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A Note on the Shape Regularity of Worsey–Farin Splits

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Abstract
We prove three-dimensional Worsey–Farin refinements inherit their parent triangulations' shape regularity.

Keywords Worsey-Farin Splits · Shape regularity

1 Introduction

Three-dimensional Worsey–Farin splits were first introduced in [15] to construct low-order C^1 splines on simplicial triangulations, and they have been extensively studied since then; see for example [12]. Recently it has been shown that smooth piecewise polynomial spaces on Worsey–Farin splits (and related ones) fit into discrete de Rham complexes. [5, 7–10]. These results are further applied to analyze convergence, stability and accuracy of numerical methods for models of incompressible fluids on theses refinements [3, 4, 11]. Therefore, it is necessary to discuss the properties of these refinements, especially in the context of approximation and stability properties of the corresponding discrete spaces. One critical geometric property for approximation theory is the shape regularity of the underlying mesh.

The shape regularity of Worsey–Farin splits are required to ensure optimal-order and uniform interpolation estimates in [12, Theorem 18.15], [1, Theorem 6.3], [14, Theorem 6.2], and [13, Theorem 8.14]. Stability estimates of a finite element method in [6] defined on Worsey–Farin splits also require regularity of the refined triangulation. The references [12, Page 515], [1, Remark 14], and [2, Page 54] explicitly conjecture that Worsey–Farin

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splits of a family of shape regular meshes remain shape regular. However, to the best of our knowledge, a proof of this result has not appeared in the literature. In this note we fill in this gap.

In [12, Lemma 4.20] and [11, Lemma 2.6], the relationship between the shape regularity constant of Powell-Sabin splits and the parent triangulations is shown. Namely, this result is proved by establishing bounds of the angles of each macro triangle. Hence, it is natural to focus on the dihedral angles in the three-dimensional Worsey–Farin case. We first prove the dihedral angles are bounded by quantities that only depend on the shape regularity of the original mesh (see Lemma 2.6 below). Using this result we prove the crucial result that the split points of each face F in the triangulation is uniformly bounded away from ∂F; see Lemma 3.3. From this result, the shape regularity of Worsey–Farin refinements is then shown.

This paper is organized as follows. In Sect. 2, we recall the Worsey–Farin refinement of a three-dimensional simplicial mesh and present some notations to better illustrate our main analysis. In Sect. 3, we show the shape regularity of Worsey–Farin splits is solely determined by the shape regularity of the parent mesh.

2 Preliminaries

2.1 Geometric Notations and Properties

We first present some basic definitions regarding the geometric properties of a tetrahedron, see [12, Definition 16.1–16.2] for more details.

Given a tetrahedron T, we denote by $\Delta_m(T)$ the set of m-dimensional simplices of T. For example, $\Delta_2(T)$ is the set of four faces of T, and $\Delta_1(T)$ is the set of six edges of T. Let ρ_T be the diameter of the inscribed sphere S_T of T, which is the largest sphere contained in T. We call the center of S_T the incenter of T, denoted by z_T, and call the radius of S_T the inradius of T, equal to $\rho_T/2$. The sphere S_T intersects each face F of T at a unique point, $z_{T,F}$. We note that $z_{T,F}$ is the orthogonal projection of the point z_T to the plane that contains F (i.e., the vector $z_T - z_{T,F}$ is normal to F). Finally, we let $h_T = \operatorname{diam}(T)$.

The following two propositions are well-known results of tetrahedra. To be self-contained we provide their proofs.

Proposition 2.1 *For a tetrahedron T, there holds*

$$\rho_T = 6|T|/\Big(\sum_{F\in\Delta_2(T)}|F|\Big).$$

Proof Consider the refinement of T obtained by connecting the incenter of T to its vertices. The resulting four subtetrahedra fill the volume of T, and thus,

$$|T| = \sum_{F\in\Delta_2(T)} \frac{1}{3}|F|\frac{\rho_T}{2},$$

which gives the result.

Proposition 2.2 *Given a tetrahedron T, let x be any vertex of T and F_x be the face of T which is opposite to x. Let P_x be the plane containing F_x, then for any point $a \in \operatorname{int}(T)$, we have*

$$\operatorname{dist}(x, P_x) > \operatorname{dist}(a, P_x). \tag{2.1}$$

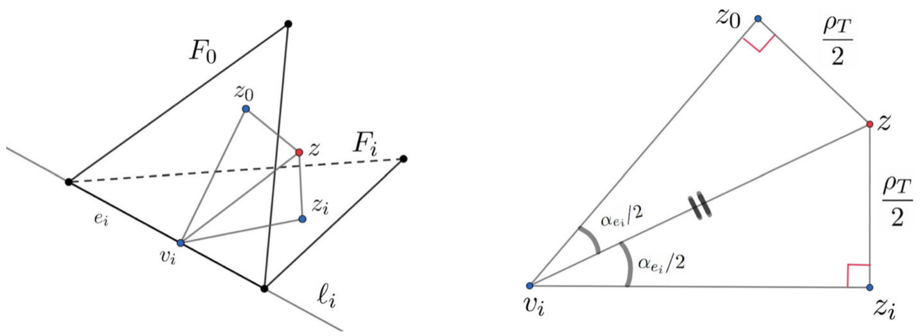

Fig. 1 A representation of the dihedral angle

In particular,

$$\mathrm{dist}(x, P_x) > \rho_T. \tag{2.2}$$

Proof Since the point $a \in \mathrm{int}(T)$ and F_x is a face of T, a and F_x form a tetrahedron $T' \subset T$. Therefore,

$$\frac{1}{3}|F_x|\,\mathrm{dist}(x, P_x) = |T| > |T'| = \frac{1}{3}|F_x|\,\mathrm{dist}(a, P_x),$$

which immediately gives (2.1). Let ℓ be the line containing z_T and z_{T,F_x}. Let ℓ intersect S_T at $a = z'_{T,F_x}$. Then $a \in \mathrm{int}(T)$ and $\mathrm{dist}(a, P_x) = |[a, z_{T,F_x}]| = \rho_T$. Hence, (2.2) follows from (2.1).

We will also need the following result that bounds $\mathrm{dist}(z_{T,F}, \partial F)$ from below using the dihedral angles.

Lemma 2.3 *Let T be a tetrahedron, and for each face $F \in \Delta_2(T)$, let $z_{T,F}$ denote the orthogonal projection of the incenter of T onto F. Let α_e be the dihedral angle of T with respect to $e \in \Delta_1(T)$. We have*

$$\min_{F\in\Delta_2(T)} \mathrm{dist}(z_{T,F}, \partial F) \geq \min_{e\in\Delta_1(T)} \frac{\rho_T}{2}\sqrt{\frac{1+\cos(\alpha_e)}{1-\cos(\alpha_e)}}. \tag{2.3}$$

Proof We use the short hand notation depicted in Fig. 1. In particular, z denotes the incenter of T and $F_i \in \Delta_2(T)$, $i = 0, \ldots, 3$ denote the faces of T. Let z_i be the orthogonal projection of z onto the plane containing F_i and note that $|[z, z_i]| = \rho_T/2$. We need to find a lower bound for $\mathrm{dist}(z_k, \partial F_k)$ $(k = 0, \ldots, 3)$ and without loss of generality we consider the case $k = 0$. To this end, let $e_i = \partial F_0 \cap \partial F_i$, $i = 1, 2, 3$ and furthermore let ℓ_i be the line containing e_i. Let γ_i be the plane determined by the points z, z_0, and z_i and let $v_i = \ell_i \cap \gamma_i$. Since $\ell_i \perp [z, z_i]$ for $j = 0, i$, we have the line ℓ_i is perpendicular to the plane γ_i, and thus $\ell_i \perp [v_i, z_j]$ for $j = 0, i$. This implies

$$\mathrm{dist}(z_j, \ell_i) = |[z_j, v_i]|, \quad j = 0, i, \quad \text{and} \quad \alpha_{e_i} := \angle z_0 v_i z_i = \angle z v_i z_0 + \angle z v_i z_i.$$

Next, note the properties $[z, z_j] \perp [z_j, v_i]$ for $j = 0, i$ and $|[z, z_j]| = \rho_T/2$ imply that the triangles $[z, v_i, z_0]$ and $[z, v_i, z_i]$ are congruent (see Fig. 1b). Consequently, $\angle z v_i z_0 =$

$\angle z v_i z_i = \alpha_{e_i}/2$ and so

$$\mathrm{dist}(z_0, \ell_i) = |[z_0, v_i]| = \frac{\rho_T/2}{\tan(\alpha_{e_i}/2)} = \frac{\rho_T}{2}\sqrt{\frac{1+\cos(\alpha_{e_i})}{1-\cos(\alpha_{e_i})}}. \tag{2.4}$$

The result now follows after using $\mathrm{dist}(z_0, \partial F_0) \geq \min_{1\leq i\leq 3} \mathrm{dist}(z_0, \ell_i)$.

2.2 Worsey–Farin Splits

Let $\mathcal{T}_h$ be a three-dimensional triangulation without hanging nodes. We recall the construction of the Worsey–Farin refinement of $\mathcal{T}_h$ in the following definition [9, 12, 15].

Definition 2.4 The Worsey–Farin refinement of $\mathcal{T}_h$, denoted by $\mathcal{T}_h^{wf}$, is defined by the following two steps:

1. Connect the incenter z_T of of each tetrahedron $T \in \mathcal{T}_h$ to its four vertices;
2. For each interior face $F = \overline{T_1} \cap \overline{T_2}$ with $T_1, T_2 \in \mathcal{T}_h$, let $m_F = L \cap F$ where $L = [z_{T_1}, z_{T_2}]$, the line segment connecting the incenter of T_1 and T_2; meanwhile, for a boundary face F with $F = \overline{T} \cap \partial\Omega$ with $T \in \mathcal{T}_h$, let m_F be the barycenter of F. We then connect m_F to the three vertices of the face F and to the incenters z_{T_1} and z_{T_2} (or z_T for the boundary case).

We see that this two-step procedure divides each $T \in \mathcal{T}_h$ into 12 subtetrahedra; we denote the set of these subtetrahedra by T^{wf}.

The result [12, Lemma 16.24] ensures that the three-dimensional Worsey–Farin refinement is well-defined; in particular, the line segment connecting the incenters of neighboring tetrahedra intersects their common face.

Definition 2.5 We define the shape regularity constant of the triangulation $\mathcal{T}_h$ as

$$c_0 = \max_{T\in\mathcal{T}_h} \frac{h_T}{\rho_T}.$$

It is well-known that shape regularity of a mesh leads to bounded dihedral angles. To be self-contained, we present a proof here.

Lemma 2.6 *Fix $T \in \mathcal{T}_h$, and let α_e denote the dihedral angle of T with respect to $e \in \Delta_1(T)$. We then have*

$$|\cos(\alpha_e)| \leq \sqrt{1 - c_0^{-2}} \quad \forall e \in \Delta_1(T). \tag{2.5}$$

Proof Write $T = [x_1, x_2, x_3, x_4]$, consider the edge $e = [x_3, x_4]$, and let ℓ be the line containing e; see Fig. 2. Let A be the orthogonal projection of x_1 onto the plane γ containing the face $[x_2, x_3, x_4]$, and let B be the point on ℓ such that $[x_1, B] \perp \ell$. Note that $[x_1, B] \perp \ell$ and $[x_1, A] \perp \gamma$ implies $[A, B] \perp \ell$. Since $|[x_1, A]| \geq \rho_T$ by Proposition 2.2 and $|[x_1, B]| \leq h_T$, the dihedral angle of e satisfies

$$\sin(\alpha_e) = \frac{|[x_1, A]|}{|[x_1, B]|} \geq \frac{\rho_T}{h_T} \geq c_0^{-1}.$$

Therefore, we have $|\cos(\alpha_e)| = \sqrt{1 - \sin^2(\alpha_e)} \leq \sqrt{1 - c_0^{-2}}$.

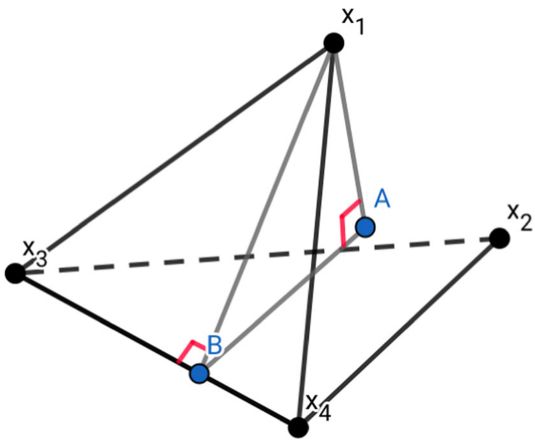

Fig. 2 Computing dihedral angles

3 Analysis of the Shape Regularity of Worsey–Farin Splits

In this section, we prove the main result of this note. We prove that the Worsey–Farin refinement $\mathcal{T}_h^{wf}$ is shape regular provided the parent triangulationn $\mathcal{T}_h$ is shape regular. To be more precise, the following theorem will be proved:

Theorem 3.1 *There exists a constant $c_1 > 0$ only depending on c_0, the shape regularity constant of $\mathcal{T}_h$ given in Definition 2.5 such that*

$$\max_{K \in \mathcal{T}_h^{wf}} \frac{h_K}{\rho_K} \leq c_1.$$

For an explicit formula of c_1, see (3.4) and (3.1).

3.1 Local Geometry

To prove the above theorem, we need to consider two cases: interior and boundary faces of $\mathcal{T}_h$. The case of boundary faces is simpler, so we first focus on the interior faces. For that case, it is sufficient to consider two adjacent elements of the mesh $\mathcal{T}_h$. To this end, let $T_1, T_2 \in \mathcal{T}_h$ be two tetrahedra that share a common face F_0. We write $T_1 = [x_1, x_2, x_3, x_4]$, $T_2 = [x_1, x_3, x_4, x_5]$, so that the common face is $F_0 = [x_1, x_3, x_4]$. We further set $F_1 = [x_2, x_3, x_4]$, and let z_i be the incenter of T_i, $i = 1, 2$ (see Fig. 3a). For $i = 1, 2$, we denote by $z_{i,0}$ the orthogonal projections of z_i onto the plane containing the face F_0 (see Fig. 3b). Likewise the orthogonal projection of z_1 onto the plane containing the face F_1 is denoted by $z_{1,1}$ (see Fig. 1a). We denote the split point of the face F_0 by m_0, i.e., m_0 is the intersection of the line $[z_1, z_2]$ and F_0.

3.2 The Position of Split Points and Bounded Dihedral Angles

The following proposition shows the relation between the split point m_0 and the projections $z_{i,0}$, $i = 1, 2$ of the incenter on the face F_0.

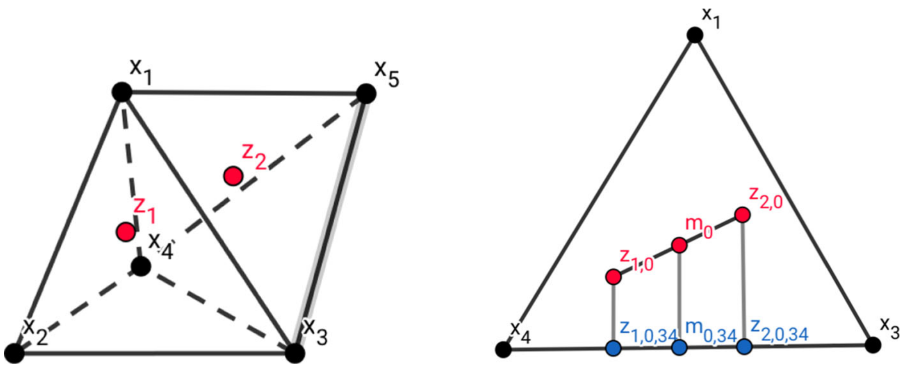

Fig. 3 A representation of the Worsey–Farin splits

Proposition 3.2 *The orthogonal projections $z_{i,0}$ $(i = 1, 2)$ lie in the interior of F_0, and the split point m_0 lies on the line segment $[z_{1,0}, z_{2,0}]$. Furthermore, we have*

$$\mathrm{dist}(m_0, \partial F_0) \geq \min_{i=1,2} \mathrm{dist}(z_{i,0}, \partial F_0).$$

Proof The proof of [12, Lemma 16.24] shows that m_0 lies on the line segment $[z_{1,0}, z_{2,0}]$ and that $z_{i,0}$ $(i = 1, 2)$ lie in the interior of F_0.

Let ℓ_i, $i = 1, 2, 3$ denote the lines that contain the three edges of F_0. Because m_0 lies on the interior of the line segment $[z_{1,0}, z_{2,0}]$, there exists a constant $\vartheta \in (0, 1)$ such that $m_0 = \vartheta z_{1,0} + (1 - \vartheta) z_{2,0}$. Then by constructing similar triangles, we have

$$\begin{aligned}
\mathrm{dist}(m_0, \partial F_0) &= \min_{1\leq i\leq 3} \mathrm{dist}(m_0, \ell_i) \\
&= \min_{1\leq i\leq 3} \vartheta\, \mathrm{dist}(z_{1,0}, \ell_i) + (1-\vartheta)\mathrm{dist}(z_{2,0}, \ell_i) \\
&\geq \vartheta \min_{1\leq i\leq 3} \mathrm{dist}(z_{1,0}, \ell_i) + (1-\vartheta) \min_{1\leq i\leq 3} \mathrm{dist}(z_{2,0}, \ell_i) = \\
&\vartheta\, \mathrm{dist}(z_{1,0}, \partial F_0) + (1-\vartheta)\mathrm{dist}(z_{2,0}, \partial F_0) \\
&\geq \min_{i=1,2} \mathrm{dist}(z_{i,0}, \partial F_0).
\end{aligned}$$

Combining Lemmas 2.3, 2.6 and Proposition 3.2, we have the following lemma which describes the position of split points. We also include the case for boundary faces.

Lemma 3.3 *Recall that m_F is the split point of F constructed by the Worsey–Farin split defined in Definition 2.4. For any face F of $\mathcal{T}_h$,*

$$\mathrm{dist}(m_F, \partial F) \geq c_2 \min_{\substack{T\in\mathcal{T}_h \\ F\in\Delta_2(T)}} h_T,$$

where

$$c_2 := \min\{\mathsf{c}_2, (3c_0)^{-1}\}, \quad \mathsf{c}_2 := (2c_0)^{-1}\sqrt{-1 + \frac{2}{1+\sqrt{1-c_0^{-2}}}}. \tag{3.1}$$

Proof (i) *F is an interior face* In this case $F \in \mathcal{F}_2(T)$ and $F \in \mathcal{F}_2(T')$ for some $T, T' \in \mathcal{T}_h$. Without loss of generality, we assume $\operatorname{dist}(z_{T,F}, \partial F) \geq \operatorname{dist}(z_{T',F}, \partial F)$. Lemma 2.3 and Proposition 3.2 tell us that

$$\begin{aligned}\operatorname{dist}(m_F, \partial F) &\geq \operatorname{dist}(z_{T,F}, \partial F)\\ &\geq \min_{e\in\mathcal{E}_1(T)} \frac{\rho_T}{2}\sqrt{\frac{1+\cos(\alpha_e)}{1-\cos(\alpha_e)}} = \min_{e\in\mathcal{E}_1(T)} \frac{\rho_T}{2}\sqrt{-1+\frac{2}{1-\cos(\alpha_e)}}.\end{aligned}$$

If $\cos(\alpha_e) \geq 0$, then $\frac{2}{1-\cos(\alpha_e)} \geq 2$, and if $\cos(\alpha_e) \leq 0$, then $\frac{2}{1-\cos(\alpha_e)} = \frac{2}{1+|\cos(\alpha_e)|} \geq \frac{2}{1+\sqrt{1-c_0^{-2}}}$ by Lemma 2.6. Consequently,

$$\operatorname{dist}(m_F, \partial F) \geq \min_{e\in\mathcal{E}_1(T)} \frac{\rho_T}{2}\sqrt{-1+\frac{2}{1-\cos(\alpha_e)}} \geq c_2 h_T.$$

(ii) *F is a boundary face* Let $T = [x_1, x_2, x_3, x_4]$ and $F = [x_1, x_3, x_4]$, and consider an arbitrary $e \in \mathcal{E}_1(F)$ with ℓ denoting the line containing e. Without loss of generality we assume $e = [x_3, x_4]$ and adopt the notation in the proof of Lemma 2.6; see Fig. 2. Because m_F is the barycenter of F, we have

$$\frac{1}{3}|F| = \frac{1}{2}\operatorname{dist}(m_F, \ell)|e|.$$

Moreover, clearly

$$|F| = \frac{1}{2}|e|\,|[x_1, B]|.$$

And therefore, since $|[x_1, B]| \geq |[x_1, A]| > \rho_T$, (where we used (2.2) and the right triangle $[x_1, A, B]$) we get

$$\operatorname{dist}(m_F, \ell) = \frac{1}{3}|[x_1, B]| \geq \frac{1}{3}\rho_T \geq (3c_0)^{-1} h_T.$$

Since $e \in \mathcal{E}_1(F)$ was arbitrary the result follows.

3.3 Proof of Theorem 3.1

Now we are ready to use Lemma 3.3 to prove Theorem 3.1.

Proof Let $K \in \mathcal{T}_h^{wf}$, and let $T \in \mathcal{T}_h$ such that $K \subset T^{wf}$. We write $T = [x_1, x_2, x_3, x_4]$, and assume, without loss of generality, that $e := [x_1, x_2]$ is an edge of both T and K. Let $F \in \mathcal{F}_2(T)$ such that the split point m_F is a vertex of K. In particular, $e \in \mathcal{E}_1(F)$ and $K = [x_1, x_2, m_F, z_T]$, where z_T is the incenter of T. We further denote by ℓ, the line containing the edge e.

We again adopt the notation in the proof of Lemma 2.6 and refer to Fig. 2. Note that $[x_1, A]$ is normal to the plane γ containing $[x_2, x_3, x_4]$, in particular, $[x_1, A] \perp [A, x_2]$. Thus $|e| = |[x_1, x_2]| > |[x_1, A]| > \rho_T$ by (2.2). Now we have $h_K \leq h_T$, $\rho_T < |e| \leq h_T$ and, by

Lemma 3.3, the volume of K is

$$
\begin{aligned}
|K| &= \frac{1}{3}\frac{\rho_T}{2} \times |[x_1, x_2, m_F]| = \frac{1}{12}\rho_T |e| \operatorname{dist}(m_F,) \\
&\geq \frac{1}{12}\rho_T^2 \operatorname{dist}(m_F, \partial F) \geq \frac{c_2}{12}\rho_T^2 \min_{\substack{T\in\mathcal{T}_h \\ F\in\Delta_2(T)}} h_T \geq \frac{c_2}{12}\rho_T^3 \geq \frac{c_2}{12 c_0^3} h_T^3 .
\end{aligned} \tag{3.2}
$$

Here we also used

$$
\min_{\substack{T\in\mathcal{T}_h \\ F\in\Delta_2(T)}} h_T \geq |e| > \rho_T.
$$

Additionally, each face of K is contained in a circle with radius $h_K/2$, and thus we have

$$
\sum_{F\in\Delta_2(K)} |F| \leq \sum_{F\in\Delta_2(K)} \frac{\pi h_K^2}{4} = \pi h_K^2. \tag{3.3}
$$

Consequently, with Proposition 2.1, (3.2) and (3.3), we have

$$
\rho_K = \frac{6|K|}{\sum_{F\in\Delta_2(K)} |F|} \geq \frac{c_2 h_T^3}{2\pi c_0^3 h_K^2} \geq \frac{c_2 h_K}{2\pi c_0^3}.
$$

Thus, setting

$$
c_1 = \frac{2\pi c_0^3}{c_2}, \tag{3.4}
$$

we have $\frac{h_K}{\rho_K} \leq c_1$. Because $K \in \mathcal{T}_h^{wf}$ was arbitrary, we conclude $\max_{K\in\mathcal{T}_h^{wf}} \frac{h_K}{\rho_K} \leq c_1$.

4 Conclusion

We have settled a conjecture concerning the shape regularity of a Worsey–Farin refinement of a parent triangulation. As described in the introduction, this is a crucial bound to obtain approximation results for splines; see for example [12, Theorem 18.15]. However, based on initial numerical calculations, the constant c_1 in Theorem 3.1 that relates the shape regularity of the parent triangulation (i.e., c_0) and its Worsey–Farin refinement is most likely not sharp. In particular, the theorem suggest that c_1 scales like c_0^5 which could be quite large even for a good quality parent triangulation. We hope that this work leads to further investigations and sharper estimates will emerge.

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Data Availability All data generated or analysed during this study are included in this article.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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