,k,n)$ and on its homology. There are two natural ways to 'grow' a triple (m,k,n) preserving the condition $X(m,k,n) \neq \varnothing$:

$$(m,k,n) \rightsquigarrow (m,k,n+1) \qquad \text{and} \qquad (m,k,n) \rightsquigarrow (m,k+m,n+1).$$

We prove that both of these growth rules yield stability results.

[1] We use singular homology with rational coefficients.

Theorem 4. *Fix integers $m, k, r \geqslant 0$. Both sequences*

$$H_d(X(m,k,n)) \qquad H_d(X(m,mn-r,n)) \qquad (n \geqslant 0) \tag{4}$$

of homology $\mathfrak{S}_n$-modules are representation stable.

Theorem 4 is a 'matroidal analogue' of configuration space stability results. The cohomology ring $H^\bullet(X(m,k,n))$ was presented by Rhoades [7], but its graded $\mathfrak{S}_n$-isomorphism type – like the $\mathfrak{S}_n$-structures of the homology representations of Theorem 4 – is unknown. Our proof of Theorem 4 does not use the cohomology presentation in [7], but rather goes through the theory of FI-modules and a geometric result on the inclusion $X(m,k,n) \subseteq \mathrm{Gr}(m,k)^n$. Our methods illustrate that a sequence of $\mathfrak{S}_n$-modules can be shown to exhibit stability even in the presence of relatively little information about these modules.

The remainder of the paper is organized as follows. In **Section 2** we give background material on affine pavings and the category of FI-modules. In **Section 3** we prove Theorem 4. This paper is an abridged and generalized version of the FPSAC 2020 extended abstract [5].

2 Affine Pavings and FI-modules

Let Z be a complex variety. An *affine paving* of Z is a chain

$$\varnothing = Z_0 \subset Z_1 \subset \cdots \subset Z_r = Z$$

of closed subvarieties of Z such that each difference $Z_i - Z_{i-1}$ is isomorphic to a disjoint union of affine spaces. If $\varnothing = Z_0 \subset Z_1 \subset \cdots \subset Z_r = Z$ is an affine paving and $0 \leqslant j \leqslant m$, let $U := Z - Z_j$. The inclusion $\iota : U \hookrightarrow Z$ induce a map $\iota_* : H_\bullet(U) \to H_\bullet(Z)$ on homology. When U arises from an affine paving of Z as above, the map ι_* is injective.

The standard affine paving of complex projective space $\mathbb{P}^{k-1}$ is given in coordinates by

$$\varnothing \subset [\star : 0 : \cdots : 0 : 0] \subset [\star : \star : \cdots : 0 : 0] \subset \cdots \subset [\star : \star : \cdots : \star : 0] \subset [\star : \star : \cdots : \star : \star] = \mathbb{P}^{k-1}$$

where the stars represent free complex entries. More generally, the Schubert decomposition of the Grassmannian $\mathrm{Gr}(m,k)$ induces an affine paving of this space. Taking the n-fold product of this paving with itself yields a paving of $\mathrm{Gr}(m,k)^n$, but this naïve paving interacts poorly with the inclusion $X(m,k,n) \subseteq \mathrm{Gr}(m,k)^n$. A nonstandard affine paving of $\mathrm{Gr}(m,k)^n$ was crucial to the presentation of the cohomology of $X(m,k,n)$ in [7].

Theorem 5. (Rhoades [7]) *There exists an affine paving $Z_0 \subset Z_1 \subset \cdots \subset Z_r$ of the Grassmann product $\mathrm{Gr}(m,k)^n$ such that $X(m,k,n) = \mathrm{Gr}(m,k)^n - Z_j$ for some j.*

In particular, if $\iota : X(m,k,n) \hookrightarrow \mathrm{Gr}(m,k)^n$ is inclusion, then $\iota_* : H_\bullet(X(m,k,n)) \hookrightarrow H_\bullet(\mathrm{Gr}(m,k)^n)$ is an injection.

Theorem 5 was a component of the presentation of $H^\bullet(X(m,k,n))$ in [7]. It will be a key tool in our proof, as well. The other ingredient we need is the category of FI-modules recalled below.

Let FI be the category whose objects are the finite sets $[n] := \{1, 2, \dots, n\}$ for $n \geqslant 0$ and whose morphisms are injective functions $f : [n] \to [p]$. Let Vect be the category of $\mathbb{Q}$-vector spaces with morphisms given by linear maps.

An FI-*module* is a covariant functor $V : \mathsf{FI} \to \mathsf{Vect}$. We write $V(n)$ instead of $V([n])$ for the vector space corresponding to $[n]$. More explicitly, an FI-module consists of

- a $\mathbb{Q}$-vector space $V(n)$ for each $n \geqslant 0$, and
- a $\mathbb{Q}$-linear map $V(f) : V(n) \to V(p)$ for each injection $f : [n] \to [p]$

such that $V(f \circ g) = V(f) \circ V(g)$ for any two injections $f : [n] \to [p]$ and $g : [p] \to [q]$ and $V(\mathrm{id}_{[n]}) = \mathrm{id}_{V(n)}$. Submodules and quotients of FI-modules are defined in the natural way. If V is an FI-module, then $V(n)$ is naturally an $\mathfrak{S}_n$-module for each n.

An FI-module V is *finitely generated* if there is a finite subset S of the disjoint union $\bigsqcup_{n \geqslant 0} V(n)$ such that no proper FI-submodule W of V satisfies $S \subseteq \bigsqcup_{n \geqslant 0} W(n)$. Finite generation of FI-modules and representation stability are equivalent notions.

Theorem 6. (Church-Ellenberg-Farb [2]) *Let $\iota_n : [n] \hookrightarrow [n+1]$ denote the standard injection, and let V be a finitely generated FI-module. Then the sequence $(V(n))_{n \geqslant 1}$ is representation stable with respect to the maps $V(\iota_n)$. Every representation stable sequence arises in this way.*

One reason Theorem 6 is useful is that finite generation in the category of FI-modules is inherited by submodules (and quotient modules, although we will not use this). Said differently, the category of FI-modules is Noetherian.

Theorem 7. (Snowden [8]; see also Church-Ellenberg-Farb [2]) *Any submodule of a finitely generated FI-module is finitely generated.*

3 Proof of Theorem 4

We need to prove the stability of two sequences of homology representations of $\mathfrak{S}_n$, namely

$$H_d(X(m,k,n)) \qquad \text{and} \qquad H_d(X(m, mn-r, n))$$

for fixed integers m, k, and r. The stability of the sequence $H_d(X(m,k,n))$ is easier to establish; we handle this case first. For any subset $I \subseteq [k]$ we let $E_I := \mathrm{span}\{e_i : i \in I\}$ be the coordinate subspace of $\mathbb{C}^k$ spanned by the corresponding basis vectors.

For any injection $f : [n] \hookrightarrow [p]$, we define an associated map $\mu_f : \mathrm{Gr}(m,k)^n \to \mathrm{Gr}(m,k)^p$ by the rule

$$\iota_f : (W_1, \dots, W_n) \mapsto (W_1', \dots, W_p') \tag{5}$$

where $W'_{f(i)} = W_i$ for $1 \leqslant i \leqslant n$ and $W'_j = E_{[m]}$ whenever $1 \leqslant j \leqslant p$ is not in the image of f. As an example, if $m = 3$, $k = 6$, and $f : [3] \to [5]$ is $f(1) = 4, f(2) = 2, f(3) = 1$ then $\mu_f : \mathrm{Gr}(3,6)^3 \to \mathrm{Gr}(3,6)^5$ is

$$\mu_f : (W_1, W_2, W_3) \mapsto (W_3, W_2, E_{[3]}, W_1, E_{[3]})$$

for all $(W_1, W_2, W_3) \in \mathrm{Gr}(3,6)^3$.

For any two injections $f : [n] \hookrightarrow [p]$ and $f : [p] \hookrightarrow [q]$ we have

$$\mu_{g \circ f} = \mu_g \circ \mu_f. \tag{6}$$

Furthermore, the map μ_f preserves the spanning condition; abusing notation, we denote the restricted map on X-spaces by the same symbol $\mu_f : X(m,k,n) \to X(m,k,p)$.

For any injection $f : [n] \hookrightarrow [p]$, the induced maps $(\mu_f)_*$ on homology fit into a commutative square

$$\begin{tikzcd}
H_\bullet(X(m,k,n)) \arrow[r, "\iota_*"] \arrow[d, "(\mu_f)_*"'] & H_\bullet(\mathrm{Gr}(m,k)^n) \arrow[d, "(\mu_f)_*"] \\
H_\bullet(X(m,k,p)) \arrow[r, "\iota_*"'] & H_\bullet(\mathrm{Gr}(m,k)^p)
\end{tikzcd}$$

where the horizontal maps are those induced by the injections $X(m,k,n) \hookrightarrow \mathrm{Gr}(m,k)^n$ and $X(m,k,p) \hookrightarrow \mathrm{Gr}(m,k)^p$. Theorem 5 implies that the horizontal arrows are injections, so that $[n] \mapsto H_d(X(m,k,n))$ is a sub-FI-module of $[n] \mapsto H_d(\mathrm{Gr}(m,k)^n)$ for each degree d. The Künneth Theorem implies

$$H_d(\mathrm{Gr}(m,k)^n) = \bigoplus_{d_1 + \cdots + d_n = d} H_{d_1}(\mathrm{Gr}(m,k)) \otimes \cdots \otimes H_{d_n}(\mathrm{Gr}(m,k)) \tag{7}$$

as graded vector spaces. Since $H_{d_i}(\mathrm{Gr}(m,k))$ is a finite-dimensional vector space for each d_i, the FI-module $[n] \mapsto H_d(\mathrm{Gr}(m,k)^n)$ is finitely generated for d fixed. Theorem 7 implies that $[n] \mapsto H_d(X(m,k,n))$ is also a finitely generated FI-module, and Theorem 6 shows that $H_d(X(m,k,n))$ is representation stable.

Now fix integers m, r and consider the sequence $X(m, mn-r, n)$ of spaces for $n \geqslant 0$. One would like to put an FI-structure on the spaces $X(m, mn-r, n)$ which is compatible with their inclusions $X(m, mn-r, n) \hookrightarrow \mathrm{Gr}(m, mn-r)^n$ into Grassmann products. However, complications arise since the ambient dimension $mn - r$ increases with n, resulting in an FI-structure which only holds when one passes to homology.

More precisely, if $f : [n] \hookrightarrow [p]$ is an injection, define $\nu_f : \mathrm{Gr}(m, mn-r)^n \hookrightarrow \mathrm{Gr}(m, mp-r)^p$ by the formula

$$\nu_f : (W_1, \ldots, W_n) \mapsto (W'_1, \ldots, W'_p) \tag{8}$$

where the spaces W'_j are determined as follows. If $f(i) = j$, we set $W'_j := W_i$, where we view W_i as a subspace of $\mathbb{C}^{mp-r}$ by means of the embedding $\mathbb{C}^{mn-r} \subseteq \mathbb{C}^{mp-r}$ along the first $mn - r$ coordinates. For the $p - n$ elements $1 \leqslant j_1 < \cdots < j_{p-n} \leqslant p$ of $[p]$ which are not in the image of f, we let $W'_{j_\ell} \subseteq \mathbb{C}^{mp-n}$ be the coordinate subspace

$$W'_{j_\ell} := E_{[m(n+\ell-1)-r+1,m(n+\ell)-r]} = \mathrm{span}\{e_i : m(n+\ell-1) - r + 1 \leqslant i \leqslant m(n+\ell) - r\}.$$

An example should clarify the definition of ν_f. Suppose $m = 2$, $n = 3$, $r = 1$, and $p = 6$. If $f : [3] \hookrightarrow [6]$ is given by $f(1) = 3$, $f(2) = 1$, $f(3) = 6$ then $\nu_f : \mathrm{Gr}(2,5)^3 \to \mathrm{Gr}(2,11)^3$ is given by

$$\nu_f : (W_1, W_2, W_3) \mapsto (W_2, E_{\{6,7\}}, W_1, E_{\{8,9\}}, E_{\{10,11\}}, W_3).$$

where we regard $W_1, W_2, W_3 \subset \mathbb{C}^{11}$ by means of the inclusion $\mathbb{C}^5 \subseteq \mathbb{C}^{11}$ along the first five coordinates. In general, whenever $\mu_f : (W_1, \ldots, W_n) \mapsto (W'_1, \ldots, W'_p)$ and $W_1 + \cdots + W_n = \mathbb{C}^{mn-r}$, we have $W'_1 + \cdots + W'_p = \mathbb{C}^{mp-r}$. The map ν_f therefore restricts to a map $X(m, mn-r, n) \hookrightarrow X(m, mp-r, p)$; we use the same symbol ν_f to refer to this restriction.

If $f : [n] \hookrightarrow [p]$ and $g : [p] \hookrightarrow [q]$ are two injections, we do **not** typically have the equality of maps $\nu_{g\circ f} = \nu_g \circ \nu_f$. For example, suppose $m = 2, r = 1$, $(n, p, q) = (3, 6, 7)$, and $f : [3] \hookrightarrow [6]$ is as above. Define $g : [6] \hookrightarrow [7]$ by $g(1) = 5, g(2) = 1, g(3) = 3, g(4) = 6, g(5) = 2, g(6) = 7$. Then

$$\nu_g \circ \nu_f : (W_1, W_2, W_3) \mapsto (E_{\{6,7\}}, E_{\{10,11\}}, W_1, E_{\{12,13\}}, W_2, E_{\{8,9\}}, W_3)$$

whereas

$$\nu_{g\circ f} : (W_1, W_2, W_3) \mapsto (E_{\{6,7\}}, E_{\{8,9\}}, W_1, E_{\{10,11\}}, W_2, E_{\{12,13\}}, W_3).$$

In general, the spaces $W_1, \ldots, W_n$ will appear in the same positions (and in the same order) in the images $(W'_1, \ldots, W'_q)$ of the tuple $(W_1, \ldots, W_n)$ under either $\nu_g \circ \nu_f$ or $\nu_{g\circ f}$, but the E's will usually appear in a different order.

For fixed injections $g : [n] \hookrightarrow [p]$ and $f : [p] \hookrightarrow [q]$, there is an invertible linear transformation $A \in \mathrm{GL}_{mq-r}(\mathbb{C})$ such that

$$\nu_g \circ \nu_f : (W_1, \ldots, W_n) \mapsto (A \cdot W'_1, \ldots, A \cdot W'_q) \text{ where } \nu_{g\circ f} : (W_1, \ldots, W_n) \mapsto (W'_1, \ldots, W'_q) \tag{9}$$

for all $(W_1, \ldots, W_n) \in \mathrm{Gr}(m, mn - r)^n$. Recall that the group $\mathrm{GL}_{mq-r}(\mathbb{C})$ is path connected. Any path from I to A in $\mathrm{GL}_{mq-r}(\mathbb{C})$ provides a homotopy between $\nu_g \circ \nu_f$ and $\nu_{g\circ f}$. In summary, we have

$$\nu_{g\circ f} \simeq \nu_g \circ \nu_f \tag{10}$$

as maps on spaces so that the induced maps $(\nu_{g\circ f})_* = (\nu_g)_* \circ (\nu_f)_*$ on homology coincide.

As before, if $f : [n] \hookrightarrow [p]$ is an injection, we have a commutative square of maps on homology

$$\begin{array}{ccc} H_\bullet(X(m,mn-r,n)) & \xrightarrow{\iota_*} & H_\bullet(\mathrm{Gr}(m,mn-r)^n) \\ (\nu_r)_* \downarrow & & \downarrow (\nu_f)_* \\ H_\bullet(X(m,mp-r,p)) & \xrightarrow[\iota_*]{} & H_\bullet(\mathrm{Gr}(m,mp-r)^p) \end{array}$$

where Theorem 5 guarantees that the horizontal maps are injective. The last paragraph shows that, for a fixed degree d, we have an FI-module $[n] \mapsto H_d(\mathrm{Gr}(m,mn-r)^n)$ with submodule $[n] \mapsto H_d(X(m,mn-r,n))$. The Künneth formula implies

$$H_d(\mathrm{Gr}(m,mn-r)^n) = \bigoplus_{d_1+\cdots+d_n=d} H_{d_1}(\mathrm{Gr}(m,mn-r)) \otimes \cdots \otimes H_{d_n}(\mathrm{Gr}(m,mn-r)). \quad (11)$$

The space $\mathrm{Gr}(m,mn-r)$ admits an affine paving whose cells (the *Schubert cells*) are indexed by partitions λ whose Young diagrams fit inside a $m \times [m(n-1)-r]$ box. The (complex) dimension of the Schubert cell corresponding to λ is the number of boxes $|\lambda|$ in λ. Consequently, the homology of $\mathrm{Gr}(m,mn-r)$ is concentrated in even degrees and for d_i even the Schubert cells induce a basis

$$\left\{\sigma_\lambda \,:\, |\lambda| = \frac{d_i}{2},\ \lambda \subseteq (m(n-1)-r)^m\right\}$$

of $H_{d_i}(\mathrm{Gr}(m,mn-r))$. Since the Schubert cells are compatible with the embeddings $\mathbb{C}^{mn-r} \subseteq \mathbb{C}^{mp-r}$, we conclude that $[n] \mapsto H_d(\mathrm{Gr}(m,mn-r)^n)$ is a finitely-generated FI-module. Theorems 6 and 7 complete the proof.

Acknowledgements

The authors thank Steven Sam and Ben Young for many helpful conversations and the anonymous referees for their comments which improved the exposition of this paper.

References

[1] T. Church. Homological stability for configuration spaces of manifolds. *Invent. Math.*, **188 (2)** (2012), 465–504.

[2] T. Church, J. S. Ellenberg, and B. Farb. FI-modules and stability for representations of symmetric groups. *Duke Math. J.* **164 (9)** (2015), 1833–1910.

[3] J. Haglund, J. Remmel, and A. T. Wilson. The Delta Conjecture. *Trans. Amer. Math. Soc.*, **370** (2018), 4029–4057.

[4] J. Haglund, B. Rhoades, and M. Shimozono. Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture. *Adv. Math.*, **329** (2018), 851–915.

[5] B. Pawlowski, E. Ramos, and B. Rhoades. Spanning Configurations and Matroidal Representation Stability. *Sém. Loth. Comb.*, Article 84B.57, (2020).

[6] B. Pawlowski and B. Rhoades. A flag variety for the Delta Conjecture. *Trans. Amer. Math. Soc.*, **374** (2019), 8195–8248.

[7] B. Rhoades. Spanning subspace configurations. *Sel. Math. New Ser.*, **27 (1)**, (2021),

[8] A. Snowden. Syzygies of Segre embeddings and Δ-modules. *Duke Math. J.*, **162 (2)** (2013), 225–277.