

# Selective Encoding Policies for Maximizing Information Freshness

Melih Bastopcu, *Graduate Student Member, IEEE*, Baturalp Buyukates<sup>id</sup>, *Graduate Student Member, IEEE*,  
and Sennur Ulukus<sup>id</sup>, *Fellow, IEEE*

**Abstract**—An information source generates independent and identically distributed status update messages from an observed random phenomenon which takes  $n$  distinct values based on a given probability mass function (PMF). These update packets are encoded at the transmitter node to be sent to a receiver node which wants to track the observed random variable with as little age as possible. The transmitter node implements a selective  $k$  encoding policy such that rather than encoding all possible  $n$  realizations, the transmitter node encodes the most probable  $k$  realizations. We consider three different policies regarding the remaining  $n - k$  less probable realizations: *highest  $k$  selective encoding* which disregards whenever a realization from the remaining  $n - k$  values occurs; *randomized selective encoding* which encodes and sends the remaining  $n - k$  realizations with a certain probability to further inform the receiver node at the expense of longer codewords for the selected  $k$  realizations; and *highest  $k$  selective encoding with an empty symbol* which sends a designated empty symbol when one of the remaining  $n - k$  realizations occurs. For all of these three encoding schemes, we find the average age and determine the age-optimal real codeword lengths, including the codeword length for the empty symbol in the case of the latter scheme, such that the average age at the receiver node is minimized. Through numerical evaluations for arbitrary PMFs, we show that these selective encoding policies result in a lower average age than encoding every realization, and find the corresponding age-optimal  $k$  values. Since we focus on real-valued codeword lengths in this paper, the resulting age value obtained in each case studied here serves as a lower bound to what can be attained by integer-valued codeword lengths in that case.

**Index Terms**—Timely source coding, age of information, information freshness, selective encoding.

## I. INTRODUCTION

**A**GE of information is a performance metric which quantifies the timeliness of information in networks. It keeps track of the time since the most recent update at the receiver has been generated at the transmitter. Age increases linearly

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The authors are with the Department of Electrical and Computer Engineering, University of Maryland, College Park, MD 20742 USA (e-mail: bastopcu@umd.edu; baturalp@umd.edu; ulukus@umd.edu).

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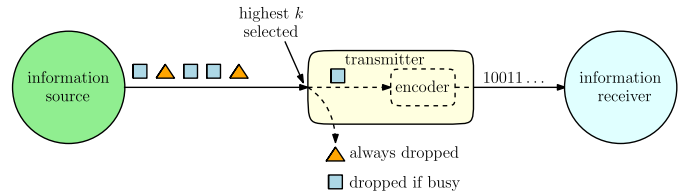


Fig. 1. An information source generates i.i.d. status updates from a random variable  $X$ . Only a portion of the realizations (shown with a square) is encoded into codewords. Update packets that come from the selected portion of the realizations that find the transmitter node idle are sent to the receiver node. Non-selected realizations (shown with a triangle) are always discarded at the transmitter node even if the transmitter node is idle.

in time such that at time  $t$  age  $\Delta(t)$  of an update packet which was generated at time  $u(t)$  is  $\Delta(t) = t - u(t)$ . When a new update packet is received, the age drops to a smaller value. Age of information has been studied in the context of queueing networks [2]–[11], scheduling and optimization [12]–[32], energy harvesting [33]–[42], reinforcement learning [43]–[47] and so on. The concept of age is applicable to a wide range of problems, e.g., in autonomous driving, augmented reality, social networks, and online gaming, as information freshness is crucial in all these and other emerging applications.

In this work, we consider a status updating system that consists of a single transmitter node and a single receiver node (see Fig. 1). The transmitter receives independent and identically distributed time-sensitive status update packets generated by an information source based on an observed random phenomenon that takes  $n$  distinct values with a known probability mass function (PMF). This observed random variable could be the position of a UAV in autonomous systems or share prices in the stock market. Arriving status update packets are encoded at the transmitter and sent to the receiver through an error-free noiseless channel. The receiver wants to acquire fresh information regarding the observed random variable, which brings up the concept of age of information.

Unlike most of the literature in which the transmission times, also referred to as service times in queueing theory, are based on a given service distribution, in this work, we design transmission times through source coding schemes by choosing the codeword lengths assigned to realizations. That is, the codeword length assigned to each realization represents the service time (transmission time) of that realization.

References that are most closely related to our work are [48]–[50] which study the timely source coding problem for a discrete-time system. References [48] and [49] study

a communication system where a source follows a zero-wait update generation model whereas in [50] status updates arrive exogenously as a Bernoulli process. In [48], the transmitter only skips the status updates that are generated while the channel is busy. Unlike the model in [48], references [49] and [50] reconstruct the entire source message stream in a lossless manner. Reference [48] finds real-relaxation of the underlying integer-valued codeword lengths using Shannon codes based on a modified version of the given PMF that achieve the optimal age with a constant gap. References [49] and [50] consider block coding and source coding problems to find age-optimal codes for FIFO queues.

Different from [48]–[50], in our model, time is not slotted and the status update packets arrive at the transmitter node following a Poisson process with a known parameter  $\lambda$ . Unlike the model in [48], we introduce an encoding mechanism where the transmitter skips not only the status updates that are generated while the channel is busy but also the least probable ones to achieve a lower average age of information at the receiver. We term this encoding mechanism *selective encoding*. In this selective encoding model, instead of encoding all possible realizations, we encode only a portion of the realizations and send to the receiver node. Specifically, we consider what we call the *highest  $k$  selective encoding* scheme in which we only encode the most probable  $k$  realizations and disregard any update packets from the remaining  $n - k$  realizations. We note that a smaller  $k$  yields shorter codeword lengths but larger interarrival times, as in this case most of the updates are not encoded. However, when  $k$  is large, codeword lengths and correspondingly the transmission times get larger even though the interarrival times get smaller. Thus, in this paper, based on the given PMF, we aim to find the optimal  $k$  which strikes a balance between these two opposing trends such that the average age at the receiver node is minimized. Due to this selective encoding scheme not every realization is sent to the receiver even if the channel is free.

Next, we consider a scenario in which the remaining  $n - k$  realizations are not completely disregarded but encoded with a certain probability which we call the *randomized selective encoding* scheme. In this scheme, in addition to the most probable  $k$  realizations, the remaining  $n - k$  less probable realizations are sometimes encoded.

A disadvantage of the highest  $k$  selective encoding scheme is the fact that the receiver node is not informed when one of the non-selected realizations occurs. For instance, during a period with no arrivals, the receiver node cannot differentiate whether there has been no arrivals or if the arrival has taken one of the non-selected values as in either case it does not receive any update packets. Thus, lastly, we take a careful look at the remaining  $n - k$  realizations and propose a modified selective encoding policy which we call the *highest  $k$  selective encoding with empty symbol* that still achieves a lower average age than encoding every realization but also informs the receiver node when one of the non-selected values is taken by the observed random variable. In this scheme, only the most probable  $k$  realizations are encoded and the remaining  $n - k$  realizations are mapped into a designated empty symbol such that in the case of these  $n - k$  non-selected realizations,

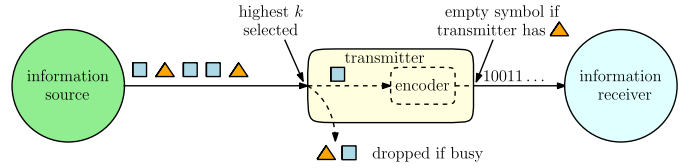


Fig. 2. Update packets that come from the selected portion of the realizations (shown with a square) that find the transmitter idle are sent to the receiver. Non-selected realizations (shown with a triangle) that find the transmitter idle are mapped into an empty symbol.

this empty symbol is sent to further inform the receiver (see Fig. 2). Thus, in such a case, the receiver at least knows that the observed random variable has taken a value from the non-selected portion even though it does not know which value was taken specifically. We consider two variations on this policy: when the empty symbol does not reset the age and when the empty symbol resets the age.<sup>1</sup>

For all three encoding schemes, we find the average age experienced by the receiver node and determine the age-optimal real codeword lengths, including the codeword length of the empty symbol in the case of the highest  $k$  selective encoding with empty symbol scheme. Through numerical evaluations for given arbitrary PMFs, we show that the proposed selective encoding policies achieve a lower average age than encoding every realization, and find the corresponding age-optimal  $k$  values. In addition, we discuss the optimality of the highest  $k$  selective encoding policy. We note that, since we focus on age-optimal real-valued codeword lengths in this paper, the obtained age values serve as lower bounds to what can be attained by integer-valued codeword lengths. Designing age-optimal integer-valued codeword lengths is not addressed in this paper and remains an interesting open problem.

Finally, a similar  $k$  out of  $n$  type of idea was used in [51]–[55] in the context of multicasting updates in networks, where each packet is transmitted until the earliest  $k$  out of  $n$  receiver nodes have received the packet. While the multicast communication problem studied in [51]–[55] and the source coding problem studied here are fundamentally different, there is an analogy between their results as follows. In [51]–[55], it was shown that sending status updates to (earliest)  $k$  out of  $n$  receivers achieves a smaller average age of information than sending status updates to every one of  $n$  receivers. Analogously, in this paper, we show that sending status updates for (most probable)  $k$  out of  $n$  realizations achieves a smaller age of information than sending status updates for every one of  $n$  realizations.

## II. SYSTEM MODEL AND PROBLEM FORMULATION

We consider a communication system in which an information source generates independent and identically distributed

<sup>1</sup>When the empty symbol  $x_e$  is received, the receiver does not know exactly which update is realized at the source. For that reason, operationally, the receiver may not reset its age when  $x_e$  is received. On the other hand, the receiver may reset its age as the empty symbol carries some *partial* information regarding the status update at the source, i.e., when the empty symbol  $x_e$  is received, the receiver knows that one of the least probable status updates is realized at the source. That is why we consider both of these scenarios in our analysis.

status update packets from an observed phenomenon that takes realizations from the set  $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$  based on a known PMF  $P_X(x_i)$  for  $i \in \{1, \dots, n\}$ .<sup>2</sup> Without loss of generality, we assume that  $P_X(x_m) \geq P_X(x_{m+1})$  for all  $m$ , i.e., the probabilities of the realizations are in a non-increasing order. Update packets arrive at the transmitter node following a Poisson process with parameter  $\lambda$ . The transmitter node implements a blocking policy in which the update packets that arrive when the transmitter node is busy are blocked and lost. Thus, the transmitter node receives only the updates that arrive when it is idle.

We consider three different encoding policies: highest  $k$  selective encoding, randomized selective encoding, and highest  $k$  selective encoding with an empty symbol.

#### A. Policy 1: Highest $k$ Selective Encoding

In the first policy, we consider a selective encoding mechanism that we call *highest  $k$  selective encoding* where the transmitter node only sends the most probable  $k$  realizations, i.e., only the realizations from set  $\mathcal{X}_k = \{x_1, \dots, x_k\}$ , which have the highest probabilities among possible  $n$  updates generated by the source, are transmitted for  $k \in \{1, \dots, n\}$ ; see Fig. 1. If an update packet from the remaining non-selected portion of the realizations arrives, the transmitter disregards that update packet and waits for the next arrival. If an update packet arrives from the selected portion of the realizations, then the transmitter encodes that update packet by using a binary alphabet with the conditional probabilities given by,

$$P_{X_k}(x_i) = \begin{cases} \frac{P_X(x_i)}{q_k}, & i = 1, 2, \dots, k \\ 0, & i = k + 1, k + 2, \dots, n, \end{cases} \quad (1)$$

where

$$q_k \triangleq \sum_{\ell=1}^k P_X(x_\ell). \quad (2)$$

The transmitter assigns codeword  $c(x_i)$  with length  $\ell(x_i)$  to realization  $x_i$  for  $i \in \{1, 2, \dots, k\}$ .

#### B. Policy 2: Randomized Selective Encoding

In the second policy, inspired by [48], we study a *randomized selective encoding* scheme. In this policy, the most probable  $k$  realizations are always encoded. However, instead of discarding the remaining  $n - k$  realizations, the transmitter node encodes them with probability  $\alpha$  and discards them with probability  $1 - \alpha$ . In other words, in this model, less likely realizations that are not encoded under the highest  $k$  selective encoding policy are sometimes transmitted to the receiver node. Thus, under this operation, codewords for each one of the  $n$  possible realizations need to be generated since every realization can be sent to the receiver node. The transmitter

assigns codeword  $c(x_i)$  with length  $\ell(x_i)$  to realization  $x_i$  for  $i \in \{1, 2, \dots, n\}$ .

The transmitter node performs encoding using the following conditional probabilities,

$$P_{X_\alpha}(x_i) = \begin{cases} \frac{P_X(x_i)}{q_{k,\alpha}}, & i = 1, 2, \dots, k \\ \alpha \frac{P_X(x_i)}{q_{k,\alpha}}, & i = k + 1, k + 2, \dots, n, \end{cases} \quad (3)$$

where

$$q_{k,\alpha} \triangleq \sum_{\ell=1}^k P_X(x_\ell) + \alpha \sum_{\ell=k+1}^n P_X(x_\ell). \quad (4)$$

#### C. Policy 3: Highest $k$ Selective Encoding With an Empty Symbol

In the third policy, we consider an encoding scheme that we call the *highest  $k$  selective encoding with an empty symbol*. In this encoding scheme, the transmitter always encodes the most probable  $k$  realizations as in the previous two policies. However, unlike the previous models, if an update packet from the remaining non-selected portion of the realizations arrives, the transmitter sends an empty status update denoted by  $x_e$  to further inform the receiver at the expense of longer codewords for the selected  $k$  realizations.

When an update packet arrives from the set  $\mathcal{X}'_k = \mathcal{X}_k \cup \{x_e\}$ , the transmitter node encodes that update packet with the binary alphabet by using the PMF given as  $\{P_X(x_1), P_X(x_2), \dots, P_X(x_k), P_X(x_e)\}$  where  $P_X(x_e) = 1 - q_k$ . Thus, in this policy, the transmitter node assigns codewords to the most probable  $k$  realizations as well as to the empty symbol  $x_e$ . That is, the transmitter assigns codeword  $c(x_i)$  with length  $\ell(x_i)$  to realization  $x_i$  for  $i \in \{1, \dots, k, e\}$ .

In this paper, we focus on the source coding aspect of timely status updating. Therefore, in all these three policies, the channel between the transmitter node and the receiver node is error-free. The transmitter node sends one bit at a unit time. Thus, if the transmitter node sends update  $x_i$  to the receiver node, this transmission takes  $\ell(x_i)$  units of time. That is, for realization  $x_i$ , the service time of the system is  $\ell(x_i)$ .

#### D. Problem Formulation

We use the age of information metric to measure the freshness of the information at the receiver node. Let  $\Delta(t)$  be the instantaneous age at the receiver node at time  $t$  with  $\Delta(0) = \Delta_0$ . Age at the receiver node increases linearly in time and drops to the age of the most recently received update upon delivery of a new update packet. We define the long term average age as,

$$\Delta = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \Delta(t) dt. \quad (5)$$

Our aim is to find the codeword lengths for each encoding policy described in Sections II-A, II-B, and II-C that minimize the long term average age for a given  $k$  such that a uniquely decodable code can be designed, i.e., the Kraft inequality is satisfied [56].

In the following section, we find an analytical expression for the long term average age  $\Delta$ .

<sup>2</sup>Even though the number of realizations,  $n$ , is a finite number, it can be arbitrarily large. We highlight that this modeling choice is well suited to many practical applications even when the underlying randomness is continuous so long as it is quantized to sufficiently many discrete levels, e.g., quantized temperature sensor measurements, trajectory commands in a UAV system such as move right, left, up, down, and so on.



### III. AVERAGE AGE ANALYSIS

As described in Section II, status update packets arrive at the transmitter as a Poisson process with rate  $\lambda$ . Update packets that arrive when the transmitter is busy are blocked from entry and dropped. Thus, upon successful delivery of a packet to the receiver, the transmitter idles until the next update packet arrives. This idle waiting period in between two arrivals is denoted by  $Z$  which is an exponential random variable with rate  $\lambda$  due to the memoryless property of exponential random variables as update interarrivals at the transmitter are exponential with  $\lambda$ .

We note that in all of the encoding policies in Section II, every packet from the set  $\mathcal{X}_k$  which successfully enters the transmitter node is always sent to the receiver. However, a packet from the remaining least probable  $n - k$  realizations which enters the transmitter might not be sent. Under the highest  $k$  selective encoding policy described in Section II-A, when one of the remaining  $n - k$  packets enters the transmitter node, the transmitter node drops the packet and proceeds to wait for the next update arrival. Under the randomized selective encoding scheme described in Section II-B, remaining  $n - k$  less likely realizations are transmitted to the receiver node with probability  $\alpha$ . Under the highest  $k$  selective encoding scheme with an empty symbol described in Section II-C, the transmitter node sends a designated empty status update to further inform the receiver about the occurrence of a realization from the remaining  $n - k$  realizations.

We denote the update packets which arrive when the transmitter node is idle and reset the age as *successful* update packets. Since the channel is noiseless and there is no preemption, these successful packets are received by the receiver node. We denote  $T_{j-1}$  as the time instant at which the  $j$ th successful update packet is received. We define update cycle denoted by  $Y_j = T_j - T_{j-1}$  as the time in between two successive successful update arrivals at the transmitter. Update cycle  $Y_j$  consists of a busy cycle and an idle cycle such that

$$Y_j = S_j + W_j, \quad (6)$$

where  $S_j$  is the service time of update  $j$  and  $W_j$  is the overall waiting time in the  $j$ th update cycle.<sup>3</sup>

Fig. 3 shows a sample age evolution at the receiver. Here,  $Q_j$  denotes the area under the instantaneous age curve in update cycle  $j$  and  $Y_j$  denotes the length of the  $j$ th update cycle as defined earlier. The metric we use, long term average age, is the average area under the age curve which is given by [9]

$$\Delta = \limsup_{n \rightarrow \infty} \frac{\frac{1}{n} \sum_{j=1}^n Q_j}{\frac{1}{n} \sum_{j=1}^n Y_j} = \frac{\mathbb{E}[Q]}{\mathbb{E}[Y]}. \quad (7)$$

By using Fig. 3, we find  $Q_j = \frac{1}{2}Y_j^2 + Y_j S_{j+1}$ , where  $Y_j$  is given in (6). Thus, using the independence of  $Y_j$  and  $S_{j+1}$ , (7) is equivalent to

$$\Delta = \frac{\mathbb{E}[Y^2]}{2\mathbb{E}[Y]} + \mathbb{E}[S]. \quad (8)$$

<sup>3</sup>We note that  $W_j$  can be thought of as the idle time following the service completion of update  $j$ .

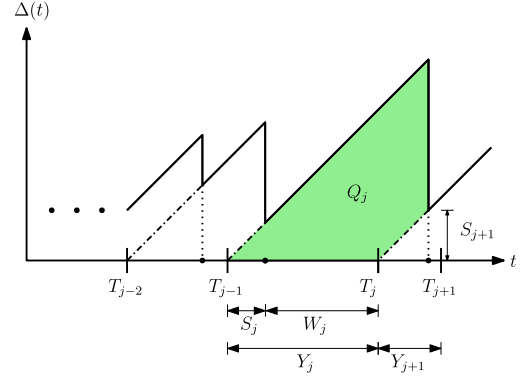


Fig. 3. Sample age evolution  $\Delta(t)$  at the receiver node. Successful updates are indexed by  $j$ . The  $j$ th successful update arrives at the server node at  $T_{j-1}$ . Update cycle at the server node is the time in between two successive arrivals and is equal to  $Y_j = S_j + W_j = T_j - T_{j-1}$ .

In the following section, we find the optimal real-valued codeword lengths for the highest  $k$  selective encoding policy described in Section II-A.

### IV. OPTIMAL CODEWORD DESIGN UNDER SELECTIVE ENCODING

In this section, we consider the highest  $k$  selective encoding policy described in Section II-A. Under this way of operation, the transmitter only sends the most probable  $k$  realizations from the set  $\mathcal{X}_k$ , and drops any update packets from the remaining  $n - k$  least probable realizations.

Proposition 1 characterizes the average age  $\Delta$  given in (8) for the encoding scheme described in Section II-A.

*Proposition 1:* Under the highest  $k$  selective encoding scheme, the average age at the receiver node is given by

$$\Delta = \frac{\mathbb{E}[L^2] + \frac{2}{q_k \lambda} \mathbb{E}[L] + \frac{2}{(q_k \lambda)^2}}{2 \left( \mathbb{E}[L] + \frac{1}{q_k \lambda} \right)} + \mathbb{E}[L], \quad (9)$$

where the first and the second moments of the codeword lengths are given by  $\mathbb{E}[L] = \sum_{i=1}^k P_{X_k}(x_i) \ell(x_i)$  and  $\mathbb{E}[L^2] = \sum_{i=1}^k P_{X_k}(x_i) \ell(x_i)^2$ .

*Proof:* With the highest  $k$  selective encoding scheme, we note that the overall waiting time  $W$  is equal to  $W = \sum_{\ell=1}^M Z_\ell$  where  $Z_\ell$ s are the i.i.d. exponential random variables with rate  $\lambda$  as discussed earlier. Here,  $M$  is a geometric random variable with parameter  $q_k$  (defined in (2)) which denotes the total number of update arrivals until the first update from the set  $\mathcal{X}_k$  is observed at the transmitter node.  $W$  is also an exponential random variable with rate  $\lambda q_k$  [57, Prob. 9.4.1]. Then, noting that the service time random variable  $S$  in (6) is the codeword length random variable  $L$ , we have

$$\mathbb{E}[Y] = \mathbb{E}[L] + \mathbb{E}[W], \quad (10)$$

$$\mathbb{E}[Y^2] = \mathbb{E}[L^2] + 2\mathbb{E}[W]\mathbb{E}[L] + \mathbb{E}[W^2], \quad (11)$$

where  $\mathbb{E}[W] = \frac{1}{q_k \lambda}$  and  $\mathbb{E}[W^2] = \frac{2}{(q_k \lambda)^2}$ . Substituting (10) and (11) in (8) yields the result in (9).<sup>4</sup> ■

<sup>4</sup>We note that the average AoI expression in (9) is aligned with [9, Theorem 1], as expected.

Thus, (9) characterizes the average age  $\Delta$  achieved at the receiver node in terms of the first and second moments of the codeword lengths for a given PMF, selected  $k$ , and update arrival rate  $\lambda$ . Next, we formulate the age minimization problem as,

$$\begin{aligned} \min_{\{\ell(x_i)\}} & \frac{\mathbb{E}[L^2] + 2a\mathbb{E}[L] + 2a^2}{2(\mathbb{E}[L] + a)} + \mathbb{E}[L] \\ \text{s.t.} & \sum_{i=1}^k 2^{-\ell(x_i)} \leq 1 \\ & \ell(x_i) \in \mathbb{R}^+, \quad i \in \{1, \dots, k\}, \end{aligned} \quad (12)$$

where the objective function is equal to the average age found in Proposition 1 with  $a = \frac{1}{\lambda q_k}$ , the first constraint is the Kraft inequality, and the second constraint represents the feasibility of the codeword lengths, i.e., each codeword length should be non-negative.

We note that the optimization problem in (12) is a nonlinear fractional problem. To solve this problem, we define the following intermediate problem, which is parameterized by  $\theta$ , similar to [22] and [36]

$$\begin{aligned} p(\theta) := & \min_{\{\ell(x_i)\}} \frac{1}{2} \mathbb{E}[L^2] + \mathbb{E}[L]^2 + (2a - \theta)\mathbb{E}[L] + a^2 - \theta a \\ \text{s.t.} & \sum_{i=1}^k 2^{-\ell(x_i)} \leq 1 \\ & \ell(x_i) \in \mathbb{R}^+, \quad i \in \{1, \dots, k\}. \end{aligned} \quad (13)$$

One can show that  $p(\theta)$  is decreasing in  $\theta$  and the optimal solution is obtained when  $p(\theta) = 0$  such that the optimal age for the problem in (12) is equal to  $\theta$ , i.e.,  $\Delta^* = \theta$  [58]. We define the Lagrangian [59] function for (13) as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \mathbb{E}[L^2] + \mathbb{E}[L]^2 + (2a - \theta)\mathbb{E}[L] + a^2 - \theta a \\ & + \beta \left( \sum_{i=1}^k 2^{-\ell(x_i)} - 1 \right), \end{aligned} \quad (14)$$

where  $\beta \geq 0$ . Next, we write the KKT conditions as

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \ell(x_i)} = & P_{X_k}(x_i) \ell(x_i) + 2\mathbb{E}[L] P_{X_k}(x_i) + (2a - \theta) P_{X_k}(x_i) \\ & - \beta (\log 2) 2^{-\ell(x_i)} = 0, \quad \forall i, \end{aligned} \quad (15)$$

and the complementary slackness condition as

$$\beta \left( \sum_{i=1}^k 2^{-\ell(x_i)} - 1 \right) = 0. \quad (16)$$

In the following lemma, we prove that the optimal codeword lengths must satisfy the Kraft inequality as an equality.

**Lemma 1:** For the age-optimal real codeword lengths, we must have  $\sum_{i=1}^k 2^{-\ell(x_i)} = 1$ .

*Proof:* Assume that the optimal codeword lengths satisfy  $\sum_{i=1}^k 2^{-\ell(x_i)} < 1$ , which implies that  $\beta = 0$  due to (16). From (15), we have

$$\begin{aligned} P_{X_k}(x_i) \ell(x_i) + 2 \left( \sum_{j=1}^k P_{X_k}(x_j) \ell(x_j) \right) P_{X_k}(x_i) \\ + (2a - \theta) P_{X_k}(x_i) = 0, \quad \forall i. \end{aligned} \quad (17)$$

Summing (17) over all  $i \in \{1, \dots, k\}$  we find

$$3\mathbb{E}[L] + 2a - \theta = 0, \quad (18)$$

where  $\mathbb{E}[L]$  is as in Proposition 1. Thus, we find  $\ell(x_i) = \frac{\theta - 2a}{3}$  for all  $i \in \{1, 2, \dots, k\}$ . Thus,  $\mathbb{E}[L] = \frac{\theta - 2a}{3}$  and  $\mathbb{E}[L^2] = \left(\frac{\theta - 2a}{3}\right)^2$  so that  $p(\theta) = -\frac{\theta^2}{6} - \frac{\theta a}{3} + \frac{a^2}{3}$ . By using  $p(\theta) = 0$ , we find  $\theta = (-1 + \sqrt{3})a$  which gives  $\ell(x_i) = \frac{(-3 + \sqrt{3})a}{3} < 0$  for  $i \in \{1, 2, \dots, k\}$ . Since the codeword lengths cannot be negative, we reach a contradiction. Thus, the optimal codeword lengths must satisfy  $\sum_{i=1}^k 2^{-\ell(x_i)} = 1$ . ■

Next, we find the optimal codeword lengths which satisfy  $\sum_{i=1}^k 2^{-\ell(x_i)} = 1$ . By summing (15) over all  $i$ , we obtain

$$\mathbb{E}[L] = \frac{\theta + \beta \log 2 - 2a}{3}. \quad (19)$$

From (15), we obtain

$$-\ell(x_i) + \frac{\beta \log 2}{P_{X_k}(x_i)} 2^{-\ell(x_i)} = 2\mathbb{E}[L] + 2a - \theta, \quad (20)$$

for  $i \in \{1, 2, \dots, k\}$ , which yields

$$\frac{\beta (\log 2)^2}{P_{X_k}(x_i)} 2^{-\ell(x_i)} e^{\frac{\beta (\log 2)^2}{P_{X_k}(x_i)} 2^{-\ell(x_i)}} = \frac{\beta (\log 2)^2}{P_{X_k}(x_i)} 2^{\frac{-\theta + 2\beta \log 2 + 2a}{3}}. \quad (21)$$

Note that (21) is in the form of  $xe^x = y$  where the solution for  $x$  is equal to  $x = W_0(y)$  if  $y \geq 0$ . Here,  $W_0(\cdot)$  denotes the principal branch of the Lambert  $W$  function [60]. Since the right hand side of (21) is always non-negative, we are only interested in  $W_0(\cdot)$  which is denoted as  $W(\cdot)$  from now on. We find the unique solution for  $\ell(x_i)$  as

$$\ell(x_i) = -\frac{\log \left( \frac{P_{X_k}(x_i)}{\beta (\log 2)^2} W \left( \frac{\beta (\log 2)^2}{P_{X_k}(x_i)} 2^{\frac{-\theta + 2\beta \log 2 + 2a}{3}} \right) \right)}{\log 2}, \quad (22)$$

for  $i \in \{1, 2, \dots, k\}$ .

In order to find the optimal codeword lengths, we solve (22) for a  $(\theta, \beta)$  pair that satisfies  $p(\theta) = 0$  and the Kraft inequality, i.e.,  $\sum_{i=1}^k 2^{-\ell(x_i)} = 1$ . Starting from an arbitrary  $(\theta, \beta)$  pair, if  $p(\theta) > 0$  (or  $p(\theta) < 0$ ), we increase (or respectively decrease)  $\theta$  in the next iteration as  $p(\theta)$  is a decreasing function of  $\theta$ . Then, we update  $\beta$  by using (19). We repeat this process until  $p(\theta) = 0$  and  $\sum_{i=1}^k 2^{-\ell(x_i)} = 1$ .

We note that the age-optimal codeword lengths found in this section are for a fixed  $k$ . Thus, depending on the selected  $k$ , different age performances are achieved at the receiver node. In Section VII, we find the age-optimal  $k$  values for some given arbitrary PMFs numerically.

Under the highest  $k$  selective encoding policy, the receiver node does not receive any update when the remaining  $n - k$  realizations occur. However, there may be scenarios in which these remaining realizations are also of interest to the receiver node. In the next section, we focus on this scenario and consider a randomized selection of the remaining  $n - k$  realizations so that these realizations are not completely ignored.

## V. OPTIMAL CODEWORD DESIGN UNDER RANDOMIZED SELECTIVE ENCODING

The selective encoding scheme discussed so far is a deterministic scheme in which a fixed number of realizations are encoded into codewords and sent to the receiver node when realized. In this section, inspired by [48], we consider a randomized selective encoding scheme where the transmitter encodes the most probable  $k$  realizations with probability 1, and encodes the remaining least probable  $n - k$  realizations with probability  $\alpha$  and thus, neglects them with probability  $1 - \alpha$ . Thus, this randomized selective encoding policy strikes a balance between encoding every single realization and the highest  $k$  selective encoding scheme discussed so far.

Theorem 1 determines the average age experienced by the receiver node under the randomized highest  $k$  selective encoding scheme.

*Theorem 1:* Under the randomized highest  $k$  selective encoding scheme, the average age at the receiver node is given by

$$\Delta_\alpha = \frac{\mathbb{E}[L^2] + \frac{2}{q_{k,\alpha}\lambda}\mathbb{E}[L] + \frac{2}{(q_{k,\alpha}\lambda)^2}}{2\left(\mathbb{E}[L] + \frac{1}{q_{k,\alpha}\lambda}\right)} + \mathbb{E}[L], \quad (23)$$

where  $\mathbb{E}[L] = \sum_{i=1}^n P_{X_\alpha}(x_i)\ell(x_i)$ , and  $\mathbb{E}[L^2] = \sum_{i=1}^n P_{X_\alpha}(x_i)\ell(x_i)^2$ .

The proof of Theorem 1 follows similarly to that of Proposition 1 by replacing  $q_k$  with  $q_{k,\alpha}$ .

Next, we formulate the age minimization problem for this case as,

$$\begin{aligned} \min_{\{\ell(x_i), \alpha\}} & \frac{\mathbb{E}[L^2] + 2\bar{a}\mathbb{E}[L] + 2\bar{a}^2}{2(\mathbb{E}[L] + \bar{a})} + \mathbb{E}[L] \\ \text{s.t.} & \sum_{i=1}^n 2^{-\ell(x_i)} \leq 1 \\ & \ell(x_i) \in \mathbb{R}^+, \quad i \in \{1, \dots, n\}, \end{aligned} \quad (24)$$

where the objective function is equal to the average age  $\Delta_\alpha$  in Theorem 1 with  $\bar{a} = \frac{1}{\lambda q_{k,\alpha}}$ , the first and second constraints follow from the Kraft inequality and the feasibility of the codeword lengths, i.e., each codeword length should be non-negative.

We first solve this problem for a fixed  $\alpha$  in this section and determine the optimal  $\alpha$  numerically for given arbitrary PMFs in Section VII. Following a similar solution technique to that in Section IV, we find

$$\ell(x_i) = -\frac{\log\left(\frac{P_{X_\alpha}(x_i)}{(\beta(\log 2)^2)W}\left(\frac{\beta(\log 2)^2}{P_{X_\alpha}(x_i)}2^{\frac{-\theta+2\beta\log 2+2\bar{a}}{3}}\right)\right)}{\log 2}, \quad (25)$$

for  $i \in \{1, 2, \dots, n\}$ . To determine the age-optimal codeword lengths  $\ell(x_i)$  for  $i \in \{1, 2, \dots, n\}$ , we then employ the algorithm described in Section IV.

In the following section, we consider the case where instead of sending the remaining least probable  $n - k$  realizations randomly, the transmitter sends an empty symbol for these updates to further inform the receiver.

## VI. OPTIMAL CODEWORD DESIGN UNDER SELECTIVE ENCODING WITH AN EMPTY SYMBOL

In this section, we calculate the average age by considering two different scenarios for the empty symbol. Operationally, the receiver may not reset its age when  $x_e$  is received as it is not a regular update packet and the receiver does not know which realization occurred specifically. On the other hand, the receiver may choose to update its age as this empty symbol carries some information, the fact that the current realization is not one of the  $k$  encoded realizations, regarding the observed random variable. Thus, in this section, we consider both of these scenarios<sup>5</sup> and find the age-optimal codeword lengths for the set  $\mathcal{X}'_k$  with the PMF  $\{P_X(x_1), P_X(x_2), \dots, P_X(x_k), P_X(x_e)\}$  in each scenario.

### A. When the Empty Symbol Does Not Reset the Age

In this way of operation, the age at the receiver is not updated when the empty status update  $x_e$  is received. Thus, sending  $x_e$  incurs an additional burden since it does not reset the age but increases the average codeword length of the selected  $k$  realizations.

The update cycle is given by (6) with

$$W = (M - 1)\ell(x_e) + \sum_{\ell=1}^M Z_\ell, \quad (26)$$

where  $M$  is defined in Section IV and denotes the total number of update arrivals until the first update from the set  $\mathcal{X}_k$  is observed at the transmitter. In other words, there are  $M - 1$  deliveries of the empty status update  $x_e$  in between two successive deliveries from the encoded set  $\mathcal{X}_k$ . As discussed earlier,  $Z$  is an exponential random variable with rate  $\lambda$  and  $M$  is a geometric random variable with parameter  $q_k$ . By using the fact that the arrival and service processes are independent, i.e.,  $S$  and  $Z$  are independent, and  $M$  is independent of  $S$  and  $Z$ , in Theorem 2, we find the average age when an empty status update does not reset the age.

*Theorem 2:* When the empty status update  $x_e$  does not reset the age, the average age under the highest  $k$  selective encoding scheme with an empty symbol at the receiver is given by

$$\Delta_e = \frac{\mathbb{E}[L^2|X'_k \neq x_e] + 2\mathbb{E}[W]\mathbb{E}[L|X'_k \neq x_e] + \mathbb{E}[W^2]}{2(\mathbb{E}[L|X'_k \neq x_e] + \mathbb{E}[W]) + \mathbb{E}[L|X'_k \neq x_e]}. \quad (27)$$

*Proof:* We note that the service time of a successful update is equal to its codeword length so that we have

$$\mathbb{E}[S] = \mathbb{E}[L|X'_k \neq x_e] = \sum_{i=1}^k P_{X_k}(x_i)\ell(x_i) \quad (28)$$

$$\mathbb{E}[S^2] = \mathbb{E}[L^2|X'_k \neq x_e] = \sum_{i=1}^k P_{X_k}(x_i)\ell(x_i)^2 \quad (29)$$

<sup>5</sup>We note that another possible scenario may be to drop the age to an intermediate level between not updating at all and updating fully, considering the partial information conveyed by the empty status update. This case is not considered in this paper.

where  $P_{X_k}(x_i)$  is defined in (1). By using the independence of  $M$  and  $Z$ , we find

$$\mathbb{E}[W] = \ell(x_e) \left( \frac{1}{q_k} - 1 \right) + \frac{1}{\lambda q_k}, \quad (30)$$

$$\mathbb{E}[W^2] = \frac{(2-q_k)(1-q_k)}{q_k^2} \ell(x_e)^2 + \frac{4(1-q_k)}{\lambda q_k^2} \ell(x_e) + \frac{2}{(\lambda q_k)^2}, \quad (31)$$

where we used  $\mathbb{E}[M] = \frac{1}{q_k}$ ,  $\mathbb{E}[M^2] = \frac{2-q_k}{q_k^2}$ , and  $Z$  has exponential distribution with rate  $\lambda$ . Substituting (28)-(31) in (8) yields the result in (27). ■

We note that  $\Delta_e$  in (27) depends on  $\ell(x_e)$  only through the overall waiting time  $W$  as the age does not change when  $x_e$  is received. Next, we write the age minimization problem as

$$\begin{aligned} \min_{\{\ell(x_i), \ell(x_e)\}} & \frac{\mathbb{E}[L^2|X'_k \neq x_e] + 2\mathbb{E}[W]\mathbb{E}[L|X'_k \neq x_e] + \mathbb{E}[W^2]}{2(\mathbb{E}[L|X'_k \neq x_e] + \mathbb{E}[W])} \\ & + \mathbb{E}[L|X'_k \neq x_e] \\ \text{s.t.} & \quad 2^{-\ell(x_e)} + \sum_{i=1}^k 2^{-\ell(x_i)} \leq 1 \\ & \quad \ell(x_i) \in \mathbb{R}^+, \quad i \in \{1, \dots, k, e\}, \end{aligned} \quad (32)$$

where the objective function is equal to the average age expression  $\Delta_e$  in (27). We note that problem (32) is not convex due to the middle term in the objective function. However, when  $\ell(x_e)$  is fixed, it is a convex problem. Thus, we first solve the problem in (32) for a fixed  $\ell(x_e)$  and then determine the optimal  $\ell(x_e)$  numerically in Section VII.

Thus, for a fixed  $\ell(x_e)$ , (32) becomes

$$\begin{aligned} \min_{\{\ell(x_i)\}} & \frac{\mathbb{E}[L^2|X'_k \neq x_e] + 2\mathbb{E}[W]\mathbb{E}[L|X'_k \neq x_e] + \mathbb{E}[W^2]}{2(\mathbb{E}[L|X'_k \neq x_e] + \mathbb{E}[W])} \\ & + \mathbb{E}[L|X'_k \neq x_e] \\ \text{s.t.} & \quad \sum_{i=1}^k 2^{-\ell(x_i)} \leq 1 - 2^{-c} \\ & \quad \ell(x_i) \in \mathbb{R}^+, \quad i \in \{1, \dots, k\}, \end{aligned} \quad (33)$$

where  $\ell(x_e) = c$ . Since the empty status update length  $\ell(x_e)$  is fixed and given, we write the Kraft inequality by subtracting the portion allocated for  $\ell(x_e)$  in the optimization problem in (33). Similar to previous sections, we define  $p(\theta)$  as

$$\begin{aligned} p(\theta) := \min_{\{\ell(x_i)\}} & \frac{1}{2} \mathbb{E}[L^2|X'_k \neq x_e] + \mathbb{E}[L|X'_k \neq x_e]^2 \\ & + (2\hat{a} - \theta) \mathbb{E}[L|X'_k \neq x_e] + \frac{d}{2} - \theta \hat{a} \\ \text{s.t.} & \quad \sum_{i=1}^k 2^{-\ell(x_i)} \leq 1 - 2^{-c} \\ & \quad \ell(x_i) \in \mathbb{R}^+, \quad i \in \{1, \dots, k\}, \end{aligned} \quad (34)$$

where  $\hat{a} = \mathbb{E}[W]$  and  $d = \mathbb{E}[W^2]$ . For a fixed and given  $\ell(x_e)$ , the optimization problem in (34) is convex. We define

the Lagrangian function as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \mathbb{E}[L^2|X'_k \neq x_e] + \mathbb{E}[L|X'_k \neq x_e]^2 \\ & + (2\hat{a} - \theta) \mathbb{E}[L|X'_k \neq x_e] + \frac{d}{2} - \theta \hat{a} \\ & + \beta \left( \sum_{i=1}^k 2^{-\ell(x_i)} + 2^{-c} - 1 \right), \end{aligned} \quad (35)$$

where  $\beta \geq 0$ . The KKT conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \ell(x_i)} = & P_{X_k}(x_i) \ell(x_i) + 2\mathbb{E}[L|X'_k \neq x_e] P_{X_k}(x_i) \\ & + (2\hat{a} - \theta) P_{X_k}(x_i) - \beta (\log 2) 2^{-\ell(x_i)} = 0, \end{aligned} \quad (36)$$

for all  $i$ , and the complementary slackness condition is

$$\beta \left( \sum_{i=1}^k 2^{-\ell(x_i)} + 2^{-c} - 1 \right) = 0. \quad (37)$$

Lemma 2 shows that the optimal codeword lengths satisfy  $\sum_{i=1}^k 2^{-\ell(x_i)} = 1 - 2^{-c}$ .

**Lemma 2:** For the age-optimal real-valued codeword lengths, we must have  $\sum_{i=1}^k 2^{-\ell(x_i)} = 1 - 2^{-c}$ .

*Proof:* Assume that the optimal codeword lengths satisfy  $\sum_{i=1}^k 2^{-\ell(x_i)} < 1 - 2^{-c}$ , which implies that  $\beta = 0$  due to (37). From (36), we have

$$\begin{aligned} P_{X_k}(x_i) \ell(x_i) + 2 \left( \sum_{j=1}^k P_{X_k}(x_j) \ell(x_j) \right) P_{X_k}(x_i) \\ + (2\hat{a} - \theta) P_{X_k}(x_i) = 0, \quad \forall i. \end{aligned} \quad (38)$$

By summing (38) over all  $i$ , we get  $\mathbb{E}[L] = \frac{\theta - 2\hat{a}}{3}$ . Then, we find  $\ell(x_i) = \frac{\theta - 2\hat{a}}{3}$  for all  $i \in \{1, \dots, k\}$  which makes  $p(\theta) = -\frac{\theta^2 + 2\hat{a}\theta + 4\hat{a}^2 - 3d}{6}$ . By using  $p(\theta) = 0$ , we find  $\theta = -\hat{a} + \sqrt{3(d - \hat{a}^2)}$  which gives  $\ell(x_i) = -\hat{a} + \sqrt{\frac{d - \hat{a}^2}{3}}$  for  $i \in \{1, \dots, k\}$ . One can show that  $\frac{\partial \theta}{\partial c} > 0$ , i.e.,  $\theta$ , hence age, is an increasing function of  $c$ . Thus, in the optimal policy, in order to minimize the average age,  $c$  must be equal to zero. However, choosing  $c = 0$  leads to  $\sum_{i=1}^k 2^{-\ell(x_i)} < 1 - 2^{-c} = 0$ . Since the sum on the left cannot be negative, we reach a contradiction. Thus, the optimal codeword lengths must satisfy  $\sum_{i=1}^k 2^{-\ell(x_i)} = 1 - 2^{-c}$ . ■

Thus, for the age-optimal codeword lengths, we have  $\sum_{i=1}^k 2^{-\ell(x_i)} = 1 - 2^{-c}$  and  $\beta \geq 0$  from (37). By summing (36) over all  $i$  and using Lemma 2 we find

$$\mathbb{E}[L|X'_k \neq x_e] = \frac{\theta + \beta \log 2 (1 - 2^{-c}) - 2\hat{a}}{3}. \quad (39)$$

From (36), we obtain

$$-\ell(x_i) + \frac{\beta \log 2}{P_{X_k}(x_i)} 2^{-\ell(x_i)} = 2\mathbb{E}[L|X'_k \neq x_e] + 2\hat{a} - \theta. \quad (40)$$

Thus, we find the unique solution for  $\ell(x_i)$  as

$$\ell(x_i) = -\frac{\log \left( \frac{P_{X_k}(x_i)}{\beta (\log 2)^2} W \left( \frac{\beta (\log 2)^2}{P_{X_k}(x_i)} 2^{-\frac{\theta + 2\beta \log 2 (1 - 2^{-c}) + 2\hat{a}}{3}} \right) \right)}{\log 2}, \quad (41)$$



for  $i \in \{1, \dots, k\}$ . To determine the age-optimal codeword lengths  $\ell(x_i)$  for  $i \in \{1, \dots, k\}$ , we then employ the algorithm described in Section IV.

We note that the average age achieved at the receiver depends on  $\ell(x_e)$ . In Section VII, we provide numerical results where we vary  $\ell(x_e)$  over all possible values and choose the one that yields the least average age for given arbitrary PMFs.

### B. When the Empty Symbol Resets the Age

In this subsection, we consider the case where the empty symbol resets the age as it carries *partial* status information as in [28], [61]. In other words, each update which arrives when the transmitter idles is accepted as a successful update.

Theorem 3 determines the average age  $\Delta_e$  when the empty symbol resets the age.

*Theorem 3:* When the empty status update  $x_e$  resets the age, the average age under the highest  $k$  selective encoding scheme at the receiver is given by

$$\Delta_e = \frac{\mathbb{E}[L^2] + 2\frac{1}{\lambda}\mathbb{E}[L] + \frac{2}{\lambda^2}}{2(\mathbb{E}[L] + \frac{1}{\lambda})} + \mathbb{E}[L]. \quad (42)$$

*Proof:* Different from the previous sections, the moments for the waiting time are equal to  $\mathbb{E}[W] = \frac{1}{\lambda}$  and  $\mathbb{E}[W^2] = \frac{2}{\lambda^2}$  as each successful symbol is able to reset the age. Thus, substituting  $\mathbb{E}[W]$  and  $\mathbb{E}[W^2]$  in (8) and noting that  $\mathbb{E}[S] = \mathbb{E}[L]$  yields the result. ■

Next, we formulate the age minimization problem as

$$\begin{aligned} \min_{\{\ell(x_i), \ell(x_e)\}} & \frac{\mathbb{E}[L^2] + 2\tilde{a}\mathbb{E}[L] + 2\tilde{a}^2}{2(\mathbb{E}[L] + \tilde{a})} + \mathbb{E}[L] \\ \text{s.t.} & \quad 2^{-\ell(x_e)} + \sum_{i=1}^k 2^{-\ell(x_i)} \leq 1 \\ & \quad \ell(x_i) \in \mathbb{R}^+, \quad i \in \{1, \dots, k, e\}, \end{aligned} \quad (43)$$

where  $\tilde{a} = \frac{1}{\lambda}$ . We follow a similar solution technique to that given in Section IV to get

$$\ell(x_i) = -\frac{\log\left(\frac{P_X(x_i)}{\beta(\log 2)^2} W\left(\frac{\beta(\log 2)^2}{P_X(x_i)} 2^{\frac{-\theta + 2\beta \log 2 + 2\tilde{a}}{3}}\right)\right)}{\log 2}, \quad (44)$$

for  $i \in \{1, \dots, k, e\}$ .

The value of  $k$  affects  $\ell(x_e)$  such that when  $k$  is close to  $n$ , the probability of the empty symbol becomes small which leads to a longer  $\ell(x_e)$ , whereas when  $k$  is small, the probability of the empty symbol becomes large which results in a shorter  $\ell(x_e)$ . In Section VII, we numerically determine the optimal  $k$  selection which achieves the lowest average age for a given arbitrary distribution.

## VII. NUMERICAL RESULTS

In this section, we provide numerical results for the optimal encoding policies that are discussed in Sections IV, V, and VI. In the first two numerical results, we perform simulations to characterize optimal  $k$  values that minimize the average age with the highest  $k$  selective encoding scheme in Section IV.

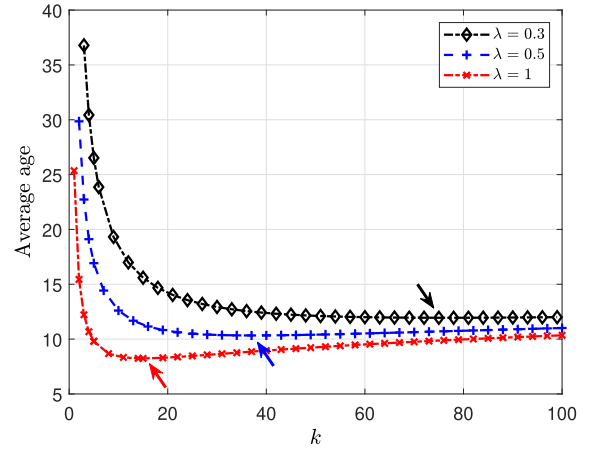


Fig. 4. The average age values with the age-optimal codeword lengths for  $\lambda \in \{0.3, 0.5, 1\}$  for the PMF provided in (45) with the parameters  $n = 100$ ,  $s = 0.4$ . We apply the highest  $k$  selective encoding scheme and vary  $k$  from 1 to  $n$  and indicate  $k$  that minimizes the average age for each  $\lambda$  with an arrow.

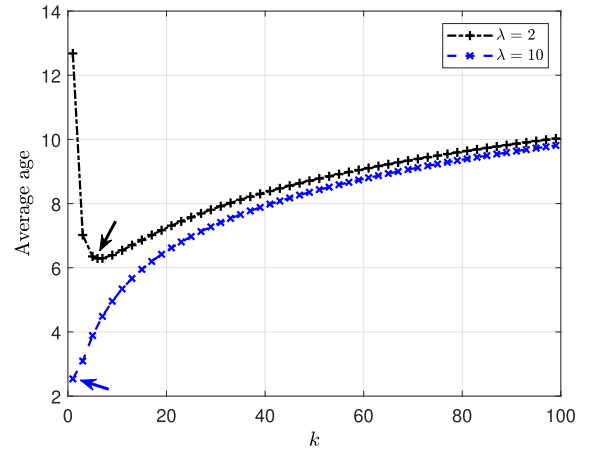


Fig. 5. The average age values with the age-optimal codeword lengths for  $\lambda \in \{2, 10\}$  for the PMF provided in (45) with the parameters  $n = 100$ ,  $s = 0.4$ . We apply the highest  $k$  selective encoding scheme and vary  $k$  from 1 to  $n$  and observe that choosing  $k = 1$  under the relatively high arrival rates ( $\lambda = 10$ ) minimizes the average age.

For these simulations, we use Zipf( $n, s$ ) distribution with the following PMF for  $n = 100$ ,  $s = 0.4$ ,

$$P_X(x_i) = \frac{i^{-s}}{\sum_{j=1}^n j^{-s}}, \quad 1 \leq i \leq n. \quad (45)$$

In Fig. 4, we show the effect of sending the most probable  $k$  realizations when the update packets arrive at the transmitter node rather infrequently, i.e., the arrival rate is low. We consider the cases in which the arrival rate is equal to  $\lambda = 0.3, 0.5, 1$ . For each arrival rate, we plot the average age as a function of  $k = 1, 2, \dots, n$ . We see that increasing the arrival rate reduces the average age as expected. In this case, optimal  $k$  is not equal to 1 since the effective arrival rate is small. In other words, the transmitter node wants to encode more updates as opposed to idly waiting for the next update arrival when the arrivals are rather infrequent. Choosing  $k$  close to  $n$  is also not optimal as the service times of the status updates with low probabilities are longer which hurts the overall age



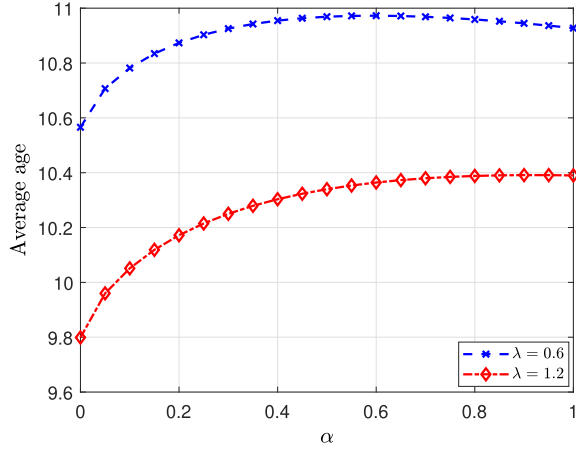


Fig. 6. The average age values with the age-optimal codeword lengths for different  $\alpha$  values with the PMF provided in (45) with  $n = 100$ ,  $s = 0.2$  for  $k = 70$  and  $\lambda = 0.6, 1.2$  when randomized highest  $k$  selective encoding is implemented.

performance. Indeed, in Fig. 4, where update arrival rates are relatively small, it is optimal to choose  $k = 76$  for  $\lambda = 0.3$ ,  $k = 37$  for  $\lambda = 0.5$ , and  $k = 15$  for  $\lambda = 1$ .

In Fig. 5, we consider a similar setting as in Fig. 4 but here update arrival rates are larger which means that updates arrive more frequently at the transmitter node. We observe that when  $\lambda = 2$ , the optimal  $k$  is still not equal to 1 (it is equal to 6 in Fig. 5) as the updates are not frequent enough. However, once updates become more available to the transmitter node, i.e., the case with  $\lambda = 10$  in Fig. 5, we observe that the transmitter node chooses to only encode the realization with the highest probability, i.e.,  $k = 1$ , and wait for the next update arrival instead of encoding more and more realizations which increases the overall codeword lengths thereby increasing the transmission times. We also observe that the average age decreases as the update arrival rate increases as in Fig. 4.

We note that when the arrival rate is high as in Fig. 5 when  $\lambda = 10$ , we observe that the age is an increasing function of  $k$  since under this arrival profile codeword lengths dominate the performance which in turn increase as  $k$  increases. On the other hand, when the arrival rate is low as in  $\lambda = 2$  in Fig. 5 and  $\lambda = 0.3, 0.5, 1$  in Fig. 4, we observe that initially the age is a decreasing function of  $k$  as the waiting time in between two successive encoded updates dominates the performance. However, when we continue to increase  $k$ , we observe that both of these opposing trends are in play and the age starts to increase with  $k$ .

For the third numerical result shown in Fig. 6, we simulate the randomized highest  $k$  selective encoding policy described in Section V with Zipf distribution in (45) with parameters  $n = 100$ ,  $s = 0.2$ . In Fig. 6, we observe two different trends depending on the update arrival frequency at the source node, even though in either case, randomization results in a higher age at the receiver node than selective encoding, i.e.,  $\alpha = 0$  case. When the arrival rate is high,  $\lambda = 1.2$  in Fig. 6, we observe that age monotonically increases with  $\alpha$  as randomization increases average codeword length, i.e., service times. Although increasing  $\alpha$  results in a higher age at the receiver node, previously discarded  $n - k$  realizations can

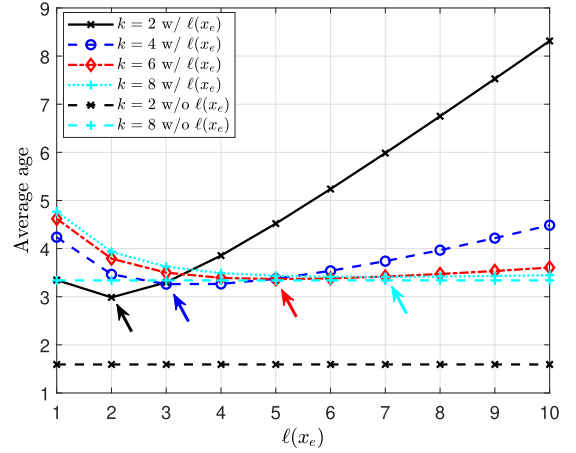


Fig. 7. Average age with the age-optimal codeword lengths with respect to  $\ell(x_e)$  with the PMF in (46) for  $n = 10$  and  $\lambda = 5$  when the empty symbol does not reset the age. Arrows indicate the age-optimal  $\ell(x_e)$  values. We also provide the optimal age without sending the empty symbol for  $k = 2$  and  $k = 8$ .

be received under this randomized selective encoding policy. Interestingly, when the arrival rate is smaller,  $\lambda = 0.6$  in Fig. 6, we observe that age initially increases with  $\alpha$  and then starts to decrease because of the decreasing waiting times as opposed to increasing codeword lengths such that when  $\alpha$  is larger than 0.3, it is better to select  $\alpha = 1$ , i.e., encoding every realization. That is, when  $\alpha$  grows beyond 0.3, encoding and sending every single realization yields a lower average age.

In the fourth and fifth numerical results, we find the optimal real-valued codeword lengths and  $k$  values that minimize the average age  $\Delta_e$  with the highest  $k$  selective encoding scheme with an empty symbol, as discussed in Section VI. For these numerical results, we use the following PMF

$$P_X(x_i) = \begin{cases} 2^{-i}, & i = 1, \dots, n-1 \\ 2^{-n+1}, & i = n. \end{cases} \quad (46)$$

In the fourth numerical result, we consider the PMF in (46) for  $n = 10$  and take  $\lambda = 5$ . We find the optimal codeword length of the empty symbol,  $\ell(x_e)$ , when the empty symbol does not reset the age (see Fig. 7). We observe that when  $k$  is small, the probability of sending the empty symbol becomes large so that a shorter codeword is preferable for  $x_e$ . For example, we observe in Fig. 7 that choosing  $\ell(x_e) = 2$  when  $k = 2$  and  $\ell(x_e) = 3$  when  $k = 4$  is optimal. Similarly, when  $k$  is larger, a longer codeword is desirable for  $x_e$ . We observe in Fig. 7 that choosing  $\ell(x_e) = 5$  when  $k = 6$  and  $\ell(x_e) = 7$  when  $k = 8$  is optimal. Further, we note in Fig. 7 that the average age increases when we send the empty symbol in the case of the remaining  $n - k$  realizations as the empty symbol increases the total waiting time for the next successful arrival as well as the codeword lengths for the encoded  $k$  realizations. For smaller  $k$  values, i.e., when  $k = 2$ , this effect is significant as the empty symbol has a large probability whereas when  $k$  is larger, i.e., when  $k = 8$ , sending an empty status update increases the age slightly (especially when  $\ell(x_e)$  is high) as the empty symbol has a small probability.

In the fifth numerical result shown in Fig. 8, we consider the case when the empty symbol  $x_e$  resets the age. We observe

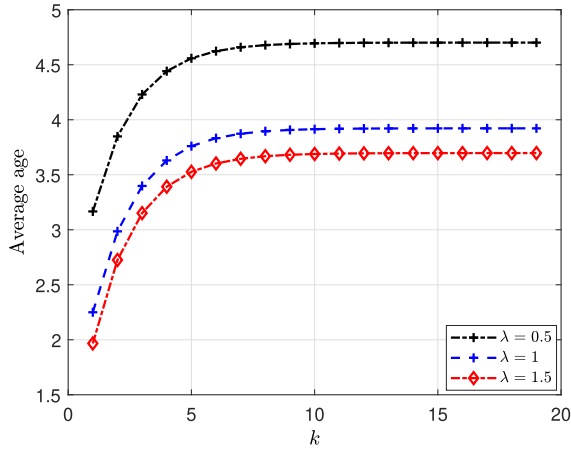


Fig. 8. Average age with the age-optimal codeword lengths for varying  $k$  with the PMF in (46) for  $n = 20$  and  $\lambda = 0.5, 1, 1.5$  when the empty symbol resets the age.

that the minimum age is achieved when  $k = 1$ , i.e., only the most probable realization is encoded. This is because the overall waiting time is independent of  $k$  and larger  $k$  values result in larger codewords which in turn increases transmission times. Thus, in this case, only the most probable realization is received separately since all others are embedded into the empty symbol. We note that this selection results in significant information loss at the receiver which is not captured by the age metric alone. This problem can be addressed by introducing a distortion constraint which measures the information loss together with the age metric which measures freshness [61].

In the sixth numerical result shown in Fig. 9, we compare the performance of the age-optimal code that we developed in Section IV with well-known codes that minimize average codeword length. For this purpose, we choose Huffman code<sup>6</sup> which takes integer-valued codeword lengths and Shannon\* code<sup>7</sup> which takes real-valued codeword lengths. We use the PMF in (45) with  $n = 10$  and  $s = 0, 3, 4$  for  $\lambda = 1$ .

We note that when  $s = 0$ , the distribution in (45) becomes a uniform distribution. We see in Fig. 9(a) that for the uniform distribution, the age-optimal real-valued codeword lengths are equivalent to Shannon\* code. This result has been observed in [48] as well. When  $k$  is equal to a power of 2 such as  $k = 2, 4, 8$  in Fig. 9(a), Huffman code becomes the same as Shannon\* code as the codeword lengths of Shannon\* code, i.e.,  $-\log_2(P_{X_k}(x_i))$ , take integer values. For the remaining  $k$  values Huffman code performs worse than Shannon\* code and the age-optimal code. When  $s = 3$ , we see in Fig. 9(b) that the age-optimal code achieves a smaller age than Huffman and Shannon\* codes. When  $k < 7$ , we see in Fig. 9(b) that Shannon\* code achieves a lower age than Huffman code whereas when  $k \geq 7$ , Huffman code achieves a lower age

than Shannon code. When  $s = 4$ , we see in Fig. 9(c) that the age-optimal code achieves the lowest age whereas Huffman code performs the worst.

Thus, we observe that when the distribution is close to a uniform distribution, i.e., when  $s$  is small, Huffman and Shannon\* codes perform similar to the age-optimal code (when the distribution is equal to uniform distribution, we see that Shannon\* code is equivalent to the age-optimal code). However, when the distribution is more polarized, i.e., when  $s$  is high, we see that the age-optimal code performs significantly better than Shannon\* and Huffman codes.

## VIII. ON THE OPTIMALITY OF THE HIGHEST $k$ SELECTIVE ENCODING

So far, we have considered only the case where the most probable  $k$  realizations are encoded and sent through the channel. Based on this selection, we found the average age and determined the age-optimal  $k$  and codeword lengths. We observed that this highest  $k$  selective encoding policy results in a lower average age than encoding every realization. However, we note that there are  $\binom{n}{k}$  selections for encoding and in this section, we discuss the optimality of the highest  $k$  selective encoding among all these different selections. We see that the average age expression in Proposition 1 depends on the PMF of  $X$  which affects the optimal codeword lengths, and the effective arrival rate. In this section, we denote the effective arrival rate as  $\lambda_e$  given by  $\lambda_e = \lambda \sum_{x \in \mathcal{X}_s} P_X(x)$  where  $\mathcal{X}_s$  is the set of arbitrarily selected  $k$  updates for encoding. Here, by choosing a different set of  $k$  realizations to encode and send, instead of the most probable  $k$  realizations, we change the effective arrival rate and codeword lengths which in turn yields a different age performance.

When the arrival rate is relatively low, we see in Fig. 4 that the average age is dominated mainly by the effective arrival rate. Thus, choosing the realizations with the highest probabilities may be desirable as this selection achieves the highest possible effective arrival rate. However, when the arrival rate is relatively high, the average age is mainly determined by the moments of the codeword lengths.

In Table I, we find the age-optimal update selections for given PMFs and arrival rates for  $k = 5$ . We use the in (46) with  $n = 10$  and Zipf distribution in (45) with parameters  $n = 10$ ,  $s = 0.2$ . In both PMFs, the updates are in decreasing order with respect to their probabilities, i.e.,  $P_X(x_i) \geq P_X(x_j)$  if  $i \leq j$ . When the arrival rate is relatively small, i.e.,  $\lambda = 0.1$  for the first PMF and  $\lambda = 0.5$  for the second PMF, we observe that choosing the realizations with the highest probabilities for encoding is optimal as this selection increases the effective arrival rate the most which is the dominating factor for the age performance when the arrivals are infrequent at the source node. That is, the optimal selection is  $\{1, 2, 3, 4, 5\}$  when  $\lambda = 0.1$  for the first PMF and when  $\lambda = 0.5$  for the second PMF. However, when the arrival rate is high, the optimal policy is to encode the realization with the highest probability and  $k - 1$  realizations with the lowest probabilities such that the optimal set is  $\{1, 7, 8, 9, 10\}$  as this selection helps to keep the moments of codewords lengths at appropriate levels which

<sup>6</sup>We acknowledge the feedback from one of the anonymous Reviewers who suggested that we compare our age-optimal code with Huffman codes.

<sup>7</sup>We note that codeword lengths of a Shannon code are equal to  $\lceil -\log_2(P_{X_k}(x_i)) \rceil$  which take integer values [56]. However, in this paper, we neglect the ceiling operator and consider real-valued codeword lengths as  $-\log_2(P_{X_k}(x_i))$  which we denote as a Shannon\* code.

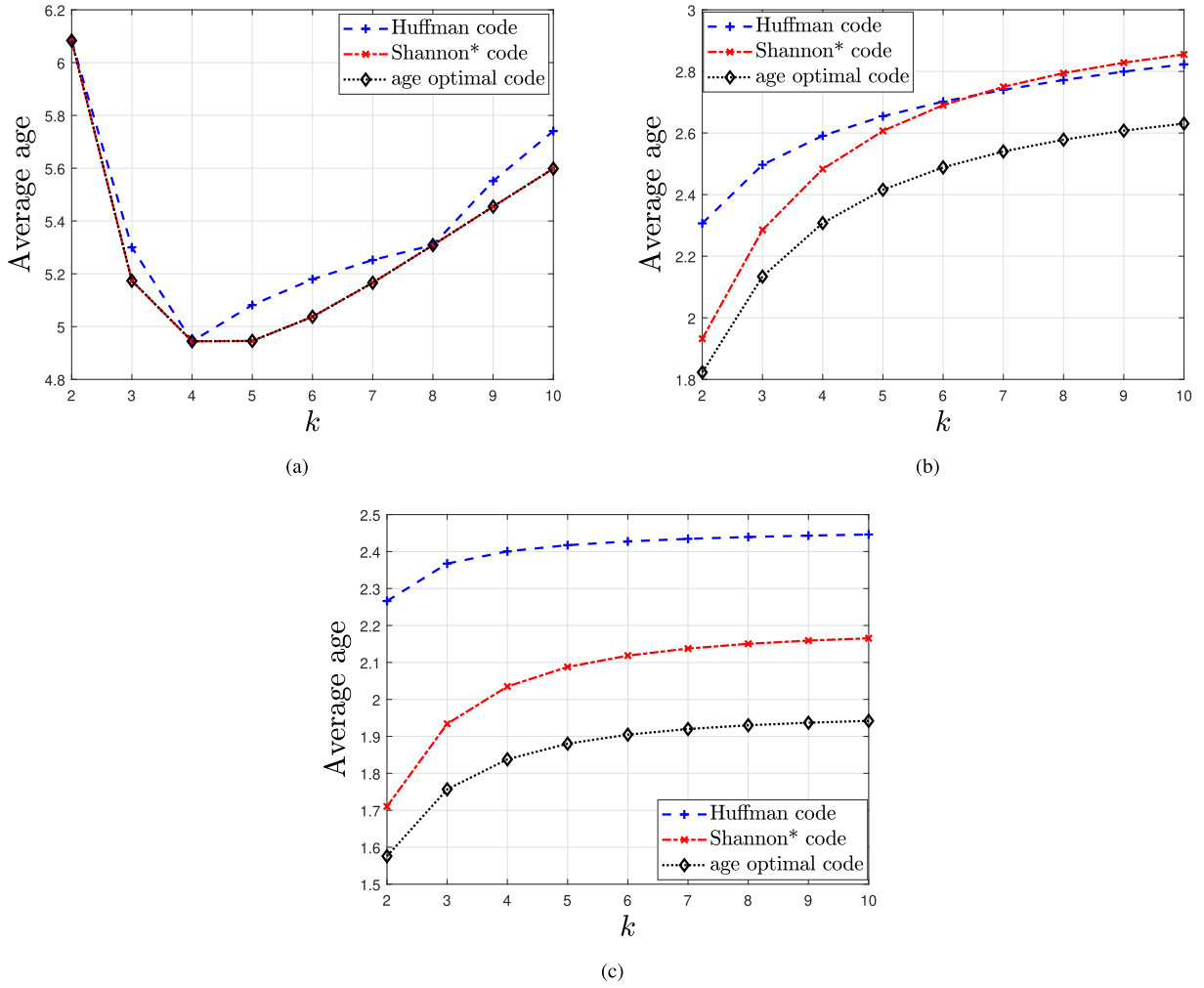


Fig. 9. The average age under Huffman code, Shannon\* code and the age-optimal code for  $\lambda = 1$  and the PMF in (45) with the parameters  $n = 10$ , (a)  $s = 0$ , (b)  $s = 3$  and (c)  $s = 4$ . We vary  $k$  from 2 to  $n$ .

TABLE I  
AGE-OPTIMAL UPDATE SELECTION FOR FIXED  $k = 5$  WITH DIFFERENT ARRIVAL RATES,  $\lambda$

PMF	$\lambda$	optimal selection	$\lambda_e$	optimal age
The PMF in (46) for $n = 10$	0.1	$\{1, 2, 3, 4, 5\}$	0.0969	12.292
	0.5	$\{1, 2, 8, 9, 10\}$	0.3789	3.867
	1	$\{1, 7, 8, 9, 10\}$	0.5156	2.4229
Zipf( $n = 10, s = 0.2$ )	0.5	$\{1, 2, 3, 4, 5\}$	0.3898	5.154
	1	$\{1, 2, 8, 9, 10\}$	0.6269	3.929
	2	$\{1, 7, 8, 9, 10\}$	1.01	3.304

are the dominating factors for the age performance when the arrivals are frequent at the source node. We see that this selection is optimal when  $\lambda = 1$  for the first PMF and when  $\lambda = 2$  for the second PMF. From these, we observe that the optimal update selection strategy is to keep the effective arrival rate as high as possible while maintaining the moments of the codeword lengths at the desired levels. We see this structure when  $\lambda = 0.5$  for the first PMF and  $\lambda = 1$  for the second PMF where the optimal selection is to choose the most probable two and the least probable three realizations, i.e., the optimal selection is  $\{1, 2, 8, 9, 10\}$ .

Thus, even though the highest  $k$  selective encoding policy improves the age performance as shown in Section VII, this selection may not necessarily be optimal for a given PMF and arrival rate among all other possible selections. In fact, in Table I we observe that, the highest  $k$  selection is optimal when the arrival rate is low. When the arrival rate is high, however, a different  $k$  selection should be implemented to get a better age performance as shown in Table I. The theoretical analysis for the optimality of the highest  $k$  selective encoding remains as a future work. Further, in some cases the realizations with lower probabilities may carry important information

that cannot be ignored. In these scenarios, an importance metric can be assigned to each realization and the encoded  $k$  realizations can be selected considering both the importance metric and the realization probabilities. We leave this problem for future work.

## IX. CONCLUSION AND DISCUSSION

We considered a status updating system in which an information source generates independent and identically distributed update packets based on an observed random variable  $X$  which takes  $n$  values based on a known PMF. We studied three different encoding schemes for the transmitter node to send the realizations to the receiver node. In all these schemes, the most probable  $k$  update realizations are always encoded. For the remaining less probable  $n - k$  realizations, we considered the case in which these realizations are completely discarded, i.e., the highest  $k$  selective encoding scheme. Next, we considered the case in which the remaining previously discarded  $n - k$  realizations are encoded into codewords randomly to further inform the receiver, i.e., randomized selective encoding scheme. Lastly, we examined the case where the remaining less probable realizations are mapped into an empty symbol to partially inform the receiver node, i.e., highest  $k$  selective encoding scheme with an empty symbol. We derived the average age for all these encoding schemes and determined the age-optimal codeword lengths. Through numerical results we showed that the proposed selective encoding scheme achieves a lower average age than encoding all the realizations, and determined the age-optimal  $k$  values for arbitrary PMFs. We investigated the optimality of the highest  $k$  selective encoding and showed through simulations that it is optimal when the arrival rate is low. We remark that, for a fixed  $k$  and given update arrival profile  $\lambda$ , selecting  $k$  out of  $n$  realizations for which the transmitter should send an update, and designing age-optimal integer codeword lengths for selected realizations, remain as open problems. Further, in our model we assume that the status updates arrive at the source node exogenously as a Poisson process. For future work, one can consider age-optimal sampling at the source instead of exogenously arriving updates at the source without any control. We remark that one can further study age along with a distortion metric that captures the information loss to design systems that assign importance to freshly generated updates without neglecting the distortion caused by skipped transmission opportunities under selective encoding mechanisms.

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**Melih Bastopcu** (Graduate Student Member, IEEE) received the B.S. degree in electrical and electronics engineering from Bilkent University, Turkey, in 2016. He is currently pursuing the Ph.D. degree with the Department of Electrical and Computer Engineering, University of Maryland, College Park. His research interests include age of information, resource allocation, low-latency systems, and communication network design.



**Baturalp Buyukates** (Graduate Student Member, IEEE) received the B.S. degree in electrical and electronics engineering from Bilkent University, Turkey, in 2016. He is currently pursuing the Ph.D. degree with the Department of Electrical and Computer Engineering, University of Maryland, College Park, MD, USA. His research interests include age of information, distributed computation, learning, and optimization in large-scale wireless communication systems.



**Sennur Ulukus** (Fellow, IEEE) received the B.S. and M.S. degrees in electrical and electronics engineering from Bilkent University and the Ph.D. degree in electrical and computer engineering from the Wireless Information Network Laboratory (WINLAB), Rutgers University.

She is the Anthony Ephremides Professor in Information Sciences and Systems with the Department of Electrical and Computer Engineering, University of Maryland (UMD), College Park, where she also holds a joint appointment with the Institute for Systems Research (ISR). Prior to joining UMD, she was a Senior Technical Staff Member at AT&T Labs-Research. She is a Distinguished Scholar-Teacher with the University of Maryland. Her research interests are in information theory, wireless communications, machine learning, signal processing and networks, with recent focus on private information retrieval, age of information, distributed coded computation, energy harvesting communications, physical layer security, and wireless energy and information transfer. She received the 2003 IEEE Marconi Prize Paper Award in Wireless Communications, the 2005 NSF CAREER Award, the 2011 ISR Outstanding Systems Engineering Faculty Award, the 2012 ECE George Corcoran Outstanding Teaching Award, and the 2019 IEEE Communications Society Best Tutorial Paper Award. She is the TPC Chair/Co-Chair of 2011 IEEE CTW, 2014 IEEE PIMRC, 2016 IEEE GLOBECOM, 2017 IEEE ISIT, 2019 IEEE ITW, and 2021 IEEE GLOBECOM. She was an Editor of IEEE TRANSACTIONS ON COMMUNICATIONS from 2003 to 2007, an Associate Editor of IEEE TRANSACTIONS ON INFORMATION THEORY from 2007 to 2010, and a Guest Editor of the IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS in 2008 and 2015, IEEE TRANSACTIONS ON INFORMATION THEORY in 2011, and JOURNAL OF COMMUNICATIONS AND NETWORKS in 2012. She was an Editor of IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS—SERIES ON GREEN COMMUNICATIONS AND NETWORKING from 2015 to 2016. She was an Area Editor of IEEE TRANSACTIONS ON GREEN COMMUNICATIONS AND NETWORKING from 2016 to 2020. She has been an Area Editor of IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS, since 2019, and a Senior Editor of IEEE TRANSACTIONS ON GREEN COMMUNICATIONS AND NETWORKING, since 2020. She was a Distinguished Lecturer of the IEEE Information Theory Society from 2018 to 2019.