



Celestial Liouville theory for Yang-Mills amplitudes

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ABSTRACT

We consider Yang-Mills theory with the coupling constant and theta angle determined by the vacuum expectation values of a dynamical (complex, zero mass) dilaton field. We discuss the tree-level N -gluon MHV scattering amplitudes in the presence of a nontrivial background dilaton field and construct the corresponding celestial amplitudes by taking Mellin transforms with respect to the light cone frame energies. In this way, we obtain two-dimensional CFT correlators of primary fields on the celestial sphere. We show that the celestial Yang-Mills amplitudes evaluated in the presence of a spherical dilaton shockwave are given by the correlation functions of primary field operators factorized into the holomorphic current operators times the “light” Liouville operators. They are evaluated in the semiclassical limit of Liouville theory (the limit of infinite central charge) and are determined by the classical Liouville field describing metrics on the celestial sphere.

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Understanding the symmetries is very important for studying physical systems. Typically, the bigger symmetry the better because many problems can be solved by pure symmetry considerations. Celestial holography [1–3] is a notable exception because conformal correlators associated with the scattering amplitudes are overconstrained by translational invariance, namely the positions of two-dimensional celestial conformal field theory (CCFT) operators associated with massless particles are constrained by the momentum conservation law. Three-point amplitudes are the extreme case because the operators are driven to the same point on the celestial sphere.¹ Similar constraints persist for any number of external particles and complicate the analysis of CCFT correlators [4].

In two recent papers [5,6], we analyzed celestial Yang-Mills amplitudes evaluated in the presence of a background dilaton field, which breaks translational invariance in a controllable way by supplying external momentum to the gluon system. In this theory, the Yang-Mills coupling constant and theta angle are determined by the vacuum expectation values of a dynamical (complex, zero mass) dilaton field [7]. A similar approach was pursued in [8,9], in the framework of self-dual gauge theories coupled to axions. In [6], we observed that the four-point correlators factorize into the “current” part and the “scalar” part. The current factor is given by the group-dependent part of the Wess-Zumino-Witten correlator of four holomorphic currents. The scalar factor can be expressed in terms of complex integrals of the Dotsenko-Fateev form [10,11], similar to the integrals describing four-point correlators in Coulomb gas models [12,13] and, more generally, in the infinite central charge limit of Liouville theory [14,15].² In this Letter, we elaborate on the connection between celestial Yang-Mills amplitudes and Liouville theory.

We begin with a brief summary of Liouville field theory (LFT), along the lines of Ref. [15]. The Lagrangian density is given by

$$\mathcal{L} = \frac{1}{\pi} \frac{\partial \phi}{\partial z} \frac{\partial \phi}{\partial \bar{z}} + \mu e^{2b\phi}, \quad (1)$$

where b is the dimensionless Liouville coupling constant and μ is the “cosmological constant” scale parameter. The “background charge at infinity,”

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¹ This problem can be avoided by changing the signature of four-dimensional metrics. This leads, however, outside the scope of a “garden variety” Euclidean CCFT in which $SL(2, \mathbb{C})$ conformal symmetry follows directly from Lorentzian $SO(1, 3)$.

² Infinite central charge of CCFT was anticipated earlier in [16], but on different grounds.

$$q = b + \frac{1}{b}, \quad (2)$$

is related to the central charge by

$$c = 1 + 6q^2. \quad (3)$$

The primary fields of LFT are the exponential operators

$$V_\alpha(z, \bar{z}) = e^{2\alpha\phi(z, \bar{z})}, \quad (4)$$

which are scalar fields with chiral dimensions

$$h(\alpha) = \bar{h}(\alpha) = \alpha(q - \alpha). \quad (5)$$

The studies of LFT focus on the correlation functions of exponential operators. Their general properties were first analyzed in [14, 15], where exact analytic expressions for the three-point functions were also derived. In general, LFT correlation functions are given by complicated expressions. There is, however, one notable exception, for a special configuration of the exponents, when

$$\sum_i \alpha_i = q - \frac{m}{b} - nb, \quad (6)$$

where m and n are non-negative integers. The correlators contain simple poles $(\sum_i \alpha_i - q + \frac{m}{b} + nb)^{-1}$, with the residues given by

$$\begin{aligned} \text{res} \int \prod_i V_{\alpha_i}(z_i, \bar{z}_i) e^{-\int d^2z \mathcal{L}[\phi]} \mathcal{D}\phi \\ = \frac{(-\mu)^n (-\tilde{\mu})^m}{m!n!} \int \left\langle \prod_i V_{\alpha_i}(z_i, \bar{z}_i) \prod_{k=1}^n V_b(u_k, \bar{u}_k) \prod_{l=1}^m V_{1/b}(v_l, \bar{v}_l) \right\rangle d^2u_1 \dots d^2u_n d^2v_1 \dots d^2v_m, \end{aligned} \quad (7)$$

where $\tilde{\mu}(\mu, b)$ is the “dual cosmological constant” written explicitly in Ref. [15], and the integrands are determined by

$$\left\langle \prod_j V_{\beta_j}(z_j, \bar{z}_j) \right\rangle = \prod_{j < j'} |z_j - z_{j'}|^{-4\beta_j \beta_{j'}}. \quad (8)$$

We are interested in the case of single integrals that appear when $n = 0, m = 1$. With the exponents parametrized as

$$\alpha_i = \sigma_i b, \quad \sum_i \sigma_i = 1, \quad (9)$$

the correlation functions are

$$\int \prod_i V_{\sigma_i b}(z_i, \bar{z}_i) e^{-\int d^2z \mathcal{L}[\phi]} \mathcal{D}\phi \sim \prod_{i < j} |z_i - z_j|^{-4\sigma_i \sigma_j b^2} \int d^2z \prod_i |z_i - z|^{-4\sigma_i}. \quad (10)$$

Note that the dimensions of primary fields are

$$d_i = h(\sigma_i b) + \bar{h}(\sigma_i b) = 2\sigma_i + 2b^2\sigma_i(1 - \sigma_i). \quad (11)$$

We are interested in CCFT primary fields with dimensions $\Delta_i = n_i + i\lambda_i$, with integer $n_i \leq 1$ and real λ_i [17,18]. The integrals encountered in the computations [5,6] of “single-valued” celestial Yang-Mills amplitudes have a form similar to Eq. (10), with $2\sigma_i = m_i + i\lambda_i$ and integer m_i . Based on this observation and on Eq. (11), we expect that if the tree-level celestial amplitudes are related to LFT, they are related to the limit $b \rightarrow 0$, that is to the $c \rightarrow \infty$ infinite central charge (3) limit of LFT. The operators with the exponents scaling as $\sigma_i b$ ($b \rightarrow 0$) are called the “light” operators [15]. The limit $b \rightarrow 0$ is the classical limit of LFT, in which the correlation functions of light operators are determined by the solutions of the Liouville equation, given by

$$2b\phi_0(z, \bar{z}) = \ln \frac{A}{\pi(1 + z\bar{z})^2} \quad (12)$$

and describing two-dimensional metrics on a sphere with area A . The computation of the correlation functions of light operators amounts to integrating over the $SL(2, \mathbb{C})$ orbits of Eq. (12) in the product $\prod_i e^{2\sigma_i b \phi_0}$.

In Refs. [5,6], we computed three- and four-gluon celestial MHV amplitudes in the presence of a specific dilaton background motivated by certain solutions of Banerjee-Ghosh (BG) equations [19]. As it is clear, however, from the discussion of Ref. [20], BG equations are satisfied by MHV amplitudes in the presence of an arbitrary dilaton source because the net effect of the source is to replace the momentum-conserving delta functions by a (Lorentz-invariant) function of the total momentum of the gluon system. The total momentum is invariant under BCFW shifts, therefore BG equations remain satisfied. As mentioned before, the celestial amplitudes obtained in [5,6] can be expressed in terms of integrals similar to Eq. (10). In this work, we will identify the background dilaton field that yields celestial amplitudes in *exactly* the same form as the Liouville integrals (10). In this way, we will connect the operators representing gluons with the Liouville operators (4) and relate CCFT to LFT.

Celestial amplitudes are obtained from the momentum space amplitudes by performing Mellin transforms with respect to the energies of external particles [21]; therefore our first goal is to rewrite the Liouville integrals (10) as Mellin transforms. Since we are interested in “mostly plus” MHV amplitudes in the helicity configurations $(- - + + \dots)$, it is natural to fulfill the condition (9) by

$$\sigma_1 = \frac{1+i\lambda_1}{2}, \quad \sigma_2 = \frac{1+i\lambda_2}{2}, \quad \sigma_k = \frac{i\lambda_k}{2} \quad (k \geq 3), \quad (13)$$

with

$$\sum_i^N \lambda_i = 0. \quad (14)$$

Hence, we focus on the integrals

$$I_N(z_1, \bar{z}_1, \dots, z_N, \bar{z}_N) = \int d^2z |z_1 - z|^{-2(1+i\lambda_1)} |z_2 - z|^{-2(1+i\lambda_2)} |z_3 - z|^{-2i\lambda_3} \dots, \quad (15)$$

which describe conformal correlators of scalar primary fields with dimensions $d_i = 2\sigma_i$. They can be evaluated in many ways [12], but for our purposes it is most convenient to follow the formalism described in [22] and express them in terms of the integrals on a “Poincaré section” of a four-dimensional embedding space $\mathbb{R}^{1,3}$.

In the embedding space, the coordinates are parametrized as $X = (X^+, X^-, X^1, X^2)$, and the Lorentzian inner product is

$$X \cdot Y = \frac{1}{2}(X^+ Y^- + X^- Y^+) - X^1 Y^1 - X^2 Y^2. \quad (16)$$

On the null-cone $X \cdot X = 0$,

$$X = \left(X^+, \frac{|z|^2}{X^+}, \frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right), \quad z \in \mathbb{C}. \quad (17)$$

The Poincaré section is constructed by quotienting the null-cone by the rescaling $X \sim \rho X$, $\rho \in \mathbb{R}$. The projective null-cone can be identified with $\mathbb{C}$ by gauge fixing this rescaling, for example by imposing the condition $X^+ = 1$. The action of $SO(1, 3)$ on $\mathbb{R}^{1,3}$ is inherited as $SL(2, \mathbb{C})$ transformations of the complex coordinates z . Note that on the Poincaré section,

$$|z_1 - z_2|^2 = 2X_1 \cdot X_2. \quad (18)$$

By using the embedding space formalism [22], the integrals (15) can be rewritten as

$$\begin{aligned} I_N &= \frac{1}{2 \text{Vol GL}(1, \mathbb{R})^+} \int_{X^+ + X^- > 0} d^4 X \delta(X \cdot X) \frac{1}{(X_1 \cdot X)^{1+i\lambda_1} (X_2 \cdot X)^{1+i\lambda_2} \prod_{k=3}^N (X_k \cdot X)^{i\lambda_k}} \\ &= \frac{C_N}{\text{Vol GL}(1, \mathbb{R})^+} \int_{X^+ + X^- > 0} d^4 X \delta(X \cdot X) \\ &\quad \times \int_{\omega_i \geq 0} e^{iX \cdot (\sum_{i=1}^N \omega_i X_i)} d\omega_1 \omega_1^{i\lambda_1} d\omega_2 \omega_2^{i\lambda_2} \prod_{k=3}^N d\omega_k \omega_k^{i\lambda_k - 1}, \end{aligned} \quad (19)$$

with the normalization constant

$$C_N = -\frac{1}{2 \Gamma(1+i\lambda_1) \Gamma(1+i\lambda_2) \prod_{k=3}^N \Gamma(i\lambda_k)}. \quad (20)$$

The volume factor $[\text{Vol GL}(1, \mathbb{R})^+]^{-1}$ cancels the divergence due to the integration over the $X \rightarrow \rho X$ rescaling “gauge” mode. The gauge can be fixed to $X^+ = 1$, with the associated Fadeev-Popov determinant equal to 1. Then

$$\begin{aligned} I_N &= C_N \int d^4 X \delta(X \cdot X) \delta(X^+ - 1) \theta(X^+ + X^-) \\ &\quad \times \int_{\omega_i \geq 0} e^{iX \cdot (\sum_{i=1}^N \omega_i X_i)} d\omega_1 \omega_1^{i\lambda_1} d\omega_2 \omega_2^{i\lambda_2} \prod_{k=3}^N d\omega_k \omega_k^{i\lambda_k - 1}. \end{aligned} \quad (21)$$

The above integral has a form of a multiple Mellin transform, with implicit regulators [17] omitted here for simplicity. Let us compare it with the Mellin transforms encountered in celestial amplitudes.

The starting point for celestial amplitudes are the momentum space amplitudes with the momenta of massless particles parametrized as

$$P = (P^+, P^-, P^1, P^2) = \omega \left(1, |z|^2, \frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right). \quad (22)$$

Note that ω is the energy in the light cone frame. Under $SL(2, \mathbb{C})$ Lorentz transformations,

$$z \rightarrow \frac{az + b}{cz + d}, \quad \omega \rightarrow \omega |cz + d|^2. \quad (23)$$

We are interested in N -gluon MHV amplitudes in the mostly plus helicity configuration $(- - + + \dots)$, evaluated in the presence of a massless dilaton source $J_\Phi(X)$ [5,6], or equivalently, in the presence of a background dilaton field $\Phi_0(X) = \square^{-1} J_\Phi$. As in Refs. [5,6], we

are considering a “weakly coupled” source, with one insertion in the amplitude. These amplitudes are converted into two-dimensional correlators of primary fields with dimensions Δ_i by performing Mellin transforms with respect to the light cone frame energies. In this way, we obtain [5,6]:

$$\mathcal{M}_N(z_1, \bar{z}_1, \dots, z_N, \bar{z}_N | \Delta_1, \dots, \Delta_N) = \mathcal{J}_N(z_i) \mathcal{S}_N(z_i, \bar{z}_i) \quad (24)$$

where

$$\mathcal{J}_N(z_i) = \sum_{\pi \in S_{N-2}} f^{a_1 a_{\pi(2)} x_1} f^{x_1 a_{\pi(3)} x_2} \dots f^{x_{N-3} a_{\pi(N-1)} a_N} \frac{z_{12}^4}{z_{1\pi(2)} z_{\pi(2)\pi(3)} \dots z_{N1}}, \quad (25)$$

is the holomorphic “soft” factor, with $(a_1, a_2, \dots, a_N)$ labeling the gluon group indices. The Mellin transforms are contained in the “scalar” part

$$\mathcal{S}_N(z_i, \bar{z}_i) = \int d^4 X \int_{\omega_i \geq 0} J_\Phi(X) \frac{e^{iX \cdot Q}}{Q^2} d\omega_1 \omega_1^{\Delta_1} d\omega_2 \omega_2^{\Delta_2} \prod_{k \geq 3}^N d\omega_k \omega_k^{\Delta_k - 2}, \quad (26)$$

where Q is the total momentum of the gluon system,

$$Q = \sum_{i=1}^N P_i \quad (27)$$

and we assumed that all gluons are outgoing, i.e., they are created by the background dilaton field.³ We see that, up to the normalization constant, the scalar part (26) matches the Liouville integral (21) with the dimensions

$$\Delta_1 = i\lambda_1 = d_1 - 1, \quad \Delta_2 = i\lambda_2 = d_2 - 1, \quad \Delta_k = 1 + i\lambda_k = d_k + 1 \quad (k \geq 3), \quad (28)$$

where $d_i = 2\sigma_i$ are the dimensions of Liouville primary fields, provided that we identify the dilaton source as

$$J_\Phi(X) = -\square[\delta(X \cdot X) \delta(X^+ - 1) \theta(X^+ + X^-)] \quad (29)$$

and choose a Poincaré section, see Eq. (17), with all $X_i^+ = 1$, so that

$$P_i = \omega_i X_i, \quad Q = \sum_{i=1}^N \omega_i X_i. \quad (30)$$

The corresponding background field is

$$\Phi_0(X) = \square^{-1} J_\Phi(X) = -\delta(X \cdot X) \delta(X^+ - 1) \theta(X^+ + X^-). \quad (31)$$

It represents a retarded (outgoing) spherical dilaton shockwave. It seems that the wave is frozen at $X^+ = 1$, which would violate Lorentz invariance. Nevertheless, as it will be clarified below, it gives rise to a covariant conformal correlator. The wave propagation is “lost” by projectivizing on the null-cone.

To further analyze Liouville integrals, we rewrite Eq. (21) as

$$\begin{aligned} I_N &= \frac{C_N}{2} \int dX^+ dX^- d^2 \vec{X} \delta(X^+ X^- - \vec{X}^2) \delta(X^+ - 1) \theta(X^+ + X^-) \\ &\quad \times \int_{\omega_i \geq 0} e^{iX \cdot Q} d\omega_1 \omega_1^{i\lambda_1} d\omega_2 \omega_2^{i\lambda_2} \prod_{k \geq 3}^N d\omega_k \omega_k^{i\lambda_k - 1} \\ &= \frac{C_N}{2} \int d^2 \vec{X} \int_{\omega_i \geq 0} \exp \left[\frac{i}{2} (Q^+ \vec{X}^2 - 2 \vec{Q} \cdot \vec{X} + Q^-) \right] d\omega_1 \omega_1^{i\lambda_1} d\omega_2 \omega_2^{i\lambda_2} \prod_{k \geq 3}^N d\omega_k \omega_k^{i\lambda_k - 1}. \end{aligned} \quad (32)$$

As a result of Gaussian integration, we obtain

$$\begin{aligned} I_N &= C_N \int_{\omega_i \geq 0} \frac{i\pi}{Q^+} \exp \left(\frac{iQ^2}{2Q^+} \right) d\omega_1 \omega_1^{i\lambda_1} d\omega_2 \omega_2^{i\lambda_2} \prod_{k \geq 3}^N d\omega_k \omega_k^{i\lambda_k - 1} \\ &= C_N \int_{\omega_i \geq 0} \frac{i\pi}{\sum_{i=1}^N \omega_i} \exp \left[\frac{i \sum_{i < j}^N \omega_i \omega_j z_{ij} \bar{z}_{ij}}{2 \sum_{i=1}^N \omega_i} \right] d\omega_1 \omega_1^{i\lambda_1} d\omega_2 \omega_2^{i\lambda_2} \prod_{k \geq 3}^N d\omega_k \omega_k^{i\lambda_k - 1}. \end{aligned} \quad (33)$$

³ Under this assumption, $Q^2 > 0$, hence we avoid crossing the singularity at $Q^2 = 0$ in Eq. (26). At this point, it is not clear how to extend our discussion to the configurations involving both outgoing and incoming particles.

It is convenient to change the integration variables to

$$\omega = \sum_{i=1}^N \omega_i, \quad y_k = \frac{\omega_k}{\sum_{i=1}^N \omega_i} \quad (k = 1, \dots, N-1). \quad (34)$$

Then

$$I_N = i\pi C_N \left(\prod_{k=1}^{N-1} \int_0^1 dy_k \right) y_1^{i\lambda_1} y_2^{i\lambda_2} \prod_{j \geq 3}^{N-1} y_j^{i\lambda_j-1} (1 - \sum_{l=1}^{N-1} y_l)^{i\lambda_N-1} \quad (35)$$

$$\begin{aligned} & \times \int_0^\infty d\omega \exp \left[\frac{i\omega}{2} \left(\sum_{i < j} y_i y_j z_{ij} \bar{z}_{ij} + (1 - \sum_{l=1}^{N-1} y_l) \sum_{i=1}^N y_i z_{iN} \bar{z}_{iN} \right) \right] \\ & = -2\pi C_N \left(\prod_{k=1}^{N-1} \int_0^1 dy_k \right) y_1^{i\lambda_1} y_2^{i\lambda_2} \prod_{j \geq 3}^{N-1} y_j^{i\lambda_j-1} (1 - \sum_{l=1}^{N-1} y_l)^{i\lambda_N-1} \\ & \times \left[\sum_{i < j}^{N-1} y_i y_j z_{ij} \bar{z}_{ij} + (1 - \sum_{l=1}^{N-1} y_l) \sum_{i=1}^N y_i z_{iN} \bar{z}_{iN} \right]^{-1}. \end{aligned} \quad (36)$$

Recall that the above integral (36) is evaluated on the support of $\sum_i \lambda_i = 0$. In order to make Lorentz covariance explicit, we multiply it by

$$2\pi \delta\left(\sum_i \lambda_i\right) = \int_0^\infty \omega^{(i \sum_{i=1}^N \lambda_i) - 1} d\omega, \quad (37)$$

and return to the original integration variables,

$$\omega_k = y_k \omega \quad (k = 1, \dots, N-1), \quad \omega_N = (1 - \sum_{k=1}^{N-1} y_k) \omega. \quad (38)$$

In this way, we obtain

$$-2\pi C_N^{-1} \delta\left(\sum_i \lambda_i\right) I_N = \int_{\omega_i \geq 0} \left(\frac{2\pi}{Q^2}\right) d\omega_1 \omega_1^{i\lambda_1} d\omega_2 \omega_2^{i\lambda_2} \prod_{k \geq 3}^N d\omega_k \omega_k^{i\lambda_k-1}, \quad (39)$$

where

$$Q^2 = \sum_{i < j}^N \omega_i \omega_j z_{ij} \bar{z}_{ij}. \quad (40)$$

The final expression (39) is manifestly Lorentz covariant under $SL(2, \mathbb{C})$ transformations (23). It matches the scalar part (26) of celestial amplitudes with the primary field dimensions written in Eq. (28) and with the scalar current

$$J_\Phi(X) = 2\pi \delta^{(4)}(X). \quad (41)$$

Note that in the presence of such a Lorentz invariant source, $SL(2, \mathbb{C})$ symmetry should remain valid for any number of insertions, as an exact symmetry of celestial amplitudes. The solution of the equation $\square \Phi_0(X) = 2\pi \delta^{(4)}(X)$, matching the shockwave (31) on the Poincaré section, is the background field

$$\Phi_0(X) = \Phi_0(r, t) = -\frac{1}{2r} \delta(r-t) \theta(t). \quad (42)$$

As a consistency check, we evaluate Liouville integrals (15) for $N = 3$ and $N = 4$ by using our method and compare with the results reported previously in the literature. For $N = 3$, we start from Eq. (39):

$$\begin{aligned} \delta\left(\sum_i \lambda_i\right) I_3 &= -C_3 \int d\omega_1 d\omega_2 d\omega_3 \omega_1^{i\lambda_1} \omega_2^{i\lambda_2} \omega_3^{i\lambda_3-1} (\omega_1 \omega_2 z_{12} \bar{z}_{12} + \omega_2 \omega_3 z_{23} \bar{z}_{23} + \omega_1 \omega_3 z_{13} \bar{z}_{13})^{-1} \\ &= -C_3 \int d\omega_1 d\omega_2 \omega_1^{i\lambda_1} \omega_2^{i\lambda_2} B(i\lambda_3, 1 - i\lambda_3) (\omega_1 z_{13} \bar{z}_{13} + \omega_2 z_{23} \bar{z}_{23})^{-i\lambda_3} (\omega_1 \omega_2 z_{12} \bar{z}_{12})^{i\lambda_3-1} \\ &= -C_3 \int d\omega_1 \omega_1^{i\lambda_1 + i\lambda_2 + i\lambda_3 - 1} B(i\lambda_3, 1 - i\lambda_3) B(-i\lambda_2, i\lambda_2 + i\lambda_3) \\ & \quad \times (z_{12} \bar{z}_{12})^{i\lambda_3-1} (z_{13} \bar{z}_{13})^{i\lambda_2} (z_{23} \bar{z}_{23})^{-i\lambda_2 - i\lambda_3} \end{aligned}$$

$$= \pi \delta \left(\sum_i \lambda_i \right) (z_{12} \bar{z}_{12})^{i\lambda_3-1} (z_{23} \bar{z}_{23})^{i\lambda_1} (z_{13} \bar{z}_{13})^{i\lambda_2} \frac{\Gamma(-i\lambda_1) \Gamma(-i\lambda_2) \Gamma(1-i\lambda_3)}{\Gamma(1+i\lambda_1) \Gamma(1+i\lambda_2) \Gamma(i\lambda_3)}. \quad (43)$$

In this way, we obtain

$$I_3 = \pi \frac{\Gamma(-i\lambda_1) \Gamma(-i\lambda_2) \Gamma(1-i\lambda_3)}{\Gamma(1+i\lambda_1) \Gamma(1+i\lambda_2) \Gamma(i\lambda_3)} (z_{12} \bar{z}_{12})^{i\lambda_3-1} (z_{23} \bar{z}_{23})^{i\lambda_1} (z_{13} \bar{z}_{13})^{i\lambda_2}, \quad (44)$$

in agreement with [15,23]. For $N = 4$, a similar computation yields

$$I_4 = (z_{12} \bar{z}_{12})^{i\lambda_3+i\lambda_4-1} (z_{13} \bar{z}_{13})^{-i\lambda_3} (z_{14} \bar{z}_{14})^{i\lambda_2+i\lambda_3} (z_{24} \bar{z}_{24})^{i\lambda_1} \\ \times \left[C_{\Delta} K_{34}^{21} \left[\frac{i\lambda_3}{2} + \frac{i\lambda_4}{2}, \frac{i\lambda_3}{2} + \frac{i\lambda_4}{2} \right] (x, \bar{x}) + C_{\tilde{\Delta}} K_{34}^{21} \left[1 - \frac{i\lambda_3}{2} - \frac{i\lambda_4}{2}, 1 - \frac{i\lambda_3}{2} - \frac{i\lambda_4}{2} \right] (x, \bar{x}) \right], \quad (45)$$

where x is the conformal invariant cross ratio,

$$x = \frac{z_{12} z_{34}}{z_{13} z_{24}}, \quad (46)$$

and $K_{34}^{21}[h, \bar{h}](x, \bar{x})$ are the two-dimensional s -channel global conformal blocks [24–28] associated with primary fields with dimensions $\Delta = h + \bar{h}$:

$$K_{34}^{21}[h, \bar{h}](x, \bar{x}) = x^{h-h_3-h_4} \bar{x}^{\bar{h}-\bar{h}_3-\bar{h}_4} {}_2F_1 \left(\begin{matrix} h-h_{12}, h+h_{34} \\ 2h \end{matrix}; x \right) {}_2F_1 \left(\begin{matrix} \bar{h}-\bar{h}_{12}, \bar{h}+\bar{h}_{34} \\ 2\bar{h} \end{matrix}; \bar{x} \right), \quad (47)$$

where $h_{ij} = h_i - h_j$, $\bar{h}_{ij} = \bar{h}_i - \bar{h}_j$. In the case under consideration, $h_i = \bar{h}_i$ and $h = \bar{h}$. We find one block with dimension $\Delta = i\lambda_3 + i\lambda_4$ and its shadow block with dimension $\tilde{\Delta} = 2 - \Delta$, with the coefficients

$$C_{\Delta} = \pi \frac{\Gamma(-i\lambda_1) \Gamma(-i\lambda_2) \Gamma(1+i\lambda_1+i\lambda_2)}{\Gamma(1+i\lambda_1) \Gamma(1+i\lambda_2) \Gamma(-i\lambda_1-i\lambda_2)}, \quad (48)$$

$$C_{\tilde{\Delta}} = \pi \frac{\Gamma(1-i\lambda_3) \Gamma(1-i\lambda_4) \Gamma(i\lambda_3+i\lambda_4-1)}{\Gamma(i\lambda_3) \Gamma(i\lambda_4) \Gamma(2-i\lambda_3-i\lambda_4)}. \quad (49)$$

Eq. (45) agrees with the expression written in Ref. [22]. Furthermore, we checked that this four-point correlator satisfies the crossing symmetry constraints. For instance, in the u -channel, in which $x \rightarrow 1-x$, one also finds a single block with dimension $\Delta = 1 + i\lambda_2 + i\lambda_3$ and a shadow block.

Now we turn to the “soft” part (25) of celestial MHV amplitudes. Here, the negative helicity -1 gluons can be associated with the holomorphic operators $\hat{J}^a(z)$ in the adjoint representation of the gauge group, with chiral weights $(h = -1, \bar{h} = 0)$, i.e., dimensions -1 , while the positive helicity $+1$ gluons to dimension $+1$ holomorphic $(h = 1, \bar{h} = 0)$ Wess-Zumino-Witten (WZW) currents $J^a(z)$.⁴ The correlators involving WZW currents are completely determined by Ward identities, which, in the context of celestial holography, follow from the leading soft gluon theorem [1–3]. Assuming

$$\langle \hat{J}^{a_1}(z_1) \hat{J}^{a_2}(z_2) \rangle = \delta^{a_1 a_2} z_{12}^2, \quad (50)$$

one obtains [8]:

$$\langle \hat{J}^{a_1}(z_1) \hat{J}^{a_2}(z_2) J^{a_3}(z_3) \cdots J^{a_N}(z_N) \rangle = \mathcal{J}_N(z_i). \quad (51)$$

At this point, we are in a position to connect the tree-level celestial MHV amplitudes to the classical limit of Liouville theory. We introduce the following operators:

$$\mathcal{O}_{\lambda}^{-a}(z, \bar{z}) = \Gamma(1+i\lambda) \hat{J}^a(z) e^{(1+i\lambda)b\phi(z, \bar{z})} \quad (52)$$

$$\mathcal{O}_{\lambda}^{+a}(z, \bar{z}) = \Gamma(i\lambda) J^a(z) e^{i\lambda b\phi(z, \bar{z})} \quad (53)$$

and consider the limit of $b \rightarrow 0$. In this limit, the dimension of $\mathcal{O}_{\lambda}^{-a}$ becomes $\Delta_- = i\lambda$ while the dimension of $\mathcal{O}_{\lambda}^{+a}$ is $\Delta_+ = 1 + i\lambda$. We associate $\mathcal{O}_{\lambda}^{\pm a}$ to gluons with helicities ± 1 , respectively, and group indices a . From our discussion, cf. Eqs. (39) and (20), it follows that

$$4\pi \delta \left(\sum_{i=1}^N \lambda_i \right) \left\langle \mathcal{O}_{\lambda_1}^{-a_1}(z_1, \bar{z}_1) \mathcal{O}_{\lambda_2}^{-a_2}(z_2, \bar{z}_2) \mathcal{O}_{\lambda_3}^{+a_3}(z_3, \bar{z}_3) \cdots \mathcal{O}_{\lambda_N}^{+a_N}(z_N, \bar{z}_N) \right\rangle \\ = \mathcal{M}_N(z_1, \bar{z}_1, \dots, z_N, \bar{z}_N | i\lambda_1, i\lambda_2, 1+i\lambda_3, \dots, 1+i\lambda_N), \quad (54)$$

where $\mathcal{M}_N$ is the celestial MHV amplitude, cf. Eqs. (24)–(26), evaluated in the shockwave background of Eqs. (41), (42). Note that the dimensions of negative helicity gluons are shifted by -1 as compared with the conventional “principal series” celestial amplitudes [21]. By using Eqs. (44) and (45), we checked explicitly that the correlators (54) satisfy BG equations for $N = 3$ and $N = 4$. We also checked that the operator product expansions extracted from the collinear limits of four-gluon amplitudes agree with Refs. [29,30].

⁴ Since the dimension of $\hat{J}^a(z)$ is negative, it is not clear how to incorporate this operator into the framework of unitary WZW models.

The fact that a two-dimensional spherical shell of the dilaton shockwave propagating in four-dimensional spacetime corresponds to the classical Liouville field describing metrics on the celestial sphere is perhaps the most natural, not to say expected, outcome of our analysis. Liouville theory has been linked before to Yang-Mills through the AGT correspondence [31], which raises the question whether our “dilaton deformation” is related in some way to Nekrasov’s Ω deformation [32,33].⁵ It would also be very interesting to connect Liouville shockwaves to the shockwaves recently studied in [36,37]. This could enable exact treatment of the background, in order to address the question whether the relation between Yang-Mills and Liouville theories remains valid beyond the weak source approximation. The most interesting and possibly related problem, however, is to understand how the quantum corrections to celestial Yang-Mills amplitudes are related to quantum Liouville theory with finite central charge. In this way, one could try to establish relations between the parameters of Yang-Mills theory and the cosmological constant μ and the Liouville coupling constant b , in a similar way as AdS/CFT correspondence relates the string coupling and string tension in the bulk to the rank of the gauge group and the Yang-Mills coupling on the boundary. Answering these questions would create a novel path towards Yang-Mills dynamics.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Tomasz Taylor reports financial support was provided by National Science Foundation.

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⁵ Although AGT correspondence applies to superconformal Yang-Mills theory, there is no difference between supersymmetric and non-supersymmetric gluon amplitudes at the tree level [34,35].

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