

PLATEAU PROBLEMS FOR MAXIMAL SURFACES IN PSEUDO-HYPERBOLIC SPACE

PROBLÈMES DE PLATEAU POUR LES SURFACES MAXIMALES DANS L'ESPACE PSEUDO-HYPERBOLIQUE

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ABSTRACT. We define and prove the existence of unique solutions of an asymptotic Plateau problem for spacelike maximal surfaces in the pseudo-hyperbolic space of signature $(2, n)$: the boundary data is given by loops on the boundary at infinity of the pseudo-hyperbolic space which are limits of positive curves. We also discuss a compact Plateau problem. The required compactness arguments rely on an analysis of the pseudo-holomorphic curves defined by the Gauß lifts of the maximal surfaces.

ABSTRACT. Nous définissons un problème de Plateau asymptotique pour les surfaces maximales de type espace dans l'espace pseudo-hyperbolique de signature $(2, n)$ dont le bord à l'infini est donné par des courbes, dites *semi-positives*, et qui sont limites de courbes positives. Nous montrons l'existence et l'unicité des solutions correspondantes et discutons le problème de Plateau compact correspondant. Les arguments de compacité utilisés requièrent l'analyse de courbes pseudo-holomorphes définies par le relevé de Gauß de surfaces maximales.

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1. INTRODUCTION

Our goal is to study Plateau problems in the pseudo-hyperbolic space $\mathbf{H}^{2,n}$, which can be quickly described as the space of negative definite lines in a vector space of signature $(2, n+1)$. As such $\mathbf{H}^{2,n}$ is a complete homogeneous pseudo-Riemannian manifold of signature $(2, n)$ and curvature -1 .

Quite naturally, $\mathbf{H}^{2,n}$ bears many resemblances to the hyperbolic plane, which corresponds to the case $n = 0$. In particular, generalising the Klein model, $\mathbf{H}^{2,n}$ may be described as one of the connected components of the complement to a quadric in the projective space of dimension $n+2$.

This quadric is classically called the *Einstein universe* and we shall denote it by $\partial_\infty \mathbf{H}^{2,n}$ [5]. Analogously to the hyperbolic case, the space $\partial_\infty \mathbf{H}^{2,n}$ carries a conformal metric of signature $(1, n)$ and we will consider it as a boundary at infinity of $\mathbf{H}^{2,n}$. Topologically, $\partial_\infty \mathbf{H}^{2,n}$ is the quotient of $\mathbf{S}^1 \times \mathbf{S}^n$ by an involution.

From the Lie group perspective, the space $\mathbf{H}^{2,n}$ has $\mathrm{PSO}(2, n+1)$ as a group of isometries and the Einstein space $\partial_\infty \mathbf{H}^{2,n}$ is the *Shilov boundary* of this rank two Hermitian group, that is the unique closed $\mathrm{PSO}(2, n+1)$ -orbit in the boundary of the symmetric domain.

Positive triples and positivity in the Shilov boundary [20] play an important role in the theory of Hermitian symmetric spaces; of notable importance are the *positive loops*. Important examples of these are spacelike curves homotopic to $\mathbf{S}^1$ and specifically the *positive circles* which are boundaries at infinity in our compactification to totally geodesic embeddings of hyperbolic planes. Then *semi-positive loops* are limits of positive loops in some natural sense (see paragraph 2.5.2 for precise definitions).

Surfaces in a pseudo-Riemannian space may have induced metrics of variable signatures. We are interested in this paper in *spacelike surfaces* in which the induced metric is positive everywhere. Among these are the *maximal surfaces* which are critical points of the area functional, for variations with compact support, see paragraph 3.3.3 for details. These maximal surfaces are the analogues of minimal surfaces in the Riemannian setting. An important case of those maximal surfaces in $\mathbf{H}^{2,n}$ are, again, the totally geodesic surfaces which are isometric to hyperbolic planes.

We refer to the first two sections of this paper for precise definitions of what we have above described only roughly.

Our main Theorem is the following.

Theorem A (ASYMPTOTIC PLATEAU PROBLEM). *Any semi-positive loop in $\partial_\infty \mathbf{H}^{2,n}$ bounds a unique complete maximal surface in $\mathbf{H}^{2,n}$.*

In this paper, a semi-positive loop is not necessarily smooth. Also note that a properly embedded surface might not be complete and so the completeness condition is not vacuous.

On the other hand, we will show in section 3 that complete spacelike surfaces limit on semi-positive loops in $\partial_\infty \mathbf{H}^{2,n}$, and so Theorem A may be understood as identifying semi-positivity as the condition on curves in $\partial_\infty \mathbf{H}^{2,n}$ that corresponds to complete maximality for surfaces in $\mathbf{H}^{2,n}$.

The uniqueness part of the theorem is strikingly different from the corresponding setting in hyperbolic space where the uniqueness of solutions of the asymptotic Plateau problem fails in general for some quasi-symmetric curves as shown by Anderson, Wang and Huang [3, 46, 29].

As a tool in the theorem above, we also prove the following result, of independent interest, on the Plateau problem with boundary in $\mathbf{H}^{2,n}$. The relevant notion for curves is that of

strongly positive curves, and among those the connected set of *deformable* ones which are defined in paragraph 3.2.

Theorem B (PLATEAU PROBLEM). *Any deformable strongly positive closed curve in $\mathbf{H}^{2,n}$ bounds a unique compact complete maximal surface in $\mathbf{H}^{2,n}$.*

One of the original motivations for this paper comes from the “equivariant situation”. Recall that $G := \mathrm{PSO}(2, n+1)$ is the isometry group of a Hermitian symmetric space M : the maximal compact subgroup of G has an $\mathrm{SO}(2)$ factor which is associated to a line bundle L over M . Thus a representation ρ of the fundamental group of a closed orientable surface S in G carries a *Toledo invariant*: the Chern class of the pull back of L by any map equivariant under ρ from the universal cover of S to M [45]. The *maximal representations* are those for which the integral of the Toledo invariant achieves its maximal value. These maximal representations have been extensively studied, from the point of view of Higgs bundles, by Bradlow, García-Prada and Gothen [14] and from the perspective of bounded cohomology, by Burger, Iozzi and Wienhard [16]. In particular, a representation is maximal if and only if it preserves a positive continuous curve [15, 16]. Then Collier, Tholozan and Toulisse have shown that there exists a unique equivariant maximal surface with respect to a maximal representation in $\mathrm{PSO}(2, n+1)$ [21]. This last result, an inspiration for our work, is now a consequence of Theorem A.

We note that maximal surfaces in $\mathbf{H}^{2,n}$ were also considered in a work by Ishihara [31] – see also Mealy [37] – and that Yang Li has obtained results for the finite Plateau problems in the Lorentzian case [38], while the codimension one Lorentzian case was studied by Bartnik and Simon in [7]. Yang Li’s paper contains many references pertinent to the flat case. Neither paper restricts to two spacelike dimensions.

Another motivation comes from the contemplation of two other rank two groups: $\mathrm{SL}(3, \mathbb{R})$ and $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$, where we notice the latter group is isogenic to $\mathrm{PSO}(2, 2)$.

Affine spheres and $\mathrm{SL}(3, \mathbb{R})$: While maximal surfaces are the natural conformal variational problem for $\mathrm{SO}(2, n)$, the analogous problem in the setting of $\mathrm{SL}(3, \mathbb{R})$ is that of affine spheres. Cheng and Yau [18], confirming a conjecture due to Calabi, proved that given any properly convex curve in the real projective space, there exists a unique affine sphere in $\mathbb{R}^3$ asymptotic to it. That result has consequences for the equivariant situation as well, due independently to Loftin and Labourie [39, 34]. Our main Theorem A may be regarded as an analogue of the Cheng–Yau Theorem: both affine spheres and maximal surfaces (for $\mathrm{SO}(2, 3)$) are lifted as holomorphic curves – known as cyclic surfaces in [35] – in G/K_1 , where G is $\mathrm{SL}(3, \mathbb{R})$ in the first case and $\mathrm{SO}(2, 3)$ in the second, and K_1 is a compact torus. Moreover these holomorphic curves finally project as minimal surfaces in the symmetric space of G .

The case $n = 1$: The case of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ and maximal surfaces in $\mathbf{H}^{2,1}$ has been extensively studied by Bonsante and Schlenker [11] and written only in the specific case of quasi-symmetric boundaries values – see also Tamburelli [42, 43] for further extensions. Our main Theorem A is thus a generalization in higher dimension of one of their main results. Also in the case of $\mathbf{H}^{2,1}$, we note that Bonsante and Seppi [12] have shown the existence, for any $K < -1$, of a unique K -surface extending a semi-positive loop in $\partial_\infty \mathbf{H}^{2,1}$.

Lorentzian asymptotic Plateau problem: There is also an interesting analogy with the work of Bonsante, Seppi and Smillie [13] in which they prove that, for every $H > 0$, any regular domain D in the $(n+1)$ -dimensional Minkowski space contains a unique entire spacelike surface of constant mean curvature H whose domain of dependence is D . Their work corresponds to the non-semisimple Lie group $\mathrm{SO}(n, 1) \ltimes \mathbb{R}^{n,1}$. The similarities between

the asymptotic behavior of their regular domains and our notion of semi-positive loops in $\partial_\infty \mathbf{H}^{2,n}$ are striking, in that both only require a non-degeneracy over 2 or 3 points.

In a subsequent paper [36], the first two authors study the analogue of the Benoist–Hulin result [8] for convex geometry and study quasisymmetric positive curves and the relation with the associated maximal surface. In contrast, Tamburelli and Wolf study the case of “polygonal curves” in the $\mathbf{H}^{2,2}$ case, whose group of isometries is $\mathrm{SO}(2, 3)$ which is isogenic to $\mathrm{Sp}(4, \mathbb{R})$ [44]; there they prove results analogous to Dumas–Wolf [23]. One goal in that work is to identify local limiting behavior of degenerating cocompact families of representations.

The proof of Theorem A follows a natural outline. We prove the uniqueness portion by relying on a version of the Omori maximum principle; the bulk of the proof is on the existence question. To that end, we approximate a semi-positive loop on $\partial_\infty \mathbf{H}^{2,n}$ by semi-positive graphs in $\mathbf{H}^{2,n}$; as maximal surfaces in our setting are stable, we solve the Plateau problem for these with a continuity method, proving compactness theorems relevant to that situation. We then need to show that these finite approximations converge, limiting on a maximal surface with the required boundary values. Thus, much of our argument comes down to obtaining compactness theorems with control on the boundary values. Some careful analysis of this setting allows us to restrict the scope of our study to disks and semi-disks. Then, the main new idea here is to use the *Gauß lift* of the surfaces, to an appropriate Grassmannian, which are shown to be pseudo-holomorphic curves. We can then use Schwarz lemmas to obtain

- (i) first a compactness theorem under a bound on the second fundamental form,
- (ii) then after a rescaling argument using a Bernstein-type theorem in the rescaled limit $\mathbf{E}^{2,n}$ of $\mathbf{H}^{2,n}$, a uniform bound on the second fundamental form.

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1.1. Structure of this article.

- (i) In section 2, we describe the geometry of the pseudo-hyperbolic space $\mathbf{H}^{2,n}$, and its boundary at infinity, the Einstein universe $\partial_\infty \mathbf{H}^{2,n}$. There we carefully define positive and semi-positive curves in $\partial_\infty \mathbf{H}^{2,n}$.
- (ii) In section 3, we discuss curves and surfaces in $\mathbf{H}^{2,n}$. In particular we introduce maximal surfaces and show that they may be interpreted as holomorphic curves. We also discuss spacelike curves and various notions related to them.
- (iii) In section 4, we prove the uniqueness part of our two main theorems.
- (iv) In section 5, we prove, using the holomorphic curve interpretation, a crucial compactness theorem for maximal surfaces. We feel this is of some independent interest.
- (v) In section 6, we describe different consequences of our main compactness Theorem, whose formulations we will use in the proof of Theorem A.
- (vi) In section 7, we prove the Plateau Theorem B by the continuity method, relying on the both the stability of the maximal surface and a compactness consequence from section 6.

- (vii) In section 8 we prove the Asymptotic Plateau Theorem A using the Plateau Theorem B, an exhaustion procedure, and the results in section 6.
- (viii) In the Appendices A and C, we describe the notion of *bounded geometry* and prove the relevant results needed for the holomorphic curve interpretation. We expect that last appendix has some independent interest.

2. PSEUDO-HYPERBOLIC GEOMETRY

In this section, we describe the basic geometry of the pseudo-hyperbolic space and its boundary, the Einstein universe. Part of the material covered here can be found in [5, 21, 22]. This section consists mainly of definitions.

2.1. The pseudo-hyperbolic space. In this paper, we will denote by E a vector space equipped with a non-degenerate quadratic form $\mathbf{q}$ of signature $(2, n + 1)$. The group $\mathbf{O}(E)$ of linear transformations of E preserving $\mathbf{q}$ has four connected components, and we will denote by $\mathbf{G} := \mathbf{SO}_0(E)$ the connected component of the identity. The group $\mathbf{G}$ is isomorphic to $\mathbf{SO}_0(2, n + 1)$.

Definition 2.1. The *pseudo-hyperbolic space* $\mathbf{H}^{2,n}$ is the space of negative definite lines in E , namely

$$\mathbf{H}^{2,n} := \mathbf{P}(\{x \in E \mid \mathbf{q}(x) < 0\}) \subset \mathbf{P}(E) \quad .$$

The pseudo-hyperbolic space $\mathbf{H}^{2,n}$ is naturally equipped with a signature $(2, n)$ pseudo-Riemannian metric $\mathbf{g}$ of curvature -1 . The group $\mathbf{G}$ acts by isometries on $\mathbf{H}^{2,n}$ and the stabilizer of a point contains a group isomorphic to $\mathbf{SO}_0(2, n)$ as an index two subgroup. In particular, $\mathbf{H}^{2,n}$ is a (pseudo-Riemannian) symmetric space of $\mathbf{G}$.

2.1.1. Geodesics and acausal sets. Complete geodesics are intersections of projective lines with $\mathbf{H}^{2,n}$. Any two distinct points (x, y) lie on a unique complete geodesic. We parametrize a geodesic by parallel tangent vectors.

A geodesic γ , which is the intersection of the projective line $\mathbf{P}(F)$ with $\mathbf{H}^{2,n}$, can be of three types:

- (i) *Spacelike geodesics*, when F has signature $(1, 1)$, or equivalently $\mathbf{q}(\dot{\gamma})$ is positive.
- (ii) *Timelike geodesics*, when F has signature $(0, 2)$, or equivalently $\mathbf{q}(\dot{\gamma})$ is negative.
- (iii) *Lightlike geodesics*, when F is degenerate, or equivalently $\mathbf{q}(\dot{\gamma}) = 0$.

A geodesic segment is the restriction of a parametrized complete geodesic to the segment $[0, 1]$. Two distinct points (x, y) are extremities of a geodesic segment, which is unique unless the corresponding complete geodesic is timelike (in which case there are exactly two such geodesic segment).

We say the pair of points (x, y) is *acausal* if they are the extremities of a spacelike geodesic segment γ . We then define its *spatial distance* as

$$\delta(x, y) := \int_0^1 \sqrt{\mathbf{q}(\dot{\gamma})} \, dt \, .$$

A subset U of $\mathbf{H}^{2,n}$ is *acausal* if every pair of distinct points in U is acausal.

2.1.2. *Hyperbolic planes.* A hyperbolic plane H in $\mathbf{H}^{2,n}$ is the intersection of $\mathbf{H}^{2,n}$ with a projective plane $\mathbf{P}(F)$ where F is a three-dimensional linear subspace of signature $(2, 1)$. The spatial distance δ restricts to the hyperbolic distance on any hyperbolic plane.

A pointed hyperbolic plane $\mathbf{P}$ is a pair (q, H) where H is a hyperbolic plane and $q \in H$. A pointed hyperbolic plane is equivalent to the datum of an orthogonal decomposition $E = L \oplus U \oplus V$ where L is a negative definite line, U a positive definite 2-plane and $V = (L \oplus U)^\perp$.

2.1.3. *The double cover.* In the sequel, we will often work with the space

$$\mathbf{H}_+^{2,n} = \{x \in E, \mathbf{q}(x) = -1\} .$$

The natural projection $\mathbf{P} : E \setminus \{0\} \rightarrow \mathbf{P}(E)$ restricts to a double cover $\mathbf{H}_+^{2,n} \rightarrow \mathbf{H}^{2,n}$.

The tangent space $\mathbf{T}_x \mathbf{H}_+^{2,n}$ is canonically identified with $x^\perp$. The restriction of $\mathbf{q}$ to $\mathbf{T}_x \mathbf{H}_+^{2,n}$ equips $\mathbf{H}_+^{2,n}$ with the signature $(2, n)$ pseudo-Riemannian metric such that the cover $\mathbf{H}_+^{2,n} \rightarrow \mathbf{H}^{2,n}$ is a local isometry. We still denote this metric by $\mathbf{g}$.

Complete geodesics in $\mathbf{H}_+^{2,n}$ are connected components of lifts of complete geodesics in $\mathbf{H}^{2,n}$. As in $\mathbf{H}^{2,n}$, we parametrize complete geodesics with parallel tangent vectors. A geodesic segment in $\mathbf{H}_+^{2,n}$ is the restriction of a (parametrized) complete geodesic to the segment $[0, 1]$. A pair of distinct points (x, y) in $\mathbf{H}_+^{2,n}$ is *acausal* if x and y are joined by a spacelike geodesic segment, and a subset U of $\mathbf{H}_+^{2,n}$ is acausal if any pairwise distinct points of U are acausal.

The incidence geometry of $\mathbf{H}_+^{2,n}$ is more subtle than that of $\mathbf{H}^{2,n}$. To describe it, first observe that the preimage of a complete geodesic $[\gamma]$ in $\mathbf{H}^{2,n}$ has one connected component if γ is timelike and two connected components otherwise. Given two distinct points x and y in $\mathbf{H}_+^{2,n}$, we denote by $[x]$ and $[y]$ their respective image in $\mathbf{H}^{2,n}$.

We distinguish the following cases:

- Case 1:* when $[x] = [y]$, that is if $x = -y$. Then any complete timelike geodesic in $\mathbf{H}^{2,n}$ passing through $[x]$ lifts to two geodesic segments between x and y . In particular, there are infinitely many geodesic segments between x and y .
- Case 2:* when $[x] \neq [y]$ and the complete geodesic passing through them is timelike. Then there is a unique geodesic segment between $[x]$ and $[y]$ having a lift with extremities x and y . In particular, there is a unique geodesic segment between x and y .
- Case 3:* If $[x] \neq [y]$ and the geodesic γ passing through them is spacelike. Then either x and y lie on the same connected component of the preimage of γ , in which case there is a unique geodesic segment between x and y , or they lie in different connected components in which case there is no geodesic segment between x and y .
- Case 4:* If $[x] \neq [y]$ and the geodesic γ passing through them is lightlike. Then either x and y lie on the same connected component of the preimage of γ , in which case there is a unique geodesic segment between x and y , or they lie in different connected components in which case there is no geodesic segment between x and y .

The different situations are easily described using the scalar product $\langle x, y \rangle$ of the points x and y associated to the quadratic form $\mathbf{q}$.

Lemma 2.2. *Consider two distinct points x and y in $\mathbf{H}_+^{2,n}$.*

- (i) *There is a spacelike geodesic segment between x and y (that is, the pair (x, y) is acausal) if and only if $\langle x, y \rangle < -1$.*
- (ii) *There is a unique timelike geodesic segment between x and y if and only if $|\langle x, y \rangle| < 1$.*
- (iii) *There is a lightlike geodesic segment between x and y if and only if $\langle x, y \rangle = -1$.*

Three points (x_1, x_2, x_3) lies in a hyperbolic plane if and only if for any $i \neq j$ we have $\langle x_i, x_j \rangle < -1$ and

$$2\langle x_1, x_2 \rangle \langle x_1, x_3 \rangle \langle x_2, x_3 \rangle + \langle x_1, x_2 \rangle^2 + \langle x_1, x_3 \rangle^2 + \langle x_2, x_3 \rangle^2 < 1. \quad (1)$$

Proof. Items (i), (ii) and (iii) correspond to cases different from *Case 1* described above. In particular, the set $\{x, y\}$ spans a 2-plane F in which the matrix of the quadratic form is given by

$$\begin{pmatrix} -1 & \langle x, y \rangle \\ \langle x, y \rangle & -1 \end{pmatrix},$$

whose determinant is equal to $\delta := 1 - |\langle x, y \rangle|^2$.

Item (ii) corresponds to *Case 2* described above. In particular, this happens if and only if the plane F has signature $(0, 2)$, that is if and only if $\delta > 0$ (the case of signature $(2, 0)$ is impossible since F contains negative definite vectors).

Item (i) is a particular situation in *Case 3*, thus a necessary condition is to have $\delta < 0$, meaning that $|\langle x, y \rangle| > 1$. The two different connected components of the preimage of the geodesic between $[x]$ and $[y]$ are distinguished by the sign of the function $\langle x, \cdot \rangle$. This sign is negative on the connected component containing x .

Item (iii) is a particular situation in *Case 4*, thus a necessary condition is to have $\delta = 0$, meaning that $|\langle x, y \rangle| = 1$. Similarly to item (i), the connected component of the preimage of the geodesic between $[x]$ and $[y]$ are distinguished by the sign of the function $\langle x, \cdot \rangle$.

For the last statement, observe that (x_1, x_2, x_3) lie in a hyperbolic plane if and only if for any $i \neq j$ the points x_i and x_j are joined by a spacelike geodesic segment and the 3-plane P spanned by x_1, x_2 and x_3 has signature $(2, 1)$. Since the subspace of P spanned by x_1 and x_2 has signature $(1, 1)$, then P has signature $(2, 1)$ if and only if $\det(\langle x_i, x_j \rangle_{1 \leq i, j \leq 3}) < 0$ which is equivalent to the condition (1). $\square$

Similarly, a (pointed) hyperbolic plane in $\mathbf{H}_+^{2,n}$ is a connected component of a lift of a (pointed) hyperbolic plane in $\mathbf{H}^{2,n}$. A pointed hyperbolic plane in $\mathbf{H}_+^{2,n}$ thus corresponds to an orthogonal decomposition $E = L \oplus U \oplus V$ where L is an oriented negative definite line, U is a positive definite plane and $V = (L \oplus U)^\perp$.

2.2. Pseudo-spheres and horospheres. We describe here the geometry of pseudo-spheres, and (pseudo)-horospheres which are counterparts in pseudo-hyperbolic space of the corresponding hyperbolic notions.

2.2.1. Pseudo-sphere. Let F be an $(n + 2)$ -dimensional real vector space equipped with a quadratic form $\mathbf{q}_F$ of signature $(2, n)$. The *pseudo-sphere* is

$$\mathbf{S}^{1,n} := \{x \in F, \mathbf{q}_F(x) = 1\}.$$

The pseudo-sphere $\mathbf{S}^{1,n}$ is equipped with a pseudo-Riemannian metric $\mathbf{g}_{\mathbf{S}^{1,n}}$ of curvature $+1$ and signature $(1, n)$. This metric is invariant under the action of the group $\mathrm{SO}_0(F)$ which is isomorphic to $\mathrm{SO}_0(2, n)$.

2.2.2. Horosphere. Let us return to the basic case where E is equipped with a signature $(2, n + 1)$ quadratic form $\mathbf{q}$. The *null-cone* of E is

$$\mathcal{N}(E) = \{v \in E \setminus \{0\}, \mathbf{q}(v) = 0\}.$$

Given a point $v \in \mathcal{N}(E)$, the set $P_v = \{x \in E, \langle x, v \rangle = -1\}$ is a (degenerate) affine hyperplane whose direction is $v^\perp$. The corresponding *horosphere* is

$$\mathcal{H}(v) = P_v \cap \mathbf{H}_+^{2,n}.$$

We also refer to the projection of $\mathcal{H}(v)$ in $\mathbf{H}^{2,n}$ as a horosphere (and denote it the same way).

Given a point $u \in \mathcal{H}(v)$, denote by F the linear subspace of E orthogonal to u and v . The restriction $\mathbf{q}|_F$ of the quadratic form $\mathbf{q}$ to F has signature $(1, n)$. Consider the map

$$f : \begin{cases} F & \mapsto E, \\ x & \mapsto u + x - \frac{\mathbf{q}_F(x)}{2} \cdot v. \end{cases}$$

One easily checks that f is a diffeomorphism between F and $\mathcal{H}(v)$.

Moreover, since $D_x f(h) = h - \langle x, h \rangle \cdot v$ and v is isotropic, the pull-back by f of the induced metric on $\mathcal{H}(v)$ (which is induced by $\mathbf{q}$) is equal to $\mathbf{q}|_F$. Thus $\mathcal{H}(v)$ is isometric to the pseudo-Euclidean space $\mathbf{E}^{1,n}$ of signature $(1, n)$.

2.2.3. Horospheres as limits of pseudo-spheres. Let x be a point in $\mathbf{H}_+^{2,n}$ (the picture is similar in $\mathbf{H}^{2,n}$). Let

$$\mathbf{T}_x^1 \mathbf{H}_+^{2,n} := \{v \in \mathbf{T}_x \mathbf{H}_+^{2,n}, \mathbf{q}(v) = 1\}.$$

Since the restriction of $\mathbf{q}$ to $\mathbf{T}_x \mathbf{H}_+^{2,n}$ has signature $(2, n)$, the space $\mathbf{T}_x^1 \mathbf{H}_+^{2,n}$ is isometric to $\mathbf{S}^{1,n}$ and its metric is $\text{Stab}_G(x)$ invariant. We will thus denote it by $\mathbf{g}_{\mathbf{S}^{1,n}}$.

For ρ positive, the exponential map $\exp_x(\rho \cdot)$ restricts to a diffeomorphism between $\mathbf{T}_x^1 \mathbf{H}_+^{2,n}$ and the hypersurface

$$\beta(x, \rho) := \{y \in \mathbf{H}_+^{2,n}, \delta(x, y) = \rho\}.$$

Because the restriction of $\mathbf{g}$ to $\beta(x, \rho)$ is also $\text{Stab}_G(x)$ -invariant, there exists a positive number $\lambda(\rho)$ such that $\exp(\rho \cdot)^* \mathbf{g} = \lambda(\rho) \mathbf{g}_{\mathbf{S}^{1,n}}$. Using the same calculation as in classical hyperbolic geometry, one sees that $\lambda(\rho) = \sinh(\rho)$. As a result, $\beta(x, \rho)$ is an umbilical hypersurface of signature $(1, n)$ whose induced metric has sectional curvature $\sinh^{-2}(\rho)$.

Let $\{\rho_k\}_{k \in \mathbb{N}}$ be a sequence of positive numbers tending to infinity, and for any k , let x_k be a point in $H_k := \beta(x, \rho_k)$. Observe that H_k , being a non-degenerate hypersurface, has a canonical normal framing, as in definition A.1. Let g_k in G map $\mathbf{T}_{x_k} H_k$ to a fixed vector space V_0 in $\mathbf{T}_{x_0} \mathbf{H}^{2,n}$. Then $g_k(H_k)$ converges to the horosphere passing through x_0 and tangent to V_0 . We will need in Proposition 8.2 the fact that this convergence is in the sense of Appendix A.1.2).

2.3. Grassmannians.

2.3.1. Riemannian symmetric space. We summarize some of the properties of the Riemannian symmetric space of G .

Proposition 2.3. *The Riemannian symmetric space of G is isometric to the Grassmannian $\text{Gr}_{2,0}(E)$ of oriented 2-planes in E of signature $(2, 0)$.*

Proof. The group G acts transitively on $\text{Gr}_{2,0}(E)$ and the stabilizer of a point is isomorphic to $\text{SO}(2) \times \text{SO}(n+1)$ which is a maximal compact subgroup of G . This realizes $\text{Gr}_{2,0}(E)$ as the Riemannian symmetric space of G . $\square$

Since the maximal compact subgroup of G contains $\text{SO}(2)$ as a factor, $\text{Gr}_{2,0}(E)$ is a Hermitian symmetric space.

The corresponding Kähler structure may be described this way. Let P be a point in $\text{Gr}_{2,0}(E)$:

- the tangent space $T_P \text{Gr}_{2,0}(E)$ at P is identified with $\text{Hom}(P, P^\perp)$. The Riemannian metric $h_P(\cdot, \cdot)$ at P is defined for $\varphi \in \text{Hom}(P, P^\perp)$ by

$$h_P(\varphi, \varphi) := -\text{tr}(\varphi^* \varphi) ,$$

where $\varphi^* : P^\perp \rightarrow P$ is the adjoint of φ using $\mathbf{q}$. Note that since $\mathbf{q}$ is negative definite on $P^\perp$, we have $\text{tr}(\varphi^* \varphi) \leq 0$.

- Since the plane P is oriented, it carries a canonical complex structure J : the rotation by angle $\pi/2$. Precomposition by J defines a *complex structure* on $\text{Hom}(P, P^\perp) = T_P \text{Gr}_{2,0}(E)$, hence a G -invariant almost complex structure on $\text{Gr}_{2,0}(E)$. This almost complex structure is the complex structure associated to the Hermitian symmetric space $\text{Gr}_{2,0}(E)$.

By a theorem of Harish-Chandra (see for instance [20]), $\text{Gr}_{2,0}(E)$ is biholomorphic to a bounded symmetric domain in $\mathbb{C}^{n+1}$.

Note that a point P in $\text{Gr}_{2,0}(E)$ gives rise to an orthogonal splitting $E = P \oplus P^\perp$. We denote by π_P the orthogonal projection from E to P . The following lemma is straightforward.

Lemma 2.4. *Given a compact set K in $\text{Gr}_{2,0}(E)$, there exists a constant C , with $C \geq 1$ such that for any Q and P in K and $v \in Q$,*

$$C^{-1} \|\pi_P(v)\| \leq \|v\| \leq \|\pi_P(v)\| .$$

Proof. The inequality on the right comes from the fact that $P^\perp$ is negative definite, so π_P is length non decreasing. The inequality on the left comes from the compactness of K . $\square$

2.3.2. Grassmannian of a pseudo-Riemannian space. In this paragraph (M, g) will be a pseudo-Riemannian manifold of signature $(2, n)$.

The Grassmannian $\mathcal{G}(M)$ of positive definite 2-planes in M is the fiber bundle $\pi : \mathcal{G}(M) \rightarrow M$ whose fiber over a point $x \in M$ is the Riemannian symmetric space $\text{Gr}_{2,0}(T_x M)$.

Observe that, $\pi : \mathcal{G}(M) \rightarrow M$ has a *horizontal distribution* given by the parallel transport, giving a splitting

$$T_{(x,P)} \mathcal{G}(M) = T_x M \oplus \text{Hom}(P, P^\perp) = P \oplus P^\perp \oplus \text{Hom}(P, P^\perp) .$$

This splitting allows us to define the *canonical Riemannian metric* g on $\mathcal{G}(M)$ given at a point $(x, P) \in \mathcal{G}(M)$ by

$$g := (g^0|_P, -g^0|_{P^\perp}, h_P) ,$$

where h_P is the Riemannian metric on the fiber described above, and g^0 is the metric on $T_x M$. Let us also define for all positive λ , the *renormalized metric*

$$g_\lambda := \left(\frac{1}{\lambda} g^0|_P, -\frac{1}{\lambda} g^0|_{P^\perp}, h_P \right) .$$

2.3.3. The Grassmannian of $\mathbf{H}^{2,n}$. When $M = \mathbf{H}^{2,n}$, we have already remarked in paragraph 2.1.2 that a point in $\mathcal{G}(\mathbf{H}^{2,n})$ is identified with an orthogonal splitting $E = L \oplus U \oplus V$ where L is a negative definite line, U a positive definite plane and $V = (L \oplus U)^\perp$. The exponential map thus naturally identifies $\mathcal{G}(\mathbf{H}^{2,n})$ with the space of pointed hyperbolic planes in $\mathbf{H}^{2,n}$. We will later on freely use this identification.

Up to an index two subgroup, the stabilizer of a point (q, P) in $\mathcal{G}(\mathbf{H}^{2,n})$ is isomorphic to $\text{SO}(2) \times \text{SO}(n)$. The projection

$$\begin{cases} \mathcal{G}(\mathbf{H}^{2,n}) & \longrightarrow \text{Gr}_{2,0}(E) , \\ L \oplus U \oplus V & \longmapsto U . \end{cases}$$

is a $\mathbf{G}$ -equivariant proper Riemannian submersion when $\mathcal{G}(\mathbf{H}^{2,n})$ is equipped with the canonical Riemannian metric.

Similarly, a point in $\mathcal{G}(\mathbf{H}_+^{2,n})$ corresponds to a pointed hyperbolic plane in $\mathbf{H}_+^{2,n}$.

2.3.4. A geometric transition. Let λ be a positive number. We denote by $\mathbf{H}_\lambda^{2,n}$ the space $\mathbf{H}^{2,n}$ equipped with the metric $\mathbf{g}_\lambda = \frac{1}{\lambda} \mathbf{g}$ where $\mathbf{g}$ is the metric on $\mathbf{H}^{2,n}$. Then we have

Proposition 2.5. (i) *The Riemannian manifold $\mathcal{G}(\mathbf{H}_\lambda^{2,n})$ is isometric to $\mathcal{G}(\mathbf{H}^{2,n})$ equipped with the normalized metric g_λ .*
(ii) *When λ tends to 0 the Riemannian manifold $\mathcal{G}(\mathbf{H}_\lambda^{2,n})$ converges in the sense of Appendix A.1.2 to $\mathcal{G}(\mathbf{E}^{2,n})$ where $\mathbf{E}^{2,n}$ is pseudo-Euclidean space of signature $(2, n)$.*

Observe that, even if the notion of convergence of Riemannian manifolds described in Appendix A.1.2 requires the choice of a point, since the manifolds $\mathcal{G}(\mathbf{H}_\lambda^{2,n})$ and $\mathcal{G}(\mathbf{E}^{2,n})$ are homogeneous, this choice is not needed here. We might write the first item in terms of our notation as stating that the two metric spaces $(\mathcal{G}(\mathbf{H}^{2,n}, \mathbf{g}_\lambda), g)$ and $(\mathcal{G}(\mathbf{H}^{2,n}, \mathbf{g}), g_\lambda)$ are isometric.

Proof. The first statement comes from the fact that the metric on $\text{Hom}(P, P^\perp)$ is a conformal invariant. The second statement is standard. $\square$

We will call $\mathcal{G}(\mathbf{H}_\lambda^{2,n})$ the *renormalized Grassmannian*.

2.4. Einstein universe. The *Einstein universe* is the boundary of $\mathbf{H}^{2,n}$ in $\mathbf{P}(E)$:

$$\partial_\infty \mathbf{H}^{2,n} := \{x \in \mathbf{P}(E), \mathbf{q}(x) = 0\}.$$

Associated is a compactification:

$$\bar{\mathbf{H}}^{2,n} := \mathbf{H}^{2,n} \cup \partial_\infty \mathbf{H}^{2,n}.$$

The group $\mathbf{G}$ acts transitively on $\partial_\infty \mathbf{H}^{2,n}$ and the stabilizer of a point in $\partial_\infty \mathbf{H}^{2,n}$ is a maximal parabolic subgroup. As for $\mathbf{H}^{2,n}$, we will often discuss the double cover of the boundary at infinity as well as the associated compactification

$$\begin{aligned} \partial_\infty \mathbf{H}_+^{2,n} &:= \{x \in \mathbf{P}_+(E), \mathbf{q}(x) = 0\}, \\ \bar{\mathbf{H}}_+^{2,n} &:= \mathbf{H}_+^{2,n} \cup \partial_\infty \mathbf{H}_+^{2,n}, \end{aligned}$$

where $\mathbf{P}_+(E) = (E \setminus \{0\})/\mathbb{R}_+$ is the set of rays in E . We will consider $\partial_\infty \mathbf{H}_+^{2,n}$ as the boundary of $\mathbf{H}_+^{2,n}$.

2.4.1. Photons, circles and lightcone. Let us first define some subsets of $\partial_\infty \mathbf{H}^{2,n}$.

- (i) A *photon* or *lightlike line* in $\partial_\infty \mathbf{H}^{2,n}$ is the projectivization of an isotropic 2-plane in E .
- (ii) A *spacelike circle* (respectively *timelike*) is the intersection of $\partial_\infty \mathbf{H}^{2,n}$ with the projectivisation of a subspace of signature $(2, 1)$ (respectively $(1, 2)$). Equivalently, a spacelike circle is the boundary of a hyperbolic plane in $\mathbf{H}^{2,n}$.

Observe that two distinct points in $\partial_\infty \mathbf{H}^{2,n}$ lie either on a photon or span a non-degenerate 2-plane in E . In the second case, we say that x and y are *transverse*.

2.4.2. Conformal structure. The tangent space $T_x \partial_\infty \mathbf{H}^{2,n}$ is identified with the space $\text{Hom}(x, x^\perp/x)$. The vector space $x^\perp/x$ inherits a signature $(1, n)$ quadratic form from $\mathbf{q}$ and E , providing $T_x \partial_\infty \mathbf{H}^{2,n}$ with a conformal class of quadratic form. As a result, $\partial_\infty \mathbf{H}^{2,n}$ is naturally equipped with a conformal structure $[g_{\text{Ein}}]$ of signature $(1, n)$.

The conformal structure then allows for the definition of timelike and lightlike vectors and curves in $\partial_\infty \mathbf{H}^{2,n}$. For instance, photons are lightlike curves, while the spacelike and timelike circles are respectively spacelike and timelike curves in $\partial_\infty \mathbf{H}^{2,n}$ in terms of the conformal structure.

2.4.3. Product structure. Let $\mathbf{P}$ be a pointed hyperbolic plane in $\mathbf{H}_+^{2,n}$, which as usual corresponds to an orthogonal splitting $E = L \oplus U \oplus V$ where U is a positive definite 2-plane, V is definite negative and L an oriented negative definite line. Let $W = L \oplus V$ and denote by $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_W$ the positive definite scalar product induced by $\pm \mathbf{q}$ on U and W respectively. Then any isotropic ray $x \in \partial_\infty \mathbf{H}_+^{2,n}$ contains a unique point $(u, w) \in U \oplus W$ with $\langle u, u \rangle_U = \langle w, w \rangle_W = 1$. This gives a diffeomorphism

$$\partial_\infty \mathbf{H}_+^{2,n} \cong \mathbf{S}^1 \times \mathbf{S}^n,$$

where $\mathbf{S}^1 \subset U$ and $\mathbf{S}^n \subset W$ are the unit spheres. In this coordinate system, the conformal metric of $\partial_\infty \mathbf{H}_+^{2,n}$ is given by

$$[g_{\text{Ein}}] = [g_{\mathbf{S}^1} \oplus -g_{\mathbf{S}^n}],$$

where $g_{\mathbf{S}^i}$ is the canonical metric on $\mathbf{S}^i$ of curvature 1 (see [24, Section 2.1]).

2.5. Positivity. We now discuss the important notion of positivity in the pseudo-hyperbolic setting.

2.5.1. Positive triples. Let τ be a triple of pairwise distinct points in the compactification $\bar{\mathbf{H}}^{2,n}$ (or in $\bar{\mathbf{H}}_+^{2,n}$). We call τ a *positive triple* if it spans a space of signature $(2, 1)$. It will be called a *negative triple* if it spans a space of signature $(1, 2)$. The positive triple is *at infinity* if all three points belong to $\partial_\infty \mathbf{H}^{2,n}$ (or in $\partial_\infty \mathbf{H}_+^{2,n}$).

Positive triples are (possibly ideal) vertices of hyperbolic triangles in $\mathbf{H}^{2,n}$. Given a positive triple τ , we will denote by b_τ the barycenter of the hyperbolic triangle spanned by τ .

We warn the reader that the terminology *positive triples*, though standard, may be confusing: being a positive triple is invariant under all permutations of the elements.

2.5.2. Semi-positive loops. We now define the notion of (semi-)positive loops in the compactification $\bar{\mathbf{H}}^{2,n}$. The definition for $\bar{\mathbf{H}}_+^{2,n}$ is similar.

Definition 2.6. Let Λ be a subset of $\bar{\mathbf{H}}^{2,n}$ homeomorphic to a circle.

- (i) Λ is a *positive loop* if any triple of points in Λ is positive.
- (ii) Λ is a *semi-positive loop* if it does not contain any negative triple, and if Λ contains at least one positive triple.

The next lemma concerns the special case of semi-positive loops in $\partial_\infty \mathbf{H}^{2,n}$. Recall that photons and transverse points are defined in Paragraph 2.4.1.

Lemma 2.7. *Let Λ be a topological circle in $\partial_\infty \mathbf{H}^{2,n}$ that does not contain any negative triple. Then Λ is a semi-positive loop if and only if it is different from a photon.*

Proof. If Λ is a photon, it does not contain any positive triple and so is not a semi-positive loop.

Conversely, let us assume that Λ does not contain any positive triple. We want to show that Λ is a photon. If Λ is not a photon, then we can find two transverse points x, y in Λ . Denote by U_x and U_y the open set of points in Λ that are transverse to x and to y respectively. Observe that U_x is contained in a photon, and the same is true for y . In fact, if not, we could find 2 points $z, t \in U_x$ such that x, z, t are pairwise transverse. In particular the triple (x, z, t) is positive.

We now claim that $\Lambda \setminus (U_x \cup U_y)$ contains at most 2 points. In fact, the complement of $U_x \cup U_y$ is contained in $(x \oplus y)^\perp$ which has signature $(1, n)$. So any triple of pairwise distinct points in $(x \oplus y)^\perp$ must be negative ($\mathbb{R}^{1,n}$ does not contain any isotropic 2-plane), so $\Lambda \setminus (U_x \cup U_y)$ cannot contain more than two points.

This implies that Λ is contained in the union of two non disjoint photons $\phi_1 \cup \phi_2$. Since two photons intersect at most in one point, $\phi_1 \cup \phi_2$ is homeomorphic to the wedge sum of two circles. The only topological circle embedded in the wedge sum of two circles is one of the circles. This implies that Λ is equal to ϕ_1 or ϕ_2 contradicting the existence of a pair of transverse points. $\square$

We have the following

Lemma 2.8. *Let Λ be a semi-positive loop in $\tilde{\mathbf{H}}^{2,n}$, τ a positive triple in Λ and b a point in the interior of the hyperbolic triangle with vertices τ . Then $b^\perp$ is disjoint from Λ . In particular, the pre-image of Λ in $\tilde{\mathbf{H}}_+^{2,n}$ has two connected components.*

Proof. Consider $\tau = (z_1, z_2, z_3)$ and b as in the proposition. Choose a lift of b in $\mathbf{H}_+^{2,n}$, and lift z_1, z_2 and z_3 to vectors in the affine hyperplane $\{x \in E, \langle x, b \rangle = -1\}$ (we denote the lift with the same letters). Since b is in the interior of the triangle with vertices z_1, z_2 and z_3 , there exists $t_1, t_2, t_3 > 0$ such that $b = \sum_{i=1}^3 t_i z_i$.

First observe that we have $\langle z_i, z_j \rangle < 0$ for any $i \neq j$. In fact, the 3-plane $P = \text{span}\{z_1, z_2, z_3\}$ splits as $P = \mathbb{R} \cdot b \oplus b^\perp$ with $b^\perp$ positive definite. Since $z_i \in P$ we can write $z_i = b + x_i$ with $x_i \in b^\perp$ and the condition $\langle z_i, z_i \rangle \leq 0$ gives $\langle x_i, x_i \rangle \leq 1$. The fact that $\langle z_i, z_j \rangle < 0$ then follows from Cauchy-Schwarz inequality together with the fact that $z_i \neq z_j$.

Consider now a vector $x \in E$ which lifts a point in Λ . We first claim that there exists at least one z_i such that $\langle x, z_i \rangle \neq 0$. In fact, if not, x would belong to the space H orthogonal to $\text{span}\{z_1, z_2, z_3\}$. Since H has signature $(0, n)$, x would be negative definite and so the space spanned by z_1, z_2 and x would have signature $(1, 2)$. This is impossible by semi-positivity.

Then we claim that there is no pair (i, j) such that $\langle x, z_i \rangle < 0$ and $\langle x, z_j \rangle > 0$. In fact, if this were the case, the matrix of $\mathbf{q}$ in the basis (z_i, z_j, x) would have the form

$$\begin{pmatrix} -\varepsilon_1 & -\alpha & -\beta \\ -\alpha & -\varepsilon_2 & \gamma \\ -\beta & \gamma & -\varepsilon_3 \end{pmatrix},$$

where $\varepsilon_i \geq 0$ and $\alpha, \beta, \gamma > 0$. The determinant of such matrix is

$$\Delta = -\varepsilon_1 \varepsilon_2 \varepsilon_3 + 2\alpha\beta\gamma + \varepsilon_3 \alpha^2 + \varepsilon_2 \beta^2 + \varepsilon_1 \gamma^2.$$

Since $\text{span}\{z_1, z_2\}$ has signature $(1, 1)$, we have $\varepsilon_1 \varepsilon_2 - \alpha^2 < 0$. In particular $\Delta > 0$ and $\text{span}\{z_1, z_2, x\}$ has signature $(1, 2)$, contradicting semi-positivity.

We thus find $\langle x, b \rangle = \sum t_i \langle x, z_i \rangle \neq 0$ and Λ is disjoint from $b^\perp$. As a result, Λ is contained in the affine chart $\mathbf{P}(E) \setminus \mathbf{P}(b^\perp)$ and its preimage in $\mathbf{P}_+(E)$ has two connected components, determined by the sign of the linear form $\langle b, \cdot \rangle$. $\square$

Lemma 2.9. *Let Λ be a semi-positive loop in $\bar{\mathbf{H}}^{2,n}$. Then*

- (i) *If Λ is contained in $\mathbf{H}^{2,n}$ and x is a point in Λ , then Λ is disjoint from $x^\perp$.*
- (ii) *If Λ is contained in $\partial_\infty \mathbf{H}^{2,n}$, then any point in Λ is contained in a positive triple of Λ .*

Proof. The first item is obvious: if $y \in \Lambda$ is orthogonal to x , then since in this case we restrict to $y \in \mathbf{H}^{2,n}$, we see that $x \oplus y$ has signature $(0, 2)$ contradicting semi-positivity.

From the second item, observe that a triple (x, y, z) in $\Lambda \subset \partial_\infty \mathbf{H}^{2,n}$ is not positive if and only if $\langle x, y \rangle \langle x, z \rangle \langle y, z \rangle = 0$. Denote by U_x the set of points y in Λ transverse to x , that is so that $\langle x, y \rangle \neq 0$. The set U_x is open and non-empty since from the previous lemma, $\langle z_i, x \rangle \neq 0$ for a positive triple (z_1, z_2, z_3) . We just have to find a pair of points y and z in U_x which are transverse to each other (as well as to x). This can always be done unless U_x is contained in a photon ϕ .

We claim that this is not possible. In fact, if U_x is different from $\Lambda \setminus \{x\}$, then its boundary in Λ would contain at least two points, and these points would be in $\phi \cap x^\perp$ which is a single point. If $U_x = \Lambda \setminus \{x\}$, then $\{x\} = \phi \cap x^\perp$ and $\Lambda = \phi$, which does not contain any positive triple. $\square$

This lemma has the following corollary.

Corollary 2.10. *Let Λ be a semi-positive loop contained either in $\partial_\infty \mathbf{H}^{2,n}$ or in $\mathbf{H}^{2,n}$ and Λ_+ be a connected component of its preimage in $\bar{\mathbf{H}}_+^{2,n}$. For any two points x and y in Λ_+ , we have $\langle x, y \rangle \leq 0$.*

Proof. If Λ is contained in $\mathbf{H}^{2,n}$, the first item of the previous lemma implies that if $x \in \Lambda_+$, the linear function $\langle x, \cdot \rangle$ never vanishes. By connectedness of Λ , the sign of $\langle x, \cdot \rangle$ is constant and must be negative because $\langle x, x \rangle = -1$.

Now assume that Λ is contained in $\partial_\infty \mathbf{H}^{2,n}$ and let z_1, z_2 and z_3 be vectors in E lifting a positive triple τ in Λ_+ whose barycenter lifts to $b = z_1 + z_2 + z_3$. As remarked in the proof of Lemma 2.8, $\langle z_i, z_j \rangle < 0$ for $i \neq j$ and for any $x \in \Lambda_+$ the sign of $\langle x, z_i \rangle$ is independent of i among those z_i with $\langle x, z_i \rangle \neq 0$. Because $\langle b, x \rangle = \sum_i \langle z_i, x \rangle < 0$, this sign must be negative. This prove the result when x is contained in a positive triple, and then for every x by the second item of the previous lemma. $\square$

We now consider the special case of semi-positive loops in $\partial_\infty \mathbf{H}_+^{2,n}$. Observe that a loop in $\partial_\infty \mathbf{H}_+^{2,n}$ is semi-positive if and only if its projection to $\partial_\infty \mathbf{H}^{2,n}$ is semi-positive.

The two-to-one cover $\partial_\infty \mathbf{H}_+^{2,n} \rightarrow \partial_\infty \mathbf{H}^{2,n}$ is nontrivial on each photon in $\partial_\infty \mathbf{H}^{2,n}$. In particular, any photon in $\partial_\infty \mathbf{H}^{2,n}$ lifts to a photon segment between two antipodal points in $\partial_\infty \mathbf{H}_+^{2,n}$. We call a *biphoton* in $\partial_\infty \mathbf{H}_+^{2,n}$ a topological circle consisting of two photon segments between antipodal points and whose projection to $\partial_\infty \mathbf{H}^{2,n}$ consists of different photons (in particular, a biphoton is not semi-positive).

We call a map f between metric spaces *strictly contracting* whenever $d(f(x), f(y)) < d(x, y)$ for x, y distinct.

Proposition 2.11. *Let Λ be a loop in $\partial_\infty \mathbf{H}_+^{2,n}$ and consider a splitting $\partial_\infty \mathbf{H}_+^{2,n} \cong \mathbf{S}^1 \times \mathbf{S}^n$ associated to a pointed hyperbolic plane.*

- (i) *The loop Λ is semi-positive if and only if it is the graph of a 1-Lipschitz map from $\mathbf{S}^1$ to $\mathbf{S}^n$ and not a biphoton, nor a photon.*
- (ii) *The loop Λ is positive if and only if it is the graph of a strictly contracting map from $\mathbf{S}^1$ to $\mathbf{S}^n$.*

The proposition will follow from the following

Lemma 2.12. *Let $\tau = (z_1, z_2, z_3)$ be a triple in $\partial_\infty \mathbf{H}_+^{2,n}$ and consider a splitting $\partial_\infty \mathbf{H}_+^{2,n} \cong \mathbf{S}^1 \times \mathbf{S}^n$ associated to a pointed hyperbolic plane. Write $z_i = (u_i, w_i)$ in this splitting.*

- (i) *We have $\langle z_i, z_j \rangle \leq 0$ for all pairs (i, j) if and only if τ is not negative and $d_{\mathbf{S}^1}(u_i, u_j) \geq d_{\mathbf{S}^n}(w_i, w_j)$ for every pair (i, j) .*
- (ii) *We have $\langle z_i, z_j \rangle < 0$ for every $i \neq j$ if and only if τ is positive and $d_{\mathbf{S}^1}(u_i, u_j) > d_{\mathbf{S}^n}(w_i, w_j)$ for any $i \neq j$.*

Proof. The determinant of the matrix with coefficients $\langle z_i, z_j \rangle$ is given by $2\langle z_1, z_2 \rangle \langle z_1, z_3 \rangle \langle z_2, z_3 \rangle$, so the condition on the sign of $\langle z_i, z_j \rangle$ implies the positivity or the non-negativity of τ .

For the condition on the distances, we use the same notation as in Subsection 2.4.3. In particular

$$\langle z_i, z_j \rangle = \langle u_i, u_j \rangle_U - \langle w_i, w_j \rangle_W.$$

For item (i), the condition $\langle z_i, z_j \rangle \leq 0$ is thus equivalent to

$$\langle u_i, u_j \rangle_U \leq \langle w_i, w_j \rangle_W \leq 1.$$

Using the formula $\langle x, y \rangle = \cos(d_{\mathbf{S}^k}(x, y))$, the previous equation holds if and only if

$$d_{\mathbf{S}^1}(u_1, u_2) \geq d_{\mathbf{S}^n}(w_1, w_2),$$

and item (i) follows. For item (ii), we replace the non-strict inequalities with strict inequalities. $\square$

Proof of Proposition 2.11. By Corollary 2.10, if Λ is semi-positive, then $\langle x, y \rangle \leq 0$ for any pair of points in Λ . In fact, Λ is a component of the pre-image of its projection to $\mathbf{P}(E)$.

By the previous Lemma, the projection of Λ to the first factor in $\partial_\infty \mathbf{H}_+^{2,n} \cong \mathbf{S}^1 \times \mathbf{S}^n$ must be injective and Λ is the graph of a 1-Lipschitz map from $\mathbf{S}^1 \rightarrow \mathbf{S}^n$. Conversely, if f is a 1-Lipschitz map, by lemma 2.12 then the image of f does not contain negative triple: if we had a negative triple (z_1, z_2, z_3) then at least one of the product $\langle z_i, z_j \rangle$ is positive. The result then follows from Lemma 2.7. The second item follows from Lemma 2.12 (ii). $\square$

We now give three important corollaries.

Corollary 2.13. *Let Λ be a semi-positive loop in $\partial_\infty \mathbf{H}_+^{2,n}$. If Λ contains two points x and y on a photon, then it contains the segment of a photon between x and y .*

Proof. The semi-positive loop Λ is the graph of a 1-Lipschitz map $f : \mathbf{S}^1 \rightarrow \mathbf{S}^n$ by Proposition 2.11. The points x and y correspond to points $(u, f(u))$ and $(v, f(v))$ with $d_{\mathbf{S}^1}^1(u, v) = d_{\mathbf{S}^n}(f(u), f(v))$.

We first observe that x and y cannot be antipodal: in fact, the pair (x, y) is antipodal if and only if the pairs (u, v) and $(f(u), f(v))$ are. Since f is 1-Lipschitz, it must map the two arcs between u and v to two geodesic arcs between $f(u)$ and $f(v)$. The graph of such a map is either a photon or a biphoton and is thus not semi-positive.

In particular, there is a unique shortest arc segment $[u, v]$ between u and v in $\mathbf{S}^1$ and f maps $[u, v]$ isometrically to an arc of geodesic in $\mathbf{S}^n$. The graph of $f|_{[u, v]}$ is the segment of a photon between x and y . $\square$

Proposition 2.11 also provides for a nice topology on the set of semi-positive loops in $\partial_\infty \mathbf{H}_+^{2,n}$: we say that a sequence $\{\Lambda_k\}_{k \in \mathbb{N}}$ converges to Λ_0 if for any splitting $\partial_\infty \mathbf{H}_+^{2,n} \cong \mathbf{S}^1 \times \mathbf{S}^n$, the sequence $\{f_k\}_{k \in \mathbb{N}}$ converges C^0 to f_0 where $\Lambda_k = \text{graph}(f_k)$ and $\Lambda_0 = \text{graph}(f_0)$.

We have the following

Corollary 2.14. *Every semi-positive loop is a limit of smooth spacelike positive loops.*

Proof. Fix a splitting $\partial_\infty \mathbf{H}_+^{2,n} \cong \mathbf{S}^1 \times \mathbf{S}^n$. By Proposition 2.11, the loop Λ is the graph of a 1-Lipschitz map $f : \mathbf{S}^1 \rightarrow \mathbf{S}^n$, and so its image is contained in a closed hemisphere H of $\mathbf{S}^n$.

For $t \in [0, 2]$, consider the geodesic isotopy $\phi_t : H \rightarrow H$ with the property that for any x , the path $(\phi_t(x))_{t \in [0, 2]}$ is the (constant speed) geodesic starting at x and ending at the center of the hemisphere. Such an isotopy is contracting for $t > 0$ and $d_{\mathbf{S}^n}(x, \phi_t(x)) \leq t$ (because H has radius $\frac{\pi}{2} < 2$).

Thus for any positive ε , there is a positive δ , such that the map $f_\varepsilon := \phi_\varepsilon \circ f$ is $(1 - 2\delta)$ -Lipschitz and is at a distance at most ε from f . Thus, by density, there is a $(1 - \delta)$ -Lipschitz smooth map g at a distance at most 2ε from f_ε . Hence g is at distance at most ε from f and its graph is a smooth positive loop. $\square$

2.5.3. Convex hulls. We want to define the convex hull of a semi-positive loop in $\mathbf{H}^{2,n}$. Note that the convex hull of a subset Λ of $\mathbf{P}(E)$ is in general not well-defined: one first needs to lift Λ to $\mathbf{P}_+(E)$, define the convex hull of the lifted cone as the intersection of all the closed half-spaces containing it, and then project down. The drawback of this construction is that it will in general depend on the lifted cone.

In our case, Lemma 2.8 implies that the convex hull of a semi-positive loop Λ in $\mathbf{H}^{2,n}$ is well-defined and will be denoted by $\mathcal{CH}(\Lambda)$. It has the following properties:

Proposition 2.15. *Let Λ be a semi-positive loop contained either in $\partial_\infty \mathbf{H}^{2,n}$ or in $\mathbf{H}^{2,n}$, and let Λ_+ a connected component of its pre-image in $\mathbf{P}_+(E)$. Then*

- (i) *The convex hull $\mathcal{CH}(\Lambda)$ is contained in $\mathbf{H}^{2,n}$.*
- (ii) *Let p be in the interior of $\mathcal{CH}(\Lambda_+)$ and q in $\mathcal{CH}(\Lambda_+)$, then $\langle p, q \rangle < 0$.*
- (iii) *Assume Λ is a positive loop in $\partial_\infty \mathbf{H}^{2,n}$. If p is a point of $\mathbf{H}_+^{2,n}$ lying in $\mathcal{CH}(\Lambda_+)$ and q is a point in Λ_+ , then $\langle p, q \rangle < 0$.*
- (iv) *Let p be in the interior of $\mathcal{CH}(\Lambda)$, then the set Λ is disjoint from $p^\perp$.*
- (v) *If Λ is contained in $\partial_\infty \mathbf{H}^{2,n}$ and p is in the interior of $\mathcal{CH}(\Lambda)$, then any geodesic ray from p to a point in Λ is spacelike.*
- (vi) *If Λ is contained in $\partial_\infty \mathbf{H}^{2,n}$, then the intersection of $\mathcal{CH}(\Lambda)$ with $\partial_\infty \mathbf{H}^{2,n}$ is equal to Λ .*

Proof of (i) Any $p \in \mathcal{CH}(\Lambda)$ can be lifted to a vector in E of the form $p_0 = \sum_{i=1}^k t_i x_i$ where $t_i > 0$ and the x_i are lifts of points in Λ_+ (actually, from a classical result of Carathéodory [17], one can take $k = \dim(E) + 2$). From Corollary 2.10 we get that $\mathbf{q}(p_0) \leq 0$.

Proof of (ii) For any vector x lifting a point in Λ_+ , the linear form $\langle x, \cdot \rangle$ is non-positive on Λ_+ by Corollary 2.10. Since p is in the interior of $\mathcal{CH}(\Lambda)$, we have $\langle x, p \rangle < 0$. Finally, any point $q \in \mathcal{CH}(\Lambda_+)$ lifts to a vector of the form $\sum_{i=1}^k t_i x_i$ with $t_i > 0$ and $x_i \in \Lambda_+$.

Proof of (iii) The vector p has the form $p = \sum_{i=1}^m t_i x_i$ with x_i (lift of rays) in Λ_+ and $t_i > 0$. For any vector q lifting a point in Λ_+ , we have $\langle p, q \rangle = \sum_{i=1}^m t_i \langle x_i, q \rangle \leq 0$ with equality if and only if $\langle x_i, q \rangle = 0$ for each i . But since Λ is positive, the only vector in Λ whose scalar product with q is 0 is q itself. Since p lies in $\mathbf{H}_+^{2,n}$, we get $\langle p, q \rangle < 0$.

Proof of (iv) As $\Lambda_+ \subset \mathcal{CH}(\Lambda_+)$, this follows from item (ii).

Proof of (v) Let x be a vector in E lifting a point in Λ_+ . By item (ii), the linear form $\langle x, \cdot \rangle$ is negative on Λ_+ and so strictly negative on the interior of $\mathcal{CH}(\Lambda_+)$. The result follows.

Proof of (vi) Let p be a vector in E lifting a point in $\mathcal{CH}(\Lambda) \cap \partial_\infty \mathbf{H}^{2,n}$. Then p can be written $\sum_{i=1}^k t_i x_i$ with $t_i > 0$ and $x_i \in \Lambda_+$. The condition $\langle p, p \rangle = 0$ thus implies that, either $k = 1$ and

$p = x_i$, or that all the x_i lie on a common photon. In this case p lies on a segment of a photon which must be contained in Λ by Corollary 2.13.

3. GRAPHS, CURVES AND SURFACES

In this section, we study the differential geometric aspects of curves and surfaces in $\mathbf{H}^{2,n}$. We define the notion of *maximal surface* and prove some important properties.

3.1. Spacelike submanifolds in pseudo-hyperbolic spaces. Recall that $\mathbf{g}$ denotes the pseudo-Riemannian metric of $\mathbf{H}^{2,n}$.

Definition 3.1 (SPACELIKE AND ACAUSAL).

- (i) A submanifold M of $\mathbf{H}^{2,n}$ is *spacelike* if the restriction of $\mathbf{g}$ to M is Riemannian. Such a submanifold is either a curve or a surface.
- (ii) A spacelike submanifold M of $\mathbf{H}^{2,n}$ is *acausal*, if every pair of distinct points in M is acausal.

3.1.1. Warped-product structure. Let $\mathbf{P} = (q, H)$ in $\mathbf{H}_+^{2,n}$ be a pointed hyperbolic plane, associated to the orthogonal decomposition $E = q \oplus U \oplus V$ where

- (i) q is an oriented negative definite line,
- (ii) U is a positive definite plane, with induced norm $\|\cdot\|$, so that $q \oplus U$ defines H .

Let $\mathbf{D}^2 \subset U$ be the unit (open) disk and $\mathbf{S}^n \subset W := q + V$ be the unit sphere. The following is proved in [21, Proposition 3.5].

Proposition 3.2. *The map*

$$\Psi : \begin{cases} \mathbf{D} \times \mathbf{S}^n & \longrightarrow \mathbf{H}_+^{2,n}, \\ (u, w) & \longmapsto \left(\frac{2}{1-\|u\|^2} u, \frac{1+\|u\|^2}{1-\|u\|^2} w \right), \end{cases} \quad (2)$$

is a diffeomorphism. Moreover, if $\mathbf{g}$ is the metric on $\mathbf{H}_+^{2,n}$, then

$$\Psi^* \mathbf{g} = \frac{4}{(1-\|u\|^2)^2} g_{\mathbf{D}} - \left(\frac{1+\|u\|^2}{1-\|u\|^2} \right)^2 g_{\mathbf{S}^n}, \quad (3)$$

where $g_{\mathbf{D}}$ and $g_{\mathbf{S}^n}$ are respectively the flat Euclidean metric on the disk and the round metric on the sphere.

Observe that the parametrization $\mathbf{H}_+^{2,n} \cong \mathbf{D} \times \mathbf{S}^n$ extends smoothly to a parametrization of $\mathbf{H}_+^{2,n} \cup \partial_\infty \mathbf{H}_+^{2,n}$ by $\overline{\mathbf{D}} \times \mathbf{S}^n$.

The diffeomorphism Ψ is called the *warped diffeomorphism* and said to define the *warped product structure* on $\mathbf{H}_+^{2,n}$.

If the preimage of q under Ψ is $(0, v)$, the preimage of H is $\mathbf{D} \times \{v\}$.

For any w in $\mathbf{S}^n$, the image of $((0, w), \mathbf{D} \times \{w\})$ is a pointed hyperbolic plane that we call *parallel* to $\mathbf{P}$. These pointed disks correspond exactly to the set of pointed hyperbolic planes whose projection to $\text{Gr}_{2,0}(E)$ is U .

The following nice fact was pointed out to us by the referee:

Lemma 3.3. *Let Ψ be as in Proposition 3.2. Identifying $\mathbf{D}$ with an hemisphere of $\mathbf{S}^2$ using the stereographic projection, the metric $\Psi^* \mathbf{g}$ is conformal to the metric $g_{\mathbf{S}^2} - g_{\mathbf{S}^n}$.*

Proof. Consider the function f from $\mathbf{D} \times \mathbf{S}^n$ to $\mathbb{R}$ sending (u, v) to $\left(\frac{1-\|u\|^2}{1+\|u\|^2}\right)^2$. We obtain that the metric $f \cdot \Psi^* \mathbf{g}$ is equal to

$$\frac{4}{(1+\|u\|^2)^2} \mathcal{G}_{\mathbf{D}} - \mathcal{G}_{\mathbf{S}^n} .$$

The result then follows from the fact that $\frac{4}{(1+\|u\|^2)^2} \mathcal{G}_{\mathbf{D}}$ is the expression of the spherical metric on an hemisphere of $\mathbf{S}^2$ in a chart given by the stereographic projection. $\square$

Definition 3.4 (WARPED PROJECTION). The *warped projection* is the map

$$\pi_{\mathbf{P}} : \mathbf{H}_+^{2,n} \longrightarrow H ,$$

corresponding (via Ψ) to the projection from $\mathbf{D} \times \mathbf{S}^n$ to $\mathbf{D} \times \{v\}$ and mapping (u, w) to (u, v) .

A *timelike sphere* is the fiber of $\pi_{\mathbf{P}}$ above q , for some pointed hyperbolic plane $\mathbf{P} = (q, H)$. It is the intersection of $\mathbf{H}_+^{2,n}$ with the subspace W of E of signature $(0, n+1)$.

Note that, given a pointed hyperbolic plane $\mathbf{P} = (q, H)$ with warped projection $\pi_{\mathbf{P}}$, the preimage by $\pi_{\mathbf{P}}$ of a point different from q is not totally geodesic, since its induced metric does not have curvature -1 .

We then have a fundamental property of $\mathbf{H}_+^{2,n}$:

Lemma 3.5 (PROJECTION INCREASES LENGTH). *The warped projection increases the length of spacelike curves. Moreover if x_1 and x_2 are two distinct points in the same fiber, then $\langle x_1, x_2 \rangle > -1$.*

Proof. The fact that the warped projection is length-increasing is a direct consequence of equation (3).

If x_1 and x_2 project onto the same point, then $\Psi^{-1}(x_i) = (u, w_i) \in \mathbf{D} \times \mathbf{S}^n$ for $i = 1, 2$. Using the expression of Ψ , we see that

$$\langle x_1, x_2 \rangle = \frac{4\|u\|^2}{(1-\|u\|^2)^2} - \left(\frac{1+\|u\|^2}{1-\|u\|^2} \right)^2 \langle w_1, w_2 \rangle_W , \quad (4)$$

where $\langle \cdot, \cdot \rangle_W$ is the positive definite scalar product induced by $-\mathbf{q}$ on $W = q \oplus V$. Since $w_i \in \mathbf{S}^n$, we have $\langle w_1, w_2 \rangle_W < 1$, thus

$$\langle x_1, x_2 \rangle > \frac{4\|u\|^2}{(1-\|u\|^2)^2} - \left(\frac{1+\|u\|^2}{1-\|u\|^2} \right)^2 = -1 . \quad (5)$$

This concludes the proof. $\square$

3.1.2. Spacelike graphs. From now on, all our surfaces are assumed to be connected and smooth up to their boundaries.

Definition 3.6. We define

- (i) A spacelike submanifold M of $\mathbf{H}_+^{2,n}$ is a *graph* if for any pointed hyperbolic plane, the restriction of the corresponding warped projection is a diffeomorphism onto its image.
- (ii) If moreover this diffeomorphism is surjective, M is an *entire graph*.

Observe that a spacelike graph is always embedded. We now use the definitions of paragraph 2.1.1. As in Lemma 3.3, we identify $\mathbf{D}$ with an hemisphere B of $\mathbf{S}^2$ using the stereographical projection.

Proposition 3.7. *Let M be a connected spacelike submanifold of $\mathbf{H}_+^{2,n}$.*

- (i) *If M is acausal then it is a graph.*

- (ii) If M is a graph, then it is the graph of a 1-Lipschitz map from a subset U of B to $\mathbf{S}^n$, in any warped product.

Proof of (i) Given a pointed hyperbolic plane $\mathbf{P}$, the corresponding warped projection $\pi_{\mathbf{P}}$ restricts to a local diffeomorphism on M . It follows from Lemma 2.2 that, since M is acausal, we have $\langle x, y \rangle \geq -1$ for any pair $x, y \in M$. Lemma 3.5 then implies that the restriction of $\pi_{\mathbf{P}}$ to M is injective and thus a diffeomorphism on its image.

Proof of (ii) Tangent vectors to the graph of f at $(x, f(x))$ have the form $(u, Df_x(u))$, where $u \in T_x D$. Using Lemma 3.3, one sees that $(u, Df_x(u))$ is spacelike if and only if

$$\|u\|^2 - \|Df_x(u)\|^2 > 0,$$

where the norms are computed using the spherical metrics on B and $\mathbf{S}^n$. It implies

$$\|Df_x\| < 1. \quad (6)$$

□

Lemma 3.8. *Let S be a connected spacelike acausal surface and $\mathbf{P}$ a pointed hyperbolic plane with associated warped projection $\pi_{\mathbf{P}}$. The restriction of $\pi_{\mathbf{P}}$ from S to $\pi_{\mathbf{P}}(S)$ increases the induced path distances.*

Proof. Let $\alpha = \pi_{\mathbf{P}}(a)$ and $\beta = \pi_{\mathbf{P}}(b)$ be points in $\pi_{\mathbf{P}}(S)$ with $a, b \in S$. For any path γ between α and β in $\pi_{\mathbf{P}}(S)$, its preimage by $\pi_{\mathbf{P}}$ in S is a curve between a and b whose length is less than that of the one of γ by Lemma 3.5. Taking the infimum over all path between α and β yields the result. □

We have several different notions of boundary:

Definition 3.9 (BOUNDARIES OF ACAUSAL SURFACES). Let S be an acausal surface in $\mathbf{H}^{2,n}$.

- (i) The *total boundary* Λ of S is $\bar{S} \setminus \text{int}(S)$, where $\bar{S}$ is the closure of S in $\bar{\mathbf{H}}^{2,n}$ and $\text{int}(S)$ is its interior.
- (ii) The *finite boundary* of S , denoted by ∂S , is the intersection of Λ with S .
- (iii) The *asymptotic boundary* of S , denoted by $\partial_{\infty} S$, is the intersection of Λ with $\partial_{\infty} \mathbf{H}^{2,n}$.
- (iv) The *free boundary* of S (or *frontier*), denoted by $\text{Fr}(S)$ is the complement of $\partial S \cup \partial_{\infty} S$ in Λ .

We will use the same notation for the corresponding objects in $\mathbf{H}_+^{2,n}$. Note that if the induced metric on S is metrically complete, then $\text{Fr}(S) = \emptyset$. If moreover S is a manifold without boundary, then $\Lambda = \partial_{\infty} S$.

Given a acausal surface S with induced metric d_I , for any point x in S , define $d_I(x, \text{Fr}(S))$ as the supremum over all R so that the closed ball of radius R and center x is complete. We also define the pseudo-distance to the frontier as

$$\check{d}(x, \text{Fr}(U)) := \inf\{\check{d}(x, z) \mid z \in \text{Fr}(U)\}.$$

Proposition 3.10 (BOUNDARY OF ACAUSAL SURFACES). *Let S be a closed spacelike surface with boundary in $\mathbf{H}_+^{2,n}$. Assume that ∂S is connected and is a graph, then S is a graph.*

Proof. Let π be a warped projection on a hyperbolic plane H . By assumption $\pi(\partial S)$ is a circle γ embedded in H . By compactness of S , $f(x) := \sharp(\pi^{-1}(x))$ is locally constant on each of the connected component of $H \setminus \gamma$. It follows (by compactness) that $f = 0$ on the unbounded component of $H \setminus \gamma$. This implies that $\pi^{-1}\gamma = \partial S$. Hence that $f = 1$ in the (interior) neighborhood of γ . Thus $f = 1$ in the bounded connected component of $H \setminus \gamma$. □

Finally, we describe entire spacelike graphs.

Proposition 3.11 (ENTIRE SPACELIKE GRAPH). *Let S be a simply connected spacelike surface without boundary.*

- (i) *If S is properly immersed or if its induced metric is complete, then S is an entire graph.*
- (ii) *If S is an entire graph, then it is acausal.*
- (iii) *If S is an entire graph, then it intersects any timelike sphere exactly once.*
- (iv) *If S is an entire graph, then its asymptotic boundary $\partial_\infty S$ is a semi-positive loop.*

Proof of (i) When the induced metric g_I on S is geodesically complete, the argument comes from [21, Proposition 3.15]. In that case, the warped projection π_P on a pointed hyperbolic plane P is length-increasing. In particular we have

$$\pi_P^* g_{H^2} \geq g_I .$$

It follows that $\pi_P^* g_{H^2}$ is also complete. As a result, the restriction of π_P to S is a proper immersion, hence a covering, and so a diffeomorphism since S is simply connected.

If S is properly immersed, the result follows from the fact that the warped projection is proper, so its restriction to S is a covering.

Proof of (ii) This was proved in [21, Lemma 3.7].

Proof of (iii) Given a timelike sphere Σ in $\mathbf{H}_+^{2,n}$ and a point $q \in \Sigma$, the orthogonal of Σ at q defines a pointed hyperbolic plane P such that $\Sigma = \pi_P^{-1}(q)$. S is then the graph of a map f and so $\Sigma \cap S = (0, f(0))$.

Proof of (iv) From Proposition 3.7, S is the graph of a 1-Lipschitz map from an hemisphere B in S^2 to S^n . Such a map extends to a 1-Lipschitz map from the equator S^1 to S^n . By Proposition 2.11, its graph $\partial_\infty S$ is semi-positive unless it is a photon or a biphoton.

To prove that $\partial_\infty S$ is not a photon nor a biphoton, observe that from item (ii) the geodesic from any point x in S to any point y in $\partial_\infty S$ is spacelike. In particular, $\partial_\infty S$ is disjoint from the hyperplane $x^\perp$. Given ϕ a photon or a biphoton, ϕ contains a pair of antipodal points (a, b) . Either a and b are contained in $x^\perp$ or they lie in different connected components of $E \setminus x^\perp$. In both cases, ϕ intersect $x^\perp$. Thus $\partial_\infty S$ is semi-positive. $\square$

Remark 3.12. We return briefly to Definition 3.6. If for a spacelike submanifold of $\mathbf{H}_+^{2,n}$, there is at least one pointed hyperbolic plane for which the warped projection is a diffeomorphism onto its image, then the proof of (ii) above shows that the submanifold is acausal. Then Proposition 3.7 implies that the submanifold is a graph over every pointed hyperbolic plane and is thus, by Definition 3.6, a spacelike graph.

Recall that $T_x f$ denotes the tangent map of f at x .

Proposition 3.13. *Let $Q = (q, Q)$ and $P = (p, P)$ be pointed hyperbolic planes. Let φ_P be the restriction of the warped projection π_P to Q . Assume that*

$$d_G(T_p P, T_q Q) < R .$$

Then

- (i) *For each such positive constant R , there exists a positive constant c so that*

$$\|T_q \varphi_P\| \leq c ,$$

- (ii) *For any b larger than 1, there exists such a positive constant R so that*

$$b^{-1} \leq \|T_q \varphi_P\| \leq b .$$

Proof. Since hyperbolic planes have complete induced metrics, for any P and Q as in the proposition, φ_P is a global diffeomorphism and so $\|\mathbb{T}_q \varphi_P\| > 0$. The result then follows from the fact that $\mathcal{G}(\mathbf{H}^{2,n})$ is locally compact and that $\varphi_P = \text{Id}$ when $P = Q$. $\square$

3.2. Strongly positive curves. Recall from 2.3.4 that $\mathbf{H}_\lambda^{2,n}$ is the space $\mathbf{H}^{2,n}$ equipped with the metric $\mathbf{g}_\lambda = \frac{1}{\lambda} \mathbf{g}$.

For a curve γ and $x \in \gamma$, the *osculating plane*, denoted $\mathbb{T}_x^{(2)}\gamma$ is (given a parametrization of γ so that $\gamma(t_0) = x$) the vector space generated by $\dot{\gamma}$ and $\nabla_{\dot{\gamma}}\dot{\gamma}$. The osculating plane is independent of the parametrization. We introduce the following properties of curves in $\mathbf{H}^{2,n}$ which are further refinements of being spacelike and positive.

Definition 3.14 (STRONGLY POSITIVE CURVES). A smooth curve γ in $\mathbf{H}_\lambda^{2,n}$ is *strongly positive* if

- (i) the curve γ positive,
- (ii) for every point x in γ , the osculating plane $\mathbb{T}_x^{(2)}\gamma$ has dimension 2 and is spacelike.
- (iii) for any pair of disjoint points x and y in γ , the totally geodesic space containing x , y and the tangent vectors in $\mathbb{T}_y\gamma$ is spacelike.

It is important to remark that since the lift of a connected positive curve to $\mathbf{H}_+^{2,n}$ has two connected components, we can use any of these to make sense of warped projection, graphs and so on. This fact will be used in the sequel.

3.2.1. Unpinched curves. Given an acausal curve γ in $\mathbf{H}^{2,n}$, we can define two distances on γ : the (extrinsic) spatial distance δ (see paragraph 2.1.1) and the distance d_γ along γ . The following notion is a comparison between those two.

Definition 3.15 (UNPINCHED CURVES). An acausal curve γ is called *unpinched* (or δ -unpinched) if there exists $\delta > 0$ such that for all x, y in γ ,

$$\frac{\delta(x, y)}{d_\gamma(x, y)} \leq \frac{1}{2} \implies \delta(x, y) > \delta.$$

A sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ is *uniformly unpinched* if there is a $\delta > 0$ such that any γ_k is δ -unpinched.

Observe also that if γ is δ -unpinched in $\mathbf{H}_\lambda^{2,n}$, it is also δ unpinched for $\mathbf{H}_\mu^{2,n}$ for $\mu \leq \lambda$.

3.2.2. Angular width. Given a curve γ , we denote by $\gamma^{(3)}$ the set of pairwise distinct triples of points in γ and by $\text{Gr}_{2,1}(E)$ the set of hyperbolic planes in $\mathbf{H}^{2,n}$. Assume γ is strongly positive and consider the map

$$\Phi_\gamma : \begin{cases} \gamma^{(3)} & \longrightarrow \text{Gr}_{2,1}(E) \\ (x, y, z) & \longmapsto x \oplus y \oplus z \end{cases}.$$

Since γ is smooth, we have

$$\lim_{\substack{x_1, x_2 \rightarrow x \\ x_1 \neq x_2}} (x_1 \oplus x_2) = x \oplus \mathbb{T}_x \gamma, \quad \lim_{\substack{x_1, x_2, x_3 \rightarrow x \\ x_1 \neq x_2 \neq x_3}} (x_1 \oplus x_2 \oplus x_3) = x \oplus \mathbb{T}_x^{(2)} \gamma.$$

In particular, Φ_γ extends to a continuous map $\overline{\Phi}_\gamma : \gamma^3 \rightarrow \text{Gr}_{2,1}(E)$ where for $x \neq y$ we have

$$\overline{\Phi}_\gamma(x, y, y) = x \oplus y \oplus \mathbb{T}_y \gamma, \quad \overline{\Phi}_\gamma(x, x, x) = x \oplus \mathbb{T}_x^{(2)} \gamma.$$

Let γ be a strongly positive curve in $\mathbf{H}^{2,n}$. Let x_0 be a point in γ . Consider the codimension 2 subspace $F := (x_0 \oplus T_{x_0}\gamma)^\perp$ of E . Observe that F has signature $(1, n)$. The projectivization of the orthogonal projection from E to F defines a map

$$\pi_0 : \mathbf{P}(E) \setminus \mathbf{P}(F^\perp) \rightarrow \mathbf{P}(F) .$$

We have

Proposition 3.16. *Let γ , x_0 , and π_0 be as above, and let y be a point in $\bar{\mathbf{H}}^{2,n}$. If $V := x_0 \oplus y \oplus T_{x_0}\gamma$ has signature $(2, 1)$, then $\pi_0(y)$ is a spacelike line in F .*

Proof. The vector $\pi_0(y)$ is in the orthogonal in V of the space W generated by x_0 and $T_{x_0}\gamma$; but W has signature $(1, 1)$, thus $\pi_0(y)$ is spacelike. $\square$

According to this proposition, π_0 maps points in $\gamma \setminus \{x_0\}$ to positive definite lines in F . If we identify the set of positive definite lines in F with the n -dimensional hyperbolic space $\mathbf{H}^n$, we obtain a curve $\pi_{x_0}(\gamma) \subset \mathbf{H}^n$ that we call *the angular projection of γ at x_0* .

Definition 3.17 (ANGULAR WIDTH). The *angular width* of a compact strongly positive curve γ is

$$w(\gamma) := \sup \{ \text{diam}(\pi_{x_0}(\gamma')) \mid x_0 \in \gamma \} ,$$

where the diameter is computed in $\mathbf{H}^n$.

Proposition 3.18 (ANGULAR WIDTH AND SPACELIKE SURFACES). *Let γ be strongly positive curve in $\mathbf{H}_+^{2,n}$. Let S be a spacelike surface with $\partial S \subset \gamma$. Assume that S is included in the convex hull of γ . Then for all points y in ∂S , we have*

$$d(T_y S, T_y^{(2)} \gamma) \leq w(\gamma) .$$

Proof. Let γ , x_0 , and π_0 be as above. Since π_0 is a linear map and thus preserves convex hulls, $\pi_0(S)$ is included in the convex hull of $\pi_0(\gamma)$. Let c be a curve in S starting from x_0 that is orthogonal to ∂S . We parametrize c by arc length so that $c(0) = x_0$. Observe first that

$$\lim_{s \rightarrow 0} \pi_0(c(s)) = \pi_0(\dot{c}(0)) .$$

It follows that $\dot{c}(0)$ belongs to the convex hull (in $\mathbf{H}^{2,n}$) of $\pi_0(\gamma)$. Similarly

$$\lim_{s \rightarrow 0} \pi_0(\gamma(s)) = \pi_0(n_0) .$$

where n_0 and $T_{x_0}\gamma$ generates $T_{x_0}^{(2)}\gamma$. It follows that

$$d_{\mathcal{G}}(T_{x_0} S, T_{x_0}^{(2)} \gamma) = d_{\mathbf{H}^n}(\dot{c}(0), n_0) \leq w(\gamma) .$$

This concludes the proof. $\square$

3.2.3. Deformation of strongly positive curves. We now introduce the class of curves for which we prove a finite Plateau problem.

Definition 3.19. A *deformation* of a strongly positive curve γ in $\mathbf{H}^{2,n}$ is an isotopy $\{\gamma_t\}_{t \in [0,1]}$ with $\gamma = \gamma_1$ such that

- (i) every curve γ_t is strongly positive,
- (ii) the curve γ_0 lies in a hyperbolic plane.

A strongly positive curve admitting a deformation is called *deformable*.

Observe that by compactness of the isotopy, if $\{\gamma_t\}_{t \in [0,1]}$ is a deformation, the angular width $w(\gamma_t)$ is uniformly bounded.

3.3. Maximal surfaces.

3.3.1. *Second fundamental form.* Consider a spacelike embedding $u : S \hookrightarrow M$, where $(M, \mathbf{g})$ is a pseudo-Riemannian manifold of signature $(2, n)$ and S a surface. The pull-back bundle u^*TM splits orthogonally as

$$u^*TM = TS \oplus NS ,$$

where the *normal bundle* NS is the orthogonal of the tangent bundle TS . We denote their induced metric by g_I and g_N respectively. Observe that g_I is positive definite, while g_N is negative definite.

We recall that *second fundamental form* Π , which is a symmetric 2-tensor on S with values in N , and the *shape operator* B which is a 1-form on S with values in $\text{Hom}(NS, TS)$ are given by

$$g_I(Y, B(X)\xi) = g_N(\Pi(X, Y), \xi) = \mathbf{g}(\nabla_X Y, \xi) ,$$

where X and Y are vector fields along S , ξ is a section of the normal bundle and ∇ the Levi-Civita connection on M . For instance the second fundamental Π_0 form on $T_x \mathbf{H}^{2,n}$, where we see $\mathbf{H}^{2,n}$ locally isometrically embedded in E is given by

$$\Pi_0(u, u) = \mathbf{g}(u, u) x .$$

Thus if γ is a geodesic in $\Sigma \subset \mathbf{H}^{2,n} \subset E$, we have

$$\left. \frac{d^2}{dt^2} \right|_{t=s} \gamma(t) = \Pi(u, u) + \mathbf{g}(u, u) x . \quad (7)$$

The *norm* of the second fundamental form Π is

$$\|\Pi\|^2 := - \max_{|v|=1} \sum_{i=1,2} \mathbf{g}(\Pi(v, e_i), \Pi(v, e_i)) , \quad (8)$$

where (e_1, e_2) is an orthonormal basis of $T_x S$.

If M_λ denotes the manifold M equipped with the metric $\mathbf{g}_\lambda = \frac{1}{\lambda} \mathbf{g}$, we have the following:

Lemma 3.20. *Let S be a spacelike surface in M , with fundamental form Π . The second fundamental form Π_λ of S in M_λ satisfies $\|\Pi_\lambda\|^2 = \lambda \|\Pi\|^2$.*

Proof. Observe that a unit vector with respect to $\mathbf{g}_\lambda$ has the form $\sqrt{\lambda}v$, where v is a unit vector for $\mathbf{g}$. The result then follows from tracking the effect of the subsequent substitutions in (8), after observing that the Levi-Civita connection is a conformal invariant. $\square$

Definition 3.21. The *mean curvature* is the normal vector field

$$H := \text{tr}_{g_I}(\Pi) = \Pi(e_1, e_1) + \Pi(e_2, e_2) .$$

3.3.2. *Variation of the area.* Given a spacelike embedding $u : S \hookrightarrow M$, one can define the *area functional*, associating to any compact set K in S the number

$$\mathcal{A}_K(u) := \int_K \text{dvol}_{g_I} ,$$

where dvol_{g_I} is the volume form of g_I . The following is classical, but we include the proof for the sake of notation.

Lemma 3.22. *A spacelike surface $u_0 : S \hookrightarrow M$ is a critical point of the area functional if and only if $H = 0$.*

Proof. Let ξ be a normal vector field with compact support. Let $\{u_t\}_{t \in (-\varepsilon, \varepsilon)}$ be a smooth deformation of u_0 so that $t \rightarrow u_t(x)$ are geodesics with initial tangent vector ξ . For ε small enough, the image $u_t(S)$ is spacelike. We respectively denote by g_t, B_t and Π_t the induced metric, shape operator and second fundamental form of u_t .

Let G and ∇ be respectively the pull-back of the metric and connection of M by $U : (t, x) \rightarrow u_t(x)$. The metric G restricts to g_t on $S \times \{t\}$. We have

$$\partial_t G(X, Y) = G(\nabla_\xi X, Y) + G(X, \nabla_\xi Y) = G(\nabla_X \xi, Y) + G(X, \nabla_Y \xi) .$$

Thus, restricting to $S \times \{0\}$, we obtain

$$\dot{g}_0(X, Y) := \frac{d}{dt} \Big|_{t=0} g_t(X, Y) = -2g_0(B_0(X)\xi, Y) = -2g_N(\Pi_0(X, Y), \xi) . \quad (9)$$

In particular, if (e_1, e_2) is an orthonormal framing of (TS, g_0) , and (e^1, e^2) its dual, we have

$$\mathrm{dvol}_{g_t} = \det \left(\mathrm{Id} - 2t \left(g_N \left(\Pi_0(e_i, e_j), \xi \right) \right)_{i,j=1,2} + o(t) \right) e^1 \wedge e^2 .$$

It follows that

$$\dot{\mathrm{dvol}}_{g_t} = -2g_N(H, \xi) \mathrm{dvol}_{g_0} . \quad (10)$$

Thus $\dot{\mathcal{A}}_K(u_0) = -2 \int_K g_N(H, \xi) \mathrm{dvol}_{g_0}$ and the result follows. $\square$

We now compute the second variation of the area functional.

Proposition 3.23. *Given a spacelike surface $u_0 : S \hookrightarrow M$ with $H = 0$, a non-zero normal deformation along $\xi \in \Omega^0(S, NS)$ and $K \subset S$ a compact subset, we have*

$$\ddot{\mathcal{A}}_K(u_0) = \int_K W(\xi) \mathrm{dvol}_{g_0} ,$$

where $W(\xi) = 2 \operatorname{tr}_{g_0}(Q_\xi)$ for the symmetric tensor Q_ξ given by

$$Q_\xi(X, Y) = g_0((R_0(\xi, X)\xi, Y) - g_0(B_0(X)\xi, B_0(Y)\xi) + g_N(\nabla_X^N \xi, \nabla_Y^N \xi)) .$$

Here R_0 is the Riemann curvature tensor of u_0^*D and D the Levi-Civita connection of M .

Proof. We use the same notation as the previous proof. By equation (10) when $H = 0$, we have

$$\ddot{\mathrm{dvol}}_{g_0} = -2g_N(\nabla_\xi H, \xi) \mathrm{dvol}_{g_0} = W(\xi) \mathrm{dvol}_{g_0} ,$$

where $W(\xi) = 2 \operatorname{tr}_{g_0}(Q_\xi)$ for the symmetric tensor Q_ξ defined by

$$Q_\xi(X, Y) = -g_0(\nabla_\xi B_0(X)\xi, Y) .$$

Our goal is to compute $Q_\xi(X, Y)$.

On the one hand, we have

$$\begin{aligned} \partial_t G(\nabla_X \xi, Y) &= G(\nabla_\xi \nabla_X \xi, Y) + G(\nabla_X \xi, \nabla_\xi Y) \\ &= G(\nabla_\xi \nabla_X \xi, Y) + G(\nabla_X \xi, \nabla_Y \xi) \\ &= G(R(\xi, X)\xi, Y) + G(\nabla_X \nabla_\xi \xi, Y) + G(\nabla_{[\xi, X]} \xi, Y) + G(\nabla_X \xi, \nabla_Y \xi) \\ &= G(R(\xi, X)\xi, Y) + G(\nabla_X \xi, \nabla_Y \xi) , \end{aligned}$$

where $R(a, b)c = \nabla_a \nabla_b c - \nabla_b \nabla_a c - \nabla_{[a, b]} c$.

Restricting to $S \times \{0\}$, we obtain

$$\frac{d}{dt} \Big|_{t=0} g_t(-B_t(X)\xi, Y) = R_0(\xi, X, \xi, Y) + g_0(B_0(X)\xi, B_0(Y)\xi) + g_N(\nabla_X^N \xi, \nabla_Y^N \xi) .$$

On the other hand, using equation (9), we thus get

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} g_t(-B_t(X)\xi, Y) &= \dot{g}_0(-B_0(X)\xi, Y) + g_0(-\nabla_\xi B_0(X)\xi, Y) \\ &= 2g_0(B_0(X)\xi, B_0(Y)\xi) + Q_\xi(X, Y) . \end{aligned}$$

This gives

$$Q_\xi(X, Y) = R_0(\xi, X, \xi, Y) - g_0(B_0(X)\xi, B_0(Y)\xi) + g_N(\nabla_X^N \xi, \nabla_Y^N \xi) . \quad (11)$$

□

Corollary 3.24. *If $u : S \hookrightarrow \mathbf{H}^{2,n}$ is a spacelike surface with $\mathbf{H} = 0$, and let ξ be a non-zero normal deformation supported on a compact $K \subset S$. Then the second variation of the area satisfies*

$$\ddot{\mathcal{A}}_K(u) \leq 4 \int_K g_N(\xi, \xi) \mathrm{dvol}_{g_0} < 0 .$$

Proof. Because $\mathbf{H}^{2,n}$ has curvature -1 , we have

$$g_0(R_0(\xi, X)\xi, Y) = g_N(\xi, \xi)g_0(X, Y) .$$

Thus, we obtain

$$\mathrm{tr}_{g_0}(Q_\xi) = 2g_N(\xi, \xi) + R ,$$

where $R = \mathrm{tr}_{g_0}(-g_0(B_0(\cdot)\xi, B_0(\cdot)\xi) + g_N(\nabla^N \xi, \nabla^N \xi)) \leq 0$. This concludes the proof. □

3.3.3. *Maximal surfaces.* Corollary 3.24 motivates the following definition

Definition 3.25. A spacelike immersion from a surface S to M is a *maximal surface* if $\mathbf{H} = 0$.

In the sequel, we will denote by Σ a maximal surface, and S any surface.

Corollary 3.24 implies the stability of maximal surfaces in $\mathbf{H}^{2,n}$ and $\mathbf{H}_+^{2,n}$: given a maximal surface u , there is no non-zero compactly supported normal deformation with $\ddot{\mathcal{A}}(u) = 0$.

Calculating the tangential part of the curvature tensor of ∇ , one obtains the following

Proposition 3.26. *If Σ is a maximal surface in M and P is a tangent plane to Σ equipped with an orthonormal frame (e_1, e_2) , then we have*

$$K_\Sigma(P) = K_M(P) - q_N(\Pi(e_1, e_1)) - q_N(\Pi(e_1, e_2)) ,$$

where K_Σ and K_M are the sectional curvatures of Σ and M , respectively, and q_N is the (negative definite) quadratic form on N .

In particular if $M = \mathbf{H}^{2,n}$, since q_N is negative definite, $K_\Sigma \geq -1$.

3.3.4. *Convex hull.* Recall from paragraph 2.5.3 that any semi-positive loop in $\mathbf{H}^{2,n}$ has a well-defined convex hull. The following was proved in [21, Proposition 3.26] for the asymptotic boundary case. The proof is the same in our case, but we include it for the sake of completeness.

Proposition 3.27. *If Σ is a maximal surface in $\mathbf{H}^{2,n}$ whose total boundary Λ (see Definition 3.9) is a semi-positive loop, then Σ is contained in the convex hull of Λ .*

Proof. Consider a connected component Σ_0 of the preimage of Σ in $\mathbf{H}_+^{2,n}$, and let φ_0 be a linear form on E which is positive on Λ and denote by φ its restriction to Σ . Now, let γ be a

geodesic on Σ that we consider as a curve in E , with $\dot{\gamma}(0) = u$, then using equation (7) in the last equality

$$\text{Hess}_x \varphi(u, u) = \left. \frac{d^2}{dt^2} \right|_{t=0} \varphi(\gamma(t)) = \varphi_0 \left(\left. \frac{d^2}{dt^2} \right|_{t=0} \gamma(t) \right) = \mathbf{q}(u) \varphi(x) + \varphi_0(\Pi(u, u)),$$

where Π is the second fundamental form of Σ and $\text{Hess}_x \varphi$ is the Hessian of φ at x . Taking the trace yields

$$\Delta \varphi = 2\varphi.$$

The classical maximum principle thus implies that φ is positive on Σ and so Σ is contained in the convex hull of Λ . $\square$

3.4. Gauß lift and holomorphic curves. Recall from Subsection 2.3 that the Grassmannian $\mathcal{G}(M)$ of positive definite 2-planes in M is the fiber bundle over M whose fiber over x is the Riemannian symmetric space $\text{Gr}_{2,0}(\mathbf{T}_x M)$. The tangent space of $\mathcal{G}(M)$ at (x, P) splits as

$$\mathbf{T}_{(x,P)} \mathcal{G}(M) = \mathbf{T}_x M \oplus \text{Hom}(P, P^\perp) = P \oplus P^\perp \oplus \text{Hom}(P, P^\perp).$$

Furthermore, the canonical Riemannian metric g on $\mathcal{G}(M)$ is given by

$$g = (g|_P, -g|_{P^\perp}, h_P),$$

where h_P is the Riemannian metric on the fiber defined in paragraph 2.3.1.

Given a spacelike surface $u : S \hookrightarrow M$, we define its *Gauß lift* by

$$\Gamma : \begin{cases} S & \longrightarrow \mathcal{G}(M) \\ x & \longmapsto (u(x), \mathbf{T}u(\mathbf{T}_x S)) \end{cases}.$$

One easily checks that in the splitting describe above, $\mathbf{T}\Gamma = (\mathbf{T}u, \Pi)$ where $\Pi \in \Omega^1(S, \text{Hom}(\mathbf{T}S, \mathbf{N}S))$ is the second fundamental form.

We denote by g_Π and d_Π the induced metric and distance respectively on S by Γ .

Proposition 3.28. *For any spacelike surface we have*

$$g_\Pi = g_I + Q \leq (1 + \|\Pi\|^2) g_I,$$

where $Q(x, y) = -\text{tr}_{g_I} (g_N (\Pi(x, \cdot), \Pi(y, \cdot)))$

Proof. As noted above, $\mathbf{T}\Gamma = (\mathbf{T}u, \Pi) \in \Omega^1(S, \mathbf{T}S \oplus \text{Hom}(\mathbf{T}S, \mathbf{N}S))$. The first term in the expression of $\Gamma^* g$ is clear, so we just have to explain the second one. We recall from subsection 2.3.1 the bilinear form $h_P(\varphi, \psi) = -\text{tr}(\varphi^* \psi)$ where $\varphi, \psi \in \mathbf{T}_P \text{Gr}_{2,0}(E)$ and $\varphi^* : P^\perp \rightarrow P$ is the adjoint of φ using the induced scalar product. In particular, the second term is given by $h_P(\Pi(x, \cdot), \Pi(y, \cdot)) = -\text{tr}_{g_I} (\Pi^*(x, \cdot) \Pi(y, \cdot))$.

Using $\Pi^* = B$, and taking an orthonormal framing (e_1, e_2) of $(\mathbf{T}S, g_I)$, the second term may then be written as

$$-\sum_{i=1,2} g_I(B(x) \Pi(y, e_i), e_i) = \sum_{i=1,2} -g_N(\Pi(x, e_i), \Pi(y, e_i)) = Q(x, y).$$

This proves the result. $\square$

Given a point x on a spacelike acausal surface Σ in $\mathbf{H}^{2,n}$, we can define three Riemannian metrics on Σ : the metric g_I induced by the metric on $\mathbf{H}^{2,n}$, the metric g_Π induced by the Gauß lift and the metric $g_H^x := \pi_x^* g_H$ where π_x is the warped projection on the pointed hyperbolic plane (x, H) tangent to Σ at x , and g_H is the hyperbolic Riemannian metric on H .

Corollary 3.29. *Let S be a spacelike surface whose second fundamental form is uniformly bounded by M ,*

- (i) the Riemannian tensors g_I, g_{II} are uniformly equivalent, with a constant only depending on M .
- (ii) For any R , the metrics g_I and g_H^x are uniformly equivalent on the ball of center x and radius R with respect to g_I with a constant only depending on M and R . Moreover the projection π_x from S equipped with d_I and d_{II} is Lipschitz.

Proof. By the previous proposition, the bound on the norm of Π_Σ implies that g_I and g_{II} are biLipschitz. Moreover, the Gauß lift of a ball B_R with respect to g_I in $\mathcal{G}(\mathbf{H}_+^{2,n})$ is contained in the closed ball of center $T_x S$ and radius $R(1 + M)$. In particular, the restriction of the warped projection π_x to B_R is infinitesimally biLipschitz by Proposition 3.13. This shows that g_I and g_H^x are uniformly equivalent. $\square$

Corollary 3.30. *If S is a properly immersed spacelike surface without boundary whose second fundamental form is uniformly bounded, then (S, g_I) is complete.*

Proof. If S is properly immersed, it is thus a global graph over each of its tangent planes. Thus by the previous corollary, for each x in S , the ball of radius 1 with center x with respect to g_I is complete, since the ball of radius 1 with center x with respect to g_H^x is complete. This shows that (S, g_I) is complete. $\square$

3.4.1. *Holonomic distribution.* Using the splitting

$$T_{(x,P)}\mathcal{G}(M) = T_x M \oplus \text{Hom}(P, P^\perp), \quad (12)$$

where $T_x M$ is the horizontal distribution we define the holonomic distribution $\mathcal{D}$ on $\mathcal{G}(M)$ by

$$\mathcal{D}_{(x,P)} := P \oplus \text{Hom}(P, P^\perp). \quad (13)$$

The following is straightforward.

Lemma 3.31. *The Gauß lift of a spacelike surface in M is tangent to the holonomic distribution.*

3.4.2. *Almost complex structure and holomorphic Gauß lift.* The holonomic distribution $\mathcal{D}$ carries a natural almost-complex structure J defined by taking the rotation i of angle $\frac{\pi}{2}$ on P and the pre-composition by i on $\text{Hom}(P, P^\perp)$:

$$J(u, A) = (iu, A \circ i). \quad (14)$$

The following is classical (see [32]).

Proposition 3.32. *A spacelike surface $u : S \hookrightarrow M$ is maximal if and only if its Gauß lift $\Gamma : S \rightarrow \mathcal{G}(M)$ is J -holomorphic when S is equipped with the complex structure j induced by g_I .*

Proof. Considering the splitting $u^*TM = TS \oplus NS$, we get $\Gamma^*\mathcal{D} = TS \oplus \text{Hom}(TS, NS)$, where $\mathcal{D}$ is the distribution on $\mathcal{G}(M)$ defined in (13). In particular, $T\Gamma \in \Omega^1(S, TS \oplus \text{Hom}(TS, NS))$ is identified with (Tu, Π) .

The first factor is clearly J -holomorphic. For the second factor, it follows from the observation that a map $\varphi : T_x S \rightarrow \text{Hom}(T_x S, N_x S)$ satisfies $\varphi \circ j = J \circ \varphi$ if and only if φ is symmetric and trace-less. $\square$

Definition 3.33 (BOUNDARY CONDITION). Let γ be a strongly positive curve (Definition 3.14).

- (i) The boundary condition associated to γ is the immersed submanifold

$$W(\gamma) := \{(x, P) \in \mathcal{G}(\mathbf{H}^{2,n}) \mid x \in \gamma, T_x \gamma \subset P\}. \quad (15)$$

- (ii) For any positive number K , the local boundary condition, is the open subset of $W(\gamma)$

$$W_K(\gamma) := \{(x, P) \in W(\gamma) \mid d(P, T_x^{(2)} \gamma) < K\}. \quad (16)$$

We have

Proposition 3.34. *Let γ be a strongly positive curve of regularity $C^{k,\alpha}$.*

- (i) $W(\gamma)$ is a submanifold of regularity $C^{k-1,\alpha}$ tangent to the holonomic distribution.
- (ii) $TW(\gamma)$ is a totally real subspace of half the dimension of the holonomic distribution.
- (iii) Finally, let ω be the orthogonal projection from $T\mathcal{G}(\mathbf{H}^{2,n})$ to the horizontal distribution, then if (x, P) belongs to $W(\gamma)$, we have $\omega(T_P W(\gamma)) = T_x \gamma$.

Proof. The first item is obvious. The last item follows from the definition of $W(\gamma)$. Let (x, P) be in $W(\gamma)$. Using the splitting (12),

$$T_{(x,P)} W(\gamma) = T_x \gamma \oplus \{A \in \text{Hom}(P, P^\perp) \mid T_x \gamma \subset \ker(A)\}.$$

It follows from the definition (13), that $T_{(x,P)} W(\gamma) \subset \mathcal{D}_{(x,P)}$. Moreover the definition (14) of the almost complex structure J implies that

$$T_{(x,P)} W(\gamma) \oplus JT_{(x,P)} W(\gamma) = \mathcal{D}_{(x,P)}.$$

This completes the proof. $\square$

4. UNIQUENESS

This section is devoted to the proof of the following result.

Theorem 4.1 (UNIQUENESS). *Let Σ be a complete maximal surface in $\mathbf{H}^{2,n}$ whose total boundary Λ is either finite and positive or asymptotic (see Definition 3.9). Then Σ is the unique complete maximal surface bounded by Λ .*

This statement was proved in [21, Theorem 3.21] in the case of a cocompact group action on Σ , and the proof relies on a maximum principle. We will adapt this maximum principle here to the non-compact case using a weak version of Omori's maximum principle. Such an adaptation was made in the case of $\mathbf{H}^{2,2}$ for polygonal surfaces in [44] and suggested to us by the first author of that work.

We work by contradiction. Suppose there exists two maximal surfaces Σ_1 and Σ_2 sharing a common boundary, denoted by Λ . By Proposition 3.27, both surfaces are contained in the convex hull $\mathcal{CH}(\Lambda)$ of Λ . Lift of $\mathcal{CH}(\Lambda)$ to $\mathbf{H}_+^{2,n}$ and recall that the scalar product of any pair of points in this lift is negative by Proposition 2.15, item (ii). This defines a lift of Σ_1 and Σ_2 in $\mathbf{H}_+^{2,n}$ that we denote the same way.

Consider the function

$$B : \begin{cases} \Sigma_1 \times \Sigma_2 & \longrightarrow \mathbb{R}, \\ (x, y) & \longmapsto \langle x, y \rangle. \end{cases}$$

We remark that since B is negative everywhere, B is bounded from above.

4.1. Lower bound on the Hessian. We prove the following estimate.

Lemma 4.2. *Let $p = (x, y)$ be a point in $\Sigma_1 \times \Sigma_2$. Then there exists two unit vectors u_0 and v_0 in $T_x \Sigma_1$ and $T_y \Sigma_2$ respectively such that, for $w_0 = (u_0, v_0)$, we have*

$$\text{Hess}_p B(w_0, w_0) \geq 2B(p) + 2. \quad (17)$$

We first compute the Hessian of B .

Lemma 4.3. *The Hessian of B at a point $p = (x, y)$ in $\Sigma_1 \times \Sigma_2$ in the direction $w = (u, v) \in T_x \Sigma_1 \times T_y \Sigma_2$ is given by*

$$\text{Hess}_p B(w, w) = (q(u) + q(v))B(p) + 2\langle u, v \rangle + \langle \Pi_1(u, u), y \rangle + \langle x, \Pi_2(v, v) \rangle.$$

Here Π_i is the second fundamental form of Σ_i .

Proof. Let γ_1 be a geodesic in Σ_1 with $\dot{\gamma}_1(0) = u$, while γ_2 is a geodesic in Σ_2 with $\dot{\gamma}_2(0) = v$, then, using equation (7) in the last equality

$$\begin{aligned} \text{Hess}_p B(w, w) &= \left. \frac{d^2}{dt^2} \right|_{t=0} \langle \gamma_1(t), \gamma_2(t) \rangle \\ &= \left\langle \left. \frac{d^2}{dt^2} \right|_{t=0} \gamma_1(t), y \right\rangle + 2\langle u, v \rangle + \left\langle x, \left. \frac{d^2}{dt^2} \right|_{t=0} \gamma_2(t) \right\rangle \\ &= \langle \mathbf{q}(u) x + \Pi_1(u, u), y \rangle + 2\langle u, v \rangle + \langle x, \mathbf{q}(v) y + \Pi_2(v, v) \rangle, \end{aligned}$$

and the result follows. $\square$

Proof of Lemma 4.2. Since the surfaces Σ_1 and Σ_2 are maximal, the quadratic forms $\langle \Pi_1(\cdot, \cdot), y \rangle$ and $\langle x, \Pi_2(\cdot, \cdot) \rangle$ have opposite eigenvalues $\pm\lambda_1$ and $\pm\lambda_2$ respectively. Thus at a given point $p = (x, y) \in \Sigma_1 \times \Sigma_2$, up to switching Σ_1 and Σ_2 , we may assume $\lambda_1 \geq \lambda_2 \geq 0$. Observe then that for any unit vector v in $v \in T_y \Sigma_2$ we have

$$\langle x, \Pi_2(v, v) \rangle \geq -\lambda_2 \geq -\lambda_1.$$

Now let us choose a unit vector $u_0 \in T_x \Sigma_1$ such that $\langle \Pi_1(u_0, u_0), y \rangle = \lambda_1$. Let $w = (u, v)$, then we obtain from lemma 4.3 that for any unit v in $T_y \Sigma_2$,

$$\text{Hess}_p B((u_0, v), (u_0, v)) = 2B(p) + 2\langle u_0, v \rangle + \langle \Pi_1(u_0, u_0), y \rangle + \langle x, \Pi_2(v, v) \rangle \geq 2B(p) + 2\langle u_0, v \rangle.$$

It is now enough in order to conclude the proof of the lemma to find a unit vector v_0 such that $\langle u_0, v_0 \rangle \geq 1$.

Let π be the orthogonal projection from E to $T_x \Sigma_1$. Since the kernel of π is negative definite, we have $\mathbf{q}(a) \leq \mathbf{q}(\pi(a))$ for any $a \in E$. Because $T_y \Sigma_2$ is positive definite, it follows that the restriction of π to $T_y \Sigma_2$ is a linear isomorphism. Let v_1 to be the unique vector such that $\pi(v_1) = u_0$ and observe that $0 < \mathbf{q}(v_1) \leq \mathbf{q}(u_0) = 1$. Finally, let v_0 the unit vector defined by $v_0 = \sqrt{\mathbf{q}(v_1)}^{-1} v_1$. Then

$$\langle u_0, v_0 \rangle = \sqrt{\mathbf{q}(v_1)}^{-1} \langle u_0, v_1 \rangle = \sqrt{\mathbf{q}(v_1)}^{-1} \langle u_0, \pi(v_1) \rangle = \sqrt{\mathbf{q}(v_1)}^{-1} \geq 1.$$

The result now follows. $\square$

4.2. A maximum principle. The following is a weaker version of Omori's maximum principle [41].

Proposition 4.4 (MAXIMUM PRINCIPLE). *Let M be a complete Riemannian manifold without boundary whose sectional curvature is bounded from below. Let f be a function of M satisfying the following:*

- (i) *The function f is of class C^2 .*
- (ii) *There are positive constants A, Λ so that, if $f(x) > A$, then there is a non-zero vector v in $T_x M$ such that*

$$\text{Hess}_x f(v, v) \geq \Lambda \|v\|^2.$$

Then either f is bounded by A , or f is unbounded.

We will denote by $B(x_0, r)$ the ball in M of center x_0 and radius r , and d_{x_0} the function distance to x_0 . Recall the classical Hessian comparison theorem (see for instance [9, Chapter 1]).

Proposition 4.5 (HESSIAN COMPARISON PRINCIPLE). *Let M be a complete Riemannian manifold whose sectional curvature is bounded below. There exists positive constants ε and λ such that for every point x_0 in M , and any vector $v \in T_{x_0}M$ with x in $B(x_0, \varepsilon)$, we have*

$$\text{Hess}_x d_{x_0}^2(v, v) \leq \lambda \|v\|^2 ,$$

Proof of Proposition 4.4. Let ε be as in Proposition 4.5 and A and Λ as in the statement of the proposition. Let $\kappa > 0$ be chosen so that for every x , we have the following inequality: for every z in $B(x, \varepsilon)$,

$$\kappa \text{Hess}_z d_y^2(v, v) \leq \frac{\Lambda}{2} \|v\|^2 .$$

To prove the proposition, it is enough to show that if there exists x_0 , with $f(x_0) > A$, then there exists y with $f(y) \geq f(x_0) + \kappa\varepsilon$. Choose x_0 so that $f(x_0) > A$. Let now y achieve the maximum of $g := f - \kappa d_{x_0}$ on the ball $B(x_0, \varepsilon)$. Observe that at this maximum, the inequality $g(y) \geq g(x_0)$ reads

$$f(y) \geq f(x_0) + \kappa d^2(x_0, y) . \quad (18)$$

In particular, $f(y) \geq f(x_0) > A$. Thus there exists v so that $\text{Hess}_y f(v, v) > \Lambda \|v\|^2$, hence

$$\text{Hess}_y g(v, v) = \text{Hess}_y f(v, v) - \kappa \text{Hess}_y d_{x_0}^2(v, v) \geq \frac{\Lambda}{2} \|v\|^2 > 0 .$$

Hence y cannot be a point in the interior of $B(x_0, \varepsilon)$. Thus $d(x_0, y) = \varepsilon$. The inequality (18) now reads

$$f(y) \geq f(x_0) + \kappa\varepsilon^2 .$$

This concludes the proof. $\square$

4.3. Proof of Theorem 4.1. We will combine two lemmas.

Lemma 4.6. *For any x in Σ_1 , the supremum M_x of B on $\{x\} \times \Sigma_2$ is greater or equal to -1 , with equality if and only if x belongs to Σ_2 .*

Proof. Consider a pointed hyperbolic plane $\mathbf{P} = (x, H)$ tangent to Σ_1 at a point x in the interior of Σ_1 . In the non-compact case, both surfaces are graphs above the entire $\mathbf{P}$ while in the compact case, they are graphs above the compact domain bounded by the image of the warped projection of their common boundary.

In particular, the fiber above x of the warped projection on $\mathbf{P}$ is a totally geodesic timelike sphere that intersects Σ_2 in a unique point y ; furthermore, the geodesic passing through x and y is timelike. This gives

$$B(x, y) = \frac{1}{2} (\langle x, x \rangle + \langle y, y \rangle - \langle x - y, x - y \rangle) \geq -1 .$$

with equality if and only if $y = x$.

Assume conversely that x belongs to both Σ_1 and Σ_2 . Then, for all y in Σ_2 , the arc $[x, y]$ is spacelike since Σ_2 is achronal. Thus

$$B(x, y) = \frac{1}{2} (\langle x, x \rangle + \langle y, y \rangle - \langle x - y, x - y \rangle) \leq -1 = B(x, x) .$$

This concludes the proof. $\square$

We now prove

Lemma 4.7. *The supremum M of B on $\Sigma_1 \times \Sigma_2$ is equal to -1 .*

Proof. We first consider the compact case. Let m be the point where the function B achieves its maximum M on $\Sigma_1 \times \Sigma_2$.

- (i) Assume first that $m = (x, y)$ where x and y belong to the interior of Σ_1 and Σ_2 respectively. At such a point m , the Hessian of B is non-positive. Then Lemma 4.2 says

$$M = B(m) \leq -1.$$

- (ii) Assume now that $m = (x, y)$ where (say) x belongs to $\partial\Sigma_1$ (the case where y belongs to $\partial\Sigma_2$ is treated in a symmetric fashion). Since x then also belongs to Σ_2 , by Lemma 4.6

$$-1 = M_x = M.$$

In both situations $M \leq -1$.

In the non-compact case, we follow a similar argument using the weak version of the Omori maximum principle Proposition 4.4. First, note that the Riemannian manifold $\Sigma_1 \times \Sigma_2$ has sectional curvature larger than -1 by Proposition 3.26. For any $x = (x_1, x_2)$ with $B(x) > -1 + \delta$ with $\delta > 0$, Lemma 4.2 implies that there exists w tangent to x with

$$\text{Hess}_x B(w, w) \geq (2B(x) + 2) \|w\|^2 \geq 2\delta \|w\|^2.$$

Since we noted after its definition that B is bounded from above, Proposition 4.4 then implies that B is bounded by $-1 + \delta$. Since this is true for all δ , the function B is bounded by -1 . Thus $M \leq -1$ in the non-compact case as well. $\square$

We can now conclude the proof of the Uniqueness Theorem 4.1. Combining the two Lemmas 4.6 and 4.7, we obtain that for all x in Σ_1 , we have $M_x = -1$. Thus, using the equality case in the same Lemma 4.6, we obtain that x belongs to Σ_2 . Thus Σ_1 is equal to Σ_2 . This concludes the proof.

5. MAIN COMPACTNESS THEOREM

The main result of this section is the following compactness theorem concerning complete surfaces.

We start with a definition.

Definition 5.1. A sequence of complete acausal surfaces $\{\Sigma_k\}_{k \in \mathbb{N}}$ with boundary converges as a graph over an open subset V of a pointed hyperbolic plane $\mathbf{P}_0$ associated to the warped projection π_0 , if, denoting $\Sigma_k^V := \Sigma_k \cap \pi_0^{-1}(V)$

- the sequence $\{U_k\}_{k \in \mathbb{N}}$, with $U_k := \pi_0(\Sigma_k^V)$ converges smoothly as open sets with smooth boundary in V to an open set with smooth boundary U_0 ,
- there exists a smooth complete acausal surface Σ_0 over $\pi_0^{-1}(U_0)$, so that $\Sigma_k^{U_k}$ converges to Σ_0 .

We will mainly use this definition of the convergence of complete acausal surfaces Σ_k in settings where the projections U_k are fixed, i.e. not varying with the parameter k . Recall the Gauß lift $\Gamma(S) \subset \mathcal{G}(M)$, described in section 3.4, of a surface in $S \subset M$ into its Grassmannian $\mathcal{G}(M)$.

Theorem 5.2 (COMPACTNESS THEOREM). *Let $\{\Sigma_k\}_{k \in \mathbb{N}}$ be a sequence of connected complete acausal maximal surfaces in $\mathbf{H}^{2,n}$, and let $\gamma_k := \partial\Sigma_k$ be the finite boundary of Σ_k . Assume that we have the following boundary conditions:*

- (i) *The sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ is strongly positive and uniformly unpinched.*
- (ii) *There is a positive constant A so that for all k , for all x in γ_k , we have $d(\mathbf{T}_{x_k} \Sigma_k, \mathbf{T}_{x_k}^{(2)} \gamma_k) \leq A$.*

- (iii) The sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ has C^∞ bounded geometry.
- (iv) There is a pointed hyperbolic plane $\mathbf{P}$ within a uniformly bounded distance of the Gauß lift $\Gamma(\Sigma_k)$.

Then, the sequence of surfaces $\{\Sigma_k\}_{k \in \mathbb{N}}$ converges as a graph on every bounded ball of $\mathbf{P}$.

The definition of bounded geometry and convergence for spacelike surfaces and strongly positive curves is given in Appendix A, the definition of uniformly unpinched is given in Definition 3.14, and the definition of finite boundary is given in Definition 3.9, while the definition of the Gauß lift is given in section 3.4.

The first three hypotheses can be thought of as a C^1 bound along the boundary, while the fourth one is an interior C^0 bound.

This theorem implies readily a uniform bound on the second fundamental form of complete acausal surfaces without boundary. Such a result is also a consequence of a result by Ishihara [31]. However the Ishihara bound is not optimal and we will improve upon it in our setting in a subsequent paper [36].

We next describe a bound on the second fundamental form in the non-complete case.

If an acausal maximal surface Σ is not complete, we define its frontier $\text{Fr}(\Sigma)$ and the distance $\delta(x, \text{Fr}(\Sigma))$ to the frontier as in paragraph 2.1.1 and Definition 3.9. We may refer to the non-complete case as the *free boundary* case. In this setting, we will have two results: we will have both a local bound on the geometry as well as a local compactness theorem.

Theorem 5.3. *Let $\{\Sigma_k\}_{k \in \mathbb{N}}$ be a sequence of connected acausal maximal surfaces in $\mathbf{H}^{2,n}$, and let $\gamma_k := \partial\Sigma_k$ be the finite boundary of Σ_k . Let also $\{x_k\}_{k \in \mathbb{N}}$ be a sequence of points so that x_k belongs to Σ_k .*

Assume that we have the following boundary conditions:

- (i) *The sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ is strongly positive and uniformly unpinched,*
- (ii) *There is a positive constant A so that for all k , for all x in γ_k , we have $d(\mathbf{T}_{x_k}\Sigma_k, \mathbf{T}_{x_k}^{(2)}\gamma_k) \leq A$.*
- (iii) *The sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ has C^∞ bounded geometry.*

Assume furthermore that $\delta(x_k, \text{Fr}(\Sigma_k))$ is bounded from below by a positive constant R . Then there exists a positive constant ε less than R so that for all k we have that

- *the second fundamental form of Σ_k is uniformly bounded on the ball Σ_k^ε (with respect to d_I) on Σ_k of center x_k and radius ε .*
- *the sequence $\{x_k, \Sigma_k^\varepsilon\}_{k \in \mathbb{N}}$ subconverges smoothly.*

In Section 6, we will describe three avatars of our compactness theorem.

5.1. Structure of the proof. The structure of the proof is as follows:

- (i) In Paragraph 5.2, we describe how to construct "good" neighborhoods of points on acausal surfaces: see Proposition 5.4.
- (ii) In Paragraph 5.3, we use this good neighborhood together with results on holomorphic curves to show local subconvergence under a uniform bound on the second fundamental form.
- (iii) In Paragraph 5.4, we extend this subconvergence globally, again under a uniform bound on the second fundamental form.
- (iv) In Paragraph 5.5, we prove a Bernstein type theorem: *complete maximal surfaces without boundary in the pseudo-Euclidean space $\mathbf{E}^{2,n}$ are spacelike planes*; we also prove a boundary version.
- (v) in Paragraph 5.6 we use the subconvergence, a renormalisation and the Bernstein type theorem to prove a uniform bound on the second fundamental form.

- (vi) We conclude the proof of the main compactness Theorem 5.2 in paragraph 5.7, also proving Theorem 5.3. .

5.2. Preliminary : constructing “good” neighborhoods. This paragraph is devoted to the proof of Proposition 5.4 below.

In order to apply the theory of holomorphic curves to prove our main compactness theorems, we want to find neighborhoods of points in an acausal surface that are homeomorphic to disks with *at most* one connected arc in the boundary. We also want to control the size of these neighborhoods: they should not be too small (with respect to the warped projection), and their Gauß lift should be uniformly bounded.

Let Σ be a spacelike surface in $\mathbf{H}_\lambda^{2,n}$. We denote by

- (i) d_I the induced metric on Σ from the metric on $\mathbf{H}_\lambda^{2,n}$, and area_I the corresponding area form,
- (ii) $\Gamma : \Sigma \rightarrow \mathcal{G}(\mathbf{H}_\lambda^{2,n})$ the Gauß lift, d_{II} the induced metric and area_{II} the corresponding area,
- (iii) $d_{\mathcal{G}}$ the metric on $\mathcal{G}(\mathbf{H}^{2,n})$.

Given a point x in Σ , let

- (i) $P_x = (x, H_x)$ be the pointed hyperbolic plane tangent to Σ at x (that is such that $T_x H_x = T_x \Sigma$),
- (ii) d_H be the metric induced on H_x by $\mathbf{H}_\lambda^{2,n}$, and U_R be the disk of center x and radius R in H_x , and
- (iii) π_x the warped projection from Σ to H_x defined by P_x ,

Proposition 5.4. *There exist constants A and δ_0 so that for any δ less than δ_0 , we have the following. Let Σ be an acausal surface in $\mathbf{H}_\lambda^{2,n}$ with $\lambda \leq 1$,*

- (i) *whose finite boundary $\partial\Sigma$ is δ -unpinched, and*
- (ii) *whose second fundamental form has norm bounded by 1.*

Then for all positive κ less than $\frac{1}{100}\delta$, any x in Σ with $d_I(x, \text{Fr}(\Sigma)) \geq \delta$ admits an open neighborhood $\mathring{\Sigma}_x$ in Σ homeomorphic to the disk with:

- (i) $\mathring{\Sigma}_x \cap \partial\Sigma$ *has at most one connected component,*
- (ii) *for all y in $\mathring{\Sigma}_x$, we have $d_{\mathcal{G}}(T_y \Sigma, T_x \Sigma) \leq A\kappa$,*
- (iii) *the subset $\mathring{\Sigma}_x$ is a graph over a subset V_x of the disk U_κ in H_x , and*
- (iv) *we have $d(x, \pi_x(\text{Fr}(\mathring{\Sigma})) \geq \frac{\kappa}{A}$.*

5.2.1. The construction: controlling projections of arcs. We assume throughout this paragraph that

Σ *is a spacelike acausal surface with second fundamental form of norm bounded by 1 and non-empty δ -unpinched boundary.*

Assuming that $U_R \cap \pi_x(\partial\Sigma)$ is non-empty, we choose w to be a closest point in $U_R \cap \pi_x(\partial\Sigma)$ to x :

$$d_H(x, w) = \inf\{(d_H(x, y) \mid y \in U_R \cap \pi_x(\partial\Sigma))\} .$$

Let ω the preimage of w .

Let $c_R = c_R(x, \Sigma)$ be the connected component of $U_R \cap \pi_x(\partial\Sigma)$ containing w , and γ_R be the preimage of c_R . The point w and the arc c_R are not uniquely chosen, and when U_R is disjoint from $\pi_x(\partial\Sigma)$, they do not exist.

We begin our approach to Proposition 5.4 by showing, under the background assumption that x is relatively distant to the image of the frontier of Σ , that short geodesic arcs in H_x may be lifted to arcs in Σ , assuming the geodesic arcs in H_x are near x .

Lemma 5.5 (LIFTING ARCS). *For any positive constants δ and ε , with $\varepsilon \leq \frac{1}{4}\delta$, and point x in Σ with $d_I(x, \text{Fr}(\Sigma)) \geq \delta$ the following holds. Let $c : [0, \varepsilon] \rightarrow H_x$ be a geodesic arc (parametrized by arclength) not intersecting $\pi_x(\partial\Sigma)$ except possibly at its extremities and such that*

$$d_I(\xi_0, x) \leq \varepsilon, \quad \pi_x(\xi_0) = c(0),$$

then c is contained in $\pi_x(\Sigma)$. Moreover, if $\pi_x(\xi_1) = c(\varepsilon)$, then

$$d_I(\xi_1, x) \leq 2\varepsilon, \quad d_G(\mathbf{T}_{\xi_1}\Sigma, \mathbf{T}_x\Sigma) \leq 4\varepsilon. \quad (19)$$

Proof. Let

$$I_0 := \{t \in I \mid \forall s \leq t, c(s) \in \pi_x(\Sigma)\}.$$

The set I_0 is open and non-empty. Let $\xi : I_0 \rightarrow \Sigma$ be the lift of c starting from ξ_0 . Since π_x is length-increasing, for all t in I_0 ,

$$d_I(\xi(t), x) \leq d_I(\xi(t), \xi_0) + d_I(\xi_0, x) \leq 2\varepsilon. \quad (20)$$

Hence we get,

$$d_I(\xi(t), \text{Fr}(\Sigma)) \geq d_I(x, \text{Fr}(\Sigma)) - d_I(\xi(t), x) \geq \frac{1}{2}\delta.$$

It follows that I_0 is closed, so $I_0 = [0, \varepsilon]$ and thus c lies in $\pi_x(\Sigma)$. The inequality (19) follows directly from the inequality (20), and the bound on the second fundamental form applied to Proposition 3.28. $\square$

Our final ingredient for the proof of Proposition 5.4 is a statement that, still assuming that x is reasonably distant from the frontier of Σ , that if the nearest component of the image of the boundary of Σ comes very near x , then that component is unique.

Lemma 5.6. *For any positive constant δ , there exists a constant $K \geq 1$ so that for any $R < \frac{1}{100}\delta$, the following holds.*

Choose x in Σ so that $c_R(x, \Sigma)$ is not empty. Assume that $d_I(x, \text{Fr}(\Sigma)) > \delta$, we have

- (i) *If the arc c_R intersects $U_{R/K}$, then c_R is the unique connected component of $\pi(\partial\Sigma) \cap U_R$ intersecting $U_{R/K}$.*
- (ii) *for all ζ in γ_R ,*

$$d_I(\zeta, x) \leq K \cdot R. \quad (21)$$

Proof. Let us prove the first assertion. Let $R' = \frac{R}{K}$ where we choose K in the sequel of the proof.

Assume that $\pi_x(\partial\Sigma)$ intersects $U_{R'}$. It follows that c_R intersects $U_{R'}$. Let w be a closest point to x in c_R . Thus w belongs to $U_{R'}$. Let ω be the preimage of w in Σ . Consider the geodesic arc from x to w whose length is less than R' and so less than $\frac{1}{4}\delta$. Applying the inequality (19), we obtain that

$$d_I(\omega, x) \leq 2R' \leq \frac{\delta}{50}.$$

We now prove

Assertion A: if $\zeta \in \partial\Sigma$ and $d_{\partial\Sigma}(\zeta, \omega) \leq 4R'$, then $\zeta \in \gamma_R$.

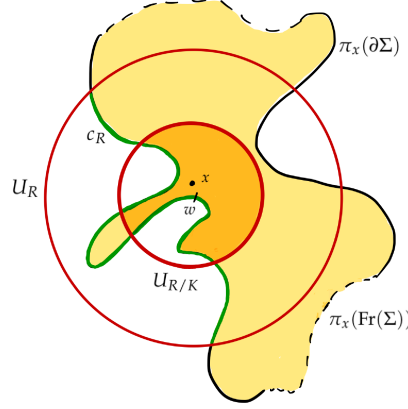


FIGURE 1. Lemma 5.6

Since $d_{\partial\Sigma}(\zeta, \omega)$ is finite, ζ and ω are by definition in the same connected component of $\partial\Sigma$. Let I be the arc along $\partial\Sigma$ from ζ to ω that we order from ω to ζ . Let F be the closed subset of I defined by

$$F = \left\{ \xi \in I \mid \forall \theta \in I, \theta \leq \xi \implies d_H(\pi_x(\theta), x) \leq \frac{R}{2} \right\}.$$

Obviously $F \subset \gamma_R$, as elements in F have projections to H that are within $\frac{R}{2}$ of x , while elements of γ_R have projections within R of x . The assertion will then follow from the fact that ζ belongs to F . We will thus show that $F = I$ using a connectedness argument.

- F is non-empty: since $d_H(\omega, x) \leq R' < \frac{R}{4}$, F contains ω . (Here we demand $K \geq 4$.)
- F is open: assume ξ belongs to F with $\xi < \zeta$. By continuity, and because elements of F project to within the smaller $\frac{R}{2}$ ball of x , while elements of γ_R project to the larger R -ball around x , we can find η in I , with $\zeta > \eta > \xi$ so that the interval J joining η to ξ lies in γ_R . By Lemma 3.29 there exists a constant k only depending on R , so that for σ in J ,

$$\begin{aligned} d_H(\pi_x(\omega), \pi_x(\sigma)) &\leq k \cdot d_I(\omega, \sigma) \\ &\leq k \cdot d_{\partial\Sigma}(\omega, \sigma) \\ &\leq k \cdot d_{\partial\Sigma}(\omega, \zeta) \leq 4k \cdot R'. \end{aligned}$$

Thus $d_H(x, \pi_x(\sigma)) \leq R'(1 + 4k)$. If we now choose $K > 2(1 + 4k)$, it follows that η belongs to F . This shows that F is open.

- Finally F is obviously closed.

We conclude that $F = I$ and so $\zeta \in F \subset V_{R/2}$, which implies that $\zeta \in \gamma_R$. This completes the proof of Assertion A.

Now, let ζ be in the intersection $\partial\Sigma \cap \pi_x^{-1}(U_R)$. The geodesic (in $\mathbf{H}^{2n}$) between ω and ζ is spacelike, and so contained in some hyperbolic plane. Since by Lemma 3.8 the warped projection increases the distance, we find

$$\delta(\omega, \zeta) \leq d_H(\pi_x(\omega), \pi_x(\zeta)) \leq 2R' \leq \delta.$$

By the δ -unpinched condition

$$d_{\partial\Sigma}(\omega, \zeta) \leq 2\delta(\omega, \zeta) \leq 4R'.$$

Using Assertion A, we can now conclude that ζ belongs to γ_R . This proves item (i).

For item (ii), consider ζ in γ_R . As above

$$\delta(\omega, \zeta) \leq d_H(\pi_x(\omega), \pi_x(\zeta)) \leq 2R \leq \delta.$$

Thus by the unpinched condition

$$d_{\partial\Sigma}(\omega, \zeta) \leq 2\delta(\omega, \zeta) \leq 4R. \quad (22)$$

It follows that

$$d_I(\zeta, x) \leq d_I(\omega, x) + d_{\partial\Sigma}(\zeta, \omega) \leq R' + 4R \leq 5R.$$

This concludes the proof. $\square$

5.2.2. Proof of Proposition 5.4. If a curve is δ -unpinched it is also κ -unpinched for all $\kappa \leq \delta$. Thus it is enough to prove the proposition for $\kappa = \delta$ and we will do so to avoid burdening the notation.

We choose $R \leq \frac{1}{100}\delta$ as in Lemma 5.6. Let K , U_R , γ_R and c_R be as in the conclusion of this lemma and the paragraph above it.

We may choose R so that the projection of $\partial\Sigma$ with $U_{R/A}$ intersects transversally. We apply Lemma 5.6 by choosing $A \geq K$ so that c_R is the unique component of $\pi_x(\partial\Sigma) \cap U_R$ intersecting $U_{R/A}$. We now apply Lemma B.1, to $\gamma' = c_R$. We thus obtain a topological disk U in U_R , whose boundary is $\partial U = \alpha_0 \cup \alpha_1$, where α_0 is a subarc of c_R , α_1 a subarc of $\partial U_{R/A}$, and so that U is contained in U_R (see Figure 2).

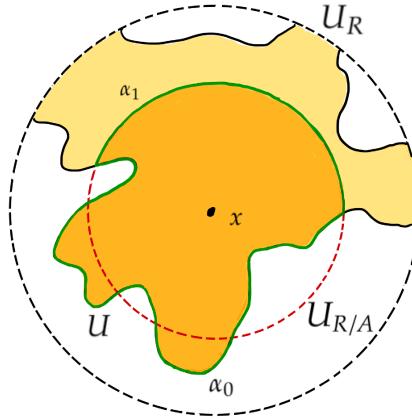


FIGURE 2. Good neighborhoods

Now let $\hat{\Sigma}$ be the preimage of U .

We first want to show that U is a subset of $\pi_x(\Sigma)$. Or in other words that any y in U has a preimage in Σ . First taking the closest point z in c_R to y , the hyperbolic geodesic arc $I := [y, z]$ lies in U . Let ζ be the preimage of z .

By Lemma 5.6, $d_I(\zeta, x) \leq K \cdot R$ where K is the constant of that lemma. Applying Lemma 5.5 with $\varepsilon = K \cdot R$, for

$$R \leq \frac{\delta}{8K}$$

we obtain that $y = \pi_x(\xi)$ with ξ in $\mathring{\Sigma}$ and

$$d_1(\xi, x) \leq 2K \cdot R, \quad d_{\mathcal{G}}(\mathsf{T}_{\xi}\Sigma, \mathsf{T}_x\Sigma) \leq 4K \cdot R. \quad (23)$$

We have just shown that $\mathring{\Sigma}$ is a graph over U , that U is a topological disk, inequality (23) holds and the boundary of $\mathring{\Sigma}$ is connected. We have thus proven that the first three items in the proposition.

The last item follows from the fact that $\pi_x(\text{Fr}(\mathring{\Sigma})) = \alpha_1$ and thus $d(x, \pi_x(\text{Fr}(\mathring{\Sigma})))$ is equal to R_1 . $\square$

5.3. Local control. Let us consider the following situation.

- (i) Let $\{\lambda_k\}_{k \in \mathbb{N}}$ be a sequence of strictly positive numbers converging to $\lambda_{\infty} \geq 0$.
- (ii) Let Σ_k be an acausal maximal surface (possibly with boundary) in $\mathbf{H}_{\lambda_k}^{2,n}$.
- (iii) Let x_k be a point in Σ_k .
- (iv) Let H_k be the totally geodesic plane containing x_k , so that $\mathsf{T}_{x_k}H_k = \mathsf{T}_{x_k}\Sigma_k$, and π_k the corresponding warped projection.

By convention, if $\lambda_{\infty} = 0$, we let $\mathbf{H}_{\lambda_{\infty}}^{2,n}$ to be $\mathbf{E}^{2,n}$, the flat pseudo-Euclidean space of signature $(2, n)$.

Definition 5.7 (LOCAL CONTROL HYPOTHESIS (*)). The sequence $\{\Sigma_k, x_k, \lambda_k\}_{k \in \mathbb{N}}$ satisfies the local control hypothesis (*), if there exist positive constants B and κ so that

- (i) Σ_k is a maximal surface in $\mathbf{H}_{\lambda_k}^{2,n}$ which is a graph over U_k , where
 - (a) the set U_k is a connected submanifold of $B(x_k, \kappa)$, the open ball of center x_k and radius κ in H_k ;
 - (b) the diameter of $\Gamma(\Sigma_k)$ is bounded by $B\kappa$;
 - (c) we have $d_H(x_k, \text{Fr}(U_k)) > \kappa B^{-1}$;
 - (d) the boundary of U_k is connected.
- (ii) We have the bound $\|\Pi_{\Sigma_k}\| \leq 1$.
- (iii) Finally let $\gamma_k = \partial\Sigma_k$ be the finite boundary of Σ_k . Assume that γ_k is strongly positive, and that
 - (a) The sequence of arcs $\{\gamma_k\}_{k \in \mathbb{N}}$ converges smoothly (in the sense of Appendix A) to a strongly positive curve γ_{∞} .
 - (b) For any point y in γ_k , we have $d_{\mathcal{G}}(\mathsf{T}_y\Sigma_k, \mathsf{T}_y^{(2)}\gamma_k) \leq B$.

Strongly positive curves in $\mathbf{H}_{\lambda}^{2,n}$ with $\lambda > 0$ are defined in Definition 3.14. For $\lambda = 0$, that is for the pseudo-Euclidean space $\mathbf{E}^{2,n}$, we apply the same definition, replacing *hyperbolic plane* by *euclidean plane* in the phrasing.

The goal of this paragraph is to show the following:

Proposition 5.8 (CONVERGENCE WITH LOCAL CONTROL). *For κ small enough, assuming the local control hypothesis (*) and the notation therein, then, after extracting a subsequence, the sequence $\{x_k, \Sigma_k, \mathbf{H}_{\lambda_k}^{2,n}\}_{k \in \mathbb{N}}$ converges in the sense of Appendix A to $\{x_{\infty}, \Sigma_{\infty}, \mathbf{H}_{\lambda_{\infty}}^{2,n}\}$ where Σ_{∞} is a maximal surface with boundary γ_{∞} .*

We prove this proposition in paragraph 5.3.3.

5.3.1. Distance and area estimates. Let $M_k = \mathsf{T}\Sigma_k$ be the Gauß lift of Σ_k in $\mathcal{G}(\mathbf{H}_{\lambda_k}^{2,n})$, and let $y_k = \mathsf{T}_{x_k}\Sigma_k$ be the lift of x_k . Then set $d_{\mathcal{G}}$ to be the Riemannian distance in $\mathcal{G}(\mathbf{H}_{\lambda_k}^{2,n})$.

Lemma 5.9. *We have the following estimates. For κ small enough, and assuming the local control hypothesis (*), there exist positive constants A, b and a depending only on κ and B so that*

$$\text{area}(M_k) \leq A, \quad (24)$$

$$\text{for all } w \text{ in } \Sigma_k, d_{\mathcal{G}}(\mathbb{T}_{x_k} \Sigma_k, \mathbb{T}_w \Sigma_k) < b\kappa, \quad (25)$$

$$\text{for all } u \text{ in } \text{Fr}(M_k), d_{\mathcal{G}}(y_k, u) > a. \quad (26)$$

Proof. The second inequality is a direct consequence of (ii)b) with $b = B$. It then follows from Corollary 3.29, that for κ small enough, the projection from M_k (equipped with d_{II}) to U_k is infinitesimally biLipschitz for some constant only depending on B . The first inequality follows.

The third estimate follows from the fact that $\text{Fr}(M_k) \subset Z$, where Z is the closed subset of $\mathcal{G}(\mathbf{H}_\lambda^{2,n})$ of points (y, P) with $\pi_k(y) \in \text{Fr}(U_k)$. So $a = \inf_{z \in Z} d(H_0, z)$ is positive and only depends on κ and A . $\square$

5.3.2. *The holomorphic translation.* Let us consider the two possible cases:

Case 1: There exists some positive ε so that $d_{\mathcal{G}}(\partial M_k, y_k) \geq \varepsilon$ for all k in $\mathbb{N}$.

Case 2: There exist $w_k \in \partial M_k$, with $\lim_{k \rightarrow \infty} d_{\mathcal{G}}(w_k, y_k) = 0$.

We refer to the notation of Appendix C: $\mathbf{D}$ denotes the open unit disk in $\mathbb{C}$, while $\mathbf{S} = \{z \in \mathbf{D}, \Re(z) \geq 0\}$ is the semi-disk. Corresponding to these two cases, we consider the following holomorphic maps

(i) in Case 1, we consider the uniformization $f_k : \mathbf{D} \rightarrow M_k \setminus \partial M_k$, so that $f_k(0) = y_k$.

(ii) in Case 2, we consider the uniformization $f_k : \mathbf{S} \rightarrow M_k$ so that $f_k(0) = w_k$.

To lighten the notation, we will write $U = \mathbf{D}$ or $\mathbf{S}$. Our hypotheses implies the following.

Lemma 5.10. *The maps f_k are holomorphic immersions. Moreover, for κ small enough we have the following bounds*

$$\text{area}(f_k(U)) \leq A, \quad (27)$$

$$\text{for all } w \text{ in } U, d(f_k(0), f_k(w)) < b\kappa, \quad (28)$$

$$\text{for all } w \text{ in } \text{Fr}(U), d(f_k(0), f_k(w)) > a, \quad (29)$$

where A, b and a only depends on κ .

Proof. This lemma is an immediate consequence of the holomorphic translation described in proposition 3.32 as well as of the bounds on M_k obtained in Lemma 5.9. $\square$

If g is a map from U to a space X , we define as in appendix C, $g(\text{Fr}(U))$ to be the set of those points x in X so that there exists a sequence $\{z_k\}_{k \in \mathbb{N}}$ tending to $\text{Fr}(U)$ with

$$\lim_{k \rightarrow \infty} g(z_k) = x.$$

Corollary 5.11. *For κ small enough, the following holds. After extracting a subsequence, the family $\{f_k\}_{k \in \mathbb{N}}$ converges to a non-constant holomorphic map f_∞ so that,*

$$f_\infty(\text{Fr}(U)) \subset \lim_{k \rightarrow \infty} f_k(\text{Fr}(U)).$$

Proof. The Lemma 5.10 guarantees that we can apply the results on pseudoholomorphic curves obtained in appendix C. More precisely, we split the discussion in the two cases described in the beginning of this paragraph:

Case 1: We are in the free boundary case and we apply Theorem C.7 to get the result.

Case 2: In this case, let us consider $W_k = W_A(\gamma_k)$ defined in Definition 3.33. By the hypotheses (iii)a and (iii)b, the totally real submanifold W_k converges smoothly to a totally real submanifold. We can now apply Theorem C.8 to get the result. $\square$

With the notation above, we have:

Lemma 5.12. *The holomorphic map f_∞ is an immersion at 0. Moreover, $f_\infty(U)$ is the Gauß lift of a maximal surface.*

Proof. We have the orthogonal splitting

$$T_{(x,P)}\mathcal{G}(\mathbf{H}^{2,n}) = P \oplus P^\perp \oplus \text{Hom}(P, P^\perp). \quad (30)$$

Let us consider the complex line subbundle V of $T\mathcal{G}$ over $\mathcal{G}(\mathbf{H}^{2,n})$ so that, in the splitting above, we have $V_{(x,P)} = P$. The orthogonal projection from $T_{(x,P)}\mathcal{G}(\mathbf{H}^{2,n})$ to P is a complex morphism, and thus gives rise to a form α in $\Omega_{\mathbb{C}}^1(\mathcal{G}, V)$.

By construction, if M is the lift of a maximal surface, then α restricted to TM is injective. Conversely, if α is non-zero restricted to a holomorphic curve M , then M is the lift of a maximal surface.

For any k , we now choose a real line bundle L_k in V so that along $W(\gamma_k)$, we have $L_k = \omega(TW(\gamma_k))$ (see Proposition 3.34).

Let v be a tangent to M_k , and $u = \omega(v)$. By Proposition 3.28,

$$\|v\|^2 = g_{\text{II}}(u, u) \leq (1 + \|\text{II}\|^2)g_{\text{I}}(u, u) \leq 2 \|\alpha(v)\|^2,$$

where in the last inequality we have used the assumption that the norm of the second fundamental form of Σ_k is bounded by 1. Thus it follows that

$$\|Tf_k\| \leq \sqrt{2} \|f_k^* \alpha\|.$$

According to Proposition C.20, this last inequality is enough to imply that the hypotheses of Theorem C.19 are all satisfied. Thus $f_\infty^* \alpha$ is non-zero, and in particular $f_\infty(U)$ is the lift of a maximal surface. $\square$

5.3.3. *Proof of Proposition 5.8.* The proposition is a consequence of Lemma 5.11 and 5.12.

5.4. Global control. Our goal in this subsection is to prove a global compactness result under assumptions that we make now precise:

- (i) Let $\{\lambda_k\}_{k \in \mathbb{N}}$ be a bounded sequence of positive numbers.
- (ii) Let $\{R_k\}_{k \in \mathbb{N}}$ be a sequence of positive number so that $\lim_{k \rightarrow \infty} R_k = \infty$.
- (iii) Let $\{\Sigma_k\}_{k \in \mathbb{N}}$ be a sequence of connected complete acausal maximal surfaces in $\mathbf{H}_{\lambda_k}^{2,n}$. Let $x_k \in \Sigma_k$ and let

$$B_k(R) := \{z \in \Sigma_k \mid d_{\text{I}}(z, x_k) \leq R\},$$

and let H_k be the totally geodesic plane tangent to Σ_k at x_k .

Definition 5.13 (GLOBAL CONTROL HYPOTHESIS (**)). The sequence $\{\Sigma_k, \lambda_k\}_{k \in \mathbb{N}}$ satisfies the *global control hypothesis* (**), if there exist positive constants A, M_0 and δ so that

- (i) The sequence $\{x_k, H_k, \mathbf{H}_{\lambda_k}^{2,n}\}_{k \in \mathbb{N}}$ converges (in the sense of Definition A.3).
- (ii) We have the bound $\|\text{II}_{\Sigma_k}\| \leq M_0$ on $B_k(R_k)$.
- (iii) The 1-dimensional manifold $\gamma_k := \partial \Sigma_k$ is strongly positive and δ -unpinched,
 - (a) The sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ has bounded geometry, and
 - (b) For any point y in γ_k , we have $d_{\mathcal{G}}(T_y \Sigma_k, T_y^{(2)} \gamma_k) \leq A$.

Our goal is now to show the

Proposition 5.14 (CONVERGENCE WITH GLOBAL CONTROL). *Assuming the global control hypothesis (**).*

- (i) *Then for any positive R , there exists ε , only depending on δ , A and $\{\gamma_k\}_{k \in \mathbb{N}}$ so that if $\{x_k\}_{k \in \mathbb{N}}$ is a sequence of points with x_k in Σ_k , with the property that $\delta(x_k, \text{Fr}(\Sigma_k))$ is bounded from below by R we have: the sequence of pointed surfaces $\{(x_k, \Sigma_k^\varepsilon)\}_{k \in \mathbb{N}}$ subconverges smoothly on the ball Σ_k^ε in Σ_k , where Σ_k^ε has center x_k and d_1 -radius ε .*
- (ii) *Assume now that Σ_k is complete, so that in particular $\text{Fr}(\Sigma_k)$ is empty. Let π_0 be the warped projection on some pointed hyperbolic plane $\mathbf{P}_0 = (H_0, x_0)$. Assume that $d_{\mathcal{G}}(\mathbf{P}_0, \Gamma(\Sigma_k))$ is uniformly bounded. Then $\{\Sigma_k, \mathbf{H}_{\lambda_k}^{2,n}\}_{k \in \mathbb{N}}$ subconverges as a graph on every compact set of H_0 (see Definition 5.1).*

We remark that by rescaling, that is replacing λ_k by $\lambda_k M_0^{-2}$ in the hypothesis and reversing in the conclusion, we can always assume that $M_0 = 1$. We will do so in the next paragraph.

5.4.1. *Getting a local control.* Let us first prove the following lemma.

Lemma 5.15. *Assume that $\{\Sigma_k, \lambda_k\}_{k \in \mathbb{N}}$ satisfies the global control hypothesis (**) with $M_0 = 1$. Then for any positive R , there exists κ and B only depending on R , A , δ and $\{\gamma_k\}_{k \in \mathbb{N}}$ so that the following is true. Let $\{x_k\}_{k \in \mathbb{N}}$ with x_k in Σ_k and $\delta(x_k, \text{Fr}(\Sigma_k))$ bounded below by R , then there exists a sequence of maximal surfaces $\{\tilde{\Sigma}_k\}_{k \in \mathbb{N}}$ so that*

- (i) *We have the inclusions $\tilde{\Sigma}_k \subset \Sigma_k$, and $\partial \tilde{\Sigma}_k \subset \partial \Sigma_k$.*
- (ii) *the sequence $\{\tilde{\Sigma}_k, x_k, \lambda_k\}_{k \in \mathbb{N}}$ satisfies the local control hypothesis (*) for κ and B .*

Proof. Let us use Proposition 5.4 to construct, given κ small enough, a $\tilde{\Sigma}_k$ and an open set U_k which will satisfy items (i)a, (i)b, (i)c, and (i)d of the local control hypothesis (*) (cf. see Definition 5.7), and thus the general item (i) is satisfied. Since all the other items are consequences of the global control hypothesis (**) (cf. see Definition 5.13), it follows that the local control hypothesis (*) is satisfied for $\tilde{\Sigma}_k$. This concludes the proof. $\square$

5.4.2. *Proof of the Global Control Proposition 5.14.* Let $\{\Sigma_k, \lambda_k\}_{k \in \mathbb{N}}$ be a sequence satisfying the global control hypothesis (**) (cf. see Definition 5.13).

Let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence of points with $x_k \in \Sigma_k$ with $\delta(x_k, \text{Fr}(\Sigma_k))$ bounded from below by a positive constant. Let $\{\tilde{\Sigma}_k\}_{k \in \mathbb{N}}$ sequence of maximal surface obtained by Lemma 5.15.

Since by Proposition 5.8, the sequence $\{x_k, \tilde{\Sigma}_k, \mathbf{H}_{\lambda_k}^{2,n}\}_{k \in \mathbb{N}}$ subconverges, it follows that there exists a constant ε , depending on the sequence $\{\Sigma_k, \lambda_k, x_k\}_{k \in \mathbb{N}}$ so that $\{x_k, \tilde{\Sigma}_k^\varepsilon, \mathbf{H}_{\lambda_k}^{2,n}\}_{k \in \mathbb{N}}$ subconverges smoothly. However, since we can choose our sequence $\{x_k\}_{k \in \mathbb{N}}$ arbitrarily, provided $\delta(x_k, \text{Fr}(\Sigma_k))$ is bounded from below, it follows that we can choose ε to depend only on δ , A and the sequence $\{\gamma_k\}_{k \in \mathbb{N}}$.

This concludes the proof of the first item of the proposition.

Let us show the second item. Let $\mathbf{P}_0$ be as in the proposition and π_0 the warped projection on $\mathbf{P}_0$. Recall that the warped projection is a dilation.

Consider y in $\mathbf{P}_0$, so that $y = \pi_0(x_k)$ with $x_k \in \Sigma_k$ and $d_{\mathcal{G}}(\mathbf{P}_0, \Gamma_{x_k} \Sigma_k)$ uniformly bounded. The first item guarantees that Σ_k converges as a graph over the ball $B_\varepsilon(y)$ of center y and radius ε in $\mathbf{P}_0$.

Let now U be the subset of $\mathbf{P}_0$, consisting of those points z so that $\{\Sigma_k\}_{k \in \mathbb{N}}$ converges as a graph over $B_\varepsilon(z)$. We focus on one such particular $z_0 \in U$; in particular, if $\{x_k\}_{k \in \mathbb{N}}$ is a sequence of points, with x_k in Σ_k so that $\{\pi_0(x_k)\}_{k \in \mathbb{N}}$ converges to an element w in $B_{\varepsilon/2}(z_0)$,

then $d_{\mathcal{G}}(T_{x_k}\Sigma_k, P_0)$ stays uniformly bounded. Thus, by the previous argument, and using that the $\text{Fr}(\Sigma_k)$ is empty, w belongs to U .

It follows that U contains the $\frac{\varepsilon}{2}$ neighborhood of itself. Thus, if U is non-empty, then $U = P_0$.

The hypothesis guarantees the existence of a sequence $\{z_k\}_{k \in \mathbb{N}}$ of points so that $d_{\mathcal{G}}(T_{z_k}\Sigma_k, P_0)$ stays bounded. It follows that $\pi_0(z_k)$ is a bounded sequence, hence subconverges to a point y which belongs to U . Hence U is not empty.

This concludes the proof of the proposition.

5.5. Bernstein type theorem. In this paragraph, we prove the following.

Theorem 5.16 (BERNSTEIN FOR MAXIMAL SURFACES). (i) *Let Σ be a complete maximal surface without boundary in $\mathbb{E}^{2,n}$. Then Σ is a totally geodesic 2-plane.*
(ii) *Let Σ be a complete maximal surface in $\mathbb{E}^{2,n}$ whose boundary $\partial\Sigma$ is a geodesic. Then Σ is a half-plane.*

The first part of this result was proved by Ishihara in [31] for p -dimensional entire graphs in the signature (p, q) pseudo-Euclidean space $\mathbb{E}^{p,q}$. Here, for completeness, we give a more direct proof for the case $\mathbb{E}^{2,n}$. The reader may wish to compare the analogous proof of Chern [19] in the classical setting.

Proof. Observe first that $\mathcal{G}(\mathbb{E}^{2,n})$ is isomorphic to $\mathbb{E}^{2,n} \times \text{Gr}_{2,0}(\mathbb{T}_x\mathbb{E}^{2,n})$ where x is any point in $\mathbb{E}^{2,n}$. By the proof of Proposition 3.32, the projection of the Gauß map $\Gamma : \Sigma \rightarrow \mathcal{G}(\mathbb{E}^{2,n})$ to the second factor of the decomposition above yields a holomorphic map $\varphi : \Sigma \rightarrow \text{Gr}_{2,0}(\mathbb{T}_x\mathbb{E}^{2,n})$.

Observe now that $\text{Gr}_{2,0}(\mathbb{E}^{2,n}) = \text{SO}_0(2, n)/(\text{SO}(2) \times \text{SO}(n))$ is the symmetric space of $\text{SO}(2, n)$ and, by a theorem of Harish-Chandra (see again [20]), is biholomorphic to a bounded domain in $\mathbb{C}^n$.

We remark that Σ is conformal to $\mathbb{C}$: by Gauß' equation, the induced metric on Σ has non-negative curvature (see Proposition 3.26): by a result of Blanc and Fiala [10] (see also [30]), we see that Σ uniformizes as the complex plane $\mathbb{C}$. By Liouville theorem φ is constant. Hence $u(\Sigma)$ is a spacelike plane.

Let us now consider the boundary case. Let us construct φ as above. Let L be a spacelike line in $\mathbb{E}^{2,n}$ and

$$W_L := \{P \in \text{Gr}_{2,0}(\mathbb{T}_x\mathbb{E}^{2,n}) \mid L \subset P\}.$$

Then W_L is totally real, and totally geodesic: the geodesic between two 2-planes with a common line L , consists of planes containing L . It follows that $\partial\Sigma$ is totally geodesic for the induced metric on Σ by φ , and thus the same argument applied to the doubling shows that S is uniformized by the half-plane.

In the corresponding symmetric domain picture we can assume that the image of W_L is a contained in real vector subspace in $\mathbb{C}^n$. Thus, again, the Liouville theorem allows us to conclude that φ is constant. $\square$

5.6. Bounds on the second fundamental form. Let $\{\Sigma_k\}_{k \in \mathbb{N}}$ be a sequence of acausal maximal surfaces which are complete with respect to d_1 . Let $\gamma_k := \partial\Sigma_k$ be the finite boundary of Σ_k . Assume that we have the following boundary conditions:

- (i) The sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ is strongly positive and uniformly unpinched.
- (ii) There is a positive constant A so that for all k , for all x in γ_k , we have $d(T_{x_k}\Sigma_k, T_{x_k}^{(2)}\gamma_k) \leq A$.
- (iii) The sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ has bounded geometry.

Our main result is now:

Proposition 5.17. *For every ε , there exists some constant M , such that if $\{x_k\}_{k \in \mathbb{N}}$ is a sequence with $x_k \in \Sigma_k$ so that $d_I(x, \text{Fr}(\Sigma_k)) > \varepsilon$, then the second fundamental form of Σ_k at x_k has norm uniformly bounded by M .*

This proposition concludes the proof of the first item of Theorem 5.3.

Proof. We first recall without proof the following folkloric result (see for instance Paragraph 1.D in [27] for a proof).

Lemma 5.18 (Λ -MAXIMUM LEMMA). *Let (X, d) be a metric space. Let η be a positive number and assume that the ball $B_y(\eta)$ of radius η centered at y is complete. Then there exists a constant $\Lambda \geq 1$ only depending on η , such that every positive locally bounded function f on X with $f(y) \geq 1$ admits a Λ -maximum on $B_y(\eta)$, that is a point x so that*

$$f(x) \geq \sup \left\{ f(y), \frac{1}{\Lambda} f(z) \mid \forall z \text{ such that } d(x, z) < \frac{1}{\Lambda \sqrt{f(x)}} \right\}. \quad (31)$$

We now argue by contradiction. For each k , let us define the function $f_k := \|\Pi_k\|$ on Σ_k . Assume that the second fundamental form of the sequence Σ_k is unbounded. Let then $\{y_k\}_{k \in \mathbb{N}}$ so that $y_k \in \Sigma_k$ and

$$\lim_{k \rightarrow \infty} f_k(y_k) = +\infty.$$

For each k , we apply the Λ -Maximum Lemma 5.18 to f_k and y_k , taking $\eta = 1$, and thus obtaining a Λ -maximum x_k of f_k in $B(y_k, 1)$. Let $\lambda_k := (\Lambda f_k(x_k))^{-2}$ so in particular we have

$$\lim_{k \rightarrow \infty} f_k(x_k) = \infty, \quad \lim_{k \rightarrow \infty} \lambda_k = 0, \quad f_k(z) \leq \frac{1}{\sqrt{\lambda_k}}.$$

for all z , with $d_I(x_k, z) \leq \lambda_k^{-\frac{1}{4}} \Lambda^{-\frac{1}{2}}$. We renormalize the metric of $\mathbf{H}^{2,n}$ by λ_k – following paragraph 2.3.4. We shall denote, by Σ_k^1 , the surface Σ_k seen as a surface in $\mathbf{H}_{\lambda_k}^{2,n}$ and denote all the geometric objects associated to Σ_k^1 with a superscript 1. Then by Lemma 3.20, $\|\Pi_k^1\| \leq 1$ for all the points z in Σ_k^1 so that

$$d_I^1(z, x_k) \leq R_k := \lambda_k^{-\frac{1}{4}} \Lambda^{-\frac{1}{2}}.$$

Moreover $\|\Pi^1(x_k)\| = 1/\Lambda$.

We can now apply the first item of the Global Control Proposition 5.14, to obtain that, after extracting a subsequence, the sequence $\{x_k, \Sigma_k^1, \mathbf{H}_{\lambda_k}^{2,n}\}_{k \in \mathbb{N}}$ converges smoothly to $(x_0, \Sigma_0, \mathbf{R}^{2,n})$, and in particular the norm of the second fundamental form of Σ_0 at x_0 is Λ^{-1} .

Observe now that since $\{\gamma_k\}_{k \in \mathbb{N}}$ has bounded geometry and is δ -unpinched, then $\{\gamma_k^1\}_{k \in \mathbb{N}}$ converges to a geodesic: more precisely, for every sequence $\{z_k\}_{k \in \mathbb{N}}$ so that $z_k \in \gamma_k^1$, then $\{z_k, \gamma_k^1, \mathbf{H}_{\lambda_k}^{2,n}\}_{k \in \mathbb{N}}$ converges to $(z_0, \gamma_0, \mathbf{R}^{2,n})$ where $z_0 \in \gamma_0$, and γ_0 is a spacelike geodesic – see the definition A.3 and the observations thereafter. Thus the boundary of Σ_0 is either empty or a geodesic. Thus by our Bernstein Theorem, the surface Σ_0 is totally geodesic. We have obtained our contradiction.

This concludes the proof of Proposition 5.17. $\square$

5.7. **Proof of Theorem 5.2 and 5.3 .** The theorems now follows from the successive use of

- Proposition 5.17 which guarantees that under the assumptions of the theorems the second fundamental form is uniformly bounded.
- Proposition 5.14 then ensures the conclusions of the theorems.

6. THREE SPECIFIC COMPACTNESS THEOREMS

Theorems 5.2 and 5.3 are quite general. They incarnate in three specific versions whose hypotheses are easier to handle.

6.0.1. *Boundary-free version.* The boundary-free version is of special interest and will be proved in Paragraph 6.3. In the next theorem, we explain three different compactness results that will be useful in this paper and its sequel.

Theorem 6.1 (COMPACTNESS THEOREM-BOUNDARY FREE). *Let $\{\Sigma_k\}_{k \in \mathbb{N}}$ be a sequence of connected complete acausal maximal surfaces without boundary. For each k , let x_k be a point in Σ_k , let Λ_k be the asymptotic boundary of Σ_k (see definition 3.9) and $\bar{\Sigma}_k := \Sigma_k \cup \Lambda_k$.*

- (i) *Assume that the sequence $\{\mathbb{T}_{x_k} \Sigma_k\}_{k \in \mathbb{N}}$ converges, then the sequence of pointed maximal surfaces $\{(x_k, \Sigma_k)\}_{k \in \mathbb{N}}$ subconverges.*
- (ii) *Assume that $\{\Lambda_k\}_{k \in \mathbb{N}}$ converges to a semi-positive loop Λ_0 . Then the sequence $\{\Sigma_k\}_{k \in \mathbb{N}}$ subconverges smoothly as a graph on every pointed hyperbolic plane to a complete maximal surface Σ_0 whose asymptotic boundary is Λ_0 .*
- (iii) *Assume that $\{(x_k, \Sigma_k)\}_{k \in \mathbb{N}}$ converges to (x_0, Σ_0) then any sequence $\{y_k\}_{k \in \mathbb{N}}$ so that y_k belongs to $\bar{\Sigma}_k$ subconverges to a point in $\Sigma_0 \cup \Lambda_0$ where Λ_0 is a semi-positive loop which is the asymptotic boundary of Σ_0*

6.0.2. *Boundary vanishing at infinity.*

Theorem 6.2 (VANISHING BOUNDARY). *Let Λ_0 be a positive loop in $\partial_\infty \mathbf{H}^{2,n}$ and $\{\Sigma_k\}_{k \in \mathbb{N}}$ be a sequence of connected complete acausal maximal surfaces.*

Assume that $\gamma_k = \partial \Sigma_k$ is compact. Assume furthermore that

- (i) *$\{\gamma_k\}_{k \in \mathbb{N}}$ converges to Λ_0 ,*
- (ii) *the sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ is a sequence of strongly positive curves of bounded geometry which are also uniformly unpinched,*
- (iii) *the angular width of γ_k is uniformly bounded.*

Then $\{\Sigma_k\}_{k \in \mathbb{N}}$ converges as a graph on any pointed hyperbolic plane to maximal surface with asymptotic boundary Λ_0 .

This will be proved in Paragraph 6.3.

6.0.3. *Finite boundary version.* We also have a finite boundary version that will be proved in Paragraph 6.4.

Theorem 6.3 (COMPACT SURFACES). *Let $\{\Sigma_k\}_{k \in \mathbb{N}}$ be a sequence of compact acausal maximal surfaces in $\mathbf{H}^{2,n}$. Let γ_k be the 1-dimensional manifold $\gamma_k := \partial \Sigma_k$. Assume that $\{\gamma_k\}_{k \in \mathbb{N}}$ are topological circles that converge smoothly to a topological circle γ which is strongly positive. Then $\{\Sigma_k\}_{k \in \mathbb{N}}$ subconverges smoothly to a maximal surface whose boundary is γ .*

6.1. Proximity of the action of $\mathbf{G}$ on $\mathbf{P}(E)$. In the first subsection, we study the dynamics on $\mathbf{P}(E)$ of a divergent sequence in $\mathbf{G}$. These dynamics will be helpful when proving the convergence of a sequence of maximal surfaces.

The main result of this section is the following:

Proposition 6.4 (PROXIMALITY). *Let $\{g_j\}_{j \in \mathbb{N}}$ be an unbounded sequence in the subgroup of $\mathbf{G}$ fixing a point x in $\mathbf{H}^{2,n}$. Then there exists a hyperplane H of degenerate signature $(1, n)$ through x , such that for any compact set C in $\mathbf{P}(E)$ not intersecting H , the orbit of C under $\{g_j\}_{j \in \mathbb{N}}$ accumulates to a point in $\partial_\infty \mathbf{H}^{2,n}$ orthogonal to x .*

6.1.1. Cartan decomposition. We call a *Barbot crown* a semi-positive loop C in $\partial_\infty \mathbf{H}^{2,n}$ made of 4 segments of photons (those objects appeared in the work of Barbot in [6] under the name *crown*). Denote the vertices of C by $\{c_+, d_+, d_-, c_-\}$, where the planes $\gamma = c_+ \oplus c_-$ and $\delta = d_+ \oplus d_-$ are non-degenerate, thus giving rise to space-like geodesics in $\mathbf{H}^{2,n}$. In particular, γ is a subset of $\delta^\perp$. In particular, $F := \text{span}\{C\}$ is a non-degenerate subspace of signature $(2, 2)$.

The group $\mathbf{A}$ which fixes $\{c_+, d_+, d_-, c_-\}$ and $F^\perp$ pointwise is a *Cartan subgroup* of $\mathbf{G}$. By construction every element of $\mathbf{A}_0$, the component of the identity of $\mathbf{A}$, is characterised by its restriction on F which, in the basis given by (c_+, d_+, d_-, c_-) is a diagonal matrix

$$a(\lambda, \mu) := \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 1/\mu & 0 \\ 0 & 0 & 0 & 1/\lambda \end{pmatrix}, \text{ with } \lambda, \mu > 0.$$

Given $\mathbf{K}$ a maximal compact subgroup of $\mathbf{G}$, the corresponding *Cartan decomposition* of $\mathbf{G}$ states that any element g in $\mathbf{G}$ may be written as $g = k'ak$ where $k', k \in \mathbf{K}$ and $a = a(\lambda, \mu)$ is in $\mathbf{A}_0$. Note that this decomposition is not unique; however, if we impose the condition $\lambda \geq \mu \geq 1$, the pair (λ, μ) is uniquely defined. We call $(\log(\lambda), \log(\mu))$ the *Cartan projection* of g .

6.1.2. Asymptotics of the action of $\mathbf{G}$. Let $\{g_j\}_{j \in \mathbb{N}}$ an unbounded sequence in $\mathbf{G}$, let $g_j = k'_j a_j k_j$ be the Cartan decomposition of g_j , and $(\log(\lambda_j), \log(\mu_j))$ the Cartan projection of g_j . We distinguish two cases:

- (i) The sequence $\{g_j\}_{j \in \mathbb{N}}$ is called *$\mathbf{P}_1$ -divergent* when the sequence $\{\lambda_j/\mu_j\}_{j \in \mathbb{N}}$ is unbounded.
- (ii) The sequence $\{g_j\}_{j \in \mathbb{N}}$ is called *non $\mathbf{P}_1$ -divergent* when the sequence $\{\lambda_j/\mu_j\}_{j \in \mathbb{N}}$ is bounded.

Here $\mathbf{P}_1$ refers to the parabolic subgroup in $\mathbf{G}$ stabilizing a point in $\partial_\infty \mathbf{H}^{2,n}$ and the terminology is inspired from [25].

Lemma 6.5 ($\mathbf{P}_1$ -DIVERGENT SEQUENCE). *Let $\{g_j\}_{j \in \mathbb{N}}$ be a $\mathbf{P}_1$ -divergent sequence in $\mathbf{G}$. Then, up to extracting a subsequence, there exist two points p and q in $\partial_\infty \mathbf{H}^{2,n}$ such that for any compact set C in $\mathbf{P}(E)$ and not intersecting $\mathbf{P}(q^\perp)$, the sequence $\{g_j\}_{j \in \mathbb{N}}$ converges uniformly to p on C .*

Proof. Let $V = c_+$ and $W = c_-^\perp$. The group $\mathbf{A}$ preserves the splitting $E = V \oplus W$. Moreover, the element a_j acts by λ_j on V and has spectral radius μ_j on W .

Let y be in $\mathbf{P}(E)$ but not in $\mathbf{P}(W)$, so that $y = [(v, w)] \in \mathbf{P}(V \oplus W)$ with $v \neq 0$. Since λ_j/μ_j is unbounded, up to extracting a subsequence, we have that $\{a_j \cdot y\}_{j \in \mathbb{N}}$ converges to $[(v, 0)]$.

Now since K is compact, the sequences $\{k'_j\}_{j \in \mathbb{N}}$ and $\{k_j\}_{j \in \mathbb{N}}$ subconverge to k'_0 and k_0 respectively. It follows that for any point y in $P(E)$ and not in $P(k_0^{-1}W)$, the sequence $\{g_j \cdot y\}_{j \in \mathbb{N}}$ subconverges to $P(k'_0V)$. Setting $q = (k_0^{-1}W)^\perp$ and $p = k'_0V$ yields the result. $\square$

Lemma 6.6 (NON P_1 -DIVERGENT SEQUENCE). *Let $\{g_j\}_{j \in \mathbb{N}}$ be a Non P_1 -divergent sequence in G . Then up to extracting a subsequence, there exist photons φ and ψ such that for any compact set C in $P(E)$ and not intersecting $P(\psi^\perp)$, the sequence $\{g_j\}_{j \in \mathbb{N}}$ converges uniformly to a point in $P(\varphi)$.*

Proof. Set $V = c_+ \oplus d_+$ and $W = (c_- \oplus d_-)^\perp$. Note that V and $W^\perp$ are isotropic planes (that is photons) preserved by A and such that $E = V \oplus W$. The spectral radius of a_j on V is λ_j and is less or equal to 1 on W .

Since $\{g_j\}_{j \in \mathbb{N}}$ is unbounded, there is a subsequence of $\{\lambda_j\}_{j \in \mathbb{N}}$ that tends to infinity.

Again, for any y in $P(E)$ not in $P(W)$, the sequence $\{a_j \cdot y\}_{j \in \mathbb{N}}$ subconverges to a point in $P(V)$. Setting $\psi := k_0^{-1}W^\perp$ and $\varphi := k'_0V$ yields the result. $\square$

6.1.3. *Proof of Proposition 6.4.* First assume that $n \geq 2$, so the group $\text{Fix}(x) = \text{Stab}(x^\perp) \cong \text{SO}(2, n)$ has rank 2. We can thus take a Barbot crown C with vertices $\{c_+, d_+, d_-, c_-\}$ in $x^\perp$, which implies that the corresponding Cartan subgroup A is in $\text{Fix}(x)$. We consider the Cartan decomposition (in $\text{Fix}(x)$) $g_j = k'_j a_j k_j$, with k'_j and k_j as elements in a maximal compact subgroup K_F of $\text{Fix}(x)$. Observe that K_F preserves the orthogonal to x .

In the P_1 -divergent case, we apply Lemma 6.5, observing that the points p and q are in $P(x^\perp)$, and the result follows with $H := P(q^\perp)$.

In the non P_1 -divergent case, we apply Lemma 6.6 and observe that φ and ψ are contained in $x^\perp$. We take any point q in ψ , and then $H := q^\perp$ so that $\psi^\perp$ is a subset of H .

Finally, if $n = 1$, the stabilizer of point is isogenic $\text{SO}(2, 1)$ and has rank 1. Thus the sequence $\{g_j\}_{j \in \mathbb{N}}$ is P_1 -divergent and we apply Lemma 6.5.

6.2. **A priori C^0 - estimates.** Both Theorems 6.1 and 6.2 will follow from some *a priori* estimate that we now state:

Proposition 6.7. *Let $\{\Sigma_k\}_{k \in \mathbb{N}}$ be a sequence of complete maximal surfaces and Λ_0 a semi-positive curve in $\partial_\infty \mathbf{H}^{2,n}$ satisfying either the hypotheses of Theorem 6.1 or those of Theorem 6.2.*

Then for any pointed hyperbolic plane $P = (q, H)$, if x_k in Σ_k is the preimage of q for the warped projection, then the sequence $\{d_{\mathcal{G}}(T_{x_k} \Sigma_k, T_q H)\}_{k \in \mathbb{N}}$ is bounded.

Proof of proposition 6.7. Suppose the result is false. Then there exists a pointed hyperbolic plane $P = (q, H)$ so that $\{d_{\mathcal{G}}(T_{x_k} H, T_{x_k} \Sigma_k)\}_{k \in \mathbb{N}}$ is unbounded where $\pi(x_k) = q$.

Since the time-like sphere

$$S =: \pi_P^{-1}(q),$$

is compact, the sequence $\{x_k\}_{k \in \mathbb{N}}$ subconverges to a point x_0 . The point x_0 belongs to $C\mathcal{H}(\Lambda_0)$ since x_k is in the convex hull of the boundary of Σ_k .

Let $\{\tau_k\}_{k \in \mathbb{N}}$ be a sequence of elements of $\text{Stab}(S)$ converging to the identity and so that $\tau_k(x_k) = x_0$, and let us write $\Sigma_k'' =: \tau_k(\Sigma_k)$.

Let H_0 be a hyperbolic plane through x_0 and let g_k in G be such that $g_k(T_{x_k} \Sigma_k) = T_{x_0} H_0$, and let $\Sigma_k' = g_k(\Sigma_k)$. Then by construction $d_{\mathcal{G}}(T_{x_0} H_0, T_{\Sigma_k'} \Sigma_k) = 0$. Observe now

- in the context of Theorem 6.1 the boundary of $\{\Sigma_k'\}_{k \in \mathbb{N}}$ is empty and all hypothesis of Theorem 5.3 are *de facto* satisfied,
- in the context of Theorem 6.2, the second hypothesis of this theorem, implies the first and third hypothesis of Theorem 5.3, moreover Proposition 3.18 and the third

hypothesis of Theorem 6.2 gives that the second hypothesis of Theorem 5.3 is satisfied.

Thus, in both cases, we can apply Theorem 5.3. Hence the sequence $\{\Sigma'_k\}_{k \in \mathbb{N}}$ subconverges on any compact to a maximal surface Σ'_0 . Here we invoke the hypotheses of Theorem 6.2 and Proposition 3.18 if the conditions on the boundary in Theorem 5.3 are not vacuously satisfied. By construction observe that $g_k(x_k) = x_0$, hence that Σ'_0 passes through x_0 .

Let finally $h_k = \tau_k \cdot g_k^{-1}$. The sequence $\{h_k\}_{k \in \mathbb{N}}$ belongs to $\text{Stab}(x_0)$, is unbounded by hypothesis, and

$$h_k(\Sigma'_k) = \Sigma''_k.$$

By Proposition 6.4, there exists

- A hyperplane H^+ of degenerate signature $(1, n)$ passing through x_0 such that for any compact set C in $\bar{\mathbf{H}}^{2,n}$ but not intersecting H^+ , the sequence $\{h_k(C)\}_{k \in \mathbb{N}}$ accumulates to a point in $\partial_\infty \mathbf{H}^{2,n} \cap \mathbf{P}(x_0^\perp)$.
- A hyperplane H^- of degenerate signature $(1, n)$ passing through x_0 such that for any compact set C in $\bar{\mathbf{H}}^{2,n}$ but not intersecting H^- , the sequence $\{h_k^{-1}(C)\}_{k \in \mathbb{N}}$ accumulates to a point in $\partial_\infty \mathbf{H}^{2,n} \cap \mathbf{P}(x_0^\perp)$.

We can now prove the proposition.

- (i) Let us first treat the case of the boundary free Theorem 6.1.

By definition a semi-positive curve contains a positive triple. Thus Λ_0 is not included in H^- , there exists a sequence $\{y_k\}_{k \in \mathbb{N}}$ in $\partial_\infty \Sigma''_k$ converging to a point y_∞ in $\Lambda_0 \setminus H^-$. Then $z_k =: h_k^{-1}(y_k)$ converges to a point z_∞ which is orthogonal to x_0 . On the other hand, since y_k belongs to $\partial_\infty \Sigma''_k$, z_k belongs to $\partial_\infty \Sigma'_k$. Thus z_∞ belongs to Λ'_0 . This contradicts [21, Lemma 3.7 (c)] which asserts that if x is a point in a complete maximal surface S and y a point in the asymptotic boundary of S , then x and y are not orthogonal.

- (ii) Let us now treat the case of Theorem 6.2, where we assume the limit Λ_0 is a positive curve.

Observe that H^+ does not contain any positive definite 2-plane, and thus does not contain Σ'_0 . Thus, there exists z_k in Σ'_k converging to z_0 in Σ'_0 such that if $y_k := h_k(z_k)$, then $\{y_k\}_{k \in \mathbb{N}}$ subconverges to a point y_∞ in $\partial_\infty \mathbf{H}^{2,n} \cap \mathbf{P}(x_0^\perp)$, which is in particular orthogonal to x_0 .

Since y_k belongs to Σ''_k , it follows that y_k converges to y_∞ which belongs to Λ_0 . We obtain a contradiction with item (iii) of Proposition 2.15.

□

6.3. Boundary-free and boundary-vanishing case. The first item of Theorem 6.3 is a direct consequence of the Compactness Theorem 5.3. Indeed in that case, the finite boundary γ_k of Σ_k is empty and the first and third hypotheses of Theorem 5.3 are fulfilled. The second hypothesis is a consequence of the hypothesis of this first item.

We now move to the proof of the second item of 6.3 In order to apply the Compactness Theorem 5.3, we first need to find a pointed hyperbolic plane at finite distance from $\text{T}\Sigma_k$. This is achieved by Proposition 6.7. This completes the proof of Theorem 6.1.

The proof of Theorem 6.2 is also immediate: we notice from Proposition 3.18 that for all x in γ_k

$$d_{\mathcal{G}}(\text{T}_x \Sigma_k, \text{T}_x^{(2)} \gamma_k) \leq w(\gamma_k).$$

Thus the uniform bounds on the angular width for γ_k guarantees that the second item in the hypothesis of Theorem 5.3 is satisfied.

Finally, let us prove the third item in Theorem 6.3 using the notation therein. Let us realize the surfaces Σ_k as graphs over a fixed pointed hyperbolic disk $\mathbf{P}$ and let π be the warped projection. We separately treat two cases.

- (i) In the case the sequence of points $\{\pi(y_k)\}_{k \in \mathbb{N}}$ stays bounded, then the sequence $\{y_k\}_{k \in \mathbb{N}}$ subconverges since $\{\Sigma_k\}_{k \in \mathbb{N}}$ subconverges as a graph over $\mathbf{P}$
- (ii) Otherwise we can assume that $\{\pi(y_k)\}_{k \in \mathbb{N}}$ converges to a point z_∞ in the boundary at infinity in $\mathbf{P}$. Recall that Σ_k is included in the convex hull Ω_k of Λ_k and that Λ_k is the graph of a 1-Lipschitz map f_k over the boundary at infinity of $\mathbf{P}$. We may thus extract a subsequence so that $\{f_k\}_{k \in \mathbb{N}}$ converges to some 1-Lipschitz map f_∞ . It then follows that $\{\Lambda_k\}_{k \in \mathbb{N}}$ converges in the Hausdorff topology to some semipositive loop Λ_∞ and thus that $\{\Omega_k\}_{k \in \mathbb{N}}$ converges in the Hausdorff topology to the convex hull of Λ_∞ . Then $\{y_k\}_{k \in \mathbb{N}}$ converges to the element $f_\infty(z_\infty)$ in Λ_∞ . This concludes the proof.

6.4. Proof of the compact boundary case: Theorem 6.3. Let us prove that the hypotheses of Theorem 5.3 are satisfied.

We first remark that the first and third boundary conditions of Theorem 5.3 are satisfied by the compactness and strong positivity of γ . Finally the second condition is satisfied due to proposition 3.18 and the fact that a maximal surface lies in the convex hull of its boundary by Proposition 3.27.

7. FINITE PLATEAU PROBLEM

In this section, we prove the following finite Plateau problem.

Theorem 7.1. *If γ is a deformable strongly positive closed curve in $\mathbf{H}_+^{2,n}$, there exists a unique complete acausal maximal spacelike surface Σ whose total boundary is γ .*

Strongly positive curves are defined in Definition 3.14, deformable ones in Definition 3.19 and total boundary in Definition 3.9. Observe also that by Proposition 3.10, such a maximal surface is a graph.

The uniqueness of Σ has been proved in Theorem 4.1, so it only remains to prove the existence.

7.1. Existence by the continuity method. Consider a deformation $\{\gamma_t\}_{t \in [0,1]}$ of γ (see Definition 3.19). By Lemma 2.8, we can lift those curves continuously to $\mathbf{H}_+^{2,n}$.

Let $\mathcal{M}$ be the space of $C^{k,\alpha}$ complete compact spacelike surfaces whose boundary is both smooth and strongly positive, which are graphs over a disk. Define

$$\mathcal{A} = \{t \in [0, 1] \mid \text{there is a complete maximal acausal surface } \Sigma \in \mathcal{M} \text{ with } \partial\Sigma = \gamma_t\}.$$

We already know that $\mathcal{A}$ is non empty since it contains $t = 0$; we also know that $\mathcal{A}$ is closed by Theorem 6.3.

Thus a connectedness argument shows that Theorem 7.1 follows from the following proposition.

Proposition 7.2. *The set $\mathcal{A}$ is open.*

The proof is a consequence of Corollary 3.24 and standard techniques of which we now give some details.

We need some preliminaries. Let H_0 be the hyperbolic plane containing γ_0 , fix a point q_0 in H_0 and consider the pointed hyperbolic plane $\mathbf{P}_0 = (q_0, H_0)$ (see Paragraph 3.1.1).

Fix t_0 in $\mathcal{A}$. The image of γ_{t_0} by the warped projection on P_0 is a smooth simple closed curve bounding a domain Ω_0 . Similarly, for any t close to t_0 , we denote by Ω_t the domain in H_0 bounded by the warped projection of γ_t . Let Σ_0 be a maximal surface with boundary γ_{t_0} , we want to prove that for all t close to t_0 , the curve γ_t bounds a maximal surface.

The next two lemmas describe coordinates on the space of (some) surfaces, and trivialisation of the associated normal bundles. These technicalities are made necessary by the non linear nature of $\mathbf{H}_+^{2,n}$.

Lemma 7.3 (CHARTS). *There exists some n -dimensional vector spaces E_1 , a neighbourhood $\mathcal{U}$ of $(t_0, 0)$ in $I \times C^{k,\alpha}(\Omega_0, E_1)$, a map*

$$\phi : \mathcal{U} \times \Omega_0 \longrightarrow \mathbf{H}_+^{2,n},$$

such that for any (t, G) in neighbourhood of $(t_0, 0)$ in $\mathcal{U}$ the map

$$\phi_{(t,G)} : x \mapsto \phi(t, G, x),$$

is a smooth parametrisation of a surface $S_{(t,G)}$ such that

- (i) $S_{(t,G)}$ is a graph over Ω_t ,
- (ii) $S_{(t_0,0)} = \Sigma_0$, and
- (iii) if G vanishes on the boundary, then $\partial S_{(t,G)} = \gamma_t$.

Proof. After possibly taking a smaller I , let us choose an immersion

$$F : I \times \Omega_0 \rightarrow \mathbf{H}_+^{2,n} = H_0 \times \mathbf{S}^n,$$

so that $\pi(F(t_0, x)) = x$ and for every t , the map from $\partial\Omega_0$ to $\mathbf{H}^{2,n}$ given by

$$x \mapsto F(t, x),$$

is a parametrisation of γ_t . Then for every t small enough, the map $x \rightarrow F(t, x)$ is a parametrisation of a spacelike surface S_t whose boundary is γ_t .

Let us now consider a trivialisation of the bundle $F^*(\mathbf{TS}^n)$ as $I \times \Omega_0 \times E_1$, and finally consider the map

$$\phi(t, G, x) = \exp_{F(t,x)}(G(x)),$$

where we have considered $G(x)$ as a vector in $\mathbf{T}_{F(t,x)}\mathbf{S}^n$ using the above trivialisation. The result follows from this construction, writing $S_{(t,G)}$ as the image of $x \mapsto \phi(t, G, x)$. $\square$

We have a similar result for the normal bundle

Lemma 7.4 (TRIVIALISATION OF THE NORMAL BUNDLE). *Similarly, there exists an n -dimensional vector space E_2 , and a smooth map*

$$\Phi : \mathcal{U} \times C^{k-2,\alpha}(\Omega_0, E_2) \times \Omega_0 \rightarrow \mathbf{TH}_+^{2,n},$$

above ϕ , such that fixing (t, G) in $\mathcal{U}$, the map

$$N_{t,G} : C^{k-2,\alpha}(E_2) \times \Omega_0 \rightarrow \mathbf{TH}_+^{2,n}$$

is a linear isomorphism with the space of $C^{k-2,\alpha}$ -section of the normal bundle to $S_{(t,G)}$.

Proof. Let us consider the bundle

$$N \rightarrow \mathcal{U} \times \Omega_0,$$

whose fiber at the point $\phi(t, G, x)$ is the normal bundle of the surface $S_{(t,G)}$ at the point $\phi(t, G, x)$.

We now trivialize this bundle as

$$\mathcal{U} \times \Omega_0 \times E_2,$$

and the result follows $\square$

Proof of Proposition 7.2. Let (t, G) in $\mathcal{U}$. We denote by $H(t, G)$ the mean curvature (vector) of the surface $\Sigma_{(t, G)}$. Using the identification in Lemma 7.4, we identify $H(t, G)$ – with the same notation – to an element of $C^{k-2, \alpha}(\Omega_0, E_2)$. Thus we consider the map

$$H : \begin{cases} \mathcal{U} & \longrightarrow C^{k-2, \alpha}(\Omega_0, E_2), \\ (t, G) & \longmapsto H(t, G). \end{cases}$$

Let $\mathcal{V}$ a neighbourhood of 0 in $C_0^{k, \alpha}(\Omega_0, E_1)$ so that $\{t_0\} \times \mathcal{V}$ is included in $\mathcal{U}$. Consider the restricted map

$$H^0 : \begin{cases} \mathcal{V} & \longrightarrow C^{k-2, \alpha}(\Omega_0, E_2), \\ G & \longmapsto H(t_0, G). \end{cases}$$

Similarly to the setting of Plateau problems in Euclidean spaces, as shown in White [47, Proposition 1.4] the map $D_0 H^0$ is Fredholm of index zero (to adapt the proof of this fact to the present setting, we need only observe that the map $D_0 H^0$ is strongly elliptic at a spacelike surface).

By Corollary 3.24, $D_0 H^0$ has zero kernel and spectrum bounded away from 0, so it is invertible with continuous inverse.

To conclude the proof, we will use an Implicit Function Theorem. Fix an extension operator $\varepsilon : C^{k, \alpha}(\partial\Omega_0, E_1) \rightarrow C^{k, \alpha}(\Omega_0, E_1)$ such that $\varepsilon(\delta)|_{\partial\Omega_0} = \delta$ for any δ . This gives an isomorphism

$$\iota : \begin{cases} (I \times C^{k, \alpha}(\partial\Omega_0, E_1)) \times C_0^{k, \alpha}(\Omega_0, E_1) & \longrightarrow I \times C^{k, \alpha}(\Omega_0, E_1), \\ (t, \delta, G) & \longmapsto (t, \varepsilon(\delta) + G). \end{cases}$$

By restricting the domain of ι to an open set $\mathcal{O}$ containing $((t_0, 0), 0)$, we can further assume ι takes value in $\mathcal{U}$.

Finally define

$$\Psi := H \circ \iota : \mathcal{O} \longrightarrow C^{k-2, \alpha}(\Omega_0, E_2).$$

The differential of Ψ at $((t_0, 0), 0)$ in the second factor — namely $C_0^{k, \alpha}(\Omega_0, E_1)$ — is just the operator $D_0 H^0$ which, as we noted, is an isomorphism. By the Implicit Function Theorem, we obtain an application θ from a neighbourhood W of $(t_0, 0)$ in $I \times C^{k, \alpha}(\partial\Omega_0, E_1)$ to a neighborhood of 0 in $C_0^{k, \alpha}(\Omega_0, E_1)$ such that for G close to 0 in $C_0^{k, \alpha}(\Omega_0, E_1)$ and (t, f) in W ,

$$\Psi((t, f), G) = 0 \text{ if and only if } G = \theta(t, f). \quad (32)$$

Observe that $\iota((t, 0), f) = (t, f)$ and let

$$\Sigma_t = S_{(t, \theta(t, 0))}.$$

By equation (32), Σ_t is maximal. Observe that since $\theta(t, f)$ belongs to $C_0^{k, \alpha}(\Omega_0, E_1)$, by the item (iii) of lemma 7.3, we have

$$\partial\Sigma_t = \partial S_t = \gamma_t.$$

Since $(t, \theta(t, 0))$ is well defined and in $\mathcal{U}$ for t in some neighborhood of t_0 , we get that $\mathcal{A}$ contains a neighbourhood of t_0 . The uniqueness part comes from equation (32) with $f = 0$. $\square$

8. ASYMPTOTIC PLATEAU PROBLEM: THEOREM A

In this section, we prove Theorem A, namely, that any semi-positive loop in $\partial_\infty \mathbf{H}^{2,n}$ is the total boundary of a unique complete maximal surface in $\mathbf{H}^{2,n}$.

In particular, in light of our previous results, what remains to prove is

Theorem 8.1. *Any (smooth) spacelike positive loop in $\partial_\infty \mathbf{H}^{2,n}$ is the total boundary of a complete maximal surface in $\mathbf{H}^{2,n}$.*

Once this case of Theorem A is established for smooth boundary loops Λ , the full theorem follows by approximation of continuous loops Λ by smooth ones, using Corollary 2.14 and Theorem 6.1.

8.1. Exhaustion. We construct here an exhaustion of a spacelike positive loop by using radial curves.

More precisely we prove

Proposition 8.2 (EXHAUSTION). *Let Λ be a spacelike positive loop in $\partial_\infty \mathbf{H}^{2,n}$. Then, there exists a sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ of closed curves in $\mathbf{H}^{2,n}$ converging to Λ satisfying the following:*

- (i) *For k large enough, the curve γ_k is strongly positive (Definition 3.14).*
- (ii) *For k large enough, the strongly positive curve γ_k is deformable (Definition 3.19).*
- (iii) *The sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ is uniformly unpinched (Definition 3.15).*
- (iv) *The sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ has bounded geometry (Appendix A).*
- (v) *The angular width of γ_k is uniformly bounded (Definition 3.17).*

We first give the construction, then prove Proposition 8.2 in paragraph 8.1.2.

8.1.1. Construction of an exhaustion. Given a point p in $\mathbf{H}^{2,n}$, recall from Subsection 2.2 that the set

$$\mathbf{T}_p^1 \mathbf{H}^{2,n} = \{v \in \mathbf{T}_p \mathbf{H}^{2,n}, \mathbf{q}(v) = 1\}$$

is isometric to the signature $(1, n)$ pseudo-sphere $\mathbf{S}^{1,n}$. We denote by $\mathbf{g}_{\mathbf{S}^{1,n}}$ its metric. Fix a sequence $\{\rho_k\}_{k \in \mathbb{N}}$ of positive real numbers tending to infinity, and for any k , set

$$\phi_k : \begin{cases} \mathbf{T}_p^1 \mathbf{H}^{2,n} & \longrightarrow \mathbf{H}^{2,n}, \\ v & \longmapsto \exp(\rho_k v). \end{cases}$$

The map ϕ_k is a diffeomorphism onto the pseudosphere $M_k := \beta(p, \rho_k) = \{x \in \mathbf{H}^{2,n}, \delta(p, x) = \rho_k\}$. We have

$$\phi_k^* g_{M_k} = \sinh(\rho_k) \mathbf{g}_{\mathbf{S}^{1,n}}. \quad (33)$$

Taking the limit as k goes to infinity, we obtain a map

$$\phi_\infty : \mathbf{T}_p^1 \mathbf{H}^{2,n} \longrightarrow \partial_\infty \mathbf{H}^{2,n}$$

which is a conformal diffeomorphism onto $\partial_\infty \mathbf{H}^{2,n} \setminus \mathbf{P}(p^\perp)$.

Let us return to our situation. If Λ is a spacelike positive curve in $\partial_\infty \mathbf{H}^{2,n}$, fix a point p in the interior of the convex hull $\mathcal{CH}(\Lambda)$. By Proposition 2.15, the set Λ is disjoint from $p^\perp$. We can thus define the smooth closed curves γ_0 in $\mathbf{T}_p^1 \mathbf{H}^{2,n}$ and γ_k in M_k respectively by

$$\gamma_0 := \phi_\infty^{-1}(\Lambda), \quad \gamma_k := \phi_k(\gamma_0).$$

In other words, γ_k is the intersection of the cone over Λ with the pseudosphere M_k .

8.1.2. *Proof of proposition 8.2.* We prove that there is a subsequence of the previously constructed sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ that satisfies the conditions of the proposition.

Proof of (i). Lift each γ_k and Λ to the hyperplane $\{x \in E, \langle p, x \rangle = -1\}$. We denote those lifts with the same notation.

Consider a smooth spacelike loop γ in E .

We may now choose an auxiliary euclidean metric h on E and let us use the arc length parametrisation with respect to this metric.

Define as in paragraph 3.2.2, the map

$$\Phi_\gamma : \begin{cases} \gamma^{(3)} & \longrightarrow \text{Gr}_3(E), \\ (x_1, x_2, x_3) & \longmapsto x_1 \oplus x_2 \oplus x_3. \end{cases}$$

where $\text{Gr}_3(E)$ is the Grassmannian manifold of 3-spaces in E .

Since γ is smooth, using $\dot{\gamma}$ and $\ddot{\gamma} := \nabla_{\dot{\gamma}} \dot{\gamma}$, the map Φ_γ extends to a continuous map, also denoted Φ_γ on γ^3 . Observe that if $\mathbf{q}(\gamma) < 0$, then the projection of γ in $\mathbf{H}^{2,n}$ is strongly positive if and only if Φ_γ takes values in $\text{Gr}_{2,1}(E)$.

We also observe that Φ_γ is Lipschitz – with respect to the induced metric by h on $\text{Gr}_3(E)$ with a Lipschitz constant that depends on the third derivatives of γ .

Take an arclength parametrization — with respect to h — of Λ . Then $\{\Phi_{\gamma_k}\}_{k \in \mathbb{N}}$ converges uniformly to Φ_Λ by Arzela–Ascoli.

To conclude the proof of this first item we just need to prove that Φ_Λ takes values in $\text{Gr}_{2,1}(E)$. Let us consider for three vectors x_1, x_2 and x_3 in E

$$\Delta(x_1, x_2, x_3) = \det(\langle x_i, x_j \rangle)_{i,j}.$$

Observe also that the map Φ_Λ does not depend on the choice of a parametrisation; we therefore choose a parametrisation of Λ so that $\mathbf{q}(\Lambda(t)) = 1$. Observe then that taking the derivatives of $\mathbf{q}(\Lambda(t)) = 0$ and $\mathbf{q}(\dot{\Lambda}(t)) = 1$ yield the following equations

$$\langle \dot{\Lambda}(t), \Lambda(t) \rangle = 0, \quad \langle \ddot{\Lambda}(t), \dot{\Lambda}(t) \rangle = 0, \quad \langle \ddot{\Lambda}(t), \Lambda(t) \rangle = -1.$$

To simplify the notation let us write

$$V(t, s, u) := \Phi_\Lambda(\Lambda(t), \Lambda(t), \Lambda(s)).$$

- (i) Since Λ is positive, $V(t, s, u)$ takes values in $\text{Gr}_{2,1}(E)$ for pairwise distinct (s, t, u) .
- (ii) For distinct s and t , $V(t, t, s)$ is the space

$$\Lambda(t) \oplus \dot{\Lambda}(t) \oplus \Lambda(s).$$

Thus, or $s \neq t$, the above equations give, letting $a = \langle \Lambda(t), \Lambda(s) \rangle$ and $b = \langle \dot{\Lambda}(t), \Lambda(s) \rangle$

$$\Delta(\Lambda(s), \Lambda(t), \dot{\Lambda}(t)) = \det \begin{pmatrix} 0 & a & b \\ a & 0 & 0 \\ b & 0 & 1 \end{pmatrix} = -a^2 < 0,$$

Thus $V(t, t, s)$ is non degenerate, hence of type $(2, 1)$ by continuity.

- (iii) Finally, $V(t, t, t)$ is the space

$$\Lambda(t) \oplus \dot{\Lambda}(t) \oplus \ddot{\Lambda}(t).$$

Letting $c = \langle \Lambda(t), \ddot{\Lambda}(t) \rangle$ we have

$$\Delta(\Lambda(t), \dot{\Lambda}(t), \ddot{\Lambda}(t)) = \det \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & c \end{pmatrix} = -1 < 0.$$

Hence $V(t, t, t)$ is non degenerate, hence of type $(2, 1)$ by continuity.

We have proved that Φ_Λ takes values in $\text{Gr}_{2,1}(E)$ and so does Φ_{γ_k} for k large enough by uniform convergence. Thus γ_k is strongly positive for k large enough.

Proof of (ii). Lift p to p_+ in $\mathbf{H}_+^{2,n}$ and let $\mathbf{P} = (p_+, H)$ be a pointed hyperbolic plane containing p_+ . This gives a decomposition $\partial_\infty \mathbf{H}_+^{2,n} = \mathbf{S}^1 \times \mathbf{S}^n$ in which the boundary of H has the form $\mathbf{S}^1 \times \{v\}$. Since Λ is disjoint from $p^\perp$, it admits a connected lift Λ_+ to $\partial_\infty \mathbf{H}_+^{2,n}$ such that $\langle p_+, \Lambda_+ \rangle < 0$. This implies that in the splitting $\mathbf{S}^1 \times \mathbf{S}^n$, the positive loop Λ_+ is the graph of a smooth contracting map $f : \mathbf{S}^1 \rightarrow \mathbf{S}^n$ (see Proposition 2.11) whose image is contained in the hemisphere $B = \{x \in \mathbf{S}^n, \langle x, v \rangle > 0\}$.

Consider the family of maps $\{\phi_t\}_{t \in [0,1]}$ from B to itself given by

$$\phi_t(x) = \frac{tx + (1-t)v}{\|tx + (1-t)v\|}.$$

Observe that each ϕ_t is smooth and contracting for $t < 1$. Thus, the maps $f_t := \phi_t \circ f$ are smooth and contracting and so the projection Λ_t of their graph to $\partial_\infty \mathbf{H}^{2,n}$ define an isotopy $\{\Lambda_t\}_{t \in [0,1]}$ such that $\Lambda_1 = \Lambda$, the loop Λ_0 is the boundary of H and for each Λ_t is a smooth positive loop.

Because p belongs to the convex hull of every Λ_t , we can apply the construction on the proof of item (i). This gives an isotopy $\{\gamma_k^t\}_{t \in [0,1]}$ for every k . By compactness of the isotopy $\{\Lambda_t\}_{t \in [0,1]}$, it follows that for k large enough, every γ_k^t is strongly positive and so γ_k is deformable.

Proof of (iii). Denote respectively by d_k and d_0 the length along γ_k and γ_0 . Given two points $x_k, y_k \in \gamma_k$, we denote by x_k^o, y_k^o the points in γ_0 such that $x_k = \phi_k(x_k^o)$ and $y_k = \phi_k(y_k^o)$. We thus have by equation (33)

$$d_k(x_k, y_k) = \sinh(\rho_k) \cdot d_0(x_k^o, y_k^o). \quad (34)$$

Since p lies in the convex hull of Λ , Item (v) of Proposition 2.15 implies that the geodesics between p and x_k and between p and y_k are spacelike: here we use that the points x_k and y_k are constructed to lie on geodesics connecting p to Λ . Since γ_k is strongly positive by the first item, the geodesic between x_k and y_k is also spacelike and so the triple (p, x_k, y_k) is positive (unless p belongs to the geodesic between x_k and y_k in which case the following still holds). Hence p, x_k and y_k are the vertices of an isosceles hyperbolic triangle T_k . Classical hyperbolic trigonometry implies that

$$\sinh\left(\frac{\delta(x_k, y_k)}{2}\right) = \sinh(\rho_k) \cdot \sin\left(\frac{\alpha(x_k, y_k)}{2}\right), \quad (35)$$

where $\alpha(x_k, y_k)$ is the angle at p in the triangle T_k . In particular, $\alpha(x_k, y_k)$ is equal to the extrinsic distance between x_k^o and y_k^o in $\mathbf{T}_p^1 \mathbf{H}^{2,n}$.

Consider a sequence $\{(x_k, y_k)\}_{k \in \mathbb{N}}$ where x_k, y_k are distinct points of γ_k such that $\delta(x_k, y_k)$ tends to 0, and let $\alpha_k := \alpha(x_k, y_k)$.

By equation (35), $\lim_{k \rightarrow \infty} \frac{\delta(x_k, y_k)}{\alpha_k \sinh(\rho_k)} = 1$. Together with equation (34), we obtain

$$\lim_{k \rightarrow \infty} \frac{\delta(x_k, y_k)}{d_k(x_k, y_k)} = \lim_{k \rightarrow \infty} \frac{\alpha_k}{d_0(x_k^o, y_k^o)} = 1,$$

where for the last equality we used the fact that γ_0 is smooth, spacelike and that α_k is equal to the extrinsic distance between x_k^o and y_k^o .

As a result, there exists a $\delta > 0$ such that for any pair of distinct points x_k, y_k on γ_k with $\delta(x_k, y_k) \leq \delta$, then $\frac{\delta(x_k, y_k)}{d_k(x_k, y_k)} > \frac{1}{2}$. So the sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ is uniformly unpinched.

Proof of (iv). In this portion of the argument, we will use some definitions and results from Appendix A. Now, the curve $\gamma_k \subset M_k$ is obtained from $\gamma_0 \subset T_p \mathbf{H}^{2,n} \cong \mathbf{S}^{1,n}$ by rescaling the metric by a factor $\sinh(\rho_k)$. In particular, the curvature of M_k and γ_k converges uniformly to 0, and so for any sequence $\{x_k\}_{k \in \mathbb{N}}$ with $x_k \in \gamma_k$, the sequence $\{x_k, \gamma_k, M_k\}_{k \in \mathbb{N}}$ converges in the sense of Appendix A to $\{x, \Delta, \mathbf{E}^{1,n}\}_{k \in \mathbb{N}}$ where Δ is a spacelike line in the pseudo-Euclidean space $\mathbf{E}^{1,n}$ and $x \in \Delta$. In other words, the sequence $\{\gamma_k, M_k\}_{k \in \mathbb{N}}$ has bounded geometry.

On the other hand, the sequence $\{M_k, \mathbf{H}^{2,n}\}_{k \in \mathbb{N}}$ has bounded geometry since, up to shifting the center of the pseudosphere, $\{M_k\}_{k \in \mathbb{N}}$ converges to an horosphere (see Subsection 2.2). It follows from Lemma A.4 that $\{\gamma_k^o, \mathbf{H}^{2,n}\}_{k \in \mathbb{N}}$ has bounded geometry when γ_k is equipped (see Definition A.1) with the normal framing γ_k^o given by $P_k(x) := T_x \gamma_k \oplus N_x M_k$, where $N_x M_k$ is the normal to M_k at x . The bounded geometry of $\{\gamma_k, \mathbf{H}^{2,n}\}_{k \in \mathbb{N}}$, when γ_k is equipped with its canonical normal framing given by $T^{(2)}\gamma_k$, will follow from a lemma.

Lemma 8.3. *Using the same notation as above, the sequence $d_{\mathcal{G}}(T^{(2)}\gamma_k, P_k)$ converges uniformly to 0.*

Proof. Parametrize γ_k by arc length and define $\ddot{\gamma}_k := \nabla_{\dot{\gamma}_k} \dot{\gamma}_k$. Then by definition, $T^{(2)}\gamma_k = \text{span}\{\dot{\gamma}_k, \ddot{\gamma}_k\}$.

For a point x in γ_k , let $n_k(x)$ be the unit vector normal to M_k pointing outward, so $P_k = \text{span}\{\dot{\gamma}_k, n_k\}$. Since the planes $P_k(x)$ and $T_x^{(2)}\gamma_k$ intersect along $T_x \gamma_k$, we have

$$d_{\mathcal{G}}(T_x^{(2)}\gamma_k, P_k(x)) = d_{\mathbf{H}^n}(n_k(x), \ddot{\gamma}_k(x)),$$

where $n_k(x)$ and $\ddot{\gamma}_k(x)$ are considered as elements in the space of positive definite lines in the signature $(1, n)$ space $(x \oplus T_x \gamma_k)^\perp$, which is identified with $\mathbf{H}^n$. In particular,

$$\cosh(d_{\mathcal{G}}(T_x^{(2)}\gamma_k, P_k(x))) = \frac{\langle n_k(x), \ddot{\gamma}_k(x) \rangle}{\|\ddot{\gamma}_k(x)\|}.$$

Let us write $\ddot{\gamma}_k =: \mu_k + \nu_k$ where μ_k belongs to $P_k(x)$ and ν_k to $P_k^\perp(x)$. So

$$\cosh(d_{\mathcal{G}}(T_x^{(2)}\gamma_k, P_k(x))) = \frac{\mu_k}{\sqrt{\mu_k^2 + \nu_k^2}}.$$

By definition, $\kappa_k := g_{M_k}(\nu_k, \nu_k)$ is the geodesic curvature of γ_k seen as a curve in M_k . Similarly, $\mu_k n_k = \Pi_k(\dot{\gamma}_k, \dot{\gamma}_k)$ where Π_k is the second fundamental form of M_k . Since M_k is umbilical with induced curvature $\sinh^{-2}(\rho_k)$, we have

$$\Pi_k(X, Y) = \coth(\rho_k) g_{M_k}(X, Y) n_k.$$

Now, the curve $\phi_k^{-1}(\gamma_k) = \gamma_0$ is fixed independently of k . Moreover, by equation (33), the induced arclength parametrization of this curve grows without bound in k , and so the

geodesic curvature κ_k of γ_k in M_k converges uniformly to zero. The result follows from the fact that $\{\mu_k\}_{k \in \mathbb{N}}$ converges to 1. $\square$

Proof of (v). For each k , let x_k be a point in γ_k such that $w(\gamma_k) = \text{diam}(\pi_{x_k}(\gamma_k))$ where $\pi_{x_k} : E \rightarrow (x_k \oplus T_{x_k}\gamma_k)^\perp$ is the orthogonal projection (see Definition 3.17). We want to prove that the limit of $\text{diam}(\pi_{x_k}(\gamma_k))$, when k tends to infinity, is finite.

Denote by H_k the hyperbolic plane containing p and $T_{x_k}\gamma_k$. Since there is a subsequence of $\{x_k\}_{k \in \mathbb{N}}$ that converges to a point $x_\infty \in \Lambda$, the sequence of pointed hyperbolic planes $P_k = (p, H_k)$ subconverges to $P_\infty = (p, H_\infty)$ where H_∞ is the hyperbolic plane containing p and $T_{x_\infty}\Lambda$, here using the smoothness of Λ .

Fix a spacelike line $L \subset T_p H_\infty$ and take $g_k \in \mathbf{G}$ such that $g_k(T_{x_k}\gamma_k) = L$, and $g_k(H_k) = H_\infty$. Denote by $\gamma'_k = g_k(\gamma_k)$, $M'_k = g_k(M_k)$ and $p'_k = g_k(p)$. From Subsection 2.2, the sequence $\{M'_k\}_{k \in \mathbb{N}}$ converges to the horosphere $\mathcal{H}$ tangent to $p_\infty := \lim p'_k$ and passing through p . Denote by $\bar{\sigma} = \sigma \cup \{p_\infty\}$ the closure in $\bar{\mathbf{H}}^{2,n}$ of the horocycle $\sigma = \mathcal{H} \cap H_\infty$.

Lemma 8.4. *Using the same notation as above, for any sequence $\{y_k\}_{k \in \mathbb{N}}$ with $y_k \in \gamma_k$, there is a subsequence of $\{g_k(y_k)\}_{k \in \mathbb{N}}$ converging to a point in $\bar{\sigma}$.*

Proof. Let $\{y_k\}_{k \in \mathbb{N}}$ be such a sequence. Up to extracting a subsequence, we can assume one of the following:

- (i) For any k we have $y_k = x_k$,
- (ii) For any k , we have $y_k \neq x_k$ but $\lim y_k = x_\infty$,
- (iii) The limit of $\{y_k\}_{k \in \mathbb{N}}$ is different from x_∞ .

We will prove that in any case, the sequence $\{g_k(y_k)\}_{k \in \mathbb{N}}$ subconverges to a point in $\bar{\sigma}$.

Case (i) is obvious since $g_k(y_k) = p \in \sigma$.

For case (ii), observe that the hyperbolic plane V_k containing p, x_k and y_k converges to the hyperbolic plane H_∞ containing p and $T_{x_\infty}\Lambda$. In fact, if x_k^o and y_k^o are the points in $\gamma_0 \subset T^\perp \mathbf{H}^{2,n}$ such that $x_k = \phi_k(x_k^o)$ and $y_k = \phi_k(y_k^o)$, then V_k is also the hyperbolic plane such that $T_p V_k$ contains x_k^o and y_k^o . By the smoothness of γ_0 , the sequence of lines through x_k^o and y_k^o converges to $T_{x_\infty^o} \gamma_0$, where $\phi_\infty(x_\infty^o) = x_\infty$.

In particular, $g_k(y_k)$ belongs to $g_k(V_k) \cap g_k(M_k)$ and so accumulates to a point in the closure of the intersection $\mathcal{H} \cap H_\infty$, that is in $\bar{\sigma}$.

For case (iii), we will use a proximality argument in the spirit of Section 6.1. For each k , write $g_k = g'_k \cdot h_k$ where $h_k(p, H_k) = (p, H_\infty)$ is such that h_k converges to the identity. Then the sequence g'_k is a sequence in $\text{Stab}(H_\infty) \cong \text{Isom}(\mathbf{H}^2)$ which by construction satisfies:

$$\lim_{k \rightarrow \infty} g'_k(p) = p_\infty, \quad \lim_{k \rightarrow \infty} (g'_k)^{-1}(p) = x_\infty.$$

The sequence $\{g_k\}_{k \in \mathbb{N}}$ is such that it approximates the sequence $\{g'_k\}_{k \in \mathbb{N}}$ which have distinct fixed points on H_∞ , is thus P_1 -divergent and by Lemma 6.5 maps any point in $\mathbf{P}(E)$ that are not in $\mathbf{P}(x_\infty^\perp)$ to a sequence converging to p_∞ . The result then follows from the fact that Λ is positive, so $\mathbf{P}(x_\infty^\perp) \cap \Lambda = x_\infty$. $\square$

Now the proof of item (v) follows. For each k , let y_k and z_k in γ_k be such that $w(\gamma_k) = \text{diam}(\pi_{x_k}(\gamma_k)) = d_{\mathbf{H}^n}(\pi_{x_k}(y_k), \pi_{x_k}(z_k))$. Since the angular width is invariant under the action of $\mathbf{G}$, we have $w(\gamma_k) = \text{diam}(\pi_p(\gamma'_k))$ where $\pi_p : E \rightarrow L^\perp$. By the previous lemma, the sequence $\{g_k(y_k)\}$ and $\{g_k(z_k)\}$ subconverge to a point in $\bar{\sigma}$. Since $\pi_p(\bar{\sigma})$ is a point, the sequence $\{w(\gamma_k)\}_{k \in \mathbb{N}}$ subconverges to 0. $\square$

8.2. Proof of Theorem 8.1. Consider $\{\gamma_k\}_{k \in \mathbb{N}}$ the sequence constructed in Proposition 8.2. By Theorem 7.1, there exists a sequence $\{\Sigma_k\}_{k \in \mathbb{N}}$ of complete acausal maximal surfaces with $\partial \Sigma_k = \gamma_k$.

Since $\{\gamma_k\}_{k \in \mathbb{N}}$ satisfies the hypothesis of Theorem 6.2, we obtain a maximal surface whose asymptotic boundary is Λ .

APPENDIX A. BOUNDED GEOMETRY IN THE PSEUDO-RIEMANNIAN SETTING

A.1. Convergence of Riemannian and pseudo-Riemannian manifolds.

A.1.1. The Riemannian setting. We say that a sequence $\{(x_k, M_k)\}_{k \in \mathbb{N}}$ of pointed Riemannian manifolds *converges* C^n to a pointed Riemannian manifold (x, M) if for every compact set K in M , there exists an integer k_0 such that, for any $k > k_0$ there is an open set U containing K in M , a diffeomorphism ϕ_k from U onto an open set in M_k , so that when k tends to infinity, we have that $\{\phi_k^{-1}(x_k)\}_{k \in \mathbb{N}}$ converges to x and the metric $\phi_k^* g_k$ converge C^n to g in U , where g_k and g are respectively the metrics on M_k and M .

Similarly, given a sequence $\{(x_k, N_k, M_k)\}_{k \in \mathbb{N}}$ such that N_k is a submanifold of M_k and $x_k \in N_k$, we say that $\{(x_k, N_k, M_k)\}_{k \in \mathbb{N}}$ *converges* C^n to (x, N, M) if for every compact set K there exists an integer k_0 and an open set U containing K , such that for $k > k_0$, there is a diffeomorphism ϕ_k from the U to an open set U_k in M_k containing x_k , so that

- (i) $\phi_k(N \cap U) = N_k \cap U_k$,
- (ii) $\{\phi_k^{-1}(x_k)\}_{k \in \mathbb{N}}$ converges to x ,
- (iii) when k tends to infinity, the sequence $\{\phi_k^* g_k\}_{k \in \mathbb{N}}$ of metrics converge C^n to g in U .

We say an estimate *only depends on the local geometry* of a Riemannian manifold (M, x) if such an estimate holds uniformly for any sequence of pointed Riemannian manifolds converging to (M, x) .

A.1.2. The pseudo-Riemannian setting. The definitions in this setting require additional data.

Let M be a pseudo-Riemannian manifold of signature (p, q) , and N a submanifold of non-degenerate signature (p', q') .

Definition A.1. A *normal framing* of N is a smooth choice, for every x in N , of a positive definite $(p - p')$ -plane of the normal space $(T_x N)^\perp$. We will denote N^o the submanifold N together with a normal framing.

Remark that if N reduces to a point x , then a normal framing x^o is the choice of a positive definite p -plane of $T_x M$.

We denote by $\mathcal{G}_p(M)$ the Grassmannian of positive definite p -dimensional subspaces of M . By definition, $\mathcal{G}_p(M)$ is a bundle over M whose fiber over x is the Grassmannian $\text{Gr}_{p,0}(T_x M)$ of positive definite p -planes in $T_x M$ (thus the fiber is identified the symmetric space of $\text{SO}(p, q)$). The tangent space of $\mathcal{G}_p(M)$ at (x, P) splits as

$$T_{(x,P)} \mathcal{G}_p(M) = T_x M \oplus \text{Hom}(P, P^\perp) = P \oplus P^\perp \oplus \text{Hom}(P, P^\perp).$$

The Riemannian metric g on $\mathcal{G}_p(M)$ is given by

$$g_{(x,P)} = g_{x|P} \oplus (-g_{x|P^\perp}) \oplus h_P,$$

where h_P is the Killing metric on the symmetric space $\text{Gr}_{p,0}(T_x M)$ evaluated at P and g the metric on M .

A normal framing of a submanifold N gives an embedding of $\mathcal{G}_{p'}(N)$ into $\mathcal{G}_p(M)$ whose image we denote by $\mathcal{G}_{p'}^o(N)$. Given a triple (x, N, M) where N is a submanifold of M with

non-degenerate induced metric and x belongs to N , we say that the normal framing (x^o, N^o, M) is *compatible* if the normal framing of x contains the normal framing of N at x .

We can thus define the notion of convergence of framed pseudo-Riemannian submanifolds by using the Riemannian metric on the Grassmannian bundle:

Definition A.2. We say that the sequence $\{(x_k^o, N_k^o, M_k)\}_{k \in \mathbb{N}}$, where x_k^o is a compatibly normally framed point of N_k , and N_k^o is a normally framed submanifold of (non-degenerate) signature (p', q') of the pseudo-Riemannian manifold M_k of signature (p, q) converges C^n , if the corresponding sequence of pointed Riemannian submanifolds $\{x_{p'}^o, \mathcal{G}_{p'}^o(N_k), \mathcal{G}_{p'}(M_k)\}_{k \in \mathbb{N}}$ converges C^n .

Note that this notion of convergence highly depends on the choice of normal framing. This choice of normal framing of N is either vacuous or natural in two important cases that are used in this paper: in the case that N is a spacelike surface in a pseudo-Riemannian manifold of signature $(2, n)$, then the framing is trivial, and it is similarly trivial for spacelike curves equipped with a (spacelike) osculating plane. Accordingly we will not explicitly describe the normal framing in our definitions below that describe the convergence in these cases.

Recall that strongly positive curves in $\mathbf{H}^{2,n}$ is in particular 2-spacelike: that is the osculating plane $T^{(2)}\gamma = \text{span}\{\dot{\gamma}, \nabla_{\dot{\gamma}}\dot{\gamma}\}$ is spacelike everywhere.

Definition A.3 (SPACELIKE SURFACES AND STRONGLY POSITIVE CURVES). Let $\{M_k\}_{k \in \mathbb{N}}$ be a sequence of pseudo-Riemannian manifolds of signature $(2, n)$,

- (i) Let N_k be spacelike surfaces in M_k . In that case, we say that

$$\{x_k, N_k, M_k\}_{k \in \mathbb{N}} \text{ converges to } \{x_\infty, N_\infty, M_\infty\}$$

$$\text{if } \{T_{x_k}N_k, TN_k, \mathcal{G}(M_k)\}_{k \in \mathbb{N}} \text{ converges to } \{T_{x_\infty}N_\infty, TN_\infty, \mathcal{G}(M_\infty)\}.$$

- (ii) Let γ_k be a spacelike curve in M_k whose osculating plane $T^{(2)}\gamma_k$ is spacelike everywhere. In that case again we say that

$$\{x_k, \gamma_k, M_k\}_{k \in \mathbb{N}} \text{ converges to } \{x, \gamma, M\}$$

$$\text{if } \{T_{x_k}^{(2)}\gamma_k, T^{(2)}\gamma_k, \mathcal{G}(M_k)\}_{k \in \mathbb{N}} \text{ converges to } \{T_x^{(2)}\gamma, T^{(2)}\gamma, \mathcal{G}(M)\}.$$

Observe that the limits are always spacelike. More precisely in the first item, since by definition $T_{x_\infty}N_\infty$ is in $\mathcal{G}(M_\infty)$, $T_{x_\infty}N_\infty$ is spacelike. The same holds for the curves in the second item: by the same argument $T_x^{(2)}\gamma$ is spacelike, hence $T\gamma$ is spacelike.

As for the Riemannian setting, we can use this notion of convergence to define the notion of bounded geometry.

A.2. Bounded Geometry. A sequence of Riemannian manifolds $\{M_k\}_{k \in \mathbb{N}}$ has C^n bounded geometry if for every sequence of points $\{x_k\}_{k \in \mathbb{N}}$ with $x_k \in M_k$, then every subsequence of $\{(M_k, x_k)\}_{k \in \mathbb{N}}$ subconverges C^n to a pointed Riemannian manifold.

A sequence $\{(N_k, M_k)\}_{k \in \mathbb{N}}$ where M_k is a Riemannian manifold and N_k a submanifold in M_k has C^n bounded geometry if for every sequence of points $\{x_k\}_{k \in \mathbb{N}}$ with $x_k \in N_k$, every subsequence of $\{(x_k, N_k, M_k)\}_{k \in \mathbb{N}}$ subconverges C^n .

In the pseudo-Riemannian setting, we say a sequence $\{M_k\}_{k \in \mathbb{N}}$ of pseudo-Riemannian manifolds has C^n bounded geometry if the sequence of Riemannian manifolds $\{\mathcal{G}_p(M_k)\}_{k \in \mathbb{N}}$ has bounded geometry.

For pseudo-Riemannian submanifolds, we use a normal framing: a sequence $\{(N_k^o, M_k)\}_{k \in \mathbb{N}}$ where N_k is a submanifold of M_k with normal framing N_k^o , has C^n bounded geometry if for any sequence $\{x_k^o\}_{k \in \mathbb{N}}$ where x_k is a point of N_k with compatible normal framing x_k^o , every subsequence of $\{x_k^o, N_k^o, M_k\}_{k \in \mathbb{N}}$ subconverges C^n .

Accordingly, using Definition A.3, we extend this definition to say that $\{(N_k, M_k)\}_{k \in \mathbb{N}}$ has bounded geometry when N_k is a spacelike surface, or a 2-spacelike curve, in a pseudo-Riemannian submanifold M_k of signature $(2, n)$.

The following is a direct consequence of the previous definitions:

Lemma A.4. *For each k , consider N_k^o a normally framed submanifold of M_k , and let γ_k^o be a normally framed submanifold of N_k . Then if $\{\gamma_k^o, N_k\}_{k \in \mathbb{N}}$ and $\{N_k^o, M_k\}_{k \in \mathbb{N}}$ have bounded geometry, then so does $\{\gamma_k^{oo}, M_k\}_{k \in \mathbb{N}}$, where γ_k^{oo} is the submanifold γ_k of M_k equipped with the normal framing induced by γ_k^o and N_k^o .*

APPENDIX B. A LEMMA IN PLANE TOPOLOGY

Lemma B.1. *Let D_1 and D_0 be two smoothly embedded closed disks in the plane so that D_1 is embedded in the interior of D_0 . Let x_1 be a point in D_1 . Let $\gamma : [0, 1] \rightarrow D_0$ be an arc smoothly embedded in D_0 with $\gamma\{0, 1\} \in \partial D_0$ and γ transverse to ∂D_1 . Then there exists a disk U embedded in the interior of D_0 , so that ,*

- (i) U contains the connected component of x_1 in $D_1 \setminus \gamma$,
- (ii) $\partial U = \gamma_1 \cup \eta$, where γ_1 is a connected sub arc of γ , and η a connected sub arc of ∂D_1

The smoothness and transversality hypothesis are quite possibly not necessary but are enough for our purpose and simplify the argument.

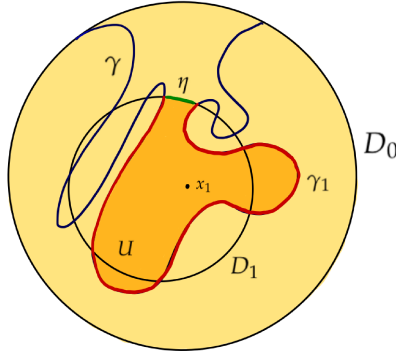


FIGURE 3. Lemma B.1

Proof. Let U_1 be the connected component of x_1 in $D_1 \setminus \gamma$. If $U_1 = D_1$ then we take $U = U_1$ and the proof is over. Otherwise, let

$$t_0 := \inf\{t \mid \gamma(t) \in \overline{U_1}\}.$$

Let $y_0 := \gamma(t_0)$, then $D_0 \setminus (\partial D_1 \cup \gamma)$ intersects a small enough disk neighbourhood of y_0 , in four connected components. Only one of these connected components, say V_0 , is included in U_1 . Let η be the connected component of $\partial D_1 \setminus \gamma$ so that a subarc of η lies in the boundary of V_0 . Let $y_1 = \gamma(s_0)$ be the other extremity of η .

By construction $\gamma[t_0, s_0] \cup \eta$ is an embedded arc, bounding an embedded disk U_2 . Since $V_0 \cap U_2$ is non-empty by construction, it follows that $U_1 \subset U_2$. We can therefore take $U = U_2$, and this concludes the proof. $\square$

APPENDIX C. HOLOMORPHIC CURVES AND GROMOV'S SCHWARZ LEMMA

We recall the basis of the theory of holomorphic curves and some results from [26] for the convergence portion and [32] for the part concerning the area. We give an improvement of these results as well as considerations of when a limit of immersions is an immersion.

C.1. Preliminaries. Recall that an *almost-complex structure* on a even dimensional manifold M is a section J of the bundle of endomorphisms of TM such that $J^2 = -\text{Id}$. When the real dimension of M is equal to 2, an almost complex structure is always integrable (that is, comes from an holomorphic atlas), and we call such a manifold M a Riemann surface.

Definition C.1. Given an almost complex manifold (M, J) , a *holomorphic curve* is a smooth map $f : (X, j) \rightarrow (M, J)$ where (X, j) is a Riemann surface, and satisfying $Tf \circ j = J \circ Tf$.

In this paper, we will be mostly interested in the case where $X = \mathbf{D}$ defined by

$$\mathbf{D} = \{z \in \mathbb{C}, |z| < 1\}.$$

The *frontier* of $\mathbf{D}$ is

$$\text{Fr}(\mathbf{D}) = \overline{\mathbf{D}} \setminus \mathbf{D} = \{z \in \mathbb{C}, |z| = 1\}.$$

Definition C.2. A *totally real submanifold* of an almost-complex manifold (M, J) (possibly with boundary) is a submanifold $W \subset M$ of half the dimension and such that for any x in W , we have $T_x M = T_x W \oplus J(T_x W)$.

Our main focus is the *semi-disk* $\mathbf{S}$ defined by

$$\mathbf{S} = \{z \in \mathbf{D}, \Re(z) \geq 0\}.$$

We denote by

$$\partial \mathbf{S} = \mathbf{S} \setminus \text{int}(\mathbf{S}) = \{z \in \mathbf{D}, \Re(z) = 0\},$$

$$\text{Fr}(\mathbf{S}) = \overline{\mathbf{S}} \setminus \mathbf{S} = \{z \in \text{Fr}(\mathbf{D}), \Re(z) \geq 0\}.$$

In this case, $\partial \mathbf{S}$ is a totally real submanifold of $\mathbf{S}$.

Definition C.3. A *holomorphic curve with boundary* in an almost complex manifold M with totally real submanifold W is a holomorphic curve f from $\mathbf{S}$ to M mapping $\partial \mathbf{S}$ to W .

Definition C.4. Let (M, J) be a almost complex manifold equipped with a Riemannian metric $\langle \cdot | \cdot \rangle$. An open set U in M is *K-calibrated* if there exists a 1-form β so that

$$\forall u \in TU, \beta(u)^2 \leq K^2 \cdot \langle u | u \rangle, \quad \langle u | u \rangle \leq K \cdot d\beta(u, Ju). \quad (36)$$

If W is a totally real, totally geodesic submanifold, we furthermore assume that β vanishes along $W \cap U$.

Then we have the following lemma

Lemma C.5 (LOCAL CALIBRATION). *There exists positive constants ε and K , only depending on the geometry of (M, x) so that the ball B of radius ε in M of center x is K -calibrated.*

We sketch a proof since the extension incorporating the totally real submanifold W is not in the original paper [40].

Proof. Let $\exp : T_x M \rightarrow M$ be the exponential map. Observe that $T_0 \exp$ is holomorphic. We choose $\beta = \exp_* \lambda$, where $\lambda \in \Omega^1(T_x M)$ is defined by $\lambda_u(v) = \langle u | Jv \rangle$. Since the preimage by $\exp$ of W is $T_x W$ since W is totally geodesic, and $T_x W$ is a totally real submanifold (actually linear) of $T_x M$ since W is, it follows that λ vanishes on $T_x W$ and thus β vanishes on W . The result follows. $\square$

Remark C.6. The notion of holomorphic curve can be extended to the following case. The (not necessarily even-dimensional) manifold M carries a distribution $\mathcal{D} \subset TM$ equipped with an almost complex structure $J : \mathcal{D} \rightarrow \mathcal{D}$. We then ask that a holomorphic curve is a map f such that Tf takes values in $\mathcal{D}$ and intertwines the almost complex structures.

In this framework, a totally real submanifold is a submanifold W of M of half the dimension of the distribution $\mathcal{D}$ and such that for any x in W we have $\mathcal{D}_x = T_x W \oplus J(T_x W)$.

All of the results described in the sequel canonically extend to this case.

C.2. Schwarz lemmas and convergence of holomorphic curves. In this subsection, we state and sketch the proofs of two of our main goals for this appendix. After a few definitions, we state the results, then collect some preliminaries before concluding with a description of the arguments.

To begin, let f be a map from $\mathbf{D}$ or $\mathbf{S}$ to M . If Z is a subset of $\text{Fr}(\mathbf{D})$ or respectively $\text{Fr}(\mathbf{S})$, we denote by $f(Z)$ the accumulation set of sequences $\{f(y_k)\}_{k \in \mathbb{N}}$ where $\{y_k\}_{k \in \mathbb{N}}$ is a sequence converging to a point in Z .

Let $\{M_k\}_{k \in \mathbb{N}}$ be a sequence of complete almost complex Riemannian manifolds with bounded geometry, let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence of points. We assume that $x_k \in M_k$ and that $\{(x_k, M_k)\}_{k \in \mathbb{N}}$ converges $C^{p,\alpha}$ as almost complex Riemannian manifolds.

Theorem C.7 (FREE BOUNDARY). *Let $\{U_k\}_{k \in \mathbb{N}}$ be a sequence, where U_k is an open set in M_k , uniformly calibrated and with bounded geometry. Let $\{f_k\}_{k \in \mathbb{N}}$ be a sequence of holomorphic maps from $\mathbf{D}$ with values in U_k . Then $\{f_k\}_{k \in \mathbb{N}}$ subconverges $C^{p,\alpha}$ on every compact set to f_0 . Assume furthermore that*

$$\sup \{\text{area}(f_k(\mathbf{D})), k \in \mathbb{N}\} < \infty. \quad (37)$$

Then for every non-empty open subset Z of $\text{Fr}(\mathbf{D})$,

$$f_0(Z) \subset \lim_{k \rightarrow \infty} f_k(Z).$$

Assume now that W_k is a totally real submanifold and totally geodesic submanifold of M_k containing x_k and that $\{(x_k, W_k, M_k)\}_{k \in \mathbb{N}}$ converges $C^{p,\alpha}$ to $\{(x_0, W_0, M_0)\}$, with W_0 totally real. In the boundary case, the following is an extension of [33].

Theorem C.8 (BOUNDARY). *Let $\{U_k\}_{k \in \mathbb{N}}$ be a sequence of open sets in M_k , uniformly calibrated and with bounded geometry. Let $\{f_k\}_{k \in \mathbb{N}}$ be a sequence of holomorphic maps from $\mathbf{S}$ with values in U_k . Then $\{f_k\}_{k \in \mathbb{N}}$ subconverges $C^{p,\alpha}$ on every compact set to f_0 . Assume furthermore that*

$$\sup \{\text{area}(f_k(\mathbf{S})), k \in \mathbb{N}\} < \infty. \quad (38)$$

Then for every non-empty open subset Z of $\text{Fr}(\mathbf{S})$, we have

$$f_0(Z) \subset \lim_{k \rightarrow \infty} f_p(Z).$$

Both these theorems represent an improvement over the corresponding earlier results which only considered the case $Z = \text{Fr}(\mathbf{D})$ or $Z = \partial \mathbf{S}$.

C.2.1. Quadrangles and extremal length. We prepare for the proof by recalling a classical construction.

A *quadrangle* $Q := (U, x_1, x_2, x_3)$ in $\mathbb{C}$ equipped with the complex structure J is a topological disk U with four marked points (x_1, x_2, x_3, x_4) in cyclic order in ∂U . The *a-rectangle* is the rectangle $R_a := (R, a, a + i, i, 0)$ of vertices $(0, a, a + i, i)$. Two quadrangles (U, x_1, x_2, x_3) and (V, y_1, y_2, y_3) are conformally equivalent if we can find a conformal

mapping Φ sending $\bar{U}$ to $\bar{V}$ so that $\phi(x_i) = y_i$. Every quadrangle is conformally equivalent to a unique a -rectangle.

Let Q be a quadrangle and Γ_Q be the set of arcs in $\bar{U}$ joining a point in the interval between x_1 to x_2 on $\text{Fr}(U)$ to a point in the interval between x_3 to x_4 on $\text{Fr}(U)$. The *extremal length* of Q is $\mathcal{L}(Q)$ where for a metric g

$$\mathcal{L}_g(Q) = \frac{\inf \{ \text{length}_g^2(\gamma) \mid \gamma \in \Gamma_Q \}}{\text{area}_g(Q)} \quad (39)$$

$$\mathcal{L}(Q) = \sup \{ \mathcal{L}_g(Q) \mid g \text{ conformal to } J \} , \quad (40)$$

where length_g and area_g denotes respectively the length and area with respect to g . By construction $\mathcal{L}(Q)$ is a conformal invariant. A classical result asserts

Proposition C.9. *We have $\mathcal{L}(R_a) = a$.*

Let Z_0 be the *standard quarter* in S^1 , that is the subarc of S^1 between 1 and i . Let Q_0 be the standard sector

$$Q_0 := \{ z \in \mathbf{D} \mid \Re(z) \geq 0, \Im(z) \geq 0, |z| < 1 \} ,$$

so that $\text{Fr}(Q_0) = Z_0$.

Let $\ell(R)$ be the Euclidean distance from 0 to a point x in $\mathbf{D}$ at hyperbolic distance R from 0. (Of course, it is classical that $\ell(R) = \tanh(R)$, but the precise formula is not important for our discussion.) We define the *R -corner quadrangle* to be the quadrangle $Q_R = (U, i\ell(R), \ell(R), 1, i)$ where $U = \{ z \in \mathbf{C} \mid \ell(R) < |z| < 1, \Re(z) > 0, \Im(z) > 0 \}$.

One then checks, for example by applying the conformal map $z \mapsto \log z$ to the domain Q_R , that

Proposition C.10. *The map $K : R \rightarrow \mathcal{L}(Q_R)$, is a decreasing homeomorphism from $(0, \infty)$ to $(0, \infty)$.*

The following lemma is a consequence of the previous discussion and is used in the sequel.

Lemma C.11. *For any positive A and ε , there exists a positive constant ρ , with the following property. Assume f is a holomorphic map from Q_0 to an almost complex manifold M . Assume that $\text{area}(f(Q_0)) \leq A$ and $\|Tf\|$ is bounded by $\frac{\varepsilon}{\rho}$ on the ball of radius ρ (in Q_0) (with respect to the hyperbolic metric on $\mathbf{D}$). Then $d(f(0), f(Z_0)) \leq 2\varepsilon$.*

Proof. We choose ρ so that $A \cdot \mathcal{L}(Q_\rho) \leq \varepsilon^2$. Let g be the induced metric by f from M . Observe that for any curve γ in Γ_{Q_ρ} ,

$$\begin{aligned} \text{length}_g(\gamma) &\geq d(f(\gamma(0)), f(\gamma(1))) \geq d(f(0), f(Z_0)) - d(f(0), f(\gamma(0))) \\ &\geq d(f(0), f(Z_0)) - \varepsilon , \end{aligned}$$

where the last inequality uses the fact that $\|Tf\|$ is bounded by $\frac{\varepsilon}{\rho}$ on $B(0, \rho)$. Thus

$$\varepsilon^2 \geq \mathcal{L}(Q_\rho) \cdot \text{area}_g(Q_\rho) \geq (d(f(0), f(Z_0)) - \varepsilon)^2 .$$

The result follows. $\square$

C.2.2. Sketch of the proof of the first part of Theorem C.7. Without the hypothesis on the area, the subconvergence is consequence of the celebrated Gromov's Schwarz Lemma [26, 40] which states that the derivatives of f_k are uniformly bounded. We sketch the argument, since we are going to sketch a modification of it. We need three preliminary lemmas. In the first two, M is a manifold equipped with an almost complex structure J and a compatible metric $\langle \cdot, \cdot \rangle$ (that is a metric for which J is an isometry). First we have (see [40])

Lemma C.12 (WEINGARTEN LEMMA). *Let Σ be a holomorphic curve in M , and let $x \in \Sigma$. Then there is a bound K only depending on the geometry of (M, x) so that the curvature of Σ at x is less than K .*

Our second lemma from [40] is

Lemma C.13 (GROMOV'S SCHWARZ LEMMA). *Let g be a conformal metric on the disk. Let g_0 be the hyperbolic metric and h the conformal factor so that $g = hg_0$. Assume that the curvature of g is bounded from above by K , and that g satisfies a linear isoperimetric inequality, that is for any disk A embedded in $\mathbf{D}$, we have*

$$\text{area}_g(A) \leq K \text{length}_g(\text{Fr}(A)) . \quad (41)$$

Then there exists a bound K_0 only depending on K so that $h \leq K_0$.

Combining these two lemmas gives the celebrated

Lemma C.14 (GROMOV'S HOLOMORPHIC SCHWARZ LEMMA). *Let K_1 be a positive constant. There exists a positive constant K_0 , only depending on the local geometry of (M, x) and K_1 so that if φ is a holomorphic map from the hyperbolic disk $\mathbf{D}$ to a K_1 -calibrated open set with bounded curvature then*

$$\|\mathbf{T}\varphi\| \leq K_0 .$$

Proof. By replacing φ by the graph map $\varphi' = (\varphi, \text{Id}) : \mathbf{D} \rightarrow M \times \mathbf{D}$ we may assume that φ is an immersion.

We consider the induced metric g by φ . By the Weingarten Lemma C.12 the curvature of g is bounded from above. From the definition of calibration, the metric g satisfies a linear isoperimetric inequality:

$$\text{area}(A) \leq K_1 \int_A d\beta = K_1 \int_{\partial A} \beta \leq K_1^2 \text{length}(\text{Fr}(A)) .$$

Thus the result follows by Gromov's Schwarz lemma C.13. $\square$

The strengthening of the first conclusion of Theorem C.7 with the hypothesis on the area is an extension of [32, Lemme 6.8]. This will be proved in the last paragraph of this section.

C.2.3. *Sketch of the proof of the first part of Theorem C.8.* We proceed as in [33, Lemme 9.1]. We have

Proposition C.15. *Let φ be a holomorphic immersion from $\mathbf{S}$ to a Riemannian almost complex manifold M equipped with a compatible metric, so that $f(\partial\mathbf{S})$ lies in a totally real totally geodesic submanifold W . Then $\partial\mathbf{S}$ is totally geodesic for the metric induced by φ .*

Proof. Let J be the complex structure of M and $\langle \cdot | \cdot \rangle$ the compatible metric. Let γ be an arc length parametrisation of $f(\partial\mathbf{S})$. The geodesic curvature of γ is $\langle \nabla_{\dot{\gamma}} \dot{\gamma} | J\dot{\gamma} \rangle$. Since $\gamma = f(\partial\mathbf{S})$ is embedded in a totally geodesic submanifold W , $\nabla_{\dot{\gamma}} \dot{\gamma}$ and $\dot{\gamma}$ lie in $\mathbf{T}W$. Since W is totally real, for all u and v in $\mathbf{T}W$, we have $\langle u | Jv \rangle = 0$, and the result now follows. $\square$

Combining this lemma with the previous arguments, we obtain

Lemma C.16 (HOLOMORPHIC SCHWARZ LEMMA WITH BOUNDARY). *There exists positive constants ε and K_0 , only depending on the geometry of (M, x) so that if φ is a holomorphic map from the semi-disk $\mathbf{S}$ into the ball of radius ε centred at x , then*

$$\|\mathbf{T}\varphi\| \leq K_0 .$$

Proof. After replacing φ by the graph map $\varphi' = (\varphi, \text{Id})$ from $\mathbf{S}$ to $M \times \mathbf{D}$ and W by $W' = W \times \partial\mathbf{S}$, we may assume that φ is an immersion.

Let g_0 be the hyperbolic metric on $\mathbf{S}$ and $g = hg_0$ the metric induced by φ . By the Weingarten Lemma, the curvature of g is bounded from above. Since $\partial\mathbf{S}$ is totally geodesic for g , we can double g to obtain a C^0 metric g_2 on $\mathbf{D}$. By the doubling argument and since $\partial\mathbf{S}$ is totally geodesic, the curvature of g_2 is also bounded from below.

To conclude the proof using Gromov's Schwarz Lemma one needs to show that g_2 satisfies a linear isoperimetric inequality. Let us use the form β obtained from the Local Calibration Lemma C.5. Let A be a disk in $\mathbf{D}$, write $A = A_0 \cup A_1$, where $A_1 = A \cap \mathbf{S}$. Then by Stokes formula

$$\text{area}(A_1) \leq K \int_{A_1} d\beta = K \int_{\text{Fr}(A_1)} \beta = K \int_{\text{Fr}(A) \cap \overline{A_1}} \beta \leq K^2 \text{length}(\text{Fr}(A)) ,$$

where the first equality follows from the fact that $\beta = 0$ on W , hence on $\partial\mathbf{S}$. Repeating the argument for A_0 leads the desired linear isoperimetric inequality for g_2 . $\square$

C.2.4. Improving regularity. Gromov's Schwarz Lemma gives uniform C^1 -bounds on the sequence $\{f_k\}_{k \in \mathbb{N}}$. We need to improve this and proves the $C^{p,\alpha}$ convergence to obtain the first part of Theorems C.7 and Theorem C.8.

This is done in two steps. As a preliminary, we choose $C^{p,\alpha}$ local coordinates on M_k so that M_k is identified with $\mathbb{C}^n$ and W_k with $\mathbb{R}^n$. Thanks to our C^1 bounds, we reduce to the case (by possibly shrinking the source) to bounded maps f_k with values in $\mathbb{C}^n$. The holomorphic curve condition then reduces to the equation

$$\partial_y f_k = J_k(f_k) \partial_x f_k , \quad (42)$$

where x and y are the coordinates on $\mathbf{D}$ or $\mathbf{S}$ and $\{J_k\}_{k \in \mathbb{N}}$ converges in $C^{p-1,\alpha}$. When present, the boundary condition is

$$f_k(\partial\mathbf{S}) \subset \mathbb{R}^n .$$

The two steps of our regularity improvement are as follows.

Uniform C^2 -bounds: we consider the 1-jet map $g_k = (f_k, \partial_y f_k)$ with values in $\mathbb{C}^{2n}$ satisfying the boundary condition

$$g_k(\partial\mathbf{S}) \subset \mathbb{R}^{2n} .$$

A derivation of (42) gives that g_k is holomorphic for a certain complex structure J'_k on $\mathbb{C}^{2n}$. Indeed, erasing for a moment the index k to have readable formulas, we claim:

$$\partial_y g = J'(g) \partial_x g ,$$

where for (s, t) in $\mathbb{C}^n \times \mathbb{C}^n$,

$$J'(s, t)(u, v) = (J(s)u, J(s)v + A(s, t)(u)) ,$$

where $A(s, t) := (D_s J)(t)$ is the derivative of J at s in the direction of t . Since $A(s, t)$ anticommutes with $J(s)$, we see that

$$J'^2(u, v) = (-u, -v + JA(u) + AJ(u)) = -(u, v) .$$

Gromov's holomorphic Schwarz Lemma then gives a uniform C^1 -bound on g_k , hence the desired C^2 -bound on f_k .

$C^{p,\alpha}$ -convergence: Now that we have C^2 bounds we can return to the equation (42) with the information that f_k is in C^2 and in particular in $C^{1,\alpha}$, knowing that J_k converges in $C^{p-1,\alpha}$.

The proposition is then

Proposition C.17. *Assume that u is in $C^{1,\alpha}$ and satisfies the equation*

$$\partial_y u = J(u) \partial_x u, \quad (43)$$

with possibly the boundary condition $u(\partial S) \subset \mathbb{R}^n$, where J is in $C^{1,\alpha}$. Then u is in $C^{2,\alpha}$. More precisely, for every positive constant A , there exist positive constants C and ε so that if the $C^{1,\alpha}$ norm of u is less than A , then the $C^{2,\alpha}$ norm of u is less than C on a ball of radius ε .

Proof. We reproduce and adapt the proof of Theorem A.2.1 in [1] in two ways, first by using $C^{1,\alpha}$ bounds rather than $W^{k,p}$ Sobolev norms, and second in assuming lower regularity of J .

In this proof, the quantities C_i will be positive constants. We may as well assume that $u(0) = 0$ and $J(0) = i$ and restate equation (43) as

$$2i\bar{\partial}u = -i(i - J(u))\partial_x u, \quad (44)$$

We use the difference quotient technique and introduce for small h

$$u^h(x, y) := \frac{1}{h}(u(x, y+h) - u(x, y)) = \int_0^1 \partial_y u(x, y+th) dt. \quad (45)$$

Moreover

$$\begin{aligned} 0 &= \partial_y u^h - (J(u)\partial_x u)^h \\ &= \partial_y u^h - J(u)\partial_x u^h - J(u)^h \partial_x u(x, y+h) \\ &= 2i\bar{\partial}u^h + i(J(u) - i)\partial_y u^h - J(u)^h \partial_x u(x, y+h). \end{aligned} \quad (46)$$

Let also β be a bell function in $\mathbb{R}$ with $\beta[0, 1/2] = 1$, $\beta[1, +\infty[= 0$, $\beta'(s) \leq 0$. We define

$$\beta_\varepsilon(x, y) = \beta\left(\frac{x^2 + y^2}{\varepsilon}\right).$$

We now obtain, from (46) and the Leibniz rule $\beta_\varepsilon \partial_y (u^h) = \partial_y (\beta_\varepsilon u^h) - \partial_y (\beta_\varepsilon) u^h$,

$$\begin{aligned} 2i\bar{\partial}(\beta_\varepsilon u^h) &= -i(J(u) - i)\partial_y (\beta_\varepsilon u^h) + i(J(u) - i)(\partial_y \beta_\varepsilon) u^h \\ &\quad + \beta_\varepsilon J(u)^h \partial_y u(x, y+h) + (\bar{\partial} \beta_\varepsilon) u^h. \end{aligned} \quad (47)$$

Let us denote by $\|v\|_{p,\alpha,\varepsilon}$ the $C^{p,\alpha}$ norm on the ball of radius ε while using the shorthand $\|v\|_{p,\alpha} = \|v\|_{p,\alpha,1}$.

Let us make a series of estimates

(i) In the equation above, let us consider the term

$$B = 2i(\bar{\partial} \beta_\varepsilon) u^h + i(J(u) - i)(\partial_y \beta_\varepsilon) u^h.$$

We have a constant C_ε depending only on ε and β , so that

$$\|B\|_{0,\alpha} \leq C_\varepsilon \|u^h\|_{0,\alpha,\varepsilon}. \quad (48)$$

(ii) Restricting ε so that $\varepsilon + h < 1$, we have

$$\|(i - J(u))(\partial_y \beta_\varepsilon u^h)\|_{0,\alpha,\varepsilon} \leq C_3 \|u\|_{0,\alpha,\varepsilon} \cdot \|\beta_\varepsilon u^h\|_{1,\alpha}. \quad (49)$$

(iii) Since J is in $C^{1,\alpha}$ and in particular uniformly Lipschitz, we have a uniform constant C_0 so that

$$|J(u)^h| \leq C_0 |u^h|.$$

Thus

$$\|\beta_\varepsilon J(u)^h \partial_y u(x+h, y)\|_{0,\alpha,\varepsilon} \leq C_1 \|u\|_{1,\alpha,\varepsilon} \cdot \|u^h\|_{0,\alpha,\varepsilon}. \quad (50)$$

Combining the estimates (48), (49) and (50) with our original equation (47), we obtain that

$$\begin{aligned} \|\bar{\partial}(\beta_\varepsilon u^h)\|_{0,\alpha,\varepsilon} &\leq C_\varepsilon \|u^h\|_{0,\alpha} + C_1 \|u\|_{1,\alpha,\varepsilon} \|u^h\|_{0,\alpha,\varepsilon} \\ &\quad + C_3 \|u\|_{0,\alpha,\varepsilon} \cdot \|u^h\|_{1,\alpha} + C_1 \|u\|_{1,\alpha,\varepsilon} \cdot \|\beta_\varepsilon u^h\|_{0,\alpha}. \end{aligned} \quad (51)$$

Recall also that from the Cauchy–Pompeiu formula (see [4, Theorem 4.7.1] for a model) we have the estimates

$$\|\beta_\varepsilon u^h\|_{1,\alpha} \leq C_4 \|\bar{\partial}(\beta_\varepsilon u^h)\|_{0,\alpha}.$$

We now fix ε , so that $C_3 C_4 \|u\|_{0,\alpha,\varepsilon} \leq \frac{1}{2}$. Then for some constant $D = D_\varepsilon$ depending on u and independent of h , we find

$$\frac{1}{2} \|u^h\|_{1,\alpha,\frac{\varepsilon}{2}} \leq \frac{1}{2} \|\beta_\varepsilon u^h\|_{1,\alpha} \leq D \|u^h\|_{0,\alpha}.$$

Observe that the same methods also yield that for all $\eta < \alpha$ we have

$$\frac{1}{2} \|u^h\|_{1,\eta,\frac{\varepsilon}{2}} \leq D \|u^h\|_{0,\eta}.$$

Let us write $\psi^h := u^h - \partial_y u$ and $M = \|u\|_{1,\alpha}$. Using the integral form for u^h in equation (45), we observe that ψ^h satisfies $\|\psi^h\|_{0,\alpha} \leq 2M$ and $|\psi^h(x)| \leq 2Mh^\alpha$.

This implies that for all $\eta < \alpha$, we have

$$|\psi^h(z) - \psi^h(w)| \leq 2M \cdot \min\{h^\alpha, |z - w|^\alpha\} \leq 4M \cdot h^{\alpha-\eta} |z - w|^\eta.$$

Thus $\|\psi^h\|_{0,\eta} \leq 4Mh^{\alpha-\eta}$. Thus u^h converges to $\partial_y u$ in $C^{0,\eta}$ for all $\eta < \alpha$, and then

$$\frac{1}{2} \|\partial_y u\|_{1,\eta,\frac{\varepsilon}{2}} \leq D \|\partial_y u\|_{0,\eta}.$$

Then taking the limit when η goes to α , and using that $\partial_y u \in C^{0,\alpha}$, we find

$$\frac{1}{2} \|\partial_y u\|_{1,\alpha,\frac{\varepsilon}{2}} \leq D \|\partial_y u\|_{0,\alpha}.$$

Thus u is in $C^{2,\alpha}$ in the ball of radius $\varepsilon/2$. $\square$

Bootstrap and regularity: We can now conclude the argument by showing that if $\{u_k\}_{k \in \mathbb{N}}$ is a sequence of solutions in $C^{p+1,\alpha}$ satisfying

$$\partial_y u_k = J_k(u_k) \partial_x(u_k),$$

where

- $\{J_k\}_{k \in \mathbb{N}}$ converges in $C^{p,\alpha}$ to J_0 for which $\mathbb{R}^n$ is totally real,
- $\{u_k\}_{k \in \mathbb{N}}$ has uniform C^1 -bounds and converges C^0 to u_0 ,

then $\{u_k\}_{k \in \mathbb{N}}$ converges in $C^{p+1,\alpha}$.

This is obtained via the bootstrap described in the first step, or equivalently as in [1, Theorem A.2.1] which immediately leads, using proposition C.17 to

Proposition C.18. *Assume that u is in $C^{k,\alpha}$ satisfies the equation (42)*

$$\partial_y u = J(u) \partial_x u,$$

with possibly the boundary condition $u(\partial\mathcal{S}) \subset \mathbb{R}^n$, where J is in $C^{p,\alpha}$, then u is in $C^{p+1,\alpha}$. More precisely, for every positive constant A , there exist positive constants C and ε so that if the $C^{p,\alpha}$ norm of u is less than A , then the $C^{k+1,\alpha}$ norm of u is less than C on a ball of radius ε .

C.2.5. *Using the hypothesis on the area.* We now show the second part of Theorem C.7 and Theorem C.8.

Proof. Using the Schwarz Lemma, we can extract in both cases a subsequence so that $\{f_k\}_{k \in \mathbb{N}}$ subconverges to f_0 .

Observe that we have a constant A so that for all subsets U of $\mathbf{D}$, or $\mathbf{S}$, then

$$\text{area}(f_0(U)) \leq A. \quad (52)$$

Let $\{y_k\}_{k \in \mathbb{N}}$ be a sequence in $\mathbf{D}$ or $\mathbf{S}$ converging to a point y_0 in an interval Z in $\text{Fr}(\mathbf{D})$.

We want to show that there exists a sequence $\{f_{n_k}\}_{k \in \mathbb{N}}$ for which we have

$$\lim_{k \rightarrow \infty} (d(f_0(y_k), f_{n_k}(Z))) = 0.$$

From the bound on the area of $f_0(\mathbf{D})$, we have that for all R

$$\lim_{k \rightarrow \infty} (\text{area}(f_0(B(y_k, R))) = 0.$$

where $B(y_k, R)$ is the ball of radius R in the hyperbolic metric.

Using the fact that $\{f_k\}_{k \in \mathbb{N}}$ converges on every compact to f_0 , we can choose a subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$, so that

$$\text{area}(f_{n_k}(B(y_k, 1))) \leq \frac{1}{k}, \quad d(f_0(y_k), f_{n_k}(y_k)) \leq \frac{1}{k}. \quad (53)$$

Let u_k be a conformal mapping of $\mathbf{D}$ that sends 0 to y_k .

Since $\{y_k\}_{k \in \mathbb{N}}$ converges to an interior point of Z , the sequence $\{u_k^{-1}(Z)\}_{k \in \mathbb{N}}$ converges to the full boundary of $\mathbf{D}$. We can thus choose for each k , a subinterval Z_k in Z so that the preimage of Z_k by u_k is a quarter of circle Z_0 .

After precomposing u_k with a rotation, we may furthermore assume that the preimage of Z_k is the standard quarter of circle.

Let then Q_0 (as defined in the beginning of paragraph C.2.1) be the standard sector. We furthermore choose Z_k so that Q_0 is a subset of $u_k^{-1}(\mathbf{S})$ in the boundary case.

To conclude the theorem it will be enough to prove

$$\lim_{k \rightarrow \infty} d(f_{n_k}(y_k), f_{n_k}(Z_k)) = 0. \quad (54)$$

Let then $g_k = f_{n_k} \circ u_k$. Assertion (54) is now restated as

$$\lim_{k \rightarrow \infty} d(g_k(0), g_k(Z_0)) = 0. \quad (55)$$

Applying Schwarz Lemma again, $\{g_k\}_{k \in \mathbb{N}}$ subconverges to some g_0 . By inequality (53), the area of $g_0(B(0, 1))$ is equal to 0, thus g_0 is constant.

Let us choose a positive ε . Let A be the bound of the area defined in inequality (52) and ρ as in Lemma C.11. Since $\{g_k\}_{k \in \mathbb{N}}$ converges uniformly on every compact to the constant map g_0 , it follows that for k large enough $\|Tg_k\|$ is bounded above by $\frac{\varepsilon}{\rho}$ on $B(0, \rho)$. We can thus conclude from Lemma C.11 that for k large enough

$$d(g_k(0), g_k(Z_0)) \leq 2\varepsilon.$$

The assertion (55) follows, hence the theorem. $\square$

C.3. Immersions. A non-constant limit of holomorphic immersions may not be immersed. We describe here certain situations in which a limit of immersions is an immersion. This result is a generalization of the case when the target is $\mathbb{C}$: roughly speaking the role of the 2-dimensional target is played by a complex line bundle $L_{\mathbb{C}}$, together with a never vanishing 1-form α with values in L . Our "immersion in $\mathbb{C}$ " is now replaced by a holomorphic map f so that $f^*\alpha$ is uniformly non-vanishing.

Let us be more precise about our hypothesis: let M be an almost complex manifold, L be a real line bundle over M , $L_{\mathbb{C}}$ the complexification of L and $\alpha \in \Omega^1(M, L_{\mathbb{C}})$ a never vanishing 1-form with values in $L_{\mathbb{C}}$.

We also choose a Hermitian metric h on $L_{\mathbb{C}}$ as well as a unitary connection ∇^0 on $L_{\mathbb{C}}$ for which L is parallel.

When we have a boundary problem defined by a totally real submanifold W we furthermore assume that $\alpha(TW) = L$. The result is the following version of a claim that a sequence of maps that are strongly immersed, in terms of the existence of one-form α which they all pull back in a non-degenerate way, limit on a map with the same immersivity property.

Theorem C.19. *Let $\{f_k\}_{k \in \mathbb{N}}$ be either (free boundary case) a sequence of holomorphic maps from $\mathbf{D}$, or (boundary case) a sequence of holomorphic maps from $\mathbf{S}$ to M so that $f_k(\partial\mathbf{S})$ is included in W . Equip (M, W) with a real line bundle L and a one-form α with values in its complexification $L_{\mathbb{C}}$ as above, together with the chosen Hermitian metric h and parallel unitary connection ∇^0 .*

Assume that

- (i) *the sequence $\{f_k\}_{k \in \mathbb{N}}$ converges to f_0 ,*
- (ii) *for all k , $f_k^*\alpha$ never vanishes,*
- (iii) *we have a constant K_0 so that for all k and*

$$\|f_k^* d^{\nabla^0} \alpha\| \leq K_0 \|f_k^* \alpha\|^2, \quad (56)$$

using the metric h and some auxiliary metric on M .

- (iv) *assume finally that $f_0^*\alpha$ is not identically zero.*

Then f_0^α never vanishes.*

We remark

Proposition C.20. *Condition (56) is satisfied in the following two cases*

$$d^{\nabla^0} \alpha = 0, \quad \text{or} \quad \|Tf_k\| \leq K_1 \|f_k^* \alpha\|.$$

Proof. The first case in Proposition C.20 is obvious. The second case follows from the remark that if $\beta \in \Omega^k(V)$, then $\|g^* \beta\| \leq \|Tg\|^k \cdot \|\beta\|$. $\square$

The first case is satisfied when $\alpha = d\pi$, where π is a submersion in $\mathbb{C}$ which maps W to a line, in which case the setting reduces to simply maps from $\mathbf{D}$ (or $\mathbf{S}$) to $\mathbb{C}$.

C.3.1. Preliminary controls. Let f be a holomorphic map from $\mathbf{D}$ to M satisfying inequality (56), and g the quadratic form defined by

$$g(X, Y) := h(\alpha(Tf(X)), \alpha(Tf(Y))),$$

Let O be the open set in $\mathbf{D}$ on which g is a metric, and let u be the continuous vector fields of norm 1 (defined up to sign) on O so that $\alpha(Tf(u)) \in L$.

Lemma C.21. *Let β be the connection form for u defined by*

$$\beta(X) = g(\nabla_X u, Ju),$$

where ∇ is the Levi-Civita connection of g . Then

$$\|\beta(X)\|^2 \leq K_1 g(X, X).$$

where K_1 only depends on K_0 , α and ∇^0 .

Proof. Let us consider the induced bundle $L_0 := f^*L$, as well as the pull-back metric h_0 , induced forms $\zeta = f^*\alpha \in \Omega^1(\mathbf{D}, L_0)$, and induced connection $D = f^*\nabla^0$. In the proof k_i will be constants only depending on α and ∇^0 . The classical formula for the Levi Civita connection tells us that

$$\begin{aligned} & 2g(\nabla_X u, Ju) \\ = & u \cdot g(X, Ju) - Ju \cdot g(X, u) + X \cdot g(u, Ju) \\ & - g(u, [X, Ju]) - g(X, [u, Ju]) + g(Ju, [X, u]) \\ = & u \cdot h_0(\zeta(X), \zeta(Ju)) - Ju \cdot h_0(\zeta(X), \zeta(u)) + X \cdot h_0(\zeta(u), \zeta(Ju)) \\ & - h_0(\zeta(u), \zeta([X, Ju])) - h_0(\zeta(X), \zeta([u, Ju])) + h_0(\zeta(Ju), \zeta([X, u])) \\ = & 2h_0(D_X(\zeta(u)), \zeta(Ju)) \\ & + h_0(\zeta(u), d^D \zeta(X, Ju)) - h_0(\zeta(Ju), d^D \zeta(X, u)) + h_0(\zeta(X), d^D \zeta(u, Ju)). \end{aligned}$$

Here of course $d^D \zeta(X, Y) = (D_X \zeta)(Y) - (D_Y \zeta)(X) - \zeta([X, Y])$.

Since L is parallel for ∇^0 , it follows that $h_0(D_X(\zeta(u)), \zeta(Ju)) = 0$. Observe now that the hypothesis (56) and the Cauchy–Schwarz inequality implies that

$$|h_0(\zeta(X), d^D \zeta(Y, Z))|^2 \leq K_0^2 g(X, X) \cdot g(Y, Y) \cdot g(Z, Z).$$

Thus, from the above inequality applied to the final three terms of the computation above of $2\beta(X) = 2g(\nabla_X u, Ju)$ and using that $g(u, u) = g(Ju, Ju) = 1$, we see that

$$\|\beta(X)\| \leq \frac{3}{2} K_0 \sqrt{g(X, X)}.$$

This concludes the proof. $\square$

Lemma C.22. Let $g = \lambda^2 g_0$, and let β_0 be the connection form of the vector u_0 proportional to u and of norm 1 for g_0 . Then

$$-(d \log \lambda) \circ J = \beta - \beta_0,$$

Proof. The connection of $g = \lambda^2 g_0$ is given by

$$\nabla = D + (d \log \lambda) \otimes \text{Id} - (d \log \lambda \circ J) \otimes J,$$

where D is the connection of g_0 . Thus, if u is the vector field of norm 1 for g , then

$$\beta(X) = g(\nabla_X u, Ju) = g(D_X u, Ju) - d \log \lambda(JX)$$

Observe now that $\lambda v = v_0$ where v_0 has norm 1 for g_0 , thus

$$g(D_X v, Jv) = \frac{1}{\lambda^2} g(D_X v_0, Jv_0) = g_0(D_X v_0, Jv_0) = \beta_0(X).$$

The result follows. $\square$

Corollary C.23. Assuming f is an immersion, let γ be either

- (i) (Free boundary case) an embedded circle γ in $\mathbf{D}$, or
- (ii) (Boundary case) or an embedded half circle so that $\partial\gamma \subset \partial\mathbf{S}$.

Then

$$\left| \int_{\gamma} (d \log \lambda) \circ J \right| \leq K_2 \int_{\gamma} \lambda ds ,$$

where ds is the arc length of γ with respect to g_0 .

Proof. Let us consider first the free boundary case: from Lemma C.22,

$$\int_{\gamma} (d \log \lambda) \circ J = \int_{\gamma} \beta - \int_{\gamma} \beta_0 = \int_{\gamma} \beta - \int_U d\beta_0 = \int_{\gamma} \beta ,$$

where U is the disk of boundary γ in the boundary free case, and boundary $\gamma \sqcup I$, where $I \subset \partial S$ in the boundary case. Observe that we have used here that

$$\int_I \beta_0 = 0 ,$$

which follows from the fact that u_0 is tangent to ∂S , so that its covariant derivative in the tangential direction is also tangential and hence orthogonal to Ju_0 . Thus the inequality follows from the bounds in Lemma C.21. For the boundary case, we first have to remark that if $X \in T\partial S$, then obviously $\beta_0(X) = 0$. Moreover, $f(\partial S)$ is a curve in a totally real and totally geodesic manifold W . Thus ∇Xu belongs to TW and Ju is orthogonal to TW . Thus

$$\beta(X) = g(\nabla Xu, Ju) = 0 .$$

The fact that β and β_0 are zero when restricted to ∂S allows us to conclude the argument. $\square$

C.3.2. *Proof of Theorem C.19.* The proof in both cases follow the same scheme. We will point out where the difference occurs. It is enough to prove that $f_0^* \alpha$ does not vanish at 0. Let g_k be the conformal metric on $\mathbf{D}$ given by

$$g_k(u, v) = h(\alpha(Tf_k(u)), \alpha(Tf_k(u))) ,$$

and λ_k be the function on $\mathbf{D}$ so that $g_k = \lambda_k^2 g_0$ where g_0 is the Euclidean metric on $\mathbf{D}$.

To prove the theorem, it is enough is to find a positive ρ so that, for all $k \in \mathbb{N}$

$$\lambda_k(0) \geq \rho .$$

Let $D(R)$ be the disk of radius R centered at 0 with respect to g_0 and $\gamma(R)$ its boundary. In the boundary case, we let $D(R)$ be the half disk centered at 0 and denote by $\gamma(R)$ the half circle which is part of $\text{Fr}(D(R))$. We denote by (r, θ) the polar coordinates on $\mathbb{C} \setminus \{0\}$. Let ω_θ is the closed form on $\mathbb{C} \setminus \{0\}$ given by

$$\omega_\theta = \frac{x dy - y dx}{x^2 + y^2} .$$

(Of course, ω_θ is usually denoted by $d\theta$ but the notation $d\theta$ –disliked by the first author – suggests that ω_θ is exact) Observe that for any 1-form α

$$\begin{aligned} \alpha \wedge \omega &= \alpha(\partial_r) dr \wedge \omega_\theta , \\ \int_{S(r)} \alpha \circ J &= \int_{S(r)} \alpha(J\partial_\theta) \omega_\theta = -r \int_{S(r)} \alpha(\partial_r) \omega_\theta . \end{aligned}$$

For $k \in \mathbb{N}$, let G_k be the function $\mathbb{R}_{>0}$ given by

$$G_k(R) := \int_{\partial D(R)} \log(\lambda_k) \cdot \omega_\theta .$$

Observe that, because λ_k does not vanish, we map apply Stokes theorem to the annulus $\{\eta < r, R\}$, and let $\eta \rightarrow 0$ to obtain

$$G_k(R) = \int_{D(R)} d \log(\lambda_k) \wedge \omega_\theta + G_k(0) = \int_0^R \left(\int_{\gamma(r)} d \log(\lambda_k)(\partial_r) \cdot \omega_\theta \right) dr + G_k(0).$$

After taking the derivatives with respect to R , we get

$$\dot{G}_k(R) = \int_{\gamma(R)} d \log(\lambda_k)(\partial_r) \cdot \omega_\theta = -\frac{1}{R} \int_{\gamma(R)} d \log(\lambda_k) \circ J.$$

Since f_k is an immersion, it follows from Corollary C.23 that

$$|\dot{G}_k(R)| \leq \frac{K_0}{R} \int_{\gamma(R)} \lambda_k ds,$$

where ds is the length with respect to g_0 . By the Schwarz lemma, we see that λ_k is uniformly bounded from above and thus $|\dot{G}_k(R)| \leq C_2$. It follows that for all R_0

$$\varepsilon \pi |\log \lambda_k(0)| = |G_k(0)| \leq |G_k(R_0)| + C_2 R_0,$$

where $\varepsilon = 2$ in the free boundary case, and $\varepsilon = 1$ in the boundary case. Thus

$$\lambda_k(0) \geq \exp \left(-\frac{1}{\varepsilon \pi} (|G_k(R_0)| + C_0 R_0) \right).$$

Then, since λ_∞ has isolated zeroes (See the similarity principle Theorem A.5.2 and Proposition A.5.3 in [1]), there exists some R_0 so that $\gamma(R_0)$ does not contain any zeroes of λ_∞ , thus there exists a positive α so that

$$\int_{\partial D(R_0)} \log(\lambda_\infty) \cdot \omega_\theta > -\alpha > -\infty.$$

Thus for k large enough, $G_k(R_0) \geq -2\alpha$. In particular, for k large enough

$$\lambda_k(0) \geq \mu := \exp \left(-\frac{1}{\varepsilon \pi} (2\alpha + C_0 R_0) \right) > 0.$$

The result follows.

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