

An Extremal Problem Arising in the Dynamics of Two-Phase Materials That Directly Reveals Information about the Internal Geometry

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Abstract

In two-phase materials, each phase having a non-local response in time, it has been found that for some driving fields the response somehow untangles at specific times, and allows one to directly infer useful information about the geometry of the material, such as the volume fractions of the phases. Motivated by this, and to obtain an algorithm for designing appropriate driving fields, we find approximate, measure independent, linear relations between the values that Markov functions take at a given set of possibly complex points, not belonging to the interval $[-1,1]$ where the measure is supported. The problem is reduced to simply one of polynomial approximation of a given function on the interval $[-1,1]$ and, to simplify the analysis, Chebyshev approximation is used. This allows one to obtain explicit estimates of the error of the approximation, in terms of the number of points and the minimum distance of the points to the interval $[-1,1]$. Assuming this minimum distance is bounded below by a number greater than $1/2$, the error converges exponentially to zero as the number of points is increased. Approximate linear relations are also obtained that incorporate a set of moments of the measure. In the context of the motivating problem, the analysis also yields bounds on the response at any particular time for any driving field, and allows one to estimate the response at a given frequency using an appropriately designed driving field that effectively is turned on only for a fixed interval of time. The approximation extends directly to Markov-type functions with a positive semidefinite operator valued measure, and this has applications to determining the shape of an inclusion in a body from boundary flux measurements at a specific time, when the time-dependent boundary potentials are suitably tailored.

Keywords: Composites, best rational approximation, volume fraction estimation, bounds on transient response, Calderon problem, Markov functions
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1 Introduction

Many systems have responses that are nonlocal in time as this is naturally a consequence of the fact that it takes time for subelements of the system to respond. Usually, this leads one to either: (1) examine the response at each, or one or more, frequencies as the convolution in time characterizing the response of the system becomes a simple multiplication in the frequency domain; or (2) examine the response to a delta function or Heaviside function as this directly reveals the integral kernel characterizing the response. In this paper we show that desired information about the system can be directly obtained from selectively designed input signals that are neither at constant frequency, nor delta or Heaviside functions.

Our initial motivation comes from the work [37, 39] where we derived microstructure-independent bounds on the viscoelastic response at a given time t of two-phase periodic composites (in antiplane shear) with prescribed volume fractions f_1 and $f_2 = 1 - f_1$ of the phases and with an applied average stress or strain prescribed as a function of time. We found that the bounds were sometimes extremely tight at particular times $t = t_0$: see Figure 1.1. This was quite a surprise because the response of each phase is nonlocal in time, yet somehow this response is untangled at these particular times. Thus, the bounds could be used in an inverse fashion to determine the volume fractions from measurements at time t_0 . Determining volume fractions of phases is important in the oil industry, where one wants to know the proportions occupied by oil and water in the rock, to finding the porosity of osteoporetic bone to detecting breast cancer, to assessing the porosity of sea-ice and other materials, and even to determining the volume of holes in Swiss cheese. Previous approaches to obtaining volume fraction information include using volume fraction dependent bounds on the complex dielectric constant at one or more frequencies in an inverse way [16, 18, 41, 42], estimation of the measure in the associated Stieltjes function whose integral gives the volume fraction [14, 15, 17], and estimation of the distributions of poles and zeros or poles and residues when the

measure is discrete or approximated by a discrete one [41, 66]. These generally require measurements at many frequencies to yield accurate estimates of the volume fraction. By contrast our approach only requires a single measurement at a specific time with an applied field that has a carefully designed variation in time.

While our bounds [39] were very tight at specific times in some examples, they were far from tight at all times in other examples: see Figure 1.2. At that time it was totally unclear as to whether appropriate input signals could produce the desired tight bounds at a specific time and, if so, what algorithm should be followed to design these input signals. The primary goal of this paper is to address this problem in the case where the input function has a finite number of frequencies. In particular, for the example of Figure 1.2, our algorithm produces the much tighter bounds of Figure 1.3. In [40] we show that one can find smooth input signals, containing a continuum of frequencies, such that the response $\text{Re}[v(t_0)]$ of the material at a specific moment of time t_0 is totally measure independent, while $\text{Re}[v(t)]$ has a smooth dependence on t , with $\text{Re}[v(t)] \rightarrow 0$ when $t \rightarrow -\infty$.

We emphasize that our results are applicable not just to determining the volume fractions of the phases in a two-phase composite but also determining the volume and shape of an inclusion in a body from exterior boundary measurements. This is shown in Sections 5.3 and 5.4. It is a classical and important inverse problem with a long history and many contributions: see [1, 3, 4, 9–11, 25, 28, 31–34, 52, 54, 55] and references therein.

A secondary goal of this paper is to solve an accompanying mathematical approximation problem, which we now outline, and which is essential to achieve the primary goal.

2 The Approximation Theory Question

It is the aim of this section to formulate and solve an approximation theory question, directly relevant to our study. Specifically, we provide bounds that provide linear correlations on the values taken by certain Markov functions, that is, Cauchy transforms of positive measures with compact support on the real line. These functions map the upper half-plane to itself and arise as compressed resolvents of self-adjoint operators. For this very reason the rational approximation theory of Markov functions was and remains a central topic of constructive function theory. Markov functions are also called, depending on the context, Herglotz functions, or Nevanlinna functions, or Stieltjes transforms.

2.1 Evaluating Markov functions

Suppose $F_\mu(z)$ is a Markov function having the integral representation

$$(2.1) \quad F_\mu(z) = \int_{-1}^1 \frac{d\mu(\lambda)}{\lambda - z},$$

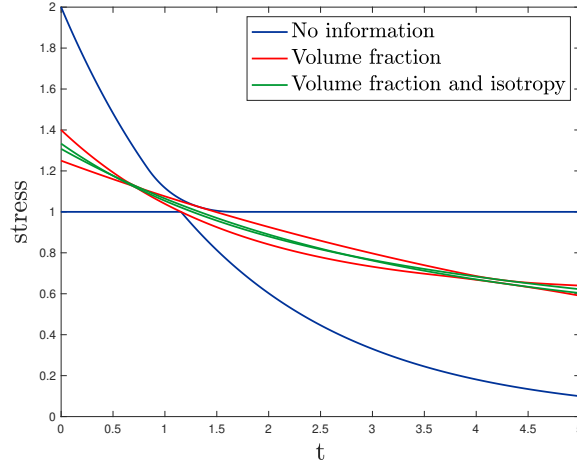


FIGURE 1.1. Comparison between the lower and upper bounds on the output average stress with an input applied average strain of $H(t)$, where $H(t)$ is the Heaviside function, 0 for $t < 0$ and 1 for $t \geq 0$. This is called a stress relaxation test. One phase is purely elastic ($G = 6000$), while the other phase is viscoelastic and modeled by the Maxwell model ($G = 12000$ and $\eta = 20000$) (the results are normalized by the response of the elastic phase). The following three cases are graphed: no information about the composite is given; the volume fraction of the components is known ($f_1 = 0.4$); and the composite is isotropic with given volume fractions. The bounds become tighter and tighter as more information on the composite structure is included, so that if color is missing from the figure the outermost pair of bounds are those with no information, the middle pair include just the volume fraction, and the innermost pair include both volume fraction and isotropy. Reproduced from Figure 6.2 in [39].

where the positive Borel measure μ has unit mass:

$$(2.2) \quad \int_{-1}^1 d\mu(\lambda) = 1.$$

Given m (possibly complex) points $z_1, z_2, \dots, z_m$ not belonging to the interval $[-1, 1]$, we are interested in finding complex constants $\alpha_1, \alpha_2, \dots, \alpha_m$ such that

$$(2.3) \quad \sum_{k=1}^m \alpha_k F_\mu(z_k) \approx 1$$

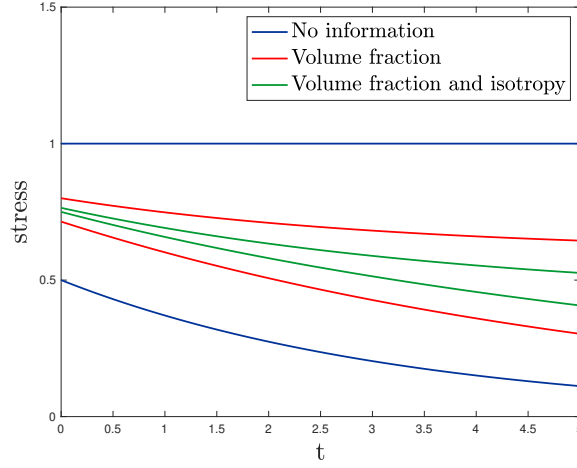


FIGURE 1.2. Comparison between the lower and upper bounds on the output stress relaxation in the “badly ordered case”, when the responses on the pure phases as a function of time do not cross with an input applied average strain of $H(t)$, where $H(t)$ is the Heaviside function. Here the purely elastic phase has shear modulus $G = 12000$, while the Maxwell parameters for the viscoelastic phase are $G = 6000$ and $\eta = 20000$ (again, the results are normalized by the response of the elastic phase). The three subcases are the same as for the previous figure. However the bounds remain quite wide except near $t = 0$. Reproduced from Figure 6.5 in [39]. The approach developed in this paper can yield tight bounds with a suitably designed input function as shown in Figure 1.3.

for all probability measures μ . Optimal bounds correlating the possible values of the m -tuple $(F_\mu(z_1), F_\mu(z_2), \dots, F_\mu(z_m))$ as μ varies over all probability measures are well-known, as derived from the well charted analysis of the Nevanlinna-Pick interpolation problem [35]. Indeed, the nonlinear constraints among the values $F_\mu(z_1), F_\mu(z_2), \dots, F_\mu(z_m)$ and standard convexity theory provide optimal bounds on the range of the left-hand side of (2.3) for given constants $\alpha_1, \alpha_2, \dots, \alpha_m$; see [35] for details. But this is not our main concern.

We would rather like to choose m points $z_1, z_2, \dots, z_m$, and find associated constants $\alpha_1, \alpha_2, \dots, \alpha_m$ for every prescribed integer m , subject to a uniform estimate

$$(2.4) \quad \sup_{\mu} \left| \sum_{k=1}^m \alpha_k F_\mu(z_k) - 1 \right| \leq \epsilon_m$$

for some computable bound ϵ_m that tends to zero as $m \rightarrow \infty$. The geometry of the locus of these points is obviously essential, and it will be detailed in the sequel. The faster the convergence, the better.

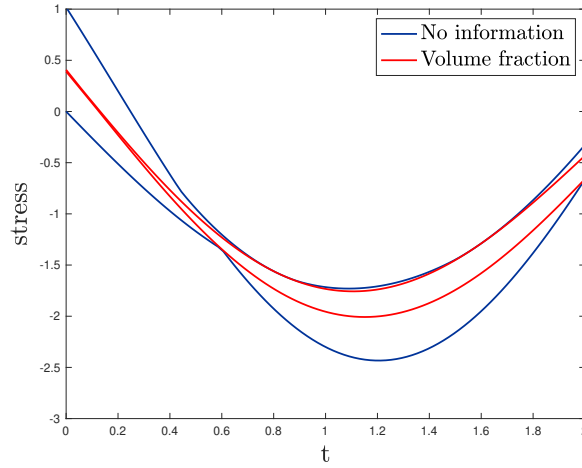


FIGURE 1.3. Comparison between the lower and upper bounds on the output stress relaxation in the “badly ordered case” ($G = 12\,000$ for the elastic phase, and $G = 6000$ and $\eta = 20,000$ for the viscoelastic Maxwell phase), when the input function is chosen accordingly to equation (2.10), which represents the main result of this paper. Specifically, equation (2.10) provides the amplitude of the applied field that gives extremely tight bounds at a chosen moment of time (here $t = 0$) when the volume fraction is known. Indeed, the bounds incorporating the volume fraction (the innermost bounds, in red) take the value 0.4 at $t = 0$, which coincides exactly with the volume fraction of the viscoelastic phase. Here, the applied loading is the sum of three time-harmonic fields with frequencies $\omega = 0.1, 0.5, 1.5$.

Since we deal with probability measures, condition (2.4) is equivalent to

$$(2.5) \quad \sup_{\lambda \in [-1, 1]} \left| \sum_{k=1}^m \frac{\alpha_k}{\lambda - z_k} - 1 \right| \leq \epsilon_m.$$

And this is good news because we turn our focus to the minimal deviation from one, on the interval $[-1, 1]$, of a rational function $R(\lambda)$ satisfying $R(\infty) = 0$ and possessing simple poles at the points $z_1, \dots, z_m$. Or equivalently, denoting $q(\lambda) = (\lambda - z_1)(\lambda - z_2) \cdots (\lambda - z_m)$ and $w(\lambda) = |q(\lambda)|^{-1}$, we aim at finding the minimal deviation from zero of a *monic polynomial* p of degree m , with respect to the weighted norm $\|p\|_w = \sup_{\lambda \in [-1, 1]} |p(\lambda)w(\lambda)|$.

Both perspectives align to well-known classical studies in approximation theory. The first one is an extremal problem in rational approximation with prescribed poles, a subject going back at least to Walsh [59]. A great deal of information in this respect was systematized in Walsh’s book [60]. The second approach is a genuine weighted Chebyshev approximation problem, and here we are on solid

ground. First, note that the functions

$$(2.6) \quad w(\lambda), \lambda w(\lambda), \lambda^2 w(\lambda), \dots$$

form a Chebyshev system on the interval $[-1, 1]$; that is, they are linearly independent and any linear combination of $w(\lambda), \lambda w(\lambda), \lambda^2 w(\lambda), \dots, \lambda^m w(\lambda)$ has at most m zeros in $[-1, 1]$. Even more, a stronger so-called Markov property of this system of functions holds. The classical Chebyshev approximation in the uniform norm theorem has an analogue for such nonorthogonal bases [29, 35]. To be more precise, there exists a unique monic polynomial p of degree m minimizing the norm $\|p w\|_\infty$: this polynomial is characterized by the fact that $|p(\lambda)|$ attains its maximal value at $m + 1$ points, and the sign of $p(\lambda)$ alternates there; see also [43]. In case $w(\lambda) = 1$, the optimal polynomial is of course the normalized Chebyshev polynomial of the first kind: $p(\lambda) = \frac{T_m(\lambda)}{2^m}$, $T_m(\cos x) = \cos(mx)$, $m \geq 0$. The constructive aspects of weighted Chebyshev approximation are rather involved; see, for instance, the early works of Werner [62–64]. In the same vein, the asymptotics of the optimal bound of our minimization problem inherently involves potential theory or operator theory concepts. We cite for a comparison basis a few remarkable results of the same flavor [6, 22, 53].

Without seeking sharp bounds and guided by the specific applications we aim at, we propose a compromise and relaxation of our extremal problem:

$$(2.7) \quad \inf_p \|p w\|_\infty \leq \|w\|_\infty \inf_p \|p\|_\infty.$$

At this point we can invoke Chebyshev original theorem and his polynomial T_m , obtaining in this way the benefit, very useful for applications, of computing in closed form the residues α_k .

2.2 Main result

The present section contains the principal estimate which provides the theoretical foundation of our explorations. As explained in the introduction, we try to balance the computational accessibility and simplicity with the loss of sharp bounds. A few comments about the versatility of the following theorem are elaborated after its proof.

THEOREM 2.1. *Let $z_1, z_2, \dots, z_m$ be mutually disjoint complex numbers, subject to the assumption that the distances from z_k to the interval $[-1, 1]$ given by*

$$(2.8) \quad d(z_k) = \min_{\lambda \in [-1, 1]} |\lambda - z_k|$$

(2.8) are bounded from below by $1/2$

$$(2.9) \quad d_{\min} = \min_k d(z_k) > 1/2.$$

Then one can find complex constants $\alpha_1, \alpha_2, \dots, \alpha_m$, each depending on m , such that the estimate (2.4) holds with $\epsilon_m \rightarrow 0$ as $m \rightarrow \infty$. In particular, the values

$$(2.10) \quad \alpha_k = -\frac{T_m(z_k)}{2^{m-1} \prod_{j \neq k} (z_k - z_j)},$$

where $T_m(z)$ is the Chebyshev polynomial of the first kind, of degree m , assure that (2.4) holds with $\epsilon_m = 2/(2d_{\min})^m$. Note that this bound converges exponentially to zero as $m \rightarrow \infty$.

PROOF. Recall that $f(z) = F_\mu(z)$ is the Cauchy transform of a probability measure supported by the interval $[-1, 1]$. From

$$(2.11) \quad \left| \sum_{k=1}^m \alpha_k f(z_k) - 1 \right| \leq \int_{-1}^1 d\mu(\lambda) \left| \sum_{k=1}^m \frac{\alpha_k}{\lambda - z_k} - 1 \right|.$$

we infer that equality is achieved in case μ is a point mass

$$(2.12) \quad \mu(\lambda) = \delta(\lambda - \lambda_0),$$

where λ_0 belongs to $[-1, 1]$. Equivalently, we note

$$(2.13) \quad \inf_{\alpha} \sup_{\mu} \left| \sum_{k=1}^m \alpha_k f(z_k) - 1 \right| = \inf_{\alpha} \sup_{\mu} \int_{-1}^1 d\mu(\lambda) \left| \sum_{k=1}^m \frac{\alpha_k}{\lambda - z_k} - 1 \right| \\ = \inf_{\alpha} \sup_{\lambda_0 \in [-1, 1]} \left| \sum_{k=1}^m \frac{\alpha_k}{\lambda_0 - z_k} - 1 \right|.$$

Therefore we seek a set of constants $\alpha_1, \alpha_2, \dots, \alpha_m$ (each dependent on m) and upper bounds ϵ_m with the property that $\epsilon_m \rightarrow 0$ as $m \rightarrow \infty$ and

$$(2.14) \quad \left| \sum_{k=1}^m \frac{\alpha_k}{\lambda - z_k} - 1 \right| \leq \epsilon_m \quad \text{for all } \lambda \in [-1, 1].$$

More clearly, direct substitution of (2.14) into (2.11) shows that relation (2.4) holds.

Write

$$(2.15) \quad \sum_{k=1}^m \frac{\alpha_k}{\lambda - z_k} = \frac{p(\lambda)}{q(\lambda)} = R(\lambda),$$

where $q(\lambda)$ is the prescribed monic polynomial

$$(2.16) \quad q(\lambda) = \prod_{j=1}^m (\lambda - z_j)$$

of degree m , and $p(\lambda)$ is a polynomial of degree at most $m - 1$ that remains to be determined. The constants α_k can then be identified with the residues at the poles

$\lambda = z_k$ of $R(\lambda)$:

$$(2.17) \quad \alpha_k = \frac{p(z_k)}{\prod_{j \neq k} (z_k - z_j)}.$$

Consequently the problem becomes one of choosing $p(\lambda)$ such that

$$(2.18) \quad \sup_{\lambda \in [-1, 1]} \left| \frac{p(\lambda)}{q(\lambda)} - 1 \right| = \sup_{\lambda \in [-1, 1]} \frac{|p(\lambda) - q(\lambda)|}{|q(\lambda)|}$$

is close to zero. Clearly, the problem is now one of polynomial approximation of the monic polynomial $q(\lambda)$ of degree m by the polynomial $p(\lambda)$. A natural choice is

$$(2.19) \quad p(\lambda) = q(\lambda) - T_m(\lambda)/2^{m-1},$$

where $T_m(\lambda)/2^{m-1}$ is the Chebyshev polynomial $T_m(\lambda)$ of degree m , normalized to be monic. This choice minimizes the sup-norm of $|p(\lambda) - q(\lambda)|$ over the interval $\lambda \in [-1, 1]$ and

$$(2.20) \quad |p(\lambda) - q(\lambda)| = |T_m(\lambda)/2^{m-1}| \leq 1/2^{m-1}$$

provides a bound on the numerator in (2.18). To bound the denominator, we have

$$(2.21) \quad |q(\lambda)| = \prod_{k=1}^m |\lambda - z_k| \geq \prod_{k=1}^m d(z_k),$$

where $d(z_k)$ is given by (2.8). Using (2.9) and the bounds (2.20) and (2.21) we see that (2.4) is satisfied with $\epsilon^m = 2/(2d_{\min})^m$. Finally, with $p(\lambda)$ given by (2.19) we see that the residues α_k at the poles $\lambda = z_k$ of $g(\lambda)$, given by (2.17) correspond to those given by (2.10). $\square$

REMARK 2.2. The use of Chebyshev polynomials is convenient as bounds on their sup-norm over the interval $[-1, 1]$ are readily available. An alternative approach, also accessible from the numerical/computational point of view, is to work with the L^2 norm and find the polynomial $p(\lambda)$ of degree $m - 1$ that approximates the given monic polynomial $q(\lambda)$ of degree m in the precise sense that

$$(2.22) \quad \int_{-1}^1 |(p(\lambda) - q(\lambda))^2| d\nu(\lambda) \quad \text{with } d\nu(\lambda) = d\lambda/|q(\lambda)|^2$$

is minimized. Subsequently, one has to invoke Bernstein-Markov's inequality which bounds an L^2 norm by uniform norm. This first step is a standard problem in the theory of orthogonal polynomials: one chooses $p(\lambda) - q(\lambda)$ to be the monic polynomial of degree m that is orthogonal to all polynomials of degree at most $m - 1$ with respect to the measure $d\nu(\lambda)$. Separating the contribution of the denominator, by selecting ν to be the measure $d\lambda/\sqrt{1 - \lambda^2}$ we recover the Chebyshev polynomials we have advocated in the proof of the main result.

REMARK 2.3. Assumption (2.9) is more than we really need. To underline the dependence on n of all data we set

$$q_n(z) = (z - z_1(n))(z - z_2(n)) \cdots (z - z_n(n)), \quad n \geq 1,$$

and

$$w_n(z) = \frac{1}{|q_n(z)|}.$$

For the proof of Theorem 2.1 above we only need

$$\limsup_n \|w_n\|_\infty^{1/n} < 2.$$

That is, there exists $r < 2$, so that for large n , the inequality

$$w_n(\lambda) \leq r^n, \quad \lambda \in [-1, 1],$$

holds true.

By taking the natural logarithm, we are led to enforce the condition

$$\limsup_n \sup_{\lambda \in [-1, 1]} \frac{1}{n} \sum_{j=1}^n \ln \frac{1}{|\lambda - z_j(n)|} < \ln 2.$$

That is, an evenly distributed probability mass on points $z_1(n), \dots, z_n(n)$ should have its logarithmic potential asymptotically bounded from above by a prescribed constant, on the interval $[-1, 1]$. Again, this turns out to be a rather typical problem of approximation theory, at least when restricting the poles of q_n to belong to some Jordan curve surrounding $[-1, 1]$. A natural choice is an ellipse with foci at ± 1 ; see also [6, 53].

2.3 Incorporating moments of the measure

Here we assume that the first n moments $M_1, M_2, \dots, M_n$ of the probability measure $d\mu$, given by (2.1), are known, in addition to $M_0 = 1$ and that m (possibly complex) points $z_1, z_2, \dots, z_m$ not on the interval $[-1, 1]$ are given. We seek complex constants $\alpha_1, \alpha_2, \dots, \alpha_m$ and $\gamma_1, \gamma_2, \dots, \gamma_n$, with say $\gamma_n = 1$ such that

$$(2.23) \quad \left| \sum_{k=1}^m \alpha_k f(z_k) - \sum_{\ell=0}^n \gamma_\ell M_\ell \right|$$

is small for all probability measures μ with the prescribed n moments. The analysis proceeds as before, only now we introduce the polynomial

$$(2.24) \quad r(\lambda) = \sum_{\ell=0}^n \gamma_\ell \lambda^\ell,$$

and set $p(\lambda)$ and $q(\lambda)$ to be the polynomials defined by (2.15) and (2.16). The goal is now to choose polynomials $p(\lambda)$ and $r(\lambda)$ of degrees $m-1$ and n , respectively,

such that $r(\lambda)$ is monic and

$$(2.25) \quad \sup_{\lambda \in [-1, 1]} \left| \frac{p(\lambda)}{q(\lambda)} - r(\lambda) \right| = \sup_{\lambda \in [-1, 1]} \frac{|p(\lambda) - q(\lambda)r(\lambda)|}{|q(\lambda)|}$$

is close to zero. We choose $p(\lambda)$ and $r(\lambda)$ such that

$$(2.26) \quad T_{m+n}(\lambda)/2^{m+n-1} = q(\lambda)r(\lambda) - p(\lambda).$$

This is simply the Euclidean division of the normalized Chebyshev polynomial $T_{m+n}(\lambda)/2^{m+n-1}$ by $q(\lambda)$ with $r(\lambda)$ being identified as the quotient polynomial and $-p(\lambda)$ being identified as the remainder polynomial. Then, assuming (2.9) and using (2.21), we have

$$(2.27) \quad \sup_{\lambda \in [-1, 1]} \left| \frac{p(\lambda)}{q(\lambda)} - r(\lambda) \right| \leq \epsilon_m^{(n)}, \quad \text{with} \quad \epsilon_m^{(n)} = \frac{2}{2^n (2d_{\min})^m}$$

satisfying $\epsilon_m^{(n)} \rightarrow 0$ as $m \rightarrow \infty$, with n being fixed. With constants α_k given by (2.17) and constants γ_ℓ being the coefficients of the polynomial $r(\lambda)$, as in (2.24), it follows that

$$(2.28) \quad \begin{aligned} & \sup_{\mu} \left| \sum_{k=1}^m \alpha_k f(z_k) - \sum_{\ell=0}^n \gamma_\ell M_\ell \right| \\ &= \sup_{\mu} \int_{-1}^1 d\mu(\lambda) \left| \sum_{k=1}^m \frac{\alpha_k}{\lambda - z_k} - \sum_{\ell=0}^n \gamma_\ell \lambda^\ell \right| \leq \epsilon_m^{(n)}. \end{aligned}$$

2.4 Operator-valued measures

Mutatis mutandis, the results exposed in the previous sections extend immediately to the resolvent of a self-adjoint operator situation, via the spectral representation

$$(2.29) \quad \mathbf{A} = \int_{\sigma(\mathbf{A})} \lambda d\mathbf{P}_\lambda,$$

where $\sigma(\mathbf{A})$ is the spectrum of $\mathbf{A}$, assumed to be contained in the interval $[-1, 1]$, and $d\mathbf{P}_\lambda$ is an orthogonal projection valued measure satisfying

$$(2.30) \quad \mathbf{I} = \int_{\sigma(\mathbf{A})} d\mathbf{P}_\lambda.$$

In this context we remark

$$\begin{aligned}
 (2.31) \quad & \inf_{\mathbf{A}} \left\| \sum_{k=1}^m \alpha_k [\mathbf{A} - z_k \mathbf{I}]^{-1} - \sum_{\ell=0}^n \gamma_\ell \mathbf{A}^\ell \right\| \\
 &= \sup_{d\mathbf{P}_\lambda} \left\| \int_{\sigma(\mathbf{A})} \left[\sum_{k=1}^m \frac{\alpha_k}{\lambda - z_k} - \sum_{\ell=0}^n \gamma_\ell \lambda^\ell \right] d\mathbf{P}_\lambda \right\| \\
 &\leq \sup_{d\mathbf{P}_\lambda} \left\| \int_{\sigma(\mathbf{A})} \left| \sum_{k=1}^m \frac{\alpha_k}{\lambda - z_k} - \sum_{\ell=0}^n \gamma_\ell \lambda^\ell \right| d\mathbf{P}_\lambda \right\|.
 \end{aligned}$$

Choosing constants $\alpha_1, \alpha_2, \dots, \alpha_m$ and $\gamma_1, \gamma_2, \dots, \gamma_n$, with $\gamma_n = 1$, as in the previous section, the bound (2.27) substituted in (2.31) implies that the desired bound

$$(2.32) \quad \inf_{\mathbf{A}} \left\| \sum_{k=1}^m \alpha_k [\mathbf{A} - z_k \mathbf{I}]^{-1} - \sum_{\ell=0}^n \gamma_\ell \mathbf{A}^\ell \right\| \leq \epsilon_m^{(n)},$$

holds with $\epsilon_m^{(n)} = 2/[2^n (2d_{\min})^m]$ which goes to zero as $m \rightarrow \infty$ provided $d_{\min} > 1/2$.

3 Relevance of the Approximation Problem to Systems with a Nonlocal Time Response and the Viscoelasticity Problem in Particular

Without going into the specific details, as these will be provided later, in many linear systems with an input function $u(t)$ varying with time t , of the form

$$(3.1) \quad u(t) = \sum_{k=1}^m \beta_k e^{-i\omega_k(t-t_0)},$$

where the ω_k are a set of (possibly complex) frequencies and t_0 is a given time, the output function $v(t)$ takes the form

$$(3.2) \quad v(t) = \sum_{k=1}^m \alpha_k a_0 F_\mu(z(\omega_k)) e^{-i\omega_k(t-t_0)},$$

in which the function $F_\mu(z)$ is given by (2.1),

$$(3.3) \quad \alpha_k = \beta_k c(\omega_k),$$

and the functions $z(\omega)$ and $c(\omega)$ depend on ω in some known way. The real constant $a_0 > 0$ and the unknown measure $d\mu$ depend on the system. In our viscoelasticity study [39] the connection with Markov functions comes from the fact that the effective shear modulus $G_*(\omega)$, which relates the average stress to the average strain at frequency ω , as a function of the shear moduli $G_1(\omega)$ and $G_2(\omega)$ of the two phases, has the property that $[(G_*/G_1) - 1]/(2f_1)$, in which f_1 is the volume fraction of phase 1 is a Markov function of $z = (G_1 + G_2)/(G_2 - G_1)$ taking the form (2.2) [7, 21, 44].

Henceforth we adopt the notational simplification

$$f(z) = F_\mu(z).$$

Thus, at time $t = t_0$, the output function is

$$(3.4) \quad v(t_0) = a_0 \sum_{k=1}^m \alpha_k f(z_k) \quad \text{with } z_k = z(\omega_k),$$

and we seek an input signal so that the output $v(t_0)$ is almost system independent with $v(t_0) \approx a_0$. According to Theorem 2.1 this will be the case if the coefficients α_k are given by (2.10) and $d_{\min} > 1/2$. So, by measuring $v(t_0)$ we can determine the system parameter a_0 . In the viscoelastic problem that we studied [37, 39], a_0 is the volume fraction f_1 (see also [7]), and it is useful to be able to determine this from indirect measurements.

Typically, one may assume the frequencies ω_k have a positive imaginary part so that the input signal $u(t)$ is essentially zero in the distant past. In (3.1) one could just take a signal with $m - 1$ frequencies ω_k , $k = 1, 2, \dots, m - 1$. Then, with the coefficients α_k being given by (2.10) and $d_{\min} > 1/2$, we have

$$(3.5) \quad v(t_0) + \alpha_m a_0 f(z(\omega_m)) = a_0 \sum_{k=1}^m \alpha_k f(z_k) \approx a_0 \quad \text{with } z_k = z(\omega_k).$$

So, if a_0 is known, a measurement of $v(t_0)$ will allow us to estimate the output $a_0 f(z(\omega_m)) e^{-i\omega_m(t-t_0)}$ at a desired (possibly real) frequency ω_m given the input $e^{-i\omega_m(t-t_0)}$.

It is often the situation, such as in the viscoelastic problem, that only the real part of $v(t)$ has a direct physical significance and, hence, one might want to find constants α_k such that, say,

$$(3.6) \quad \begin{aligned} 2 \operatorname{Re}[v(t_0)] &= a_0 \left(\sum_{k=1}^m \alpha_k f(z_k) + \sum_{k=1}^m \overline{\alpha_k} \overline{f(z_k)} \right) \\ &= a_0 \left(\sum_{k=1}^m \alpha_k f(z_k) + \sum_{k=1}^m \overline{\alpha_k} f(\overline{z_k}) \right) \approx a_0, \end{aligned}$$

where the overline denotes complex conjugation. This, again, reduces to a problem of the form (2.3) where, after renumbering, the complex values of z_k come in pairs, z_k and $z_{k+1} = \overline{z_k}$, and we may take $\alpha_{k+1} = \overline{\alpha_k}$ so that the left-hand side of (2.3) is real.

We can gain more flexibility in the choice of the input signal if we replace $T_m(z_k)$ in the formula (2.10) for the residues α_k with $(z_k - z_0) T_{m-1}(z_k)$, where z_0 is a prescribed real zero of $p(\lambda) - q(\lambda) = (\lambda - z_0) T_{m-1}(\lambda)$. In particular, we

may choose z_0 to, say, minimize

$$(3.7) \quad \max_{t \leq t_0} |v(t)|/|v(t_0)| \approx \max_{t \leq t_0} \left| \sum_{k=1}^m \alpha_k f(z_k) e^{-i\omega_k(t-t_0)} \right|$$

to help ensure that the output signal is not too wild. If we are only interested in $\operatorname{Re}[v(t)]$ so that the z_k come in complex conjugate pairs, then we may replace $T_m(z_k)$ in (2.10) with $(z_k - z_0)(z_k - \bar{z}_0)T_{m-2}(z_k)$, and choose z_0 to, say, minimize

$$(3.8) \quad \max_{t \leq t_0} |\operatorname{Re}[v(t)]|/|\operatorname{Re}[v(t_0)]| \approx \max_{t \leq t_0} \left| \operatorname{Re} \left[\sum_{k=1}^m \alpha_k f(z_k) e^{-i\omega_k(t-t_0)} \right] \right|.$$

In the first case, note that the signal $u(t)$ (3.1) is linear in z_0 while in the second case it is linear in the real coefficients of the quadratic $(\lambda - z_0)(\lambda - \bar{z}_0)$. So in either case we have a linear space of possible signals (though $|z_0|$ should not be too large for the approximation to hold at time t_0). Also $\alpha_k \rightarrow 0$ as $z_0 \rightarrow z_k$ so in this limit the frequency ω_k is absent from the input and output signals. More generally, to help minimize (3.7) or (3.8) one might replace $T_m(z_k)$ with $s_M(z_k)T_{m-M}(z_k)$ where $s_M(\lambda)$ is a polynomial of fixed degree $M < m$.

The results of Section 2.3 allow us to determine a relation between the n moments $M_1, M_2, \dots, M_n$ and a_0 if $v(t_0)$ is measured. This can be useful when the moments have a physical significance: in the viscoelastic problem, for instance, M_1 depends only on the volume fraction f_1 if one assumes that the composite has sufficient symmetry to ensure that its response remains invariant as the material is rotated [7]. So, incorporating the moment M_1 and measuring the response at time t_0 then allows us to obtain tighter bounds on f_1 , in a similar way to that done in [37, 39].

The relevance of our inequality (2.32) for operator-valued measures is that in many linear systems with an input field $\mathbf{u}(t)$ varying with time t , of the form

$$(3.9) \quad \mathbf{u}(t) = \sum_{k=1}^m \beta_k e^{-i\omega_k(t-t_0)} \mathbf{u}_0,$$

the output field $\mathbf{v}(t)$ takes the form

$$(3.10) \quad \mathbf{v}(t) = \sum_{k=1}^m \alpha_k e^{-i\omega_k(t-t_0)} a_0 [\mathbf{A} - z(\omega_k) \mathbf{I}]^{-1} \mathbf{u}_0 \quad \text{with } \alpha_k = \beta_k c(\omega_k),$$

where the real constant a_0 and the self-adjoint operator $\mathbf{A}$ characterize the response of the system, and the system parameters $z(\omega)$ and $c(\omega)$ depend on the frequency ω in some known way. Then, the bound (2.32) implies

$$(3.11) \quad \left| \mathbf{v}(t_0) - a_0 \sum_{\ell=0}^n \gamma_\ell \mathbf{A}^\ell \mathbf{u}_0 \right| \leq a_0 \epsilon_m^{(n)} |\mathbf{u}_0|.$$

4 Applications of the Approximation Scheme

With the mathematical and physical backgrounds in place, we return to the bounds of Markov functions. The notations are the same as in the approximation theory section.

4.1 Bounds on the output function $v(t)$ at any time t

Supposing any constants $\alpha_1, \alpha_2, \dots, \alpha_m$ are given, it is easy to get bounds on $v(t)$ given by (3.2) at any time t that incorporate the n known moments $M_1, M_2, \dots, M_n$. One introduces an angle θ and Lagrange multipliers $\gamma_1, \gamma_2, \dots, \gamma_n$ and takes the minimum value of

$$(4.1) \quad \int_{-1}^1 d\mu(\lambda) \operatorname{Re} \left[e^{i\theta} \sum_{k=1}^m \frac{\alpha_k e^{-i\omega_k(t-t_0)}}{\lambda - z(\omega_k)} + \sum_{\ell=1}^n \gamma_\ell \lambda^\ell \right],$$

as μ varies over all probability measures supported on $[-1, 1]$ with unconstrained moments. The minimum will be achieved by the point masses $\mu = \delta(\lambda - \lambda_0)$, where λ_0 may take one or more values. Typically we will need to choose the Lagrange multipliers $\gamma_1, \gamma_2, \dots, \gamma_n$ (that depend on θ) so that the minimum is achieved at n values $\lambda_0 = \lambda_0^{(\ell)}$, $\ell = 1, 2, \dots, n$, and then adjust the measure to be distributed at these points

$$(4.2) \quad d\mu(\lambda) = \sum_{\ell=1}^n w_\ell \delta(\lambda - \lambda_0^{(\ell)}),$$

with the nonnegative weights w_ℓ , that sum to 1, chosen so that the moments take their desired values. Then with this measure we obtain the bound

$$(4.3) \quad \operatorname{Re}[e^{i\theta} v(t)] \geq a_0 \sum_{\ell=1}^n w_\ell \operatorname{Re} \left[e^{i\theta} \sum_{k=1}^m \frac{\alpha_k e^{-i\omega_k(t-t_0)}}{\lambda_0^{(\ell)} - z(\omega_k)} \right].$$

By varying θ from 0 to 2π we obtain bounds that confine $v(t)$ to a convex region in the complex plane. Of course, if we are only interested in bounding $\operatorname{Re}[v(t)]$, then it suffices to take $\theta = 0$ or π .

Figure 4.1 and Figure 4.2 depict the lower and upper bounds on $\operatorname{Re}[v(t)]$ for two systems ($z(\omega) = 2 + i/\omega$ in Figure 4.1, thus mimicking the low-frequency dielectric response of a lossy dielectric material, and $z(\omega) = 2 - 2/\omega^2$ in Figure 4.2, thus mimicking the dielectric response of a plasma), when the coefficients α_k in (4.3) are chosen such that the bounds are extremely tight at $t_0 = 0$, according to (2.10). For both systems, the bounds on $\operatorname{Re}[v(t)]$ are tighter the higher the amount of pieces of information on the system is incorporated. Notice that the bounds colored in black (the largest ones) correspond to the case where only the zeroth-order moment M_0 of the measure is known but not the value of a_0 : in such a case, as shown by the zoomed graph in the blue box, at $t = 0$, the upper bound takes value 1 and the lower bound takes value 0, which are the smallest and the largest values a_0 can take. On the other hand, when a_0 is assigned, the value that the corresponding

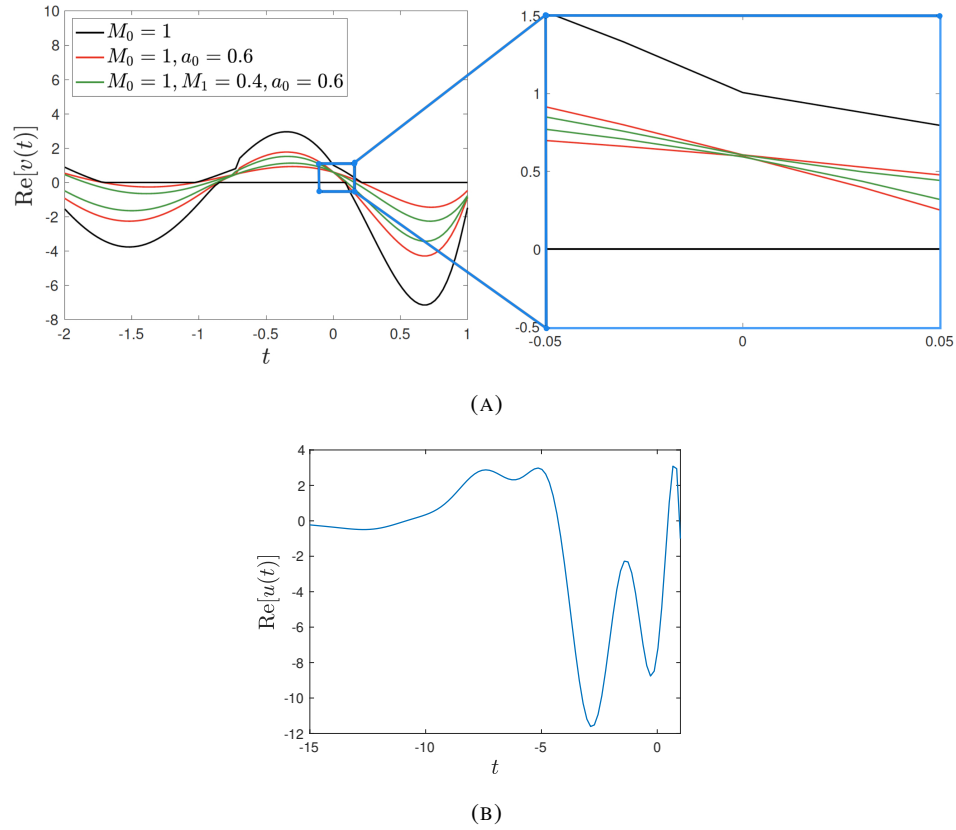


FIGURE 4.1. (a) Bounds on the real part of the response of the system, $\text{Re}[v(t)]$ (4.3), when the system is such that $z(\omega) = 2 + i/\omega$ and the input signal $\text{Re}[u(t)]$ is the one depicted in (b) (with $c(\omega)=1$). We choose the frequencies ω_k to be $[1 + 1i; 0.5 + 0.3i; 2 + 0.5i]$, and we select the coefficients α_k according to (2.10) so that the bounds are extremely tight at $t_0 = 0$, whereas the point masses $\lambda_0^{(\ell)}$ and the weights w_ℓ are chosen for each moment of time t such that the minimum value of (4.1) is attained while the moments of the measure take their desired values. Specifically, the bounds on $\text{Re}[v(t)]$ are plotted for three different scenarios, as shown by the legend.

bounds take at $t = 0$ is exactly $a_0 = 0.6$, as shown by the zoomed graph in the blue box. The graphs show clearly that, in order to estimate the system parameter a_0 , one has just to measure the response of the system at a specific moment of time t_0 (if the applied field is carefully chosen).

These are the type of bounds used in [39] to bound the temporal response of two-phase composites in antiplane elasticity. It is not yet clear whether those bounds can be derived from variational principles. In general, in the theory of composites,

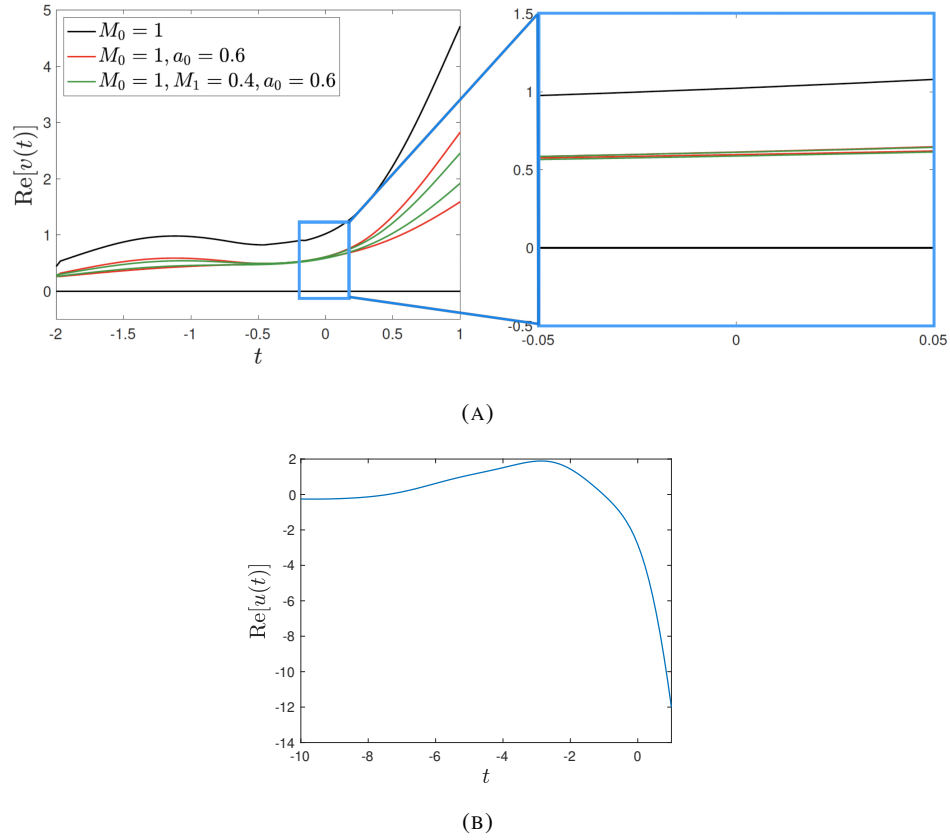


FIGURE 4.2. (a) Bounds on the real part of the response of the system, $\text{Re}[v(t)]$ (4.3), when the system is such that $z(\omega) = 2 - 2/\omega^2$ and the input signal $\text{Re}[u(t)]$ is the one depicted in (b) (with $c(\omega)=1$). We choose the frequencies ω_k to be $[1 + 1i; 0.5 + 0.3i; 2 + 0.5i]$, as in the case depicted in Figure 4.1.

variational methods have proven to be more powerful than analytic approaches. Variational methods produce tighter bounds that often easily extend to multiphase composites: see the books [2, 13, 46, 56, 57] and references therein. For example, the variational approach gives tighter bounds on the complex permittivity at constant frequency of two-phase lossy composites [30] than the bounds obtained by the analytic approach [8, 44]. It also produces bounds on the complex effective bulk and shear moduli of viscoelastic composites [20, 51]. An exception is bounds that correlate the complex effective dielectric constant at more than two frequencies [45] that have yet to be obtained by a systematic variational approach. Variational bounds in the time domain are available [12, 38], but these are nonlocal in time.

4.2 Using an appropriate input signal to predict the response at a given frequency

Naturally, if one is interested in the response $v_0(t)$ at a given (possibly complex) frequency ω_0 , the easiest solution is to take an input signal $u_0(t)$ at that frequency. However, it might not be easy to experimentally generate a signal at that frequency or it might not be easy to measure the response at that frequency. The problem becomes: find complex constants $\alpha_1, \alpha_2, \dots, \alpha_m$ such that

$$(4.4) \quad \sup_{\lambda \in [-1, 1]} \left| \sum_{k=1}^m \frac{\alpha_k}{\lambda - z_k} - \frac{1}{\lambda - z_0} \right| \leq \epsilon_m$$

with $z_k = z(\omega_k)$, $k = 0, 1, \dots, m$. Defining the polynomials $p(\lambda)$ and $q(\lambda)$ as in (2.15) and (2.16) one needs to find $p(\lambda)$ of degree $m - 1$ such

$$(4.5) \quad \sup_{\lambda \in [-1, 1]} \left| \frac{(\lambda - z_0)p(\lambda) - q(\lambda)}{(\lambda - z_0)q(\lambda)} \right| \leq \epsilon_m.$$

Proceeding as before we choose

$$(4.6) \quad (\lambda - z_0)p(\lambda) = q(\lambda) - b_m T_{m-1}(\lambda) \quad \text{with } b_m = q(z_0)/T_{m-1}(z_0),$$

where b_m has been chosen so that the polynomial $q(\lambda) - b_m T_{m-1}(\lambda)$ has a factor of $(\lambda - z_0)$. Then the residues of $R(\lambda) = p(\lambda)/q(\lambda)$ are given by

$$(4.7) \quad \begin{aligned} \alpha_k &= -b_m \frac{T_{m-1}(z_k)}{(z_k - z_0) \prod_{j \neq k} (z_k - z_j)} \\ &= -\frac{T_{m-1}(z_k) \prod_{j \neq 0} (z_0 - z_j)}{T_{m-1}(z_0) (z_k - z_0) \prod_{j \neq 0, k} (z_k - z_j)} \end{aligned}$$

and

$$(4.8) \quad \sup_{\lambda \in [-1, 1]} |(\lambda - z_0)p(\lambda) - q(\lambda)| = \sup_{\lambda \in [-1, 1]} |b_m T_{m-1}(\lambda)| = |b_m|,$$

so that (4.4) holds with

$$(4.9) \quad \epsilon_m = \frac{|b_m|}{d_0 \inf_{\lambda \in [-1, 1]} |q(\lambda)|},$$

where d_0 denotes the distance from z_0 to the interval $[-1, 1]$. Joukowski's map yields

$$(4.10) \quad z_0 = \frac{1}{2} \left(\zeta_0 + \frac{1}{\zeta_0} \right) \quad \text{with } R = |\zeta_0| > 1,$$

whence

$$(4.11) \quad T_{m-1}(z_0) = \frac{1}{2} \left(\zeta_0^{m-1} + \frac{1}{\zeta_0^{m-1}} \right).$$

Moreover, since ζ_0 runs over a circle of radius R , we have

$$(4.12) \quad d_0 = \inf_{\lambda \in [-1, 1]} \frac{|\zeta_0^2 - 2\zeta_0\lambda + 1|}{2} \geq \frac{(R-1)^2}{2R}$$

and

$$(4.13) \quad |T_{m-1}(z_0)| \geq \frac{1}{2}(R^{m-1} - R^{1-m}), \quad m \geq 2,$$

implying

$$(4.14) \quad |b_m| \leq \frac{2|q(z_0)|}{R^{m-1} - R^{1-m}}.$$

All in all, the relevant bound ϵ_m satisfies

$$(4.15) \quad |\epsilon_m| \leq \frac{4R}{(R-1)^2} \frac{1}{R^{m-1} - R^{1-m}} \sup_{\lambda \in [-1, 1]} \frac{|q(z_0)|}{|q(\lambda)|}.$$

We obtain an exponential decay $\epsilon_m \rightarrow 0$ as $m \rightarrow \infty$ provided the geometry of the loci $z_1, z_2, \dots, z_m$ is subject to the following condition: for a positive constant $r < R$, each $z_j \in H(r) = H_1(r) \cup H_2(r) \cup H_3(r)$ where

$$\begin{aligned} H_1(r) &= \left\{ z : \left| \frac{z - z_0}{z + 1} \right| \leq r, \operatorname{Re} z \leq -1 \right\}, \\ H_2(r) &= \left\{ z : \left| \frac{z - z_0}{\operatorname{Im} z} \right| \leq r, \operatorname{Re} z \in [-1, 1] \right\}, \\ H_3(r) &= \left\{ z : \left| \frac{z - z_0}{z - 1} \right| \leq r, \operatorname{Re} z \geq 1 \right\}. \end{aligned}$$

In other words, all of the z_j must be close to z_0 in the precise sense that $z_j \in H(r)$. Note that, as shown in Figure 4.3a, in case $r < 1$, H_1 and H_3 are sectors of disks, while H_2 is a portion of an ellipse. For $r \in (1, R)$ these regions are complements of disks/ellipse, containing the point z_0 , as shown in Figure 4.3c. Some of these regions can be empty, depending on the position of z_0 .

A conservative choice would be $r = 1$ (see Figure 4.3b), in which situation H_1 and H_3 are bounded by straight lines, while H_2 is a parabola. To fix ideas, let us assume $z_0 = x_0 + iy_0$ with $x_0 \geq 0$ and $y_0 \geq 0$, all other cases being symmetrical. Then the euclidean region $H(1)$ where $z_1, z_2, \dots, z_m$ are allowed consists of points $z = x + iy$ subject to the constraints:

$$(4.16) \quad x \geq 1 \quad \text{and} \quad \operatorname{dist}(z, z_0) \leq \operatorname{dist}(z, 1),$$

union with

$$(4.17) \quad x \in [-1, 1] \quad \text{and} \quad (x - x_0)^2 + y_0^2 \leq 2y_0y.$$

If $y_0 = 0$, then necessarily $x_0 > 1$, and H is simply the right half-plane $x > \frac{1+x_0}{2}$, while in the case $y_0 > 0$, $H(1)$ is the interior of a parabola with vertex at $(x_0, \frac{y_0}{2})$, within the band $|x| \leq 1$, union with the polygonal region defined by the first distance inequality (in $x \geq 1$).

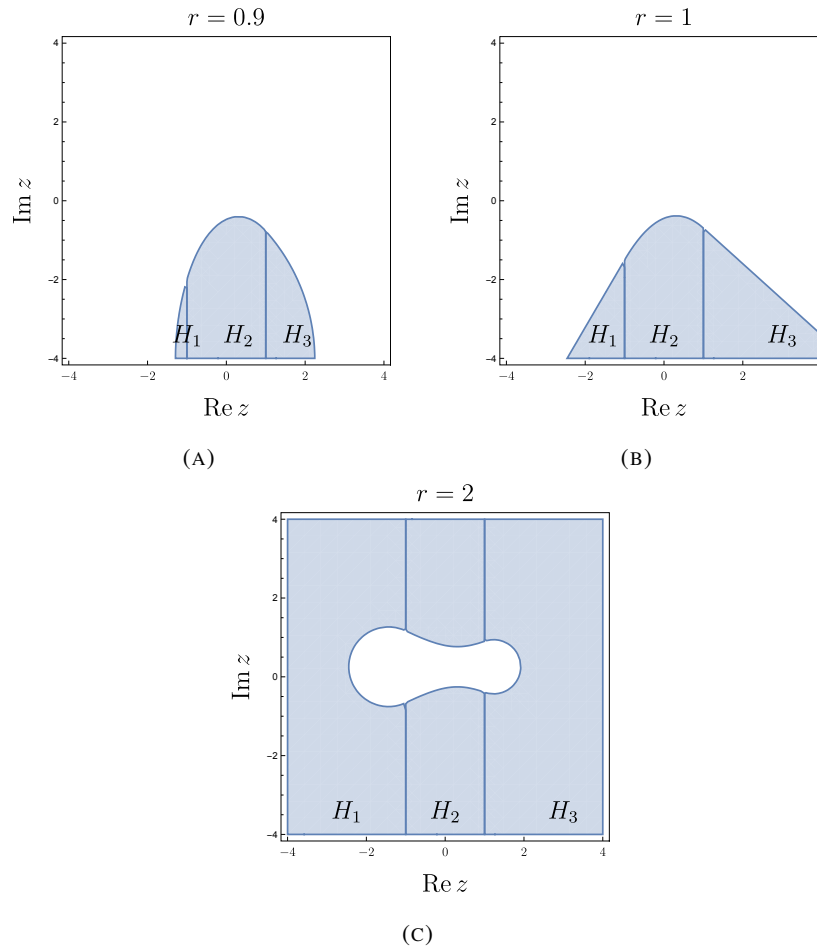


FIGURE 4.3. Representation of the loci z_k for a system for which $z_0 = 0.308824 - 0.764706i$ and $R = 2.061$.

Now with an input signal of the form (3.1), with $\beta_k = \alpha_k/c(\omega_k)$, generating the output function $v(t)$ given by (3.2), (4.4) implies the bound

$$(4.18) \quad |v(t_0) - v_0(t_0)| \leq a_0 \epsilon_m,$$

where

$$(4.19) \quad v_0(t_0) = a_0 F_\mu(z(\omega_0))$$

is the response at time t_0 to the single frequency input signal

$$(4.20) \quad u_0(t) = e^{-i\omega_0(t-t_0)}/c(\omega_0).$$

Of course, because this response $v_0(t)$ is for a single frequency, $v_0(t_0)$ determines $v_0(t)$ for all t .

In Figure 4.4 we depict the response $v_0(t)$ of a given system subject to an input signal at the frequency ω_0 and we compare the value it takes at $t_0 = 0$ with the value taken by the bounds on the response $v(t)$ of a system having the same values of the moments of the measure but subject to a multiple-frequency signal with amplitudes α_k chosen such that the bounds are extremely tight at $t_0 = 0$: $v_0(t_0)$ lies, as expected, between the bounds on $v(t)$ at $t = t_0$,

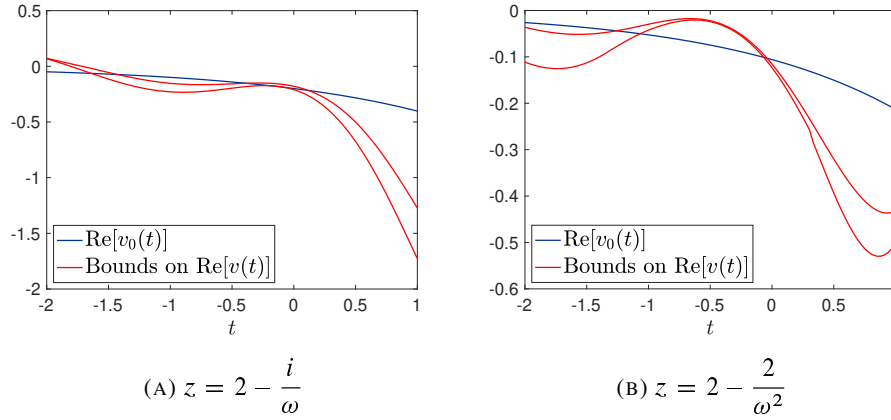


FIGURE 4.4. Comparison between the response $v_0(t)$ of a given system with point masses at -0.5 and 0.5 , due to an input at the frequency $\omega_0 = 0.7i$, and the upper and lower bounds on the response $v(t)$ of a system having the same value of the moments of the measure M_i ($M_0 = 1$ and $M_1 = 0.4$) and subject to an input signal of the type (3.1), with ω_k given by $[1 + 1i; 0.5 + 0.3i; 2 + 0.5i]$ and coefficients β_k chosen accordingly to (3.3) and (4.7). Notice that in both cases the value of $v_0(t)$ at $t_0 = 0$ lies between the bounds on $v(t)$ at $t_0 = 0$.

REMARK 4.1. The analysis is easily extended to the case where the response $v_0(t)$ is known for a given ω_0 but one wants to predict the derivative

$$(4.21) \quad \frac{v_0(t_0)}{d\omega_0} = a_0 \frac{dF_\mu(z)}{dz} \Big|_{z=z(\omega_0)} \frac{dz(\omega_0)}{d\omega_0}.$$

As

$$(4.22) \quad \frac{dF_\mu(z)}{dz} = \int_{-1}^1 \frac{d\mu(\lambda)}{(\lambda - z)^2},$$

the problem becomes: find complex constants $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$(4.23) \quad \sup_{\lambda \in [-1, 1]} \left| \sum_{j=1}^m \frac{\alpha_j}{\lambda - z_j} + \frac{\alpha_0}{\lambda - z_0} - \frac{1}{(\lambda - z_0)^2} \right| \leq \epsilon_m.$$

Defining the polynomials $p(\lambda)$ and $q(\lambda)$ as in (2.15) and (2.16) one needs to find $p(\lambda)$ of degree $m - 1$ such that

$$(4.24) \quad \sup_{\lambda \in [-1, 1]} \left| \frac{(\lambda - z_0)^2 p(\lambda) - q(\lambda)[1 - \alpha_0(\lambda - z_0)]}{(\lambda - z_0)^2 q(\lambda)} \right| \leq \epsilon_m.$$

We now choose

$$(4.25) \quad (\lambda - z_0)^2 p(\lambda) = q(\lambda)[1 - \alpha_0(\lambda - z_0)] - b_m T_{m-1}(\lambda),$$

with

$$(4.26) \quad b_m = q(z_0)/T_{m-1}(z_0), \quad \alpha_0 = \frac{q'(\lambda) - b_m T'_{m-1}(\lambda)}{q(z_0)},$$

selected so that the polynomial on the right-hand side of (4.25) has a factor of $(\lambda - z_0)^2$, in which $q'(\lambda) = dq(\lambda)/d\lambda$ and $T'_{m-1}(\lambda) = dT_{m-1}(\lambda)/d\lambda$. So the residues α_k , for $k \neq 0$, are now given by

$$(4.27) \quad \begin{aligned} \alpha_k &= -b_m \frac{T_m(z_k)}{(z_k - z_0)^2 \prod_{j \neq k} (z_k - z_j)} \\ &= -\frac{T_m(z_k) \prod_{j \neq 0} (z_0 - z_j)}{T_m(z_0) (z_k - z_0)^2 \prod_{j \neq 0, k} (z_k - z_j)}, \end{aligned}$$

where b_m is still given by (4.6) and

$$(4.28) \quad \begin{aligned} &\sup_{\lambda \in [-1, 1]} |(\lambda - z_0)^2 p(\lambda) - q(\lambda)[1 - \alpha_0(\lambda - z_0)]| \\ &= \sup_{\lambda \in [-1, 1]} |b_m T_{m-1}(\lambda)| = |b_m| \end{aligned}$$

so that (4.23) holds with

$$(4.29) \quad \epsilon_m = \frac{|b_m|}{d_0^2 \inf_{\lambda \in [-1, 1]} |q(\lambda)|}.$$

Apart from an extra factor of d_0 , this is exactly the same as the formula (4.9), and so the convergence $\epsilon_m \rightarrow 0$ as $m \rightarrow \infty$ is assured provided for a positive constant $r < R$, with each $z_j \in H(r) = H_1(r) \cup H_2(r) \cup H_3(r)$.

5 A General Framework for a Wide Variety of Time-Dependent Problems

The second part of the present article deals with a sketch of a unifying framework that allows us to treat the conductivity or antiplane viscoelastic response of bodies containing an inclusion in a matrix where one is interested in estimating the volume and/or shape of the inclusion.

5.1 The general framework

Suppose that, in some Hilbert (or vector space) $\mathcal{H}$, one is interested in solving for $\mathbf{J}$ the equations

$$(5.1) \quad \mathbf{J} = \mathbf{L}\mathbf{E}, \quad \mathbf{Q}\mathbf{J} = \mathbf{J}, \quad \mathbf{Q}\mathbf{E} = \mathbf{E}_0,$$

for a prescribed field $\mathbf{E}_0$, where $\mathbf{L} : \mathcal{H} \rightarrow \mathcal{H}$ is an operator satisfying appropriate boundedness and coercivity conditions, and $\mathbf{Q}$ is a self-adjoint projection onto a subspace $\mathcal{S}$ of $\mathcal{H}$, so that both $\mathbf{E}_0$ and $\mathbf{J}$ lie in $\mathcal{S}$. Note that we can rewrite (5.1) as

$$(5.2) \quad \mathbf{J} = \mathbf{L}\mathbf{E}' - \mathbf{s}, \quad \Gamma_1 \mathbf{E}' = \mathbf{E}', \quad \Gamma_1 \mathbf{J} = 0,$$

with $\mathbf{E}' = \mathbf{E} - \mathbf{E}_0$, $\mathbf{s} = -\mathbf{L}\mathbf{E}_0$ being the source term, and $\Gamma_1 = \mathbf{I} - \mathbf{Q}$ being the projection onto the orthogonal complement of $\mathcal{S}$ in $\mathcal{H}$. These equations arise in the extended abstract theory of composites and apply to an enormous plethora of linear continuum equations in physics: see, for example, the books [19, 46] and the articles [47–50].

The simplest example is for electrical conductivity (and equivalent equations), where one has

$$(5.3) \quad \begin{aligned} \mathbf{j}'(\mathbf{x}) &= \sigma(\mathbf{x})\mathbf{e}(\mathbf{x}) - \mathbf{s}(\mathbf{x}), \quad \Gamma_1 \mathbf{e} = \mathbf{e}, \quad \Gamma_1 \mathbf{j}' = 0, \\ \text{with } \Gamma_1 &= \nabla(\nabla^2)^{-1}\nabla, \end{aligned}$$

where $\sigma(\mathbf{x})$ is the conductivity tensor, while $\nabla \cdot \mathbf{s}$, $\mathbf{j} = \mathbf{j}' + \mathbf{s}$, and $\mathbf{e}$ are the current source, current and electric field, and $(\nabla^2)^{-1}$ is the inverse Laplacian (there is obviously considerable flexibility in the choice of $\mathbf{s}(\mathbf{x})$, the only constraints being square integrability and that $\nabla \cdot \mathbf{s}$ equals the current source). As current is conserved, $\nabla \cdot \mathbf{j} = \nabla \cdot \mathbf{s}$, implying $\nabla \cdot \mathbf{j}' = 0$, which is clearly equivalent to $\Gamma_1 \mathbf{j}' = 0$. In Fourier space $\Gamma_1(\mathbf{k}) = \mathbf{k} \otimes \mathbf{k} / k^2$, and $\Gamma_1 \mathbf{e} = \mathbf{e}$ implies the Fourier components $\hat{\mathbf{e}}(\mathbf{k})$ of $\mathbf{e}$ satisfy $\hat{\mathbf{e}} = -i\mathbf{k}(i\mathbf{k} \cdot \hat{\mathbf{e}}) / k^2$. So $\mathbf{e}$ is the gradient of a potential with Fourier components $-i\mathbf{k} \cdot \hat{\mathbf{e}} / k^2$. In antiplane elasticity one takes a material with a cross-section in the (x_1, x_2) -plane that is independent of x_3 , applies shearing in the x_3 -direction and observes warping of the cross-section. The displacement $u_3(\mathbf{x})$ in the x_3 -direction that is associated with this warping satisfies a conductivity-type equation $\nabla \cdot G \nabla u_3 = \nabla s$, where ∇s is a shearing source term (dependent on (x_1, x_2)), $G(x_1, x_2)$ is the shear modulus, and correspondingly $\mathbf{e} = -\nabla u_3$ and $\mathbf{j} = G \nabla u_3$. The antiplane response also governs the warping of rods under torsion for rods that have a noncircular cylindrical shape and are composed of long fibers aligned with the cylinder axis and embedded in a matrix such that the fiber separation is much less than the cylinder circumference.

One approach to solving (5.1) is to apply $\mathbf{Q}$ to both sides of the relation $\mathbf{E} = \mathbf{L}^{-1}\mathbf{J}$ to obtain $\mathbf{E}_0 = \mathbf{Q}\mathbf{L}^{-1}\mathbf{Q}\mathbf{J}$, giving

$$(5.4) \quad \mathbf{J} = [\mathbf{Q}\mathbf{L}^{-1}\mathbf{Q}]^{-1}\mathbf{E}_0,$$

where the inverse is on the subspace $\mathcal{S}$. In general, the operator $\mathbf{L}$ depends on the frequency ω and $\mathbf{E}_0$ could depend on ω too. Then the response at this frequency is

$$(5.5) \quad \widehat{\mathbf{J}}(\omega) = [\mathbf{Q}(\mathbf{L}(\omega))^{-1}\mathbf{Q}]^{-1}\widehat{\mathbf{E}}_0(\omega).$$

We are interested in the response in the time domain when $\widehat{\mathbf{E}}_0(\omega) = \beta(\omega)\mathbf{E}_0$ for some complex amplitude $\beta(\omega)$ and $\mathbf{E}_0 \in \mathcal{S}$ does not depend on ω . In particular, for a sum of a finite number of (possibly complex) frequencies in the time domain the input signal is

$$(5.6) \quad \mathbf{E}_0(t) = \sum_{k=1}^m e^{-i\omega_k(t-t_0)}\widehat{\mathbf{E}}_0(\omega_k) = \sum_{k=1}^m \beta_k e^{-i\omega_k(t-t_0)}\mathbf{E}_0$$

with $\beta_k = \beta(\omega_k)$.

The resulting field $\mathbf{J}(t)$ is then

$$(5.7) \quad \mathbf{J}(t) = \sum_{k=1}^m \beta_k e^{-i\omega_k(t-t_0)}[\mathbf{Q}(\mathbf{L}(\omega_k))^{-1}\mathbf{Q}]^{-1}\mathbf{E}_0,$$

and we want this to have a simple approximate formula at time t_0 .

To make progress we use another approach to solving (5.1). We introduce a “reference medium” $\mathbf{L}_0 = c_0\mathbf{I}$ where the real constant c_0 is chosen so that $\mathbf{L} - \mathbf{L}_0$ is coercive and introduce the so-called “polarization field”

$$(5.8) \quad \mathbf{G} = (\mathbf{L} - \mathbf{L}_0)\mathbf{E} = (\mathbf{L} - c_0\mathbf{I})\mathbf{E} = \mathbf{J} - c_0\mathbf{E}.$$

Applying the projection $\mathbf{I} - \mathbf{Q}$ to this equation gives

$$(5.9) \quad (\mathbf{I} - \mathbf{Q})\mathbf{G} = -c_0(\mathbf{E} - \mathbf{E}_0) = c_0\mathbf{E}_0 - c_0(\mathbf{L} - c_0\mathbf{I})^{-1}\mathbf{G},$$

and solving this for $\mathbf{G}$ yields

$$(5.10) \quad \mathbf{G} = c_0[(\mathbf{I} - \mathbf{Q}) + c_0(\mathbf{L} - c_0\mathbf{I})^{-1}]^{-1}\mathbf{E}_0.$$

Finally, applying $\mathbf{Q}$ to both sides gives

$$(5.11) \quad \mathbf{J} = c_0\{\mathbf{Q} + \mathbf{Q}[(\mathbf{I} - \mathbf{Q}) + c_0(\mathbf{L} - c_0\mathbf{I})^{-1}]^{-1}\mathbf{Q}\}\mathbf{E}_0.$$

By comparing (5.4) and (5.10) we have

$$\begin{aligned} [\mathbf{Q}\mathbf{L}^{-1}\mathbf{Q}]^{-1} &= c_0\mathbf{Q} + c_0\mathbf{Q}[(\mathbf{I} - \mathbf{Q}) + c_0(\mathbf{L} - c_0\mathbf{I})^{-1}]^{-1}\mathbf{Q} \\ &= c_0\{\mathbf{Q} - 2\mathbf{Q}[\Psi - (\mathbf{L} + c_0\mathbf{I})(\mathbf{L} - c_0\mathbf{I})^{-1}]^{-1}\mathbf{Q}\}, \end{aligned}$$

where $\Psi = 2\mathbf{Q} - \mathbf{I}$ has eigenvalues ± 1 . It is not obvious at all that the right-hand side of (5.12) is independent of c_0 but the preceding derivation shows this. This type of solution using a reference medium $\mathbf{L}_0$ (that need not be proportional to $\mathbf{I}$) is well-known in the theory of composites: see, for example, chapter 14 of [46], [65], and references therein.

Now assume $\mathbf{L}$ takes the form

$$(5.12) \quad \mathbf{L} = c_1\mathbf{P} + c_2(\mathbf{I} - \mathbf{P}),$$

where $\mathbf{P}$ is a projection operator onto a subspace $\mathcal{P}$ of $\mathcal{H}$. In the theory of composites for two phase composites, one frequently has

$$(5.13) \quad \mathbf{L} = c_1 \mathbf{I} \chi(\mathbf{x}) + c_2 \mathbf{I}(1 - \chi(\mathbf{x})),$$

where the characteristic function $\chi(\mathbf{x})$ is 1 in phase 1 and 0 in phase 2, and c_1 and c_2 could be the material moduli. For the antiplane elasticity problem one has $c_1 = G_1$ and $c_2 = G_2$, where G_1 and G_2 are the shear moduli of the phases. We take the limit $c_0 \rightarrow c_2$ and then (5.12) becomes

$$(5.14) \quad [\mathbf{Q}\mathbf{L}^{-1}\mathbf{Q}]^{-1} = c_2 \mathbf{Q} + 2c_2 \mathbf{Q}\mathbf{P}[\mathbf{P}\Psi\mathbf{P} - z\mathbf{P}]^{-1}\mathbf{P}\mathbf{Q},$$

where the operator inverse is to be taken on the subspace $\mathcal{P}$ and

$$(5.15) \quad z = \frac{c_1 + c_2}{c_1 - c_2}.$$

Note that $\mathbf{P}\Psi\mathbf{P}$, like Ψ , has norm at most 1. In general, the two moduli c_1 and c_2 depend on the frequency ω and hence z defined by (5.15) will also, i.e., $z = z(\omega)$. Given an input field of the form (5.6) and letting

$$(5.16) \quad \mathbf{J}_2(t) = \mathbf{Q} \sum_{k=1}^m \beta_k c_2(\omega_k) e^{-i\omega_k(t-t_0)} \mathbf{E}_0$$

denote the response when $\mathbf{P} = 0$, i.e., when $\mathbf{L}(\omega) = c_2(\omega)\mathbf{I}$, the corresponding output field can be taken to be

$$(5.17) \quad \mathbf{v}(t) = \mathbf{J}(t) - \mathbf{J}_2(t) = \mathbf{Q} \sum_{k=1}^m \alpha_k e^{-i\omega_k(t-t_0)} 2\mathbf{P}[\mathbf{P}\Psi\mathbf{P} - z_k \mathbf{P}]^{-1} \mathbf{P}\mathbf{E}_0,$$

with

$$(5.18) \quad z_k = z(\omega_k) = \frac{c_1(\omega_k) + c_2(\omega_k)}{c_1(\omega_k) - c_2(\omega_k)}, \quad \alpha_k = \beta_k c_2(\omega_k),$$

and we arrive back at the problem we have been studying. In particular, with constants α_k given by (2.10) the inequality (2.32) with $n = 0$ implies

$$(5.19) \quad |\mathbf{J}(t_0) - \mathbf{J}_2(t_0) - 2\mathbf{Q}\mathbf{P}\mathbf{E}_0| \leq 4|\mathbf{P}\mathbf{E}_0|/(2d_{\min})^m.$$

Alternatively, we could have chosen $c_0 = c_1$ and let

$$(5.20) \quad \mathbf{J}_1(t) = \mathbf{Q} \sum_{k=1}^m \beta_k c_1(\omega_k) e^{-i\omega_k(t-t_0)} \mathbf{E}_0$$

denote the response when $\mathbf{P} = \mathbf{I}$, i.e. when $\mathbf{L}(\omega) = c_1(\omega)\mathbf{I}$. Then, similarly to (5.17), we would have

$$(5.21) \quad \mathbf{J}(t) - \mathbf{J}_1(t) = \mathbf{Q} \sum_{k=1}^m \alpha_k e^{-i\omega_k(t-t_0)} 2\mathbf{P}_{\perp}[(\mathbf{P}_{\perp}\Psi\mathbf{P}_{\perp} + z_k \mathbf{P}_{\perp})^{-1} \mathbf{P}_{\perp} \mathbf{E}_0],$$

where z_k is still given by (5.18), but now with $\alpha_k = \beta_k c_1(\omega_k)$, where $\mathbf{P}_{\perp} = \mathbf{I} - \mathbf{P}$ is the projection onto the subspace perpendicular to $\mathcal{P}$. The problem, with $n = 0$

and with the same choice of coefficients α_k , requires a different input signal, i.e., a different choice of the β_k given by $\beta_k = \beta_k/c_1(\omega_k)$, to ensure that

$$(5.22) \quad |\mathbf{J}(t_0) - \mathbf{J}_1(t_0) + 2\mathbf{Q}\mathbf{P}_\perp \mathbf{E}_0| \leq 4|\mathbf{P}\mathbf{E}_0|/(2d_{\min})^m.$$

5.2 Implementation to the theory of composites and its generalizations

In the theory of composites and its generalizations, one can identify a subspace of $\mathcal{S}$ that we call $\mathcal{U}$ of “source free” fields, and we may wish to confine $\mathbf{E}_0$ to this subspace. Then (5.1) can be rewritten as

$$(5.23) \quad \mathbf{J} = \mathbf{L}\mathbf{E}, \quad \mathbf{\Gamma}_2 \mathbf{E} = 0, \quad \mathbf{\Gamma}_1 \mathbf{J} = 0, \quad \mathbf{\Gamma}_0 \mathbf{E} = \mathbf{E}_0,$$

where $\mathbf{\Gamma}_0$ is the projection onto $\mathcal{U}$, $\mathbf{\Gamma}_1$ is the projection onto $\mathcal{E}$, defined as the orthogonal complement of $\mathcal{S}$, and $\mathbf{\Gamma}_2$ is the projection onto $\mathcal{J}$, defined as the orthogonal complement of $\mathcal{U}$ in the subspace $\mathcal{S}$. Then $\mathbf{Q} = \mathbf{\Gamma}_0 + \mathbf{\Gamma}_2$ and the Hilbert space $\mathcal{H}$ has the decomposition

$$(5.24) \quad \mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J},$$

and the projections onto these three subspaces are respectively $\mathbf{\Gamma}_0$, $\mathbf{\Gamma}_1$, and $\mathbf{\Gamma}_2$.

In particular, as observed independently in sections 2.4 and 2.5 of [23] and in chapter 3 of [19], the Dirichlet-Neumann problem can be reformulated as a problem in the theory of composites. In the simplest case of electrical conductivity, where one has an inclusion D (not necessarily simply connected) of (isotropic) conductivity c_1 in a simply connected body Ω having smooth boundary, with c_2 being the (isotropic) conductivity of $\Omega \setminus D$, we may take $\mathcal{H}$ as the space of vector fields that are square integrable with the usual normalized L^2 inner product,

$$(5.25) \quad (\mathbf{A}_1, \mathbf{A}_2) = \frac{1}{|\Omega|} \int_{\Omega} \mathbf{A}_1(\mathbf{x}) \cdot \overline{\mathbf{A}_2(\mathbf{x})} d\mathbf{x},$$

where $|\Omega|$ is the volume of Ω , and take

- $\mathcal{U}$ to consist of gradients of harmonic fields $\mathbf{u}_0 = -\nabla V$ with $\nabla^2 V = 0$ in Ω ,
- $\mathcal{E}$ to consist of gradients $\mathbf{e} = -\nabla V$ with $V = 0$ on the boundary $\partial\Omega$ of Ω ,
- $\mathcal{J}$ to consist of divergent free vector fields $\mathbf{j}$ with $\nabla \cdot \mathbf{j} = 0$ and $\mathbf{j} \cdot \mathbf{n} = 0$ on $\partial\Omega$, where $\mathbf{n}$ is the outwards normal to $\partial\Omega$.

The conductivity of the body may be identified with $\mathbf{L}$ given by (5.12) where $\mathbf{P}$ is the projection onto those fields that are zero outside D . As we are considering time-dependent problems in the quasistatic limit, where the body is small compared to the wavelength and attenuation lengths of electromagnetic waves at the frequencies ω_k , the moduli c_1 and c_2 and the fields are typically complex and frequency-dependent. The fields in $\mathcal{U}$ can be identified either by the values that V takes on the boundary $\partial\Omega$ or by the values that the flux $\mathbf{n} \cdot \nabla V$ takes on the boundary $\partial\Omega$. Thus the equations (5.23) are nothing other than the Dirichlet problem in

the body Ω ,

$$(5.26) \quad \begin{aligned} \mathbf{j} &= \mathbf{L}\mathbf{e}, \quad \mathbf{e} = -\nabla V, \quad \nabla \cdot \mathbf{j} = 0, \quad \mathbf{e}_0 = -\nabla V_0, \\ \nabla^2 V_0 &= 0, \quad V = V_0, \quad \text{on } \partial\Omega, \end{aligned}$$

and the mapping from $\Gamma_0 \mathbf{e}$ to $\Gamma_0 \mathbf{j}$ is nothing other than the Dirichlet to Neumann map giving $\mathbf{n} \cdot \mathbf{j}$ in terms of V on $\partial\Omega$.

For periodic two-phase conducting composites, with unit cell Ω , the framework is similar. We take $\mathcal{H}$ as the space of vector fields that are Ω -periodic with the usual normalized L^2 inner product, given by (5.25), and take

- $\mathcal{U}$ to consist of gradients of constant fields $\mathbf{u}_0$ (that do not depend on $\mathbf{x}$),
- $\mathcal{E}$ to consist of gradients $\mathbf{e} = -\nabla V$ with V being an Ω -periodic potential,
- $\mathcal{J}$ to consist of Ω -periodic divergent-free vector fields $\mathbf{j}$ with $\nabla \cdot \mathbf{j} = 0$, having zero average over Ω .

The conductivity of the body may be identified with $\mathbf{L}$ given by (5.12) where $\mathbf{P}$ is the projection onto those fields in $\mathcal{H}$ that are zero outside phase 1, and c_1 is the (isotropic) conductivity of phase 1 while c_2 is the (isotropic) conductivity of phase 2.

REMARK 5.1. More generally, the conductivity in the periodic composite could be anisotropic, with the conductivity tensor having the special form

$$(5.27) \quad \mathbf{L}(\omega) = c_1(\omega) \mathbf{L}_0 \mathbf{P} + c_2(\omega) \mathbf{L}_0 (\mathbf{I} - \mathbf{P}),$$

where $\mathbf{L}_0$ is a constant positive definite tensor. As $\mathbf{L}_0$ commutes with Γ_0 and $\mathbf{P}$, we can define new orthogonal spaces

$$(5.28) \quad \mathcal{E}' = \mathbf{L}_0^{1/2} \mathcal{E}, \quad \mathcal{J}' = \mathbf{L}_0^{-1/2} \mathcal{J}, \quad \mathcal{U}' = \mathbf{L}_0^{1/2} \mathcal{U} = \mathbf{L}_0^{-1/2} \mathcal{U} = \mathcal{U},$$

and rewrite (5.23) in the form

$$(5.29) \quad \mathbf{J}' = \mathbf{L}' \mathbf{E}', \quad \Gamma'_2 \mathbf{E}' = 0, \quad \Gamma'_1 \mathbf{J}' = 0, \quad \Gamma'_0 \mathbf{E}' = \mathbf{E}'_0,$$

where

$$\begin{aligned} \mathbf{J}' &= \mathbf{L}_0^{-1/2} \mathbf{J}, \quad \mathbf{E}' = \mathbf{L}_0^{1/2} \mathbf{E}, \quad \mathbf{E}'_0 = \mathbf{L}_0^{1/2} \mathbf{E}_0, \\ \mathbf{L}' &= \mathbf{L}_0^{-1/2} \mathbf{L} \mathbf{L}_0^{-1/2} = c_1(\omega) \mathbf{P} + c_2(\omega) (\mathbf{I} - \mathbf{P}), \end{aligned}$$

and

$$(5.30) \quad \Gamma'_0 = \Gamma_0, \quad \Gamma'_1 = \mathbf{L}_0^{-1/2} \Gamma_1 (\Gamma_1 \mathbf{L}_0 \Gamma_1)^{-1}, \quad \Gamma'_2 = \mathbf{I} - \Gamma'_1 - \Gamma'_2$$

are the projections onto $\mathcal{U}' = \mathcal{U}$, $\mathcal{E}'$, and $\mathcal{J}'$, in which the inverse in the formula for Γ'_1 is to be taken on the subspace $\mathcal{E}$. As $\mathbf{L}'$ now takes the same form as (5.12) we are back to the same problem.

Similarly, in a body where the conductivity tensor has the special form (5.27) we may take

- $\mathcal{U}'$ to consist of gradients of fields $\mathbf{u}_0 = -\mathbf{L}_0^{1/2} \nabla V$ with $\nabla \cdot \mathbf{L}_0 \nabla V = 0$ in Ω ,

- $\mathcal{E}'$ to consist of fields $\mathbf{e}' = -\mathbf{L}_0^{1/2} \nabla V$ with $V = 0$ on the boundary $\partial\Omega$ of Ω ,
- $\mathcal{J}'$ to consist of fields $\mathbf{j}'$ with $\nabla \cdot \mathbf{L}_0^{1/2} \mathbf{j}' = 0$ and $(\mathbf{L}_0^{1/2} \mathbf{j}') \cdot \mathbf{n} = 0$ on $\partial\Omega$, where $\mathbf{n}$ is the outwards normal to $\partial\Omega$

as our three orthogonal subspaces. Letting Γ_0 , Γ_1 , and Γ_2 denote the projections onto these three subspaces, respectively, and setting $\mathbf{L}' = c_1(\omega)\mathbf{P} + c_2(\omega)(\mathbf{I} - \mathbf{P})$, the equations (5.28) hold and we may proceed as before.

5.3 Application to solving the Calderon problem with time varying fields

Let us now use ideas from the Calderon problem to solve the inverse problem of finding the inclusion D from boundary measurements on $\partial\Omega$. With Ω being a three-dimensional body, we can take

$$(5.31) \quad V_0 = e^{i\mathbf{k} \cdot \mathbf{x}} \quad \text{with } k_1, k_2 \text{ real} \quad \text{and} \quad k_3 = i\sqrt{k_1^2 + k_2^2},$$

where the last condition implies $\mathbf{k} \cdot \mathbf{k} = 0$, which ensures that V_0 is harmonic. Then (5.19) implies

$$(5.32) \quad (\mathbf{J}(t_0) - \mathbf{J}_1(t_0) + 2\mathbf{QPE}_0, \nabla e^{i\mathbf{k}' \cdot \mathbf{x}}) \leq 4|\mathbf{k}||\mathbf{k}'|/(2d_{\min})^m$$

for all real or complex $\mathbf{k}'$. We now choose $\mathbf{k}'$ with

$$(5.33) \quad k'_3 = -k_3, \quad k'_1, k'_2 \text{ real and with } (k'_1)^2 + (k'_2)^2 = k_1^2 + k_2^2$$

to ensure that $e^{i\mathbf{k}' \cdot \mathbf{x}}$ is harmonic, and so that

$$(5.34) \quad \begin{aligned} & (2\mathbf{QPE}_0, \nabla e^{i\mathbf{k}' \cdot \mathbf{x}}) \\ &= 2(\mathbf{PE}_0, \mathbf{Q} \nabla e^{i\mathbf{k}' \cdot \mathbf{x}}) = 2(\mathbf{PE}_0, \nabla e^{i\mathbf{k}' \cdot \mathbf{x}}) \\ &= 2(k_1 k'_1 + k_2 k'_2 - k_1^2 - k_2^2) \frac{1}{|\Omega|} \int_D e^{i(k_1 - k'_1)x_1 + i(k_2 - k'_2)x_2} d\mathbf{x} \end{aligned}$$

only depends on the Fourier coefficients of the characteristic function associated with D . Then, using integration by parts,

$$(5.35) \quad (\mathbf{J}(t_0) - \mathbf{J}_1(t_0), \nabla e^{i\mathbf{k}' \cdot \mathbf{x}}) = \frac{1}{|\Omega|} \int_{\partial\Omega} [\mathbf{J}(t_0) - \mathbf{J}_1(t_0)] \cdot \mathbf{n} e^{i\mathbf{k}' \cdot \mathbf{x}} dS,$$

where $\mathbf{J}(t_0) \cdot \mathbf{n}$ can be measured, while $\mathbf{J}_1(t_0) \cdot \mathbf{n}$ can be computed. As there is nothing special about the x_3 -axis, we may rotate the Cartesian coordinates to get estimates of other Fourier coefficients of the characteristic function associated with D . We may also take $\mathbf{E}_0$ as constant and replace $\nabla e^{i\mathbf{k}' \cdot \mathbf{x}}$ by $\mathbf{E}_0$ to get

$$(5.36) \quad \begin{aligned} (\mathbf{J}(t_0) - \mathbf{J}_1(t_0) + 2\mathbf{QPE}_0, \mathbf{E}_0) &= (\mathbf{J}(t_0) - \mathbf{J}_1(t_0), \mathbf{E}_0) + |\mathbf{E}_0|^2 |D|/|\Omega| \\ &\leq 4|\mathbf{E}_0|^2/(2d_{\min})^m, \end{aligned}$$

thus giving an estimate of the volume fraction $|D|/|\Omega|$ that D occupies in the body (i.e., the Fourier coefficient at $\mathbf{k} = 0$).

With Ω being a two-dimensional body, the situation is similar. We take

$$(5.37) \quad k_1 \text{ real and } k_2 = i k_1, \quad k'_1 = -k_1, \quad k'_2 = -i k_1,$$

and (5.34) is replaced by

$$(5.38) \quad (2\mathbf{QPE}_0, \nabla e^{i\mathbf{k}' \cdot \mathbf{x}}) = -4k_1^2 \int_D e^{2ik_1 x_1} d\mathbf{x},$$

while (5.35) and (5.36) still hold. Again we approximately recover the Fourier coefficients of the characteristic function associated with D from measurements of $\mathbf{J}(t_0) \cdot \mathbf{n}$ and computations of $\mathbf{J}_1(t_0) \cdot \mathbf{n}$.

In the usual Calderon problem one solves the inverse problem by taking $|\mathbf{k}|$ to be very large, according to the so-called complex geometric optics approach [55]. Here we see that there is no need to take $|\mathbf{k}|$ to be very large if we allow time-dependent applied fields. For electromagnetism in nonmagnetic media, the measurements are difficult as the time response is typically extremely rapid (from table 7.7.1 in [24] we see that electromagnetic relaxation times in seconds for copper, distilled water, corn oil, and mica are 1.5×10^{-19} , 3.6×10^{-6} , 0.55, and 5.1×10^4 , respectively, and measurements would need to be taken on these time scales). On the other hand, for the equivalent magnetic permeability, fluid permeability, or antiplane elasticity problems, the relaxation times are much more reasonable [5, 26, 36] and measurements in the time domain become feasible. Even in electrical systems one can get long relaxation times, such as the time to charge a capacitor.

From an experimental perspective, even for antiplane elasticity, it would be difficult to obtain the high-order Fourier coefficients of D as the boundary fields needed to retrieve this information have a very fast spatial decay which would be difficult to generate and measure.

REMARK 5.2. Instead of taking $\mathbf{E}_0(t) = \Gamma_0 \mathbf{E}(t)$ and $\mathbf{J}(t)$ as our input and output fields, one could take $\mathbf{J}_0(t) = \Gamma_0 \mathbf{J}(t)$ and $\mathbf{E}(t)$ as our input and output fields. Then one has

$$(5.39) \quad \mathbf{E} = \mathbf{L}^{-1} \mathbf{J}, \quad \Gamma_1 \mathbf{J} = 0, \quad \Gamma_2 \mathbf{E} = 0, \quad \Gamma_0 \mathbf{J} = \mathbf{J}_0,$$

which is exactly of the same form as (5.23), but with $\mathbf{L}$ replaced by $\mathbf{L}^{-1}$ and the roles of Γ_1 , Γ_2 , and $\mathbf{E}$ and $\mathbf{J}$, and $\mathbf{E}_0$ and $\mathbf{J}_0$ interchanged. So all the preceding analysis immediately applies to this dual problem too.

5.4 Generalizations

In many problems of interest, the fields in $\mathcal{H}$ take values in a, say, s -dimensional tensor space $\mathcal{T}$ and the operator $\mathbf{L} : \mathcal{H} \rightarrow \mathcal{H}$ in (5.1), appropriately defined, is frequency dependent with the properties that

- $\mathbf{L}(\omega)$ is an analytic function of ω in the upper half-plane $\text{Im}(\omega) > 0$,
- $\text{Im}[\omega \mathbf{L}(\omega)] \geq 0$ when $\text{Im}(\omega) > 0$,
- $\overline{\mathbf{L}(\omega)} = \mathbf{L}(-\bar{\omega})$ when $\text{Im}(\omega) > 0$,

where the overline denotes complex conjugation. By appropriately defined we mean that $\mathbf{L}(\omega)$ (and accordingly $\mathbf{J}$) may need to be multiplied by a function of ω , for example i , ω , or $i\omega$, to achieve these properties. In the case of materials where $\mathbf{L}$ acts locally in real space, i.e., if $\mathbf{Q} = \mathbf{L}\mathbf{P}$, then $\mathbf{Q}(\mathbf{x}) = \mathbf{L}(\mathbf{x})\mathbf{P}(\mathbf{x})$ for some $\mathbf{L}(\mathbf{x})$, the first property is a consequence of causality, the second a consequence of passivity (that the material does not generate energy—see, for example, [61]), and the third a consequence of $\mathbf{L}(\omega)$ being the Fourier transform of a real kernel. It follows that $\mathbf{L}$ is an analytic function of $-\omega^2$ with spectrum on the negative real $-\omega^2$ axis (corresponding to real values of ω) having the implied properties that

- $\text{Im}(\mathbf{L}) \geq 0$ when $\text{Im}(-\omega^2) \leq 0$,
- $\mathbf{L}$ is real and $\mathbf{L} \geq 0$ when ω^2 is real and $-\omega^2 \geq 0$.

In other words, $\mathbf{L}(\omega)$ is an operator-valued Stieltjes function of $-\omega^2$. The operator $\mathbf{B} = [\mathbf{Q}\mathbf{L}^{-1}\mathbf{Q}]^{-1}$ entering (5.4) has the property that it is an analytic function of $\mathbf{L}$ with

$$(5.40) \quad \begin{aligned} &\text{Im}(\mathbf{B}) \geq 0 \quad \text{when } \text{Im}(\mathbf{L}) \geq 0, \\ &\mathbf{B} \text{ is real and } \mathbf{B} \geq 0 \quad \text{when } \mathbf{L} \text{ is real and } \mathbf{L} \leq 0. \end{aligned}$$

Hence, the Stieltjes properties of $\mathbf{L}$ as a function of $-\omega^2$ pass to those of $\mathbf{B}$ as a function of $-\omega^2$:

$$(5.41) \quad \begin{aligned} &\text{Im}(\mathbf{B}) \geq 0 \quad \text{when } \text{Im}(-\omega^2) \leq 0, \\ &\mathbf{B} \text{ is real and } \mathbf{B} \geq 0 \quad \text{when } \omega^2 \text{ is real and } -\omega^2 \geq 0. \end{aligned}$$

Introducing

$$(5.42) \quad z = \frac{\omega^2 - c}{\omega^2 + c} = 1 - \frac{2c}{\omega^2 + c},$$

for some real $c > 0$, ensures that the spectrum of $\mathbf{B}(z)$ is on the interval $[-1, 1]$ and

$$(5.43) \quad \begin{aligned} &\text{Im}(\mathbf{B}(z)) \geq 0 \quad \text{when } \text{Im}(z) \geq 0, \\ &\mathbf{B} \text{ is real and } \mathbf{B} \geq 0 \quad \text{when } z \text{ is real and } z > 1 \text{ or } z < -1. \end{aligned}$$

Note that this choice of z is quite different to that in (5.15), and not restricted to two-phase composites. Thus, $\mathbf{B}(z)$ has the integral representation

$$(5.44) \quad \mathbf{B}(z) = \mathbf{B}_0 + \int_{-1}^1 \frac{d\mathbf{M}(\lambda)}{\lambda - z},$$

where $\mathbf{B}_0$ is a positive definite operator and $d\mathbf{M}(\lambda)$ is a positive definite real operator-valued measure satisfying the constraint

$$(5.45) \quad \int_{-1}^1 \frac{d\mathbf{M}(\lambda)}{1 - \lambda} \leq \mathbf{B}_0.$$

To begin, suppose we are only interested in the quadratic form $(\mathbf{B}\mathbf{E}_0, \mathbf{E}_0)$ associated with $\mathbf{B}$. Then,

$$(5.46) \quad \begin{aligned} (\mathbf{B}(z)\mathbf{E}_0, \mathbf{E}_0) &= k_0 \left[1 + \int_{-1}^1 \frac{(1-\lambda)d\eta(\lambda)}{\lambda-z} \right] \\ &= k_0 \left\{ 1 + \int_{-1}^1 \left[-1 + \frac{1-z}{\lambda-z} \right] d\eta(\lambda) \right\}, \end{aligned}$$

where $k_0 = (\mathbf{B}_0\mathbf{E}_0, \mathbf{E}_0)$ is real and positive and

$$(5.47) \quad d\eta(\lambda) = (d\mathbf{M}(\lambda)\mathbf{E}_0, \mathbf{E}_0)/[k_0(1-\lambda)]$$

is a positive real-valued measure, satisfying the constraint

$$(5.48) \quad \int_{-1}^1 d\eta(\lambda) \leq 1.$$

Note that k_0 can be identified with $(\mathbf{B}(z)\mathbf{E}_0, \mathbf{E}_0)$ in the limit $z \rightarrow \infty$, i.e., as $\omega \rightarrow i\sqrt{c}$.

If we are interested in finding complex coefficients ξ_k , $k = 1, 2, \dots, m$, such that

$$(5.49) \quad \begin{aligned} &(\mathbf{B}(z_0)\mathbf{E}_0, \mathbf{E}_0) - \sum_{k=1}^m \xi_k (\mathbf{B}(z_k)\mathbf{E}_0, \mathbf{E}_0) \\ &= k_0 \left\{ \left(1 - \sum_{k=1}^m \xi_k \right) \left[1 - \int_{-1}^1 d\eta(\lambda) \right] \right. \\ &\quad \left. + \int_{-1}^1 \left[\frac{1-z_0}{\lambda-z_0} - \sum_{k=1}^m \frac{\xi_k(1-z_k)}{\lambda-z_k} \right] d\eta(\lambda) \right\} \end{aligned}$$

is small, we require that

$$(5.50) \quad \sup_{\lambda \in [-1, 1]} \left| \frac{1}{\lambda-z_0} - \sum_{k=1}^m \frac{\xi_k(1-z_k)/(1-z_0)}{\lambda-z_k} \right| \leq \epsilon_m.$$

In particular, with $\lambda = 1$, this implies

$$(5.51) \quad \left| 1 - \sum_{k=1}^m \xi_k \right| \leq |1-z_0|\epsilon_m,$$

and so we obtain

$$(5.52) \quad (\mathbf{B}(z_0)\mathbf{E}_0, \mathbf{E}_0) - \sum_{k=1}^m \xi_k (\mathbf{B}(z_k)\mathbf{E}_0, \mathbf{E}_0) \leq 2k_0|1-z_0|\epsilon_m.$$

By setting $\alpha_k = \xi_k(1-z_k)/(1-z_0)$ we see this is exactly the problem encountered in Section 6, and we may take the coefficients α_k to be given by (4.7). The motivation for studying this problem is that the response at special frequencies can sometimes directly reveal information about the geometry. This is the case for

elastodynamics in the quasistatic limit when only two materials are present. The material parameters are the bulk moduli $\kappa_1(\omega)$, $\kappa_2(\omega)$ and shear moduli $\mu_1(\omega)$, $\mu_2(\omega)$ of the two phases. It may happen that $\mu_1(\omega_0) = \mu_2(\omega_0)$ for certain complex frequencies ω_0 and if $\kappa_1(\omega_0) \neq \kappa_2(\omega_0)$ the response at frequency ω_0 can reveal the volume fraction of phase 1 in a composite, or more generally in a two-phase body.

REMARK 5.3. It is not much more difficult to treat bilinear forms. Then we have

$$\begin{aligned}
 & (\mathbf{B}(z)\mathbf{E}_0, \mathbf{E}'_0) \\
 &= (\mathbf{B}(z)(\mathbf{E}_0 + \mathbf{E}'_0), \mathbf{E}_0 + \mathbf{E}'_0) - (\mathbf{B}(z)(\mathbf{E}_0 - \mathbf{E}'_0), \mathbf{E}_0 - \mathbf{E}'_0) \\
 (5.53) \quad &= k_0^{(1)} \left\{ 1 + \int_{-1}^1 \left[-1 + \frac{1-z}{\lambda-z} \right] d\eta_1(\lambda) \right\} \\
 &\quad - k_0^{(2)} \left\{ 1 + \int_{-1}^1 \left[-1 + \frac{1-z}{\lambda-z} \right] d\eta_2(\lambda) \right\},
 \end{aligned}$$

where

$$(5.54) \quad k_0^{(1)} = (\mathbf{B}_0(\mathbf{E}_0 + \mathbf{E}'_0), \mathbf{E}_0 + \mathbf{E}'_0), \quad k_0^{(2)} = (\mathbf{B}_0(\mathbf{E}_0 - \mathbf{E}'_0), \mathbf{E}_0 - \mathbf{E}'_0),$$

are both real and positive, while

$$\begin{aligned}
 (5.55) \quad & d\eta_1(\lambda) = (d\mathbf{M}(\lambda)(\mathbf{E}_0 + \mathbf{E}'_0), \mathbf{E}_0 + \mathbf{E}'_0) / [k_0^{(1)}(1-\lambda)], \\
 & d\eta_2(\lambda) = (d\mathbf{M}(\lambda)(\mathbf{E}_0 - \mathbf{E}'_0), \mathbf{E}_0 - \mathbf{E}'_0) / [k_0^{(2)}(1-\lambda)]
 \end{aligned}$$

are positive real-valued measures, satisfying the constraints that

$$(5.56) \quad \int_{-1}^1 d\eta_1(\lambda) \leq 1, \quad \int_{-1}^1 d\eta_2(\lambda) \leq 1.$$

We seek complex coefficients ξ_k , $k = 1, 2, \dots, m$, such that

$$\begin{aligned}
 & (\mathbf{B}(z_0)\mathbf{E}_0, \mathbf{E}'_0) - \sum_{k=1}^m \xi_k (\mathbf{B}(z_k)\mathbf{E}_0, \mathbf{E}'_0) \\
 (5.57) \quad &= k_0^{(1)} \left\{ \left(1 - \sum_{k=1}^m \xi_k\right) \left[1 - \int_{-1}^1 d\eta_1(\lambda)\right] + \int_{-1}^1 \left[\frac{1-z_0}{\lambda-z_0} - \sum_{k=1}^m \frac{\xi_k(1-z_k)}{\lambda-z_k} \right] d\eta_1(\lambda) \right\} / 4 \\
 &\quad - k_0^{(2)} \left\{ \left(1 - \sum_{k=1}^m \xi_k\right) \left[1 - \int_{-1}^1 d\eta_2(\lambda)\right] + \int_{-1}^1 \left[\frac{1-z_0}{\lambda-z_0} - \sum_{k=1}^m \frac{\xi_k(1-z_k)}{\lambda-z_k} \right] d\eta_2(\lambda) \right\} / 4
 \end{aligned}$$

is small. Using the bounds (5.50) we obtain

$$(5.58) \quad \left| (\mathbf{B}(z_0)\mathbf{E}_0, \mathbf{E}'_0) - \sum_{k=1}^m \xi_k (\mathbf{B}(z_k)\mathbf{E}_0, \mathbf{E}'_0) \right| \leq (k_0^{(1)} + k_0^{(2)}) |1 - z_0| \epsilon_m / 2.$$

REMARK 5.4. Noting that

$$(5.59) \quad \begin{aligned} \frac{d}{dz}(\mathbf{B}(z)\mathbf{E}_0, \mathbf{E}_0) &= k_0 \int_{-1}^1 \frac{(1-\lambda)d\eta(\lambda)}{(\lambda-z)^2} \\ &= k_0 \int_{-1}^1 \frac{1}{\lambda-z} \left[-1 + \frac{1-z}{\lambda-z} \right] d\eta(\lambda), \end{aligned}$$

we can easily obtain bounds that correlate this derivative at z_0 with the values of $(\mathbf{B}(z_k)\mathbf{E}_0, \mathbf{E}_0)$, $k = 0, 1, 2, \dots, m$. We seek complex constants γ_k , $k = 0, 1, 2, \dots, m$, such that

$$(5.60) \quad \begin{aligned} &(\mathbf{B}(z_0)\mathbf{E}_0, \mathbf{E}'_0) - \sum_{k=1}^m \xi_k (\mathbf{B}(z_k)\mathbf{E}_0, \mathbf{E}'_0) \\ &= k_0^{(1)} \left\{ \left(1 - \sum_{k=1}^m \xi_k \right) \left[1 - \int_{-1}^1 d\eta_1(\lambda) \right] \right. \\ &\quad \left. + \int_{-1}^1 \left[\frac{1-z_0}{\lambda-z_0} - \sum_{k=1}^m \frac{\xi_k(1-z_k)}{\lambda-z_k} \right] d\eta_1(\lambda) \right\} / 4 \\ &\quad - k_0^{(2)} \left\{ \left(1 - \sum_{k=1}^m \xi_k \right) \left[1 - \int_{-1}^1 d\eta_2(\lambda) \right] \right. \\ &\quad \left. + \int_{-1}^1 \left[\frac{1-z_0}{\lambda-z_0} - \sum_{k=1}^m \frac{\xi_k(1-z_k)}{\lambda-z_k} \right] d\eta_2(\lambda) \right\} / 4 \end{aligned}$$

is small. Using the bounds (5.50) we obtain

$$(5.61) \quad \left| (\mathbf{B}(z_0)\mathbf{E}_0, \mathbf{E}'_0) - \sum_{k=1}^m \xi_k (\mathbf{B}(z_k)\mathbf{E}_0, \mathbf{E}'_0) \right| \leq (k_0^{(1)} + k_0^{(2)}) |1 - z_0| \epsilon_m / 2.$$

is small, and this is ensured if

$$(5.62) \quad \sup_{\lambda \in [-1, 1]} \left| \sum_{k=0}^m \frac{\xi_k(1-z_k)/(1-z_0)}{\lambda-z_k} + \frac{\xi_0 + [1/(1-z_0)]}{\lambda-z_0} - \frac{1}{(\lambda-z_0)^2} \right| \leq \epsilon_m,$$

and $\epsilon_m \rightarrow 0$ as $m \rightarrow \infty$. Observe that (5.62) with $\lambda = 1$ implies

$$(5.63) \quad \left| \sum_{k=0}^m \xi_k \right| \leq |1 - z_0| \epsilon_m.$$

Comparing (5.62) with (4.23) we see that we should choose

$$(5.64) \quad \xi_0 = \alpha_0 - [1/(1-z_0)], \quad \xi_k = \alpha_k(1-z_0)/(1-z_k),$$

and then, with b_m and coefficients α_k given by (4.26) and (4.27), (5.62) holds with ϵ_m given by (4.29). Then

$$(5.65) \quad \left| \left[\frac{d}{dz_0} (\mathbf{B}(z_0) \mathbf{E}_0, \mathbf{E}_0) \right] - \sum_{k=0}^m \xi_k (\mathbf{B}(z_k) \mathbf{E}_0, \mathbf{E}_0) \right| \leq 2|k_0| |1 - z_0| \epsilon_m,$$

holds, and similarly one has

$$(5.66) \quad \left| \left[\frac{d}{dz_0} (\mathbf{B}(z_0) \mathbf{E}_0, \mathbf{E}'_0) \right] - \sum_{k=0}^m \xi_k (\mathbf{B}(z_k) \mathbf{E}_0, \mathbf{E}'_0) \right| \leq (k_0^{(1)} + k_0^{(2)}) |1 - z_0| \epsilon_m.$$

The convergence of ϵ_m to zero as $m \rightarrow \infty$ is again ensured provided for a positive constant $r < R$, where R is defined by (4.10), each $z_k \in H(r) = H_1(r) \cup H_2(r) \cup H_3(r)$, where the regions H_i , $i = 1, 2, 3$, are given by (4.16). The motivation for studying this problem is that the response may be trivial at certain frequencies ω_0 while the derivative of the response with respect to ω at $\omega = \omega_0$ directly reveals some information about the body. This is the case for electromagnetism when only two nonmagnetic materials are present (with magnetic permeabilities $\mu_1 = \mu_2 = \mu_0$ where μ_0 is the permeability of the vacuum). It may happen that the electric permittivities of the two phases satisfy $\varepsilon_1(\omega_0) = \varepsilon_2(\omega_0)$ for certain complex frequencies ω_0 . At this frequency ω_0 the body is homogeneous and its response can be easily calculated. Using perturbation theory and assuming $d\varepsilon_1(\omega_0)/d\omega_0 \neq d\varepsilon_2(\omega_0)/d\omega_0$ the derivative of the response with respect to ω at $\omega = \omega_0$ reveals information about the distribution of the two phases in the body.

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