Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/era

ERA, 30(6): 2121–2137.
DOI: 10.3934/era.2022107
Received: 14 October 2021
Revised: 29 December 2021
Accepted: 16 January 2022
Published: 14 April 2022

Research article

Maximal and minimal weak solutions for elliptic problems with nonlinearity on the boundary

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Abstract: This paper deals with the existence of weak solutions for semilinear elliptic equation with nonlinearity on the boundary. We establish the existence of a maximal and a minimal weak solution between an ordered pair of sub- and supersolution for both monotone and nonmonotone nonlinearities. We use iteration argument when the nonlinearity is monotone. For the nonmonotone case, we utilize the surjectivity of a pseudomonotone and coercive operator, Zorn's lemma and a version of Kato's inequality.

Keywords: elliptic problem; nonlinear boundary conditions; maximal and minimal weak solution; pseudomonotone operator; Kato's inequality

1. Introduction

We consider an elliptic equation with nonlinear boundary condition of the form

$$\begin{cases} -\Delta u + u &= 0 \quad \text{in} \quad \Omega; \\ \frac{\partial u}{\partial \eta} &= f(x,u) \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with $C^{2,\alpha}$ $(0 < \alpha < 1)$ boundary $\partial\Omega$, and $\partial/\partial\eta := \eta(x) \cdot \nabla$ denotes the outer normal derivative on the boundary $\partial\Omega$. Here $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $f(\cdot, u)$ is measurable for each u and $f(x, \cdot)$ is continuous for a.e. $x \in \partial\Omega$.

In this paper, we investigate the existence of maximal and minimal weak solutions (to be clarified later) between an ordered pair of sub- and supersolution of (1.1) for both monotone and nonmonotone nonlinearities. We use monotone iteration procedure when the nonlinearity is monotone. The nonmonotone case required a careful use of the surjectivity of a bounded, pseudomonotone and coercive operator, Zorn's lemma and a version of Kato's inequality up to the boundary. This proof, for the nonmonotone case, is motivated by the works in [1] and [2].

Elliptic equations with nonlinear boundary conditions have attracted a lot of attention over the last decades, see for instance [3–9] and references therein. Motivation to study equations with nonlinear boundary conditions stems from the fact that, when the reaction near the boundary depends on the density itself, linear boundary conditions (Dirichlet, Neumann, or Robin) are often inadequate to study chemical, biological, or ecological processes, see [10–13] and references therein, for specific applications.

The existence of a solution between an ordered pair of sub- and supersolution of elliptic boundary value problems has been studied extensively. For the linear boundary conditions, the sub–supersolution method for classical solutions were developed in [14–16] to study the solvability of quasi-linear and semi-linear equations using monotone iteration method. This method also yields the existence of a maximal and a minimal solution. These iterative methods can be thought as a generalization of the Perron arguments on sub- and superharmonic functions for existence of solutions of the boundary value problem. For relatively recent results on the existence of maximal and minimal solutions, for the linear boundary conditions, we refer readers to [2, 17, 18] for the Laplacian case, and [1, 19] for the p-Laplacian case.

For the nonlinear boundary case, see [20] and [13, Ch. 4] where the existence of maximal and minimal classical solutions was established for the monotone case. To the best of our knowledge, our results concerning the existence of maximal and minimal weak solutions are new for both monotone and nonmonotone cases.

We begin with the definitions of weak solution and weak sub- and supersolution. For this, we make use of the real Lebesgue space $L^r(\partial\Omega)$ and the Sobolev space $H^1(\Omega)$.

Definition 1.1. *We say that a function $u \in H^1(\Omega)$ is a weak solution to* (1.1) *whenever:*

(i) $f(.,u(.)) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$ *if* $N > 2$, *and* $f(.,u(.)) \in L^r(\partial\Omega)$ *for* $r > 1$ *if* $N = 2$, *and*

(ii) $\displaystyle\int_\Omega (\nabla u \nabla \psi + u\psi) = \int_{\partial\Omega} f(x,u)\psi \qquad$ *for all* $\psi \in H^1(\Omega)$.

Definition 1.2. *We say that a function $\overline{u} \in H^1(\Omega)$ is a weak supersolution to* (1.1) *whenever:*

(i) $f(.,\overline{u}(.)) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$ *if* $N > 2$, *and* $f(.,\overline{u}(.)) \in L^r(\partial\Omega)$ *for* $r > 1$ *if* $N = 2$, *and*

(ii) $\displaystyle\int_\Omega (\nabla \overline{u} \nabla \psi + \overline{u}\psi) \geq \int_{\partial\Omega} f(x,\overline{u})\psi \quad$ *for all* $0 \leq \psi \in H^1(\Omega)$.

A weak subsolution $\underline{u}$ is defined by reversing the inequality in (ii) *above.*

Remark 1.3. Let $\Gamma : H^1(\Omega) \to L^r(\partial\Omega)$ be the trace operator given by $\Gamma u = u|_{\partial\Omega}$. It is known that, see e.g. [21], [22, Thm 2.79], and [23, Chapter 6], Γ is continuous (compact) if

$$\begin{cases} 1 \leq r \leq \frac{2(N-1)}{N-2} \ \left(1 \leq r < \frac{2(N-1)}{N-2}\right) \text{ if } N > 2 \\ r \geq 1 \quad (r \geq 1) \text{ if } N = 2\,. \end{cases} \tag{1.2}$$

Therefore, the integrals on the right hand side of (ii) of Definition 1.1 and Definition 1.2 make sense since (i) holds, and $\frac{2(N-1)}{N}$ is the conjugate of $\frac{2(N-1)}{N-2}$ when $N > 2$.

We state and prove our results for the case $N > 2$, since the case $N = 2$ follows clearly using (1.2). We state our first result concerning maximal and minimal solutions for the monotone case.

Theorem 1.4. *Suppose there exists a pair of weak sub- and supersolution $\underline{u}$ and $\overline{u}$, respectively, satisfying $\underline{u} \le \overline{u}$ in $\overline{\Omega}$. Assume that*

(H1) *there exists $k \ge 0$ such that the map $s \mapsto f(x,s) + ks$ is nondecreasing for all $\underline{u} \le s \le \overline{u}$, and for all $x \in \partial\Omega$.*

Then, there exist a minimal weak solution u_* and a maximal weak solution u^* to (1.1)*, in the sense that if u is any weak solution to* (1.1) *such that $\underline{u} \le u \le \overline{u}$, then $u_* \le u \le u^*$.*

Next, we note that if f is locally Lipschitz with respect to the second variable u, and the interval $[\underline{u}, \overline{u}]$ is bounded, then f satisfies the hypothesis **(H1)**. For functions f that do not satisfy the monotonicity condition given in **(H1)**, we have the following existence result.

Theorem 1.5. *Suppose there exists a pair of weak sub- and supersolution $\underline{u}$ and $\overline{u}$, respectively, satisfying $\underline{u} \le \overline{u}$ in $\overline{\Omega}$. Assume that*

(H2) *there exists a $K \in L^r(\partial\Omega)$, $r > \frac{2(N-1)}{N}$, such that $|f(x,s)| \le K(x)$ a.e. $x \in \partial\Omega$, for all s satisfying $\underline{u}(x) \le s \le \overline{u}(x)$.*

Then (1.1) *has at least one weak solution u such that $\underline{u} \le u \le \overline{u}$.*

Finally, we state a result that guarantees the existence of a maximal and a minimal weak solution without assuming monotonicity condition **(H1)** on the nonlinearity f.

Theorem 1.6. *Assume hypotheses of Theorem 1.5 hold. Then, there exist a minimal weak solution u_* and a maximal weak solution u^* to* (1.1)*, in the sense that if u is any weak solution to* (1.1) *such that $\underline{u} \le u \le \overline{u}$, then $u_* \le u \le u^*$.*

In [24], Hess proved the existence of a solution, assuming that

$$\int_{\partial\Omega} \sup_{\underline{u}(x) \le s \le \overline{u}(x)} |f(x,s)|^q < \infty, \tag{1.3}$$

for $q = 2$. Our result, Theorem 1.4, is sharper, needing only that condition (1.3) hold for $q = \frac{2(N-1)}{N} = 2 - \frac{2}{N} < 2$.

In Section 2, we collect some known results that will be helpful in the sequel. We also state and prove a version of Kato's inequality for our setting, see Theorem 2.4 and Corollary 2.5. In Section 3, we prove Theorem 1.4 using monotone iteration method. In Section 4, we prove Theorem 1.5 by showing that an appropriately defined operator is surjective. We also prove Theorem 1.6 in Section 4 by utilizing Theorem 1.5, Zorn's Lemma and Theorem 2.4. In Section 5, we discuss applications of our results.

2. Preliminaries and auxiliary results

Here we collect some results that we use in the sequel. First, we recall an existence and uniqueness result for a linear problem.

Proposition 2.1. *([4, 8]) Let $h \in L^q(\partial\Omega)$ for $q \geq 1$. Then, the linear problem*

$$\begin{cases} -\Delta v + v &= 0 \quad \text{in} \quad \Omega\,; \\ \frac{\partial v}{\partial \eta} &= h \quad \text{on} \quad \partial\Omega \end{cases}$$

has a unique solution $v \in W^{1,m}(\Omega)$ and

$$\|v\|_{W^{1,m}(\Omega)} \leq C\|h\|_{L^q(\partial\Omega)}, \qquad \textit{where} \quad 1 \leq m \leq Nq/(N-1)\,.$$

In particular, if $q = \frac{2(N-1)}{N}$, then $u \in H^1(\Omega)$.

Next, let X be a reflexive Banach space and $A : X \to X^*$. We say that the operator A is *coercive* if

$$\frac{\langle A(\psi), \psi\rangle}{\|\psi\|_X} \to \infty \text{ as } \|\psi\|_X \to \infty\,.$$

We say that A is *pseudomonotone*, whenever

$$v_n \rightharpoonup v \quad \text{in} \quad X \quad \text{and} \quad \limsup_{n\to\infty}\langle A(v_n), v_n - v\rangle \leq 0 \quad \text{imply}$$

$$\liminf_{n\to\infty}\langle A(v_n), v_n - \psi\rangle \geq \langle A(v), v - \psi\rangle \quad \text{for any} \quad \psi \in X. \tag{2.1}$$

We will utilize the following surjectivity result in the proof of Theorem 1.5.

Proposition 2.2. *([25, Thm. II. 2.8], [22, Thm. 2.99]) Let X be a reflexive Banach space. If $A : X \to X^*$ is a bounded, pseudomonotone and coercive operator, then for each $b \in X^*$, $Au = b$ has a solution.*

Finally, we say that a subset Y of a partially ordered set $(X, \leq)$ is a chain if $x \leq y$ or $y \leq x$ for every $x, y \in Y$. Then, to prove Theorem 1.6, we use the following version of Zorn's lemma (see [22]):

Proposition 2.3 (Zorn's lemma)**.** *If in a partially ordered set $(X, \leq)$, every chain Y has an upper bound, then X possesses a maximal element.*

2.1. A version of Kato's inequality

In [26], authors established Kato's inequality up to the boundary for a function $u \in W^{1,1}(\Omega)$. Here, we state and prove a version of Kato's inequality up to the boundary, that is necessary in the proof of Theorem 1.6. This result can be rephrased as *the maximum of two weak subsolutions is also a weak subsolution*. In particular, *the maximum of two weak solutions is a weak subsolution*.

Theorem 2.4. *Let u_1 and u_2 be functions in $H^1(\Omega)$ such that there exist f_1 and f_2 in $L^r(\partial\Omega)$, for $r \geq \frac{2(N-1)}{N}$, satisfying*

$$\int_\Omega (\nabla u_i \nabla\psi + u_i\psi) \leq \int_{\partial\Omega} f_i\psi \qquad \textit{for all } 0 \leq \psi \in H^1(\Omega)\,, \tag{2.2}$$

for $i = 1, 2$. Then, $u := \max\{u_1, u_2\}$ satisfies

$$\int_\Omega (\nabla u\nabla\psi + u\psi) \leq \int_{\partial\Omega} f\psi \qquad \textit{for all } 0 \leq \psi \in H^1(\Omega)\,,$$

where $f(x) := \begin{cases} f_1(x) & \textit{if } u_1(x) > u_2(x) \\ f_2(x) & \textit{if } u_1(x) \leq u_2(x), \end{cases}$ *a.e.* $x \in \partial\Omega$.

Proof. Define

$$\Omega_1 := \{x \in \Omega : u_1(x) > u_2(x)\} \text{ and } \Omega_2 := \Omega \setminus \Omega_1$$

and

$$\Gamma_1 := \{x \in \partial\Omega : u_1(x) > u_2(x)\} \text{ and } \Gamma_2 := \partial\Omega \setminus \Gamma_1 .$$

Fix $0 \le \psi \in H^1(\Omega)$. Then,

$$\begin{aligned} I &= \int_\Omega \nabla u \nabla \psi + \int_\Omega u\psi \\ &= \underbrace{\int_{\Omega_1} (\nabla u_1 \nabla\psi + u_1\psi)}_{I_1} + \underbrace{\int_{\Omega_2} (\nabla u_2 \nabla\psi + u_2\psi)}_{I_2} . \end{aligned}$$

Consider a sequence $\xi_n \in C^1(\mathbb{R})$ such that

$$\xi_n(t) := \begin{cases} 1 & \text{if } t \ge 1/n \\ 0 & \text{if } t \le 0, \end{cases}$$

and $\xi_n' > 0$ on $(0, 1/n)$. Then, define the sequence of functions

$$r_n(x) := \xi_n((u_1 - u_2)(x)) \qquad \text{for } x \in \overline{\Omega} .$$

Observe that $r_n \in H^1(\Omega)$ and r_n converges pointwise to $\chi_{\Omega_1\cup\Gamma_1}$, where the characteristic function is defined as $\chi_{\Omega_1\cup\Gamma_1}(x) := \begin{cases} 1 & \text{if } x \in \Omega_1 \cup \Gamma_1 \\ 0 & \text{if otherwise} . \end{cases}$ Moreover, $\|r_n\|_{L^\infty(\Omega)\cap L^\infty(\partial\Omega)} \le 1$ and $\operatorname{supp}(\nabla r_n) \subset \overline{D}_n$, where $D_n := \{x \in \Omega : 0 < u_1(x) - u_2(x) < \frac{1}{n}\}$. Then, using Lebesgue Dominated Convergence Theorem, we have that

$$I_1 = \lim_{n\to\infty} \left[\int_\Omega r_n \nabla u_1 \nabla\psi + \int_\Omega r_n u_1 \psi \right].$$

Since $r_n \in H^1(\Omega) \cap L^\infty(\Omega) \cap L^\infty(\partial\Omega)$, it follows that $r_n\psi \in H^1(\Omega)$ for any test function $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$. Recalling that $\nabla r_n = 0$ on $\Omega \setminus D_n$, and that u_1 satisfies (2.2), we can write

$$\begin{aligned} \int_\Omega r_n \nabla u_1 \nabla\psi + r_n u_1 \psi &= \int_\Omega \nabla u_1 \nabla(r_n\psi) + u_1(r_n\psi) - \int_{D_n} \psi \nabla u_1 \nabla r_n \\ &\le \int_{\partial\Omega} f_1 r_n \psi - \int_{D_n} \psi \nabla u_1 \nabla r_n . \end{aligned} \tag{2.3}$$

Taking the limit as $n \to \infty$ in the first term of the right-hand side of (2.3), using the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_{\partial\Omega} f_1 r_n \psi = \int_{\Gamma_1} f_1 \psi .$$

Likewise, for I_2 we have

$$I_2 = \lim_{n\to\infty} \left[\int_\Omega (1 - r_n) \nabla u_2 \nabla\psi + \int_\Omega (1 - r_n) u_2 \psi \right],$$

and

$$\int_\Omega (1-r_n)\nabla u_2\nabla\psi + \int_\Omega (1-r_n)u_2\psi$$
$$= \int_\Omega \nabla u_2\nabla[(1-r_n)\psi] + u_2(1-r_n)\psi + \int_{D_n}\psi\nabla u_2\nabla r_n$$
$$\leq \int_{\partial\Omega} f_2(1-r_n)\psi + \int_{D_n}\psi\nabla u_2\nabla r_n. \tag{2.4}$$

Taking the limit as $n\to\infty$ in the first term of the right-hand side of (2.4) and using the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty}\int_{\partial\Omega} f_2(1-r_n)\psi = \int_{\Gamma_2} f_2\psi\,.$$

Using the fact that $\nabla r_n = \xi_n'(u_1-u_2)\nabla(u_1-u_2)$, the sum of the second terms of the right-hand side of (2.3) and (2.4) yields

$$-\int_{D_n}\psi\nabla u_1\nabla r_n + \int_{D_n}\psi\nabla u_2\nabla r_n = -\int_{D_n}\psi\nabla(u_1-u_2)\nabla r_n$$
$$= -\int_{D_n}\psi\xi_n'(u_1-u_2)|\nabla(u_1-u_2)|^2 \leq 0, \tag{2.5}$$

since $\psi\geq 0$ and $\xi_n'\geq 0$. Adding (2.3) and (2.4), taking the limit, and using (2.5), we get

$$I = I_1 + I_2 \leq \int_{\Gamma_1} f_1\psi + \int_{\Gamma_2} f_2\psi = \int_{\partial\Omega} f\psi\,.$$

Thus, $u := \max\{u_1,u_2\}$ satisfies

$$\int_\Omega (\nabla u\nabla\psi + u\psi) \leq \int_{\partial\Omega} f\psi \qquad \text{for all } 0\leq\psi\in H^1(\Omega)\,,$$

completing the proof of Theorem 2.4. □

Likewise, we have a result for the minimum of two supersolutions.

Corollary 2.5. *Let u_1 and u_2 be functions in $H^1(\Omega)$ such that there exist f_1 and f_2 in $L^r(\partial\Omega)$, for $r\geq\frac{2(N-1)}{N}$, satisfying*

$$\int_\Omega (\nabla u_i\nabla\psi + u_i\psi) \geq \int_{\partial\Omega} f_i\psi \qquad \textit{for all } 0\leq\psi\in H^1(\Omega)\,,$$

for $i=1,2$. Then, $u:=\min\{u_1,u_2\}$ satisfies

$$\int_\Omega (\nabla u\nabla\psi + u\psi) \geq \int_{\partial\Omega} f\psi, \qquad \textit{for all } 0\leq\psi\in H^1(\Omega)\,,$$

where

$$f(x) := \begin{cases} f_1(x) & if\ u_1(x) < u_2(x) \\ f_2(x) & if\ u_1(x)\geq u_2(x), \end{cases} \qquad a.e.\ x\in\partial\Omega\,.$$

Proof. Using the fact that $\min\{u_1,u_2\} = \max\{-u_1,-u_2\}$, the proof follows from Theorem 2.4. □

3. Proof of Theorem 1.4

We will construct a monotone operator, and show that the iterative scheme starting with a weak subsolution (supersolution) will converge to a minimal (maximal) weak solution.

Let $J := \{u \in H^1(\Omega) : \underline{u} \leq u \leq \overline{u}\}$. Define the linear map $T : J \to H^1(\Omega)$ by $T(u) = v$, where v satisfies

$$\begin{cases} -\Delta v + v = 0 & \text{in } \Omega\,; \\ \frac{\partial v}{\partial \eta} + kv = f(x,u) + ku & \text{on } \partial\Omega\,. \end{cases}$$

Step 1. T is well-defined and maps J into itself.
For every $u \in J$, we have $\underline{u} \leq u \leq \overline{u}$. Then using **(H1)** and the fact that $\underline{u}$ and $\overline{u}$ are sub and supersolutions, we get

$$f(x,\underline{u}) + k\underline{u} \leq f(x,u) + ku \leq f(x,\overline{u}) + k\overline{u},$$

and

$$0 \leq |u| \leq \max\{|\underline{u}|, |\overline{u}|\} \leq |\underline{u}| + |\overline{u}|.$$

Taking into account the definitions of $\underline{u}$ and $\overline{u}$, we have that $f(.,\underline{u}(.)),\ f(.,\overline{u}(.))$ are in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Since $\underline{u},\ \overline{u} \in H^1(\Omega)$, then by the continuity of the trace operator (1.2) and the embedding of $L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$ into $L^{\frac{2(N-1)}{N}}(\partial\Omega)$, for every $u \in J$, we have

$$\begin{aligned} &\|f(x,u) + ku\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \\ &\quad \leq \|f(x,\underline{u}) + k\underline{u}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} + \|f(x,\overline{u}) + k\overline{u}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \leq C\,. \end{aligned} \tag{3.1}$$

Therefore, $f(.,u(.)) + ku(.) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Then, Proposition 2.1 implies that $v = T(u) \in H^1(\Omega)$ is unique. Thus, the map T is well-defined.

Further, if $u, w \in J$ with $u \leq w$, then by the weak maximum principle and the fact that f satisfies **(H1)**, $T(u) \leq T(w)$, that is, the map T is nondecreasing. Moreover, repeating the argument and using Definition 1.2(ii), it follows that

$$\underline{u} \leq T(\underline{u}) \leq T(\overline{u}) \leq \overline{u}. \tag{3.2}$$

Hence, T maps J to J.

Step 2. There exist weakly convergent monotone sequences in $H^1(\Omega)$.
Let's construct monotone sequences $\{u_n\}$ and $\{w_n\}$ successively from the (linear) iteration process

$$u_n = T(u_{n-1}) \text{ with } u_0 = \underline{u} \text{ and } w_n = T(w_{n-1}) \text{ with } w_0 = \overline{u}\,.$$

Using (3.2) and the monotonicity of T, we get

$$\underline{u} = u_0 \leq \cdots \leq u_n \leq \cdots \leq w_n \leq \cdots \leq w_0 = \overline{u}. \tag{3.3}$$

We show that $\{u_n\}$ is convergent. The proof for $\{w_n\}$ is analogous. We see that $u_n = T(u_{n-1})$ satisfies

$$\int_\Omega (\nabla u_n \nabla\psi + u_n\psi) + k\int_{\partial\Omega} u_n\psi = \int_{\partial\Omega} (f(x,u_{n-1}) + ku_{n-1})\psi,$$

for all $\psi \in H^1(\Omega)$. Letting $u_n = T(u_{n-1})$ as a test function, we get

$$\int_\Omega \left(|\nabla u_n|^2 + u_n^2\right) + k \int_{\partial\Omega} u_n^2 = \int_{\partial\Omega} (f(x, u_{n-1}) + k u_{n-1}) u_n. \tag{3.4}$$

Since $u_{n-1}, u_n \in J$, using Hölder's inequality in (3.4), and the bound (3.1), we have

$$\begin{aligned} \|u_n\|^2_{H^1(\Omega)} &\le \|u_n\|^2_{H^1(\Omega)} + k\|u_n\|^2_{L^2(\partial\Omega)} \\ &\le \|f(x, u_{n-1}) + k u_{n-1}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \|u_n\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)} \\ &\le C\left(\|\overline{u}\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)} + \|\underline{u}\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)}\right). \end{aligned}$$

Hence, there exists a uniform constant $C' > 0$, depending on Ω, f, k, $\underline{u}$ and $\overline{u}$, such that

$$\|u_n\|_{H^1(\Omega)} \le C'. \tag{3.5}$$

By the reflexivity of $H^1(\Omega)$, (3.5), there is a subsequence (relabeled) u_n which converges weakly to u_* in $H^1(\Omega)$.

Step 3. $f(x, u_n) + k u_n$ converges weakly to $f(x, u_*) + k u_*$ in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$.

Since the sequence u_n in Step 2 is nondecreasing and bounded (see (3.3)), it converges pointwise to u_*, that is,

$$u_*(x) = \lim_{n\to\infty} u_n(x) \in J. \tag{3.6}$$

Using the fact that f is continuous in the second variable u for a.e $x \in \partial\Omega$ and (3.6), we have that

$$f(x, u_*(x)) + k u_* = \lim_{n\to\infty} f(x, u_n(x)) + k u_n(x).$$

By (3.1), $f(x, u_n) + k u_n$ is bounded in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Then, Lebesgue Dominated Convergence Theorem yields

$$\|(f(x, u_n) + k u_n) - (f(x, u_*) + k u_*)\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \to 0 \qquad \text{as } n \to \infty.$$

Therefore, $f(x, u_n) + k u_n$ converges strongly (hence weakly) to $f(x, u_*) + k u_*$ in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Thus, for all $\psi \in H^1(\Omega) \subset L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$, we have

$$\lim_{n\to\infty} \int_{\partial\Omega} (f(x, u_n) + k u_n)\psi = \int_{\partial\Omega} (f(x, u_*) + k u_*)\psi. \tag{3.7}$$

Step 4. u_* is a weak solution to (1.1).

First, since $u_* \in H^1(\Omega)$, the continuity of the trace (1.2) and $L^{\frac{2(N-1)}{N-2}}(\partial\Omega) \hookrightarrow L^{\frac{2(N-1)}{N}}(\partial\Omega)$, imply that $u_* \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Therefore, for some positive constant C'', we have

$$\begin{aligned} \|f(x, u_*)\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega),} &= \|f(x, u_*) + k u_* - k u_*\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega),} \\ &\le \|f(x, u_*) + k u_*\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega),} + \|k u_*\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega),} \\ &\le C''. \end{aligned}$$

Second, from the monotone iteration, we know that $u_n = T(u_{n-1})$ satisfies

$$\int_{\Omega} (\nabla u_n \nabla \psi + u_n \psi) + k \int_{\partial\Omega} u_n \psi = \int_{\partial\Omega} (f(x, u_{n-1}) + k u_{n-1}) \psi.$$

Observe that u_n converges weakly to u_* in $H^1(\Omega)$, strongly in $L^2(\partial\Omega)$ (see Step 2) and $f(x, u_n) + ku_n$ converges weakly to $f(x, u_*) + ku_*$ in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$ (see Step 3). Then taking the limit as $n \to \infty$ and using (3.7), we get for any $\psi \in H^1(\Omega)$

$$\begin{aligned} \int_{\Omega} (\nabla u_* \nabla \psi + u_* \psi) + \int_{\partial\Omega} k u_* \psi &= \lim_{n\to\infty} \left(\int_{\Omega} (\nabla u_n \nabla \psi + u_n \psi) + \int_{\partial\Omega} k u_n \psi \right) \\ &= \lim_{n\to\infty} \left(\int_{\partial\Omega} (f(x, u_{n-1}) + k u_{n-1}) \psi \right) \\ &= \int_{\partial\Omega} (f(x, u_*) + k u_*) \psi \,. \end{aligned}$$

Hence,

$$\int_{\Omega} (\nabla u_* \nabla \psi + u_* \psi) = \int_{\partial\Omega} f(x, u_*) \psi \qquad \text{for all } \psi \in H^1(\Omega).$$

Moreover, we also have $f(x, u_*) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Thus u_* is a weak solution to (1.1).

Step 5. u_* is the minimal weak solution in the interval $[\underline{u}, \overline{u}]$.
Let v be a weak solution to (1.1) with $\underline{u} \le v \le \overline{u}$. Then v is a weak supersolution, and $\underline{u} \le v$. Repeating the above iteration procedure with $u_0 = \underline{u}$, we get $\underline{u} \le u_* \le v$. Thus u_* is a weak minimal solution.

Similarly, we can construct the maximal weak solution u^* from the sequence $\{w_n\}$ with $w_0 = \overline{u}$. This completes the proof of Theorem 1.4. □

4. Proofs of Theorem 1.5 and Theorem 1.6

We prove Theorem 1.5 by applying Proposition 2.2 to an appropriate operator related to our problem (1.1). Then, Theorem 1.6 is proved by using Zorn's lemma and Theorem 2.4. Theorem 1.5 guarantees that the set defined for the Zorn's lemma in the proof of Theorem 1.6 is nonempty.

4.1. Proof of Theorem 1.5

Let us consider a modified problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega\,; \\ \frac{\partial u}{\partial \eta} = g(x, u) & \text{on } \partial\Omega\,, \end{cases} \tag{4.1}$$

where

$$g(x, s) := \begin{cases} f(x, \underline{u}(x)), & s < \underline{u}(x), \\ f(x, s), & \underline{u}(x) \le s \le \overline{u}(x), \\ f(x, \overline{u}(x)), & s > \overline{u}(x) \end{cases} \tag{4.2}$$

is the truncated function. We observe that g is a Carathéodory function, since f is a Carathéodory function. We note that a weak solution u of (4.1) is a weak solution of (1.1) whenever $\underline{u} \le u \le \overline{u}$.

Our plan is to establish the existence of a weak solution u of (4.1), and verify that $\underline{u} \le u \le \overline{u}$. For the existence part, we use Proposition 2.2. For this, we define the map $B\colon H^1(\Omega) \to (H^1(\Omega))^*$ given by

$$\langle B(v), \psi\rangle := \int_\Omega (\nabla v \nabla \psi + v\psi) - \int_{\partial\Omega} g(x, v)\psi, \tag{4.3}$$

for all $\psi \in H^1(\Omega)$.

First, we show that B is *well-defined and bounded.* The first integral of (4.3) is well-defined since $v, \psi \in H^1(\Omega)$. By the Hölder's inequality combined with the continuity of trace operator (1.2) and hypothesis **(H2)**, we get

$$\int_{\{\underline{u}\le v\le\overline{u}\}} |f(x, v)\psi| \le \|K\|_{L^r(\partial\Omega)}\|\psi\|_{L^{r'}(\partial\Omega)}, \tag{4.4}$$

where $r' < \frac{2(N-1)}{N-2}$ is the conjugate of r. Then, the definition of g given in (4.2), Definition 1.2(i), and (4.4) yield

$$\begin{aligned}\left|\int_{\partial\Omega} g(x, v)\psi\right| &\le \int_{\{v<\underline{u}\}} |f(x, \underline{u})\psi| + \int_{\{\underline{u}\le v\le\overline{u}\}} |f(x, v)\psi| + \int_{\{v>\overline{u}\}} |f(x, \overline{u})\psi| \\ &\le C_2\|\psi\|_{H^1(\Omega)}\,, \end{aligned} \tag{4.5}$$

where the last inequalities of (4.5) follow by (4.4) and (1.2), and the constant C_2 depends only on K and Ω.

Second, we show that B is *pseudomonotone*, see definition (2.1). For this, we set $B = L - G$, where $L, G : H^1(\Omega) \to (H^1(\Omega))^*$ are defined by

$$\langle L(v), \psi\rangle := \int_\Omega (\nabla v \nabla \psi + v\psi) \text{ and } \langle G(v), \psi\rangle := \int_{\partial\Omega} g(x, v)\psi\,,$$

for all $\psi \in H^1(\Omega)$. Then we show that B is pseudomonotone in the following steps. Let $v_n \rightharpoonup v$ in $H^1(\Omega)$.

Step 1: $Lv_n \to Lv$ in $(H^1(\Omega))^*$.
Since $v_n \rightharpoonup v$, $\langle L(v_n) - L(v), \psi\rangle \to 0$ as $n \to \infty$ for all $\psi \in H^1(\Omega)$. Hence,

$$\|L(v_n) - L(v)\|_{(H^1(\Omega))^*} = \sup_{\|\psi\|_{H^1(\Omega)}\le 1} |\langle L(v_n) - L(v), \psi\rangle| \to 0 \text{ as } n \to \infty\,,$$

as desired.

Step 2: $G(v_n) \to G(v)$ in $(H^1(\Omega))^*$.
Suppose that $v_n \rightharpoonup v$ in $H^1(\Omega)$ but $G(v_n) \not\to G(v)$ in $(H^1(\Omega))^*$. Then there exists $\varepsilon_0 > 0$ and a subsequence $\{v_{n_j}\}$ such that

$$\|G(v_{n_j}) - G(v)\|_{(H^1(\Omega))^*} \ge \varepsilon_0. \tag{4.6}$$

Using the fact that $\{v_{n_j}\}$ is bounded in $H^1(\Omega)$ and the compactness of the trace operator (1.2), there exists a subsequence $\{v'_{n_j}\}$ such that $v'_{n_j} \to v$ in $L^{r'}(\partial\Omega)$, where $r' < \frac{2(N-1)}{N-2}$. By [27, Theorem 4.9], there exists a subsequence $\{v''_{n_j}\}$ such that

$$v''_{n_j}(x) \to v(x) \text{ a.e. } x \in \partial\Omega\,.$$

Since $g(x,.)$ is continuous for a.e. $x \in \partial\Omega$, then $g(x, v''_{n_j}(x)) \to g(x, v(x))$ a.e. $x \in \partial\Omega$ and $g(x, v''_{n_j}(x))$ is bounded in $L^r(\partial\Omega)$ by **(H2)**. Using the Lebesgue Dominated Convergence Theorem, we get

$$\|g(., v''_{n_j}(.)) - g(., v(.))\|_{L^r(\partial\Omega)} \to 0 \text{ as } n \to \infty\,.$$

By the Hölder's inequality, for all $\psi \in H^1(\Omega)$, we get

$$\langle G(v''_{n_j}) - G(v), \psi\rangle \to 0 \text{ as } j \to \infty\,.$$

Therefore, $\|G(v''_{n_j}) - G(v)\|_{(H^1(\Omega))^*} = \sup_{\|\psi\|_{H^1(\Omega)}\le 1} |\langle G(v''_{n_j}) - G(v), \psi\rangle| \to 0$ as $j \to \infty$. Hence, $G(v''_{n_j}) \to G(v)$ in $(H^1(\Omega))^*$ as $j \to \infty$, a contradiction to (4.6).

Step 3: B is pseudomonotone.
Let $v_n \rightharpoonup v$ in $H^1(\Omega)$. Using *Step 1-Step 2*, we get that

$$B(v_n) \to B(v) \qquad \text{in} \quad (H^1(\Omega))^*\,.$$

Therefore, $\langle B(v_n), \psi\rangle \to \langle B(v), \psi\rangle$ as $n \to \infty$ for all $\psi \in H^1(\Omega)$. Furthermore, by [27, Proposition 3.5 (iv)], $\langle B(v_n), v_n\rangle \to \langle B(v), v\rangle$ as $n \to \infty$. Hence,

$$\langle B(v_n), v_n - \psi\rangle \to \langle B(v), v - \psi\rangle \text{ as } n \to \infty\,,$$

establishing that B is pseudomonotone.

Finally, we show that B is *coercive*, i.e., $\langle B(\psi), \psi\rangle/\|\psi\|_{H^1(\Omega)} \to \infty$ as $\|\psi\|_{H^1(\Omega)} \to \infty$. For any $\psi \in H^1(\Omega)$, using (4.5) in the definition of the operator B, we have

$$\langle B(\psi), \psi\rangle \ge \|\psi\|^2_{H^1(\Omega)} - C_2\|\psi\|_{H^1(\Omega)} \ge \frac{1}{2}\|\psi\|^2_{H^1(\Omega)} - C_3.$$

Hence B is *coercive*. Thus B satisfies the hypotheses of Proposition 2.2 with $X = H^1(\Omega)$. Therefore, for $b = 0 \in (H^1(\Omega))^*$, there exists $u \in H^1(\Omega)$ such that

$$\langle B(u), \psi\rangle = 0 \quad \forall\psi \in H^1(\Omega)\,.$$

Moreover, $g(x,.)$ is bounded in $L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$ by **(H2)**, and therefore in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$ by continuous embedding of $L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$ into $L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Hence u is a weak solution of (4.1). It remains to prove that u is a weak solution of (1.1). For this, we will show that $\underline{u} \le u \le \overline{u}$ in $\overline{\Omega}$, so that $g = f$ in (4.1).

Clearly, $(u - \overline{u})_+ := \max\{0, u - \overline{u}\} \in H^1(\Omega)$ and $(\underline{u} - u)_+ := \max\{0, \underline{u} - u\} \in H^1(\Omega)$. Then, using the weak formulation of (4.1) with the test function $\psi := (u - \overline{u})_+ \ge 0$ in $H^1(\Omega)$, and the facts that $\overline{u}$ is a supersolution of (1.1) and $(u - \overline{u})_+ = 0$ in $\{u \le \overline{u}\}$, we have

$$\int_\Omega (\nabla u \nabla (u - \overline{u})_+ + u(u - \overline{u})_+) = \int_{\partial\Omega} g(x, u)(u - \overline{u})_+ \tag{4.7}$$

$$
\begin{aligned}
&= \int_{\{u>\overline{u}\}} f(x,\overline{u})(u-\overline{u})_+ \\
&= \int_{\partial\Omega} f(x,\overline{u})(u-\overline{u})_+ \\
&\leq \int_{\Omega} \nabla\overline{u}\,\nabla(u-\overline{u})_+ + \int_{\Omega} \overline{u}(u-\overline{u})_+ .
\end{aligned}
$$

Then, (4.7) yields

$$
\begin{aligned}
0 &\leq \int_{\Omega} |\nabla(u-\overline{u})_+|^2 + \int_{\Omega} |(u-\overline{u})_+|^2 \\
&= \int_{\Omega} \nabla(u-\overline{u})\nabla(u-\overline{u})_+ + \int_{\Omega} (u-\overline{u})(u-\overline{u})_+ \\
&\leq 0,
\end{aligned}
$$

which implies that $\|(u-\overline{u})_+\|_{H^1(\Omega)} = 0$. That is, $u \leq \overline{u}$ a.e. in Ω. Using the the continuity of the trace operator (1.2), we get that $\|(u-\overline{u})_+\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)} = 0$. Hence, $u \leq \overline{u}$ a.e. in $\overline{\Omega}$.

Analogously, taking the test function $\psi := (\underline{u}-u)_+ \geq 0$ and using the fact that $\underline{u}$ is a subsolution of (1.1), we obtain that

$$
\begin{aligned}
0 &\leq \int_{\Omega} |\nabla(\underline{u}-u)_+|^2 + \int_{\Omega} |(\underline{u}-u)_+|^2 \\
&= \int_{\Omega} \nabla(\underline{u}-u)\nabla(\underline{u}-u)_+ + \int_{\Omega} (\underline{u}-u)(\underline{u}-u)_+ \leq 0,
\end{aligned}
$$

Therefore, $\underline{u} \leq u$ a.e. in $\overline{\Omega}$, and hence $\underline{u} \leq u \leq \overline{u}$ a.e. in $\overline{\Omega}$. Thus, u is a weak solution of (1.1), completing the proof of Theorem 1.5. □

4.2. Proof of Theorem 1.6

We will use Zorn's Lemma and Proposition 2.3, to prove our result. Consider the set

$$
\begin{aligned}
\mathcal{A} := \{u \in H^1(\Omega) \colon\ & \underline{u}(x) \leq u(x) \leq \overline{u}(x) \text{ a.e. in } \overline{\Omega} \\
& \text{and } u \text{ is a weak solution of (1.1)}\},
\end{aligned}
$$

and we note that $\mathcal{A}$ is nonempty by Theorem 1.5. Let $\{u_i\}_{i\in I} \subseteq \mathcal{A}$ be a family of *chain*. Since u_i is a weak solution of (1.1), taking u_i as the test function and using (4.4), we get

$$
\|u_i\|_{H^1(\Omega)} = \int_{\Omega} \left(|\nabla u_i|^2 + u_i^2\right) = \int_{\partial\Omega} f(x,u_i)u_i \leq C ,
$$

where C depends on $\underline{u}$, $\overline{u}$, K, $\partial\Omega$ but independent of $i \in I$. By the separability and reflexivity of $H^1(\Omega)$, there exists an increasing sequence u_n such that

$$
u_n \rightharpoonup u := \sup_{i\in I} u_i \text{ in } H^1(\Omega) .
$$

Clearly, u is an upper bound of the chain $\{u_i\}_{i\in I}$. It suffices to show that $u \in \mathcal{A}$. Since $\{u_n\}$ is nondecreasing and $\underline{u} \le u_n \le \overline{u}$, we have that $u_n(x) \to u(x)$, and $u_n(x) \le u(x)$ for all n, and $\underline{u}(x) \le u(x) \le \overline{u}(x)$ pointwise a.e. in $\overline{\Omega}$. Furthermore, since f is Carathéodory, we have that

$$f(x, u_n(x)) \to f(x, u(x)) \text{ as } n \to \infty\,.$$

This, in conjunction with **(H2)**, and the Lebesgue Dominated Convergence Theorem yields $\|f(x, u_n) - f(x, u)\|_{L^r(\partial\Omega)} \to 0$ as $n \to \infty$. Therefore, using Hölder's inequality, we deduce that

$$\begin{aligned}\left|\int_{\partial\Omega} f(x, u_n)\psi - \int_{\partial\Omega} f(x, u)\psi\right| &\le \int_{\partial\Omega} |f(x, u_n) - f(x, u)|\ |\psi| \\ &\le \|f(x, u_n) - f(x, u)\|_{L^r(\partial\Omega)}\, \|\psi\|_{L^{r'}(\partial\Omega)} \to 0\,,\end{aligned}$$

which yields

$$\lim_{n\to\infty} \int_{\partial\Omega} f(x, u_n)\psi = \int_{\partial\Omega} f(x, u)\psi \quad \text{for all } \psi \in H^1(\Omega).$$

Taking the limit as $n \to \infty$, we get for any $\psi \in H^1(\Omega)$

$$\begin{aligned}\int_{\Omega} (\nabla u \nabla\psi + u\psi) &= \lim_{n\to\infty} \int_{\Omega} (\nabla u_n \nabla\psi + u_n\psi) \\ &= \lim_{n\to\infty} \int_{\partial\Omega} f(x, u_n)\psi = \int_{\partial\Omega} f(x, u)\psi\,.\end{aligned}$$

Hence, u is a weak solution of (1.1), thus concluding that $u \in \mathcal{A}$.

By Zorn's Lemma, there exists a maximal element $u^* \in \mathcal{A}$. It remains to show that u^* is maximal in the sense that if $\hat{u}$ is any other weak solution of (1.1) between $\underline{u}$ and $\overline{u}$, then $\hat{u} \le u^*$. So, let $\hat{u}$ be a weak solution of (1.1) between $\underline{u}$ and $\overline{u}$, and u^* is the maximal element of $\mathcal{A}$. By Proposition 2.4, $u = \max\{\hat{u}, u^*\}$ is a subsolution of (1.1). Then, by Theorem 1.5, there exists a weak solution u_0 of (1.1) satisfying

$$\underline{u} \le u \le u_0 \le \overline{u}\,.$$

Thus, $u_0 \in \mathcal{A}$. On the other hand, $u^* \le \max\{\hat{u}, u^*\} = u \le u_0$. But u^* is maximal element of $\mathcal{A}$, so necessarily $u^* = u_0$. Therefore, we readily see that $\hat{u} \le u \le u_0 = u^*$, and hence

$$\underline{u} \le \hat{u} \le u^* \le \overline{u}\,,$$

as desired. The existence of a minimal element u_* of $\mathcal{A}$ is proved analogously. This completes the proof of Theorem 1.6. □

5. Examples

In this section, we apply our existence results, Theorem 1.4 and Theorem 1.5, to problems involving sublinear nonlinearities. In particular, in each case we construct an ordered pair of weak sub- and supersolution. We apply Theorem 1.4 to establish Theorem 5.1 and Theorem 1.5 in Remark 5.2 below.

Theorem 5.1. *Consider*

$$\begin{cases} -\Delta u + u = 0 & \text{in} \quad \Omega\,; \\ \frac{\partial u}{\partial \eta} = \lambda f(u) & \text{on} \quad \partial\Omega\,, \end{cases} \tag{5.1}$$

where $\lambda > 0$ *parameter and* $f : [0, \infty) \to [0, \infty)$ *is locally Lipschitz continuous function satisfying*

(i) $f(0) = 0$ *with* $f'(0) > 0$*, and*
(ii) $\lim_{s\to\infty} \frac{f(s)}{s} = 0$.

Then (5.1) *has a positive weak solution for* $\lambda > \frac{\mu_1}{f'(0)}$*, where* $\mu_1 > 0$ *is the first eigenvalue of the Steklov eigenvalue problem*

$$\begin{cases} -\Delta\varphi_1 + \varphi_1 = 0 & \text{in } \Omega\,; \\ \frac{\partial\varphi_1}{\partial\eta} = \mu_1\varphi_1 & \text{on } \partial\Omega\,, \end{cases} \tag{5.2}$$

and $0 < \varphi_1 \in H^1(\Omega)$ *is the corresponding eigenfunction.*

Proof. Let $\lambda > \frac{\mu_1}{f'(0)}$ be fixed. Using hypothesis (i), we verify that $\underline{u} := \epsilon\varphi_1$ is a subsolution of (5.1) for $\epsilon \approx 0$. Indeed, we observe that since $\lambda > \frac{\mu_1}{f'(0)}$ is fixed, $\xi(s) := \mu_1 s - \lambda f(s)$ satisfies $\xi(0) = 0$ and $\xi'(0) < 0$, then $\xi(s) < 0$ for $s \approx 0$. Therefore, for all $0 \le \psi \in H^1(\Omega)$, the following holds for $\epsilon \approx 0$

$$\int_\Omega \nabla\underline{u}\nabla\psi + \int_\Omega \underline{u}\psi = \mu_1 \int_{\partial\Omega} (\epsilon\varphi_1)\psi \le \lambda \int_{\partial\Omega} f(\epsilon\varphi_1)\psi = \lambda \int_{\partial\Omega} f(\underline{u})\psi\,.$$

Next, using hypothesis (ii), we show that there exists $M_\lambda > 0$ such that $\overline{u} := Me$ is a weak supersolution of (5.1) for all $M \ge M_\lambda$, where e is the unique positive solution of

$$\begin{cases} -\Delta e + e = 0 & \text{in } \Omega\,; \\ \frac{\partial e}{\partial\eta} = 1 & \text{on } \partial\Omega\,. \end{cases}$$

We observe that while f is not assumed to be nondecreasing, $\overline{f}(t) := \max_{s\in[0,t]} f(s)$ is nondecreasing, and $f(t) \le \overline{f}(t)$ for all $t \ge 0$. Moreover, due to hypothesis (ii), $\overline{f}$ satisfies the sublinear condition at infinity

$$\lim_{t\to+\infty} \frac{\overline{f}(t)}{t} = 0\,.$$

Therefore, there exists $M_\lambda > 0$ such that for all $M \ge M_\lambda$

$$\frac{\overline{f}(M\|e\|_{L^\infty(\partial\Omega)})}{M\|e\|_{L^\infty(\partial\Omega)}} \le \frac{1}{\lambda\|e\|_{L^\infty(\partial\Omega)}} \text{ or equivalently } \lambda\overline{f}(M\|e\|_{L^\infty(\partial\Omega)}) \le M\,.$$

Then $\overline{u} = Me \in H^1(\Omega)$ satisfies

$$\begin{aligned} \int_\Omega \nabla\overline{u}\nabla\psi + \int_\Omega \overline{u}\psi &= M \int_{\partial\Omega} \psi \\ &\ge \lambda \int_{\partial\Omega} \overline{f}(M\|e\|_{L^\infty(\partial\Omega)})\psi \\ &\ge \lambda \int_{\partial\Omega} \overline{f}(Me)\psi \\ &\ge \lambda \int_{\partial\Omega} f(Me)\psi = \lambda \int_{\partial\Omega} f(\overline{u})\psi \end{aligned}$$

for all $0 \le \phi \in H^1(\Omega)$. Therefore, $\overline{u}$ is a weak supersolution of (1.1) for each $\lambda > \frac{\mu_1}{f'(0)}$. Clearly $\overline{u} = Me \ge \epsilon(\lambda)\varphi_1 = \underline{u}$ a.e. in $\overline{\Omega}$. We remark that since f is locally Lipschitz and $[\underline{u}, \overline{u}]$ is bounded, f satisfies hypothesis **(H1)** of Theorem 1.4. Hence, there exists a positive weak solution u of (5.1) such that $\epsilon\varphi_1 \le u \le Me$ a.e. in $\overline{\Omega}$ for any $\lambda > \frac{\mu_1}{f'(0)}$. This completes the proof. □

Remark 5.2. On the other hand, if f is continuous (not necessarily Lipschitz), satisfies hypothesis (ii) of Theorem 5.1 and $f(s) > 0$ for $s \geq 0$, the problem (5.1) has a positive weak solution for each $\lambda > 0$. Indeed, it is easy to see that $\underline{u} \equiv 0$ is a strict weak subsolution and for each $\lambda > 0$, there exists $M_\lambda > 0$ such that $\overline{u} = Me$ is a weak supersolution for all $M \geq M_\lambda$, as in the proof of Theorem 5.1. Then, the result follows by Theorem 1.5.

Acknowledgements

We would like to thank the referee for carefully reading the manuscript and providing valuable suggestions. We would like to thank Professor Jesús Jaramillo for helpful discussions about Step 2 in the proof of Theorem 1.5. The fifth author is supported by grants PID2019-103860GB-I00, MICINN, Spain, and by UCM-BSCH, Spain, GR58/08, Grupo 920894. All authors acknowledge MSRI for bringing this group together for collaboration.

Conflict of interest

The authors declare there is no conflicts of interest.

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