Development and Application of a Novel High-Order Fully Actuated System Approach—Part I: 3-DOF Quadrotor Control

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***Abstract*—The quadrotor hierarchical control design (position-attitude) based on the state-space modeling has been widely applied in the past. Although the state-space representation, based on a group of first-order differential equations, is effective in modeling many dynamic systems, inherent high-order dynamics and control of quadrotor systems may not be properly handled by the state-space modeling. This letter proposes a modified high-order fully actuated (HOFA) theory for a group of high-order dynamic systems, including the quadrotor system, without relying on pseudo strict-feedback forms required by the original HOFA approach. Hence, the quadrotor model can be essentially converted into two HOFA subsystems. A nonlinear 3-DOF quadrotor modeling and control is applied as an example to demonstrate the effectiveness of the proposed approach, which can achieve arbitrarily assignable eigenstructure like a stabilized linear system.**

***Index Terms*—Quadrotor control, high order fully actuated, modeling and control of nonlinear systems.**

I. Introduction

OVER the last decades, unmanned aerial vehicles have been the focus of much research because of their broad educational, commercial, industrial, and military applications [1], [2], [3]. The rotorcraft UAVs have several merits over fixed-wing UAVs, such as hovering at a particular place, vertical takeoff, landing, and agile maneuvering. The quadrotor is a unique rotorcraft and does not require complex mechanical linkages commonly applied on other types of rotorcrafts such as helicopters and tiltrotors. It is a highly coupled, under-actuated system with four rotor inputs (mixed into the roll, pitch, yaw, and throttle) and six outputs (3-DOF translational position and 3-DOF orientation). As a differentially flat nonlinear system [1], [14], [17], generally, the 3-DOF translational positions plus the yaw angle of a quadrotor with respect to the earth frame are tracked by the flight controller. The roll and pitch angles are not independently controlled with four inputs.

To achieve desired flying performance, model-based control via the state space modeling was developed and applied in a hierarchical control architecture for quadrotors. The hierarchical architecture is widely used to separate different time-scale control between the fast inner loop (attitude) and slow outer loop (position). Linear control systems such as Classical PID algorithms have been demonstrated successfully due to their simplicity and proven reliability in practice [4]. Multi-loop PID architectures derived for a particular flight profile over a trimmed model have performed even better [11]. Nonlinear controls such as Euler angle-based quadrotor control design techniques with rigorous mathematical foundations include feedback linearization and backstepping Lyapunov functions, sliding mode control, robust adaptive tracking control [5], [6], [13]. Further investigation on nonlinear manifolds such as the quaternion [7], [8] and the special Euclidean group SE(3) [10] based controllers are developed with desirable close-loop properties that are almost global. In [12], the Lagrange formulation is explored, providing a more straightforward derivation of drag compensation with hierarchical structures. However, for all these hierarchical controllers with first-order state space representation, the position tracking performance will be unsatisfactory if the quadrotor cannot track the desired attitude quickly and precisely [14].

Different from all these hierarchical designs, a controller design method based on the high-order model without the assumption that the attitude dynamics are much faster than position control in the horizontal plane was firstly presented in [14]. A high-order state observer is designed to estimate the thrust's first and second-order time derivatives used in the controller designed for the high-order system model. Although a high-order system modeling was applied, the modeling and control design was not organized in a systematic manner.

Recently, a new system modeling technique, namely the high-order fully actuated model (HOFA), was developed to eliminate nonlinear terms using the full actuation [20], [21]. After cancellation, a globally stabilized linear system is obtained in a controllable form. The HOFA approach is relatively general and can be applied to systems that the well-known backstepping and feedback linearization methods are not applicable. One key step of the HOFA approach is about the derivation of a HOFA model for the studied system under different circumstances. The crucial procedure is to convert the nonlinear system into a pseudo strict-feedback form [20]. Once the set of HOFA models is obtained, the controllers of the subsystems are then immediately written out. Finally, the

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closed-loop system consists of several independent constant linear subsystems with arbitrarily assignable eigenstructures. Conducting the HOFA approach on the quadrotor platform would provide the advantage over the conventional hierarchical designs mentioned above. Unfortunately, a group of nonlinear systems, including quadrotors, cannot be converted into a pseudo strict-feedback form based on [22].

This letter will provide a modified HOFA approach, for the first time, in the application of quadrotor modeling and control. This modified HOFA formulation does not require the model to be converted into a pseudo strict-feedback form. Because the planar quadrotor model is a typical non-minimum phase dynamical system that has been heavily featured in the literature of acrobatic robots [16], [25], [26], a 2nd-order 3-DOF planar quadrotor model is applied as an example to verify the effectiveness of the modified HOFA approach. The formulated planar quadrotor system is divided into two high-order subsystems. Then, inputs of each subsystem are directly written, and each closed-loop subsystem has desired eigenstructures. The HOFA approach for a complete mixed-order 6-DOF quadrotor is discussed in the authors' paper with experimental results [23].

This letter is structured as follows. The existing HOFA approach is reviewed in Section II. A modified HOFA approach, which does not require the pseudo strict-feedback form in the existing method, is presented in Section III. A 3-DOF planar quadrotor model and the HOFA formulation are presented in Section IV. Simulation results are discussed to verify the proposed approach and control performance in Section V, and this letter is concluded in Section VI.

II. Preliminaries of HOFA Modeling and Control

This section briefly reviews modeling and control of HOFA systems with some standard terminologies and mathematic expressions. For more details on the HOFA approach, the literature [19], [20], [21] and references therein are referred.

For $x \in \mathbb{R}^n$, and $A_i \in \mathbb{R}^{n\times n}(i = 1, 2, \ldots, m)$, frequently used symbols in this letter are defined as follows,

$$\begin{aligned} x^{(0\sim m)} &\triangleq \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(m)} \end{bmatrix}, \\ x_{i\sim j}^{(0\sim m)} &\triangleq \begin{bmatrix} x_i^{(0\sim m)} \\ x_{i+1}^{(0\sim m)} \\ \vdots \\ x_j^{(0\sim m)} \end{bmatrix}, j \geq i, \\ A_{0\sim m} &\triangleq \begin{bmatrix} A_0 & A_1 & \cdots & A_m \end{bmatrix}. \end{aligned} \tag{1}$$

A. Modeling and Control of HOFA Systems

Consider a general nonlinear system in the following affine form,

$$x^{(m)} = f\left(x^{(0\sim m-1)}\right) + B\left(x^{(0\sim m-1)}\right)u, \tag{2}$$

where $x \in \mathbb{R}^n$ is the system state vector, $u \in \mathbb{R}^r$ is the system input vector, $f(x^{(0\sim m-1)}) \in \mathbb{R}^n$ is a sufficiently differentiable vector function, and $B(x^{(0\sim m-1)}) \in \mathbb{R}^{n\times r}$ is a matrix function. Note that the upper index m of x represents the mth time derivative of state $x(t)$, without the symbol of "$\sim$". Other terms with "$\sim$" represent vectors defined in (1).

Given $\mathrm{rank}(B(x^{(0\sim m-1)})) = r = n$, the system (2) is called fully actuated [20]. If $r > n$, the system (2) is called over-actuated, and such systems can be similarly treated as fully actuated ones in terms of control.

Proposition 1 [20]: Let $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, m-1$, be a set of given matrices, then the following controller

$$\begin{cases} u = -B^{-1}\left(x^{(0\sim m-1)}\right)\left(A_{0\sim m-1}x^{(0\sim m-1)} + u^*\right) \\ u^* = f\left(x^{(0\sim m-1)}\right) - v, \end{cases} \tag{3}$$

for the fully actuated system (2) with $r = n$ will give the constant linear closed-loop system,

$$x^{(m)} + A_{0\sim m-1}x^{(0\sim m-1)} = v. \tag{4}$$

For the linear system (4), the eigenvalues or eigenstructures can be arbitrarily assigned by selecting appropriate A_i matrices and external signal v [20]. This desired property for control design makes modeling a HOFA system attractive for the control design of general nonlinear systems. In fact, many nonlinear systems can be (re)formulated into HOFA systems.

B. HOFA Approach for Underactuated Systems

Given the system (2) with $r < n$ the system is called underactuated [17], [18]. Without loss of generality, we consider the case $n = r + n_0$ with $0 < n_0 < r$. To convert an underactuated system into a HOFA form, a basic procedure was provided in [20] and reviewed here as a baseline to propose the new approach in the next section.

The first step is to find a unimodular matrix Q with $z = [z_1\ z_2]^T = Qx$, $z_1 \in \mathbb{R}^{n-r}$ satisfying,

$$QB = \begin{bmatrix} 0 \\ G\left(x^{(0\sim m-1)}\right) \end{bmatrix}, \ G \in \mathbb{R}^{r\times r}. \tag{5}$$

Multiplying Q on both sides of (2) and replacing x with z give the pseudo strict-feedback form,

$$\begin{bmatrix} z_1^{(m)} \\ z_2^{(m)} \end{bmatrix} = \begin{bmatrix} f_1\left(z^{(0\sim m-1)}\right) \\ f_2\left(z^{(0\sim m-1)}\right) \end{bmatrix} + \begin{bmatrix} 0 \\ G\left(z^{(0\sim m-1)}\right) \end{bmatrix}u. \tag{6}$$

Assumption 1 [20]: $z_1^{(m)} = f_1(z^{(0\sim m-1)})$ is in the following form,

$$z_1^{(m)} = g\left(z_1^{(0\sim m-1)}\right) + B_1 z_2. \tag{7}$$

where $g \in \mathbb{R}^{n-r}$ is a vector function. This step is crucial and restricts the subsystem (7) to be linear with respect to z_2. $B_1 \in \mathbb{R}^{(n-r)\times r}$ is a full-row rank matrix.

This assumption restricts that $z_1^{(m)}$ has a direct relation with z_2 but not its derivatives. Taking the m-order derivatives on both sides of (7) and substituting $z_2^{(m)}$ from the second subsystem of (6), give,

$$z_1^{(2m)} = g^{(m)}(\cdot) + B_1(f_2(\cdot) + G(\cdot)u), \tag{8}$$

which can be rewritten as

$$z_1^{(2m)} = \psi(\cdot) + B_1 G(\cdot)u, \tag{9}$$

with

$$\psi(\cdot) = \psi\left(z_1^{(0\sim 2m-1)}, z_2^{(0\sim m-1)}\right) = g^{(m)}(\cdot) + B_1 f_2(\cdot). \tag{10}$$

To further expand (9), we divide the control vector as,

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \ u_1 \in \mathbb{R}^{n-r}. \tag{11}$$

Partitioning $G(\cdot)$ as $G(z^{(0\sim m-1)}) = [G_1 G_2]$, $G_1 \in \mathbb{R}^{r\times(n-r)}$, (9) can be rewritten as,

$$z_1^{(2m)} = \psi(\cdot) + B_1 G_1\ u_1 + B_1 G_2 u_2. \quad (12)$$

Assuming $B_1 G_1$ is nonsingular [20], we can design the controller for the first HOFA subsystem as,

$$u_1 = -(B_1G_1)^{-1}\begin{pmatrix} A^z_{0\sim 2m-1} z_1^{(0\sim 2m-1)} + \psi(\cdot) \\ + B_1 G_2\ u_2 - v_1 \end{pmatrix}, \quad (13)$$

which gives the following linear high-order system,

$$z_1^{(2m)} + A^z_{0\sim 2m-1} z_1^{(0\sim 2m-1)} = v_1, \quad (14)$$

where $v_1 \in \mathbb{R}^{n-r}$ is an external signal. z_2 in (6) also needs to be handled. Let P be a nonsingular matrix such that (M is nonsingular),

$$B_1 P = \left[M_{(n-r)\times(n-r)}\ 0_{(n-r)\times(2r-n)}\right]. \quad (15)$$

We can define the transformation,

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^{-1} z_2,\ y_1 \in \mathbb{R}^{n-r}, \quad (16)$$

and correspondingly partition P^{-1} as,

$$P^{-1} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix},\ N_1 \in \mathbb{R}^{(n-r)\times r}. \quad (17)$$

Then we multiply P^{-1} on both sides of the second part of (6),

$$\begin{aligned} P^{-1} z_2^{(m)} &= P^{-1} f_2\left(z^{(0\sim m-1)}\right) + P^{-1} G\left(z^{(0\sim m-1)}\right) u, \\ &\Downarrow \\ y_1^{(m)} &= N_1 f_2\left(z_1^{(0\sim m-1)}, y^{(0\sim m-1)}\right) \\ &\quad + N_1 G\left(z_1^{(0\sim m-1)}, y^{(0\sim m-1)}\right) u, \\ y_2^{(m)} &= N_2 f_2\left(z_1^{(0\sim m-1)}, y^{(0\sim m-1)}\right) \\ &\quad + N_2 G\left(z_1^{(0\sim m-1)}, y^{(0\sim m-1)}\right) u. \end{aligned} \quad (18)$$

Note that from (15),

$$\begin{aligned} B_1 z_2^{(m)} &= B_1 P P^{-1} z_2^{(m)} \\ &= \left[M_{(n-r)\times(n-r)} 0_{(n-r)\times(2r-n)}\right] y^m = M y_1^{(m)}. \end{aligned} \quad (19)$$

Based on (7), (8), and (19), y_1 has contributions to $z_1^{(2m)}$ and y_2 does not. To derive the control of y_2, $y_2^{(m)}$ can be rewritten from (18),

$$y_2^{(m)} = N_2 f_2(\cdot) + N_2 G_1\ u_1 + N_2 G_2\ u_2. \quad (20)$$

Assume $N_2 G_2$ is nonsingular, we can design the controller for (20) as follows,

$$u_2 = -(N_2G_2)^{-1}\begin{pmatrix} A^y_{0\sim m-1} y_2^{(0\sim m-1)} + N_2 f_2(\cdot) \\ + N_2 G_1 u_1 - v_2 \end{pmatrix}, \quad (21)$$

which produces the following linear closed-loop system

$$y_2^{(m)} + A^y_{0\sim m-1} y_2^{(0\sim m-1)} = v_2 \quad (22)$$

If $\det(N_2 G_2) \neq 0$ is not met, (20) needs to be reorganized into an affine form and then handled similarly [20].

Remark 1: Although the above HOFA derivation is organized systematically in [20], the related process cannot be directly applied to model and control a quadrotor system because the required unimodular matrix Q may not exist to formulate the pseudo strict-feedback form (proof in Appendix). Inspired by the dynamic properties of quadrotor systems, we propose to modify the HOFA approach for the application without requiring a pseudo strict-feedback form.

III. Modified HOFA Approach

To apply the HOFA approach to quadrotor modeling and control, we present a modified HOFA approach that does not require converting the system (2) into a pseudo strict-feedback form, which are feasible for more high-order nonlinear systems.

First, we need to find a unimodular matrix Q with $z = [z_1\ z_2]^T = Qx$, $z_1 \in \mathbb{R}^{n-r}$ and a nonsingular matrix $P \in \mathbb{R}^{r\times r}$, such that $u = Pu_z = P[u_{z,1}\ u_{z,2}]^T$, $u_{z,2} \in \mathbb{R}^{n-r}$ and,

$$QBP = \begin{bmatrix} G_1 & 0_{(n-r)\times(n-r)} \\ & G_2 \end{bmatrix}, \quad G_1 \in \mathbb{R}^{(n-r)\times(2r-n)},\ G_2 \in \mathbb{R}^{r\times r}. \quad (23)$$

Then we have the following derivations starting by multiplying Q on both sides of (2) with $Qf = [f_1\ f_2]^T$,

$$\begin{aligned} Qx^m &= Qf\left(x^{(0\sim m-1)}\right) + QB\left(x^{(0\sim m-1)}\right) Pu_z \\ &\Downarrow \\ z_1^{(m)} &= f_1\left(z^{(0\sim m-1)}\right) + G_1\left(z^{(0\sim m-1)}\right) u_{z,1} \\ z_2^{(m)} &= f_2\left(z^{(0\sim m-1)}\right) + G_2\left(z^{(0\sim m-1)}\right) u_z. \end{aligned} \quad (24)$$

Assumption 2: $z_1^{(m)}$ is in the following form:

$$z_1^{(m)} = f_1\left(z_1^{(0\sim m-1)}, z_2\right) + G_1\left(z_1^{(0\sim m-1)}, z_2\right) u_{z,1}, \quad (25)$$

Remark 2: Although the Assumption 2 restricts the subsystem to have a non-derivative term of z_2, the same as Assumption 1, z_1 is not required to be affine with z_2. Moreover, the coefficient matrix G_1 in front of control $u_{z,1}$ is more general than a constant matrix.

We now take m-order derivatives on both sides of (25), and apply the general Leibniz rule to have

$$z_1^{(2m)} = f_1^{(m)}\left(z_1^{(0\sim m-1)}, z_2\right) + \sum_{k=0}^{m} \binom{m}{k} G_1^{(k)} u_{z,1}^{(m-k)}. \quad (26)$$

Let $f_1^{(m)}(z_1^{(0\sim m-1)}, z_2) = h(z_1^{(0\sim 2m-1)}, z_2^{(0\sim m)})$, (26) is written as,

$$\begin{aligned} z_1^{(2m)} &= h\left(z_1^{(0\sim 2m-1)}, z_2^{(0\sim m)}\right) \\ &\quad + \sum_{k=0}^{m-1} \binom{m}{k} G_1^{(k)} u_{z,1}^{(m-k)} + G_1^{(m)} u_{z,1}. \end{aligned} \quad (27)$$

The 1st and the 3rd terms of (27) can be written as,

$$\begin{aligned} h\left(z_1^{(0\sim 2m-1)}, z_2^{(0\sim m)}\right) &= h_1\left(z_1^{(0\sim 2m-1)}, z_2^{(0\sim m-1)}\right) \\ &\quad + h_2\left(z_1^{(0\sim m-1)}, z_2\right) z_2^{(m)}, \end{aligned} \quad (28)$$

$$\begin{aligned} G_1^{(m)}\left(z_1^{(0\sim m-1)}, z_2\right) u_{z,1}^{(0)} &= (H_1\left(z_1^{(0\sim 2m-1)}, z_2^{(0\sim m-1)}\right) \\ &\quad + H_2\left(z_1^{(0\sim m-1)}, z_2\right) z_2^{(m)}) u_{z,1}, \end{aligned} \quad (29)$$

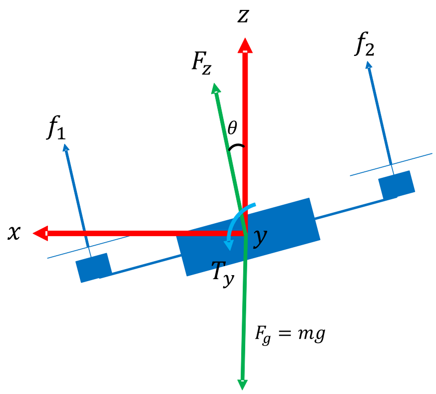

Fig. 1. 3-DOF quadrotor for a planar motion.

Denote $\sum_{k=0}^{m-1}\binom{m}{k}G_1^{(k)}u_{z,1}^{(m-k)} = L_1(\cdot)$, $h_1(\cdot)+H_1(\cdot)u_{z,1}+L_1(\cdot)=\psi(\cdot)$, and $h_2(\cdot)+H_2(\cdot)u_{z,1}=\zeta(\cdot)$, and substitute the expression of $z_2^{(m)}$ in (6), (28), and (29) into (27),

$$z_1^{(2m)} = \psi(\cdot)+\zeta(\cdot)(f_2\left(z^{(0\sim m-1)}\right) + G_2\left(z^{(0\sim m-1)}\right)u_z). \tag{30}$$

Partitioning

$$G_2\left(z^{(0\sim m-1)}\right)=\left[G_{2-1}\ \ G_{2-2}\right],\ G_{2-2}\in\mathbb{R}^{r\times(n-r)}, \tag{31}$$

we have

$$z_1^{(2m)}= \psi(\cdot)+\zeta(\cdot)\big(f_2(\cdot)+G_{2-1}u_{z,1}+G_{2-2}u_{z,2}\big). \tag{32}$$

Assume $\zeta(\cdot)G_{2-2}$ is nonsingular (singularity can be avoided [24], see an UAV example in Section IV), we can design the controller for the above HOFA subsystem as

$$\begin{aligned} u_{z,2} &= -(\zeta(\cdot)G_{2-2})^{-1}\begin{pmatrix} A^z_{0\sim 2m-1}z_1^{(0\sim 2m-1)} \\ +\psi(\cdot)+u^*_{z,2}-v_1\end{pmatrix}, \\ u^*_{z,2} &= \zeta(\cdot)\big(f_2(\cdot)+G_{2-1}u_{z,1}\big), \end{aligned} \tag{33}$$

where v_1 is an external signal of dimension $n-r$. Finally, we have the following linear closed-loop system

$$z_1^{(2m)}+A^z_{0\sim 2m-1}z_1^{(0\sim 2m-1)}=v_1 \tag{34}$$

Thus, we have completed the derivation of the first HOFA subsystem. The process of separating the leftover system from $z_2^{(m)}$ and using $u_{z,1}$ as the input remains the same as the process from (15) to (22).

IV. HOFA Applications: 3-DOF Quadrotor Control

Consider a 3-DOF planar quadrotor system in Figure 1. The quadrotor is confined to move within the $x-z$ plane. The mathematic model is given below [22],

$$\begin{cases} \ddot{x}=\frac{f_1+f_2}{m}\sin\theta \\ \ddot{z}=\frac{f_1+f_2}{m}\cos\theta-g \\ \ddot{\theta}=\frac{f_2-f_1}{J}, \end{cases} \tag{35}$$

where θ is the pitch angle, f_1, f_2 are rotor thrusts, m is the quadrotor mass, J is the yaw moment of inertia, and g is the gravity acceleration.

To control the positions of x and z with the modified HOFA approach, we select the system inputs as $u_1=\frac{f_1+f_2}{m}$, $u_2=\frac{f_2-f_1}{J}$, $Q=I_{3\times 3}$, and $P=I_{2\times 2}$. We have

$$z_1=x, z_2=\left[z_{2-1}\ \ z_{2-2}\right]^T=\left[z\ \ \theta\right]^T. \tag{36}$$

The system (35) is rewritten as,

$$\begin{bmatrix} z_1^{(2)} \\ z_{2-1}^{(2)} \\ z_{2-2}^{(2)} \end{bmatrix}=\begin{bmatrix} 0 \\ -g \\ 0\end{bmatrix}+\underbrace{\begin{bmatrix} \sin Z_{2-2} & 0 \\ \cos Z_{2-2} & 0 \\ 0 & 1\end{bmatrix}}_{B}\begin{bmatrix} u_1 \\ u_2\end{bmatrix}. \tag{37}$$

This system cannot be reformed with the existing HOFA approach (see proof in Appendix), namely in the form of (7). However, the B matrix in (37) is already in the form of (23), and the sub-system z_1 in (37) is already in the form of (25). Thus, we can take the 2nd order derivative of the first equation in (37) yields,

$$\begin{aligned} z_1^{(4)} &= \ddot{u}_1\sin z_{2-2}+2\dot{u}_1\dot{z}_{2-2}\cos z_{2-2} \\ &\quad -\dot{z}^2_{2-2}u_1\sin z_{2-2}+\ddot{z}_{2-2}u_1\cos z_{2-2}. \end{aligned} \tag{38}$$

Substituting $\ddot{z}_{2-2}=u_2$ from (37), we can design u_2 as,

$$\begin{aligned} u_2 &= \frac{u_2^*+A^{z_1}_{0\sim 3}z_1^{(0\sim 3)}-v_1}{-u_1\cos z_{2-2}},\ u_2^*=\ddot{u}_1\sin z_{2-2} \\ &\quad +2\dot{u}_1\dot{z}_{2-2}\cos z_{2-2}-\dot{z}^2_{2-2}u_1\sin z_{2-2}, \end{aligned} \tag{39}$$

which gives the following closed-loop system (translational position $x=z_1$)

$$z_1^{(4)}+A^{z_1}_{0\sim 3}z_1^{(0\sim 3)}=v_1. \tag{40}$$

Regarding the leftover system z_{2-1}, it is trivial to have

$$u_1=\frac{g-A^{z_2}_{0\sim 1}z_{2-1}^{(0\sim 1)}+v_2}{\cos z_{2-2}}, \tag{41}$$

which results in the following closed-loop system (altitude $z=z_{2-1}$)

$$z_{2-1}^{(2)}+A^{z_2}_{0\sim 1}z_{2-1}^{(0\sim 1)}=v_2 \tag{42}$$

Remark 3: To avoid singularity in (39) and (41), constraints at near-zero interval are needed for u_1 and $\cos z_{2-2}$, which are trivial and work for large-angle fighting maneuvers [24].

$A^{z_1}_{0\sim 3}\in\mathbb{R}^{1\times 4}$, $A^{z_2}_{0\sim 1}\in\mathbb{R}^{1\times 2}$ and v_1, v_2 are external signals. All the variables used in the controller are all available, and the states of the original system (35) are all measurable. Clearly, once the set of HOFA models is obtained, the controllers of the subsystems (x and z) are immediately written out, and the obscure tunning procedure in the conventional hierarchical method is skipped. Admittedly, the high-order terms give the challenge to noise and disturbance attenuation [14]. A well-designed observer may be needed.

This method is currently based on the model with a fixed order. Assuming the angular velocities are obtained from the derivatives of the Euler angles, this approach can also be used for a second-order 6-DOF quadrotor model. The derivation of the HOFA model for the complete mixed-order 6-DOF quadrotor system shown in Figure 2 and experimental results are presented in the authors' paper [23].

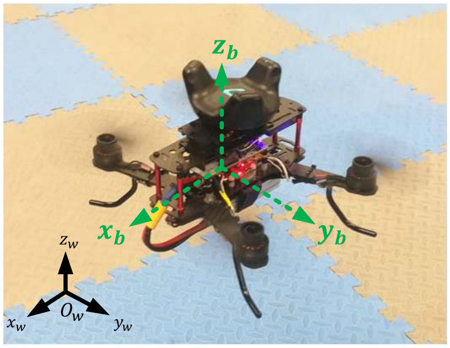

Fig. 2. 6-DOF quadrotor system.

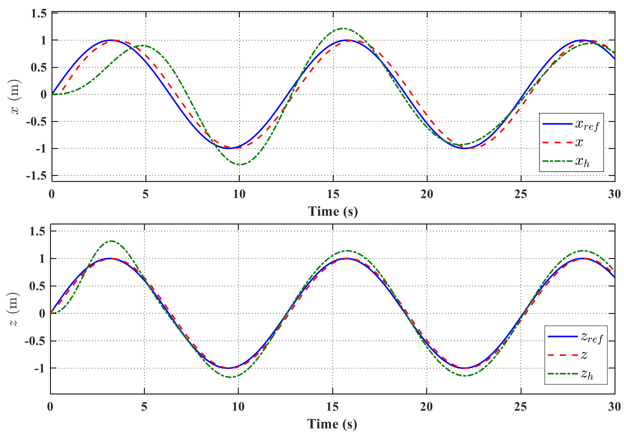

Fig. 3. Tracking results comparison between the HOFA approach and one conventional controller.

V. Simulation Results and Discussions

This section presents simulation results in MATLAB/Simulink to validate the proposed modified HOFA approach for the control of a planar quadrotor.

The quadrotor parameters are set as $m = 1.0$ kg, $= 0.01$ kg$\cdot m^2$, and $g = 9.8\ m/s^2$. The control parameters are,

$$\begin{aligned} A_{0\sim3}^{z_1} &= [24,\ 216,\ 864,\ 1296], \\ v_1 &= 36z_{1,\text{ref}}^{(2)}+432z_{1,\text{ref}}^{(1)}+1296z_{1,\text{ref}}, \\ A_{0\sim1}^{z_2} &= [16,\ 64], \\ v_2 &= 8z_{2-1,\text{ref}}^{(1)}+64z_{2-1,\text{ref}}. \end{aligned} \tag{43}$$

The closed-loop system can also be written in the frequency domain with a capital Z as the state,

$$\begin{bmatrix} Z_1 \\ Z_{2-1} \end{bmatrix} = \begin{bmatrix} \frac{36}{s^2+12s+36} & 0 \\ 0 & \frac{8}{s+8} \end{bmatrix} \begin{bmatrix} Z_{1,\text{ref}} \\ Z_{2-1,\text{ref}} \end{bmatrix}. \tag{44}$$

Naturally, the designed bandwidth of the altitude ($z = z_{2-1}$) tracking is faster than that of the translational position ($x = z_1$) tracking. Sinusoidal signals are applied as the reference inputs and tracking control results and errors are presented in Figure 3 and Figure 4, respectively.

We first compare tracking results with a conventional controller (e.g., a PID controller with a hierarchical structure [4]) to show advantage of HOFA controller on parameter tuning. x_h and z_h are outputs of the conventional controller. The comparison represents a common scenario as the conventional controller may not be well tuned due to the obscure tuning procedure. From Figure 3 and Figure 4, it is clear to see that tracking errors of x and z from the modified HOFA approach are smaller than the conventional method.

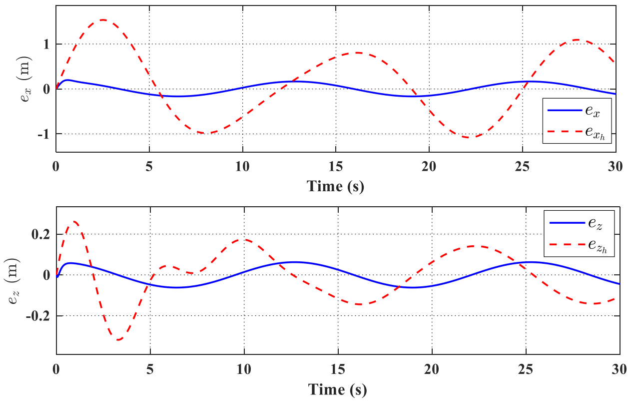

Fig. 4. Tracking errors comparison between the HOFA approach and one conventional controller.

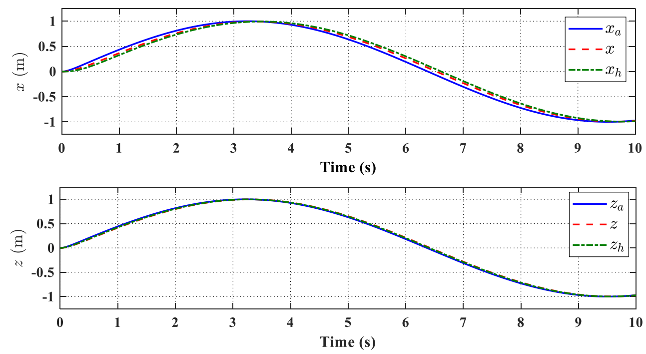

Fig. 5. Tracking results comparison among two HOFA controllers and one well-tuned conventional controller.

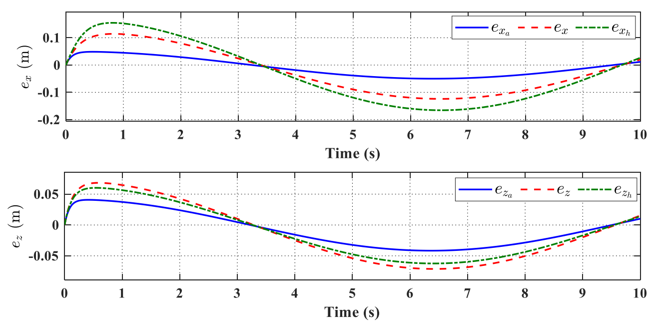

Fig. 6. Tracking error comparison among two HOFA controllers and one well-tuned conventional controller.

Denote x_a and z_a as the outputs of an alternative HOFA controllers with higher bandwidth, which can be easily obtained based on the feature of the HOFA approach without tuning efforts,

$$\begin{bmatrix} X_a \\ Z_a \end{bmatrix} = \begin{bmatrix} \frac{10}{s+10} & 0 \\ 0 & \frac{12}{s+12} \end{bmatrix} \begin{bmatrix} X_{a,\text{ref}} \\ Z_{a,\text{ref}} \end{bmatrix} \tag{45}$$

Different HOFA controllers (or control parameters) for different close-loop system performances are presented in Figure 5 and Figure 6 to compare with the conventional controller. The outputs of the conventional controller are well tuned in this case for comparison. To get the desired closed-loop system response, the conventional method typically requires extensive work and experience of controller tuning. However, the HOFA approach can easily achieve desired control performance by arbitrarily assigning desired eigenvalues. Even if a well-tuned conventional controller is acquired to get a similar performance to that defined in (44), it is much easier to obtain a better and faster close-loop performance defined in (45) with the HOFA approach.

VI. Conclusion

Unlike conventional methods based on the universal state-space representation, Lyapunov methods, and hierarchical structure, the HOFA approach can obtain a constant linear closed-loop system from nonlinear systems. However, the HOFA system reformulation from certain nonlinear systems is sometimes difficult based on the existing HOFA approach. This letter proposes a modified HOFA approach for the quadrotor control or a family of nonlinear systems that cannot be directly converted into a pseudo strict-feedback form. The theoretical development and simulation results show the effectiveness of the proposed method.

Appendix

*Proof [Infeasibility of **Q** Matrix for Quadrotor Systems]:* For the system (35), we assume the unimodular matrix Q exists with $QB = \begin{bmatrix} 0 \\ G \end{bmatrix}$, $G \in \mathbb{R}^{2\times 2}$. Denote $u_1 = \frac{f_1+f_2}{m}$, $u_2 = \frac{f_2-f_1}{J}$, and we have,

$$\begin{bmatrix} \ddot{x} \\ \ddot{z} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} \sin\theta & 0 \\ \cos & 0 \\ 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{46}$$

Denote

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}, \tag{47}$$

and we have

$$QB = \begin{bmatrix} q_{11}\sin\theta + q_{12}\cos\theta \; q_{13} \\ q_{21}\sin\theta + q_{32}\cos\theta \; q_{23} \\ q_{31}\sin\theta + q_{32}\cos\theta \; q_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ G_{2\times 2} \end{bmatrix}. \tag{48}$$

Considering the first element in (48),

$$q_{11}\sin\theta + q_{12}\cos\theta = 0 \tag{49}$$

We have

$$q_{12} = -q_{11}\tan\theta. \tag{50}$$

Thus, Q contains one element that is a variable of the pitch angle θ. Based on the definition of a unimodular matrix [27], this implies it is infeasible to find a unimodular matrix Q to convert the system (37) into a pseudo strict-feedback form.

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