Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2022-0002s

Vol. **15**, No. 4, pp. 903-937
November 2022

A Nonlocal Stokes System with Volume Constraints

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Received 10 December 2020; Accepted (in revised version) 19 January 2022

Abstract. In this paper, we introduce a nonlocal model for linear steady Stokes system with physical no-slip boundary condition. We use the idea of volume constraint to enforce the no-slip boundary condition and prove that the nonlocal model is well-posed. We also show that and the solution of the nonlocal system converges to the solution of the original Stokes system as the nonlocality vanishes.

AMS subject classifications: 45P05, 45A05, 35A23, 46E35

Key words: Nonlocal Stokes system, nonlocal operators, smoothed particle hydrodynamics, incompressible flows, well-posedness, local limit.

1. Introduction

Recently, nonlocal models and corresponding numerical methods have attracted much attention due to many successful applications. For example, in solid mechanics, the theory of peridynamics [38] has been used as a possible alternative to conventional models of elasticity and fracture mechanics. Many numerical methods have also been developed to simulate nonlocal models like peridynamics based on rigorous mathematical analysis [10–12, 30, 31, 39, 43]. Nonlocal methods are also successfully applied in image processing and data analysis [2, 4, 6, 19, 20, 22, 23, 29, 33–35, 41]. The idea of integral approximation is also applied to derive numerical scheme for solving PDEs on point cloud [25, 26].

In this paper, we study the nonlocal analog of the Stokes system in fluid mechanics. Previously, nonlocal Stokes models have been proposed in [13,24] and analyzed subject to periodic boundary condition. In this paper, we consider the case of a nonlocal no-slip

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boundary condition. More precisely, for the conventional, local linear Stokes system on a domain $\Omega \subset \mathbb{R}^n$,

$$\begin{cases} \Delta \boldsymbol{u}(\boldsymbol{x}) - \nabla p(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \nabla \cdot \boldsymbol{u}(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega \end{cases} \tag{1.1}$$

the no-slip boundary condition on the boundary $\partial\Omega$ is

$$\boldsymbol{u} = 0 \quad \text{at } \partial\Omega. \tag{1.2}$$

For the pressure, we impose average zero condition

$$\int_\Omega p(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 0. \tag{1.3}$$

The no-slip boundary condition is a Dirichlet type boundary condition and it is often used in many real world applications. However, the theoretical study with no-slip boundary condition is also much more difficult. The first question is how to enforce no-slip boundary condition in the nonlocal approach. Recently, Du *et al.* [10] proposed volume constraint to deal with the boundary condition in the nonlocal diffusion problem by enforcing the condition over a nonlocal region adjacent to the boundary. Adopting this idea, in the nonlocal Stokes system, we extend the no-slip condition to a small layer as shown in Fig. 1.

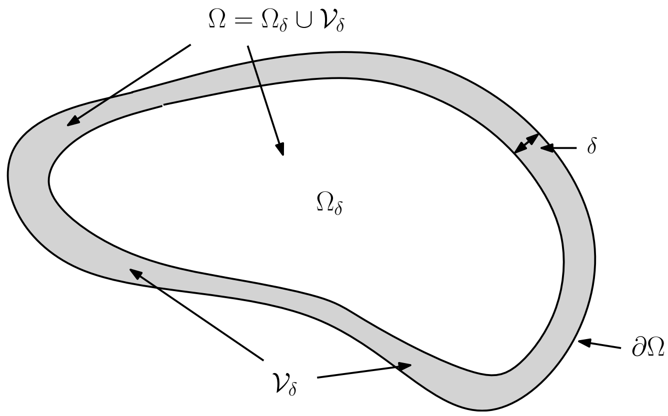

Figure 1: Computational domain in non-local Stokes model.

For a nonlocal problem involving nonlocal interactions on the range of $\delta > 0$, the whole computational domain Ω is decomposed to two parts. $\Omega = \mathcal{V}_\delta \bigcup \Omega_\delta$ as shown in Fig. 1 and $\boldsymbol{u}$ is enforced to be zero in $\mathcal{V}_\delta$, i.e.

$$\boldsymbol{u}_\delta(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathcal{V}_\delta. \tag{1.4}$$

Definition of Ω_δ and $\mathcal{V}_\delta$ will be given in (2.1). The parameter δ is often called the nonlocal horizon parameter [9, 38]. In Ω_δ, the Stokes equation is approximated is formulated as

$$\begin{cases} \mathcal{L}_\delta \boldsymbol{u}_\delta(\boldsymbol{x}) - \mathcal{G}_\delta p_\delta(\boldsymbol{x}) = \displaystyle\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, & \boldsymbol{x}\in\Omega_\delta, & (1.5a)\\ \mathcal{D}_\delta \boldsymbol{u}_\delta(\boldsymbol{x}) - \bar{\mathcal{L}}_\delta p_\delta(\boldsymbol{x}) = 0, & \boldsymbol{x}\in\Omega. & (1.5b)\end{cases}$$

The nonlocal integral operators used in (1.5) represent the nonlocal diffusion (Laplacian) $\mathcal{L}_\delta$, nonlocal gradient $\mathcal{G}_\delta$ and nonlocal divergence $\mathcal{D}_\delta$ respectively as in [13] and the references cited therein. An additional operator $\bar{\mathcal{L}}_\delta$ is also used, which is a rescaled nonlocal diffusion operator. The particular forms of the operators adopted here are given by

$$\mathcal{L}_\delta \boldsymbol{u}(\boldsymbol{x}) = \frac{1}{\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x})\big)\mathrm{d}\boldsymbol{y}, \tag{1.6}$$

$$\mathcal{G}_\delta p(\boldsymbol{x}) = \frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{y}-\boldsymbol{x})p(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \tag{1.7}$$

$$\mathcal{D}_\delta \boldsymbol{u}(\boldsymbol{x}) = \frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \tag{1.8}$$

$$\bar{\mathcal{L}}_\delta p(\boldsymbol{x}) = \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(p(\boldsymbol{y}) - p(\boldsymbol{x})\big)\mathrm{d}\boldsymbol{y} \tag{1.9}$$

for some nonnegative and smooth kernels $R_\delta(\boldsymbol{x},\boldsymbol{y})$ and $\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})$ specified later.

Finally, we also need average zero condition for the pressure

$$\int_\Omega p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0. \tag{1.10}$$

(1.4), (1.5) and (1.10) form a complete nonlocal formulation of the Stokes system.

As pointed out in the literature on nonlocal modeling (e.g. [9, 13]), nonlocal integral approximations are closely related to many numerical schemes of computational fluid dynamics, such as the smoothed particle hydrodynamics (SPH) [18, 27, 28, 32], vortex methods [1, 7] and others [3, 5, 15, 21, 40]. Analysis to the linear steady Stokes equation in this paper could give some new understanding to the theoretical foundation of these methods.

The Stokes system (1.1) is well-known to be a saddle point problem. This remains the case for the nonlocal Stokes system given in [13] subject to periodic boundary conditions. Here, different from [13], we add a relaxation term, $\bar{\mathcal{L}}_\delta p_\delta(\boldsymbol{x})$, in Eq. (1.5b). It mimics the classical technique of stabilizing the approximation of incompressibility by adding a positive definite block to the original saddle point system. Although this results in a slightly compressible system, the stabilization term vanishes as $\delta \to 0$ so that it does not destroy the approximation of the nonlocal formulation to the local limit. Yet, this additional term is crucial for the stability and well-posedness in our case where smooth nonlocal kernels are used to define the nonlocal operators. Indeed, the well-posed study in [13] showed that, without extra relaxation, it is necessary to use singular kernels. A remedy was provided in [24] by incorporating non-radial nonlocal interactions. The addition of the relaxation term enables the use of smooth kernels in the definition of the associated nonlocal operators which may allow more flexible practical implementation such as more conventional quadratures for smooth functions. For the Fourier analysis of a related formulation with periodic boundary conditions, we refer to [42].

The rest of the paper is organized as follows. We give the formulation of the nonlocal linear Stokes system in Section 2 together with some related assumptions and estimates. Then the well-posedness of the nonlocal model is established in Section 3. The vanishing nonlocality limit is analyzed in Section 4. In Section 5, we conclude with a summary and a discussion on future research.

2. Nonlocal Stokes system with related assumptions and estimates

In this section we present the nonlocal Stokes model in more details, together with some basic assumptions on the geometry and kernel functions used to define the model, along with some related estimates.

2.1. Notation and assumptions

First, we let Ω_δ and $\mathcal{V}_\delta$ be subsets of Ω defined as

$$\Omega_\delta = \big\{\boldsymbol{x} \in \Omega : B(\boldsymbol{x}, 2\delta) \cap \partial\Omega = \emptyset\big\}, \quad \mathcal{V}_\delta = \Omega \backslash \Omega_\delta. \tag{2.1}$$

The relation of Ω, $\partial\Omega$, Ω_δ and $\mathcal{V}_\delta$ are showed in Fig. 1.

Next, we state the following assumptions on the domain Ω and a kernel function $R(r)$.

Assumption 2.1.

- Assumptions on the computational domain: $\Omega \in \mathbb{R}^n$ is open, bounded and connected. $\partial\Omega$ is C^2 smooth.
- Assumptions on the kernel function $R(r)$:
  (a) (regularity) $R \in C^1[0,1]$;
  (b) (positivity and compact support) $R(r) \geq 0$ and $R(r) = 0$ for $\forall r > 1$;
  (c) (nondegeneracy) $\exists \gamma_0 > 0$ so that $R(r) \geq \gamma_0$ for $0 \leq r \leq \frac{1}{2}$.[†]

Then, the rescaled kernels used in the definitions of the nonlocal operators have the form

$$\begin{aligned} R_\delta(\boldsymbol{x}, \boldsymbol{y}) &= C_\delta R\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{4\delta^2}\right), \\ \bar{R}_\delta(\boldsymbol{x}, \boldsymbol{y}) &= C_\delta \bar{R}\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{4\delta^2}\right), \end{aligned} \tag{2.2}$$

[†]Here $\frac{1}{2}$ can be replaced by any constant in $(0,1)$.

where

$$\bar{R}(r) = \int_r^{+\infty} R(s)\mathrm{d}s = \int_r^1 R(s)\mathrm{d}s, \tag{2.3}$$

which satisfies obviously

$$\bar{R}'(r) = \frac{d}{dr}\bar{R}(r) = R(r), \quad \forall r \in \mathbb{R}^+,$$

$$\bar{R}(r) = 0, \quad \forall r > 1.$$

The constant $C_\delta = \alpha_n \delta^{-n}$ in (2.2) is a normalization factor so that

$$\int_{\mathbb{R}^n} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \alpha_n S_n \int_0^1 \bar{R}\left(\frac{r^2}{4}\right) r^{n-1}\mathrm{d}r = 1 \tag{2.4}$$

with S_n denotes area of the unit sphere in $\mathbb{R}^n$. With this normalization factor, the local limits of $\mathcal{L}_\delta$, $\mathcal{G}_\delta$ and $\mathcal{D}_\delta$ recover the classical Laplacian Δ, gradient and divergence operators respectively as δ goes to 0. Moreover, $\bar{\mathcal{L}}_\delta$ also behaves like a nonlocal analog of $\beta_n\delta^2\Delta$, that is, a scaled nonlocal Laplacian that vanishes in the local limit.

2.2. Nonlocal Stokes system with volume constraint

By combining the volume constraint boundary condition of $\boldsymbol{u}$ and the average zero condition of p, we have the nonlocal Stokes model given as follows:

$$\begin{cases} \mathcal{L}_\delta \boldsymbol{u}_\delta(\boldsymbol{x}) - \mathcal{G}_\delta p_\delta(\boldsymbol{x}) = \displaystyle\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x}\in\Omega_\delta, & \text{(2.5a)}\\ \mathcal{D}_\delta \boldsymbol{u}_\delta(\boldsymbol{x}) - \bar{\mathcal{L}}_\delta p_\delta(\boldsymbol{x}) = 0, \quad \boldsymbol{x}\in\Omega, & \text{(2.5b)}\\ \boldsymbol{u}_\delta(\boldsymbol{x}) = 0, \boldsymbol{x}\in\mathcal{V}_\delta, & \text{(2.5c)}\\ \displaystyle\int_\Omega p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0. & \text{(2.5d)} \end{cases}$$

The integral operators have been defined in (1.6)-(1.9). A formal derivation of the nonlocal model is given in the Appendix A.

Formally, the choices of normalization specified in this paper further imply that the local limits of $\mathcal{L}_\delta$, $\mathcal{G}_\delta$ and $\mathcal{D}_\delta$ recover the classical Laplacian Δ, gradient and divergence operators respectively as δ goes to 0 [9,31]. Moreover, $\bar{\mathcal{L}}_\delta$ also behaves like a nonlocal analog of $\beta_n\delta^2\Delta$, that is, a scaled nonlocal Laplacian that vanishes in the local limit. Thus, we may see (2.5) as a nonlocal extension of the local Stokes model (1.1)-(1.2).

Remark 2.1. For the study of the nonlocal model with periodic boundary condition on $\Omega = (0,1)^n$, we can use Fourier transform to get the Fourier symbols of the nonlocal operators, see the discussion in [42].

2.3. Related estimates

Next, we list several technical results of the kernel functions which will be used in the subsequent analysis.

Lemma 2.1. *Let $R = R(r)$ be a kernel function satisfying Assumption 2.1 and R_δ, $\bar{R}_\delta$ be given by (2.2) and (2.3) respectively.*

(i) There exists a constant $C > 0$, independent of δ such that

$$\begin{aligned}&\left|\nabla_{\boldsymbol{x}} \bar{R}_\delta(\boldsymbol{x}, \boldsymbol{y})\right| \leq \frac{C}{\delta} R_\delta(\boldsymbol{x}, \boldsymbol{y}), \\ &\left|\nabla_{\boldsymbol{y}} \nabla_{\boldsymbol{x}} \bar{R}_\delta(\boldsymbol{x}, \boldsymbol{y})\right| \leq \frac{C}{\delta^2}\Big(\left|R_\delta'(\boldsymbol{x}, \boldsymbol{y})\right| + \left|R_\delta(\boldsymbol{x}, \boldsymbol{y})\right|\Big)\end{aligned}$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, where

$$R_\delta'(\boldsymbol{x}, \boldsymbol{y}) = C_\delta R'\left(\frac{\|\boldsymbol{x} - \boldsymbol{y}\|^2}{4\delta^2}\right), \quad R'(r) = \frac{\mathrm{d}R(r)}{\mathrm{d}r}.$$

(ii) Let $\tilde{R}$ be a kernel function satisfying the Assumption 2.1(a),(b) and

$$\tilde{R}_\delta(\boldsymbol{x}, \boldsymbol{y}) = \alpha_n \delta^{-n} \tilde{R}\left(\frac{\|\boldsymbol{x} - \boldsymbol{y}\|^2}{4\delta^2}\right).$$

There exists a constant $\eta_0 > 0$ only dependent on Ω and $\tilde{R}$ such that for $\delta \leq \eta_0$

$$\frac{\tilde{\omega}_n}{3} < \tilde{\omega}_\delta(\boldsymbol{x}) := \int_\Omega \tilde{R}_\delta(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\boldsymbol{y} \leq \tilde{\omega}_n := \alpha_n S_n \int_0^1 \tilde{R}\left(\frac{r^2}{4}\right) r^{n-1} \mathrm{d}r.$$

(iii) Let

$$K_\delta(\boldsymbol{y}, \mathbf{z}) = \int_\Omega \left|\tilde{R}_\delta(\boldsymbol{x}, \mathbf{z}) \nabla_{\boldsymbol{x}} \tilde{R}_\delta(\boldsymbol{x}, \boldsymbol{y})\right| \mathrm{d}\boldsymbol{x}$$

for any $\boldsymbol{y}, \mathbf{z} \in \mathbb{R}^n$. There exists $C > 0$ independent on δ such that

$$K_\delta(\boldsymbol{y}, \mathbf{z}) \leq C R\left(\frac{\|\boldsymbol{y} - \mathbf{z}\|^2}{32\delta^2}\right).$$

Proof. (i) Can be checked directly.

(ii) This estimate is classical for smooth mollifiers. For the sake of completeness, we give a brief proof here. The upper bound is easy to prove using the non-negativity of $\tilde{R}_\delta$

$$\tilde{\omega}_\delta(\boldsymbol{x}) = \int_\Omega \tilde{R}_\delta(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\boldsymbol{y} \leq \int_{\mathbb{R}^n} \tilde{R}_\delta(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\boldsymbol{y} = \tilde{\omega}_n.$$

To prove the lower bound, we need to use the condition that $\partial\Omega$ is C^2 and $\tilde{R}_\delta$ is continuous and bounded. Then for $\boldsymbol{x} \in \partial\Omega$,

$$\lim_{\delta\to 0} \tilde{\omega}_\delta(\boldsymbol{x}) = \lim_{\delta\to 0} \int_\Omega \tilde{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \alpha_n \int_{\boldsymbol{x}+\mathbb{R}^n_+} \tilde{R}\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{4}\right)\mathrm{d}\boldsymbol{y} = \frac{\tilde{\omega}_n}{2},$$

where $\mathbb{R}^n_+ = \{\boldsymbol{y} = (y_1,\cdots,y_n) \in \mathbb{R}^n : y_1 \geq 0\}$. On the other hand, for $\boldsymbol{x} \in \Omega$, since Ω is open,

$$\lim_{\delta\to 0} \tilde{\omega}_\delta(\boldsymbol{x}) = \lim_{\delta\to 0} \int_\Omega \tilde{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \alpha_n \int_{\mathbb{R}^n} \tilde{R}\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{4}\right)\mathrm{d}\boldsymbol{y} = \tilde{\omega}_n.$$

So, for any $\boldsymbol{x} \in \bar{\Omega} = \Omega \cup \partial\Omega$, there exists $\eta_{\boldsymbol{x}} > 0$ such that for any $\delta \leq \eta_{\boldsymbol{x}}$, we have $\tilde{\omega}_\delta(\boldsymbol{x}) > \tilde{\omega}_n/3$. Using the compactness of $\bar{\Omega}$, there exists $\eta_0 > 0$ such that for any $\boldsymbol{x} \in \bar{\Omega}$, $\delta \leq \eta_0$, we have $\tilde{\omega}_\delta(\boldsymbol{x}) > \tilde{\omega}_n/3$.

(iii) When $\|\boldsymbol{y}-\mathbf{z}\| \geq 4\delta$, we have $\max\{\|\boldsymbol{x}-\mathbf{z}\|, \|\boldsymbol{x}-\boldsymbol{y}\|\} \geq 2\delta$, then using condition (a) and (b) in Assumption 2.1,

$$\tilde{R}_\delta(\boldsymbol{x},\mathbf{z})\nabla_{\boldsymbol{x}}\tilde{R}_\delta(\boldsymbol{x},\boldsymbol{y}) = 0, \quad \forall \boldsymbol{x} \in \Omega.$$

This gives that if $\|\boldsymbol{y}-\mathbf{z}\| \geq 4\delta$,

$$K_\delta(\boldsymbol{y},\mathbf{z}) = 0.$$

When $\|\boldsymbol{y}-\mathbf{z}\| < 4\delta$, we have

$$\frac{\|\boldsymbol{y}-\mathbf{z}\|^2}{32\delta^2} < \frac{1}{2}.$$

Using condition (c) in Assumption 2.1,

$$\begin{aligned}
K_\delta(\boldsymbol{y},\mathbf{z}) &= \int_\Omega \left|\tilde{R}_\delta(\boldsymbol{x},\mathbf{z})\nabla_{\boldsymbol{x}}\tilde{R}_\delta(\boldsymbol{x},\boldsymbol{y})\right|\mathrm{d}\boldsymbol{x} \\
&\leq \frac{1}{4\delta^2}\int_\Omega \|\boldsymbol{x}-\boldsymbol{y}\|\left|\tilde{R}_\delta(\boldsymbol{x},\mathbf{z})\right|\left|\tilde{R}'_\delta(\boldsymbol{x},\boldsymbol{y})\right|\mathrm{d}\boldsymbol{x} \\
&\leq \frac{1}{2\delta}\int_\Omega \left|\tilde{R}_\delta(\boldsymbol{x},\mathbf{z})\right|\left|\tilde{R}'_\delta(\boldsymbol{x},\boldsymbol{y})\right|\mathrm{d}\boldsymbol{x} \\
&\leq \frac{C_\delta^2}{2\delta}\int_{\Omega\cap B(\frac{\boldsymbol{y}+\mathbf{z}}{2},2\delta)} \left|\tilde{R}\left(\frac{\|\boldsymbol{x}-\mathbf{z}\|^2}{4\delta^2}\right)\right|\left|\tilde{R}'\left(\frac{\|\boldsymbol{x}-\mathbf{z}\|^2}{4\delta^2}\right)\right|\mathrm{d}\boldsymbol{x} \\
&\leq \frac{\tilde{M}C_\delta^2}{2\delta}\left|B\left(\frac{\boldsymbol{y}+\mathbf{z}}{2},2\delta\right)\right| \\
&\leq \frac{C\tilde{M}}{\delta\gamma_0}C_\delta\gamma_0 \leq \frac{C\tilde{M}}{\delta\gamma_0}C_\delta R\left(\frac{\|\boldsymbol{y}-\mathbf{z}\|^2}{32\delta^2}\right),
\end{aligned}$$

where $\tilde{M} = \max_{r\in[0,1]}|\tilde{R}(r)\tilde{R}'(r)|$, γ_0 is the constant in condition (c) in Assumption 2.1, C_δ is the normalization factor in (2.2). □

3. Well-posedness of the nonlocal Stokes system (2.5)

In this section, we prove the well-posedness of the nonlocal Stokes system (2.5). More precisely, we show the following theorem.

Theorem 3.1. *Suppose that the Assumption* 2.1 *is satisfied. For any* $\boldsymbol{f} \in H^{-1}(\Omega)$*, there exits one and only one pair* $(\boldsymbol{u}, p)$ *such that*

(a) $\boldsymbol{u} \in H^1(\Omega_\delta)$*,* $p \in L^2(\Omega)$*. In addition,*

$$\|\boldsymbol{u}\|_{H^1(\Omega_\delta)} + \|p\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)},$$

where $C > 0$ *is a constant that only depends on* Ω *and kernel function* R*.*

(b) The pair $(\boldsymbol{u}, p)$ *satisfies the nonlocal Stokes system* (2.5).

In the proof of the well-posedness, we need several technical lemmas.

Lemma 3.1 ([37])**.** *If* δ *is small enough, for any function* $u \in L^2(\Omega)$*, there exists a constant* $C > 0$*, independent of* δ *and* u *such that*

$$\begin{aligned}&\int_\Omega \int_\Omega R\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{32\delta^2}\right)\big(u(\boldsymbol{x}) - u(\boldsymbol{y})\big)^2 \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \\ &\leq C \int_\Omega \int_\Omega R\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{4\delta^2}\right)\big(u(\boldsymbol{x}) - u(\boldsymbol{y})\big)^2 \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}.\end{aligned}$$

Similar results concerning the scaling of the nonlocal interaction neighborhood like the above one can also be found in [14] for other types of kernels including fractional ones.

Next, we consider an extension to a similar result shown in [36].

Lemma 3.2. *For any function* $u \in L_2(\mathbb{R}^n)$ *vanishing outside of* Ω_δ*, i.e.* $u(\boldsymbol{x}) = 0$ *for* $\boldsymbol{x} \in \mathbb{R}^n \backslash \Omega_\delta$*, there exists a constant* $C > 0$ *independent on* δ *such that*

$$\begin{aligned}&\frac{1}{\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta} R_\delta(\boldsymbol{x}, \boldsymbol{y})\big(u(\boldsymbol{x}) - u(\boldsymbol{y})\big)^2 \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \\ &\quad + \frac{1}{\delta^2}\int_{\Omega_\delta} u^2(\boldsymbol{x}) \left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x}, \boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\ &\geq C\int_\Omega |\nabla v|^2 \mathrm{d}\boldsymbol{x},\end{aligned}$$

where

$$\begin{aligned}v(\boldsymbol{x}) &= \frac{1}{\tilde{w}_\delta(\boldsymbol{x})}\int_{\Omega_\delta} \tilde{R}_\delta(\boldsymbol{x}, \boldsymbol{y})u(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \frac{1}{\tilde{w}_\delta(\boldsymbol{x})}\int_\Omega \tilde{R}_\delta(\boldsymbol{x}, \boldsymbol{y})u(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \\ \tilde{w}_\delta(\boldsymbol{x}) &= \int_\Omega \tilde{R}_\delta(\boldsymbol{x}, \boldsymbol{y})\,\mathrm{d}\boldsymbol{y},\end{aligned}$$

and

$$\tilde{R}_\delta(\boldsymbol{x},\boldsymbol{y}) = C_\delta \tilde{R}\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\delta^2}\right),$$

and $\tilde{R}$ is a kernel function satisfying condition (a)-(c) in Assumption 2.1.

Proof. For any $\boldsymbol{x} \in \Omega$, we have

$$\begin{aligned}\nabla v(\boldsymbol{x}) &= \frac{1}{\tilde{w}_\delta(\boldsymbol{x})}\int_\Omega \nabla_{\boldsymbol{x}} \tilde{R}_\delta(\boldsymbol{x},\boldsymbol{y}) u(\boldsymbol{y})\mathrm{d}\boldsymbol{y} - \frac{\nabla \tilde{w}_\delta(\boldsymbol{x})}{\tilde{w}_\delta^2(\boldsymbol{x})}\int_\Omega \big(\tilde{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big)u(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\ &= \frac{1}{\tilde{w}_\delta^2(\boldsymbol{x})}\int_\Omega\int_\Omega \tilde{R}_\delta(\boldsymbol{x},\mathbf{z})\nabla_{\boldsymbol{x}}\tilde{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{y})-u(\mathbf{z})\big)\mathrm{d}\boldsymbol{y}\mathrm{d}\mathbf{z}.\end{aligned}$$

This leads to

$$\begin{aligned}\int_\Omega |\nabla v(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} &= \int_\Omega \frac{1}{\tilde{w}_\delta^4(\boldsymbol{x})}\left|\int_\Omega\int_\Omega \tilde{R}_\delta(\boldsymbol{x},\mathbf{z})\nabla_{\boldsymbol{x}}\tilde{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{y})-u(\mathbf{z})\big)\mathrm{d}\boldsymbol{y}\mathrm{d}\mathbf{z}\right|^2 \mathrm{d}\boldsymbol{x} \\ &\le \frac{1}{\tilde{\omega}_{\min}^4}\int_\Omega \left(\int_\Omega\int_\Omega \big|\tilde{R}_\delta(\boldsymbol{x},\mathbf{z})\nabla_{\boldsymbol{x}}\tilde{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big|\mathrm{d}\boldsymbol{y}\mathrm{d}\mathbf{z}\right) \\ &\quad\times\left(\int_\Omega\int_\Omega \big|\tilde{R}_\delta(\boldsymbol{x},\mathbf{z})\nabla_{\boldsymbol{x}}\tilde{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big|\big(u(\boldsymbol{y})-u(\mathbf{z})\big)^2\mathrm{d}\boldsymbol{y}\mathrm{d}\mathbf{z}\right)\mathrm{d}\boldsymbol{x} \\ &\le \frac{C}{\delta\tilde{\omega}_{\min}^4}\int_\Omega\int_\Omega K_\delta(\boldsymbol{y},\mathbf{z})\big(u(\boldsymbol{y})-u(\mathbf{z})\big)^2\mathrm{d}\boldsymbol{y}\mathrm{d}\mathbf{z}\end{aligned}$$

with

$$\tilde{\omega}_{\min} = \frac{1}{3}\alpha_n S_n \int_0^1 \tilde{R}\left(\frac{r^2}{4}\right) r^{n-1}\mathrm{d}r$$

given in Lemma 2.1 and

$$K_\delta(\boldsymbol{y},\mathbf{z}) = \int_\Omega \big|\tilde{R}_\delta(\boldsymbol{x},\mathbf{z})\nabla_{\boldsymbol{x}}\tilde{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big|\mathrm{d}\boldsymbol{x}.$$

In the last inequality, we use the estimate

$$\begin{aligned}&\int_\Omega\int_\Omega \big|\tilde{R}_\delta(\boldsymbol{x},\mathbf{z})\big|\big|\nabla_{\boldsymbol{x}}\tilde{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big|\mathrm{d}\boldsymbol{y}\mathrm{d}\mathbf{z} \\ &\le \frac{1}{4\delta^2}\int_\Omega\int_\Omega \|\boldsymbol{x}-\boldsymbol{y}\|\big|\tilde{R}'_\delta(\boldsymbol{x},\boldsymbol{y})\big|\big|\tilde{R}_\delta(\boldsymbol{x},\mathbf{z})\big|\mathrm{d}\boldsymbol{y}\mathrm{d}\mathbf{z} \\ &\le \frac{1}{2\delta}\int_\Omega\int_\Omega \big|\tilde{R}'_\delta(\boldsymbol{x},\boldsymbol{y})\big|\big|\tilde{R}_\delta(\boldsymbol{x},\mathbf{z})\big|\mathrm{d}\boldsymbol{y}\mathrm{d}\mathbf{z} \le \frac{C}{\delta}\end{aligned}$$

with

$$\tilde{R}'_\delta(\boldsymbol{x},\boldsymbol{y}) = C_\delta \tilde{R}'\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\delta^2}\right), \quad \tilde{R}'(r) = \frac{\mathrm{d}\tilde{R}}{\mathrm{d}r}.$$

Finally, Lemma 2.1 (ii) gives that

$$\begin{aligned}\int_{\Omega_\delta}|\nabla v(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x} &\le \frac{C}{\delta^2}\int_\Omega\int_\Omega C_\delta R\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{32\delta^2}\right)\big(u(\boldsymbol{x})-u(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\ &\le \frac{C}{\delta^2}\int_\Omega\int_\Omega C_\delta R\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\delta^2}\right)\big(u(\boldsymbol{x})-u(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\ &= \frac{C}{\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{x})-u(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\ &\quad+\frac{2C}{\delta^2}\int_{\mathcal{V}_\delta}\left(\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}.\end{aligned}$$

The second inequality comes from Lemma 3.1. □

Using above lemma, it is easy to get a nonlocal Poincáre inequality for the special kernels, Lemma 3.3.

Lemma 3.3. *For any function $u \in L_2(\mathbb{R}^n)$ vanishing outside of Ω_δ, there exists a constant $C > 0$ independent on δ such that*

$$\begin{aligned}&\frac{1}{\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{x})-u(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\ &\quad+\frac{1}{\delta^2}\int_{\Omega_\delta} u^2(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\ &\ge C\|u\|^2_{L_2(\Omega_\delta)}\end{aligned}$$

if δ is small enough.

Proof. Let

$$v(\boldsymbol{x}) = \frac{1}{w_\delta(\boldsymbol{x})}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \frac{1}{w_\delta(\boldsymbol{x})}\int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})\mathrm{d}\boldsymbol{y}.$$

Using the definition of Ω_δ,

$$v(\boldsymbol{x}) = 0, \quad \forall \boldsymbol{x}\in\partial\Omega.$$

Then Lemma 3.2 and the Poincáre inequality imply that

$$\begin{aligned}\|v\|^2_{L^2(\Omega)} \le \frac{C}{\delta^2}\bigg(&\int_{\Omega_\delta}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{x})-u(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\ &+\int_{\Omega_\delta} u^2(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\bigg).\end{aligned}$$

On the other hand, for $\boldsymbol{x}\in\Omega_\delta$

$$u(x) - v(x) = \frac{1}{w_\delta(\boldsymbol{x})}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{x})-u(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}.$$

Therefore,

$$\begin{aligned}\|u-v\|^2_{L^2(\Omega_\delta)} &\le \frac{1}{\omega_{\min}^2}\int_\Omega\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{x})-u(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\
&\le \frac{\omega_{\max}}{\omega_{\min}^2}\int_\Omega\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{x})-u(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\
&= \frac{\omega_{\max}}{\omega_{\min}^2}\Bigg(\int_{\Omega_\delta}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{x})-u(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&\qquad\qquad + 2\int_{\Omega_\delta} u^2(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\Bigg),\end{aligned}$$

where

$$\omega_{\min}=\frac{1}{3}\alpha_n S_n\int_0^1 R\left(\frac{r^2}{4}\right)r^{n-1}\mathrm{d}r,\quad \omega_{\max}=\alpha_n S_n\int_0^1 R\left(\frac{r^2}{4}\right)r^{n-1}\mathrm{d}r$$

as given in Lemma 2.1. □

Remark 3.1. Support of $\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}$ is a narrow band adjacent to $\partial\Omega$ with the width of 4δ. So the second term in Lemma 3.3, $\frac{1}{\delta^2}\int_{\Omega_\delta} u^2(\boldsymbol{x})(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y})\mathrm{d}\boldsymbol{x}$, is used to control $u(\boldsymbol{x})$ near the boundary while the first term controls the fluctuation in the interior. Lemma 3.3 is actually very natural following the spirit of the Poincáre inequality. For more general discussions, we refer to, e.g., [9,30,31] and the references cited therein.

Lemma 3.4. *For any function $p\in L_2(\Omega)$ with $\int_{\Omega_\delta} p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=0$, there exists a constant $C>0$ independent on δ such that*

$$\frac{1}{\delta^2}\int_\Omega\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(p(\boldsymbol{x})-p(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\ge C\|p\|^2_{L_2(\Omega)}$$

if δ is small enough.

Proof. For p with $\int_{\Omega_\delta} p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=0$, we also have nonlocal Poincáre inequality [37],

$$\|p\|^2_{L^2(\Omega_\delta)}\le\frac{C}{\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(p(\boldsymbol{x})-p(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}.$$

Using nondegeneracy assumption in Assumption 2.1, it is easy to verify that for any $\boldsymbol{x}\in\Omega$,

$$\int_{\Omega_\delta}\bar{R}_{4\delta}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\ge c_0>0,$$

where

$$\bar{R}_{4\delta}(\boldsymbol{x},\boldsymbol{y})=C_\delta\bar{R}\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{4(4\delta)^2}\right),$$

and C_δ is the normalization factor in (2.2)

$$\begin{aligned}
\|p\|^2_{L^2(\Omega)} &\le C\int_\Omega |p(\boldsymbol{x})|^2 \left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&\le C\int_\Omega \left(\int_{\Omega_\delta} |p(\boldsymbol{x})|^2 \bar{R}_{4\delta}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&\le C\int_\Omega \left(\int_{\Omega_\delta} |p(\boldsymbol{x})-p(\boldsymbol{y})|^2 \bar{R}_{4\delta}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&\quad + C\int_\Omega \left(\int_{\Omega_\delta} |p(\boldsymbol{y})|^2 \bar{R}_{4\delta}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&\le C\int_\Omega\int_\Omega |p(\boldsymbol{x})-p(\boldsymbol{y})|^2 \bar{R}_{4\delta}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x} + C\int_{\Omega_\delta}|p(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x} \\
&\le C\int_\Omega\int_\Omega |p(\boldsymbol{x})-p(\boldsymbol{y})|^2 \bar{R}_{\delta}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x} + C\|p\|^2_{L^2(\Omega_\delta)}.
\end{aligned}$$

The proof is complete. □

Now we can prove the main theorem in this section, Theorem 3.1.

Proof. First, in the nonlocal Stokes system, we replace the condition

$$\int_\Omega p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0$$

by

$$\int_{\Omega_\delta} p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0$$

and denote the pressure in the original nonlocal Stokes system as $\bar{p}_\delta$. It is obvious that

$$\bar{p}_\delta = p_\delta - \frac{1}{|\Omega|}\int_\Omega p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{3.1}$$

The existence and uniqueness of the solution to the nonlocal Stokes system is a direct implication of Lax-Milgram theorem by introducing the bilinear form in $\boldsymbol{L}^2_\delta(\Omega)\times\boldsymbol{L}^2_\delta(\Omega)$

$$\begin{aligned}
a\big([\boldsymbol{u},p],[\boldsymbol{v},q]\big) &= \frac{1}{2\delta^2}\int_\Omega\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{u}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{y})\big)\cdot\big(\boldsymbol{v}(\boldsymbol{x})-\boldsymbol{v}(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \\
&\quad + \frac{1}{2\delta^2}\int_\Omega\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{y}-\boldsymbol{x})\cdot\big(\boldsymbol{v}(\boldsymbol{x})p(\boldsymbol{y})-\boldsymbol{u}(\boldsymbol{x})q(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \\
&\quad + \int_\Omega\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(p(\boldsymbol{x})-p(\boldsymbol{y})\big)\big(q(\boldsymbol{x})-q(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y},
\end{aligned} \tag{3.2}$$

where

$$\boldsymbol{L}^2_\delta(\Omega) = \left\{[\boldsymbol{u},p] : \boldsymbol{u}\in L^2(\Omega)^n,\ p\in L^2(\Omega), \mathrm{supp}(\boldsymbol{u})\subset\Omega_\delta, \int_{\Omega_\delta} p(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0\right\}.$$

To apply Lax-Milgram theorem, we need to check the continuity and coercivity of the bilinear form, i.e. for any $[\boldsymbol{u},p],[\boldsymbol{v},q]\in \boldsymbol{L}^2_\delta(\Omega)$,

$$\begin{aligned}&\big|a\big([\boldsymbol{u},p],[\boldsymbol{v},q]\big)\big|\leq C\big(\|\boldsymbol{u}\|_{L^2(\Omega)}+\|p\|_{L^2(\Omega)}\big)\big(\|\boldsymbol{v}\|_{L^2(\Omega)}+\|q\|_{L^2(\Omega)}\big),\\&a\big([\boldsymbol{u},p],[\boldsymbol{u},p]\big)\geq C\left(\|\boldsymbol{u}\|^2_{L^2(\Omega)}+\|p\|^2_{L^2(\Omega)}\right)\end{aligned}$$

with $C>0$ independent on $[\boldsymbol{u},p]$ and $[\boldsymbol{v},q]$. Note that here constant C may depend on δ.

The continuity is easy to check and coercivity can be given by Lemmas 3.3 and 3.4. Then, the existence and uniqueness of the solution is given by Lax-Milgram theorem [16, Section 6.2.1].

In the rest of the proof, we will devote to get the uniform upper bound of $\|\boldsymbol{u}\|^2_{H^1(\Omega_\delta)}+\|p\|^2_{L^2(\Omega)}$. Multiplying $\boldsymbol{u}_\delta$ on Eq. (2.5a) and multiplying p on Eq. (2.5b) and integrating over Ω and adding them together, we can get

$$\begin{aligned}&\frac{1}{\delta^2}\int_\Omega\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}_\delta(\boldsymbol{x})-\boldsymbol{u}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\&\quad+\int_\Omega\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(p_\delta(\boldsymbol{x})-p_\delta(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\&=-2\int_\Omega\left(\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\cdot\boldsymbol{u}_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.\end{aligned}\tag{3.3}$$

From (3.3), using Lemma 3.3, we have

$$\|\boldsymbol{u}_\delta\|^2_{L^2(\Omega)}\leq C\int_\Omega\left(\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\cdot\boldsymbol{u}_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

and

$$\left|\int_\Omega\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\cdot\boldsymbol{u}_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right|\leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\|\tilde{\boldsymbol{u}}_\delta\|_{H^1(\Omega)}$$

with

$$\tilde{\boldsymbol{u}}_\delta(\boldsymbol{y})=\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

Notice that $\boldsymbol{u}_\delta(\boldsymbol{y})=0,\boldsymbol{y}\in\mathcal{V}_\delta$ and $\int_\Omega\nabla_{\boldsymbol{y}}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{x}=0,\boldsymbol{y}\in\Omega_\delta$, so

$$\boldsymbol{u}_\delta(\boldsymbol{y})\int_\Omega\nabla_{\boldsymbol{y}}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{x}=0,\quad \boldsymbol{y}\in\Omega.$$

Then we have

$$\begin{aligned}\|\nabla\tilde{\boldsymbol{u}}_\delta\|^2_{L^2(\Omega)}&=\int_\Omega\left|\int_\Omega\nabla_{\boldsymbol{y}}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2\mathrm{d}\boldsymbol{y}\\&=\int_\Omega\left|\int_\Omega\nabla_{\boldsymbol{y}}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{u}_\delta(\boldsymbol{x})-\boldsymbol{u}_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{x}\right|^2\mathrm{d}\boldsymbol{y}\end{aligned}$$

$$\leq \int_\Omega \left(\int_\Omega |\nabla_{\boldsymbol{y}} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})|\mathrm{d}\boldsymbol{x}\right) \int_\Omega |\nabla_{\boldsymbol{y}} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})| |\boldsymbol{u}_\delta(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$
$$\leq \frac{C}{\delta^2} \int_\Omega \int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y}) |\boldsymbol{u}_\delta(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$

to get the last inequality, Lemma 2.1 is used.

Moreover, it is easy to see that

$$\|\tilde{\boldsymbol{u}}_\delta\|^2_{L^2(\Omega)} \leq C\|\boldsymbol{u}_\delta\|^2_{L^2(\Omega)} \leq \frac{C}{\delta^2} \int_\Omega \int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y}) |\boldsymbol{u}_\delta(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}.$$

Putting above estimates together, we have

$$\begin{aligned} \|\tilde{\boldsymbol{u}}_\delta\|^2_{H^1(\Omega)} &\leq \frac{C}{\delta^2} \int_\Omega \int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y}) |\boldsymbol{u}_\delta(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \\ &\leq C \left| \int_\Omega \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{f}(\boldsymbol{y}) \cdot \boldsymbol{u}_\delta(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \right|. \end{aligned}$$

It follows that

$$\left| \int_\Omega \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{f}(\boldsymbol{y}) \cdot \boldsymbol{u}_\delta(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \right| \leq C\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)}.$$

Hence, we get

$$\|\boldsymbol{u}_\delta\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)}. \tag{3.4}$$

In addition, from Eq. (2.5a), $\boldsymbol{u}_\delta$ has following expression, for any $\boldsymbol{x} \in \Omega_\delta$:

$$\begin{aligned} \boldsymbol{u}_\delta(\boldsymbol{x}) = &\frac{1}{w_\delta(\boldsymbol{x})} \int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{u}_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \\ &+ \frac{1}{2w_\delta(\boldsymbol{x})} \int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y}) p_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \\ &- \frac{\delta^2}{w_\delta(\boldsymbol{x})} \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{f}(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}. \end{aligned} \tag{3.5}$$

Using Lemma 3.2, (3.3) and (3.4), we have

$$\begin{aligned} &\left\| \nabla \left(\frac{1}{w_\delta(\boldsymbol{x})} \int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{u}_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right) \right\|^2_{L^2(\Omega_\delta)} \\ \leq &\frac{C}{\delta^2} \int_\Omega \int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y}) |\boldsymbol{u}_\delta(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \leq C\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)}. \end{aligned} \tag{3.6}$$

Notice that for any $\boldsymbol{x} \in \Omega_\delta, w_\delta(\boldsymbol{x})$ is a positive constant. Then we have

$$\left\| \nabla \left(\frac{1}{2w_\delta(\boldsymbol{x})} \int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y}) p_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right) \right\|^2_{L^2(\Omega_\delta)}$$

$$
\begin{aligned}
&\leq C\int_{\Omega_\delta}\left|\int_\Omega \nabla_{\boldsymbol{x}} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right|^2\mathrm{d}\boldsymbol{x}\\
&\quad + C\int_{\Omega_\delta}\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x}\\
&\leq \frac{C}{\delta^2}\int_{\Omega}\left|\int_\Omega |R'_\delta(\boldsymbol{x},\boldsymbol{y})||\boldsymbol{x}-\boldsymbol{y}|^2|p_\delta(\boldsymbol{y})|\mathrm{d}\boldsymbol{y}\right|^2\mathrm{d}\boldsymbol{x}\\
&\quad + C\int_{\Omega}\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x}\\
&\leq C\int_{\Omega}\left(\int_\Omega |R'_\delta(\boldsymbol{x},\boldsymbol{y})||p_\delta(\boldsymbol{y})|\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x}\\
&\quad + C\int_{\Omega}\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x} \leq C\|p_\delta\|^2_{L^2(\Omega)},
\end{aligned}
\tag{3.7}
$$

where

$$
R'_\delta(\boldsymbol{x},\boldsymbol{y}) = C_\delta R'\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\delta^2}\right),\quad R'(r)=\frac{\mathrm{d}}{\mathrm{d}r}R(r).
$$

In addition, direct calculation gives that

$$
\left\|\nabla\left(\frac{\delta^2}{w_\delta(\boldsymbol{x})}\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\right\|_{L^2(\Omega_\delta)} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)}. \tag{3.8}
$$

For any $v\in L^2(\Omega_\delta)$,

$$
\begin{aligned}
&\int_{\Omega_\delta} v(\boldsymbol{x})\nabla_{\boldsymbol{x}}\left(\frac{\delta^2}{w_\delta(\boldsymbol{x})}\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&=\int_\Omega\left(\int_{\Omega_\delta} v(\boldsymbol{x})\nabla_{\boldsymbol{x}}\left(\frac{\delta^2}{w_\delta(\boldsymbol{x})}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\right)\mathrm{d}\boldsymbol{x}\right)\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\\
&\leq \|\boldsymbol{f}\|_{H^{-1}(\Omega)}\|\tilde{v}\|_{H^1(\Omega_\delta)},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{v}(\boldsymbol{y}) &= \int_{\Omega_\delta} v(\boldsymbol{x})\nabla_{\boldsymbol{x}}\left(\frac{\delta^2}{w_\delta(\boldsymbol{x})}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\right)\mathrm{d}\boldsymbol{x}\\
&= \int_{\Omega_\delta} v(\boldsymbol{x})\frac{\delta^2}{w_\delta(\boldsymbol{x})}\nabla_{\boldsymbol{x}}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Here we use the fact that $\omega_\delta(\boldsymbol{x})$ is a constant over Ω_δ.

Using Lemma 2.1, it is easy to check that

$$
\|\tilde{v}\|_{H^1(\Omega_\delta)} \leq C\|v\|_{L^2(\Omega_\delta)}.
$$

Then (3.8) is obtained. Putting (3.4) and (3.6)-(3.8) together, we obtain

$$
\|\boldsymbol{u}_\delta\|_{H^1(\Omega_\delta)} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + C\|p_\delta\|_{L^2(\Omega)}. \tag{3.9}
$$

Next, we turn to estimate the pressure p. First, considering the problem

$$\nabla \cdot \boldsymbol{v}(\boldsymbol{x}) = p_\delta(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega_\delta. \tag{3.10}$$

It is well known (e.g. [17, Section 3.3]) that if Ω_δ satisfies cone condition, there exists at least one solution of (3.10), denoted by $\boldsymbol{v}$ such that

$$\boldsymbol{v} \in H_0^1(\Omega_\delta), \quad \|\boldsymbol{v}\|_{H^1(\Omega_\delta)} \le c\|p_\delta\|_{L^2(\Omega_\delta)} \tag{3.11}$$

with $c > 0$ independent on δ. Proof of (3.11) can be found in Appendix B. Then, we extend $\boldsymbol{v}$ to Ω by assigning the value on $\mathcal{V}_\delta$ to be 0 and denote the new function also by $\boldsymbol{v}$. Obviously, we have

$$\boldsymbol{v} \in H_0^1(\Omega_\delta) \cap H_0^1(\Omega), \quad \|\boldsymbol{v}\|_{H^1(\Omega)} \le c\|p_\delta\|_{L^2(\Omega_\delta)}. \tag{3.12}$$

On the other hand, using Eq. (2.5b), for any $\boldsymbol{x} \in \Omega$ we have

$$\begin{aligned}
\bar{w}_\delta(\boldsymbol{x})p_\delta(\boldsymbol{x}) &= \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{u}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&= \int_{\Omega_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\nabla\cdot \boldsymbol{v}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{2\delta^2}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{u}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&\quad + \frac{1}{2\delta^2}\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{u}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&= -\frac{1}{2\delta^2}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot \bar{\boldsymbol{v}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y},
\end{aligned} \tag{3.13}$$

where

$$\bar{w}_\delta(\boldsymbol{x}) = \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \quad \bar{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{u}_\delta.$$

Then, it follows that

$$\begin{aligned}
&\frac{1}{2\delta^2}\int_{\Omega_\delta} \bar{\boldsymbol{v}}(\boldsymbol{x})\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&= -\frac{1}{2\delta^2}\int_\Omega p_\delta(\boldsymbol{x})\left(\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\bar{\boldsymbol{v}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega p_\delta^2(\boldsymbol{x})\bar{w}_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_\Omega p_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{3.14}$$

The first term is positive, thus a good term. The second term becomes

$$\begin{aligned}
&-\int_\Omega p_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega p_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&\quad - \int_\Omega p_\delta^2(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{3.15}$$

The second term of (3.15) can be controlled by the first term of (3.14). And the first term is bounded by

$$\begin{aligned}
&\int_{\Omega} p_\delta(\boldsymbol{x}) \left(\int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) \big(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y})\big) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \\
=& \frac{1}{2} \int_{\mathcal{V}_\delta} \int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) \big(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y})\big)^2 \mathrm{d}\boldsymbol{y} \mathrm{d}\boldsymbol{x} \\
&+ \int_{\Omega_\delta} p_\delta(\boldsymbol{x}) \left(\int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) \big(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y})\big) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \\
\geq& \frac{1}{2} \int_{\mathcal{V}_\delta} \int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) \big(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y})\big)^2 \mathrm{d}\boldsymbol{y} \mathrm{d}\boldsymbol{x} \\
&+ \int_{\Omega_\delta} \left(\int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) \big(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y})\big)^2 \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \\
&- \left| \int_{\Omega_\delta} \left(\int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) \big(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y})\big) p_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \right| \\
\geq& \frac{1}{2} \int_{\Omega} \left(\int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) \big(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y})\big)^2 \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \\
&- \frac{1}{2} \int_{\mathcal{V}_\delta} p_\delta^2(\boldsymbol{x}) \left(\int_{\Omega_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{3.16}$$

Combining (3.14)-(3.16), we get

$$\begin{aligned}
&\frac{1}{2\delta^2} \int_{\Omega_\delta} \bar{\boldsymbol{v}}(\boldsymbol{x}) \left(\int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y}) (\boldsymbol{x} - \boldsymbol{y}) p_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \\
\geq& \int_{\Omega_\delta} p_\delta^2(\boldsymbol{x}) \left(\int_{\Omega_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{3.17}$$

Now, we are ready to get the estimate of p_δ. Multiplying Eq. (2.5a) by $\bar{\boldsymbol{v}}$ and integrating over Ω_δ, using the fact that $\bar{\boldsymbol{v}}(\boldsymbol{x}) = 0, \boldsymbol{x} \in \mathcal{V}_\delta$, we have

$$\begin{aligned}
&-\frac{1}{2\delta^2} \int_{\Omega} \int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y}) \big(\boldsymbol{u}_\delta(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{y})\big) \cdot \big(\bar{\boldsymbol{v}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{y})\big) \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{y} \\
&+\frac{1}{2\delta^2} \int_{\Omega_\delta} \bar{\boldsymbol{v}}(\boldsymbol{x}) \left(\int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y}) (\boldsymbol{x} - \boldsymbol{y}) p_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \\
=& \int_{\Omega_\delta} \bar{\boldsymbol{v}}(\boldsymbol{x}) \left(\int_{\Omega_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{f}(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{3.18}$$

Using (3.3), (3.9), (3.12), (3.18) and (3.17), we have

$$\begin{aligned}
\frac{1}{2} \|p\|^2_{L^2(\Omega_\delta)} \leq& \left(\frac{1}{2\delta^2} \int_{\Omega} \int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y}) |\boldsymbol{u}_\delta(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{y} \right)^{\frac{1}{2}} \\
&\times \left(\frac{1}{2\delta^2} \int_{\Omega} \int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y}) |\bar{\boldsymbol{v}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{y} \right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+\|\bar{\boldsymbol{v}}\|_{H^1(\Omega_\delta)}\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\\
\le&\left(\frac{1}{2\delta^2}\int_\Omega\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}_\delta(\boldsymbol{x})-\boldsymbol{u}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{\frac{1}{2}}\\
&\times\left(\left(\frac{1}{2\delta^2}\int_\Omega\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{v}(\boldsymbol{x})-\boldsymbol{v}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{\frac{1}{2}}\right.\\
&\qquad\left.+\left(\frac{1}{2\delta^2}\int_\Omega\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}_\delta(\boldsymbol{x})-\boldsymbol{u}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{\frac{1}{2}}\right)\\
&+\big(\|\boldsymbol{v}\|_{H^1(\Omega_\delta)}+\|\boldsymbol{u}_\delta\|_{H^1(\Omega_\delta)}\big)\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\\
\le&\|\boldsymbol{u}_\delta\|_{H^1(\Omega_\delta)}\|\boldsymbol{f}\|_{H^{-1}(\Omega)}+\|\boldsymbol{u}_\delta\|^{\frac{1}{2}}_{H^1(\Omega_\delta)}\|\boldsymbol{f}\|^{\frac{1}{2}}_{H^{-1}(\Omega)}\|\boldsymbol{v}\|_{H^1(\Omega_\delta)}\\
&+C\big(\|p_\delta\|_{L^2(\Omega_\delta)}+\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\big)\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\\
\le&C\big(\|p_\delta\|_{L^2(\Omega_\delta)}+\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\big)\|\boldsymbol{f}\|_{H^{-1}(\Omega)}.
\end{aligned}
\tag{3.19}
$$

Using (3.3), (3.9), (3.19) and Lemma 3.4 yields

$$
\begin{aligned}
\|p_\delta\|^2_{L^2(\Omega)}&\le C\|\boldsymbol{u}_\delta\|_{L^2(\Omega)}\|\boldsymbol{f}\|_{H^{-1}(\Omega)}+C\|p_\delta\|^2_{L^2(\Omega_\delta)}\\
&\le C\big(\|p_\delta\|_{L^2(\Omega)}+\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\big)\|\boldsymbol{f}\|_{H^{-1}(\Omega)},
\end{aligned}
$$

which implies

$$
\|p_\delta\|_{L^2(\Omega)}\le C\|\boldsymbol{f}\|_{H^{-1}(\Omega)}. \tag{3.20}
$$

This also gives the H^1 estimate of $\boldsymbol{u}_\delta$ using (3.9),

$$
\|\boldsymbol{u}_\delta\|_{H^1(\Omega_\delta)}\le C\|\boldsymbol{f}\|_{H^{-1}(\Omega)} \tag{3.21}
$$

and using (3.1),

$$
\|\bar{p}_\delta\|_{L^2(\Omega)}\le\|p_\delta\|_{L^2(\Omega)}+\frac{1}{|\Omega|}\left|\int_\Omega p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|\le C\|\boldsymbol{f}\|_{H^{-1}(\Omega)}. \tag{3.22}
$$

Note that in the above, the fact that

$$
\frac{1}{|\Omega|}\left|\int_\Omega p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|\le\frac{1}{\sqrt{|\Omega|}}\|p_\delta\|_{L^2(\Omega)}
$$

is used. □

4. Vanishing nonlocality

Besides the well-posedness, we are also interested in the limiting behavior of the nonlocal Stokes system (2.5) as the nonlocality vanishes, i.e. $\delta\to 0$. In this section, under some assumptions, we prove that solutions of the nonlocal Stokes system converge to the solution of the Stokes system as $\delta \to 0$. Furthermore, we give an estimate on the convergence rate. The result is summarized in Theorem 4.2.

Before stating the main theorem, we give several technical results that are used to prove the main theorem.

We also need the following theorem on the order of the nonlocal approximation which can be proved via simple Taylor expansion.

Theorem 4.1. *Let*

$$r(\boldsymbol{x}) = -\frac{1}{\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{x})-u(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y} - \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\Delta u(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \quad \forall \boldsymbol{x}\in\Omega_\delta.$$

There exist constants C, T_0 depending only on Ω such that for any $\delta \le T_0$, for $u \in H^3(\Omega)$,

$$\|r(\boldsymbol{x})\|_{L^2(\Omega_\delta)} \le C\delta\|u\|_{H^3(\Omega)}, \tag{4.1}$$

$$\|\nabla r(\boldsymbol{x})\|_{L^2(\Omega_\delta)} \le C\|u\|_{H^3(\Omega)}. \tag{4.2}$$

We then have the main result of this section regarding the convergence of the nonlocal Stokes system as the nonlocality vanishes.

Theorem 4.2. *Let $\boldsymbol{u}(\boldsymbol{x}), p(\boldsymbol{x})$ be solution of Stokes system* (1.1) *and $\boldsymbol{u}_\delta(\boldsymbol{x}), p_\delta(\boldsymbol{x})$ be solution of nonlocal Stokes system* (2.5) *with $\boldsymbol{f}\in H^1(\Omega)$. There exists a constant $C>0$ that only depends on Ω and R such that*

$$\|\boldsymbol{u}-\boldsymbol{u}_\delta\|_{H^1(\Omega_\delta)} + \|p-p_\delta\|_{L^2(\Omega)} \le C\sqrt{\delta}\|f\|_{H^1(\Omega)}.$$

Proof. Let

$$\boldsymbol{e}_\delta(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{x})$$

and

$$d_\delta = p - p_\delta - \frac{1}{|\Omega_\delta|}\int_{\Omega_\delta}\big(p(\boldsymbol{x})-p_\delta(\boldsymbol{x})\big)\mathrm{d}\boldsymbol{x},$$

then $\boldsymbol{e}_\delta$ and d_δ satisfy

$$\begin{cases} -\dfrac{1}{\delta^2}\displaystyle\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y} \\ \quad +\dfrac{1}{2\delta^2}\displaystyle\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \boldsymbol{r}_{\boldsymbol{u}}(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega_\delta, & (4.3\text{a}) \\ \boldsymbol{e}_\delta(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathcal{V}_\delta, & (4.3\text{b}) \\ \dfrac{1}{2\delta^2}\displaystyle\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\ \quad -\displaystyle\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y} = r_p(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega, & (4.3\text{c}) \\ \displaystyle\int_{\Omega_\delta} d_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0, & (4.3\text{d}) \end{cases}$$

where

$$\boldsymbol{r}_{\boldsymbol{u}}(\boldsymbol{x}) = \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\Delta \boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{u}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}, \quad \forall \boldsymbol{x}\in\Omega_\delta, \tag{4.4}$$

$$r_p(\boldsymbol{x}) = -\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(p(\boldsymbol{x})-p(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}, \quad \forall \boldsymbol{x}\in\Omega. \tag{4.5}$$

First, we focus on the following estimate:

$$\begin{aligned}
&\frac{1}{\delta^2}\int_{\Omega_\delta} \boldsymbol{e}_\delta(\boldsymbol{x})\cdot\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\
=&\frac{1}{\delta^2}\int_{\Omega_\delta} \boldsymbol{e}_\delta(\boldsymbol{x})\cdot\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\
&+\frac{1}{\delta^2}\int_{\Omega_\delta} \boldsymbol{e}_\delta(\boldsymbol{x})\cdot\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\
=&\frac{1}{2\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&+\frac{1}{\delta^2}\int_{\Omega_\delta} \boldsymbol{e}_\delta(\boldsymbol{x})\cdot\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{4.6}$$

The second term of the right-hand side of (4.6) can be calculated as

$$\begin{aligned}
&\frac{1}{\delta^2}\int_{\Omega_\delta} \boldsymbol{e}_\delta(\boldsymbol{x})\cdot\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\
=&\frac{1}{\delta^2}\int_{\Omega_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&-\frac{1}{\delta^2}\int_{\Omega_\delta}\boldsymbol{e}_\delta(\boldsymbol{x})\cdot\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{4.7}$$

Here we use the definition of $\boldsymbol{e}_\delta$ and the volume constraint condition $\boldsymbol{u}_\delta(\boldsymbol{x})=0, \boldsymbol{x}\in\mathcal{V}_\delta$ to get that $\boldsymbol{e}_\delta(\boldsymbol{x}) = u(\boldsymbol{x}), \boldsymbol{x}\in\mathcal{V}_\delta$.

The first term is positive which is good for us. We only need to bound the second term of (4.7). First, the second term can be bounded as following:

$$\begin{aligned}
&\frac{1}{\delta^2}\left|\int_{\Omega_\delta}\boldsymbol{e}_\delta(\boldsymbol{x})\cdot\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right|\\
\le&\frac{1}{\delta^2}\int_{\Omega_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)^{\frac{1}{2}}\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)^{\frac{1}{2}}\mathrm{d}\boldsymbol{x}\\
\le&\frac{1}{\delta^2}\left(\int_{\Omega_\delta}\frac{1}{2}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}+2\int_{\Omega_\delta}\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right)\\
\le&\frac{1}{2\delta^2}\int_{\Omega_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}+\frac{2}{\delta^2}\int_{\mathcal{V}_\delta}|\boldsymbol{u}(\boldsymbol{y})|^2\left(\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{x}\right)\mathrm{d}\boldsymbol{y}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2\delta^2}\int_{\Omega_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}+\frac{C}{\delta^2}\int_{\mathcal{V}_\delta}|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\\
&\le \frac{1}{2\delta^2}\int_{\Omega_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}+C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)}. \qquad (4.8)
\end{aligned}
$$

Here we use Lemma B.1 in Appendix B to get the last inequality. By substituting (4.8), (4.7) in (4.6), we get

$$
\begin{aligned}
&\left|\frac{1}{\delta^2}\int_{\Omega_\delta}\boldsymbol{e}_\delta(\boldsymbol{x})\cdot\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\right|\\
\ge{}& \frac{1}{2\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&+\frac{1}{2\delta^2}\int_{\Omega_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}-C\|\boldsymbol{f}\|^2_{H^1(\Omega)}\delta. \qquad (4.9)
\end{aligned}
$$

This is the key estimate to show the convergence.

We also need the following bound:

$$
\begin{aligned}
&\left|\frac{1}{\delta^2}\int_{\Omega_\delta}\boldsymbol{e}_\delta(\boldsymbol{x})\cdot\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right.\\
&\qquad\left.+\frac{1}{\delta^2}\int_\Omega d_\delta(\boldsymbol{x})\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right|\\
={}&\left|\frac{1}{\delta^2}\int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right|\\
\le{}&\frac{1}{\delta}\int_\Omega\left(\int_{\mathcal{V}_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})|d_\delta(\boldsymbol{x})||\boldsymbol{u}(\boldsymbol{y})|\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
\le{}&\frac{1}{\delta}\left[\int_\Omega\left(\int_{\mathcal{V}_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})|d_\delta(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\int_\Omega\left(\int_{\mathcal{V}_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right]^{\frac{1}{2}}\\
\le{}&C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}. \qquad (4.10)
\end{aligned}
$$

Multiplying $\boldsymbol{e}_\delta(\boldsymbol{x}), d_\delta(\boldsymbol{x})$ on both sides of Eqs. (4.3a) and (4.3c) and integrating over Ω_δ,Ω respectively and adding them together, using (4.9), (4.10), we have

$$
\begin{aligned}
&\frac{1}{\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&\quad+\frac{1}{2\delta^2}\int_{\Omega_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&\quad+\int_\Omega\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})|d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
\le{}&\big(\|\boldsymbol{r}_{\boldsymbol{u}}\|_{L^2(\Omega_\delta)}\big)\|\boldsymbol{e}_\delta\|_{L^2(\Omega_\delta)}+\|r_p\|_{L^2(\Omega)}\|d_\delta\|_{L^2(\Omega)}\\
&\quad+C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}+C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)}. \qquad (4.11)
\end{aligned}
$$

To simplify the notation, we denote the right hand side of (4.11) as Q^2.

It is well known (e.g. [17, Section 3.3]) that with the condition that

$$\int_{\Omega_\delta} d_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0,$$

there exists at least one function $\boldsymbol{\psi} \in H_0^1(\Omega_\delta)$ such that

$$\nabla \cdot \boldsymbol{\psi}(\boldsymbol{x}) = d_\delta(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega_\delta, \quad \text{and} \quad \|\boldsymbol{\psi}\|_{H^1(\Omega_\delta)} \le c\|d_\delta\|_{L^2(\Omega_\delta)} \tag{4.12}$$

and c is a constant independent on δ, the proof can be found in Appendix C.

Then, we extend $\boldsymbol{\psi}$ to Ω by assigning the value on $\mathcal{V}_\delta$ to be 0 and denote the new function also by $\boldsymbol{\psi}$. Obviously, we have

$$\boldsymbol{\psi} \in H_0^1(\Omega_\delta) \cap H_0^1(\Omega), \quad \|\boldsymbol{\psi}\|_{H^1(\Omega)} \le c\|d_\delta\|_{L^2(\Omega_\delta)}. \tag{4.13}$$

Using Eq (4.3c), we have

$$\begin{aligned}
\bar{w}_\delta(\boldsymbol{x})d_\delta(\boldsymbol{x}) &= \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} - r_p(\boldsymbol{x}) \\
&= \int_{\Omega_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\nabla\cdot\boldsymbol{\psi}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{2\delta^2}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&\quad + \frac{1}{2\delta^2}\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} - r_p(\boldsymbol{x}) \\
&= -\frac{1}{2\delta^2}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\bar{\boldsymbol{\psi}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{2\delta^2}\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&\quad + \int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} - r_p(\boldsymbol{x}),
\end{aligned} \tag{4.14}$$

where

$$\bar{w}_\delta(\boldsymbol{x}) = \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \quad \bar{\boldsymbol{\psi}} = \boldsymbol{\psi} - \boldsymbol{e}_\delta.$$

Then, it follows that

$$\begin{aligned}
&\frac{1}{2\delta^2}\int_{\Omega_\delta} \bar{\boldsymbol{\psi}}(\boldsymbol{x})\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&= -\frac{1}{2\delta^2}\int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\bar{\boldsymbol{\psi}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega d_\delta^2(\boldsymbol{x})\bar{w}_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&\quad - \int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} + \int_\Omega d_\delta(\boldsymbol{x})r_p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{4.15}$$

The first term is positive which is a good term. The second term becomes

$$\begin{aligned}
&-\int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
=&\int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&-\frac{1}{2\delta^2}\int_\Omega d_\delta^2(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}. \qquad (4.16)
\end{aligned}$$

The second term of (4.16) can be controlled by the first term of (4.15). And the first term is bounded by

$$\begin{aligned}
&\int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
=&\frac{1}{2}\int_{\mathcal{V}_\delta}\int_{\mathcal{V}_\delta}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\
&+\int_{\Omega_\delta} d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
\geq&\frac{1}{2}\int_{\mathcal{V}_\delta}\int_{\mathcal{V}_\delta}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\
&+\int_{\Omega_\delta}\left(\int_{\mathcal{V}_\delta}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&-\left|\int_{\Omega_\delta}\left(\int_{\mathcal{V}_\delta}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})\big)d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right|\\
\geq&\frac{1}{2}\int_\Omega\left(\int_{\mathcal{V}_\delta}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\big(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})\big)^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&-\frac{1}{2}\int_{\mathcal{V}_\delta} d_\delta^2(\boldsymbol{x})\left(\int_{\Omega_\delta}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}. \qquad (4.17)
\end{aligned}$$

Combining (4.15)-(4.17), we get

$$\begin{aligned}
&\frac{1}{2\delta^2}\int_{\Omega_\delta}\bar{\boldsymbol{\psi}}(\boldsymbol{x})\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
\geq&\int_{\Omega_\delta} d_\delta^2(\boldsymbol{x})\left(\int_{\Omega_\delta}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&-\frac{1}{2\delta^2}\int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&+\int_\Omega d_\delta(\boldsymbol{x})r_p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \qquad (4.18)
\end{aligned}$$

In addition, we have

$$\left|\frac{1}{2\delta^2}\int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right|$$

$$
\begin{aligned}
&\le \frac{1}{2\delta}\int_\Omega |d_\delta(\boldsymbol{x})|\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}(\boldsymbol{y})|\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&\le \frac{1}{2\delta}\left[\int_\Omega |d_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\int_\Omega\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right]^{\frac{1}{2}}\\
&\le C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)} \qquad (4.19)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\int_\Omega d_\delta(\boldsymbol{x})r_p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right| &= \left|\int_\Omega d_\delta(\boldsymbol{x})\left(\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(p(\boldsymbol{x})-p(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right|\\
&\le C\delta\|p\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)} \le C\delta\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}. \qquad (4.20)
\end{aligned}
$$

Multiplying Eq. (4.3a) by $\bar{\boldsymbol{\psi}}$ and using (4.18)-(4.20), we have

$$
\begin{aligned}
\|d_\delta\|^2_{L^2(\Omega_\delta)} \le\, & \frac{1}{\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})\big)\cdot\big(\bar{\boldsymbol{\psi}}(\boldsymbol{x})-\bar{\boldsymbol{\psi}}(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&+\frac{1}{\delta^2}\int_{\Omega_\delta}\bar{\boldsymbol{\psi}}(\boldsymbol{x})\cdot\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&+\|\bar{\boldsymbol{\psi}}\|_{L^2(\Omega_\delta)}\big(\|\boldsymbol{r}_{\boldsymbol{u}}\|_{L^2(\Omega_\delta)}\big)+C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}. \qquad (4.21)
\end{aligned}
$$

The first term can be bounded as

$$
\begin{aligned}
&\left|\frac{1}{\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})\big)\cdot\big(\bar{\boldsymbol{\psi}}(\boldsymbol{x})-\bar{\boldsymbol{\psi}}(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right|\\
\le&\left(\frac{1}{\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{\frac{1}{2}}\\
&\times\left(\frac{1}{\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\bar{\boldsymbol{\psi}}(\boldsymbol{x})-\bar{\boldsymbol{\psi}}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{\frac{1}{2}}\\
\le&\left(\frac{1}{\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{\frac{1}{2}}\\
&\times\left(\left(\frac{1}{\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{\psi}(\boldsymbol{x})-\boldsymbol{\psi}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{\frac{1}{2}}\right.\\
&\qquad\left.+\left(\frac{1}{\delta^2}\int_{\Omega_\delta}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{\frac{1}{2}}\right)\\
\le&\, Q^2+CQ\|\boldsymbol{\psi}\|_{H^1(\Omega_\delta)}\le Q^2+CQ\|d_\delta\|_{L^2(\Omega_\delta)}. \qquad (4.22)
\end{aligned}
$$

The estimate of the second term of (4.21) is more involved. First

$$
\left|\frac{1}{\delta^2}\int_{\Omega_\delta}\bar{\boldsymbol{\psi}}(\boldsymbol{x})\cdot\left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right|
$$

$$
\begin{aligned}
&\le \left| \frac{1}{\delta^2} \int_{\Omega_\delta} \left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) \big(\bar{\boldsymbol{\psi}}(\boldsymbol{x}) - \bar{\boldsymbol{\psi}}(\boldsymbol{y})\big) \cdot \big(\boldsymbol{e}_\delta(\boldsymbol{x}) - \boldsymbol{e}_\delta(\boldsymbol{y})\big) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \right| \\
&\quad + \left| \frac{1}{\delta^2} \int_{\mathcal{V}_\delta} \boldsymbol{u}(\boldsymbol{x}) \cdot \left(\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) \big(\boldsymbol{e}_\delta(\boldsymbol{x}) - \boldsymbol{e}_\delta(\boldsymbol{y})\big) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \right| \\
&\le \left[\left(\frac{1}{\delta^2} \int_{\Omega_\delta} \left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) |\bar{\boldsymbol{\psi}}(\boldsymbol{x}) - \bar{\boldsymbol{\psi}}(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left(\frac{1}{\delta^2} \int_{\mathcal{V}_\delta} |\boldsymbol{u}(\boldsymbol{x})|^2 \left(\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{2}} \right] \\
&\quad \times \left(\frac{1}{\delta^2} \int_{\Omega_\delta} \left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) |\boldsymbol{e}_\delta(\boldsymbol{x}) - \boldsymbol{e}_\delta(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{2}} \\
&\le C\big(\|\boldsymbol{\psi}\|_{H^1(\Omega)} + \sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\big) \\
&\quad \times \left(\frac{1}{\delta^2} \int_{\Omega_\delta} \left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) |\boldsymbol{e}_\delta(\boldsymbol{x}) - \boldsymbol{e}_\delta(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{2}} \\
&\quad + \frac{C}{\delta^2} \int_{\Omega_\delta} \left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) |\boldsymbol{e}_\delta(\boldsymbol{x}) - \boldsymbol{e}_\delta(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x}. \qquad (4.23)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\frac{1}{\delta^2} \int_{\Omega_\delta} \left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) |\boldsymbol{e}_\delta(\boldsymbol{x}) - \boldsymbol{e}_\delta(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \\
&\le \frac{2}{\delta^2} \int_{\Omega_\delta} \left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) |\boldsymbol{e}_\delta(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} + \frac{2}{\delta^2} \int_{\Omega_\delta} \left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) |\boldsymbol{e}_\delta(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \\
&\le \frac{2}{\delta^2} \int_{\Omega_\delta} |\boldsymbol{e}_\delta(\boldsymbol{x})|^2 \left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} + \frac{2}{\delta^2} \int_{\Omega_\delta} \left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) |\boldsymbol{u}(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \\
&\le Q^2 + C\delta \|\boldsymbol{f}\|^2_{H^1(\Omega)}. \qquad (4.24)
\end{aligned}
$$

Combining (4.23) and (4.24), we get

$$
\begin{aligned}
&\left| \frac{1}{\delta^2} \int_{\Omega_\delta} \bar{\boldsymbol{\psi}}(\boldsymbol{x}) \cdot \left(\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) \big(\boldsymbol{e}_\delta(\boldsymbol{x}) - \boldsymbol{e}_\delta(\boldsymbol{y})\big) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \right| \\
&\le \big(\|d_\delta\|_{L^2(\Omega_\delta)} + \sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\big)\big(Q + \sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\big) + Q^2 + \delta\|\boldsymbol{f}\|^2_{H^1(\Omega)}. \qquad (4.25)
\end{aligned}
$$

Substituting (4.22) and (4.25) in (4.21),

$$
\begin{aligned}
\|d_\delta\|^2_{L^2(\Omega_\delta)} &\le Q^2 + C\big(\|d_\delta\|_{L^2(\Omega_\delta)} + \sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\big)\big(Q + \sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\big) \\
&\quad + \|\bar{\boldsymbol{\psi}}\|_{L^2(\Omega_\delta)} \|\boldsymbol{r_u}\|_{L^2(\Omega)} + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)} \|d_\delta\|_{L^2(\Omega)} \\
&\le Q^2 + C\big(\|d_\delta\|_{L^2(\Omega_\delta)} + \sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\big)\big(Q + \sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\big) \\
&\quad + \big(\|d_\delta\|_{L^2(\Omega_\delta)} + \|\boldsymbol{e}_\delta\|_{L^2(\Omega_\delta)}\big) \big(\|\boldsymbol{r_u}\|_{L^2(\Omega_\delta)}\big) \\
&\quad + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)} \|d_\delta\|_{L^2(\Omega)}. \qquad (4.26)
\end{aligned}
$$

On the other hand, using Lemma 3.4, we have

$$\|d_\delta\|^2_{L^2(\Omega)} \le C\int_\Omega\int_\Omega |d_\delta(\boldsymbol{x}) - d_\delta(\boldsymbol{y})|^2 R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x} + C\|d_\delta\|^2_{L^2(\Omega_\delta)}. \tag{4.27}$$

Then it follows from (4.11) and above inequality

$$\begin{aligned}\|d_\delta\|^2_{L^2(\Omega)} \le\, & Q^2 + C\big(\|d_\delta\|_{L^2(\Omega)} + \sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\big)\big(Q + \sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\big)\\ & + \big(\|d_\delta\|_{L^2(\Omega)} + \|\boldsymbol{e}_\delta\|_{L^2(\Omega_\delta)}\big)\big(\|\boldsymbol{r_u}\|_{L^2(\Omega_\delta)}\big).\end{aligned} \tag{4.28}$$

Theorem 4.1 gives that

$$\|\boldsymbol{r_u}\|_{L^2(\Omega)} \le C\delta\|\boldsymbol{f}\|_{H^1(\Omega)}, \quad \|r_p\|_{L^2(\Omega)} \le C\delta\|\boldsymbol{f}\|_{H^1(\Omega)}. \tag{4.29}$$

Following Lemma 3.3 and (4.11), we have

$$\begin{aligned}\|\boldsymbol{e}_\delta\|^2_{L^2(\Omega_\delta)} \le Q^2 \le\, & C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|\boldsymbol{e}_\delta\|_{L^2(\Omega_\delta)} + C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)}\\ & + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)},\end{aligned}$$

which implies that

$$\|\boldsymbol{e}_\delta\|^2_{L^2(\Omega_\delta)} \le C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)} + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}. \tag{4.30}$$

Consequently, Q^2 is bounded by

$$Q^2 \le C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)} + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}. \tag{4.31}$$

Now, we have the bound of $\|d_\delta\|_{L^2(\Omega)}$ from (4.28) and (4.31),

$$\begin{aligned}\|d_\delta\|^2_{L^2(\Omega)} \le\, & C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)} + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}\\ & + \big(\|d_\delta\|_{L^2(\Omega)} + \sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\big)Q\\ \le\, & C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)} + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}\\ & + \left(\frac{1}{2}\|d_\delta\|^2_{L^2(\Omega)} + \delta\|\boldsymbol{f}\|^2_{H^1(\Omega)}\right).\end{aligned}$$

Therefore

$$\|d_\delta\|^2_{L^2(\Omega)} \le C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)} + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}. \tag{4.32}$$

Then the bound of $\|d_\delta\|_{L^2(\Omega)}$ is obtained

$$\|d_\delta\|_{L^2(\Omega)} \le C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}. \tag{4.33}$$

The bound of $\|\boldsymbol{e}_\delta\|_{L^2(\Omega_\delta)}$ follows from (4.30) and (4.33),

$$\|\boldsymbol{e}_\delta\|_{L^2(\Omega_\delta)} \le C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)} \tag{4.34}$$

and

$$\|p - p_\delta\|_{L^2(\Omega)} \le \|d_\delta\|_{L^2(\Omega)} + |\bar{d}_\delta| \le C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}, \tag{4.35}$$

where

$$\bar{d}_\delta = \frac{1}{|\Omega|}\int_\Omega d_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

and we use the fact that

$$|\bar{d}_\delta| = \frac{1}{|\Omega|}\left|\int_\Omega d_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right| \le \frac{1}{\sqrt{|\Omega|}}\|d_\delta\|_{L^2(\Omega)}.$$

Finally, the bound of $\|\boldsymbol{e}_\delta\|_{H^1(\Omega_\delta)}$ can be derived from

$$\begin{aligned}\boldsymbol{e}_\delta(\boldsymbol{x}) = &\frac{1}{w_\delta(\boldsymbol{x})}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\ &+ \frac{1}{2w_\delta(\boldsymbol{x})}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} - \delta^2\boldsymbol{r}_{\boldsymbol{u}}(\boldsymbol{x}).\end{aligned} \tag{4.36}$$

We are left with estimating the three terms on the right hand side one by one. The third term is easy to bound using Theorem 4.1,

$$\left\|\delta^2\nabla\boldsymbol{r}_{\boldsymbol{u}}(\boldsymbol{x})\right\|_{L^2(\Omega_\delta)} \le \delta^2\|\boldsymbol{f}\|_{H^1(\Omega)}.$$

Notice that for any $\boldsymbol{x}\in\Omega_\delta$, $w_\delta(\boldsymbol{x})$ is a positive constant. Then we have

$$\begin{aligned}&\left\|\nabla\left(\frac{1}{2w_\delta(\boldsymbol{x})}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\right\|^2_{L^2(\Omega_\delta)} \\ \le\, &C\int_{\Omega_\delta}\left|\int_\Omega \nabla_{\boldsymbol{x}}R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right|^2\mathrm{d}\boldsymbol{x} + C\int_{\Omega_\delta}\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x} \\ \le\, &\frac{C}{\delta^2}\int_\Omega\left|\int_\Omega |R'_\delta(\boldsymbol{x},\boldsymbol{y})||\boldsymbol{x}-\boldsymbol{y}|^2 d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right|^2\mathrm{d}\boldsymbol{x} + C\int_\Omega\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x} \\ \le\, &C\int_\Omega\left(\int_\Omega |R'_\delta(\boldsymbol{x},\boldsymbol{y})|d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x} + C\int_\Omega\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x} \\ \le\, &C\|d_\delta\|^2_{L^2(\Omega)} \le C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)},\end{aligned}$$

where

$$R'_\delta(\boldsymbol{x},\boldsymbol{y}) = C_\delta R'\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\delta^2}\right), \quad R'(r) = \frac{\mathrm{d}}{\mathrm{d}r}R(r).$$

The first term of (4.36) can be split into two terms

$$\begin{aligned}&\frac{1}{w_\delta(\boldsymbol{x})}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\ =\, &\frac{1}{w_\delta(\boldsymbol{x})}\int_{\Omega_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{w_\delta(\boldsymbol{x})}\int_{\mathcal{V}_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}.\end{aligned}$$

Using Lemma B.1,

$$\left\|\nabla\left(\frac{1}{w_\delta(\boldsymbol{x})}\int_{\mathcal{V}_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\right\|_{L^2(\Omega_\delta)}\le C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}.$$

And it follows from Lemma 3.2 and (4.11),

$$\left\|\nabla\left(\frac{1}{w_\delta(\boldsymbol{x})}\int_{\Omega_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\right\|_{L^2(\Omega_\delta)}\le C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}.$$

Hence, the proof is complete. □

5. Discussion and conclusion

In this paper, we propose a nonlocal model for linear steady Stokes equation with no-slip boundary condition. The main idea is to use volume constraint to enforce the no-slip boundary condition and add a relaxation term in the divergence free condition to maintain the well-posedness of the nonlocal system. As the nonlocal horizon parameter δ approaches 0, the solution of the nonlocal system converges to the solution of the original Stoke equation, assuming that the solution to the latter is sufficiently smooth.

In terms of future work, one may examine the convergence with minimal regularity assumptions on the local systems. It is also interesting to consider the numerical discretizations. From the nonlocal system, we can derive a numerical scheme for the original Stokes system on point cloud. Assume we are given a set of sample points P sampling the domain Ω and a subset $S\subset P$ sampling the boundary of Ω. In addition, assume we are given one vector $\mathbf{V}=(V_1,\cdots,V_n)^t$ where V_i is an volume weight of $\boldsymbol{x}_i$ in Ω, so that for any C^1 function f on Ω, $\int_\Omega f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ can be approximated by $\sum_{\boldsymbol{x}_i\in\Omega}f(\boldsymbol{x}_i)V_i$.

Then, the nonlocal Stokes system (2.5) can be discretized as following:

$$\begin{aligned}&-\frac{1}{\delta^2}\sum_{\boldsymbol{x}_j\in\Omega}R_\delta(\boldsymbol{x}_i,\boldsymbol{x}_j)(\boldsymbol{u}_i-\boldsymbol{u}_j)V_j+\frac{1}{2\delta^2}\sum_{\boldsymbol{x}_j\in\Omega}R_\delta(\boldsymbol{x}_i,\boldsymbol{x}_j)(\boldsymbol{x}_i-\boldsymbol{x}_j)p_jV_j\\&=\sum_{\boldsymbol{x}_j\in\Omega}\bar{R}_\delta(\boldsymbol{x}_i,\boldsymbol{x}_j)\boldsymbol{f}_jV_j,\quad \boldsymbol{x}_i\in\Omega_\delta,\\&\frac{1}{2\delta^2}\sum_{\boldsymbol{x}_j\in\Omega}R_\delta(\boldsymbol{x}_i,\boldsymbol{x}_j)(\boldsymbol{x}_i-\boldsymbol{x}_j)\boldsymbol{u}_jV_j-\sum_{\boldsymbol{x}_j\in\Omega}\bar{R}_\delta(\boldsymbol{x}_i,\boldsymbol{x}_j)(p_i-p_j)V_j=0,\quad \boldsymbol{x}_i\in\Omega,\\&\boldsymbol{u}_i=0,\quad \boldsymbol{x}_i\in\mathcal{V}_\delta.\end{aligned}$$

This scheme is very simple and easy to implement. However, the accuracy is relatively low. We can show that the error of above scheme is $\mathcal{O}(\frac{h}{\delta^2}+\delta)$, where h is the average distance among the sample points in P. The first term $\frac{h}{\delta^2}$ comes from the error of the numerical integral and the second term δ is from error between nonlocal system and the original Stoke equation. Further improvement and studies of asymptotically compatible scheme [39] are interesting questions to be explored further.

Appendix A. Formal derivation of the nonlocal Stokes model

Based on Assumptions 2.1 on the nonlocal kernels, we give some formal derivation of the nonlocal Stokes model from its local counterpart.

First, for $\boldsymbol{x} \in \Omega_\delta$, we multiply $\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})$ on both sides of the first equation of the Stokes system (1.1) evaluated at $\boldsymbol{y} \in \Omega$ and taking integral with respect to $\boldsymbol{y}$ over Ω,

$$\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\Delta\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} - \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\nabla p(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x} \in \Omega_\delta.$$

For the left hand side, we apply integration by parts and using the property $\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) = 0$ for $\boldsymbol{y} \in \partial\Omega$ and the relation between $\bar{R}$ and R,

$$\begin{aligned}&\frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{y}-\boldsymbol{x})\cdot\nabla\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} - \frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{y}-\boldsymbol{x})p(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\\ &= \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x} \in \Omega_\delta. \end{aligned} \tag{A.1}$$

For the first term of the left-hand side, the derivation in [36] proceeds with an approximation by Taylor expansion for $\boldsymbol{x} \in \Omega_\delta$,

$$\begin{aligned}&\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\Delta\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\\ &= -\frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\nabla\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\\ &= -\frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\left(\boldsymbol{u}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{y})-\frac{1}{2}\sum_{i,j=1}^n (x_i-y_i)(x_j-y_j)\frac{\partial^2\boldsymbol{u}(\boldsymbol{y})}{\partial y_i\partial y_j}\right)\mathrm{d}\boldsymbol{y}+\mathcal{O}(\delta)\\ &= -\frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{u}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}+\frac{1}{2}\sum_{i,j=1}^n\int_\Omega\frac{\partial}{\partial y_i}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(x_j-y_j)\frac{\partial^2\boldsymbol{u}(\boldsymbol{y})}{\partial y_i\partial y_j}\mathrm{d}\boldsymbol{y}+\mathcal{O}(\delta)\\ &= -\frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{u}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}+\frac{1}{2}\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\Delta\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}+\mathcal{O}(\delta)\\ &= -\frac{1}{\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{u}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}+\mathcal{O}(\delta).\end{aligned}$$

By dropping $\mathcal{O}(\delta)$ term, we obtain

$$\begin{aligned}&-\frac{1}{\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{u}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}+\frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})p(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\\ &= \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x} \in \Omega_\delta.\end{aligned}$$

From the derivation, it would appear that the error in the approximation of the left-hand side is formally of order $\mathcal{O}(\delta)$.

The derivation of Eq. (1.5b) is much easier. We also multiply $\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})$ in the divergence free equation and carry out integration by parts over Ω

$$\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}=0.$$

Then a stabilization term that mimics a nonlocal analog of the multiple of $\delta^2\Delta p$ is added to the above to obtain Eq. (1.5b): We remark that the stabilization term is $\mathcal{O}(\delta^2)$ so that its presence does not affect the order of the overall approximation.

Appendix B. Some basic estimates on the local Stokes system

Lemma B.1. *Let $\boldsymbol{u}(\boldsymbol{x})$ be the solution of the Stokes system* (1.1) *and $\boldsymbol{f}\in H^1(\Omega)$, then there are generic constants $C>0$ and $T_0>0$, depending only on Ω and $\partial\Omega$ such that for any $\delta<T_0$,*

$$\int_{\mathcal{V}_\delta}|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\le C\delta^3\|\boldsymbol{f}\|^2_{L^2(\Omega)}.$$

Proof. Since $\partial\Omega$ is compact and C^∞ smooth. Consequently, it is well known that $\partial\Omega$ has positive reaches [8], which means that there exists $T_0>0$ only depends on $\partial\Omega$, if $t<T_0,\mathcal{V}_\delta$ can be parameterized as $(\mathbf{z}(\boldsymbol{y}),\tau)\in\partial\Omega\times[0,1]$, where

$$\boldsymbol{y}=\mathbf{z}(\boldsymbol{y})+\tau(\mathbf{z}'(\boldsymbol{y})-\mathbf{z}(\boldsymbol{y}))$$

and

$$\left|\det\left(\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}(\mathbf{z}(\boldsymbol{y}),\tau)}\right)\right|\le C\delta$$

and $C>0$ is a constant only depends on Ω and $\partial\Omega$. Here $\mathbf{z}'(\boldsymbol{y})$ is the intersection point of $\partial\Omega'$ and the line determined by $\mathbf{z}(\boldsymbol{y})$ and $\boldsymbol{y}$. The parametrization is illustrated in Fig. 2.

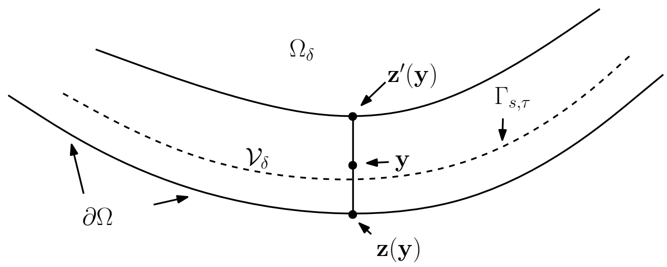

Figure 2: Parametrization of $\mathcal{V}_\delta$.

First, we have

$$\int_{\mathcal{V}_\delta}|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}=\int_{\mathcal{V}_\delta}|\boldsymbol{u}(\boldsymbol{y})-\boldsymbol{u}(\mathbf{z}(\boldsymbol{y}))|^2\mathrm{d}\boldsymbol{y}$$

$$
\begin{aligned}
&= \int_{\mathcal{V}_\delta} \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \boldsymbol{u}(\boldsymbol{y} + s(\mathbf{z}(\boldsymbol{y}) - \boldsymbol{y}))\mathrm{d}s \right|^2 \mathrm{d}\boldsymbol{y} \\
&= \int_{\mathcal{V}_\delta} \left| \int_0^1 (\mathbf{z}(\boldsymbol{y}) - \boldsymbol{y}) \cdot \nabla \boldsymbol{u}(\boldsymbol{y} + s(\mathbf{z}(\boldsymbol{y}) - \boldsymbol{y}))\mathrm{d}s \right|^2 \mathrm{d}\boldsymbol{y} \\
&\le C\delta^2 \int_{\mathcal{V}_\delta} \int_0^1 |\nabla \boldsymbol{u}(\boldsymbol{y} + s(\mathbf{z}(\boldsymbol{y}) - \boldsymbol{y}))|^2 \,\mathrm{d}s\mathrm{d}\boldsymbol{y} \\
&\le C\delta^2 \sup_{0\le s\le 1} \int_{\mathcal{V}_\delta} |\nabla \boldsymbol{u}(\boldsymbol{y} + s(\mathbf{z}(\boldsymbol{y}) - \boldsymbol{y}))|^2 \,\mathrm{d}\boldsymbol{y}.
\end{aligned}
$$

Here, we use the fact that $\|\mathbf{z}(\boldsymbol{y}) - \boldsymbol{y}\|_2 \le 2\delta$ to get the second last inequality.

Then, the proof can be completed by following estimation.

$$
\begin{aligned}
&\int_{\mathcal{V}_\delta} \big|\nabla \boldsymbol{u}\big(\boldsymbol{y} + s(\mathbf{z}(\boldsymbol{y}) - \boldsymbol{y})\big)\big|^2 \mathrm{d}\boldsymbol{y} \\
&\le C\delta \int_0^1 \int_{\partial\Omega} \big|\nabla \boldsymbol{u}\big(\mathbf{z}(\boldsymbol{y}) + (1-s)\tau(\mathbf{z}'(\boldsymbol{y}) - \mathbf{z}(\boldsymbol{y}))\big)\big|^2 \mathrm{d}\mathbf{z}(\boldsymbol{y})\mathrm{d}\tau \\
&\le C\delta \sup_{0\le\tau\le 1} \int_{\partial\Omega} \big|\nabla \boldsymbol{u}\big(\mathbf{z} + (1-s)\tau(\mathbf{z}' - \mathbf{z})\big)\big|^2 \mathrm{d}\mathbf{z} \\
&\le C\delta \sup_{0\le\tau\le 1} \int_{\Gamma_{s,\tau}} |\nabla \boldsymbol{u}(\tilde{\mathbf{z}})|^2 \,\mathrm{d}\tilde{\mathbf{z}} \\
&\le C\delta \|\boldsymbol{u}\|^2_{H^2(\Omega)} \le C\delta \|\boldsymbol{f}\|^2_{L^2(\Omega)},
\end{aligned}
$$

where $\Gamma_{s,\tau}$ is a $k-1$ dimensional manifold given by

$$
\Gamma_{s,\tau} = \big\{\mathbf{z} + (1-s)\tau(\mathbf{z}' - \mathbf{z}) : \mathbf{z} \in \partial\Omega\big\}.
$$

We use the trace theorem to get the second last inequality and the last inequality is due to that $\boldsymbol{u}$ is the solution of the Stokes system (1.1). □

Appendix C. Divergence estimation (3.11) (4.12)

Theorem C.1 ([17, Theorem III.3.1]). *Let Ω be a bounded domain of $\mathbb{R}^n, n \ge 2$ such that*

$$
\Omega = \bigcup_{k=1}^N \Omega_k, \quad N \ge 1,
$$

where each Ω_k is star-shaped with respect to some open ball B_k with $\bar{B}_k \subset \Omega_k$. Then, given $f \in L^q(\Omega), 1 < q < \infty$, satisfying $\int_\Omega f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0$, there exists at least one solution $\boldsymbol{v} \in W_0^{1,q}(\Omega)$ to

$$
\nabla \cdot \boldsymbol{v}(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,
$$

and

$$\|\boldsymbol{v}\|_{1,q} \le c\|f\|_q.$$

Furthermore, the constant c *admits the following estimate:*

$$c \le c_0 C \left(\frac{d(\Omega)}{R_0}\right)^n \left(1 + \frac{d(\Omega)}{R_0}\right),$$

where R_0 *is the smallest radius of the balls* $B_k, d(\Omega)$ *is the diameter of* $\Omega, c_0 = c_0(n,q)$ *and* C *is an upper bound for the constants* C_k *given as following:*

$$C_1 = 1 + \frac{|\Omega_1|^{1-\frac{1}{q}}}{|F_1|^{1-\frac{1}{q}}},$$
$$C_k = \left(1 + \frac{|\Omega_k|^{1-\frac{1}{q}}}{|F_k|^{1-\frac{1}{q}}}\right) \prod_{i=1}^{k-1} \left(1 + |F_i|^{-(1-\frac{1}{q})} |D_i - \Omega_i|^{1-\frac{1}{q}}\right), \quad k \ge 2,$$

where $F_i = \Omega_i \cap D_i$ *and* $D_i = \bigcup_{s=i+1}^{N} \Omega_s$.[‡]

Based on above theorem, to get the constant independent on δ in (4.12), we need to find decomposition for $\Omega_\eta, 0 \le \eta \le \delta_0$ such that corresponding R_0 and $|F_i|$ both have uniform lower bound independent on η with some $\delta_0 > 0$. Next, we will give an explicit way to construct the decomposition of Ω_η.

Under the assumption that the boundary $\partial\Omega$ is C^2 smooth, as shown in Fig. 3, for any point $\boldsymbol{x} \in \partial\Omega$, there exists $\delta_{\boldsymbol{x}} > 0$ such that

$$U_{\boldsymbol{x}} = \left\{\mathbf{z} \in \Omega : |\mathbf{z} - \boldsymbol{x}| < \delta_{\boldsymbol{x}}\right\}$$

is star-shaped with respect to open ball $B(\boldsymbol{y}, \delta_{\boldsymbol{x}}/4)$ with $\boldsymbol{y} = \boldsymbol{x} - \frac{2}{3}\delta_{\boldsymbol{x}}\mathbf{n}(\boldsymbol{x}), \mathbf{n}(\boldsymbol{x})$ is the outer normal of $\partial\Omega$ at $\boldsymbol{x}$.

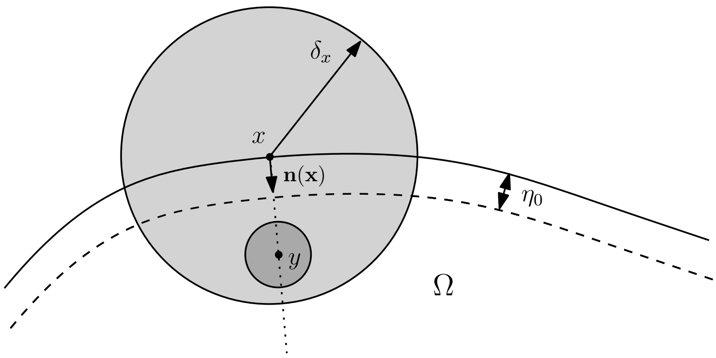

Figure 3: Cover of $\partial\Omega$.

[‡]Since Ω is connected, we can always label sets F_i in such a way that $|F_i| \neq 0, i = 1, \dots, N$.

$\bigcup_{\boldsymbol{x}\in\partial\Omega} U_{\boldsymbol{x}}$ is an open cover of $\partial\Omega$. Since $\partial\Omega$ is compact, there exist $\boldsymbol{x}_k \in \partial\Omega, k = 1,\ldots,N$ such that

$$\partial\Omega \subset \bigcup_{k=1}^{N} U_{\boldsymbol{x}_k}.$$

Compactness of $\partial\Omega$ also implies that there exists $\eta_0 \in (0, \frac{1}{2}\min_{1\le k\le N}\delta_{\boldsymbol{x}_k})$ such that

$$\mathcal{V}_{\eta_0} \subset \bigcup_{k=1}^{N} U_{\boldsymbol{x}_k}.$$

Recall that $\mathcal{V}_{\eta_0} = \{\boldsymbol{x}\in\Omega : \mathrm{dist}(\boldsymbol{x},\partial\Omega)\le\eta_0\}$.

For any $0\le\eta\le\eta_0/2$,

$$U^{\eta}_{\boldsymbol{x}_k} = \left\{\mathbf{z}\in\Omega_\eta : |\mathbf{z}-\boldsymbol{x}_k| < \delta_{\boldsymbol{x}_k}\right\}, \quad k=1,\ldots,N$$

are also star-shaped with respect to $B(\boldsymbol{y}_k, \delta_{\boldsymbol{x}_k}/4)$ with $\boldsymbol{y}_k = \boldsymbol{x}_k - \frac{2}{3}\delta_{\boldsymbol{x}_k}\mathbf{n}(\boldsymbol{x}_k), \mathbf{n}(\boldsymbol{x}_k)$ is the outer normal of $\partial\Omega$ at $\boldsymbol{x}_k$. On the other hand, compactness of $\bar{\Omega}_{\eta_0}$ gives $\mathbf{z}_1,\cdots,\mathbf{z}_M\in\bar{\Omega}_{\eta_0}$ such that

$$\bar{\Omega}_{\eta_0} \subset \bigcup_{k=1}^{M} B\left(\mathbf{z}_k, \frac{\eta_0}{2}\right).$$

For any $0\le\eta\le\eta_0/2$,

$$\Omega_\eta = \left(\bigcup_{k=1}^{N} U^{\eta}_{\boldsymbol{x}_k}\right) \bigcup \left(\bigcup_{k=1}^{M} B\left(\mathbf{z}_k, \frac{\eta_0}{2}\right)\right),$$

$U^{\eta}_{\boldsymbol{x}_k}$ is star-shaped with respect to $B(\boldsymbol{y}_k, \eta_0/2)$ and $B(\mathbf{z}_k, \eta_0/2)$ is star-shaped with respect to itself. It is easy to check based on above decomposition, Theorem C.1 implies (3.11) and (4.12).

Acknowledgments

The research of Z.S. was supported in part by grant NSFC 12071244. The research of Q.D. was supported in part by NSF DMS-2012562 and DMS-1937254.

References

[1] J. T. Beale and A. Majda, *High order accurate vortex methods with explicit velocity kernels*, J. Comput. Phys. 58 (1985), 188–208.

[2] M. Belkin and P. Niyogi, *Laplacian eigenmaps for dimensionality reduction and data representation*, Neural Comput. 15 (2003), 1373–1396.

[3] A. Chertock, *A practical guide to deterministic particle methods*, Handb. Numer. Anal. 18 (2017), 177–202.

[4] P. T. Choi, K. C. Lam, and L. M. Lui, *Flash: Fast landmark aligned spherical harmonic parameterization for genus-0 closed brain surfaces*, SIAM J. Imaging Sci. 8 (2015), 67–94.
[5] A. Cohen and B. Perthame, *Optimal approximations of transport equations by particle and pseudoparticle methods*, SIAM J. Math. Anal. 32 (2000), 616–636.
[6] R. R. Coifman, S. Lafon, A. B. Lee, M. Maggioni, F. Warner, and S. Zucker, *Geometric diffusions as a tool for harmonic analysis and structure definition of data: Diffusion maps*, Proc. Natl. Acad. Sci. USA (2005), 7426–7431.
[7] G. H. Cottet and P. Koumoutsakos, *Vortex Methods Theory and Practice*, Cambridge University Press, 2000.
[8] T. K. Dey, J. Sun, and Y. Wang, *Approximating cycles in a shortest basis of the first homology group from point data*, Inverse Problems 27 (2011), 124004.
[9] Q. Du, *Nonlocal Modeling, Analysis, and Computation*, SIAM, 2019.
[10] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou, *Analysis and approximation of nonlocal diffusion problems with volume constraints*, SIAM Rev. 54 (2012), 667–696.
[11] Q. Du, L. Ju, L. Tian, and K. Zhou, *A posteriori error analysis of finite element method for linear nonlocal diffusion and peridynamic models*, Math. Comp. 82 (2013), 1889–1922.
[12] Q. Du, T. Li, and X. Zhao, *A convergent adaptive finite element algorithm for nonlocal diffusion and peridynamic models*, SIAM J. Numer. Anal. 51 (2013), 1211–1234.
[13] Q. Du and X. Tian, *Mathematics of smoothed particle hydrodynamics: A study via nonlocal Stokes equations*, Found. Comput. Math. 20 (2020), 801–826.
[14] Q. Du, X. Tian, C. Wright, and Y. Yu, *Nonlocal trace spaces and extension results for nonlocal calculus*, arXiv:2107.00177, 2021.
[15] J. Eldredge, A. Leonard, and T. Colonius, *A general deterministic treatment of derivatives in particle methods*, J. Comput. Phys. 180 (2002), 686–709.
[16] L. C. Evans, *Partial Differential Equations*, AMS, 1997.
[17] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problem*, Springer, 2011.
[18] R. A. Gingold and J. J. Monaghan, *Smoothed particle hydrodynamics: theory and application to non-spherical stars*, Mon. Notices Royal Astron. Soc. 181 (1977), 375–389.
[19] X. Gu, Y. Wang, T. F. Chan, P. M. Thompson, and S.-T. Yau, *Genus zero surface conformal mapping and its application to brain surface mapping*, IEEE TMI 23 (2004), 949–958.
[20] C.-Y. Kao, R. Lai, and B. Osting, *Maximization of Laplace-Beltrami eigenvalues on closed Riemannian surfaces*, ESAIM CONTR OPTIM CA 23 (2017), 685–720.
[21] P. Koumoutsakos, *Multiscale flow simulations using particles*, Annu. Rev. Fluid Mech. 37 (2005), 457–487.
[22] R. Lai, Z. Wen, W. Yin, X. Gu, and L. Lui, *Folding-free global conformal mapping for genus-0 surfaces by harmonic energy minimization*, J. Sci. Comput. 58 (2014), 705–725.
[23] R. Lai and H. Zhao, *Multi-scale non-rigid point cloud registration using robust Sliced-Wasserstein distance via Laplace-Beltrami eigenmap*, SIAM J. Imaging Sci. 10 (2017), 449–483.
[24] H. Lee and Q. Du, *Nonlocal gradient operators with a nonspherical interaction neighborhood and their applications*, ESAIM: Math. Model. Numer. 54 (2020), 105–128.
[25] Z. Li, Z. Shi, and J. Sun, *Point integral method for solving Poisson-type equations on manifolds from point clouds with convergence guarantees*, Commun. Comput. Phys. 22 (2017), 228–258.
[26] Z. Li, Z. Shi, and J. Sun, *Point integral method for elliptic equations with variable coefficients on point cloud*, Commun. Comput. Phys. 26 (2019), 506–530.

[27] M. B. LIU AND G. R. LIU, *Smoothed particle hydrodynamics (sph): an overview and recent developments*, Arch Comput Methods Eng. 17 (2010), 25–76.

[28] L. B. LUCY, *A numerical approach to the testing of the fission hypothesis*, Astron J. 82 (1977), 1013–1024.

[29] T. W. MENG, P. T. CHOI, AND L. M. LUI, *Tempo: Feature-endowed teichmuller extremal mappings of point clouds*, SIAM J. Imaging Sci. 9 (2016), 1582–1618.

[30] T. MENGESHA AND Q. DU, *Nonlocal constrained value problems for a linear peridynamic Navier equation*, J. Elasticity 116 (2014), 27–51.

[31] T. MENGESHA AND Q. DU, *Characterization of function spaces of vector fields and an application in nonlinear peridynamics*, Nonlinear Anal. 140 (2016), 82–111.

[32] J. J. MONAGHAN, *Smoothed particle hydrodynamics*, Rep. Prog. Phys. 68 (2005), 1703–1759.

[33] S. OSHER, Z. SHI, AND W. ZHU, *Low dimensional manifold model for image processing*, SIAM J. Imaging Sci. 10 (2017), 1669–1690.

[34] G. PEYRÉ, *Manifold models for signals and images*, Comput. Vis. Image Underst. 113 (2009), 248–260.

[35] M. REUTER, F. E. WOLTER, AND N. PEINECKE, *Laplace-Beltrami spectra as 'shape-dna' of surfaces and solids*, Computer Aided Design 38 (2006), 342–366.

[36] Z. SHI, *Enforce the Dirichlet boundary condition by volume constraint in point integral method*, Commun. Math. Sci. 15 (2017), 1743–1769.

[37] Z. SHI AND J. SUN, *Convergence of the point integral method for Poisson equation on point cloud*, Res. Math. Sci. 4 (2017).

[38] S. A. SILLING, *Reformulation of elasticity theory for discontinuities and long-range forces*, J. Mech. Phys. Solids 48 (2000), 175–209.

[39] X. TIAN AND Q. DU, *Asymptotically compatible schemes and applications to robust discretization of nonlocal models*, SIAM J. Numer. Anal. 52 (2014), 1641–1665.

[40] A. TORNBERG AND B. ENGQUIST, *Numerical approximations of singular source terms in differential equations*, J. Comput. Phys. 200 (2004), 462–488.

[41] T. W. WONG, L. M. LUI, X. GU, P. THOMPSON, T. CHAN, AND S.-T. YAU, *Instrinic feature extraction and hippocampal surface registration using harmonic eigenmap*, Technical Report, UCLA CAM Report 11–65, 2011.

[42] Y. ZHANG, Z. SHI, AND Q. DU, *Nonlocal Stokes equation with relaxation on the divergence free equation*, Preprint, 2021.

[43] K. ZHOU AND Q. DU, *Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions*, SIAM J. Numer. Anal. 48 (2010), 1759–1780.