Stability of invertible, frustration-free ground states against large perturbations

Sven Bachmann[1], Wojciech De Roeck[2], Brecht Donvil[3,4], and Martin Fraas[5]

[1]Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada

[2]Institute of Theoretical Physics, K.U. Leuven, 3001 Leuven, Belgium

[3]Institute for Complex Quantum Systems and Center for IQST, Ulm University, 89069 Ulm, Germany

[4]Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland

[5]Department of Mathematics, University of California, Davis, Davis, CA, 95616, USA

A gapped ground state of a quantum spin system has a natural length scale set by the gap. This length scale governs the decay of correlations. A common intuition is that this length scale also controls the spatial relaxation towards the ground state away from impurities or boundaries. The aim of this article is to take a step towards a proof of this intuition. We assume that the ground state is frustration-free and invertible, i.e. it has no long-range entanglement. Moreover, we assume the property that we are aiming to prove for one specific kind of boundary condition; namely open boundary conditions. This assumption is also known as the "local topological quantum order" (LTQO) condition. With these assumptions we can prove stretched exponential decay away from boundaries or impurities, for any of the ground states of the perturbed system. In contrast to most earlier results, we do not assume that the perturbations at the boundary or the impurity are small. In particular, the perturbed system itself can have long-range entanglement.

1   Informal statement of the result

Since a full ab initio statement of our assumptions and theorem requires quite some setup and definitions, we first state some simplified assumptions and the result, freely using terminology that is probably known to most of our readers. We consider a spin system on a finite discrete set Γ, say, a subset of $\mathbb{Z}^d$. We define a Hamiltonian $H + J$ where both terms H and J are local Hamiltonians, i.e. sums of local terms $H = \sum_{X \subset \Gamma} h_X, J = \sum_{X \subset \Gamma} j_X$ with $||h_X||, ||j_X||$ decaying rapidly in $\text{diam}(X)$. We assume moreover the following properties:

1. The spatial support of J is confined to a region Γ_j with arbitrary size. Crucially, we do not assume that the terms j_X are small.

2. The Hamiltonian H has a ground state Ω that is *invertible*. This means there is an auxiliary state Ω' such that $\Omega \otimes \Omega'$ is connected to a product state by a locally generated unitary, i.e. it is automorphically equivalent to a product state.

3. H is frustration-free, i.e. the local terms h_X are all minimized by the state Ω.

4. The open boundary restrictions (OBC) $H_Z = \sum_{X \subset Z} h_Z$ for balls Z with radius r have a spectral gap $\gamma(r)$ above the ground state sector, that decays no faster than an inverse polynomial in r, as $r \to \infty$.

5. The ground states of the open boundary restrictions H_Z for balls Z satisfy the so-called 'local topological quantum order' (LTQO) condition: Their local restrictions to a set X approach the local restriction of the state Ω, quasi-exponentially fast, as $\text{dist}(X, Z^c) \to \infty$.

The result is that local restrictions of any ground state of $H + J$ approach the local restriction of Ω fast, as a function of the distance to the impurity or boundary region Γ_j, see also Figure 1.

Informal claim on local stability: *There is a stretching exponent $\beta > 0$ and finite constant C such that, for any ground state Φ of $H + J$, and for any local observable O_X supported in $X \subset \Gamma$ with $\text{diam}(X) \leq \text{dist}(X, \Gamma_j)^\beta$,*

$$|\langle \Phi, O_X \Phi \rangle - \langle \Omega, O_X \Omega \rangle| \leq C ||O_X|| e^{-(\text{dist}(X, \Gamma_j))^\beta}$$
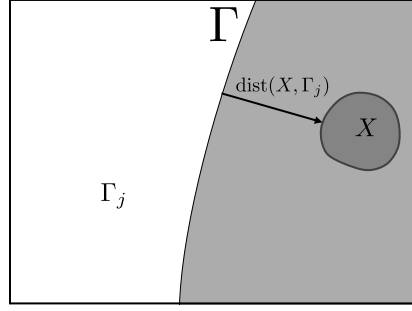
Figure 1: The informal claim on local stability concerns ground state expectation values of observables supported in the region X, far away from Γ_j, the support of J.

As far as we can see, there is no natural sense in which Φ resembles Ω, other than what is expressed in this claim. In particular, Φ is not necessarily the unique ground state, it has in general no gap[1], no short range entanglement properties and it is not frustration-free. To illustrate this, in particular the possibility of long-range entanglement, we consider $\Gamma = \{1, 2, \ldots, L\}$ and we imagine that at each site there is a qubit, i.e. a two-dimensional Hilbert space, with base states $|\downarrow\rangle, |\uparrow\rangle$. The Hamiltonian is $H = \sum_{i \in \Gamma} \sigma_i^z$ where σ_i^z acts at site i, as $\sigma_i^z |\uparrow\rangle_i = |\uparrow\rangle_i$ and $\sigma_i^z |\downarrow\rangle_i = -|\downarrow\rangle_i$. This Hamiltonian has a unique ground state that is a product, namely $\otimes_i |\downarrow\rangle_i$. The perturbation J is chosen as $J = -\sigma_1^z - \sigma_L^z + e^{-cL} j_{1,L}$ with some $c > 0$ and $j_{1,L}$ a two-qubit operator acting on sites 1 and L. Because of the very small prefactor e^{-cL}, this perturbation J satisfies any reasonable locality requirement. By taking $j_{1,L} = 0$, the perturbed system $H + J$ has 4-fold ground state degenerary. By taking $j_{1,L}$ to be minus the rank-1 projector on the entangled Bell state $\frac{1}{\sqrt{2}} (|\uparrow\rangle_1 \otimes |\uparrow\rangle_L + |\downarrow\rangle_1 \otimes |\downarrow\rangle_L)$, the perturbed system $H + J$ has a unique ground state that has maximal entanglement between qubits 1 and L.

1.1 Discussion

1.1.1 Spectral stability of the ground state sector

The above result should be contrasted with recent work on the stability of the spectral gap against locally small perturbations. That recent work can be split in two classes. The first class concerns systems where the unperturbed ground state is a product, see [1, 2, 3, 4, 5, 6, 7]. The second class is based on the so-called Bravyi-Hastings-Michalakis (BHM) argument which applies to frustration-free ground states, see [8, 9, 10, 11]. Both classes are eventually based on some form of spectral perturbation theory applied to the ground state sector. In all of the above cases, one also obtains some form of locality for the action of small perturbations, captured by the slogan 'local perturbations perturb locally', see also [12, 13].

Our result is different in spirit, as our perturbations are not assumed to be small, but we obtain only information about the perturbed ground states in regions far away from the region Γ_j where the perturbation acts. On the other hand, and in contrast to the quoted works, our result relies crucially on the variational principle and so it holds only for ground states (or ceiling states). One should realize that it is not only the proof of the results in [8, 9, 10] that does not apply in our case, but also the results should not be expected, as already stressed above through an example.

1.1.2 Role of the invertibility assumption

If one drops the invertibility assumption, then the claim on local stability in Section 1 can no longer hold. To see this, one can consider the two-dimensional toric code model [14, 15] on a large square Γ. This model is frustration-free and with appropriate boundary conditions, there is a unique groundstate Ω. The OBC restrictions always have a spectral gap that is bounded below by a positive constant, and the topological order condition holds. However, the state Ω is not invertible, regardless of boundary conditions. The lack of invertibility is associated with anyonic excitations that are created in pairs. By choosing a boundary condition that fixes a single anyon excitation at the boundary of a region Γ, one forces a partner anyon to be present somewhere in the interior of Γ, see Figure 2. In this case, there will be multiple ground states, labelled by the position x of the partner anyon. Since x can be arbitrarily far from the boundary, the claim on local stability does not hold. Of course, the large degeneracy of ground states is not a robust feature

---

[1]Actually, our assumptions do not explicitly require a global gap for H either, but such a gap is there in spirit because of the invertibility assumption, and we have left it in the abstract to fix thoughts.
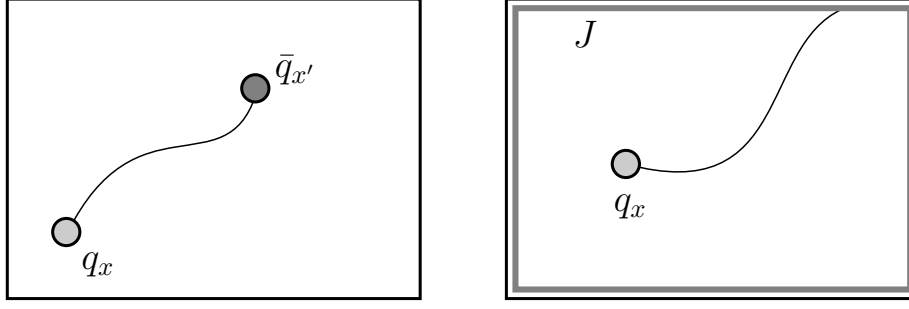
Figure 2: Left: a pair of anyons at sites x, x' connected by a (locally non-observable) string. This is an excited state of the toric code Hamiltonian. Right: one of the anyons is fixed at the boundary by the boundary condition. This constrains the partner anyon to be present in the ground state, at some arbitrary site x.

of the model. Indeed, if one adds a small global perturbation to the model, which will in general give the hitherto static anyons some mobility (see e.g. [16]), then the degeneracy will be lifted and the unique ground state will correspond to a delocalized anyon. In this case, the claim on local stability is still violated because the ground state now differs locally from Ω by an amount that is an inverse polynomial in the volume $|\Gamma|$.

1.1.3 Remaining assumptions, extensions, and related results

As argued above, the invertibility assumption can not be dropped. We can also not drop the LTQO condition. Consider for example $H = H_0 + B$ acting on the chain of qubits (as the example immediately below the informal claim). We take H_0 the classical Ising Hamiltonian $H_0 = -\sum_{i=1}^{L-1} \sigma_i^z \sigma_{i+1}^z$ such that its ground states exhibits spontaneous symmetry breaking, with ground states $\otimes_i |\downarrow\rangle_i$ and $\otimes_i |\uparrow\rangle_i$. The term $B = \sigma_1^z$ breaks the symmetry at the boundary, so that H has a unique ground state $\Omega = \otimes_i |\downarrow\rangle_i$. In that case our result does not hold, as we see by taking $J = -B$. The LTQO assumption fails to hold since in the interior of Γ, the symmetry remains unbroken and the OBC restrictions in the interior also have $\otimes_i |\uparrow\rangle_i$ as ground states. Of course, one could imagine replacing the LTQO condition by some form of translation invariance of the Hamiltonian H, ruling out the above example.

Apart from this, we do not see that the other assumptions are fundamentally necessary, though they are necessary for our proof. However, a mild extension of our result seems within reach: we hope that techniques introduced within the BHM argument, see Section 1.1.1, could allow to generalize our result in the following way: If we were to tighten our assumptions so that the local gap $\gamma(r) \geq \gamma > 0$, i.e. the local gap is uniformly bounded from below, then the result should also hold for Hamiltonians of the type $H' = H + \epsilon K$ where H and its ground state satisfy the tightened assumptions, K is a Hamiltonian with the same locality restrictions as H and J, and $\epsilon \ll 1$ is a small parameter.

Results similar to ours have recently been obtained independently in [17], as a corollary to [4]. The result of [17, 4] yields full exponential decay and requires no explicit frustration freeness, but it is restricted to weakly interacting spins, i.e. perturbations of products. Instead, our result relies on the automorphic equivalence of the unperturbed ground state to a product state, possibly upon adjoining an auxiliary state. Both results need hence an underlying product structure. Another line of research that is loosely connected to ours, concerns stability at nonzero temperature in spatial dimension 1, see [18, 19].

Acknowledgements

2 Setup

2.1 Preliminaries

2.1.1 Spatial structure

We consider a finite graph Γ, equipped with the graph distance $\text{dist}(\cdot, \cdot)$. Let $B_r(x) = \{y : \text{dist}(x, y) \leq r\}$ be the ball of radius r centered at $x \in \Gamma$. The graph is assumed to have a finite dimension $d < \infty$, i.e. there is a $C_\Gamma < \infty$ such that

$$\sup_{x \in \Gamma} |B_r(x)| \leq 1 + C_\Gamma r^d.$$

For reasons of recognizability, we refer to vertices as 'sites'. To every site x is associated a finite-dimensional Hilbert space $\mathcal{H}_x$ and we define the total Hilbert space $\mathcal{H} = \mathcal{H}_\Gamma = \otimes_{x \in \Gamma} \mathcal{H}_x$ and the algebra $\mathcal{A} = \mathcal{A}_\Gamma = \mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. For any $X \subset \Gamma$, we have the Hilbert space $\mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x$ and the algebra $\mathcal{A}_X = \mathcal{B}(\mathcal{H}_X)$. We use the usual embedding $\mathcal{A}_X \to \mathcal{A}$ given by $O_X \to O_X \otimes \mathbb{1}_{X^c}$. As is customary, we will identify $O_X \otimes \mathbb{1}_{X^c}$ with O_X and say that O_X is supported in X.

2.1.2 Locality

We write $\mathbb{N}^+$ for the strictly positive naturals and we let $\mathcal{M}$ be a class of functions $m : \mathbb{N}^+ \to \mathbb{R}^+$ of quasi-exponential decay. The class $\mathcal{M}$ is defined by the following two conditions.

1. m is non-increasing.
2. For every $0 < \alpha < 1$, there exists $C_\alpha, c_\alpha > 0$ such that $m(r) \leq C_\alpha e^{-c_\alpha r^\alpha}$.

Note that the same class of functions is obtained by setting $c_\alpha = 1$, which is used often in the proofs. To express the locality properties of Hamiltonians, we consider collections q of local terms $(q_X)_{X \subset \Gamma}, q_X \in \mathcal{A}_X$, sometimes called 'interactions', and we endow them with a family of norms, parametrized by $m \in \mathcal{M}$:

$$||q||_m := \sup_{x \in \Gamma} \sum_{X \ni x} \frac{||q_X||}{m(1 + \text{diam}(X))}$$

The locality property is then expressed by the finiteness of $||q||_m$ for some $m \in \mathcal{M}$.

2.1.3 Trace norms

We denote by $\text{tr}^{(X)}$ the trace on the Hilbert space $\mathcal{H}_X$, and we abbreviate $\text{tr} = \text{tr}^{(\Gamma)}$, i.e. the trace on the global Hilbert space $\mathcal{H}$. We recall the trace norm $|| \cdot ||_{1,X}$ on $\mathcal{A}_X = \mathcal{B}(\mathcal{H}_X)$, defined by

$$||O||_{1,X} = \text{tr}^{(X)} \sqrt{OO^*}, \qquad O \in \mathcal{A}_X$$

We then denote by tr_X the partial trace $\text{tr}_X : \mathcal{A} \mapsto \mathcal{A}_{X_c}$, satisfying $\text{tr}^{(X)} \text{tr}_{X^c} O = \text{tr} O$ for any $O \in \mathcal{A}$. Since we will often use trace norms of operators resulting from the partial trace we introduce a short-hand notation:

$$|O|_X := || \text{tr}_{X^c} O ||_{1,X}.$$

Nonnegative operators ρ on $\mathcal{H}$ that have unit trace $\text{tr} \rho = 1$, are called density matrices and we write $\langle O \rangle_\rho = \text{tr}(\rho O)$ for $O \in \mathcal{A}$. For $\Psi \in \mathcal{H}$, we write ρ_Ψ for the pure density matrix equal to the orthogonal projector on $\mathbb{C}\Psi$.

2.2 Spaces and Hamiltonian

The Hamiltonian H and the perturbation J are written as

$$H = \sum_{X \subset \Gamma} h_X, \qquad J = \sum_{X \subset \Gamma} j_X \qquad h_X = h_X^* \in \mathcal{A}_X, \quad j_X = j_X^* \in \mathcal{A}_X \tag{1}$$

The perturbation J is spatially restricted to a region $\Gamma_j \subset \Gamma$ in a mild sense

$$j_X = 0 \quad \text{unless} \quad X \cap \Gamma_j \neq \emptyset$$

Then, we assume, for the collections h, j,

$$||h||_{m_h}, ||j||_{m_j} < \infty$$

for some $m_h, m_j \in \mathcal{M}$.

2.3   Assumptions

Our first assumption states that the Hamiltonian H has a frustration free ground state Ω, i.e. Ω is an eigenvector of H with eigenvalue equal to $\min(\mathrm{spec}(H))$ ($\mathrm{spec}(\cdot)$ is the spectrum). Our conventions are such that this eigenvalue is 0.

Assumption 1 (Frustration-free ground state)**.** *All the terms h_X are nonnegative: $h_X \geq 0$. There is a $\Omega \in \mathcal{H}, ||\Omega|| = 1$ such that $\Omega \in \mathrm{Ker}(h_X)$ for any X. We denote by $\mu = \rho_\Omega$ the corresponding density matrix.*

We note that the frustration free property depends on the way the Hamiltonian is written as a sum of local terms, i.e. on the interaction. Next, we introduce 'open boundary condition' (OBC) restrictions of H,

$$H_Z = \sum_{X \subset Z} h_X.$$

Just as the frustration free property, the notion of OBC restriction depends on the interaction. We let P_Z be the orthogonal projector on $\mathrm{Ker}(H_Z)$. Since $h_X \geq 0$, it holds that

$$\mathrm{Ker}(H_Z) = \underset{X \subset Z}{\cap} \mathrm{Ker}(h_X)$$

and hence also

$$P_Z P_{Z'} = P_{Z'}, \qquad Z \subset Z'.$$

The following assumption has come to be known as 'Local Topological Quantum Order' (LTQO) but, in our case, it is better described as the property that the density matrix of any ground state of the OBC Hamiltonian H_Z looks similar to the global ground state μ in the deep interior of Z. Recall that $B_r(x) \subset \Gamma$ is the ball or radius r centered at x.

Assumption 2 (OBC-regularity)**.** *There is $m_O \in \mathcal{M}$ and $d_O < \infty$ such that, for any $x \in \Gamma$ and $\Psi \in \mathrm{Ker}(H_{B_r(x)})$,*

$$|\rho_\Psi - \mu|_{B_{r-k}(x)} \leq r^{d_O} m_O(k), \qquad k < r$$

Let us briefly comment on the precise form of the above bound. We have in mind, roughly, that any $\Psi \in \mathrm{Ker}(H_{B_r(x)})$ differs from Ω only through the presence of boundary modes. One realization of this would be that any such Ψ is of the form $e^{iF}\Omega$ with F a sum of terms supported near the boundary of $B_r(x)$. This would indeed lead to the bound above with d_O chosen such that $|\partial B_r(x)| \leq Cr^{d_O}$.

The next assumption concerns the local gap of OBC Hamiltonians. Let

$$\gamma(Z) = \min(\mathrm{spec}(H_Z) \setminus \{0\})$$

be the spectral gap of H_Z.

Assumption 3 (Local Gap)**.** *There are $C_\gamma, d_\gamma < \infty$ such that, for any $x \in \Gamma$*

$$\frac{1}{\gamma(B_r(x))} \leq C_\gamma r^{d_\gamma}.$$

This assumption 3 might be misleading. In most examples of models with gapped frustration-free ground states, that we are aware of, OBC restrictions $H_Z = \sum_{X \subset Z} h_Z$ have a gap that is actually bounded below in Z, and the physical edge modes are found as ground states of H_Z, i.e. the eigenvalue at 0 will typically be degenerate. In that case, the genuine restriction is not so much assumption 3 but rather assumption 2 which describes the kernel $\mathrm{Ker}(H_{B_r(x)})$.

2.3.1   Invertibility

Let $\mathcal{H}'$ be $\mathcal{H}' = \mathcal{H}'_\Gamma = \otimes_{x \in \Gamma} \mathcal{H}'_x$ with $\mathcal{H}'_x$ finite-dimensional Hilbert spaces. We will now consider the tensor product $\mathcal{H} \otimes \mathcal{H}'$ which we denote by

$$\widetilde{\mathcal{H}} = \mathcal{H} \otimes^{\mathrm{aux}} \mathcal{H}'.$$

The superscript $\otimes^{\mathrm{aux}}$ reminds us of the fact that this is not a tensor product between disjoint spatial regions, but between the 'original' Hilbert space and an 'auxiliary' Hilbert space. This is helpful because we also view $\widetilde{\mathcal{H}}$ again as a tensor product over sites

$$\widetilde{\mathcal{H}} = \otimes_{x \in \Gamma} \widetilde{\mathcal{H}}_x, \qquad \widetilde{\mathcal{H}}_x = \mathcal{H}_x \otimes^{\mathrm{aux}} \mathcal{H}'_x$$
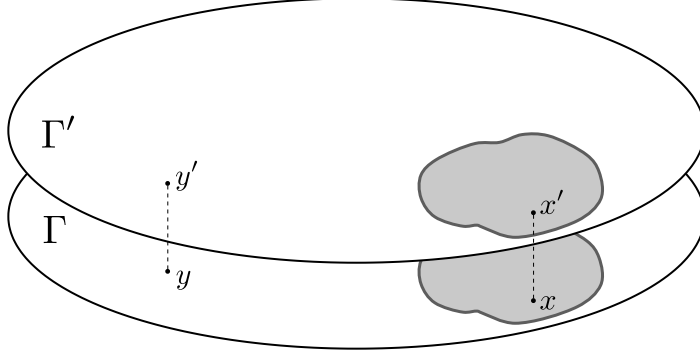
Figure 3: The invertibility assumption uses an auxiliary Hilbert space $\mathcal{H}'$ that has the same spatial structure as the original space $\mathcal{H}$, i.e. Γ' is a copy of Γ.

The local structure of the space $\widetilde{\mathcal{H}}$ is analogous to the one of $\mathcal{H}$, except that now the dimension of the on-site spaces is larger, see Figure 3. We consistently put a prime on algebras derived from $\mathcal{H}'$, $\mathcal{A}' = \mathcal{B}(\mathcal{H}')$ and we put a tilde on algebras derived from $\widetilde{\mathcal{H}}$, i.e. $\widetilde{\mathcal{A}} = \mathcal{B}(\widetilde{\mathcal{H}})$ and $\mathcal{A}'_X \subset \mathcal{A}'$, $\widetilde{\mathcal{A}}_X \subset \widetilde{\mathcal{A}}$ are subalgebras of operators supported in X. We also copy the definition of collections q of local operators $(q_X)_X$ and their norms $||\cdot||_m$, simply replacing $\mathcal{A}_X$ by $\widetilde{\mathcal{A}}_X$. The next assumption expresses that the state Ω has no *long-range entanglement*. Loosely speaking, one could describe this assumption as 'upon adjoining an auxiliary state Ω', $\Omega \otimes \Omega'$ is automorphically equivalent to a product state' [12, 20].

Assumption 4 (Invertibility). *There are collections $q(s)$ of local operators $(q_X(s))_{X \subset \Gamma}$, indexed by $s \in [0, 1]$, such that*

1. *$q_X(s) = q_X^*(s) \in \widetilde{\mathcal{A}}_X$ and $s \mapsto q_X(s)$ is measurable, for any X.*
2. *$\sup_{s \in [0,1]} ||q(s)||_{m_I} \leq C_I < \infty$ for some $m_I \in \mathcal{M}$.*
3. *There is a product state $\Pi = \otimes_{x \in \Gamma} \Pi_x \in \widetilde{\mathcal{H}}$ and a state $\Omega' \in \mathcal{H}'$ such that*

$$\Omega \otimes^{\mathrm{aux}} \Omega' = U(1)\Pi, \qquad U(s) = \mathbb{1} + i \int_0^s du\, H_q(u) U(u)$$

where $H_q(s) = \sum_{X \subset \Gamma} q_X(s)$.

We say that Ω' is an 'inverse' to Ω.

The 'invertibility' assumption roughly corresponds to states that do not have anyonic excitations. Examples of invertible states are: symmetry protected topological states (SPT's) [21, 22, 23], states characterized by an integer quantum Hall effect [24], etc.. We refer to the extensive literature for a more thorough discussion. The motivation for considering the type of spatial decay expressed by the class $\mathcal{M}$ is because this corresponds to the decay one can prove for the spectral flow connecting ground states of a uniformly gapped Hamiltonian, by the technique of (quasi-)adiabatic continuation, see [20, 12].

2.4 Result

We say that a unit vector $\Phi \in \mathcal{H}_\Gamma$ is a ground state of the perturbed Hamiltonian $H + J$ if Φ is an eigenvector of $H + J$ with eigenvalue equal to $\min \operatorname{spec}(H + J)$. Recall that μ is the density matrix associated to Ω, the distinguished ground state of the unperturbed Hamiltonian H, and ρ_Φ is the density matrix associated to Φ. By a 'constant', we mean a quantity that can depend only on $m_h, m_j, m_I \in \mathcal{M}$, the numbers C_γ, C_Γ and d, d_O, d_γ. In particular, constants can not depend on the sizes $|\Gamma|$ and $|\Gamma_j|$.

Theorem 1. *For any $w > 0$, there is a constant $C(w)$ such that, for any ground state Φ of $H + J$, and any $x \in \Gamma$, with $R := \operatorname{dist}(x, \Gamma_j)$,*

$$|\rho_\Phi - \mu|_{B_{(R^p/2)}(x)} \leq C(w) e^{-R^{p-w}}, \qquad p = \frac{1}{d + d_\gamma + 2}$$

To connect this theorem to the informal claim on stability in Section 1, note that the left hand side is equal to $\sup_{O \in \mathcal{A}_X, ||O||=1} |\langle \Phi, O\Phi \rangle - \langle \Omega, O\Omega \rangle|$ with $X = B_{(R^p/2)}(x)$.

3 Proof of Theorem 1

We define some additional notation to be used in this section. For regions $Z \subset \Gamma$, we define the r fattening

$$(Z)_r := \{x \in \Gamma, \operatorname{dist}(x, Z) \leq r\}$$

(we will simply write $(Z)_r = Z_r$ when no confusion is possible) and the boundary

$$\partial Z = (Z)_1 \cap (Z^c)_1$$

We use constants C, c as introduced just before Theorem 1, i.e. depending on a number of fixed parameters. Sometimes we write $C(a)$ to indicate that C can additionally depend on a parameter a. We also use the generic notation m for a function in $\mathcal{M}$ that depends possibly on those same fixed parameters. Just as the constants C, c, the precise function m can change from line to line. In the same vein, we also use p to denote a polynomial that depends only on the fixed parameters.

3.1 Stitching maps

Recall that μ is the global ground state density matrix of the unperturbed Hamiltonian H. We say that a family of maps Σ_Z, indexed by $Z \subset \Gamma$, are stitching maps if the they satisfy the following properties.

Definition 1. *Stitching maps Σ_Z with $Z \subset \Gamma$ are trace-preserving completely positive maps $\mathcal{A} \to \mathcal{A}$ iff. there is an $m \in \mathcal{M}$ such that, for any $Z, X \subset \Gamma$ and any density matrices ρ, ω on $\mathcal{H}$,*

1. $|\Sigma_Z(\rho) - \operatorname{tr}_{Z^c} \mu \otimes \operatorname{tr}_Z \rho|_X \leq |X| m(\operatorname{dist}(\partial Z, X))$
2. $|\Sigma_Z(\rho) - \Sigma_Z(\omega)|_X \leq |\rho - \omega|_{(X)_r} + |X| m(r)$
3. $\Sigma_Z(\mu) = \mu$

To state this in a rough way, stitching maps are such that $\Sigma_Z(\mu) = \mu$ and further

$$\Sigma_Z(\rho) = \begin{cases} \mu & \text{deep inside } Z \\ \rho & \text{far outside } Z \end{cases}$$

The stitching maps will be used for regions Z that are far away from the perturbation region Γ_j.

The purpose of the invertibility assumption 4 is precisely to ensure the existence of stitching maps, as we show now. Recall that $\widetilde{\mathcal{H}}$ is the enlarged Hilbert space.

Proposition 1. *For any region $Z \subset \Gamma$, there are unitaries $V = V_Z$ acting on $\widetilde{\mathcal{H}}$ such that the following family of CP maps $\Sigma = \Sigma_Z$ are stitching maps in the sense of definition 1:*

$$\Sigma(\rho) = \operatorname{tr}_{\mathcal{H}'} \left[V^* \left(\operatorname{tr}_{Z^c}(V \mu \otimes^{\operatorname{aux}} \mu' V^*) \otimes \operatorname{tr}_Z(V \rho \otimes^{\operatorname{aux}} \mu' V^*) \right) V \right] \tag{2}$$

where $\mu = \rho_\Omega$ and $\mu' = \rho_{\Omega'}$ are density matrices on $\mathcal{A}$ and $\mathcal{A}'$ and $\Omega' \in \mathcal{H}'$ is an 'inverse state' to Ω, see assumption 4.

We first construct the appropriate unitaries V featuring in Proposition 1. To that end, we introduce the truncated interactions $\hat{q}(s)$ by

$$\hat{q}_X(s) = \begin{cases} q_X(s) & X \subset Z \quad \text{or} \quad X \subset Z^c \\ 0 & \text{otherwise} \end{cases}$$

with $q(s)$ as given in assumption 4. We let $\hat{U}(s)$ be the unitary defined analogously to $U(s)$ in the invertibility assumption 4, but with $H_q(s)$ replaced by $H_{\hat{q}}(s) = \sum_X \hat{q}_X(s)$. We then set

$$V(s) = \hat{U}(s) U^*(s), \qquad V = V(1)$$

and we observe that

$$V \Omega \otimes^{\operatorname{aux}} \Omega' = \Omega_Z \otimes \Omega_{Z^c} \tag{3}$$

for some $\Omega_Z \in \widetilde{\mathcal{H}}_Z, \Omega_{Z^c} \in \widetilde{\mathcal{H}}_{Z^c}$. This shows that Σ_Z satisfies property 3) of definition 1. The full proof of Proposition 1, i.e. the verifications of properties 1) and 2) of definition 1, uses some standard terminology and locality estimates that are not needed in the rest of our proof, hence we postpone them to the appendices, but we do state the relevant locality property of the unitary operators V. To that end, let us introduce the conditional expectation, for any $Z \subset \Gamma$,

$$\mathbb{E}_{Z^c} : \widetilde{\mathcal{A}} \mapsto \widetilde{\mathcal{A}}_Z : O \mapsto \frac{1}{N} \operatorname{tr}_{Z^c}(O), \qquad N = \dim(\widetilde{\mathcal{H}}_{Z^c}).$$

Lemma 1. *For any regions X, Z and $O \in \widetilde{\mathcal{A}}_X$, with $V = V_Z$ as defined above,*

$$||V^*OV - \mathbb{E}_{(X_r)^c}(V^*OV)|| \leq ||O|||X|m(r) \tag{4}$$

and

$$||V^*OV - O|| \leq ||O|||X|m(\text{dist}(X, \partial Z)) \tag{5}$$

The same estimates hold • as well if we exchange V and V^*.

The proof of Lemma 1 follows from well-known considerations based on Lieb-Robinson bounds, and it is sketched in Appendix B. Finally, the proof of properties 1) and 2) of definition 1 is an intuitive consequence of Lemma 1. This proof is given in Appendix A.

3.1.1 Action of a stitching map Σ_Z at the stitch ∂Z

We will now establish a property of stitching maps, which we will henceforth denote by Σ_Z, that relies crucially on the OBC-regularity assumption 2. It makes explicit that stitching maps are 'seamless', i.e. they do not introduce errors at the cut ∂Z. Recall that P_X is the ground state projector corresponding to the OBC restriction H_X and let $\bar{P}_X = \mathbb{1} - P_X$.

Proposition 2. *Let $\sigma = \rho_\Psi$ be the pure density matrix associated to some $\Psi \in \mathcal{H}$. Consider a set Z and balls $B_r = B_r(x)$ for some site x. Then (recall that $p(\cdot)$ is a polynomial),*

$$\langle \bar{P}_{B_{r-k}} \rangle_{\Sigma_Z(\sigma)} \leq 3\langle \bar{P}_{B_r} \rangle_\sigma + p(r)m(k).$$

Proof. Since Z will be fixed, we drop it and write $\Sigma = \Sigma_Z$. Let us denote $\epsilon := \langle \bar{P}_{B_r} \rangle_\sigma$ and remark that $||P_{B_r}\Psi - \Psi|| \leq \sqrt{\epsilon}$. If $\epsilon = 1$, then the required bound is trivial, hence we assume $\epsilon < 1$. We split

$$\sigma = \sum_{i=1}^4 \zeta_i = P_{B_r}\sigma P_{B_r} + P_{B_r}\sigma \bar{P}_{B_r} + \bar{P}_{B_r}\sigma P_{B_r} + \bar{P}_{B_r}\sigma \bar{P}_{B_r}.$$

We will treat these terms separately, i,e. find bounds on

$$\text{tr}\,\bar{P}_{B_{r-k}}\Sigma(\zeta_i), \qquad i = 1, 2, 3, 4.$$

The term $\zeta_1 = P_{B_r}\sigma P_{B_r}$
Since $\epsilon < 1$, $\text{tr}\,\zeta_1 > 0$ and we can define the pure density matrix $\zeta = \zeta_1/\text{tr}\,\zeta_1$, satisfying $\langle \bar{P}_B \rangle_\zeta = 0$. By the properties of definition 1, we get the equality and first inequality in

$$|\Sigma(\zeta) - \mu|_{B_{r-k}} = |\Sigma(\zeta) - \Sigma(\mu)|_{B_{r-k}} \leq |\zeta - \mu|_{B_{r-k/2}} + p(r)m(k) \leq p(r)m(k)$$

whereas the last inequality follows from the OBC regularity assumption 2. By the frustration-free property, $\text{tr}\,\bar{P}_{B_{r-k}}\mu = 0$, and thus the above inequality yields $\text{tr}\,\bar{P}_{B_{r-k}}\Sigma(\zeta) \leq p(r)m(k)$. Since $\zeta_1 \leq \zeta$, this implies

$$\text{tr}\,\bar{P}_{B_{r-k}}\Sigma(\zeta_1) \leq p(r)m(k)$$

The term $\zeta_4 = \bar{P}_{B_r}\sigma \bar{P}_{B_r}$
Here we have $\text{tr}\,\zeta_4 = \text{tr}\,\bar{P}_{B_r}\sigma = \epsilon$ and since Σ is trace-preserving,

$$\text{tr}\,\bar{P}_{B_{r-k}}\Sigma(\zeta_4) \leq \epsilon.$$

The term $\zeta_2 = P_{B_r}\sigma \bar{P}_{B_r}$
Since σ is pure, we can write

$$\zeta_2 = P_{B_r}\sigma \bar{P}_{B_r} = \langle \bar{P}_{B_r} \rangle_\sigma^{1/2}|a\rangle\langle b|, \qquad ||a|| \leq ||b|| = 1, \qquad P_{B_r}a = a$$

and we note that $|a\rangle\langle a| = \zeta_1$. To estimate $\text{tr}\,\bar{P}_{B_{r-k}}\Sigma(|a\rangle\langle b|)$, we use Lemma 2 below to get

$$|\text{tr}\,\bar{P}_{B_{r-k}}\Sigma(\zeta_2)| \leq \sqrt{\langle \bar{P}_B \rangle_\sigma}\sqrt{\text{tr}(\bar{P}_{B_{r-k}}\Sigma(\zeta_1))} \leq \sqrt{\epsilon}\sqrt{p(r)m(k)} \leq \epsilon + p(r)m(k)$$

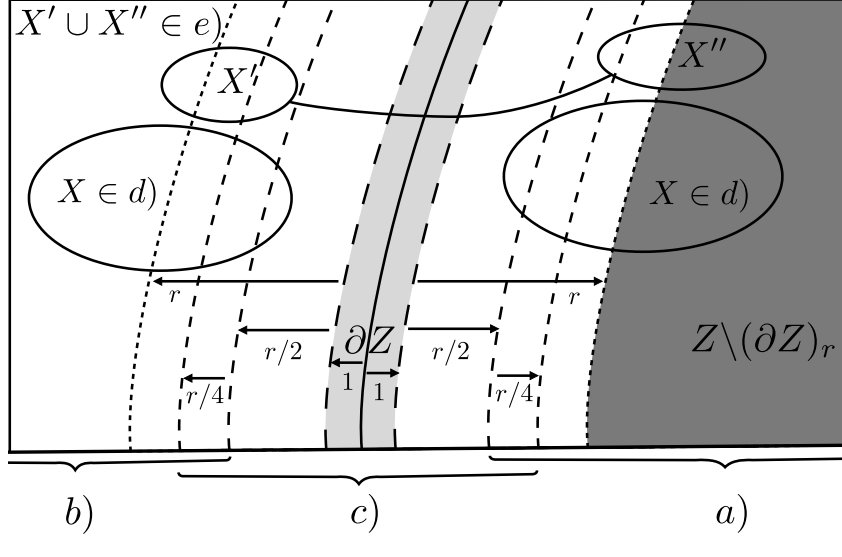The second inequality follows from the bound for the term ζ_1 above.

Figure 4: The classes of sets X in the proof of Lemma 3.

Lemma 2. *For an orthogonal projector P, a trace-conserving CP map Γ and $||a||, ||b|| \leq 1$, we have*

$$|\operatorname{tr}(P\Gamma(|a\rangle\langle b|))| \leq \sqrt{\operatorname{tr}(P\Gamma(|a\rangle\langle a|))\operatorname{tr}(P\Gamma(|b\rangle\langle b|))} \leq \sqrt{\operatorname{tr}(P\Gamma(|a\rangle\langle a|))}$$

Proof. Introducing the adjoint map Γ^*, the first inequality reads

$$|\langle b, \Gamma^*(P)a\rangle| \leq \sqrt{\langle a, \Gamma^*(P)a\rangle\langle b, \Gamma^*(P)b\rangle}.$$

It follows from positivity of $\Gamma^*(P)$ and the Cauchy-Schwarz inequality. □

The term $\zeta_3 = \bar{P}_{B_r}\sigma P_{B_r}$
The bound and its proof are analogous to the case of ζ_2. The claim of the lemma follows by adding the bounds on $\operatorname{tr}\bar{P}_{B_{r-k}}\Sigma(\zeta_i)$ for $i = 1, 2, 3, 4$. □

3.2 A bound on the energy increase due to stitching

If we apply the stitching map Σ_Z to a pure density matrix σ, we expect its energy (associated to H) to decrease in the far interior of Z and to remain unchanged far outside Z. Around the stitch ∂Z, the energy can of course increase. In the present section we prove that this energy increase is upper bounded by the energy of σ around the stitch. We make this precise in Lemma 3. The proof is based largely on Proposition 2 that allows us to convert a bound on the energy to a bound on the local deviation from μ.

Lemma 3. *Let σ be a pure density matrix, i.e. $\sigma = \rho_\Psi$ for some $\Psi \in \mathcal{H}$, and let $\Sigma = \Sigma_Z$ for some region $Z \subset \Gamma$. Then*

$$\sum_{X \not\subset Z\setminus(\partial Z)_r} \left(\langle h_X\rangle_{\Sigma(\sigma)} - \langle h_X\rangle_\sigma\right) \leq Cr^{d+d_\gamma}\langle H_{(\partial Z)_r}\rangle_\sigma + m(r)|\partial Z|$$

Proof. We split the class of X contributing the left hand side[2] in disjoint classes a, b, c, d, e:

 a) $X \subset Z \setminus (\partial Z)_{r/2}$

 b) $X \subset Z^c \setminus (\partial Z)_{r/2}$

 c) $X \cap (\partial Z)_{r/2} \neq \emptyset$ and $\operatorname{diam}(X) < r/4$.

 d) $X \cap (\partial Z)_{r/2} \neq \emptyset$ and $\operatorname{diam}(X) \geq r/4$.

 e) $X \subset ((\partial Z)_{r/2})^c$ and $X \cap Z \neq \emptyset$ and $X \cap Z^c \neq \emptyset$.

We will now estimate, with $x = a, b, c, d, e$,

$$\sum_{X \in \text{class } x} \left(\langle h_X\rangle_{\Sigma(\sigma)} - \langle h_X\rangle_\sigma\right) \tag{6}$$

---

[2]The condition $X \not\subset Z \setminus (\partial Z)_r$ appearing in the statement of the lemma, is actually irrelevant for its validity.

class a

$$\sum_{X\in\text{class }a}\left(\langle h_X\rangle_{\Sigma(\sigma)}-\langle h_X\rangle_\sigma\right)\leq\sum_{X\subset Z\setminus(\partial Z)_{r/2}}\langle h_X\rangle_{\Sigma(\sigma)}$$

$$\leq\sum_{X\subset Z\setminus(\partial Z)_{r/2}}\left(\langle h_X\rangle_\mu+||h_X|||X|m(\text{dist}(X,\partial Z))\right)$$

$$\leq\sum_{x:\text{dist}(x,\partial Z)\geq r/2}m(\text{dist}(x,\partial Z))\sum_{X\ni x}||h_X|||X|\leq|\partial Z|m(r)$$

The first inequality follows from non-negativity of h_X. The second from property 1) of definition 1 and the third from $\langle h_X\rangle_\mu=0$. Then we use $\sup_x\sum_{X\ni x}||h_X|||X|\leq C$ and the rapid decay of m to absorb a polynomial factor in r.

class b

$$\sum_{X\in\text{class }b}\left(\langle h_X\rangle_{\Sigma(\sigma)}-\langle h_X\rangle_\sigma\right)\leq\sum_{X\subset Z^c\setminus(\partial Z)_{r/2}}||h_X|||X|m(\text{dist}(X,Z))$$

$$\leq\sum_{x:\text{dist}(x,Z)\geq r/2}m(\text{dist}(x,Z))\sum_{X\ni x}||h_X|||X|\leq|\partial Z|m(r)$$

The first inequality is property 1) of definition 1, the rest is analogous to case a above.

class c

$$\sum_{X\in\text{class }c}\left(\langle h_X\rangle_{\Sigma(\sigma)}-\langle h_X\rangle_\sigma\right)\leq\sum_{x\in(\partial Z)_{r/2}}\sum_{X\ni x,\text{diam}(X)\leq r/4}\langle h_X\rangle_{\Sigma(\sigma)}$$

$$\leq\sum_{x\in(\partial Z)_{r/2}}\sum_{X\ni x,\text{diam}(X)\leq r/4}||h_X||\langle\bar{P}_X\rangle_{\Sigma(\sigma)}$$

$$\leq\sum_{x\in(\partial Z)_{r/2}}C\langle\bar{P}_{B_{r/4}(x)}\rangle_{\Sigma(\sigma)}$$

$$\leq\sum_{x\in(\partial Z)_{r/2}}C\langle\bar{P}_{B_{r/2}(x)}\rangle_\sigma+m(r)$$

$$\leq\sum_{x\in(\partial Z)_{r/2}}Cr^{d_\gamma}\langle H_{B_{r/2}(x)}\rangle_\sigma+m(r)$$

$$\leq Cr^{d+d_\gamma}\langle H_{(\partial Z)_r}\rangle_\sigma+|(\partial Z)_{r/2}|m(r)$$

$$\leq Cr^{d+d_\gamma}\langle H_{(\partial Z)_r}\rangle_\sigma+|\partial Z|m(r)$$

The first and second inequality is by nonnegativity of h_X. The third inequality uses $\bar{P}_X\leq\bar{P}_{X'}$ for $X\subset X'$, i.e. the frustration freeness of assumption 1, and $\sum_{X\ni x}||h_X||<C$. The fourth inequality uses Proposition 2. The fifth inequality is by the local gap assumption 3. The sixth inequality follows because any $X\subset(\partial Z)_r$ is included in at most a number Cr^d of balls with radius $r/2$. For the last inequality, we use $|(\partial Z)_{r/2}|\leq p(r)|\partial Z|$ and we absorbed the polynomial in m.

class d

$$\sum_{X\in\text{class }d}\left(\langle h_X\rangle_{\Sigma(\sigma)}-\langle h_X\rangle_\sigma\right)\leq\sum_{x\in(\partial Z)_{r/2}}\sum_{X\ni x:\text{diam}(X)\geq r/4}2||h_X||$$

$$\leq\sum_{x\in(\partial Z)_{r/2}}m(r)\leq|\partial Z|m(r)$$

The first inequality follows from $\langle h_X\rangle_{\Sigma(\sigma)},\langle h_X\rangle_\sigma\leq||h_X||$. Then we use analogous reasoning as for **class** a.

class e

$$\sum_{X\in\text{class }e}\left(\langle h_X\rangle_{\Sigma(\sigma)}-\langle h_X\rangle_\sigma\right)\leq\sum_{x:\text{dist}(x,\partial Z)\geq r/2}\sum_{X\ni x:\text{diam}(X)\geq\text{dist}(x,\partial Z)}2||h_X||\leq|\partial Z|m(r)$$

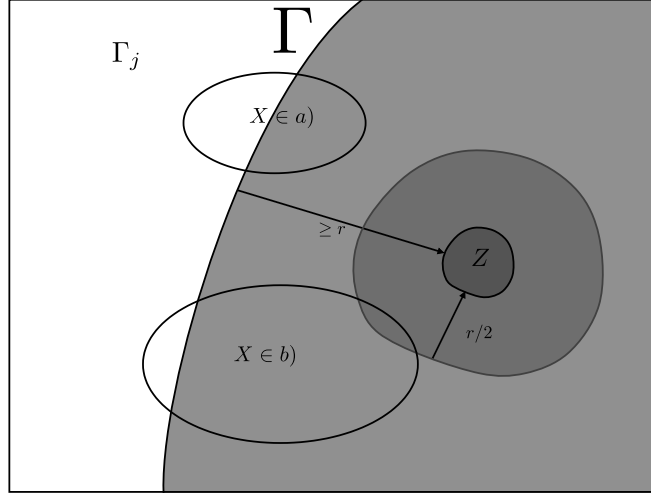The reasoning is analogous to that for **class** d and **class** a.

Figure 5: The classes of sets X in the proof of Lemma 4. We only need to consider X such that $X \cap \Gamma_j \neq \emptyset$, by the definition of j.

$\square$

The previous Lemma 3 was only concerned with the unperturbed Hamiltonian H. We now state an analogous estimate for the Hamiltonian J.

Lemma 4. *Let σ be a density matrix and let Z be a set such that* $\operatorname{dist}(Z, \Gamma_j) \geq r$. *Then*

$$|\langle J \rangle_{\Sigma_Z(\sigma)} - \langle J \rangle_\sigma| \leq m(r)|\partial Z| \qquad (7)$$

Proof. We estimate

$$|\langle J \rangle_{\Sigma_Z(\sigma)} - \langle J \rangle_\sigma| \leq \sum_X |\langle j_X \rangle_{\Sigma_Z(\sigma)} - \langle j_X \rangle_\sigma|$$

. We split the sets X into two disjoint classes:

a) $\operatorname{dist}(X, Z) \geq r/2$

b) $\operatorname{dist}(X, Z) < r/2$

We estimate the contribution of each class and we get in each case the desired bound.

class a

$$\sum_{X \in \text{class a}} |\langle j_X \rangle_{\Sigma_Z(\sigma)} - \langle j_X \rangle_\sigma| \leq \sum_{x: \operatorname{dist}(x,Z) > r/2} m(\operatorname{dist}(x,Z)) \sum_{X \ni x} ||j_X||$$

$$\leq C \sum_{x: \operatorname{dist}(x,Z) > r/2} m(\operatorname{dist}(x,Z)) \leq m(r)|\partial Z|$$

We use property 1 of definition 1 to get the first inequality. The last inequality follows by the rapid decay of m and the fact that Γ has finite dimension d.

class b

$$\sum_{X \in \text{class b}} |\langle j_X \rangle_{\Sigma_Z(\sigma)} - \langle j_X \rangle_\sigma| \leq \sum_{k \geq r/2} \sum_{x: \operatorname{dist}(x,\Gamma_j)=k} \sum_{X \ni x: \operatorname{diam}(X) \geq k} 2||j_X||$$

$$\leq \sum_{k \geq r/2} \sum_{x: \operatorname{dist}(x,\Gamma_j)=k} m(k) \leq \sum_{k \geq r/2} Ck^d|\partial Z|m(k) \leq m(r)|\partial Z|$$

The first inequality follows because $j_X = 0$ unless $X \cap \Gamma_j \neq \emptyset$. In the last inequality, we again absorbed a polynomial factor in m.

$\square$

3.3 Isoperimetry

From now on, we assume that $\sigma = \rho_\Phi$ with Φ a ground state of $H + J$. In particular, this means that the following inequality holds:

$$\langle H + J \rangle_{\Sigma_Z(\sigma)} - \langle H + J \rangle_\sigma \geq 0. \tag{8}$$

We will use Lemma's 3 and 4 to relate the energy in a region Z to the energy around ∂Z, which explains the title of this subsection. We again choose the region Z to be far away from Γ_j: $\text{dist}(Z, \Gamma_j) \geq r$ with $r \gg 1$. By Lemma 4, we can discard J in the above inequality at small cost, and we obtain

$$\langle H \rangle_{\Sigma_Z(\sigma)} - \langle H \rangle_\sigma + m(r)|\partial Z| \geq 0 \tag{9}$$

We split this inequality as

$$\langle H_{Z \setminus (\partial Z)_r} \rangle_{\Sigma_Z(\sigma)} - \langle H_{Z \setminus (\partial Z)_r} \rangle_\sigma + \sum_{X \not\subset Z \setminus (\partial Z)_r} \left(\langle h_X \rangle_{\Sigma_Z(\sigma)} - \langle h_X \rangle_\sigma \right) + m(r)|\partial Z| \geq 0 \tag{10}$$

from which we get

Lemma 5.

$$\langle H_{Z \setminus (\partial Z)_r} \rangle_\sigma \leq C r^{d+d_\gamma} \langle H_{(\partial Z)_r} \rangle_\sigma + m(r)|\partial Z|$$

Proof. We start from inequality (10). By the same easy reasoning as used in the proof of lemma 3 for **class a**, we bound the term $\langle H_{Z \setminus (\partial Z)_r} \rangle_{\Sigma(\sigma)}$ by $m(r)|\partial Z|$. Then the upper bound of Lemma 3 gives us immediately the desired claim. $\qquad \square$

Let us now put Lemma 5 to use. We will consider a sequence of regions Z_i chosen as concentric balls $Z_i = B_{R_i+r} = B_{R_i+r}(x)$, with x fixed and R_i to be specified. We note that

$$B_{R_i} = Z_i \setminus (\partial Z_i)_{r-1}$$

and we abbreviate

$$E_i = \langle H_{B_{R_i}} \rangle_\sigma, \qquad \delta_i = \langle H_{(\partial B_{R_i+r})_{r-1}} \rangle_\sigma$$

Then, the inequality in Lemma 5 reads

$$E_i \leq C r^{d+d_\gamma} \delta_i + m(r)|\partial B_{R_i+r}| \tag{11}$$

Note also that $E_{i+1} \geq E_i + \delta_i$ provided that $R_{i+1} \geq R_i + 2r$. Our strategy will be to establish a lower bound on E_i, depending on E_1, and then eventually use the a priori upper bound

$$E_i = \langle H_{B_{R_i}} \rangle_\sigma \leq ||H_{B_{R_i}}|| \leq C R_i^d$$

to get an upper bound on E_1. Proceeding in this way, we obtain

Lemma 6. *Let $R = \text{dist}(x, \Gamma_j)$ with x the center of the balls above. Then, for any $w > 0$*

$$\langle H_{B_{R^p}} \rangle_\sigma \leq C(w) e^{-R^{p-w}}, \qquad p = \frac{1}{d + d_\gamma + 2}.$$

Proof. We choose $R_i = (2i - 1)r$ for $i = 1, \ldots, i_*$ with i_* the largest integer that is smaller than $R/(2r)$. Using inequality (11) and $E_{i+1} \geq E_i + \delta_i$ we obtain

$$E_{i+1} \geq a E_i - b_i, \qquad a = (1 + c r^{-d-d_\gamma}), \qquad b_i = m(r)|\partial B_{R_i+r}| \tag{12}$$

where we also updated the function m compared to (11). We will now use a Grönwall-type inequality. Multiplying the inequality (12) by a^{-i} and summing over i yields

$$\sum_{i=1}^k E_{i+1} a^{-i} \geq \sum_{i=1}^k a^{-i+1} E_i + \sum_{i=1}^k b_i a^{-i},$$

which implies

$$E_i a^{-i+1} \geq E_1 - \sum_{j=1}^{i-1} b_j a^{-j}.$$

 12

and hence, choosing $i = i_*$,

$$E_1 \leq E_{i_*} a^{-i_* + 1} + \sum_{j=1}^{i_* - 1} b_j a^{-j}.$$

To estimate the sum on the right hand side, we bound $|\partial B_{R_j + r}| \leq C(3jr)^d$ and estimate

$$\sum_{j=1}^{i_* - 1} b_j a^{-j} \leq m(r) r^d \sum_{j=1}^{\infty} j^d a^{-j} \leq m(r).$$

Using in addition the a priori bound $E_{i_*} \leq \langle H_{B_R} \rangle_\sigma < CR^d$, we get

$$E_1 \leq CR^d a^{1 - \frac{R}{2r}} + m(r) \leq CR^d e^{-cRr^{-(d + d_\gamma + 1)}} + m(r).$$

To optimize the inequality we let r grow with R, namely $r = R^p$. For $m \in \mathcal{M}$ and $w > 0$, we can find $C(w)$ such that $m(r) \leq C(w)e^{-r^{1 - w/p}}$ and we get hence

$$E_1 \leq C(w)e^{-R^{p-w}}.$$

which is the claim of the lemma, since $R_1 = r = R^p$.

$\square$

Proof of Theorem 1. We recall that $\sigma = \rho_\Phi$ with Φ a ground state of $H + J$. By the local gap assumption 3, we have

$$\langle \bar{P}_{B_r} \rangle_\sigma \leq Cr^{d_\gamma} \langle H_{B_r} \rangle_\sigma$$

and by Lemma 6, we then get, for any $w > 0$,

$$\langle \bar{P}_{B_r} \rangle_\sigma \leq \delta = C(w)r^{d_\gamma} e^{-R^{p-w}}, \qquad r = R^p \tag{13}$$

From (13) we get, with $P = P_{B_r}$ and for R large enough, $P\Phi \neq 0$, and setting $\Theta = \frac{P\Phi}{||P\Phi||}$, we get $|\langle \Theta, \Phi \rangle| \geq \sqrt{1 - \delta}$. From this one bounds ($|\cdot|_\Gamma$ is the global trace norm)

$$|\rho_\Phi - \rho_\Theta|_\Gamma \leq 2\sqrt{\delta}$$

Since $\Theta \in \mathrm{Ker}(H_{B_r})$, the OBC regularity assumption 2 allows us to bound

$$|\sigma - \mu|_{B_{r-k}} \leq |\sigma - \rho_\Theta|_{B_{r-k}} + |\rho_\Theta - \mu|_{B_{r-k}} \leq 2\sqrt{\delta} + p(r)m(k)$$

with $p(\cdot)$ a polynomial. We choose $k = r/2$. Then we get, for any $w > 0$

$$|\sigma - \mu|_{B_{R^p/2}} \leq C(w)e^{-R^{p-w}}$$

This yields the statement of Theorem 1.

$\square$

A Appendix: Locality properties of CP maps

We establish some vocabulary that is helpful for the proof of Proposition 1.

A.1 Completely positive maps

We consider completely positive (CP) maps acting from $\mathcal{B}(\mathcal{G})$ to $\mathcal{B}(\mathcal{G}')$, with $\mathcal{G}, \mathcal{G}'$ finite-dimensional Hilbert spaces. We need identity-preserving CP maps and trace-preserving CP maps, which are dual to each other. Let $\Upsilon : \mathcal{B}(\mathcal{G}) \mapsto \mathcal{B}(\mathcal{G}')$ be an identity preserving CP map, then the adjoint trace-preserving CP map $\Upsilon^* : \mathcal{B}(\mathcal{G}') \mapsto \mathcal{B}(\mathcal{G})$ is defined by

$$\mathrm{tr}^{(\mathcal{G}')}(S\Upsilon(O)) = \mathrm{tr}^{(\mathcal{G})}(\Upsilon^*(S)O), \qquad O \in \mathcal{B}(\mathcal{G}), S \in \mathcal{B}(\mathcal{G}').$$

From the Russo–Dye theorem, it follows that identity preserving CP maps are contracting w.r.t. operator norm: $||\Upsilon(O)|| \leq ||O||$ and trace-preserving CP maps are contracting w.r.t trace norm $||\Upsilon^*(S)||_1 \leq ||S||_1$ with $||S||_1 = \mathrm{tr}\left(\sqrt{SS^*}\right)$, cf. the notation in Section 2.1.3.

A.2 Conditional expectations

Let us consider a bipartite finite-dimensional Hilbert space $\mathcal{G} = \mathcal{G}_a \otimes \mathcal{G}_b$. For any density matrix σ_b on $\mathcal{G}_b$, we can define the conditional expectation

$$\mathbb{E}_{\sigma_b} : \mathcal{B}(\mathcal{G}) \mapsto \mathcal{B}(\mathcal{G}) : O \mapsto \mathrm{tr}_{\mathcal{G}_b}((\mathbb{1}_a \otimes \sigma_b)O) \otimes \mathbb{1}_b.$$

Note that $\mathbb{E}_{\sigma_b}$ is an identity-preserving CP map and that its range is $\mathcal{B}(\mathcal{G}_a) \otimes \mathbb{1}_b$ which is naturally identified with $\mathcal{B}(\mathcal{G}_a)$. Its adjoint is

$$\mathbb{E}_{\sigma_b}^* : \mathcal{B}(\mathcal{G}) \mapsto \mathcal{B}(\mathcal{G}) : S \mapsto \mathrm{tr}_{\mathcal{G}_b}(S) \otimes \sigma_b. \tag{14}$$

Let ι_b be the density matrix $\frac{1}{\dim(\mathcal{G}_b)} \mathbb{1}_b$ on $\mathcal{G}_b$, i.e. ι_b corresponds to the tracial state on $\mathcal{B}(\mathcal{G}_b)$. The case where $\mathcal{G}_b$ is $\mathcal{H}_Z$ or $\widetilde{\mathcal{H}}_Z$ for some $Z \subset \Gamma$, and $\sigma_b = \iota_b$ is of special signficance to us and we denote the corresponding conditional expectation simply by $\mathbb{E}_Z$. This is consistent with the usage of the symbol $\mathbb{E}_Z$ in Lemma 1.

A.3 Almost locality preserving CP maps

We consider identity-preserving CP maps Υ, acting between (subalgebras of) $\widetilde{\mathcal{A}}$, i.e. there is a spatial structure. We say that Υ preserves almost locality if there exists $m \in \mathcal{M}$ such that,

$$||\Upsilon(O) - \mathbb{E}_{(X_r)^c} \circ \Upsilon(O)|| \le ||O|||X|m(r), \qquad \forall O \in \mathcal{A}_X, \forall X \subset \Gamma, \tag{15}$$

where $\mathbb{E}_{(X_r)^c}$ is the conditional expectation using the tracial state on $(X_r)^c$, as discussed above. A useful property is that, if Υ, Υ' preserve almost locality (say, with functions m_1, m_2) then so does $\Upsilon' \circ \Upsilon$. Indeed, for $O \in \mathcal{A}_X$,

$$\Upsilon' \circ \Upsilon(O) = \Upsilon' \left(\mathbb{E}_{(X_{r/2})^c} \circ \Upsilon(O) + E_1 \right) \tag{16}$$

$$= \mathbb{E}_{(X_r)^c} \circ \Upsilon' \circ \mathbb{E}_{(X_{r/2})^c} \circ \Upsilon(O) + E_2 + \Upsilon'(E_1) \tag{17}$$

$$= \mathbb{E}_{(X_r)^c} \circ \Upsilon' \circ \Upsilon(O) - \mathbb{E}_{(X_r)^c} \circ \Upsilon'(E_1) + E_2 + \Upsilon'(E_1) \tag{18}$$

Using that Υ, Υ' are norm-contracting and almost locality preserving, and that the image of $\mathbb{E}_{(X_{r/2})^c}$ is in $\mathcal{A}_{X_{r/2}}$, the error terms $E_{1,2}$ are bounded as

$$||E_1|| \le ||O|||X|m_1(r/2)), \qquad ||E_2|| \le ||O|||X_{r/2}|m_2(r/2)).$$

Since $|X_{r/2}| \le C|X|r^d$ and also $\mathbb{E}_{(X_r)^c}$ is norm-contracting, we verify that the overall error term is of the form $||O||m(r)|X|$, and hence $\Upsilon' \circ \Upsilon$ indeed preserves almost locality.

If an identity-preserving CP map Υ preserves almost locality, then its adjoint Υ^*, satisfies the following property, for any density matrix ρ,

$$|\Upsilon^*(\rho) - \Upsilon^* \circ \mathbb{E}_{(X_r)^c}^*(\rho)|_X \le |X|m(r) \tag{19}$$

with $\mathbb{E}_{(X_r)^c}^*(\rho) = \mathrm{tr}_{(X_r)^c} \rho \otimes \iota_{(X_r)^c}$ This follows from (15) by duality, using (14).

A.4 Verification of properties 1) and 2) of definition 1

We study the the trace-preserving map $\Sigma = \Sigma_Z$, defined in Proposition 1, via its adjoint Σ^*, an identity-preserving CP map. We decompose $\Sigma^* = \Upsilon_1 \circ \ldots \circ \Upsilon_6$ with Υ_i the identity-preserving CP maps given below.

Υ_6: $\mathcal{A} \to \widetilde{\mathcal{A}} : O \mapsto O \otimes^{\mathrm{aux}} \mathbb{1}_{\mathcal{A}'}$

Υ_5: $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}} : O \mapsto VOV^*$

Υ_4: $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}_{Z^c} : O \mapsto \mathbb{E}_\kappa(O)$ with the density matrix $\kappa = \mathrm{tr}_{Z^c}(V\mu \otimes^{\mathrm{aux}} \mu'V^*)$ on $\mathcal{H}_Z$.

Υ_3: $\widetilde{\mathcal{A}}_{Z^c} \to \widetilde{\mathcal{A}} : O \mapsto \mathbb{1}_Z \otimes O$

Υ_2: $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}} : O \mapsto V^*OV$

Υ_1: $\widetilde{\mathcal{A}} \to \mathcal{A} : O \mapsto \mathbb{E}_{\mu'}(O)$ with the density matrix μ' on $\mathcal{H}'$.

Each of the maps Υ_j preserves almost locality, i.e. satisfies (15). This is immediate for Υ_i with $i = 1, 3, 4, 6$, whereas for $i = 2, 5$ it is a consequence of the bound (4) in Lemma 1. Hence, by duality, we obtain that the locality property (19) is satisfied for $\Upsilon^* = \Sigma$, the map from Proposition 1. We then write, for any density matrices ρ, ω,

$$\Sigma(\rho) - \Sigma(\omega) = \left(\Sigma(\rho) - \Sigma \circ \mathbb{E}^*_{(X_r)^c}(\rho) \right) - \left(\Sigma(\omega) - \Sigma \circ \mathbb{E}^*_{(X_r)^c}(\omega) \right) + \Sigma \circ \mathbb{E}^*_{(X_r)^c}(\rho - \omega) \qquad (20)$$

The first two terms on the right-hand side are bounded in $|\cdot|_X$-norm by (19) with $\Upsilon^* = \Sigma$. The last term is bounded as

$$|\Sigma \circ \mathbb{E}^*_{(X_r)^c}(\rho - \omega)|_X \le |\operatorname{tr}_{(X_r)^c}(\rho - \omega) \otimes \iota_{(X_r)^c}|_X \le |\rho - \omega|_{X_r}$$

where we used in particular contractivity of Σ. This yields property 2) of definition 1. To check property 1) of definition 1, we compare the action of Υ_2, Υ_5 on $\mathcal{A}_X$ with X far from ∂Z with the action of the identity map. By the bound (5) in Lemma 1, we have,

$$||\Upsilon_j(O) - O|| \le ||O|| m(\operatorname{dist}(X, \partial Z))|X|, \qquad O \in \mathcal{A}_X, \qquad j = 2, 5.$$

The verification of property 1) now follows by similar reasoning as for property 2).

B Appendix: Locality estimates

We review the standard propagation bounds that are necessary for the proof of Lemma 1. These bounds go back to [25] but we use the recent formulation in [26]. There are a few differences in our treatment compared to [26]. Most importantly, we have an underlying finite graph Γ equipped with the graph distance, and we always assume that this graph has a finite spatial dimension d as defined in Section 2.1.1.

B.1 Norms on interactions

In the main text, we defined a family of norms $||\cdot||_m$ with $m \in \mathcal{M}$. In [26], as in many preceding works, another family of norms is used, which we introduce now. Namely, one defines so-called F-functions F as functions satisfying

1. $F : \mathbb{N} \to \mathbb{R}^+$ is non-increasing.

2. There is $C_F < \infty$ such that, for all $x, y \in \Gamma$,

$$\sum_{z \in \Gamma} F(\operatorname{dist}(x, z)) F(\operatorname{dist}(z, y)) \le C_F F(\operatorname{dist}(x, y)).$$

3. There is C'_F such that, for each $x \in \Gamma$, $\sum_{y \in \Gamma} F(\operatorname{dist}(x, y)) \le C'_F$.

Strictly speaking, the latter two properties are of course empty for a finite graph Γ, but we have in mind that these properties holds for constants C_F, C'_F that can be chosen uniform in $|\Gamma|$. This is consistent with the conventions used in the main text, as explained in Section 2.4.

Then we can define a family of corresponding norms on interactions.

$$|||z|||_F = \sup_{x,y \in \Gamma} \sum_{S \supset \{x,y\}} \frac{||z_S||}{F(\operatorname{dist}(x, y))}$$

(Note that we use the symbol $|||\cdot|||_F$ to distinghuish these norms from the norms $||\cdot||_m$ in the main text). If we work with functions decaying faster than polynomials, one can switch back and forth between the norms $|||\cdot|||_F$ and $||\cdot||_m$, as we state in the next lemma. In stating the comparison, we write $F \in \mathcal{M}$ meaning that $F|_{\mathbb{N}^+} \in \mathcal{M}$ (The functions m in the main text are defined on $\mathbb{N}^+ = \{1, 2, \ldots\}$ whereas the functions F are defined on $\mathbb{N} = \{0, 1, 2, \ldots\}$.)

Lemma 7. *For any $m \in \mathcal{M}$, there is an F-function F in $\mathcal{M}$ such that*

$$|||z|||_F \le ||z||_m$$

For any F-function $F \in \mathcal{M}$, there is a $m \in \mathcal{M}$ such that

$$||z||_m \le |||z|||_F$$

To prove this, we need the following observation

Lemma 8. *Let $f : \mathbb{N}^+ \to \mathbb{R}^+$ be a function such that $\lim_{r\to\infty} e^{r^\alpha} f(r) = 0$ for any $0 < \alpha < 1$, then there is an F-function $F \in \mathcal{M}$ such that $f(r) \leq F(r)$ for any $r \in \mathbb{N}^+$.*

Proof. We say a function $f : \mathbb{N}^+ \to \mathbb{R}^+$ is logarithmically superadditive if

$$f(r_1)f(r_2) \leq f(r_1 + r_2) \qquad r_1, r_2 \in \mathbb{N}^+. \tag{21}$$

Following [27], we define the following transformation of functions $f : \mathbb{N}^+ \to \mathbb{R}^+$:

$$\mathrm{S}(f)(r) = \sup_{\ell \in \mathbb{N}^+} \sup_{(r_1,\ldots,r_\ell)\in(\mathbb{N}^+)^\ell : \sum_i r_i = r} \prod_i f(r_i)$$

This transformation has the following properties

1. $\mathrm{S}(f)$ is logarithmically superadditive.
2. If $f_1 \leq f_2$, then $\mathrm{S}(f_1) \leq \mathrm{S}(f_2)$. In particular, $f \leq \mathrm{S}(f)$.
3. If f is logarithmically superadditive, then $\mathrm{S}(f) = f$.

Let now f satisfy the assumption of the lemma and put $f'(r) = c_0(1 + r^{d+2})f(r)$ with c_0 such that $f' < 1$. Then, we put

$$F(r) = \frac{1}{c_0(1 + r^{d+2})}\mathrm{S}(f'(r)), \qquad r \in \mathbb{N}^+,$$

and $F(0) = F(1)$. Then F is an F-function (such properties are extensively discussed in [26]) and, by property 2) above, $F(r) \geq f(r)$. It remains to show that F has sufficiently rapid decay. Since $f' < 1$ and $f'e^{r^\alpha} \to 0$, we can find $c_\alpha > 0$ such that $f'(r) \leq f_\alpha(r) = e^{-(c_\alpha r)^\alpha}$. Since f_α, is logarithmically superadditive, we have

$$\mathrm{S}(f') \leq \mathrm{S}(f_\alpha) = f_\alpha.$$

Therefore, $\mathrm{S}(f')$ and hence $F(r)$ have indeed sufficiently rapid decay and we conclude that $F \in \mathcal{M}$. $\square$

Proof of Lemma 7. To get the first claim we observe

$$|||z|||_F = \sup_{x,y} \frac{1}{F(\mathrm{dist}(x,y))} \sum_{S : S \supset \{x,y\}} ||z_S|| \tag{22}$$

$$\leq \sup_{x,y} \sum_{S : S \supset \{x,y\}} \frac{||z_S||}{F(\mathrm{diam}(S))} \tag{23}$$

$$\leq \sup_{x} \sum_{S : S \ni x} \frac{||z_S||}{F(\mathrm{diam}(S))} \tag{24}$$

In order to have the last expression dominated by $||z||_m$, we need to choose the F-function F such that $F(r) \geq m(r+1)$. The possibility of doing this was argued in Lemma 8. For the second claim, we write

$$||z||_m = \sup_x \sum_{S : S \ni x} \frac{1}{m(\mathrm{diam}(S)+1)} ||z_S||$$

$$= \sup_x \sum_{k=0} \sum_{S : S \ni x, \mathrm{diam}(S)=k} \frac{1}{m(k+1)} ||z_S||$$

$$\leq \sup_x \sum_{k=0} \sum_{y,z : \mathrm{dist}(y,x) \leq k, \mathrm{dist}(y,z)=k} \frac{1}{m(k+1)} \sum_{S : S \ni \{y,z\}} ||z_S||$$

$$\leq \sum_{k=0} \frac{(1+C_\Gamma k^d)^2}{m(k+1)} \sup_{y,z : \mathrm{dist}(y,z)=k} \sum_{S : S \ni \{y,z\}} ||z_S||$$

$$= \sum_{k=0} \frac{1}{1+k^2} \frac{1}{f(k+1)} \sup_{y,z : \mathrm{dist}(y,z)=k} \sum_{S : S \ni \{y,z\}} ||z_S||, \qquad f(k+1) = \frac{m(k+1)}{(1+k^2)(1+C_\Gamma k^d)^2}$$

$$\leq C_0 \sup_{y,z} \frac{1}{f(\mathrm{dist}(y,z)+1)} \sum_{S : S \ni \{y,z\}} ||z_S||, \qquad C_0 = \sum_{k=0} \frac{1}{1+k^2}$$

In order to have the last expression dominated by $|||z|||_F$, we need to choose $m \in \mathcal{M}$ such that, for all $r \in \mathbb{N}^+$, $f(r) \geq C_0 F(r-1)$, and hence,

$$m(r) \geq g(r) = C_0 (1 + (r-1)^2)(1 + C_\Gamma (r-1)^d)^2 F(r-1).$$

The function g is not necessarily non-increasing, which is remedied by setting

$$m(r) = \sup_{r' \geq r} g(r').$$

We can now check that $m \in \mathcal{M}$ as it inherits the rapid decay from F. $\qquad\qquad\square$

We call $\mathfrak{B}_F$ the set of interactions with finite $||| \cdot |||_F$-norm and $\mathfrak{B}_F(I)$, with $I \subset \mathbb{R}$ a (possibly infinite) interval, the set of time-dependent interactions $I \ni s \mapsto z(s)$, with finite norm

$$|||z|||_F = \sup_{s \in I} |||z(s)|||_F$$

and such that $s \mapsto z_S(s)$ is measurable for all $S \subset \Gamma$. Pointwise addition of interactions $(z_1 + z_2)_S(s) = (z_1)_S(s) + (z_2)_S(s)$ makes $\mathfrak{B}_F$ and $\mathfrak{B}_F(I)$ into Banach spaces w.r.t. the norm $||| \cdot |||_F$.

B.2 Locally generated automorphisms

Let $z \in \mathfrak{B}_F(I)$ and let $s, t \in I$ with $s \leq t$. We let $\alpha_z(t,s)[\cdot]$ be the dynamics (a family of automorphisms $\mathcal{A} \to \mathcal{A}$ or $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$) generated by the family $z(\cdot)$ acting from time s to time t, in the sense that

$$\alpha_z(t,s)[A] = A + i \int_s^t du\, \alpha_z(u,s)([H_z(u), A]), \qquad H_z(u) = \sum_{S \subset \Gamma} z_S(u).$$

Since Γ is finite, the existence and uniqueness of this dynamics follows from elementary facts on matrix-valued ODE's. For $s = 0$, one often abbreviates

$$\alpha_z(t) = \alpha_z(t,0)$$

as we did in the main text. We first state a version of the Lieb-Robinson bound, using the language introduced in Section 2.1.

Theorem 2 (Lieb-Robinson bound). *Let $z \in \mathfrak{B}_F(I)$, and let $s, t \in I$ with $s \leq t$. Let $A \in \mathcal{A}_{X_A}, B \in \mathcal{A}_{X_B}$ with $X_A \cap X_B = \emptyset$. Then*

$$||[\alpha_z(t,s)[A], B]|| \leq \frac{||A||\,||B||}{C_F} e^{C_F |t-s| |||z|||_F} \left(\sum_{x \in X_A, y \in X_B} F(\mathrm{dist}(x,y)) \right)$$

This is a slightly weakened form of Theorem 3.1 in [26], except for two details. Our graph Γ is finite and we allow our time-dependent interaction z to be composed of measurable functions $t \mapsto z_S(t)$ for any S. Inspection of the proof in [26] confirms that this is all what is needed.

The Lieb-Robinson bound does not directly address the question how to write the evolved observable $\alpha_z(t,s)[A]$ as a sum of local terms. A natural way to do this is to use the conditional expectation $\mathbb{E}_X$ as defined in Section 3.1. If $A \in \mathcal{A}_X$, then we can write

$$\alpha_z(t,s)[A] = \sum_{k=0}^{\infty} \Delta_{X_k}(\alpha_z(t,s)[A]) \tag{25}$$

where the maps Δ_{X_k} are defined as

$$\Delta_{X_k} = \begin{cases} \mathbb{E}_{X^c} & k = 0 \\ \mathbb{E}_{(X_k)^c} - \mathbb{E}_{(X_{k-1})^c} & k > 0 \end{cases}$$

and we recall that $X_k = \{\mathrm{dist}(X, \cdot) \leq k\}$. We see that the terms corresponding to k in the sum in (25) are supported on the fattened sets X_k. The Lieb-Robinson bound can then be used to provide bounds on the right-hand side of (25) via the following standard lemma, whose proof we omit.

Lemma 9. *For any $A \in \mathcal{A}$,*

$$||A - \mathbb{E}_{Z^c}[A]|| \leq \sup_{O \in \mathcal{A}_{Z^c}, ||O||=1} ||[O, A]||.$$

B.2.1 Evolution of interactions

Let g be an interaction and recall that we associate to it a Hamiltonian H_g

$$H_g = \sum_{S \subset \Gamma} g_S$$

Let now $z = (z(s))_{s \in I}$ be as above, i.e. a time-dependent interaction, determining a family of autormorphisms $\alpha_z(t,s)$. We can use these autormorphisms to obtain a new Hamiltonian by transforming H_g into $\alpha_z(t,s)[H_g]$. It is very intuitive that this Hamiltonian can again be expressed as a sum of local terms with good decay properties in the size of the support, i.e. we can express the Hamiltonian as the Hamiltonian corresponding to an new interaction that we then view as a time-evolved interaction $\alpha_z(t,s)[g]$. A priori, there are multiple ways to define $\alpha_z(t,s)[g]$ since there are multiple ways to represent an operator as a sum of local terms. For convenience, we follow [26] and set

$$(\alpha_z(t,s)[g])_S = \sum_{X \subset \Gamma, k \in \mathbb{N}: X_k = S} \Delta_{X_k}(\alpha_z(t,s)[g_X]),$$

using the operators Δ_{X_k} introduced above. This definition has the following convenient properties

1. $H_{\alpha_z(t,s)[g]} = \alpha_z(t,s)[H_g]$.

2. If g is anchored in a set $\Gamma_g \subset \Gamma$, then $\alpha_z(t,s)[g]$ is also anchored in Γ_g.

3. If the F-functions F_g, F_z are in $\mathcal{M}$, then there is an F-function $F' \in \mathcal{M}$ depending only on F_g, F_z, such that

$$|||\alpha_z(t,s)[g]|||_{F'} \leq e^{C_{F_z}|t-s| |||z|||_{F_z}} |||g|||_{F_g}. \tag{26}$$

The first two properties follow immediately from the definitions. The third property is derived starting from the Lieb-Robinsin bound. For details we refer to [26], where one can find explicit choices for F'.

This enables us to define many new time-dependent interactions in Lemma 10. To set the stage, we first define the "commutator" of two interactions z_1, z_2 as

$$(i[z_1, z_2])_S = \sum_{S_1, S_2: S_1 \cup S_2 = S, S_1 \cap S_2 \neq \emptyset} i[z_{S_1}, z_{S_2}]$$

We note that if either one of z_1, z_2 is anchored in a set Γ_z, then the commutator is also anchored in Γ_z. Then we can establish

Lemma 10. *Let z_0, z_1, z_2 be time-dependent interactions in $\mathfrak{B}_{F_{0,1,2}}(I)$, respectively, with $F_{0,1,2} \in \mathcal{M}$. Then the following interactions are in $\mathfrak{B}_F(I)$, with $F \in \mathcal{M}$ and F depending only on F_0 (for item 1)) and on F_1, F_2 (for items 2,3),*

1. *$\bar{z}_0(s) = -\alpha_{z_0}(s)[z_0(s)]$*

2. *$i[z_1, z_2](s)$*

3. *$w(s) = z_2(s) - z_1(s) + i(\alpha_{z_2}(s))^{-1} \left[\int_0^s \alpha_{z_2}(u) \Big\{ [z_2(u) - z_1(u), \alpha_{z_1}(s,u)[z_1(s)]] \Big\} du \right]$*

The time-dependent interaction $\bar{z}_0$ in item 1) satisfies

$$\alpha_{\bar{z}_0}(t,s) = (\alpha_{z_0}(t,s))^{-1} \tag{27}$$

The time-dependent interaction w in item 3) satisfies

$$\alpha_w(t,s) = (\alpha_{z_1}(t,s))^{-1} \circ \alpha_{z_2}(t,s) \tag{28}$$

Sketch of proof. First, one verifies that the newly defined time-dependent interactions satisfy the measurability condition, using that $z_{0,1,2}$ do so. Then, the bound on $\bar{z}_0$ follows directly from (26). The identities (27) and (28) follow by direct computation. The bound on w follows by combining all the previous items, i.e. the fact that commutators, time-evolution and inverse time-evolution map $\mathfrak{B}_F(I)$ into $\mathfrak{B}_{F'}(I)$ for certain F'. $\qquad \square$

On our way to prove Lemma 1, we first state the following observation concerning the unitary family $V_Z(s) = V(s)$ introduced in the proof of Proposition 1 in the main text.

Lemma 11. *There is an F-function $F \in \mathcal{M}$ and a time-dependent interaction $l \in \mathfrak{B}_F([0,1])$ such that*

$$V(s)OV^*(s) = \alpha_l(s)[O], \qquad O \in \widetilde{\mathcal{A}} \tag{29}$$

Moreover, l is anchored in ∂Z.

Proof. We recall the time-dependent interactions $q, \hat{q}$. These were defined as having a finite $||\cdot||_m$ norm, with $m \in \mathcal{M}$. By Lemma 7, we find an F-function F_0 such that $q, \hat{q}$ have a finite $|||\cdot|||_{F_0}$-norm. From the definition of $q, \hat{q}$, we deduce that

$$V^*(s)OV(s) = (\alpha_q(s))^{-1} \circ \alpha_{\hat{q}}(s)[O]$$

We set

$$\tilde{q}(s) = q(s) - \hat{q}(s)$$

and we note that $\tilde{q}(s)$ is anchored in ∂Z, for any s. Now we invoke item 3) of Lemma 10, with $z_1 = q, z_2 = \hat{q}$, to conclude that there is $l \in \mathfrak{B}([0,1])$, given by

$$l(s) = -\tilde{q}(s) - i(\alpha_{\hat{q}}(s))^{-1} \left[\int_0^s \alpha_{\hat{q}}(u) \Big\{ \big[\tilde{q}(u), \alpha_q(s,u)[q(s)] \big] \Big\} du \right],$$

such that $V(s)OV^*(s) = \alpha_l(s)[O]$. Since commutators and time-evolution preserve the property of being anchored in a set, we conclude that l, just as $\tilde{q}$, is anchored in ∂Z. $\square$

B.3 Proof of Lemma 1

For convenience we restate Lemma 1:

Lemma 12 (repetition of Lemma 1)**.** *For any regions X, Z and $O \in \widetilde{\mathcal{A}}_X$, with $V = V_Z$ as in Lemma 11 above,*

$$||V^*OV - \mathbb{E}_{(X_r)^c}(V^*OV)|| \leq ||O|||X|m(r) \tag{30}$$

and

$$||V^*OV - O|| \leq ||O|||X|m(\mathrm{dist}(X, \partial Z)) \tag{31}$$

The same estimates hold as well if we exchange V and V^*.

To get the bound (30), we invoke Lemma 11 and Lemma 9 to relate conditional expectations to commutators. Then, we use the Lieb-Robinson bound (Theorem 2) for the dynamics α_l:

$$||V^*OV - \mathbb{E}_{(X_r)^c}(V^*OV)|| \leq \sup_{A \in \widetilde{\mathcal{A}}_{(X_r)^c}, ||A|| \leq 1} |||[A, \alpha_l(1)[O]|||$$

$$\leq ||O||e^{C_{F_l}|||z_l|||_{F_l}} \left(\sum_{x \in X, y \in (X_r)^c} F_l(\mathrm{dist}(x,y)) \right)$$

The last expression between brackets is estimated as $|X|m(r)$, since $F_l \in \mathcal{M}$ and the spatial dimension d is finite.

Let us now turn to (31). By the Heisenberg equation,

$$V^*OV - O = i \int_0^1 \alpha_l(u)[H_l(u), O]du.$$

Therefore, we can bound

$$||V^*OV - O|| \le \int_0^1 du \sum_S ||\alpha_l(u)[l_S(u), O]||$$

$$= \int_0^1 du \sum_S |||[l_S(u), O]||$$

$$\le 2||O|| \sup_{u\in[0,1]} \sum_{S:S\cap\partial Z\neq\emptyset, S\cap X\neq\emptyset} ||l_S(u)||$$

$$\le 2||O|| \sup_{u\in[0,1]} \sum_{x\in X, y\in\partial Z} \sum_{S:S\supset\{x,y\}} ||l_S(u)||$$

$$\le 2||O|| \sup_{u\in[0,1]} |||l(u)|||_{F_l} \sum_{x\in X, y\in\partial Z} F_l(\text{dist}(x,y))$$

$$= 2||O|| |||l|||_{F_l} \sum_{x\in X, y\in\partial Z} F_l(\text{dist}(x,y))$$

The third line follows because O is supported in X and l is anchored in ∂Z. The expression in the last line is dominated by $||O|| |X| m(\text{dist}(X, \partial Z))$ upon using that $F_l \in \mathcal{M}$, that the spatial dimension d is finite, and bounding $|||l|||_{F_l}$ by a constant.

To get the bounds (30) and (31) for VOV^* instead of V^*OV, we note that

$$VOV^* = (\alpha_l(1))^{-1}[O] = \alpha_{\bar{l}}(1)[O]$$

with $\bar{l}$ related to l as in Lemma 10. By Lemma 10, $\bar{l} \in \mathfrak{B}_{F_{\bar{l}}}([0,1])$ for some F-function $F_{\bar{l}} \in \mathcal{M}$. We can therefore copy the arguments given above for $V^*OV = \alpha_l(1)[O]$, replacing l by $\bar{l}$.

References

[1] Wojciech De Roeck and Marius Schütz. "An exponentially local spectral flow for possibly non-self-adjoint perturbations of non-interacting quantum spins, inspired by kam theory". Letters in Mathematical Physics **107**, 505–532 (2017).

[2] Simone Del Vecchio, Jürg Fröhlich, Alessandro Pizzo, and Stefano Rossi. "Lie-schwinger block-diagonalization and gapped quantum chains: analyticity of the ground-state energy". Journal of Functional Analysis **279**, 108703 (2020).

[3] Juerg Froehlich and Alessandro Pizzo. "Lie–schwinger block-diagonalization and gapped quantum chains". Communications in Mathematical Physics **375**, 2039–2069 (2020).

[4] DA Yarotsky. "Ground states in relatively bounded quantum perturbations of classical lattice systems". Communications in mathematical physics **261**, 799–819 (2006).

[5] Nilanjana Datta, Roberto Fernández, and Jürg Fröhlich. "Low-temperature phase diagrams of quantum lattice systems. i. stability for quantum perturbations of classical systems with finitely-many ground states". Journal of statistical physics **84**, 455–534 (1996).

[6] Christian Borgs, R Koteckỳ, and D Ueltschi. "Low temperature phase diagrams for quantum perturbations of classical spin systems". Communications in mathematical physics **181**, 409–446 (1996).

[7] Matthew F Lapa and Michael Levin. "Stability of ground state degeneracy to long-range interactions" (2021). arXiv:2107.11396.

[8] Sergey Bravyi, Matthew B Hastings, and Spyridon Michalakis. "Topological quantum order: stability under local perturbations". Journal of mathematical physics **51**, 093512 (2010).

[9] Spyridon Michalakis and Justyna P Zwolak. "Stability of frustration-free hamiltonians". Communications in Mathematical Physics **322**, 277–302 (2013).

[10] Bruno Nachtergaele, Robert Sims, and Amanda Young. "Quasi-locality bounds for quantum lattice systems. part ii. perturbations of frustration-free spin models with gapped ground states". In Annales Henri Poincaré. Volume 23, pages 393–511. Springer (2022).

[11] Bruno Nachtergaele, Robert Sims, and Amanda Young. "Stability of the bulk gap for frustration-free topologically ordered quantum lattice systems" (2021). arXiv:2102.07209.

Accepted in ⟨ Quantum 2022-06-11, click title to verify. Published under CC-BY 4.0.

20

[12] Sven Bachmann, Spyridon Michalakis, Bruno Nachtergaele, and Robert Sims. "Automorphic equivalence within gapped phases of quantum lattice systems". Communications in Mathematical Physics **309**, 835–871 (2012).

[13] Wojciech De Roeck and Marius Schütz. "Local perturbations perturb—exponentially–locally". Journal of Mathematical Physics **56**, 061901 (2015).

[14] Alexei Kitaev. "Anyons in an exactly solved model and beyond". Annals of Physics **321**, 2–111 (2006).

[15] Alexei Kitaev and Chris Laumann. "Topological phases and quantum computation". Exact methods in low-dimensional statistical physics and quantum computing, Lecture Notes of the Les Houches Summer SchoolPages 101–125 (2009). url: arxiv.org/pdf/0904.2771.pdf.

[16] Bruno Nachtergaele and Nicholas E Sherman. "Dispersive toric code model with fusion and defusion". Physical Review B **101**, 115105 (2020).

[17] Joscha Henheik, Stefan Teufel, and Tom Wessel. "Local stability of ground states in locally gapped and weakly interacting quantum spin systems". Letters in Mathematical Physics **112**, 1–12 (2022).

[18] Matthew B Hastings. "Quantum belief propagation: An algorithm for thermal quantum systems". Physical Review B **76**, 201102 (2007).

[19] Kohtaro Kato and Fernando GSL Brandao. "Quantum approximate markov chains are thermal". Communications in Mathematical Physics **370**, 117–149 (2019).

[20] Matthew B Hastings and Xiao-Gang Wen. "Quasiadiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance". Physical review b **72**, 045141 (2005).

[21] Daniel S Freed. "Anomalies and invertible field theories". In Proc. Symp. Pure Math. Volume 88, pages 25–46. (2014). url: arxiv.org/pdf/1404.7224.pdf.

[22] A. Kitaev. "On the classification of short-range entangled states". http://scgp.stonybrook.edu/video_portal/video.php?id=2010.

[23] Zheng-Cheng Gu and Xiao-Gang Wen. "Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order". Physical Review B **80**, 155131 (2009).

[24] Anton Kapustin and Nikita Sopenko. "Hall conductance and the statistics of flux insertions in gapped interacting lattice systems". Journal of Mathematical Physics **61**, 101901 (2020).

[25] E.H. Lieb and D.W. Robinson. "The finite group velocity of quantum spin systems". Commun. Math. Phys. **28**, 251–257 (1972).

[26] Bruno Nachtergaele, Robert Sims, and Amanda Young. "Quasi-locality bounds for quantum lattice systems. i. lieb-robinson bounds, quasi-local maps, and spectral flow automorphisms". Journal of Mathematical Physics **60**, 061101 (2019).

[27] A. Bruckner. "Minimal superadditive extensions of superadditive functions". Pacific J. Math. **10**, 1155–1162 (1960). url: msp.org/pjm/1960/10-4/pjm-v10-n4-s.pdf#page=51.