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RESONANT SOLUTIONS FOR ELLIPTIC SYSTEMS WITH NEUMANN BOUNDARY CONDITIONS

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Honoring the memory of Professor John W. Neuberger

ABSTRACT. We consider a sublinear perturbation of an elliptic eigenvalue system with homogeneous Neumann boundary conditions. For oscillatory nonlinearities and using bifurcation from infinity, we prove the existence of an unbounded sequence of turning points and an unbounded sequence of resonant solutions.

1. INTRODUCTION

We consider the nonlinear elliptic system with homogeneous Neumann boundary conditions

$$\begin{gathered}-\Delta u_1 + u_1 = \lambda(a_1u_1 + a_2u_2) + f_1\big(x, (u_1, u_2)\big),\\ -\Delta u_2 + u_2 = \lambda(a_2u_1 + a_3u_2) + f_2\big(x, (u_1, u_2)\big), \text{in } \Omega\,,\\ \frac{\partial u_1}{\partial \eta} = \frac{\partial u_2}{\partial \eta} = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain, $\partial/\partial\eta := \eta(x) \cdot \nabla$ denotes the outer normal derivative on $\partial\Omega$, $a_i > 0$, $i = 1, 2, 3$ are fixed, and $\lambda \in \mathbb{R}$ is a bifurcation parameter. The nonlinearity $f = (f_1, f_2)$, where $f_i \colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$ are Carathéodory functions, that is, $f_i = f_i\big(x, s\big)$ are measurable in $x \in \Omega$ and continuous with respect to $s = (s_1, s_2) \in \mathbb{R}^2$.

Roughly speaking, we assume that the nonlinearity satisfies

(a) $f(x, s) = o(|s|)$ at infinity, and
(b) f is oscillatory.

Assumption (a), by a mechanism of parametric resonance, produces unbounded branches of solutions when λ approaches certain eigenvalue in the sense of Rabinowitz [14]. Assumption (b) transfers the oscillatory behavior to the solutions branch, yielding infinitely many *turning points* and infinitely many *resonant solutions* (see definitions 1.1-1.2).

We assume that f satisfies the following hypothesis:

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(H1) There exists $h \in L^r(\Omega)$ with $r > N/2$ and a continuous function $\tilde{f}\colon \mathbb{R}^2 \to \mathbb{R}^+$ satisfying
$$|f_i(x,s)| \le h(x)\tilde{f}(s), \quad i = 1,2,$$
with
$$\lim_{|s|\to\infty} \frac{\tilde{f}(s)}{|s|} = 0, \quad \text{where } |s| = \big|(s_1,s_2)\big| := \sqrt{s_1^2 + s_2^2}.$$

(H2) There exists a function $B \in L^r(\Omega)$ with $r > N/2$, $\alpha < 1$ and $s_0 > 0$ such that we have
$$\frac{|f_i(x,s)|}{|s|^\alpha} \le B(x), \quad \text{for } |s| > s_0,\ x \in \Omega, \text{ and } i = 1,2.$$

(H3) $f(x,s)$ is differentiable in $s = (s_1,s_2)$, $\frac{\partial f_i}{\partial s_j}(\cdot,\cdot) \in C(\Omega \times \mathbb{R}^2;\mathbb{R})$, and for all $i,j = 1,2$,
$$\sup_{|s|\ge M} \big\|\frac{\partial f_i}{\partial s_j}(\cdot,s)\big\|_{L^\infty(\Omega)} \to 0 \quad \text{as } M \to \infty.$$

By a weak solution to (1.1) we mean a pair $(\lambda,u) \in \mathbb{R} \times \big(H^1(\Omega)\big)^2$ such that for all $\psi = (\psi_1,\psi_2) \in H^1(\Omega)^2$,
$$\begin{aligned}
\int_\Omega \nabla u_1 \nabla\psi_1 + u_1\psi_1 &= \lambda \int_\Omega (a_1u_1 + a_2u_2)\psi_1 + \int_\Omega f_1(x,(u_1,u_2))\psi_1,\\
\int_\Omega \nabla u_2 \nabla\psi_2 + u_2\psi_2 &= \lambda \int_\Omega (a_2u_1 + a_3u_2)\psi_2 + \int_\Omega f_2(x,(u_1,u_2))\psi_2.
\end{aligned} \tag{1.2}$$

Because of (H1), weak solutions of (1.1) lie in the space $\big(W^{2,r}(\Omega)\big)^2$, $r > N/2$, which is continuously embedded in $\big(C(\overline{\Omega})\big)^2$. Therefore, we consider $\mathbb{R} \times \big(C(\overline{\Omega})\big)^2$ as our underlying space. Throughout this work, we consider the Banach space $\big(C(\overline{\Omega})\big)^2$ equipped with the norm
$$\|u\| = \Big(\|u_1\|^2_{C(\overline{\Omega})} + \|u_2\|^2_{C(\overline{\Omega})}\Big)^{1/2}.$$

Note that system (1.1) can be rewritten in matrix form as
$$(-\Delta + \mathbf{I})u = \lambda A u + f(x,u), \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \eta} = 0, \quad \text{on } \partial\Omega, \tag{1.3}$$
where
$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}, \quad f(x,u) = \begin{pmatrix} f_1\big(x,(u_1,u_2)\big) \\ f_2\big(x,(u_1,u_2)\big) \end{pmatrix}, \tag{1.4}$$
$\mathbf{I}$ is the 2×2 identity matrix, and $(-\Delta + \mathbf{I})u = \big((-\Delta + I)u_1, (-\Delta + I)u_2\big)^T$. Note that the matrix A is symmetric.

It is straightforward to show that the eigenvalues of the matrix A are the numbers
$$\mu_\pm = \frac{a_1 + a_3}{2} \pm \sqrt{\big(\frac{a_1 - a_3}{2}\big)^2 + a_2^2}, \tag{1.5}$$
with $\mu_+ > 0$, and $\mu_- \lesseqgtr 0$ if and only if $\det(A) \lesseqgtr 0$. Let us denote by $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ the eigenvector associated to μ_+,
$$Ab = \mu_+ b, \quad \text{normalized so that} \quad |b| = (b_1^2 + b_2^2)^{1/2} = 1. \tag{1.6}$$

Since A is symmetric and $a_i > 0$, $i = 1, 2, 3$, then, $b^T A = \mu_+ b^T$ and $b_1, b_2 > 0$. Moreover b is the only eigenvector of A with both components positive.

Using a Rabinowitz result [14], we prove that the branch of solutions bifurcating from infinity at $\lambda = 1/\mu^+$, denoted by $\mathscr{D} \subset \mathbb{R} \times \left(C(\overline{\Omega})\right)^2$, forms a continuum, that is, the closed and connected set

$$\{(\lambda, u) \in \mathbb{R} \times \left(C(\overline{\Omega})\right)^2 \colon (\lambda, u) \text{ is a weak solution to (1.1)}\}.$$

$\mathscr{D}$ contains large positive solutions (in both components), and large negative solutions to (1.1), see Theorem 2.2. Let $\mathscr{D}^+$ denote the continuum of positive solutions bifurcating from infinity at $\lambda = 1/\mu_+$ (resp. $\mathscr{D}^-$ for negative solutions). As a mater of fact, solutions can be expressed as

$$u = tb + w,$$

where $t > 0$, $w = o(|t|)$ as $t \to +\infty$, and $w \in (\operatorname{span}\{b\})^\perp$, see Proposition 3.2. We will see that $(\lambda_n, u_n) = (\lambda_n, t_n b + w_n) \to (1/\mu_+, \infty)$ whenever $t_n \to \infty$.

Definition 1.1. We say that $(\lambda_n^*, u_n^*) \in \mathscr{D}^+$ is a *turning point* if there is a neighborhood of (λ^*, u^*) in $\mathbb{R} \times (C(\overline{\Omega}))^2$ such that there are no solutions (λ, u) close to (λ^*, u^*) for $\lambda > \lambda^*$ or for $\lambda < \lambda^*$.

It is well known that the Neumann Laplacian operator has a discrete spectrum of infinitely many non-negative eigenvalues with no finite accumulation point, so

$$\sigma(-\Delta + I) = \{\lambda_i \colon 1 = \lambda_1 < \lambda_2 \le \dots \le \lambda_n \dots,\ \lambda_n \to \infty \text{ as } n \to \infty\}. \tag{1.7}$$

Let $b_+ = b$, and b_- be the eigenvectors associated to μ_-; observe that if $(-\Delta + I)\phi = \lambda\mu_\pm\phi$ in Ω, and $\frac{\partial\phi}{\partial\eta} = 0$ on $\partial\Omega$, then $(-\Delta + \mathbf{I})b_\pm\phi = \lambda\mu_\pm(b_\pm\phi) = \lambda A(b_\pm\phi)$, $\frac{\partial}{\partial\eta}b_\pm\phi = 0$. In other words, $\lambda\mu_\pm \in \sigma(-\Delta + I)$, implies that $0 \in \sigma(-\Delta + \mathbf{I} - \lambda A)$. In particular, if u is a solution to (1.1) corresponding to $\lambda = 1/\mu_+$, we will say that u is a resonant solution to (1.1).

Definition 1.2. If $\lambda\mu_\pm \in \sigma(-\Delta + I)$, we say that the system (1.1) is *resonant*.

One interesting question is whether the bifurcating branch $\mathscr{D}^+$ is subcritical or supercritical. That is, if it is formed only with solutions (λ, u) with $\lambda < 1/\mu_+$ or $\lambda > 1/\mu_+$ respectively.

We can easily see how to determine whether the bifurcation of positive solutions emanating from the first eigenvalue is sub- or super-critical. To do this, let $(\lambda_n, u_n) = (\lambda_n, t_n b + w_n)$ solve (1.1) for $\lambda_n \to 1/\mu_+$ and $t_n \to +\infty$. Multiplying (1.3) by b^T on the left, integrating the result, using that $\int_\Omega b^T u_n = t_n|\Omega|$, and the symmetry of the matrix A, so $b^T A = \mu_+ b^T$, we obtain

$$t_n|\Omega|(1 - \lambda_n\mu_+) = \int_\Omega b_1 f_1(\cdot, t_n b + w_n) + b_2 f_2(\cdot, t_n b + w_n). \tag{1.8}$$

Hence, the sign of $1 - \lambda\mu_+$ is the same as that of the right hand side. Hence, if the right-hand side is greater than 0, the bifurcation of positive solutions will be subcritical and if right-hand side is less than 0, it will be supercritical.

Let us define the quantities

$$\underline{\mathbf{F}}_+ := \int_\Omega \liminf_{s \to +\infty} \frac{s \sum_{i=1}^2 b_i f_i(\cdot, s)}{|s|^{1+\alpha}}, \quad \underline{\mathbf{F}}_+ := \int_\Omega \limsup_{s \to +\infty} \frac{s \sum_{i=1}^2 b_i f_i(\cdot, s)}{|s|^{1+\alpha}}.$$

From the above, and Fatou's Lemma, if $\underline{\mathbf{F}}_+ > 0$, then $\mathscr{D}^+$ is subcritical, while if $\underline{\mathbf{F}}_+ < 0$, then $\mathscr{D}^+$ is supercritical. In this work, we will consider nonlinearities satisfying

$$\underline{\mathbf{F}}_+ < 0 < \underline{\mathbf{F}}_+. \tag{1.9}$$

The above-mentioned condition means that the bifurcating continuum $\mathscr{D}^+$ is neither sub-critical nor super-criticall, and hence Landesman-Lazer type conditions do not hold. The main purpose of this article is to establish the existence of infinitely many resonant solutions at $\lambda = 1/\mu_+$ in the absence of Landesman-Lazer type conditions. We note that condition (1.9) reflects the oscillatory behavior of $\mathscr{D}^+$ near infinity around the bifurcation point $\lambda = 1/\mu_+$, yielding infinitely many resonant solutions.

Hypothesis (H1) guarantees that the continuum of positive solutions or negative solutions bifurcates from infinity at $\lambda = 1/\mu_+$ in the sense of Rabinowitz [14], see Theorem 2.2. Hypothesis (H2) provides the estimates $|\lambda - 1/\mu_+| = O(t^{\alpha-1})$ and $\|w\| = O(t^\alpha)$ in Proposition 3.2. We try to unveil the sign of the right-hand side in (1.8), just looking at the signs of

$$\liminf_{n\to+\infty} \int_\Omega \frac{t_n \sum_{i=1}^2 b_i f_i(\cdot, t_n b)}{|t_n|^{1+\alpha}}, \quad \limsup_{n\to+\infty} \int_\Omega \frac{t_n \sum_{i=1}^2 b_i f_i(\cdot, t_n b)}{|t_n|^{1+\alpha}}.$$

This is done in Lemma (3.4). Hypothesis (H3) helps establishing the inequalities (3.14)–(3.15). With these tools, in Theorem 1.3 we take two sequences $\{t_n\}$ and $\{t'_n\}$ satisfying (1.10), and from here we obtain the existence of unbounded sequences of sub- and super-critical solutions of (1.1) in $\mathscr{D}^+$. In particular, we prove the following result.

Theorem 1.3. *Let* (H1)–(H3) *hold. Suppose that there exists two increasing sequences* $\{t_n\}$ *and* $\{t'_n\}$ *tending to* $+\infty$ *and satisfying*

$$-\infty < \lim_{n\to\infty} \frac{t'_n \sum_{i=1}^2 b_i \int_\Omega f_i(\cdot, t'_n b)}{|t'_n|^{1+\alpha}} < 0 < \lim_{n\to\infty} \frac{t_n \sum_{i=1}^2 b_i \int_\Omega f_i(\cdot, t_n b)}{|t_n|^{1+\alpha}} < +\infty, \tag{1.10}$$

where b is the positive eigenfunction of the matrix A defined in (1.4). *Then, the following assertions hold:*

- (i) *There exist two sequences $\{(\lambda_n, u_n)\}$ and $\{(\lambda'_n, u'_n)\}$ in $\mathscr{D}^+$ approaching to $(1/\mu_+, +\infty)$ as $n \to \infty$, with $\lambda_n < 1/\mu_+$ (subcritical) and $\lambda'_n > 1/\mu_+$ (supercritical), respectively.*
- (ii) *There is a sequence of turning points $\{(\lambda^*_n, u^*_n)\}$ in $\mathscr{D}^+$ approaching to $(1/\mu_+, +\infty)$ as $n \to \infty$.*

  Furthermore, one can choose two sequences of turning points, one of them subcritical $\lambda^*_{2n+1} < 1/\mu_+$ and the other supercritical $\lambda^*_{2n} > 1/\mu_+$.
- (iii) *There is a sequence of resonant solutions. That is, there are infinitely many solutions $\{(1/\mu_+, \hat{u}_n)\} \in \mathscr{D}^+$ such that $\|\hat{u}_n\|_{C(\overline{\Omega})} \to \infty$ as $n \to \infty$.*

Similar results have been considered for the single equation in [9], and for the single equation case with nonlinear boundary conditions, in [4, 5, 3, 8], and in [7] for bifurcation from zero results.

In Section 2 we will see that hypothesis (H1) guarantees that the unbounded continuum of positive and negative bifurcates from infinity at $\lambda = 1/\mu_+$. More precisely, the solutions have the form (2.8) and (2.9), respectively. In Section 3 we will prove a series of technical results involving hypotheses (H1)–(H3), needed for proving Theorem 1.3 in Section 4.

2. Bifurcation from infinity

Let us consider the solid cone $P = \{v \in C(\overline{\Omega})\colon v \geq 0 \text{ in } \overline{\Omega}\}$, and let $\mathscr{K} : L^r(\Omega) \to \big\{W^{2,r}(\Omega)\colon \frac{\partial v}{\partial \eta} = 0\big\}$ be the linear operator defined as

$$\mathscr{K} h := v, \tag{2.1}$$

where v is the solution to the Neumann problem

$$\begin{gathered} (-\Delta + I)v = h, \quad \text{in } \Omega, \\ \frac{\partial v}{\partial \eta} = 0, \quad \text{in } \partial\Omega, \end{gathered} \tag{2.2}$$

and $\|v\|_{W^{2,r}(\Omega)} \leq C\|h\|_{L^r(\Omega)}$, (see [10, p. 162]). $\mathscr{K}$ is known as the resolvent operator for the Neumann problem (2.2). For $r > N/2$, and by Rellich-Kondrachov theorem on compact embedding of Sobolev spaces [6], $W^{2,r}(\Omega) \hookrightarrow W^{1,q}(\Omega)$ if $q < r^*$ (where $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{N} < \frac{1}{N}$), moreover $W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$ if $q > N$. Consequently, $\mathscr{K} : L^r(\Omega) \to C(\overline{\Omega})$ is a compact operator.

By the maximum principle [2, Theorem 4.1], for all $h \geq 0$, $h \neq 0$, we have $v = \mathscr{K} h \in \mathring{P}$, where $\mathring{P} = \{u \in C(\overline{\Omega})\colon u > 0 \text{ in } \overline{\Omega}\}$. In other words, $\mathscr{K}$ is *strongly positive*. By the Krein-Rutman theorem [1], the *spectral radius*, denoted by $r(\mathscr{K})$, and defined as the supremum of the modulus of the elements in the spectrum, is a simple eigenvalue with a positive normalized eigenfunction $\phi \in \mathring{P}$. Furthermore, there is no other eigenvalue with a positive eigenfunction. Moreover,

$$r(\mathscr{K}) = 1, \quad \text{and} \quad \phi = 1. \tag{2.3}$$

Using the operator $\mathscr{K}$, equation (1.1) (or its matrix form (1.3)) can be rewritten as

$$u = \lambda A \mathscr{K} u + \mathscr{K} f(x, u), \quad \text{in } \Omega. \tag{2.4}$$

We prove that Rabinowitz's bifurcation results [13, 14] remain valid in nonlinear systems with homogeneous Neumann boundary conditions such as (1.1).

We start with a technical Proposition, characterizing the behavior of a blowing up sequence of solutions. The bifurcation from infinity point is reached when $\lambda = 1/\mu^+$, and there exists a subsequence of solutions such that, when normalized, converges to b in $\big(C^\gamma(\overline{\Omega})\big)^2$ for some $0 < \gamma < 1$.

Proposition 2.1. *Let* (H1) *hold. Let* (λ_n, u_n) *be a sequence of solutions of* (1.1)*, where* $u_n = (u_{1,n}, u_{2,n})^T$ *are positive in both components,* $\lambda_n \to \lambda_0$ *and* $\|u_n\|_{(C(\overline{\Omega}))^2} \to \infty$*. Then* $\lambda_0 = 1/\mu^+$ *and there exists a sub-sequence, again denoted by* $\{u_n\}$*, such that*

$$\lim_{n\to\infty} \frac{u_n}{\|u_n\|_{(C(\overline{\Omega}))^2}} = b,$$

where $b \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *is defined by* (1.6) *in* $\big(C^\gamma(\overline{\Omega})\big)^2$ *for some* $\gamma \in (0,1)$*.*

Proof. Let $v_n := \big(\frac{u_{1,n}}{\|u_n\|_{(C(\overline{\Omega}))^2}}, \frac{u_{2,n}}{\|u_n\|_{(C(\overline{\Omega}))^2}}\big)^T$. Since $u_n \in \big(W^{2,r}(\Omega)\big)^2$ with $r > N/2$, by the compact embedding theorem, $u_n \in \big(C^\gamma(\overline{\Omega})\big)^2$ for some $\gamma > 0$. By (H1), we obtain that $\|v_n\|_{(C^\gamma(\overline{\Omega}))^2} \leq C$.

Using the compact embedding, $\big(C^\gamma(\overline{\Omega})\big)^2 \hookrightarrow \big(C^{\gamma'}(\overline{\Omega})\big)^2$ for $0 < \gamma' < \gamma$, there exists a convergent sub-sequence, namely v_n, such that $\lambda_n \to \lambda_0$, and $v_n \to \phi =$

$(\phi_1,\phi_2)^T$, with $\phi_i \geq 0$, in $\big(C^{\gamma'}(\overline{\Omega})\big)^2$. Since $v_{i,n}\geq 0$, $i=1,2$, and $\|v_n\|_{(C(\overline{\Omega}))^2}=1$, it is easy to see that $\phi\not\equiv(0,0)^T$, and since $\phi_i\geq 0$, also $\big(\int_\Omega\phi_1,\int_\Omega\phi_2\big)\neq(0,0)$. Moreover, v_n satisfies

$$\begin{gathered}(-\Delta+\mathbf{I})v_n=\lambda_nAv_n+\frac{f(x,u_n)}{\|u_n\|_{(C(\overline{\Omega}))^2}},\quad \text{in } \Omega\\ \frac{\partial v_n}{\partial\eta}=0,\quad \text{on } \partial\Omega.\end{gathered}\tag{2.5}$$

By (H1) and the continuity of $\tilde{f}$, we obtain that

$$\lim_{n\to\infty}\frac{f_i(x,u_n)}{\|u_n\|_{(C(\overline{\Omega}))^2}}=0,\quad i=1,2.$$

Taking the limit in the weak formulation of the system (2.5) we obtain

$$\begin{gathered}(-\Delta+\mathbf{I})\phi=\lambda_0A\phi,\quad \text{in } \Omega,\\ \frac{\partial\phi_1}{\partial\eta}=\frac{\partial\phi_2}{\partial\eta}=0,\quad \text{on } \partial\Omega,\end{gathered}\tag{2.6}$$

or equivalently

$$\lambda_0A\mathscr{K}\phi=\phi.\tag{2.7}$$

Integrating (2.6) in Ω, and since the divergence theorem, $\int_\Omega\Delta\phi_i=\int_{\partial\Omega}\frac{\partial\phi_i}{\partial\eta}=0$, so

$$(\lambda_0a_1-1)\int_\Omega\phi_1+\lambda_0a_2\int_\Omega\phi_2=0,\quad \lambda_0a_2\int_\Omega\phi_1+(\lambda_0a_3-1)\int_\Omega\phi_2=0.$$

From $\int_\Omega\phi_i\geq 0$, for $i=1,2$, and $\big(\int_\Omega\phi_1,\int_\Omega\phi_2\big)\neq(0,0)$, we have $0\in\sigma(\lambda_0A-I)$ with an associated eigenvector of nonnegative components. In other words, $\lambda_0\neq 0$, $1/\lambda_0\in\sigma(A)$, and $1/\lambda_0=\mu_+$, which implies that (2.7) reduces to

$$A\mathscr{K}\phi=\mu_+\phi,\quad \text{in } \Omega.$$

By (2.13), $r(A\mathscr{K})=\mu_+$, and $\phi=b=\begin{pmatrix}b_1\\ b_2\end{pmatrix}$ is the first normalized eigenfunction of $A\mathscr{K}$. □

Next we prove a bifurcation from infinity result.

Theorem 2.2. *If f satisfies* (H1)*, then the set of solutions of* (1.1) *possesses an unbounded component bifurcating from infinity at $\lambda=1/\mu_+$, namely $\mathscr{D}\subset\mathbb{R}\times(C(\overline{\Omega}))^2$, where μ_+ is defined by* (1.5)*.*

Moreover, the set of solutions bifurcating from infinity at $\lambda=1/\mu_+$ contains large positive solutions or large negative solutions of (1.1)*, namely $\mathscr{D}^+$ and $\mathscr{D}^-$ respectively. Also we have:*

(i) *There exists a neighborhood $\mathscr{O}^+$ of $(1/\mu_+,\infty)$ such that $(\lambda,u)\in\mathscr{D}^+\cap\mathscr{O}^+$ and $(\lambda,u)\neq(1/\mu_+,\infty)$ implies that this solutions can be expressed as*

$$(\lambda,u)=(\lambda,tb+w)\,,\tag{2.8}$$

where $t>0$, and $w=o(|t|)$ as $t\to+\infty$.

(ii) *There exists a neighborhood $\mathscr{O}^-$ of $(1/\mu_+,\infty)$ such that $(\lambda,u)\in\mathscr{D}^-\cap\mathscr{O}^-$ and $(\lambda,u)\neq(1/\mu_+,\infty)$ implies that this solutions can be expressed as*

$$(\lambda,u)=(\lambda,tb+w)\,,\tag{2.9}$$

where $t<0$, and $w=o(|t|)$ as $t\to-\infty$.

Proof. Let $\mathscr{K}$ be the resolvent operator to the Neumannn problem (2.1)-(2.2). Recall that (2.4) is equivalent to (1.1).

To use the Rabinowitz bifurcation from infinity result [14, Theorem 1.6] it is necessary to verify the following four conditions:

(a) $A\mathscr{K}$ is compact on $(C(\overline{\Omega}))^2$.

(b) $\mathscr{K} f$ is continuous with
$$\mathscr{K} f(u) = o(\|u\|) \quad \text{as } \|u\| \to \infty. \tag{2.10}$$

(c) $\|u\|^2 \mathscr{K} f\big(\frac{u}{\|u\|^2}\big)$ is a compact operator.

(d) μ_+ is a simple eigenvalue of $A\mathscr{K}$ and its corresponding normalized eigenfunction is b.

(a) It follows from the compactness of the operator $\mathscr{K}$.

(b) Given $u \in \big(C(\overline{\Omega})\big)^2$, by the continuity of $\mathscr{K} : L^r(\Omega) \to C(\overline{\Omega})$ and using (H1) we have that for each $\varepsilon > 0$ there exist $C_1, C_2 > 0$ such that
$$\begin{aligned}
\big\|\mathscr{K} f(\cdot, u(\cdot))\big\|_{(C(\overline{\Omega}))^2} &\le C_1 \big\|f(\cdot, u(\cdot))\big\|_{(L^r(\Omega))^2} \\
&\le C_1 \|h\|_{L^r(\Omega)} \big\|\tilde f(u(\cdot))\big\|_{C(\overline{\Omega})} \\
&\le C_1 \|h\|_{L^r(\Omega)} \big(\varepsilon\|u\| + C_2\big)
\end{aligned}$$
which implies (2.10).

(c) Let us define $\mathscr{H} : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}^2$ as follows
$$\mathscr{H}(x, u) = \|u\|^2 \mathscr{K} f\Big(x, \frac{u}{\|u\|^2}\Big). \tag{2.11}$$
Let us consider the closed subset $\overline{\Omega} \times B_\delta$ with
$$B_\delta = \big\{u \in \big(C(\overline{\Omega})\big)^2 : \|u\|_{(C(\overline{\Omega}))^2} \le \delta\big\}.$$
It is sufficient to prove that the image of $\overline{\Omega} \times B_\delta$ under $\mathscr{H}$ is relatively compact in $\big(C(\overline{\Omega})\big)^2$ for some $\delta > 0$ sufficiently small.

Let us consider the map $u \mapsto f(\cdot, u)$ from $\big(C(\overline{\Omega})\big)^2$ to $\big(L^r(\Omega)\big)^2$, $r > N/2$. Using (H1) and the continuity of $\tilde f$, for each $\varepsilon > 0$ there exists $C > 0$ such that
$$\int_\Omega |f_i(x, u)|^r \le \int_\Omega h(x)^r \tilde f(u)^r \, dx \le \int_\Omega h(x)^r \big(\varepsilon|u(x)| + C\big)^r \, dx.$$
Thus,
$$\|f(\cdot, u)\|_{(L^r(\Omega))^2} \le \|h\|_{L^r(\Omega)} \big(\varepsilon\|u\|_{(C(\overline{\Omega}))^2} + C\big), \quad \forall u \in \big(C(\overline{\Omega})\big)^2. \tag{2.12}$$
Let $v = (v_1, v_2)^T \in B_\delta$. Applying (2.12) to $v/\|v\|^2_{(C(\overline{\Omega}))^2}$ and taking $\varepsilon = 1$, we readily obtain
$$\|v\|^2 \,\big\|f\big(\cdot, \frac{v}{\|v\|^2}\big)\big\|_{(L^r(\Omega))^2} \le \|h\|_{L^r(\Omega)} \big(\|v\| + C\|v\|^2\big) \le \|h\|_{L^r(\Omega)} (\delta + C\delta^2).$$
By the continuity of $\mathscr{K} : L^r(\Omega) \to W^{2,r}(\Omega)$, there exists $C_3 > 0$ such that
$$\|\mathscr{H}(\cdot, v)\| \le C_3 \|h\|_{L^r(\Omega)} \, (\delta + C\delta^2),$$
where $\mathscr{H}$ is defined in (2.11), which implies that the map $\mathscr{H}$ sends closed sets into bounded sets.

(d) By (2.3) we know that $r(\mathscr{K}) = 1$ is a simple eigenvalue of $\mathscr{K}$ and its corresponding eigenfunction is equal to 1. Notice that the hypothesis $a_i > 0$ for $i =$

1, 2, 3, guarantee that $A\mathscr{K}$ is a strongly positive compact operator. Again by the Krein-Rutman theorem [1], there exists a unique positive eigenfunction associated to $r(A\mathscr{K})$ up to a multiplicative constant.

Now, observe that $A\mathscr{K}b = Ab = \mu_+ b$, so $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is the first normalized eigenfunction of $A\mathscr{K}$. That is,

$$r(A\mathscr{K}) = \mu_+, \text{ and } b \text{ is its corresponding eigenfunction,} \tag{2.13}$$

as desired. □

3. Functional framework and auxiliary results

We analyze the associated linear system, the nonhomogeneous Neumann problem

$$\begin{gathered} (-\Delta + \mathbf{I})u = \lambda A u + g(x), \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

where

$$\lambda \in J := \begin{cases} \big(-\infty, \min\big\{\frac{1}{\mu_-}, \frac{\lambda_2}{\mu_+}\big\}\big), & \text{if } \mu_- > 0, \\ \big(-\infty, \frac{\lambda_2}{\mu_+}\big), & \text{if } \mu_- = 0, \\ \big(\frac{1}{\mu_-}, \frac{\lambda_2}{\mu_+}\big), & \text{if } \mu_- < 0, \end{cases}$$

where λ_2 is defined in (1.7), and $g \in \big(L^r(\Omega)\big)^2$.

By the Fredholm alternative, (3.1) has a unique solution $u \in \big(W^{2,r}(\Omega)\big)^2$ if $\lambda \neq 1/\mu_+$, see [11]. Moreover, by the compact embedding theorem $u \in \big(C(\overline{\Omega})\big)^2$, since $r > N/2$.

Let us consider the orthogonal decomposition

$$\big(L^r(\Omega)\big)^2 = \operatorname{span}\{b\} \oplus \Big\{\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \big(L^r(\Omega)\big)^2 : \int_\Omega b_1\phi_1 + b_2\phi_2 = 0\Big\}. \tag{3.2}$$

For $g \in \big(L^r(\Omega)\big)^2$, there exists $g^{(1)} \in \big(L^r(\Omega)\big)^2$ such that $g(\cdot) = \tilde{a}_1\, b + g^{(1)}(\cdot)$, where $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ and $g^{(1)} = \begin{pmatrix} g_1^{(1)} \\ g_2^{(1)} \end{pmatrix}$ satisfy

$$\tilde{a}_1 = \frac{1}{|\Omega|} \int_\Omega b_1 g_1(x) + b_2 g_2(x) \quad \text{and} \quad \int_\Omega b_1 g_1^{(1)}(x) + b_2 g_2^{(1)}(x) = 0. \tag{3.3}$$

By the Fredholm alternative, the linear elliptic system (3.1) has a unique solution if $\lambda \neq 1/\mu_+$ and does not have a solution if $\lambda = 1/\mu_+$ and $\tilde{a}_1 \neq 0$. Consequently, for $\lambda \neq 1/\mu_+$, the solution $u = u(\lambda)$ of (3.1) belongs to $\big(W^{2,r}(\Omega)\big)^2 \subset \big(L^r(\Omega)\big)^2$, and

$$u = \frac{\tilde{a}_1}{1 - \lambda\mu_+} b + w, \quad \text{with} \int_\Omega b_1 w_1(x) + b_2 w_2(x)\, dx = 0. \tag{3.4}$$

Moreover, $w = w(\lambda)$ satisfies

$$\begin{gathered} (-\Delta + \mathbf{I})w = \lambda A w + g^{(1)}(x), \quad \text{in } \Omega, \\ \frac{\partial w}{\partial \eta} = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.5}$$

where $g^{(1)}$ is defined by (3.3).

If $\lambda = 1/\mu_+$, by the Fredholm alternative there exists a function $v \in \big(W^{2,r}(\Omega)\big)^2$ such that $v+cb$ solves (3.5) for any $c \in \mathbb{R}$. Fixing c_0 such that $\int_\Omega b_1v_1+b_2v_2+c_0 = 0$, and defining $w(1/\mu_+) = v + c_0b$, then, $w(1/\mu_+) \in (\operatorname{span}\{b\})^\perp$.

Next, we estimate the norm of the solution to the linear system (3.5) in $\big(C(\overline{\Omega})\big)^2$ whenever $g \in \big(L^r(\Omega)\big)^2$. From now on, throughout this paper C (possibly with indices) denotes several constants independent of the solution u, and which may change from line to line.

Lemma 3.1. *For each compact set $K \subset J$, there exists a constant $C = C(K) > 0$, independent of $\lambda \in K$ such that*

$$\|w(\lambda)\|_{(C(\overline{\Omega}))^2} \le C\|g^{(1)}(\cdot)\|_{(L^r(\Omega))^2},$$

where $w = (w_1, w_2)^T$ satisfies (3.5) *with $\int_\Omega b_1w_1+b_2w_2 = 0$, and $g^{(1)} \in (\operatorname{span}\{b\})^\perp$.*

Proof. By the above discussion, $w = w(\lambda)$ satisfying (3.4) and (3.5) is well defined.

Suppose that there exists a sequence $\lambda_n \to 1/\mu_+$ such that $\|w(\lambda_n)\|_{(C(\overline{\Omega}))^2} \to \infty$. Therefore,

$$\frac{w(\lambda_n)}{\|w(\lambda_n)\|_{(C(\overline{\Omega}))^2}} \to b,$$

which contradicts that $\int_\Omega b_1w_1(\lambda_n,\cdot)+b_2w_2(\lambda_n,\cdot) = 0$. Therefore, there exists $C > 0$ and $\delta > 0$ such that $\|w(\lambda)\|_{(C(\overline{\Omega}))^2} < C$ independent of λ for any $|\lambda - 1/\mu_+| < \delta$.

Let $\lambda \in K \setminus (1/\mu_+ - \delta, 1/\mu_+ + \delta)$. By the Fredholm alternative, $w(\lambda) \in (W^{2,r}(\Omega))^2$ is the unique solution to (3.5). Using the L^r-estimate and the embedding $(W^{2,r}(\Omega))^2 \hookrightarrow \big(C(\overline{\Omega})\big)^2$, we have that

$$\|w(\lambda)\|_{(C(\overline{\Omega}))^2} \le C\|w\|_{(W^{2,r}(\Omega))^2} \le C\|g^{(1)}(\cdot)\|_{(L^r(\Omega))^2} < \infty.$$

Now, let $\lambda \in K$ and let us define a family of bounded operators as follows

$$\begin{gathered} T_\lambda\colon \Big\{g_1 \in \big(L^r(\Omega)\big)^2 : \int_\Omega b_1g_1^{(1)} + b_2g_2^{(1)} = 0\Big\} \to \big(C(\overline{\Omega})\big)^2, \\ T_\lambda g_1 = w(\lambda), \end{gathered}$$

where $w(\lambda)$ is the solution to (3.5). More precisely, for every $\lambda \in K$, T_λ is a continuous operator. Moreover, $\sup_{\lambda\in K}\|T_\lambda g_1\|_{(C(\overline{\Omega}))^2} < \infty$. Finally, the Uniform Boundedness Principle implies the result. $\square$

Proposition 3.2. *Let* (H1) *and* (H2) *hold. Then, there exists a neighborhood of $(1/\mu_+, \infty) \times \big(C(\overline{\Omega})\big)^2 \subset \mathbb{R} \times \big(C(\overline{\Omega})\big)^2$ given by*

$$\mathscr{O} := \Big\{(\lambda, u) \in \mathbb{R} \times \big(C(\overline{\Omega})\big)^2 : |\lambda - 1/\mu_+| < \delta_0,\ u_i > 0,\ i = 1, 2,\ \|u\|_{(C(\overline{\Omega}))^2} > M_0\Big\},$$

for some small δ_0 and large M_0, such that the following hold:

(i) *There exist positive constants C_1, C_2 (independent of λ) such that if $(\lambda, u) \in \mathscr{D}^+ \cap \mathscr{O}$ and $\big(\lambda, \|u\|_{(C(\overline{\Omega}))^2}\big) \ne (1/\mu_+, \infty)$ then*

$$u = tb + w, \quad \text{where } t > 0, \text{ and } \int_\Omega b_1w_1 + b_2w_2 = 0, \tag{3.6}$$

$$\|w\|_{(C(\overline{\Omega}))^2} \le C_1\|B\|_{L^r(\Omega)}t^\alpha \quad \text{as } t \to \infty, \tag{3.7}$$

$$|\lambda - 1/\mu_+| \le C_2t^{\alpha-1} \ \text{as}\ \ t \to \infty. \tag{3.8}$$

(ii) *There exists $t_0 > 0$ such that for all $t \geq t_0$ there exists $(\lambda, u) \in \mathscr{D}^+ \cap \mathscr{O}$ satisfying $u = tb + w$ with $w = (w_1, w_2)$ satisfying $\int_\Omega b_1 w_1 + b_2 w_2 = 0$.*

Proof. By Theorem 2.2 it is clear that $\mathscr{D}^+ \cap \mathscr{O} \neq \emptyset$. Let $(\lambda, u) \in \mathscr{D}^+ \cap \mathscr{O}$, by (3.2) u can be written as

$$u = tb + w, \quad \text{where } t > 0, \quad \int_\Omega b_1 w_1 + b_2 w_2 = 0.$$

Let $(\lambda, u) = (\lambda, tb + w)$ solve (1.1). Multiplying (1.3) by b^T on the left, integrating the result, using that $\int_\Omega b^T u = t|\Omega|$, and the symmetry of the matrix A, so $b^T A = \mu_+ b^T$, we obtain

$$t|\Omega|(1 - \lambda\mu_+) = \int_\Omega b^T f\left(\cdot, tb + w\right),$$

or equivalently

$$(1 - \lambda\mu_+) = \frac{1}{|\Omega|} \int_\Omega \frac{b_1 f_1(\cdot, tb + w) + b_2 f_2(\cdot, tb + w)}{t}. \tag{3.9}$$

Now, using (H1) and since $w = o(|t|)$ as $|t| \to \infty$, we have that

$$\frac{|f(x, tb + w)|}{|t|} = \frac{|f(x, tb + w)|}{|tb + w|} |b + \frac{w}{t}| \to 0 \quad \text{as } t \to \infty. \tag{3.10}$$

By (H1), (3.9)-(3.10), and the Lebesgue dominated convergence theorem, we readily obtain that $\lambda \to 1/\mu_+$ as $t \to \infty$.

By (H2) we deduce that

$$|f(x, tb + w)| = |t|^\alpha \frac{|f(x, tb + w)|}{|tb + w|^\alpha} \left|b + \frac{w}{t}\right|^\alpha \leq |t|^\alpha B(x) \left|b + \frac{w}{t}\right|^\alpha. \tag{3.11}$$

Substituting (3.11) into (3.9),

$$|\lambda - 1/\mu_+| \leq \frac{|t|^{\alpha - 1}}{|\Omega|\mu_+} \int_\Omega B(x) \left|b + \frac{w}{t}\right|^\alpha \leq C\|B\|_{L^r(\Omega)} |t|^{\alpha - 1},$$

and assertion (3.8) readily follows.

To complete the proof of part (i), it only remains to verify that (3.7) holds. Because of $f(\cdot, u(\cdot)) \in \left(L^r(\Omega)\right)^2$, by (3.2) and (3.3) there exists a unique orthogonal decomposition

$$f(x, u) = \tilde{a}_1 b + f^{(1)}(x, u),$$

where $\tilde{a}_1 = (1/|\Omega|) \int_\Omega b_1 f_1(x, u) + b_2 f_2(x, u)$ and $f^{(1)} = \begin{pmatrix} f_1^{(1)} \\ f_2^{(1)} \end{pmatrix} \in \left(\operatorname{span} b\right)^\perp$, that is, $\int_\Omega b_1 f_1^{(1)} + b_2 f_2^{(1)} = 0$. By Lemma 3.1, we obtain

$$\|w\|_{(C(\overline{\Omega}))^2} \leq C\|f^{(1)}\|_{(L^r(\Omega))^2} \leq C'\|f\|_{(L^r(\Omega))^2}.$$

From estimate (3.11), $w = o(|t|)$, and Hölder inequality, we have

$$\|w\|_{(C(\overline{\Omega}))^2} \leq C\|B\|_{L^r(\Omega)} |t|^\alpha, \quad \text{as } t \to \infty.$$

(ii) Since $\mathscr{D}^+$ bifurcates from infinity at $\lambda = 1/\mu_+$, we have that despite of $\mathscr{D}^+ \cap \mathscr{O}$ is not necessarily connected, it contains an unbounded connected component, namely $\mathscr{G}$ such that if $(\lambda, u) \in \mathscr{G} \subset \mathscr{D}^+ \cap \mathscr{O}$, then

$$u = tb + w, \quad \text{with } \int_\Omega b_1 w_1 + b_2 w_2 = 0 \text{ and } t = \frac{1}{|\Omega|} \int_\Omega b_1 u_1 + b_2 u_2. \tag{3.12}$$

By the continuity of the projection of $\mathscr{G}$ on span$\{b\}$, the set

$$\{t \in \mathbb{R}\colon (1.1) \text{ has a solution satisfying (3.12)}\}$$

also contains an unbounded connected set as desired. □

A straightforward consequence of Proposition 3.2 is the following result.

Corollary 3.3. *Assume* (H1) *and* (H2) *hold. Let* $\{(\lambda_n, u_n)\} \subset \mathscr{D}^+ \cap \mathscr{O}$ *be such that* $\lambda_n \to 1/\mu_+$ *and* $u_n = t_n b + w_n$*, with* $\int_\Omega b_1 w_{1,n} + b_2 w_{2,n} = 0$ *and* $t_n = \frac{1}{|\Omega|}\int_\Omega b_1 u_{1,n} + b_2 u_{2,n} \to \infty$*. Then*

$$\lim_{n\to\infty} \frac{u_n}{\|u_n\|_{(C(\overline{\Omega}))^2}} = b, \quad \textit{uniformly in } \overline{\Omega},$$
$$\lim_{n\to\infty} \frac{u_n}{t_n} = b, \quad \textit{uniformly in } \overline{\Omega},$$
$$\lim_{n\to\infty} \frac{\|u_n\|_{(C(\overline{\Omega}))^2}}{t_n} = 1, \quad \textit{uniformly in } \overline{\Omega}.$$

Hypothesis (H3) will be key for proving the following result. Later on, the main theorem of this paper, Theorem 1.3, we will see that the next lemma together with hypothesis (1.10) are sufficient to conclude the existence of subcritical solutions ($\lambda < 1/\mu_+$) and supercritical solutions ($\lambda > 1/\mu_+$), respectively, in the unbounded continuum $\mathscr{D}^+$.

Lemma 3.4. *Let* f *satisfy* (H3). *Suppose there exists* $\alpha < 1$ *and a function* $B \in L^1(\Omega)$ *such that for* $x \in \Omega$*, and for all* (λ, s) *close to the bifurcation point* $(1/\mu_+, +\infty)$*, we have*

$$\frac{|f(x,s)|}{|s|^\alpha} \le B_1(x). \tag{3.13}$$

Let $\lambda_n \to 1/\mu_+$ *and* $w_n \in L^\infty(\Omega)$ *are such that* $\|w_n\|_{L^\infty(\Omega)} = O(|t_n|^\alpha)$ *as* $n \to \infty$*. Then*

$$\liminf_{n\to\infty} \sum_{i=1}^2 b_i \int_\Omega \frac{f_i(\cdot, t_n b + w_n)}{|t_n b + w_n|^\alpha} \ge \liminf_{n\to\infty} \frac{t_n \sum_{i=1}^2 b_i \int_\Omega f_i(\cdot, t_n b)}{|t_n|^{1+\alpha}}, \tag{3.14}$$
$$\limsup_{n\to\infty} \sum_{i=1}^2 b_i \int_\Omega \frac{f_i(\cdot, t_n b + w_n)}{|t_n b + w_n|^\alpha} \le \limsup_{n\to\infty} \frac{t_n \sum_{i=1}^2 b_i \int_\Omega f_i(\cdot, t_n b)}{|t_n|^{1+\alpha}}. \tag{3.15}$$

Proof. Let $w = (w_1, w_2) \in (L^\infty(\Omega))^2$, $t > 0$ and $\|w\| < t/2$. By the Mean Value Theorem,

$$f_i(x, tb + w) - f_i(x, tb) = \int_0^1 \Big(\frac{\partial f_i}{\partial s_1}(x, tb + \tau w) w_1 + \frac{\partial f_i}{\partial s_2}(x, tb + \tau w) w_2\Big)\, d\tau, \tag{3.16}$$

for all $x \in \Omega$ and $i = 1, 2$. Consequently, we can write (3.16) in matrix notation as

$$f(x, tb + w) - f(x, tb) = \Big(\int_0^1 Jf(x, tb + \tau w)\, d\tau\Big) w,$$

where

$$Jf := \begin{pmatrix} \frac{\partial f_1}{\partial s_1} & \frac{\partial f_1}{\partial s_2} \\ \frac{\partial f_2}{\partial s_1} & \frac{\partial f_2}{\partial s_2} \end{pmatrix}$$

denotes the Jacobian matrix of $f = (f_1, f_2)$ and the integral of a matrix is taken component-wise. Consequently,

$$\begin{aligned} &\int_\Omega |f(\cdot, tb + w) - f(\cdot, tb)|\, dx \\ &\le C\|w\|_{(L^\infty(\Omega))^2} \sum_{i,j=1}^{2} \sup_{\tau\in[0,1]} \big\|\frac{\partial f_i}{\partial s_j}(\cdot, tb + \tau w)\big\|_{L^\infty(\Omega)} . \end{aligned} \tag{3.17}$$

If $\|w\|_{(L^\infty(\Omega))^2} = O(|t|^\alpha)$, using (3.17) and (H3), we obtain

$$\begin{aligned} &\int_\Omega \frac{|f(\cdot, tb + w) - f(\cdot, tb)|}{|t|^\alpha}\, dx \\ &\le C \sum_{i,j=1}^{2} \sup_{|s|\ge M} \big\|\frac{\partial f_i}{\partial s_j}(\cdot, s)\big\|_{L^\infty(\Omega)} \frac{\|w\|_{(L^\infty(\Omega))^2}}{|t|^\alpha} \to 0, \end{aligned} \tag{3.18}$$

as $M \to \infty$. Now, let us consider $t_n \to \infty$ and $w_n = O(|t_n|^\alpha)$ as $n \to \infty$. By (3.18) and the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} &\liminf_{n\to\infty} \frac{\sum_{i=1}^2 b_i \int_\Omega f_i(\cdot, t_n b + w_n)}{|t_n|^\alpha} \\ &\ge \lim_{n\to\infty} \frac{\sum_{i=1}^2 b_i \int_\Omega f_i(\cdot, t_n b + w_n) - f_i(\cdot, t_n b)}{|t_n|^\alpha} + \liminf_{n\to\infty} \frac{\sum_{i=1}^2 b_i \int_\Omega f_i(\cdot, t_n b)}{|t_n|^\alpha} \\ &= \liminf_{n\to\infty} \frac{\sum_{i=1}^2 b_i \int_\Omega f_i(\cdot, t_n b)}{|t_n|^\alpha}. \end{aligned} \tag{3.19}$$

Now, note that

$$\frac{\sum_{i=1}^2 b_i f_i(\cdot, t_n b + w_n)}{|t_n|^\alpha} = \frac{\sum_{i=1}^2 b_i f_i(\cdot, t_n b + w_n)}{|t_n b + w_n|^\alpha} \big|b + \frac{w_n}{t_n}\big|^\alpha. \tag{3.20}$$

Combining (3.19) and (3.20), and since $|b| = 1$, we readily obtain

$$\begin{aligned} \liminf_{n\to\infty} \frac{\sum_{i=1}^2 b_i \int_\Omega f_i(\cdot, t_n b)}{|t_n|^\alpha} &\le \liminf_{n\to\infty} \sum_{i=1}^{2} b_i \int_\Omega \frac{f_i(\cdot, t_n b + w_n)}{|t_n b + w_n|^\alpha} \big|b + \frac{w_n}{t_n}\big|^\alpha \\ &\le \liminf_{n\to\infty} \sum_{i=1}^{2} b_i \int_\Omega \frac{f_i(\cdot, t_n b + w_n)}{|t_n b + w_n|^\alpha}. \end{aligned}$$

By (3.13), the right-hand side above is well-defined, which implies (3.14). The proof of (3.15) is analogous. □

4. Proof of Theorem 1.3

Proof. (i) Let $(\lambda_n, u_n), (\lambda_n', u_n') \in \mathscr{D}^+ \cap \mathscr{O}$ such that $(\lambda_n, \|u_n\|), (\lambda_n', \|u_n'\|) \to (1/\mu_+, \infty)$ provided by Proposition 3.2. Therefore

$$\begin{aligned} u_n = t_n b + w_n \quad &\text{with} \int_\Omega b_1 w_{1,n} + b_2 w_{2,n} = 0 \text{ and } t_n := \frac{1}{|\Omega|}\int_\Omega b_1 u_{1,n} + b_2 u_{2,n}, \\ u_n' = t_n' b + w_n' \quad &\text{with} \int_\Omega b_1 (w_{1,n}') + b_2 (w_{2,n}') = 0 \text{ and } t_n' := \frac{1}{|\Omega|}\int_\Omega b_1 u_{1,n}' + b_2 u_{2,n}'. \end{aligned}$$

By (3.9) we have

$$|\Omega|\, t_n \left(1-\lambda_n\mu_+\right) = \int_\Omega b_1 f_1(\cdot,u_n) + b_2 f_2(\cdot,u_n).$$

Dividing by $\|u_n\|^\alpha_{(C(\overline{\Omega}))^2}$, Corollary 3.3 yields

$$\liminf_{n\to\infty} \frac{|\Omega|\big(1-\lambda_n\mu_+\big)}{\|u_n\|^{\alpha-1}_{(C(\overline{\Omega}))^2}} = \liminf_{n\to\infty} \int_\Omega \frac{b_1 f_1(\cdot,u_n) + b_2 f_2(\cdot,u_n)}{\|u_n\|^\alpha_{(C(\overline{\Omega}))^2}}.$$

Furthermore,

$$\begin{aligned}
&\sum_{i=1}^2 b_i \int_\Omega \frac{f_i(\cdot,u_n)}{\|u_n\|^\alpha_{(C(\overline{\Omega}))^2}} \\
&= \sum_{i=1}^2 b_i \int_\Omega \frac{f_i(\cdot,u_n)}{|u_n|^\alpha}\Big(\frac{|u_n|}{\|u_n\|_{(C(\overline{\Omega}))^2}}\Big)^\alpha \\
&= \sum_{i=1}^2 b_i \int_\Omega \frac{f_i(\cdot,u_n)}{|u_n|^\alpha}\Big[\Big(\frac{|u_n|}{\|u_n\|_{(C(\overline{\Omega}))^2}}\Big)^\alpha - 1\Big] + \sum_{i=1}^2 b_i \int_\Omega \frac{f_i(\cdot,u_n)}{|u_n|^\alpha}.
\end{aligned}$$

Now, by (H3) and Corollary 3.3,

$$\sum_{i=1}^2 b_i \int_\Omega \Big|\frac{f_i(\cdot,u_n)}{|u_n|^\alpha}\Big[\Big(\frac{|u_n|}{\|u_n\|_{(C(\overline{\Omega}))^2}}\Big)^\alpha - 1\Big]\Big| \le \int_\Omega B(x)\Big[\Big(\frac{|u_n|}{\|u_n\|_{(C(\overline{\Omega}))^2}}\Big)^\alpha - 1\Big] \to 0,$$

as $n\to\infty$. Consequently,

$$\liminf_{n\to\infty} \frac{|\Omega|\big(1-\lambda_n\mu_+\big)}{\|u_n\|^{\alpha-1}_{(C(\overline{\Omega}))^2}} \ge \liminf_{n\to\infty} \sum_{i=1}^2 b_i \int_\Omega \frac{f_i(\cdot,u_n)}{|u_n|^\alpha}.$$

Now, Lemma 3.4 implies

$$\begin{aligned}
\liminf_{n\to\infty} \frac{|\Omega|\big(1-\lambda_n\mu_+\big)}{\|u_n\|^{\alpha-1}_\infty} &\ge \liminf_{n\to\infty} \sum_{i=1}^2 b_i \int_\Omega \frac{f_i(\cdot,t_n b + w_n)}{|t_n b + w_n|^\alpha} \\
&\ge \liminf_{n\to\infty} \frac{t_n \sum_{i=1}^2 b_i \int_\Omega f_i(\cdot,t_n b)}{|t_n|^{1+\alpha}} > 0\,,
\end{aligned}$$

where the positivity comes from the hypothesis (1.10). Therefore, $\lambda_n < 1/\mu_+$ is subcritical for n sufficiently large. Likewise, $\lambda'_n > 1/\mu_+$ is supercritical for n sufficiently large.

(ii) Let $\{t_n\}$ and $\{t'_n\}$ be sequences such that $t_n, t'_n > 0$ and $t_n, t'_n \to \infty$ as $n\to\infty$. Then, we can choose $t_n < t'_n < t_{n+1}$ for all $n\ge 1$ and $t_n, t'_n \ge t_0$, where t_0 is as defined in Proposition 3.2, (ii). Then there exists (λ_n, u_n), $(\lambda'_n, u'_n) \in \mathscr{D}^+ \cap \mathscr{O}$ such that

$$\begin{gathered}
u_n = t_n b + w_n \quad \text{with } \int_\Omega b_1 w_{1,n} + b_2 w_{2,n} = 0, \\
u'_n = t'_n b + w'_n \quad \text{with } \int_\Omega b_1 (w'_{1,n}) + b_2 (w'_{2,n}) = 0.
\end{gathered}$$

By part (i) we obtain that $\lambda_n < 1/\mu_+$ and $\lambda'_n > 1/\mu_+$ for n sufficiently large. If $(\lambda, u) \in \mathscr{D}^+ \cap \mathscr{O}$ and $t = \frac{1}{|\Omega|}\int_\Omega b_1 u_1 + b_2 u_2 > t_0$, Proposition 3.2 (i) yields for t_0

sufficiently large:

$$\begin{gathered} \|u\|_{(C(\overline{\Omega}))^2} = \|tb + w\|_{(C(\overline{\Omega}))^2} \le \left(1 + C\|B\|_{L^r(\Omega)}|t_0|^{\alpha-1}\right) t \le 2t, \\ |\lambda - 1/\mu_+| < Ct^{\alpha-1} \le Ct_0^{\alpha-1}. \end{gathered} \tag{4.1}$$

Let us define

$$K_n := \big\{(\lambda, u) \in \mathscr{D}^+ \cap \mathscr{O} : t = \frac{1}{|\Omega|}\int_\Omega b_1u_1 + b_2u_2 \text{ and } t_n \le t \le t_{n+1}\big\}.$$

We will show that K_n is a compact set in $\mathbb{R} \times \big(C(\overline{\Omega})\big)^2$. Let $\{(\mu_k, v_k)\} \subset K_n$. If $t_n \le \frac{1}{|\Omega|}\int_\Omega b_1v_1^{(k)} + b_2v_2^{(k)} \le t_{n+1}$ for all k. By (4.1), we easily obtain that $\|v_k\|_{(C(\overline{\Omega}))^2} \le 2t_{n+1}$ for all k. By [12, Th. 2.4], there exists a constant C_n such that

$$\|v_k\|_{(C^\alpha(\overline{\Omega}))^2} \le C_1(1 + \|v_k\|_{(C(\overline{\Omega}))^2}) \le C_n.$$

Using the compact embedding $C^\alpha(\overline{\Omega}) \hookrightarrow C^\beta(\overline{\Omega})$ for some $\beta \in (0, \alpha)$, we infer that there exists $u^* \in (C^\beta(\overline{\Omega}))^2$ such that $v_k \to u^*$ in $(C^\beta(\overline{\Omega}))^2$ up to a subsequence and also $(\mu_k, v_k) \to (\mu^*, u^*)$. By definition of K_n, we have that (μ_k, v_k) satisfies

$$\begin{gathered} (-\Delta + \mathbf{I})v_k = \mu_k A v_k + f(x, v_k), \quad \text{in } \Omega, \\ \frac{\partial v_k}{\partial \eta} = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

By assumption f is Carathéodory, hence $f(\mu_k, \cdot, v_k) \to f(\mu^*, \cdot, u^*)$ pointwise. Consequently, $f(\mu_k, \cdot, v_k) \to f(\mu^*, \cdot, u^*)$ in $\big(L^r(\Omega)\big)^2$. Passing to the limit in the weak formulation (1.2), we can see that u^* is a weak solution to

$$\begin{gathered} (-\Delta + \mathbf{I})u^* = \mu^* A u^* + f(\mu_k, x, u^*), \quad \text{in } \Omega, \\ \frac{\partial u^*}{\partial \eta} = 0, \quad \text{on } \partial\Omega. \end{gathered}$$

Using the continuity of the projection onto $\operatorname{span} b$ implies that

$$t_0 \le t_n \le t^* := \frac{1}{|\Omega|}\int_\Omega b_1u_1^* + b_2u_2^* \le t_{n+1}.$$

Thus, $(\mu^*, u^*) \in K_n$, which establishes the compactness of K_n.

Since $t_n < t_n' < t_{n+1}$, by part (i) there exists $(\lambda_n', u_n') \in K_n$ such that $u_n' = t_n' b + w_n'$ and $\int_\Omega b_1(w_{1,n}') + b_2(w_{2,n}') = 0$ and $\lambda_n' > 1/\mu_+$. We define

$$\lambda_n^* := \sup\{\lambda : (\lambda, u) \in K_n\}$$

Then, $\lambda_n^* \ge \lambda_n' > 1/\mu_+$ (supercritical). Analogously to the previous limiting argument, there exists $u_n^* = \big(u_{1,n}^*, u_{2,n}^*\big)$ such that $(\lambda_n^*, u_n^*) \in K_n$. That is,

$$t_n \le \frac{1}{|\Omega|}\int_\Omega b_1u_{1,n}^* + b_2u_{2,n}^* \le t_{n+1},$$

recalling that t_n and t_{n+1} are associated to $\lambda_n < 1/\mu_+$ and $\lambda_{n+1} < 1/\mu_+$, respectively. We can observe that there is no solution (λ, u) close to (λ_n^*, u_n^*) with $\lambda > \lambda_n^*$. Indeed, if there is a solution (λ, u) close to (λ_n^*, u_n^*) with $\lambda > \lambda_n^*$, by the continuity of the projection, we have $t_n \le \frac{1}{|\Omega|}\int_\Omega b_1u_1 + b_2u_2 \le t_{n+1}$, so $(\lambda, u) \in K_n$, which contradict the definition of λ_n^*. Therefore, (λ_n^*, u_n^*) is a supercritical turning point.

Similarly, letting

$$K_n' := \big\{(\lambda, u) \in \mathscr{D}^+ \cap \mathscr{O} : t' = \frac{1}{|\Omega|}\int_\Omega b_1u_1 + b_2u_2 \text{ and } t_n' \le t' \le t_{n+1}'\big\},$$

$$\lambda_{*,n} = \inf \{\lambda : (\lambda, u) \in K'_n\}.$$

We can guarantee the existence of a sequence $\{(\lambda_{*,n}), u_{*,n}\} \subset K'_n$, of subcritical turning points with $\lambda_{*,n} \leq \lambda_n < 1/\mu_+$. Lastly, combining the sequences $\{\lambda^*_n\}$ and $\{\lambda_{*,n}\}$ and relabeling, we obtain two subsequences of turning points, one subcritical ($\lambda^*_{2n+1} < 1/\mu_+$) and the other supercritical ($\lambda^*_{2n} > 1/\mu_+$) as we desired.

(iii) Now, we prove the existence of a sequence of resonant solutions. That is, solutions u of (1.1) corresponding to $\lambda = 1/\mu_+$. It is sufficient to show that there exists $n_0 \in \mathbb{N}$ such that K_n and K'_n contain resonant solutions of the form $(1/\mu_+, u)$ for each $n \geq n_0$. Suppose on the contrary that there exists a sequence of integers numbers $n_j \to \infty$ such that K_{n_j} does not contain any resonant solutions. Let us define the compact sets

$$K^+_{n_j} := \{(\lambda, u) \in K_{n_j} : \lambda > 1/\mu_+\},$$

which can be rewritten as

$$\begin{aligned} K^+_{n_j} = \mathscr{D}^+ \cap \mathscr{O} \cap \Big\{&(\lambda, u) \in \mathbb{R} \times \big(C(\overline{\Omega})\big)^2 : \lambda > 1/\mu_+, \\ &t_{n_j} \leq \frac{1}{|\Omega|} \int_\Omega b_1 u_1 + b_2 u_2 \leq t_{n_j+1}\Big\}. \end{aligned}$$

Thus, $K^+_{n_j}$ contains at least one nonempty connected component of $\mathscr{D}^+$. In view of $t_n < t'_n < t_{n+1}$ by construction, we easily obtain that $(t_{n_j}, t_{n_j+1}) \cap (t_{n_j+1}, t_{n_j+2}) = \emptyset$, then $K^+_{n_j} \cap K^+_{n_j+1} = \emptyset$ for all $j \in \mathbb{N}$. Since $\mathscr{D}^+$ is a continuum (closed connected set) in $\mathbb{R} \times \big(C(\overline{\Omega})\big)^2$, the contradiction comes from the fact that a continuum cannot contain two nonempty disjoint connected components. A similar argument applied to the sets K'_n also generates a sequence of resonant solutions u corresponding to $\lambda = 1/\mu_+$. □

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