



A Type B analog of the Whitehouse representation

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Abstract

We give a Type B analog of Whitehouse's lifts of the Eulerian representations from S_n to S_{n+1} by introducing a family of B_n -representations that lift to B_{n+1} . As in Type A , we interpret these representations combinatorially via a family of orthogonal idempotents in the Mantaci-Reutenauer algebra, and topologically as the graded pieces of the cohomology of a certain $\mathbb{Z}_2$ -orbit configuration space of $\mathbb{R}^3$. We show that the lifted B_{n+1} -representations also have a configuration space interpretation, and further parallel the Type A story by giving analogs of many of its notable properties, such as connections to equivariant cohomology and the Varchenko-Gelfand ring.

Keywords Configuration spaces · Orbit configuration spaces · Eulerian idempotents · Solomon's descent algebra · The Mantaci-Reutenauer algebra · The hyperoctahedral group · Equivariant cohomology

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1 Introduction

Let V be a representation of a finite group H ; then V is said to have a *lift* to a group G containing H if there is a representation of G that restricts to V . The goal of this paper is to (1) identify a family of representations of the hyperoctahedral group B_n that decompose the regular representation $\mathbb{Q}[B_n]$ and lift to B_{n+1} , and (2) interpret these representations combinatorially and topologically.

1.1 Type A motivation

Our work is inspired by the well-documented Type A story of a family of S_n -representations lifting to representations of S_{n+1} studied by Whitehouse [52], Early–Reiner [23], Mathieu [37], Getzler–Kapranov [28], Moseley–Proudfoot–Young [40], and others. These S_n -representations and their lifts arose from two distinct perspectives. The first is via a family of orthogonal idempotents $\{\epsilon_k\}_{0 \leq k \leq n-1}$ known as the *Eulerian idempotents*. The ϵ_k lie in *Solomon’s descent algebra* $\Sigma[S_n]$, the subalgebra of $\mathbb{Q}[S_n]$ generated by sums of permutations $\sigma = (\sigma_1, \dots, \sigma_n)$ with the same descent set

$$\text{Des}(\sigma_1, \dots, \sigma_n) := \{i \in [n-1] : \sigma_i > \sigma(i+1)\}.$$

The Eulerian idempotents have been extensively researched in the world of algebraic combinatorics, and generate the *Eulerian representations* $E_n^{(k)} := \epsilon_k \mathbb{Q}[S_n]$, which lift to a family of S_{n+1} -representations called the *Whitehouse representations* $F_{n+1}^{(k)}$ [52].

The second viewpoint comes from the study of the configuration space

$$\text{Conf}_n(\mathbb{R}^d) := \{(x_1, \dots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j\}.$$

As a ring, the cohomology $H^* \text{Conf}_n(\mathbb{R}^d)$ has an elegant description due to Arnol’d [4] (for $d = 2$), F. Cohen [18] (for $d \geq 2$), and Varchenko–Gelfand (for $d = 1$) [50]. When d is even, $H^* \text{Conf}_n(\mathbb{R}^d)$ is the Type A *Orlik–Solomon algebra*. The relevant scenario here will be when d is odd, which can be split into two cases:

- $d = 1$: The space $\text{Conf}_n(\mathbb{R})$ is the complement of the Braid arrangement, and its cohomology $H^* \text{Conf}_n(\mathbb{R})$ is a disjoint union of $n!$ contractible pieces. As a representation, $H^* \text{Conf}_n(\mathbb{R})$ is the regular representation $\mathbb{Q}[S_n]$. Varchenko–Gelfand gave a presentation of $H^* \text{Conf}_n(\mathbb{R})$ in terms of certain combinatorial functions called *Heaviside functions*; these functions impose an ascending filtration by degree with associated graded ring $\text{gr}(H^* \text{Conf}_n(\mathbb{R}))$ [50]. For $0 \leq k \leq n - 1$, denote by $\text{gr}(H^* \text{Conf}_n(\mathbb{R}))_k$ the k -th graded piece of $\text{gr}(H^* \text{Conf}_n(\mathbb{R}))$.
- $d \geq 3$ and odd: In this case, $H^* \text{Conf}_n(\mathbb{R}^d)$ is concentrated in degrees $0, d - 1, 2(d - 1), \dots, (n - 1)(d - 1)$. The presentation for $H^* \text{Conf}_n(\mathbb{R}^d)$ and the representations carried on each graded piece are the same for all odd $d \geq 3$, and in this way one obtains a family of n representations of S_n which decompose $\mathbb{Q}[S_n]$.

When $d = 1$ and $d = 3$, one obtains lifts from S_n to S_{n+1} of the representations carried by $H^* \text{Conf}_n(\mathbb{R}^d)$. Recall that $U(1)$, the unitary group, is homeomorphic to the 1-sphere $\mathbb{S}^1$ (e.g. the 1-point compactification of $\mathbb{R}$) and SU_2 , the group of 2×2 unitary matrices of determinant 1 over $\mathbb{C}$, is homeomorphic to $\mathbb{S}^3$ (e.g. the 1-point compactification of $\mathbb{R}^3$). Our lifts will come from the quotient spaces

$$\mathcal{V}_{n+1}^1 := \text{Conf}_{n+1}(U(1))/U(1), \quad \mathcal{V}_{n+1}^3 := \text{Conf}_{n+1}(SU_2)/SU_2; \quad (1.1.1)$$

in both cases the quotient is by (left) diagonal multiplication.

As in the case of $\text{Conf}_n(\mathbb{R})$, the cohomology of $\mathcal{V}_{n+1}^1$ has a presentation in terms of *cyclic Heaviside functions* due to Moseley–Proudfoot–Young [40], and again the cyclic Heaviside functions give rise to an ascending filtration with associated graded ring $\text{gr}(H^* \mathcal{V}_{n+1}^1) = \bigoplus_{k=0}^{n-1} \text{gr}(H^* \mathcal{V}_{n+1}^1)_k$. The presentation of $H^* \mathcal{V}_{n+1}^3$ was computed by Early–Reiner in [23].

Though not obvious, these viewpoints—the Eulerian and Whitehouse representations on one hand and “ d odd” configuration space cohomology on the other—turn out to be equivalent and serve as a beautiful link between classical combinatorial objects and important topological ones. In particular they are connected by the following representation isomorphisms:

$$E_n^{(n-1-k)} \cong_{S_n} H^{2k} \text{Conf}_n(\mathbb{R}^3) \cong_{S_n} \text{gr}(H^* \text{Conf}_n(\mathbb{R}))_k \quad (1.1.2)$$

$$F_{n+1}^{(n-1-k)} \cong_{S_{n+1}} H^{2k} \mathcal{V}_{n+1}^3 \cong_{S_{n+1}} \text{gr}(H^* \mathcal{V}_{n+1}^1)_k. \quad (1.1.3)$$

Note that each term in (1.1.2) simultaneously lifts to the corresponding term in (1.1.3). The first isomorphism in (1.1.2) was proved by the author in [16] in the context of Coxeter groups, while the second is a result of Moseley [39]. The isomorphisms in (1.1.3) are due to Early–Reiner [23] and Moseley–Proudfoot–Young [40], respectively. In Sect. 2.1.3, we summarize the notable properties of the Eulerian and Whitehouse representations, including a recursion relating them, a description of $H^* \mathcal{V}_{n+1}^3$ as an induced representation, and connections to equivariant cohomology. Our main results provide a Type B analog of each of these properties.

1.2 Type B analog

Our goal is to construct an analog to both Type A perspectives discussed above for Type B. Perhaps the most obvious generalization of the Type A descent algebra $\Sigma[S_n]$ is the Type B descent algebra $\Sigma[B_n]$, with Coxeter length used to describe $\text{Des}(\sigma)$ and Type B Eulerian idempotents¹ defined by Bergeron–Bergeron in [8]. However, it turns out that the

¹ In fact, in [9], Bergeron–Bergeron–Howlett–Taylor defined analogous idempotents for any finite Coxeter group.

corresponding Eulerian representations of B_n (studied by the author in [16] for instance) do *not* lift to B_{n+1} !

Instead, the right analogy in the context of lifts is to replace Solomon's descent algebra by the *Mantaci-Reutenauer algebra* $\Sigma'[B_n]$, a combinatorially defined subalgebra of $\mathbb{Q}[B_n]$ generalizing $\Sigma[S_n]$ and containing $\Sigma[B_n]$. It was introduced by Mantaci–Reutenauer in [35] and has a basis indexed by signed integer compositions $(a_1, \dots, a_\ell)$, where $a_i \in \mathbb{Z} \setminus \{0\}$ and $|a_1| + \dots + |a_\ell| = n$. The Mantaci–Reutenauer algebra has rich combinatorics and representation theory studied by Aguiar–Bergeron–Nyman [2], Bonnafé–Hohlweg [15] and Douglass–Tomlin [22], much of which generalizes properties of $\Sigma[S_n]$.

The role of the Eulerian idempotents will be played by a family of orthogonal idempotents $\{\mathfrak{g}_k\}_{0 \leq k \leq n}$, obtained as a sum of idempotents $\mathfrak{g}_{(\lambda^+, \lambda^-)}$ in $\Sigma'[B_n]$ introduced by Vazirani [51]:

$$\mathfrak{g}_k := \sum_{\substack{(\lambda^+, \lambda^-) \\ \ell(\lambda^+) = k}} \mathfrak{g}_{(\lambda^+, \lambda^-)}. \quad (1.2.1)$$

The above analogies are quite natural in the following sense. Let $\tau : B_n \rightarrow S_n$ be the projection which forgets the signs of $\sigma \in B_n$. In [2], Aguiar–Bergeron–Nyman study the properties of τ and show that it extends to a surjective algebra homomorphism $\tau : \Sigma'[B_n] \rightarrow \Sigma[S_n]$. In Proposition 6.8, we show that $\tau(\mathfrak{g}_0) = 0$ and $\tau(\mathfrak{g}_k) = \mathfrak{e}_{k-1}$ for $0 < k \leq n$.

The B_n -representations of interest are then defined for $0 \leq k \leq n$ to be

$$G_n^{(k)} := \mathfrak{g}_k \mathbb{Q}[B_n].$$

We obtain analogs of the “hidden” action² spaces $\mathcal{V}_{n+1}^1$ and $\mathcal{V}_{n+1}^3$ by considering the $\mathbb{Z}_2$ -orbit configuration spaces on $U(1)$ and SU_2 induced from the antipodal action on the sphere, and taking their respective quotients by the relevant diagonal action:

$$\mathcal{Y}_{n+1}^1 := \text{Conf}_{n+1}^{\mathbb{Z}_2}(U(1))/U(1), \quad \mathcal{Y}_{n+1}^3 := \text{Conf}_{n+1}^{\mathbb{Z}_2}(SU_2)/SU_2. \quad (1.2.2)$$

Both $\mathcal{Y}_{n+1}^1$ and $\mathcal{Y}_{n+1}^3$ carry a natural action by B_{n+1} (e.g. permutation and negation of each coordinate), and we show that there are B_n -equivariant homeomorphisms $\mathcal{Y}_{n+1}^d \cong \mathcal{Z}_n^d$ for $d = 1, 3$, where

$$\mathcal{Z}_n^d := \text{Conf}_n^{(\varphi)}(\mathbb{R}^d \setminus \{0\}) := \left\{ (x_1, \dots, x_n) \in (\mathbb{R}^d \setminus \{0\})^n : x_i \neq x_j \neq -\frac{x_j}{|x_j|^2} \right\}.$$

The strangeness in the definition of $\mathcal{Z}_n^d$ (first studied by Feichtner–Ziegler in [24]) comes from the fact that the composition of the antipodal map $x \mapsto -x$ with stereographic projection $\pi : \mathbb{S}^d \rightarrow \mathbb{R}^d$ gives a map φ sending x to $-x/|x|^2$. (Here we take $|x|$ to be the magnitude of $x \in \mathbb{R}^d \setminus \{0\}$ and $\mathbb{S}^d$ to be embedded in $\mathbb{R}^{d+1}$ by placing its south pole at the origin of $\mathbb{R}^{d+1}$.)

Nonetheless, many of the properties of $\mathcal{Z}_n^d$ mirror those of $\text{Conf}_n(\mathbb{R}^d)$ for odd d . In particular, we again have two cases:

- $d = 1$: In direct parallel with Type A, the spaces $\mathcal{Y}_{n+1}^1$ and $\mathcal{Z}_n^1$ are disjoint unions of $2^n n!$ contractible pieces. In Theorems 3.7 and 3.10, we give a presentation for $H^* \mathcal{Y}_{n+1}^1$ and $H^* \mathcal{Z}_n^1$, respectively. Our presentation is in terms of *signed cyclic Heaviside functions* and *signed Heaviside functions*; again, these combinatorial functions impose an ascending

² The word hidden refers to the fact that there is not an obvious S_{n+1} action on $\text{Conf}_n(\mathbb{R}^d)$.

filtration on both rings by degree, with corresponding associated graded rings

$$\mathrm{gr}(H^* \mathcal{Y}_{n+1}^1) = \bigoplus_{k=0}^n \mathrm{gr}(H^* \mathcal{Y}_{n+1}^1)_k, \quad \mathrm{gr}(H^* \mathcal{Z}_n^1) = \bigoplus_{k=0}^n \mathrm{gr}(H^* \mathcal{Z}_n^1)_k. \quad (1.2.3)$$

- $d \geq 3$ and odd: Once more, the analysis of $H^* \mathcal{Z}_n^d$ is identical for all odd $d \geq 3$, so assume $d = 3$. Feichtner-Ziegler prove that $H^* \mathcal{Z}_n^3$ is concentrated in degrees $0, 2, 4, \dots, 2n$; in Theorem 4.7 and Corollary 5.15 we give presentations for $H^* \mathcal{Z}_n^3$ and $H^* \mathcal{Y}_{n+1}^3$, respectively, that bear a strong resemblance to the F. Cohen [18] presentation for $H^* \mathrm{Conf}_n(\mathbb{R}^3)$ and Early-Reiner [23] presentation for $H^* \mathcal{V}_{n+1}^3$.

Our main results give an analogy of the Type A story:

Theorem 1.1 *For $0 \leq k \leq n$, there are B_n -representation isomorphisms:*

$$G_n^{(n-k)} \cong H^{2k} \mathcal{Z}_n^3 \cong \mathrm{gr}(H^* \mathcal{Z}_n^1)_k, \quad (1.2.4)$$

lifting to B_{n+1} -representation isomorphisms

$$H^{2k} \mathcal{Y}_{n+1}^3 \cong \mathrm{gr}(H^* \mathcal{Y}_{n+1}^1)_k. \quad (1.2.5)$$

We further

- show that the circle group $U(1)$ acts on both $\mathcal{Z}_n^3$ and $\mathcal{Y}_{n+1}^3$ (Proposition 5.8), and compute presentations for their $U(1)$ -equivariant cohomologies (Theorem 5.12);
- prove that for both $H^* \mathcal{Z}_n^1$ and $H^* \mathcal{Y}_{n+1}^1$ the associated graded ring with respect to the filtration induced by the $U(1)$ -equivariant cohomology coincides with (1.2.3) (Theorem 5.13);
- compute a recursion relating $H^* \mathcal{Z}_n^3$ and $H^* \mathcal{Y}_n^3$ (Corollary 4.4); and
- show that as ungraded rings,

$$H^* \mathcal{Z}_n^3 \cong_{B_n} \mathbb{Q}[B_n], \quad H^* \mathcal{Y}_{n+1}^3 \cong_{B_{n+1}} \mathrm{Ind}_{(c)}^{B_{n+1}} \mathbb{1},$$

where c is a Coxeter element of B_{n+1} and $\mathbb{1}$ is the trivial representation. (Theorem 5.10).

In addition to drawing upon and generalizing techniques of Berget [10], Moseley [39], and Moseley–Proudfoot–Young [40], the primary novelties in our methodology will be to (1) study the “lifted” spaces $\mathcal{Y}_{n+1}^d$ to deduce information about the spaces $\mathcal{Z}_n^d$ and (2) to utilize various filtrations in the signed Heaviside and signed cyclic Heaviside functions.

1.3 Outline of the paper

The remainder of the paper proceeds as follows:

- Section 2 first gives a more detailed description of the Type A motivation including a “property wishlist” for the Type B analog (§ 2.1.3). We then discuss a general framework for obtaining hidden action spaces from orbit configuration spaces (Proposition 2.10). Our Type B work will serve as the primary example of this framework. We conclude with a review of properties of B_n and its representation theory.
- Section 3 focuses on the spaces $\mathcal{Y}_{n+1}^1$ and $\mathcal{Z}_n^1$, and their cohomology. We introduce the signed cyclic Heaviside functions and use their combinatorial properties to give presentations for $H^* \mathcal{Y}_{n+1}^1$ (Theorem 3.7) and $H^* \mathcal{Z}_n^1$ (Theorem 3.10), as well as their respective associated graded rings (Corollaries 3.8 and 3.11.)

- Section 4 then considers the $d = 3$ case and the spaces $\mathcal{Y}_{n+1}^3$ and $\mathcal{Z}_n^3$; we prove a recursion relating their cohomologies (Corollary 4.4), compute the presentation of $H^*\mathcal{Z}_n^3$, and show that it has a bi-grading (Corollary 3.15).
- Section 5 turns to equivariant cohomology. We compute $H_{U(1)}^*\mathcal{Z}_n^3$ (Theorem 5.12), as well as the ungraded representations $H^*\mathcal{Z}_n^3$ and $H^*\mathcal{Y}_{n+1}^3$ (Theorem 5.10). This allows us to fully understand the relationship between the $d = 1$ and $d = 3$ cases by showing the filtrations by cyclic Heaviside functions and equivariant cohomology coincide (Theorem 5.14).
- Section 6 introduces the idempotents $\mathfrak{g}_{(\lambda^+, \lambda^-)}$ and $\mathfrak{g}_k$, as well as the representations of B_n they generate. From here we are able to conclude Theorem 1.1 (Corollary 6.27) and more specifically, relate the $\mathfrak{g}_{(\lambda^+, \lambda^-)}$ to $H^*\mathcal{Z}_n^3$.

2 Background

2.1 Type A revisited

We begin by fleshing out the Type A motivation described in Sect. 1.1.

One way to define the Eulerian idempotents $\mathfrak{e}_k$ is via the generating function due to Garsia–Reutenauer [26]:

$$\sum_{k=0}^{n-1} t^{k+1} \mathfrak{e}_k = \sum_{\sigma \in S_n} \binom{t-1+n-\text{des}(\sigma)}{n} \sigma. \quad (2.1.1)$$

For more equivalent definitions, see Aguiar–Mahajan [1], Loday [34], Gerstenhaber–Schack [27], and Saliola [44].

Example 2.1 When $n = 3$, the Eulerian idempotents are

$$\begin{aligned} \mathfrak{e}_0 &= \frac{1}{6}(2 - (12) - (23) - (123) - (132) + 2(13)), \\ \mathfrak{e}_1 &= \frac{1}{2}(1 - (13)), \\ \mathfrak{e}_2 &= \frac{1}{6}(1 + (12) + (23) + (13) + (123) + (132)). \end{aligned}$$

Definition 2.2 The k -th Eulerian representation is the right ideal

$$E_n^{(k)} := \mathfrak{e}_k \mathbb{Q}[S_n].$$

The Whitehouse lifts $F_{n+1}^{(k)}$ of the Eulerian representations are obtained by introducing an idempotent $f_{n+1}^{(k)}$ in $\mathbb{Q}[S_{n+1}]$ as follows. View $S_n \leq S_{n+1}$ as the subgroup fixing the element $n+1$, let w_{n+1} be the $n+1$ cycle $(12 \dots (n+1)) \in S_{n+1}$, and define

$$\mathcal{W}_{n+1} := \frac{1}{n+1} \sum_{i=0}^n (w_{n+1})^i.$$

Whitehouse shows the element $f_{n+1}^{(k)} := \mathcal{W}_{n+1} e_n^{(k)}$ is an idempotent in $\mathbb{Q}[S_{n+1}]$, generating a family of representations

$$F_{n+1}^{(k)} := f_{n+1}^{(k)} \mathbb{Q}[S_{n+1}]$$

which we will call the *Whitehouse representations*. She then proves that the $F_{n+1}^{(k)}$ are lifts of the $E_n^{(k)}$ [52, Prop 1.4].

Example 2.3 ($n = 3$) Denote by S^λ the irreducible symmetric group representation indexed by the partition λ . Then the S_3 Eulerian representations and their S_4 lifts are

$$\begin{aligned} E_3^{(0)} &= S^{(2,1)} & F_4^{(0)} &= S^{(2,2)} \\ E_3^{(1)} &= S^{(2,1)} \oplus S^{(1,1,1)} & F_4^{(1)} &= S^{(2,1,1)} \\ E_3^{(2)} &= S^{(3)} & F_4^{(2)} &= S^{(4)}. \end{aligned}$$

Each $F_4^{(k)}$ restricts to the representation $E_3^{(k)}$ via the symmetric group branching rules.

2.1.1 The spaces $\text{Conf}_n(\mathbb{R}^d)$ and their cohomology

We begin with the case that $d = 1$. Recall that the space $\text{Conf}_n(\mathbb{R})$ is a disjoint union of $n!$ contractible pieces. Each piece is parametrized by a relative ordering of $x_1, \dots, x_n$ in $\mathbb{R}$, and so $H^* \text{Conf}_n(\mathbb{R}, \mathbb{Z}) = H^0 \text{Conf}_n(\mathbb{R}, \mathbb{Z})$ can be understood as the space of $\mathbb{Z}$ -valued functions on the set of connected components of $\text{Conf}_n(\mathbb{R})$.

Varchenko–Gelfand give a combinatorial set of generators for $H^0 \text{Conf}_n(\mathbb{R})$ called *Heaviside functions*, defined by

$$u_{ij}(x_1, \dots, x_n) := \begin{cases} 1 & x_i < x_j \\ 0 & x_i > x_j \end{cases}$$

for $i \neq j \in [n] := \{1, \dots, n\}$. A permutation $\sigma \in S_n$ naturally acts on u_{ij} :

$$\sigma \cdot u_{ij} = u_{\sigma(i)\sigma(j)}.$$

The space of such Heaviside functions forms a $\mathbb{Z}$ -algebra, where the u_{ij} add and multiply pointwise as functions, so that multiplication is given by

$$u_{ij} \cdot u_{kl}(x_1, \dots, x_n) = \begin{cases} 1 & x_i < x_j \text{ and } x_k < x_l \\ 0 & \text{otherwise.} \end{cases}$$

This implies certain natural relations, for example that $u_{ij}^2 = u_{ij}$. Similarly, one can deduce that $1 - u_{ij} = u_{ji}$, so that $u_{ij} \cdot u_{jk} \cdot (1 - u_{ik}) = 0$, since it is impossible that $x_i < x_j < x_k$ but $x_i > x_k$. This is the essential idea behind Theorem 2.4.

Theorem 2.4 (Varchenko–Gelfand [50]) *The ring $H^* \text{Conf}_n(\mathbb{R})$ has presentation $\mathbb{Z}[u_{ij}]/\mathcal{Q}$, where $\mathcal{Q}$ is generated by*

$$\begin{aligned} (i) \quad & u_{ij}^2 = u_{ij}, \quad (ii) \quad u_{ij} = (1 - u_{ji}), \\ (iii) \quad & u_{ij}u_{jk}(1 - u_{ik}) - (1 - u_{ij})(1 - u_{jk})u_{ik} = 0, \end{aligned}$$

for distinct $i, j, k \in [n]$.

Call the ring $\mathbb{Z}[u_{ij}]/\mathcal{Q}$ the *Varchenko–Gelfand ring*. The presentation in Theorem 2.4 imposes an ascending filtration on the Varchenko–Gelfand ring obtained from the natural degree grading on $\mathbb{Z}[u_{ij}]/\mathcal{Q}$: the m^{th} layer in the filtration is the span of monomials in the variables

u_{ij} having degree at most m . The associated graded ring from this filtration has presentation $\mathbb{Z}[u_{ij}]/\text{indeg}(\mathcal{Q})$ with $\text{indeg}(\mathcal{Q})$ generated by³

$$(i) \quad u_{ij}^2 = 0, \quad (ii) \quad u_{ij} = -u_{ji}, \quad (iii) \quad u_{ij}u_{jk} - u_{ij}u_{ik} - u_{jk}u_{ik} = 0.$$

When $d \geq 2$, the space $\text{Conf}_n(\mathbb{R}^d)$ is no longer comprised of contractible, disjoint pieces but nonetheless has an elegant presentation due to F. Cohen ($d \geq 2$) and Ar'nold ($d = 2$).

Theorem 2.5 (F. Cohen [18], Ar'nold [4]) *For $d \geq 2$, the ring $H^* \text{Conf}_n(\mathbb{R}^d)$ has presentation $\mathbb{Z}\langle u_{ij} \rangle / \mathcal{Q}'$ where $\mathcal{Q}'$ is generated by the relations $u_{ij}u_{k\ell} = (-1)^{d+1}u_{k\ell}u_{ij}$ and*

$$(i) \quad u_{ij}^2 = 0 \quad (ii) \quad u_{ij} = (-1)^d u_{ji}, \quad (iii) \quad u_{ij}u_{jk} + u_{jk}u_{ki} + u_{ki}u_{ij} = 0$$

for distinct $i, j, k, \ell \in [n]$.

As in the $d = 1$ case, S_n acts on the u_{ij} by permuting coordinates.

The generator u_{ij} lies in $H^{d-1} \text{Conf}_n(\mathbb{R}^d)$, which together with the relations in $\mathcal{Q}'$ implies that $H^* \text{Conf}_n(\mathbb{R}^d)$ is concentrated in degrees $0, (d-1), 2(d-1), \dots, (n-1)(d-1)$. Direct comparison of the above presentations shows that $\mathbb{Z}[u_{ij}]/\text{in}(\mathcal{Q}) \cong H^* \text{Conf}_n(\mathbb{R}^d)$ for $d \geq 3$ and odd.

Example 2.6 (Lie_n) The top degree cohomology of $\text{Conf}_n(\mathbb{R}^d)$ has a particularly nice description:

$$H^{(d-1)(n-1)} \text{Conf}_n(\mathbb{R}^d) \cong_{S_n} \begin{cases} \text{Lie}_n & d \geq 3 \text{ and odd} \\ \text{Lie}_n \otimes \text{sgn} & d \text{ even,} \end{cases}$$

where Lie_n is the *multilinear component of the free Lie algebra on n generators*; see Reutenauer [42] for more on the many wonderful properties of Lie_n . A result of Kraskiewicz–Weyman [33] says that Lie_n can be described as the induced representation

$$\text{Lie}_n = \text{Ind}_{\mathbb{Z}_n}^{S_n} \omega,$$

where $\mathbb{Z}_n$ is the cyclic group generated by an n -cycle in S_n and ω is the character sending the generator of $\mathbb{Z}_n$ to $e^{2\pi i/n}$.

The relations in Theorem 2.5 imply that Lie_n has basis

$$u_{1i_1}u_{i_1i_2}u_{i_2i_3} \cdots u_{i_{n-1}i_n}$$

for distinct $i_1, i_2, \dots, i_n \in [n]$, and is therefore $(n-1)!$ dimensional. There are other notable bases for Lie_n , in particular using Lyndon words; see Barcelo [6].

We will return to Lie_n in Sect. 6, where it will play a key role in describing $H^* \mathcal{Z}_n^3$.

2.1.2 Hidden actions and lifts

Assume that $d = 1, 3$. The fact that $H^* \text{Conf}_n(\mathbb{R}^d)$ lifts to an S_{n+1} -representation comes from the fact that the spaces $\text{Conf}_n(\mathbb{R}^d)$ have *hidden* (e.g. non-obvious) actions of S_{n+1} via the S_n -equivariant maps

$$f_A^d : \mathcal{V}_{n+1}^d \xrightarrow{\cong} \text{Conf}_n(\mathbb{R}^d) \quad (2.1.2)$$

$$(p_0, \dots, p_{n-1}, p_n) \mapsto (\pi(p_0^{-1}p_1), \dots, \pi(p_0^{-1}p_n)) \quad (2.1.3)$$

³ The notation $\text{indeg}(\mathcal{Q})$ refers to the fact that $\text{indeg}(\mathcal{Q})$ is an initial ideal for some degree (partial) ordering on monomials. This will be rigorously defined and discussed in Sect. 3.1.

where $\pi : \mathbb{S}^d \setminus \{\infty\} \rightarrow \mathbb{R}^d$ is the stereographic projection, and as in the introduction

$$\mathcal{V}_{n+1}^1 := \text{Conf}_{n+1}(U(1))/U(1), \quad \mathcal{V}_{n+1}^3 := \text{Conf}_{n+1}(SU_2)/SU_2.$$

Here we identify the element 1 in SU_2 and $U(1)$ with ∞ (the point at infinity) in $\mathbb{S}^3$ and $\mathbb{S}^1$, so that π is a homeomorphism between $SU_2 \setminus \{1\} \rightarrow \mathbb{R}^3$ and $U(1) \setminus \{1\} \rightarrow \mathbb{R}$.

One can thus define an S_{n+1} -action on $\text{Conf}_n(\mathbb{R}^d)$ as the natural S_{n+1} -action by coordinate permutation on $\mathcal{V}_{n+1}^d$. Let

$$[n]_0 := \{0, 1, \dots, n\}.$$

We will think of S_{n+1} as the group of permutations on the set $[n]_0$. The homeomorphism (2.1.2) implies that one recovers the standard S_n -action on $\text{Conf}_n(\mathbb{R}^d)$ by permuting only the last n coordinates of $\mathcal{V}_{n+1}^d$. We will discuss a more general framework for obtaining hidden actions in Sect. 2.2.

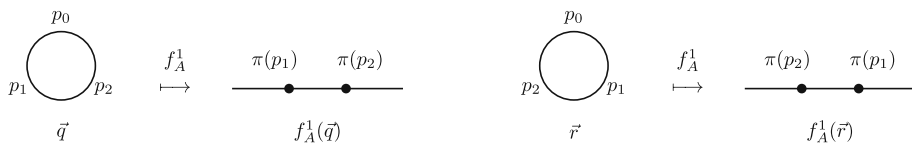
The space $\mathcal{V}_{n+1}^1$ can seem unwieldy, but it actually has an intuitive description; $\mathcal{V}_{n+1}^1$ has representatives $(1, p_1, \dots, p_n)$ for $p_i \neq p_j \neq 1$ and like $\text{Conf}_n(\mathbb{R})$, is comprised of $n!$ contractible pieces. Each disjoint piece of $\mathcal{V}_{n+1}^1$ is parametrized by a relative ordering of $p_0, \dots, p_n$ around the circle. These disjoint pieces (S_n -equivariantly) biject with the pieces of $\text{Conf}_n(\mathbb{R})$. To move from $\mathcal{V}_{n+1}^1$ to $\text{Conf}_n(\mathbb{R})$, read the ordering of $p_1, \dots, p_n$ around $U(1)$ counter-clockwise beginning after p_0 .

When we move to cohomology, the Heaviside functions u_{ij} also lift to *cyclic Heaviside functions* $v_{ijk} \in \mathcal{V}_{n+1}^1$, defined in [40] by Moseley–Proudfoot–Young as:

$$v_{ijk}(p_0, \dots, p_n) := \begin{cases} 1 & p_i < p_j < p_k \text{ in counter-clockwise order on } U(1) \\ 0 & \text{otherwise,} \end{cases}$$

where now $i, j, k \in [n]_0$.

Example 2.7 Consider the two representatives $\vec{q}$ and $\vec{r}$ of $\mathcal{V}_3^1$ and their images under f_A^1 :



Note that $v_{123}(\vec{q}) = u_{12}(f_A^1(\vec{q})) = 1$, while $v_{123}(\vec{r}) = u_{12}(f_A^1(\vec{r})) = 0$. On the other hand $v_{213}(\vec{q}) = u_{21}(f_A^1(\vec{q})) = 0$ and $v_{213}(\vec{r}) = u_{21}(f_A^1(\vec{r})) = 1$.

The v_{ijk} again form a $\mathbb{Z}$ -algebra and provide an elegant combinatorial description for the ring $H^* \mathcal{V}_{n+1}^1$. As in the case of $H^* \text{Conf}_n(\mathbb{R})$, the degree grading on $H^* \mathcal{V}_{n+1}^1$ from the v_{ijk} imposes an ascending filtration with associated graded ring $\text{gr}(H^* \mathcal{V}_{n+1}^1)$.

For both $d = 1, 3$, an explicit presentation for $H^* \mathcal{V}_{n+1}^d$ can be recovered from the presentation in Theorem 2.4 via the induced isomorphism f_A^* sending u_{ij} to v_{0ij} , along with the additional relation due to Early–Reiner [23, Thm 3] (see also Moseley–Proudfoot–Young [40, Rmk 2.9] and Matherne–Miyata–Proudfoot–Ramos [36, Thm. A.4]):

$$v_{ijk} - v_{ij\ell} + v_{ik\ell} - v_{j\ell k} = 0.$$

Remark 2.8 (Cyclic operads) Kontsevich in [32] showed there was a hidden action of S_{n+1} on Lie_n . In [28], Getzler–Kapranov then formalized his techniques to introduce *cyclic operads* more generally. Using constructions of $H_* \text{Conf}_n(\mathbb{R}^d)$ and Poisson operads (see for instance Sinha [45]), one can apply the work of Getzler–Kapranov ([28, Prop 3.11]) to show that there is a cyclic operad structure on all graded pieces of the homology of $\text{Conf}_n(\mathbb{R}^3)$. It would be interesting to relate this cyclic operad structure to the bases of $H^* \text{Conf}_n(\mathbb{R}^3)$ and $H_* \text{Conf}_n(\mathbb{R}^3)$ more directly.

2.1.3 Property wishlist

We will finish the Type *A* summary by providing a “property wishlist” which will serve as the inspiration and guiding motivation of our Type *B* work. In particular, we highlight the following:

- There is an isomorphism of S_n -representations⁴ for $0 \leq k \leq n-1$:

$$E_n^{(n-1-k)} \cong_{S_n} H^{2k} \text{Conf}_n(\mathbb{R}^3). \quad (2.1.4)$$

This was first deduced by comparing a result of Sundaram and Welker for subspace arrangements [48, Thm 4.4(iii)] with descriptions of the characters of $E_n^{(k)}$ by Hanlon [29], and was later proved in the context of Coxeter groups by the author in [16].

- Equation (2.1.4) lifts to an isomorphism of S_{n+1} -representations [23, Prop. 2]:

$$F_{n+1}^{(n-1-k)} \cong_{S_{n+1}} H^{2k}(\mathcal{V}_{n+1}^3). \quad (2.1.5)$$

- There is a recursion⁵ relating the Eulerian and Whitehouse representations of S_n :

$$E_n^{(k)} = F_n^{(k-1)} \oplus \left(S^{(n-1,1)} \otimes F_n^{(k)} \right), \quad (2.1.6)$$

where $S^{(n-1,1)}$ is the irreducible reflection representation of S_n [23, Prop. 1].

- There is an S_{n+1} -representation isomorphism

$$\sum_{k=0}^{n-1} F_{n+1}^{(k)} \cong \text{Ind}_{\mathbb{Z}_{n+1}}^{S_{n+1}} \mathbb{1},$$

where $\mathbb{Z}_{n+1}$ is the cyclic group generated by an $(n+1)$ -cycle in S_{n+1} and $\mathbb{1}$ is the trivial representation [52, Prop 1.5].

- The circle group $U(1)$ acts on $\mathbb{R}^3$ by rotation around the x -axis, thereby inducing an action on $\text{Conf}_n(\mathbb{R}^3)$. The filtration induced from the $U(1)$ -equivariant cohomology $H_{U(1)}^* \text{Conf}_n(\mathbb{R}^3)$ implies an isomorphism of S_n -modules for $0 \leq k \leq n-1$ [39]:

$$\text{gr}(H^* \text{Conf}_n(\mathbb{R}))_k \cong_{S_n} H^{2k} \text{Conf}_n(\mathbb{R}^3), \quad (2.1.7)$$

where $\text{gr}(H^* \text{Conf}_n(\mathbb{R}))$ coincides with the associated graded ring coming from the filtration by Heaviside functions [39].

- Equation (2.1.7) also lifts to an S_{n+1} -module isomorphism for $0 \leq k \leq n-1$ [40]:

$$\text{gr}(H^* \mathcal{V}_{n+1}^1)_k \cong_{S_{n+1}} H^{2k}(\mathcal{V}_{n+1}^3). \quad (2.1.8)$$

⁴ In fact (2.1.4) holds for any odd $d \geq 3$ by replacing $H^{2k} \text{Conf}_n(\mathbb{R}^d)$ with $H^{(d-1)k} \text{Conf}_n(\mathbb{R}^d)$.

⁵ We think of this as a recursion in the sense that the formula relates the representation $E_n^{(k)}$ (which lifts to $F_{n+1}^{(k)}$) to the representation $F_n^{(k)}$ (which restricts to $E_{n-1}^{(k)}$).

where again (2.1.8) comes from a $U(1)$ action on $\mathcal{V}_{n+1}^3$ and subsequent computation of $H_{U(1)}^* \mathcal{V}_{n+1}^3$. The grading on the left-hand-side also coincides with the associated graded ring coming from the filtration by cyclic Heaviside functions.

The remainder of this paper is devoted to obtaining an analogous Type B statement for each of the properties listed above.

2.2 Orbit configuration spaces and hidden actions

We now introduce a general framework with which to construct spaces with hidden actions. Our Type B spaces will emerge from this general framework (see Example 2.12).

Definition 2.9 For a group H acting freely on a topological space X , the n -th ordered orbit configuration space is

$$\mathrm{Conf}_n^H(X) := \{(x_1, \dots, x_n) \in X^n : h \cdot x_i \cap h \cdot x_j = \emptyset \text{ for } i \neq j \text{ and any } h \in H\}.$$

Orbit configuration spaces were first defined by Xicoténcatl in [53], and have proved integral to the study of universal covers of certain configuration spaces [53], hyperplane arrangements associated to root systems by Bibby [12] and Moci [38], and equivariant loop spaces by Xicoténcatl [55]. Computing the cohomology of orbit configuration spaces is an active area of study developed by Casto [17], Denham–Suciu [19], Feichtner–Ziegler [24], Bibby–Nadish [11, 13] and others.

If H is the trivial group, Definition 2.9 recovers the classical configuration space of X . More generally any orbit configuration space $\mathrm{Conf}_n^H(X)$ has a natural action by the wreath product $H \wr S_n$. Recall that the wreath product $H \wr S_n$ is the group whose elements are of the form

$$H \wr S_n = \{(h_1, \dots, h_n, \sigma) : h_i \in H, \sigma \in S_n\}$$

with multiplication defined by

$$(h'_1, h'_2, \dots, h'_n, \sigma') \cdot (h_1, h_2, \dots, h_n, \sigma) = (h'_{\sigma(1)} h_1, h'_{\sigma(2)} h_2, \dots, h'_{\sigma(n)} h_n, \sigma' \sigma).$$

For instance if $H = \mathbb{Z}_2$, this wreath product $\mathbb{Z}_2 \wr S_n$ is B_n , and if $H = \mathbb{Z}_r$ for $r \geq 2$, one obtains the complex reflection group $G(r, 1, n)$ (sometimes referred to as a *generalized symmetric group*).

Our goal is to study orbit configuration spaces equipped with a hidden action. In other words, given an orbit configuration space $\mathrm{Conf}_n^H(X)$, we would like to identify a space Y on which $H \wr S_{n+1}$ acts, and an $H \wr S_n$ -equivariant homeomorphism

$$\mathrm{Conf}_n^H(X) \cong Y.$$

If such a homeomorphism exists, we say that $\mathrm{Conf}_n^H(X)$ has a *hidden action* by $H \wr S_{n+1}$.

Proposition 2.10 *Let X be a topological space with*

- *A transitive (left)-action by a group G and*
- *A free (left)-action by a group H in the center of G .*

Then there is an $H \wr S_n$ -equivariant homeomorphism

$$\mathrm{Conf}_{n+1}^H(X)/G \cong \mathrm{Conf}_n^H(X - \mathcal{O}_H(x_0))/G_{x_0},$$

where both quotients are by left-diagonal multiplication, $\mathcal{O}_H(x_0)$ is the H -orbit of $x_0 \in X$ and G_{x_0} is the G -stabilizer of x_0 .

Proof Since H acts freely on X it will also act freely on the space $X \setminus \mathcal{O}_H(x_0)$ obtained by removing the H -orbit of some point x_0 from X . The G -stabilizer of x_0 , denoted G_{x_0} , then acts on $X \setminus \mathcal{O}_H(x_0)$ and since H necessarily commutes with G_{x_0} , one also obtains a well-defined space $\text{Conf}_n^H(X - \mathcal{O}_H(x_0))/G_{x_0}$.

Fix a point $(p_0, p_1, \dots, p_n) \in \text{Conf}_{n+1}^H(X)/G$; because G acts transitively on X , there is some $g \in G$ sending x_0 to p_0 . Thus in $\text{Conf}_{n+1}^H(X)/G$,

$$(p_0, \dots, p_n) \sim (x_0, g^{-1}p_1, \dots, g^{-1}p_n).$$

Then the maps back-and-forth giving inverse homeomorphisms are

$$\begin{aligned} \text{Conf}_{n+1}^H(X)/G &\longrightarrow \text{Conf}_n^H(X - \mathcal{O}_H(x_0))/G_{x_0} \\ (p_0, p_1, \dots, p_n) &\longmapsto (g^{-1}p_1, \dots, g^{-1}p_n) \end{aligned}$$

$$\begin{aligned} \text{Conf}_n^H(X - \mathcal{O}_H(x_0))/G_{x_0} &\longrightarrow \text{Conf}_{n+1}^H(X)/G \\ (q_1, \dots, q_n) &\longmapsto (x_0, q_1, \dots, q_n). \end{aligned}$$

To check that the first map is well-defined, note that p_i and p_j are in the same H -orbit if and only if the same is true for gp_i and gp_j . Thus $(g^{-1}p_1, \dots, g^{-1}p_n)$ is indeed in $\text{Conf}_n^H(X - \mathcal{O}_H(x_0))/G_{x_0}$. Further, if $g_1x_0 = p_0$, then for any $g_2 \in G$ one has $g_2p_0 = g_2g_1x_0$ and

$$\begin{aligned} (g_2p_0, \dots, g_2p_n) &\longmapsto (g_1^{-1}g_2^{-1}g_2p_n, \dots, g_1^{-1}g_2^{-1}g_2p_n) = (g_1^{-1}p_1, \dots, g_1^{-1}p_n) \\ (p_0, \dots, p_n) &\longmapsto (g_1^{-1}p_1, \dots, g_1^{-1}p_n). \end{aligned}$$

One can similarly check that the second map is well-defined and that their composition gives the identity. The $H \wr S_n$ equivariance follows from the fact that H commutes with G . $\square$

We will be interested in the special case that G acts simply transitively on X so that G_{x_0} is trivial, in which case one obtains the isomorphism

$$\text{Conf}_n^H(X - \mathcal{O}_H(x_0)) \cong \text{Conf}_{n+1}^H(X)/G.$$

Example 2.11 Take H to be the trivial group.

- Let $X = G = SU_2$ and $x_0 = 1$, the identity of SU_2 . Then Proposition 2.10 recovers the S_n -equivariant homeomorphism

$$\text{Conf}_{n+1}(SU_2)/SU_2 \cong \text{Conf}_n(\mathbb{R}^3),$$

where the map here is precisely (2.1.2), since $SU_2 \setminus \{1\} \cong \mathbb{R}^3$.

- Let $X = G = U(1)$ and set $x_0 = 1 \in U(1)$, so that $U(1) \setminus \{1\} \cong \mathbb{R}$. Then Proposition 2.10 (again via (2.1.2)) gives

$$\text{Conf}_{n+1}(U(1))/U(1) \cong \text{Conf}_n(\mathbb{R}).$$

One way of framing all subsequent work in this paper is to ask the question: what happens to Example 2.11 when H is replaced by $\mathbb{Z}_2$, acting via the antipodal map?

Example 2.12 As suggested above, take $H = \mathbb{Z}_2$ acting as the antipodal map (e.g. by -1) on SU_2 and $U(1)$. Then

- When $X = G = SU_2$ and $x_0 = 1$, the orbit of 1 is ± 1 . Recall that for $x \in \mathbb{R}^d \setminus \{0\}$, $\varphi(x) := -x/|x|^2$ and that $B_n \cong \mathbb{Z}_2 \wr S_n$. Proposition 2.10 then implies there is a B_n -equivariant homeomorphism

$$\begin{aligned} \text{Conf}_{n+1}^{\mathbb{Z}_2}(SU_2)/SU_2 &=: \mathcal{Y}_{n+1}^3 \cong_{B_n} \mathcal{Z}_n^3 := \text{Conf}_n^{(\varphi)}(\mathbb{R}^3 \setminus \{0\}) \\ &= \{(x_1, \dots, x_n) \in \mathbb{R}^3 \setminus \{0\} : x_i \neq x_j, \varphi(x_j)\}. \end{aligned}$$

- When $X = G = U(1)$, then Proposition 2.10 says that

$$\begin{aligned} \text{Conf}_{n+1}^{\mathbb{Z}_2}(U(1))/U(1) &=: \mathcal{Y}_{n+1}^1 \cong_{B_n} \mathcal{Z}_n^1 := \text{Conf}_n^{(\varphi)}(\mathbb{R} \setminus \{0\}) \\ &= \{(x_1, \dots, x_n) \in \mathbb{R} \setminus \{0\} : x_i \neq x_j, \varphi(x_j)\}. \end{aligned}$$

The space $\mathcal{Z}_n^d$ arises naturally in the work of Feichtner–Ziegler in [24] as part of the fiber sequence

$$\mathcal{Z}_n^d \longrightarrow \text{Conf}_{n+1}^{\mathbb{Z}_2}(\mathbb{S}^d) \longrightarrow \mathbb{S}^d,$$

where the last map sends a point $(x_1, \dots, x_{n+1})$ to x_{n+1} . They then use the above sequence to compute the cohomology of $\text{Conf}_{n+1}^{\mathbb{Z}_2}(\mathbb{S}^d)$. In this sense, $\mathcal{Z}_n^d$ is the natural analog of $\text{Conf}_n(\mathbb{R}^d)$, which arises in the fiber sequence

$$\text{Conf}_n(\mathbb{R}^d) \longrightarrow \text{Conf}_{n+1}(\mathbb{S}^d) \longrightarrow \mathbb{S}^d.$$

Remark 2.13 A priori, one might expect that to parallel the Type A story, the analog of $\text{Conf}_n(\mathbb{R}^d)$ would be the more “standard” $\mathbb{Z}_2$ -orbit configuration space

$$\text{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^d) = \{(x_1, \dots, x_n) \in \mathbb{R}^{dn} : x_i \neq \pm x_j \neq 0\}$$

whose cohomology was studied by Xicoténcatl in [53], Moseley in [40] and related to the Type B Eulerian idempotents in [16] by the author. However, this is *not* the case because there is no B_n -equivariant map between $\text{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^d)$ and $\mathcal{Y}_{n+1}^d$.

That being said, the space $\text{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^d)$ and its cohomology will not be completely irrelevant. In particular, we will see that

- some relations in $H^* \text{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^3)$ will help us recover relations in $H^* \mathcal{Z}_n^3$;
- the spaces $H^* \text{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^d)$ and $H^* \mathcal{Z}_n^d$ are isomorphic as *vector spaces* but not as B_n modules; and
- when d is odd, both $H^* \mathcal{Z}_n^d$ and $H^* \text{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^d)$ have total representation $\mathbb{Q}[B_n]$, despite carrying different representations on each graded piece.

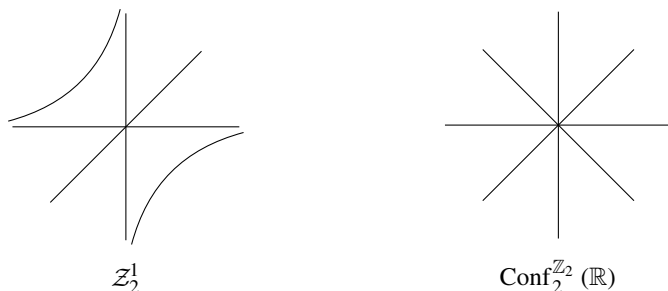
Example 2.14 It is helpful to see a concrete example of the space $\mathcal{Z}_n^d$. Consider for instance

$$\mathcal{Z}_2^1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\} : x_1 \neq x_2, x_1 \neq \varphi(x_2) = -\frac{x_2}{|x_2|^2} \right\},$$

(shown on the left below), compared to to the space

$$\text{Conf}_2^{\mathbb{Z}_2}(\mathbb{R}) := \{(x_1, x_2) \in \mathbb{R}^2 \setminus \{0\} : x_1 \neq x_2, x_1 \neq -x_2\},$$

which is the complement of the reflection arrangement associated to B_2 (shown on the right below).



Importantly, $\mathcal{Z}_2^1$ is not the complement of linear subspaces (and the same is true of $\mathcal{Z}_n^1$ in general).

Remark 2.15 In both Examples 2.11 and 2.12, the existence of a hidden action at the level of topological spaces depends upon the sphere $\mathbb{S}^d$ having a realization as a Lie group. There is no such description for $\mathbb{S}^d$ in the case that d is even. In [25], Giaffi obtains a hidden action of S_{n+1} on the space $H^* \text{Conf}_n(\mathbb{R}^2)$, but he shows that this action is on the level of cohomology rather than topological spaces. It would be interesting to investigate whether the same is true in Type B —that is, determine whether there is some hidden action on $H^* \mathcal{Z}_n^2$ but not on $\mathcal{Z}_n^2$.

2.3 The hyperoctahedral group

We have seen that B_n is isomorphic to the wreath product $\mathbb{Z}_2 \wr S_n$. Now, we will view B_n combinatorially as the group of signed permutations, and more abstractly as a Coxeter group. In what follows, we will discuss conventions for both perspectives, and then discuss the representation theory of B_n .

2.3.1 B_n as the group of signed permutations

Recall that $[n] = \{1, 2, \dots, n\}$, and define the following sets:

- $[n]^- := \{\bar{1}, \dots, \bar{n}\}$, and
- $[n]^\pm := \{1, \bar{1}, \dots, n, \bar{n}\}$.

We adopt the convention that $\bar{i}$ behaves like $-i$, so that $\bar{\bar{i}} = i$ and $|\bar{i}|$ (e.g. the absolute value of $\bar{i}$) is i . We will first think of B_n as a subgroup of the group of permutations of $[n]^\pm$:

Definition 2.16 The *hyperoctahedral group* B_n is the group of bijective maps from $[n]^\pm$ to $[n]^\pm$, subject to the condition that for any $\sigma \in B_n$, if $\sigma : i \mapsto j$, then $\sigma : \bar{i} \mapsto \bar{j}$ for $i, j \in [n]^\pm$.

A permutation in S_n can be written as a permutation matrix, in one-line notation, or in cycle notation. The same is true of signed permutations. That is, $\sigma \in B_n$ can be written

- as a *signed permutation matrix*; in this case, we realize σ as an $n \times n$ matrix $M(\sigma)$ with i, j -th entry

$$M(\sigma)_{ij} = \begin{cases} 1 & \sigma : j \mapsto i \\ -1 & \sigma : j \mapsto \bar{i} \\ 0 & \text{otherwise;} \end{cases}$$

- in *one-line notation*, meaning that σ is written as $(\sigma_1, \dots, \sigma_n)$ where $i \mapsto \sigma_i$ for $\sigma_i \in [n]^\pm$;
- in *cycle notation*, meaning that σ is factored as a product of positive and negative cycles. A negative cycle, written as $(\sigma_1 \sigma_2 \dots \sigma_\ell)^-$ for $\sigma_i \in [n]^\pm$ means that σ sends $\sigma_i \mapsto \sigma_{i+1}$ for $1 \leq i \leq \ell - 1$ and $\sigma_\ell \mapsto \overline{\sigma_1}$. On the other hand, a positive cycle $(\sigma_1 \sigma_2 \dots \sigma_\ell)$ sends $\sigma_i \mapsto \sigma_{i+1}$ for $1 \leq i \leq \ell - 1$ and $\sigma_\ell \mapsto \sigma_1$.

We will make clear from context whether we are using one-line or cycle notation.

Example 2.17 Consider $\sigma \in B_7$ sending

$$1 \mapsto 2 \quad 2 \mapsto \bar{3} \quad 3 \mapsto 1 \quad 4 \mapsto 7 \quad 5 \mapsto \bar{6} \quad 6 \mapsto 5 \quad 7 \mapsto 4.$$

Then σ can be written

- as the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

- in one-line notation as $(2, \bar{3}, 1, 7, \bar{6}, 5, 4)$ and
- in cycle notation as

$$(12\bar{3})^- (47) (56)^-.$$

We will see in Sect. 6 (for instance, Definition 6.1) that one-line notation is useful for defining combinatorial statistics on B_n .

The primary use of cycle notation is to describe the conjugacy classes of B_n via *cycle type*. Signed permutations have cycle type described by *signed partitions*, which are ordered pairs of partitions (λ^+, λ^-) where $|\lambda^+| + |\lambda^-| = n$. Note that we allow λ^+ or λ^- to be $\emptyset$, the unique (empty) partition of zero.

Definition 2.18 The signed permutation $\sigma \in B_n$ is said to have *cycle type*

$$(\lambda^+, \lambda^-) = (\underbrace{\lambda_1, \dots, \lambda_k}_{\lambda^+}, \underbrace{\lambda_{k+1}, \dots, \lambda_\ell}_{\lambda^-})$$

if σ can be factored into cycles

$$\sigma = c_1 \cdots c_k \cdot d_1 \cdots d_\ell,$$

where the c_i are positive cycles of length λ_i for $1 \leq i \leq k$ and the d_j are negative cycles of length λ_j for $k+1 \leq j \leq \ell$.

Define $\mathcal{C}_{(\lambda^+, \lambda^-)}$ to be the conjugacy class comprised of elements in B_n of cycle type (λ^+, λ^-) .

For instance, the permutation $(12\bar{3})^- (47) (56)^-$ from Example 2.17 has cycle type $((2), (3, 2))$.

2.3.2 B_n as a Coxeter group

Importantly, B_n is also a *Coxeter group*, meaning that it has a presentation of the form

$$W = \langle r_1, \dots, r_k : (r_i r_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. The elements $r_1, \dots, r_k$ are called *simple reflections*. We will only briefly survey Coxeter groups here; for a comprehensive treatment, the reader should consult Björner–Brenti [14], Humphreys [30], or Kane [31].

When $W = B_n$, the simple reflections are (in cycle notation) the adjacent transpositions

$$s_i := (i, i + 1)$$

for $1 \leq i \leq n - 1$ and the element

$$t_n := (n)^-.$$

More generally, it will be useful to think about the element $t_i = (i)^-$ for $1 \leq i \leq n$.

Any element of B_n can be written as a product of the s_i and t_n (with repeats allowed), and σ is said to be a *reduced expression* if there is no way to rewrite σ using fewer generators. In this case, the number of generators used to write σ is the *length* of σ . Every Coxeter group has a unique longest word, w_0 . In the case of B_n , this is (in one-line notation)

$$w_0 = (\bar{1}, \dots, \bar{n}) = -1.$$

In cycle notation, $w_0 = (1)^-(2)^- \cdots (n)^- = t_1 t_2 \cdots t_n$.

We will also be interested in the *Coxeter elements* of B_n . For any Coxeter group W , a Coxeter element c is a product of each simple reflection, without repeats:

$$c = r_{i_1} \cdots r_{i_k}.$$

Coxeter elements are not unique but they are all conjugate to each other. In the case of B_n , Coxeter elements are of the form (in cycle notation)

$$(i_1 i_2 i_3 \cdots i_n)^-.$$

for distinct $i_1, \dots, i_n \in [n]$. Thus all Coxeter elements have cycle type $(\emptyset, (n))$, and are in the conjugacy class $\mathcal{C}_{\emptyset, (n)}$.

2.3.3 Representation theory of the hyperoctahedral group

This section will closely follow Stembridge's write-up in [46]. Recall that $\tau : B_n \rightarrow S_n$ is the surjection which forgets the signs of permutations (e.g. $\tau(s_i) = s_i$ and $\tau(t_n) = 1$). Let $S = \{s_1, \dots, s_{n-1}, t_n\}$ and for $\sigma \in B_n$ and $s \in S$, let $c_s(\sigma)$ count the number of instances of s in a reduced expression for w . Then define

$$\delta_s(\sigma) = (-1)^{c_s(\sigma)}.$$

Like conjugacy classes, irreducible representations of B_n are indexed by signed partitions (λ^+, λ^-) .

We define the irreducible characters $\chi^{\lambda, \mu}$ of B_n as follows:

- $\chi^{\lambda, \emptyset}$ is obtained from the irreducible character χ^λ of S_n by pulling back along τ ; in other words

$$\chi^{\lambda, \emptyset}(s_i) = \chi^\lambda(s_i), \quad \chi^{\lambda, \emptyset}(t_n) = 1.$$

Table 1 The 1-dimensional representations of B_n

Representation	Character of $\sigma \in B_n$
$\chi^{(n),\emptyset}$	1
$\chi^{\emptyset,(n)}$	$\delta_{t_n}(\sigma)$
$\chi^{(1^n),\emptyset}$	$\delta_{s_i}(\sigma)$
$\chi^{\emptyset,(1^n)}$	$\delta_{t_n} \otimes \delta_{s_i}(\sigma)$.

Table 2 The character table for B_1

	$\mathcal{C}_{(1),\emptyset} = \{1\}$	$\mathcal{C}_{\emptyset,(1)} = \{t_1\}$
$\chi^{(1),\emptyset}$	1	1
$\chi^{\emptyset,(1)}$	1	-1

- $\chi^{\emptyset,\lambda}$ is obtained from the tensor product

$$\chi^{\emptyset,\lambda} = \chi^{\lambda,\emptyset} \otimes \delta_{t_n}.$$

- $\chi^{\lambda,\mu}$ is defined by the induction product:

$$\chi^{\lambda,\mu} = \chi^{\lambda,\emptyset} \cdot \chi^{\emptyset,\mu} := \text{Ind}_{B_{|\lambda|} \times B_{|\mu|}}^{B_n} (\chi^{\lambda,\emptyset} \times \chi^{\emptyset,\mu}).$$

The induction product further implies that one inherits branching rules for $\chi^{\lambda,\mu} \cdot \chi^{\nu,\psi}$ from the Type A Littlewood-Richardson Rule by rewriting

$$\begin{aligned} \chi^{\lambda,\mu} \cdot \chi^{\nu,\psi} &= (\chi^{\lambda,\emptyset} \cdot \chi^{\nu,\emptyset}) \cdot (\chi^{\emptyset,\mu} \cdot \chi^{\emptyset,\psi}) \\ &= \left(\sum_{\alpha} c_{\lambda,\nu}^{\alpha} \chi^{\alpha,\emptyset} \right) \cdot \left(\sum_{\alpha} c_{\mu,\psi}^{\alpha} \chi^{\emptyset,\alpha} \right) \\ &= \sum_{\alpha,\beta} c_{\lambda,\nu}^{\alpha} c_{\mu,\psi}^{\beta} \chi^{\alpha,\beta}, \end{aligned}$$

where $c_{\lambda,\nu}^{\alpha}$, $c_{\mu,\psi}^{\beta}$ are the standard Littlewood-Richardson coefficients.

Example 2.19 There are four 1-dimensional representations of B_n , shown in Table 1 below. Note that $\chi^{(1^n),\emptyset}$ has character δ_{s_i} for any $i \in [n-1]$ because all such s_i are conjugate. The representation $\chi^{(n),\emptyset}$ is the trivial representation and $\chi^{\emptyset,(1^n)}$ is the sign representation.

Example 2.20 Consider the character tables for B_1 (Table 2) and B_2 (Table 3), which will be useful in the proof of Proposition 4.5.

3 The $d = 1$ case in Type B

Our first goal is to determine a presentation for the cohomology of $\mathcal{Z}_n^1$ and $\mathcal{Y}_{n+1}^1$ with coefficients in $\mathbb{Z}$; later in Sect. 4 we will change to having coefficients in $\mathbb{Q}$ (and indicate as much). In contrast to Type A, we will begin with $\mathcal{Y}_{n+1}^1$.

Table 3 The character table for B_2

	$\mathcal{C}_{(1,1),\emptyset=\{1\}}$	$\mathcal{C}_{(2),\emptyset=\{s_1, t_2 s_1 t_2\}}$	$\mathcal{C}_{(1),(1)}=\{t_2, s_1 t_2 s_1\}$	$\mathcal{C}_{\emptyset,(2)}=\{s_1 t_2, t_2 s_1\}$	$\mathcal{C}_{\emptyset,(1,1)}=\{(s_1 t_2)^2 = w_0\}$
$\chi^{(2),\emptyset}$	1	1	1	1	1
$\chi^{\emptyset,(1,1)}$	1	-1	-1	1	1
$\chi^{(1,1),\emptyset}$	1	-1	1	-1	1
$\chi^{\emptyset,(2)}$	1	1	-1	-1	1
$\chi^{(1),(1)}$	2	0	0	0	-2

3.1 Signed cyclic Heaviside functions and $\mathcal{Y}_{n+1}^1$

The space $\mathcal{Y}_{n+1}^1$ is comprised of $2^n n!$ contractible pieces, parametrized by arrangements of $p_0, \dots, p_n$ and their antipodal points $-p_0, \dots, -p_n$ on $U(1)$. Given a point $\vec{p} = (p_0, \dots, p_n) \in \mathcal{Y}_{n+1}^1$, write $C(\vec{p}) = C(p_0, \dots, p_n)$ as the arrangement of $\vec{p}$ with antipodes on $U(1)$ and $-p_i$ as $p_{\bar{i}}$. By convention $\bar{\bar{i}} = i$. Recall that

$$[n] = \{1, \dots, n\}, \quad [n]_0 = \{0, 1, 2, \dots, n\}, \quad [n]^- = \{\bar{1}, \dots, \bar{n}\}, \quad [n]^\pm = [n] \cup [n]^-.$$

Further define

$$[n]_0^- := \{\bar{0}, \bar{1}, \dots, \bar{n}\}, \quad [n]_0^\pm := \{0, \bar{0}, 1, \bar{1}, \dots, n, \bar{n}\}.$$

When considering $C(\vec{p})$, we will identify the point p_i with i and $-p_i = p_{\bar{i}}$ with $\bar{i}$, so that $C(\vec{p})$ has entries in $[n]_0^\pm$. We take B_{n+1} to be the group of signed permutations on $[n]_0^\pm$, so that $\sigma \in B_{n+1}$ acts on elements of $C(\vec{p})$ by sending $i \in [n]_0^\pm$ to $\sigma(i)$. We will also consider the restricted action of $B_n \leq B_{n+1}$, where B_n is thought of as the subgroup of B_{n+1} fixing $\bar{0}$ and 0.

Example 3.1 Let $\vec{p} = (x_0, x_1, x_2, x_3)$, where $x_j = e^{i\zeta_j}$ and $x_{\bar{j}} = e^{i\bar{\zeta}_j}$, where $\bar{\zeta}_j = \zeta_j + \pi$. Suppose that

$$0 \leq \zeta_0 < \zeta_1 < \bar{\zeta}_2 < \zeta_3 < \bar{\zeta}_0 < \bar{\zeta}_1 < \zeta_2 < \bar{\zeta}_3 < 2\pi.$$

Then we can write $C(\vec{p}) = (0, 1, \bar{2}, 3, \bar{0}, \bar{1}, 2, \bar{3})$, which is drawn below. Note that we are only interested in the relative order of the x_j , rather than their values.

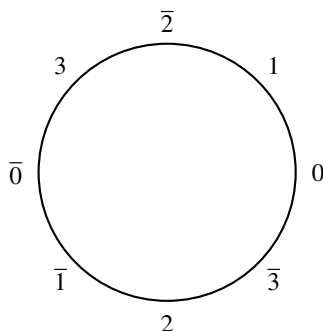


Table 4 Signed cyclic Heaviside functions evaluating the 8 representatives in $\mathcal{Y}_3^1$

$C(\vec{p})$	$y_{0\bar{0}1}$	$y_{0\bar{0}2}$	y_{012}	$y_{01\bar{2}}$	$y_{0\bar{1}2}$	$y_{0\bar{1}\bar{2}}$
$(0, 1, 2, \bar{0}, \bar{1}, \bar{2})$	0	0	1	1	0	1
$(0, 2, 1, \bar{0}, \bar{2}, \bar{1})$	0	0	0	1	0	0
$(0, \bar{1}, 2, \bar{0}, \bar{1}, \bar{2})$	1	0	0	1	1	1
$(0, 2, \bar{1}, \bar{0}, \bar{2}, 1)$	1	0	0	0	0	1
$(0, 1, \bar{2}, \bar{0}, \bar{1}, 2)$	0	1	1	1	1	0
$(0, \bar{2}, 1, \bar{0}, 2, \bar{1})$	0	1	1	0	0	0
$(0, \bar{1}, \bar{2}, \bar{0}, 1, 2)$	1	1	1	0	1	1
$(0, \bar{2}, \bar{1}, \bar{0}, 2, 1)$	1	1	0	0	1	0

We define *signed cyclic Heaviside functions* y_{ijk} for distinct $i, j, k \in [n]_0^\pm$ as

$$y_{ijk}(\vec{p}) := \begin{cases} 1 & p_i < p_j < p_k \text{ counter-clockwise in } C(\vec{p}) \\ 0 & \text{otherwise.} \end{cases}$$

Once again, the y_{ijk} form a $\mathbb{Z}$ -algebra with multiplication given by

$$y_{ijk} \cdot y_{qrs}(\vec{p}) := \begin{cases} 1 & p_i < p_j < p_k \text{ and } p_q < p_r < p_s \text{ counter-clockwise in } C(\vec{p}) \\ 0 & \text{otherwise.} \end{cases}$$

The B_{n+1} action on $\mathcal{Y}_{n+1}^1$ induces an action on the signed cyclic Heaviside functions, where $\sigma \in B_{n+1}$ acts by

$$\sigma \cdot y_{ijk} = y_{\sigma(i)\sigma(j)\sigma(k)}.$$

Example 3.2 Consider the 8 representatives of $\mathcal{Y}_3^1$ and their evaluation via select cyclic Heaviside functions shown in Table 4. Note that if y_{ijk} is included, there is no need to include y_{jik} since $y_{jik} = 1 - y_{ijk}$. Each entry in Table 4 indicates the value of a signed cyclic Heaviside function (columns) evaluated at a given representative in $\mathcal{Y}_{n+1}^1$ (rows).

Proposition 3.3 *The B_{n+1} action on the set of connected components of $\mathcal{Y}_{n+1}^1$ is isomorphic to the coset action*

$$B_{n+1}/\langle c \rangle,$$

where c is a Coxeter element of B_{n+1} . The B_n action on the set of connected components of $\mathcal{Y}_{n+1}^1$ is simply transitive.

Proof It is not difficult to see that the B_{n+1} action on the connected components of $\mathcal{Y}_{n+1}^1$ is transitive. Consider a “typical” point $C(\vec{p}) = (0, 1, 2, \dots, n, \bar{0}, \bar{1}, \dots, \bar{n}) \in \mathcal{Y}_{n+1}^1$; it is fixed by the cyclic subgroup generated by the Coxeter element

$$c = (012 \cdots n)^-.$$

This proves the first claim.

For the second, since B_n fixes 0 and $\bar{0}$, any $\sigma \in B_n$ will act on our typical point $C(\vec{p})$ as

$$\sigma \cdot (0, 1, 2, \dots, n, \bar{0}, \bar{1}, \dots, \bar{n}) = (0, \sigma(1), \sigma(2), \dots, \sigma(n), \bar{0}, \sigma(\bar{1}), \dots, \sigma(\bar{n})),$$

and is thus fixed-point free. This B_n action is also transitive, and so the second claim follows. $\square$

Table 5 Relation (iv) on all possible cyclic orderings of elements i, j, k, ℓ

	y_{ijk}	$y_{ij\ell}$	$y_{ik\ell}$	$y_{jk\ell}$	$y_{ijk} - y_{ij\ell} + y_{ik\ell} - y_{jk\ell}$
(i, j, k, ℓ)	1	1	1	1	0
(i, k, j, ℓ)	0	1	1	0	0
(j, i, k, ℓ)	0	0	1	1	0
(j, k, i, ℓ)	1	0	0	1	0
(k, i, j, ℓ)	1	1	0	0	0
(k, j, i, ℓ)	0	0	0	0	0

The next step is to give a presentation of $H^*\mathcal{Y}_{n+1}^1$ using the signed cyclic Heaviside functions as generators. First we determine relations that hold in $H^*\mathcal{Y}_{n+1}^1$.

Proposition 3.4 *The following relations hold in $H^*\mathcal{Y}_{n+1}^1$ for $i, j, k, \ell \in [n]_0^\pm$:*

$$\begin{aligned}
 (i) \quad & y_{ijk}(1 - y_{ijk}) = 0, \quad (ii) \quad y_{ijk} = 1 - y_{ikj}, \quad (iii) \quad y_{\bar{i}jk} = y_i\bar{j}\bar{k}, \\
 (iv) \quad & y_{ijk} - y_{ij\ell} + y_{ik\ell} - y_{jk\ell} = 0, \quad (v) \quad y_{ijk}y_{ik\ell}(1 - y_{ij\ell}) \\
 & + (1 - y_{ijk})(1 - y_{ik\ell})y_{ij\ell} = 0.
 \end{aligned}$$

Proof Relations (i) follows from the fact that y_{ijk} and $(1 - y_{ijk})$ have disjoint support. Relation (ii) holds because y_{ijk} and y_{ikj} have disjoint support and $y_{ijk} + y_{ikj} = 1$. Relation (iii) comes from noting that by definition of the antipode, $(\bar{i}, j, k)$ appears in $\vec{p}$ if and only if $(i, \bar{j}, \bar{k})$ appears in $\vec{p}$. For relation (iv), consider the six possible relative cyclic orderings of distinct elements i, j, k and ℓ , and their evaluations on $y_{ijk}, y_{ij\ell}, y_{ik\ell}$ and $y_{jk\ell}$, shown in Table 5.

Relation (v) follows because $y_{ijk}, y_{ik\ell}$ and $(1 - y_{ij\ell})$ have disjoint support, implying that $(1 - y_{ijk}), (1 - y_{ik\ell})$ and $y_{ij\ell}$ do as well. $\square$

Note that relations (i), (ii) and (v) deal only with the two of the three coordinates in the y_{ijk} (and will therefore be useful in the restriction to $H^*\mathcal{Z}_n^1$), whereas relations (iii) and (iv) use all three coordinates, and so are only useful to $H^*\mathcal{Z}_n^1$ in specific cases, discussed in Remark 3.5.

Remark 3.5 Relation (iv) has several noteworthy special cases that will be important in computing the presentation for $H^*\mathcal{Y}_{n+1}^1$ (Theorem 3.7) and $H^*\mathcal{Z}_n^1$ (Theorem 3.10) in § 3.2.

(1) Taking $i = 0$ gives a method of rewriting any $y_{jk\ell}$ as

$$y_{jk\ell} = y_{0jk} - y_{0j\ell} + y_{0k\ell}.$$

This allows us to write any signed cyclic Heaviside function in terms of y_{ijk} with $i = 0$, which we will use in the proof of Theorem 3.7 and when restricting from $H^*\mathcal{Y}_{n+1}^1$ to $H^*\mathcal{Z}_n^1$.

(2) Taking $j = 0, k = \bar{i}$ and $\ell = \bar{0}$ gives

$$\begin{aligned}
 0 &= y_{0\bar{i}\bar{i}} - y_{\bar{0}i\bar{0}} + y_{\bar{0}\bar{i}\bar{i}} - y_{\bar{0}\bar{0}\bar{i}} \\
 &= y_{\bar{0}\bar{i}\bar{i}} - y_{\bar{0}\bar{0}\bar{i}} + y_{\bar{0}\bar{i}\bar{i}} - y_{\bar{0}\bar{0}\bar{i}} \\
 &= 2(y_{\bar{0}\bar{i}\bar{i}} - y_{\bar{0}\bar{0}\bar{i}}),
 \end{aligned}$$

where the second line follows by using Relation (iii). This relation will prove to be a useful reduction in the proof of Theorem 3.7.

(3) Taking $j = \bar{i}$ gives

$$\begin{aligned} 0 &= y_{i\bar{i}k} - y_{i\bar{i}\ell} + y_{ik\ell} - y_{\bar{i}k\ell} \\ &= y_{i\bar{i}k} - y_{i\bar{i}\ell} + y_{ik\ell} - y_{i\bar{k}\ell}. \end{aligned}$$

This case is most relevant when $i = 0$, in which case

$$0 = y_{0\bar{0}k} - y_{0\bar{0}\ell} + y_{0k\ell} - y_{0\bar{k}\ell}.$$

Similarly, using relations (iii) and (ii),

$$\begin{aligned} 0 &= y_{0\bar{0}k} - y_{0\bar{0}\ell} + y_{0k\bar{\ell}} - y_{0\bar{k}\ell} \\ &= y_{0\bar{0}k} - y_{0\bar{0}\ell} + y_{0k\bar{\ell}} - y_{0\bar{k}\ell} \\ &= y_{0\bar{0}k} - (1 - y_{0\bar{0}\ell}) + y_{0k\bar{\ell}} - y_{0\bar{k}\ell}. \end{aligned}$$

In addition to appearing as a reduction in the proof of Theorem 3.7, these relations will be instrumental in computing a presentation for $H^* \mathcal{Z}_n^1$ in Theorem 3.10 and will appear again in Sect. 4 (Proposition 4.6).

Next, we show that the relations in Proposition 3.4 are sufficient to describe $H^* \mathcal{Y}_{n+1}^1$. In order to do so, we will employ a Gröbner basis lemma used by Dorpalen-Barry in [21]. For vectors

$$\begin{aligned} (\mathbf{i}, \mathbf{j}, \mathbf{k}) &:= (i_1, j_1, k_1), (i_2, j_2, k_2), \dots, (i_\ell, j_\ell, k_\ell), \\ \mathbf{a} &:= (a_1, \dots, a_\ell) \end{aligned}$$

write

$$\mathbf{y}_{(i,j,k)}^{\mathbf{a}} := y_{i_1 j_1 k_1}^{a_1} y_{i_2 j_2 k_2}^{a_2} \cdots y_{i_\ell j_\ell k_\ell}^{a_\ell}.$$

Then given a polynomial

$$f = \sum c_a \mathbf{y}_{(i,j,k)}^a,$$

with $c_a \in \mathbb{Z}$, the *degree* of f is

$$\deg(f) := \max \left\{ \sum_i a_i : c_a \neq 0 \right\}$$

and the *degree-initial form* of f is

$$\text{indeg}(f) := \sum_{\sum_i a_i = \deg(f)} c_a \mathbf{y}_{(i,j,k)}^a.$$

In other words, $\text{indeg}(f)$ picks off the top degree terms of f . Given a polynomial ideal $\mathcal{Q}$, let $\text{indeg}(\mathcal{Q})$ be the ideal generated by $\text{indeg}(f)$ for $f \in \mathcal{Q}$.

A similar notion exists for any total ordering $<$ on monomials provided that $<$ is a *well-ordering*, meaning that if $f < f'$, then for any monomial g , one has $fg < f'g$. Suppose $<$ is a well-ordering on monomials in $\mathbb{Z}[y_{ijk}]$ for $i, j, k \in [n]_0^\pm$. Then given a polynomial

$$f = \sum c_a \mathbf{y}_{(i,j,k)}^a \in \mathbb{Z}[y_{ijk}],$$

define $\text{in}_{\prec}(f)$ to be the $\prec$ -leading monomial in f . The ordering $\prec$ is said to be a *degree order* if it is compatible with the natural degree ordering on f in the sense that $\text{in}_{\prec}(f) = \text{in}_{\prec}(\text{in}_{\deg}(f))$.

Our goal is to define a degree order $\prec$ on the set of polynomials $\mathbb{Z}[y_{ijk}]$ for any distinct $i, j, k \in [n]_0^{\pm}$. To do so, first order $[n]_0^{\pm}$:

$$\{0 < \bar{0} < 1 < \bar{1} < \cdots < n < \bar{n}\}. \quad (3.1.1)$$

Given y_{ijk} with $i, j, k \in [n]_0^{\pm}$, order the indices so that $i < j, k$ in (3.1.1); this is always possible because the y_{ijk} are equivalent up to cyclic rotation. From there, define $\prec$ by lexicographically ordering the indices in y_{ijk} using the total order in (3.1.1). One can check that $\prec$ does indeed define a well-ordering on $\mathbb{Z}[y_{ijk}]$ for $i, j, k \in [n]_0^{\pm}$. We will use the ordering $\prec$ to argue that the relations in Proposition 3.4 are the only ones needed to generate $H^*\mathcal{Y}_{n+1}^1$.

Given a set of polynomials $\mathcal{I} = \{f_i\}_{i \in I}$ indexed by a set I , the polynomials which are *not* divisible by the terms $\text{in}_{\prec}(f_i)$ for each f_i are called the $\text{in}_{\prec}(\mathcal{I})$ -standard monomials. For example, if $\mathcal{I} = \{y_{ijk}^2 - y_{ijk}\}$, then

$$\text{in}_{\prec}(y_{ijk}^2 - y_{ijk}) = y_{ijk}^2,$$

and so in this case y_{ijk} is an $\text{in}_{\prec}(\mathcal{I})$ -standard monomial. To foreshadow, when $\mathcal{I}$ is the set of relations in Proposition 3.4, the set of $\text{in}_{\prec}(\mathcal{I})$ -standard monomials will be a $\mathbb{Z}$ -basis for $\mathcal{Y}_{n+1}^1$.

Finally, recall that an *ascending filtration* of a commutative ring R is a nested sequence of $\mathbb{Z}$ -submodules $F_0 \subseteq F_1 \subseteq \cdots$ such that for $f \in F_i$ and $f' \in F_j$, one has $f \cdot f' \in F_{i+j}$. The *associated graded ring* of R with respect to this filtration is then defined to be

$$\text{gr}(R) := \bigoplus_{i \geq 0} F_i / F_{i-1}.$$

In our case, the ring we are interested in is $H^*\mathcal{Y}_{n+1}^1$ and the filtration is the natural filtration by degree, so that F_d consists of polynomials of degree d or less.

From these definitions we may now state the relevant lemma. Let $S = \mathbb{Z}[e_1, \dots, e_r]$, where the e_i are standard basis vectors.

Lemma 3.6 (Dorpalen-Barry, [21] Lemma 8) *Let R be a free $\mathbb{Z}$ -module of rank r . Given a surjection $\gamma : S \twoheadrightarrow R$, let $\mathcal{I} = \{f_i\}_{i \in I} \subset S$ be a set of polynomials such that*

- (1) *Each f_i is monic (e.g. the coefficient of $\text{in}_{\deg}(f_i)$ is ± 1),*
- (2) *$\mathcal{I} \subset \ker(\gamma)$, and*
- (3) *the set of $\text{in}_{\prec}(\mathcal{I})$ -standard monomials $\mathcal{N} = \{m_1, \dots, m_t\}$ has cardinality $t \leq r$.*

Then

- (1) *$R \cong S/(\mathcal{I})$ as $\mathbb{Z}$ -modules, where $(\mathcal{I})$ is the ideal generated by $\mathcal{I}$ and*
- (2) *the cardinality of $\mathcal{N}$ is r , so that $\gamma(m_1), \dots, \gamma(m_r)$ is a $\mathbb{Z}$ -basis for R .*

In the case that $\prec$ is a degree-ordering, then there is a further $\mathbb{Z}$ -module isomorphism

$$S / \text{in}_{\deg}(\mathcal{I}) \cong \text{gr}(R).$$

The Hilbert series for $\text{gr}(R)$ is then given by

$$\text{Hilb}(\text{gr}(R), t) = \sum_{i=1}^r t^{\deg(m_i)}.$$

We will see that Lemma 3.6 is precisely what we need to give a presentation for $H^*\mathcal{Y}_{n+1}^1$. Define the sets

$$\underbrace{\{y_{0\bar{0}1}\}}_{h_1}, \underbrace{\{y_{0\bar{0}2}, y_{012}, y_{01\bar{2}}\}}_{h_2}, \dots, \underbrace{\{y_{0\bar{0}n}, y_{01n}, y_{01\bar{n}}, \dots, y_{0(n-1)n}, y_{0(n-1)\bar{n}}\}}_{h_n},$$

and let $\mathcal{N}$ be the set of monomials obtained by multiplying at most one term in each h_i . Intuitively, think of each h_i as a hand and the elements in h_i as its fingers; then $\mathcal{N}$ is the set of monomials obtained by picking at most one finger from each hand⁶.

Theorem 3.7 *Let $\mathcal{I}$ be the ideal generated by the relations in Proposition 3.4. Then*

$$H^*\mathcal{Y}_{n+1}^1 = \mathbb{Z}[y_{ijk}]/\mathcal{I}$$

where i, j, k are distinct elements in $[n]_0^\pm$. Further, the set $\mathcal{N}$ is a basis for $H^*\mathcal{Y}_{n+1}^1$.

Proof Suppose m is an $\text{in}_<(\mathcal{I})$ -standard monomial. We make the following reductions:

- By Proposition 3.4 (i), we may assume m is square free;
- Using Proposition 3.4 (iv) and setting $i = 0$, we may also assume that m is comprised of generators of the form y_{0jk} ;
- Furthermore, using Proposition 3.4 (iv) and Remark 3.5 (2) we have that m cannot contain $y_{0\bar{i}i}$ for $i \in [n]$;
- Using 3.4 (ii), we may assume further for y_{0jk} that $j < k$ with respect to the ordering in (3.1.1);
- By Remark 3.5 (3) we have that m cannot contain any generator of the form $y_{0\bar{j}k}$ or $y_{0j\bar{k}}$ for $j, k \in [n]$.

Finally, we must use relation (v) to show that m is in fact in $\mathcal{N}$. Note that (v) can be understood as a choice of three elements $i < j < k \in [n]_0^\pm \setminus \{0\}$, corresponding to the three monomials y_{0ij} , y_{0jk} and y_{0ik} . Applying relation 3.4 (v) to these generators implies that the term $y_{0ik}y_{0jk}$ cannot divide m . We use this logic to underline terms that are not divisible by m for $i, j, k \in [n]$:

- $(i < j < k)$:

$$0 = y_{0ij}y_{jk} - y_{0ij}y_{0ik} - \underline{y_{0ik}y_{jk}} + y_{0ik}.$$

- $(i < j < \bar{k})$:

$$0 = y_{0ij}y_{j\bar{k}} - y_{0ij}y_{0i\bar{k}} - \underline{y_{0i\bar{k}}y_{j\bar{k}}} + y_{0i\bar{k}}.$$

- $(\bar{0} < i < j)$:

$$0 = y_{0\bar{0}i}y_{0ij} - y_{0\bar{0}i}y_{0\bar{0}j} - \underline{y_{0ij}y_{0\bar{0}j}} + y_{0\bar{0}j}.$$

- $(\bar{0} < i < \bar{j})$: Noting that $y_{0\bar{0}j} = y_{\bar{0}0j} = (1 - y_{0\bar{0}j})$ by 3.4 (iii) and 3.4 (ii):

$$\begin{aligned} 0 &= y_{0\bar{0}i}y_{0i\bar{j}} - y_{0\bar{0}i}y_{0\bar{0}\bar{j}} - y_{0i\bar{j}}y_{0\bar{0}\bar{j}} - y_{0\bar{0}\bar{j}} \\ &= y_{0\bar{0}i}y_{0i\bar{j}} - y_{0\bar{0}i}(1 - y_{0\bar{0}j}) - \underline{y_{0i\bar{j}}(1 - y_{0\bar{0}j})} - (1 - y_{0\bar{0}j}). \end{aligned}$$

⁶ The terminology of “hands” and “fingers” originates in Hélène Barcelo’s thesis [5, Thm 2.1] and was later used in Barcelo–Goupil [7], both in the context of describing an *nbc-basis*. Such bases arise in the study of matroids. While we are not in the matroid (e.g. hyperplane) setting, because our description of $\mathcal{N}$ uses the hand/finger description, we may refer to it basis as an *nbc-basis* nonetheless.

- $(i < j < \bar{j})$: Noting that $y_{0j\bar{j}} = 1 - y_{0\bar{j}j} = 1 - y_{0\bar{0}j}$ by 3.4 (iii) and 3.4 (ii):

$$\begin{aligned} 0 &= y_{0ij}y_{0j\bar{j}} - y_{ij}y_{0i\bar{j}} - y_{0j\bar{j}}y_{0i\bar{j}} \\ &= y_{0ij}(1 - y_{0\bar{0}j}) - \underline{y_{ij}y_{0i\bar{j}}} - (1 - y_{0\bar{0}j})y_{0i\bar{j}}. \end{aligned}$$

- $(i < \bar{j} < k)$: By 3.4 (iv), Remark 3.5 (3) one has $y_{0\bar{j}k} = y_{0\bar{0}j} + y_{0\bar{0}k} + y_{0j\bar{k}} - 1$, and so

$$\begin{aligned} 0 &= y_{0i\bar{j}}y_{\bar{j}k} - y_{\bar{j}k}y_{ik} - y_{0i\bar{j}}y_{ik} + y_{ik} \\ &= y_{0i\bar{j}}(y_{0\bar{0}j} + y_{0\bar{0}k} + y_{0j\bar{k}} - 1) - (y_{0\bar{0}j} + y_{0\bar{0}k} + \underline{y_{0j\bar{k}}} - 1)y_{ik} - y_{0i\bar{j}}y_{ik} + y_{ik}. \end{aligned}$$

- $(\bar{i} < j < k)$: Again, we have $y_{0\bar{i}j} = y_{0\bar{0}i} + y_{0\bar{0}j} + y_{0i\bar{j}} - 1$, and $y_{0\bar{i}k} = y_{0\bar{0}i} + y_{0\bar{0}k} + y_{0i\bar{k}} - 1$. Hence

$$\begin{aligned} 0 &= y_{0\bar{i}j}y_{0jk} - y_{0\bar{i}j}y_{0\bar{i}k} - y_{0jk}y_{0\bar{i}k} + y_{0\bar{i}k} \\ &= (y_{0\bar{0}i} + y_{0\bar{0}j} + y_{0i\bar{j}} - 1)y_{0jk} - (y_{0\bar{0}i} + y_{0\bar{0}j} + y_{0i\bar{j}} - 1)(y_{0\bar{0}i} + y_{0\bar{0}k} + y_{0i\bar{k}} - 1) \\ &\quad - \underline{y_{0jk}}(y_{0\bar{0}i} + y_{0\bar{0}k} + \underline{y_{0i\bar{k}}}) - 1 + (y_{0\bar{0}i} + y_{0\bar{0}k} + y_{0i\bar{k}} - 1). \end{aligned}$$

It follows that $m \in \mathcal{N}$, because all of the terms violating the definition of $\mathcal{N}$ (e.g. picking multiple generators from the same set h_i) are in blue above. Thus the hypotheses in Lemma 3.6 are satisfied, and the claim follows. $\square$

Note that $<$ is also a degree ordering, which by Lemma 3.6 implies the following.

Corollary 3.8 *The associated graded ring of $H^*\mathcal{Y}_{n+1}^1$ with respect to the filtration by degree has presentation*

$$\mathrm{gr}(H^*\mathcal{Y}_{n+1}^1) \cong \mathbb{Z}[y_{ijk}]/\mathrm{indeg}(\mathcal{I}),$$

for distinct $i, j, k \in [n]_0^\pm$, where $\mathrm{indeg}(\mathcal{I})$ is generated by the relations

$$\begin{aligned} (i) \ y_{ijk}^2, \quad (ii) \ y_{ijk} - y_{ij\ell} + y_{ik\ell} - y_{j\ell k} \quad (iii) \ y_{i\bar{j}k} = y_{i\bar{j}\bar{k}} \\ (iv) \ y_{ijk} = -y_{ikj} \quad (v) \ y_{ijk}y_{ik\ell} - y_{ijk}y_{ij\ell} - y_{ik\ell}y_{ij\ell}. \end{aligned}$$

Further,

$$\mathrm{Hilb}(\mathrm{gr}(H^*\mathcal{Y}_{n+1}^1), t) = (1+t)(3+t)(5+t) \cdots ((2n-1)+t).$$

Proof The relations generating $\mathrm{indeg}(\mathcal{I})$ come from computing $\mathrm{indeg}(f_i)$ for each $f_i \in \mathcal{I}$ in Theorem 3.7. Note that expanding relation 3.4 (v) gives

$$\begin{aligned} y_{ijk}y_{ik\ell}(1 - y_{ij\ell}) + (1 - y_{ijk})(1 - y_{ik\ell})y_{ij\ell} \\ = y_{ijk}y_{ik\ell} - y_{ijk}y_{ik\ell}y_{ij\ell} + y_{ij} - y_{ijk}y_{ik\ell} - y_{ik\ell}y_{ij\ell} + y_{ijk}y_{ik\ell}y_{ij\ell} \\ = y_{ijk}y_{ik\ell} + y_{ij} - y_{ijk}y_{ik\ell} - y_{ik\ell}y_{ij\ell}. \end{aligned}$$

Removing the single degree 1 term here gives a homogeneous degree 2 relation. The Hilbert series follows from a standard counting argument using the set $\mathcal{N}$. $\square$

Example 3.9 Suppose $n+1=3$. Then a basis for $H^*\mathcal{Y}_3^1$, separated by degree in the y_{ijk} is

$$\begin{aligned} \mathrm{deg}(0) : & 1 \\ \mathrm{deg}(1) : & y_{0\bar{0}1}, \ y_{012}, \ y_{01\bar{2}}, \ y_{0\bar{0}2}, \\ \mathrm{deg}(2) : & y_{0\bar{0}1}y_{012}, \ y_{0\bar{0}1}y_{01\bar{2}}, \ y_{0\bar{0}1}y_{0\bar{0}2} \end{aligned}$$

The Hilbert series for $\mathrm{gr}(H^* \mathcal{Y}_3^1)$ is

$$1 + 4t + 3t^2 = (1 + t)(3 + t).$$

3.2 A presentation for $H^* \mathcal{Z}_n^1$

The B_n module isomorphism induced from Proposition 2.10 (applied as in Example 2.12) recovers a presentation for $H^* \mathcal{Z}_n^1$ and $\mathrm{gr}(H^* \mathcal{Z}_n^1)$. The isomorphism can be described as follows: for each Heaviside function y_{ijk} , fix $i = 0$ and define for $j, k \in [n]^\pm$,

$$\begin{aligned} z_{jk} &:= y_{0jk} \\ z_j &:= y_{00j}. \end{aligned}$$

These z_{jk} and z_j inherit a B_n action: $\sigma \in B_n$ acts by

$$\begin{aligned} \sigma \cdot z_{jk} &= z_{\sigma(j)\sigma(k)}, \\ \sigma \cdot z_j &= z_{\sigma(j)}. \end{aligned}$$

The elements z_{jk} and z_j are (by construction) functionals on $\mathcal{Z}_n^1$. The latter generator has a simple description:

$$z_j(x_1, \dots, x_n) = \begin{cases} 1 & x_j > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The z_{jk} are a bit more complicated; if $j, k \in [n]$, then

$$z_{jk}(x_1, \dots, x_n) = \begin{cases} 1 & x_j < x_k \\ 0 & \text{otherwise,} \end{cases}$$

and

$$z_{j\bar{k}}(x_1, \dots, x_n) = \begin{cases} 1 & x_j < \varphi(x_k) \\ 0 & \text{otherwise,} \end{cases}$$

where $\varphi(x_j) = -x_j/|x_j|^2$. The generator $z_{\bar{j}k}$ can analogously be interpreted as 1 when $\varphi(x_j) < x_k$, and the generator $z_{\bar{j}\bar{k}}$ is 1 when $\varphi(x_j) < \varphi(x_k)$. Note that $\varphi(x_j) < \varphi(x_k)$ is *not* equivalent to $-x_j < -x_k$.

We now obtain from Theorem 3.7 a presentation for $H^* \mathcal{Z}_n^1$.

Theorem 3.10 *There is an isomorphism of $\mathbb{Z}$ -modules*

$$H^* \mathcal{Z}_n^1 = \mathbb{Z}[z_{ij}, z_i]/\mathcal{J}$$

for $i \neq j \in [n]^\pm$, where $\mathcal{J}$ is generated by the relations:

$$(i) \quad z_{ij}(1 - z_{ij}) \quad (ii) \quad z_i(1 - z_i) \quad (iii) \quad z_i - z_j + z_{ij} - z_{i\bar{j}} \quad (iv) \quad z_i - (1 - z_{\bar{i}})$$

and

$$\begin{aligned} (v) \quad & z_{ij}z_{jk}(1 - z_{ik}) + (1 - z_{ij})(1 - z_{jk})z_{ik} \\ (vi) \quad & z_{ij}z_i(1 - z_j) + (1 - z_{ij})(1 - z_i)z_j \\ (vii) \quad & z_jz_{i\bar{j}}(1 - z_{ij}) + (1 - z_j)(1 - z_{i\bar{j}})z_{ij}. \end{aligned}$$

The “extra” relations (vi) and (vii) compared to $H^*\mathcal{Y}_{n+1}^1$ are necessary because unlike the y_{ijk} , there are now two types of generators, z_j and z_{jk} .

Proof The isomorphism between $H^*\mathcal{Y}_{n+1}^1$ and $H^*\mathcal{Z}_n^1$ implies that $H^*\mathcal{Z}_n^1$ has a $\mathbb{Z}$ -basis consisting of monomials that are a product of at most one element from each set

$$\{z_1\}, \{z_2, z_{12}, z_{1\bar{2}}\}, \dots, \{z_n, z_{1n}, \dots, z_{(n-1)n}, z_{1\bar{n}}, \dots, z_{(n-1)\bar{n}}\}.$$

That Relations (i)—(v) hold in $H^*\mathcal{Z}_n^1$ follows immediately from the identification of variables discussed above. Relation (vi) can be seen directly: for $i, j \in [n]$, the function $z_{ij}z_i(1 - z_j)$ is nonzero on $(x_1, \dots, x_n)$ if $x_i < x_j$, $x_i > 0$, and $x_j < 0$, which is impossible. Similarly, $(1 - z_{ij})(1 - z_i)z_j$ is always 0, so the sum must be 0 as well. The relation is then closed under the action by B_n .

An analogous argument holds for (vii); $z_jz_i\bar{j}(1 - z_{ij})$ is non-zero on $(x_1, \dots, x_n)$ if $x_i > x_j$, $x_i < \varphi(x_j)$ and $x_j > 0$. However if $x_j > 0$, this means that $\varphi(x_j) < 0$, and so $x_i < 0$, contradicting the assumption that $x_i > x_j$. Analogously, the second summand in (vii) is 0. A similar argument as the proof of Theorem 3.7 (e.g. using the ordering induced from (3.1.1) and Lemma 3.6) then shows that these relations are sufficient to generate $H^*\mathcal{Z}_n^1$. $\square$

Using Theorem 3.8, we obtain a presentation for $\text{gr}(H^*\mathcal{Z}_n^1)$ as well.

Corollary 3.11 *The associated graded ring of $H^*\mathcal{Z}_n^1$ with respect to the filtration by degree is given by*

$$\text{gr}(H^*\mathcal{Z}_n^1) = \mathbb{Z}[z_{ij}, z_i] / \text{indeg}(\mathcal{J})$$

for distinct $i, j \in [n]^\pm$ where $\text{indeg}(\mathcal{J})$ is generated by

$$\begin{aligned} & \text{(i)} \quad z_{ij}^2 \quad \text{(ii)} \quad z_i^2 \quad \text{(iii)} \quad z_i - z_j + z_{ij} - z_{i\bar{j}} \quad \text{(iv)} \quad z_i + z_{\bar{i}} \\ & \text{(v)} \quad z_{ij}z_{jk} - z_{ij}z_{ik} - z_{jk}z_{ik} \quad \text{(vi)} \quad z_iz_{ij} - z_{ij}z_j - z_iz_j \quad \text{(vii)} \quad z_jz_{i\bar{j}} - z_{i\bar{j}}z_{ij} - z_jz_{ij}. \end{aligned}$$

Example 3.12 ($n = 2$) The above restriction tells us that when $n = 2$, the basis for $\text{gr}(H^*\mathcal{Z}_n^1)$ is

$$\begin{aligned} \text{deg}(0) &: 1 \\ \text{deg}(1) &: z_1, \quad z_{12}, \quad z_{1\bar{2}}, \quad z_2, \\ \text{deg}(2) &: z_1z_{12}, \quad z_1z_{1\bar{2}}, \quad z_1z_2, \end{aligned}$$

and the Hilbert series of $\text{gr}(H^*\mathcal{Z}_2^1)$ is the same as that of $\text{gr}(H^*\mathcal{Y}_3^1)$.

3.2.1 A bi-grading on $H^*\mathcal{Z}_n^1$

Finally, we introduce a further filtration on $\text{gr}(H^*\mathcal{Z}_n^1)$.

Proposition 3.13 *Let P_ℓ be the ideal in $\text{gr}(H^*\mathcal{Z}_n^1)$ with monomials of degree ℓ or higher in the z_i for $i \in [n]^\pm$. Then there is a descending filtration on $\text{gr}(H^*\mathcal{Z}_n^1)$ that is stable under the B_n -action on $\text{gr}(H^*\mathcal{Z}_n^1)$:*

$$P_n \subset P_{n-1} \subset \dots \subset P_1 \subset P_0.$$

Proof Consider first the polynomial ring $\mathbb{Z}[z_{ij}, z_i]$ for $i, j \in [n]^\pm$; it is clear that it has a filtration by z_i degree, and that this filtration respects the action by B_n . One can further check that the ideal $\text{indeg}(\mathcal{J})$ in Corollary 3.11 also respects the filtration because each relation (and its B_n orbit) is in either P_0 , P_1 or P_2 . $\square$

Definition 3.14 Define the associated graded ring of $\text{gr}(H^*Z_n^1)$ with respect to the filtration in Proposition 3.13 to be

$$\mathcal{G} = \bigoplus_{0 \leq k \leq n} \mathcal{G}_k = \bigoplus_{0 \leq \ell \leq k \leq n} \mathcal{G}_{k,\ell},$$

where $\mathcal{G}_k$ consists of monomials of degree k in the variables z_{ij} and z_i for $i, j \in [n]^\pm$ and $\mathcal{G}_{k,\ell}$ consists of monomials in $\mathcal{G}_k$ which are degree ℓ in the z_i variables.

Corollary 3.15 As a bi-graded ring, $\mathcal{G}$ has a presentation

$$\mathbb{Z}[z_{ij}, z_i]/\mathcal{L}$$

for $i, j \in [n]^\pm$, where $\mathcal{L}$ is generated by $z_{ij}^2 = z_i^2 = 0$ and

$$(i) \ z_{ij} + z_{ji} \quad (ii) \ z_i + z_{\bar{i}} \quad (iii) \ z_{i\bar{j}} - z_{\bar{i}j} \quad (iv) \ z_{ij}z_{i\bar{j}} \quad (v) \ z_{ij}z_{jk} - z_{ij}z_{ik} - z_{jk}z_{ik}.$$

Remark 3.16 The fact that $\mathbb{Q}[B_n]$ is semisimple means that for any filtration stable under the action of B_n , passing to the associated graded ring will not change the isomorphism type of the representation. Hence we will study representations on $\mathcal{G}_{k,\ell}$ and use this to deduce information about $H^*Z_n^1$ and $H^*Z_n^3$.

Remark 3.17 The filtration in Proposition 3.13 (using the identification $z_i \leftrightarrow y_{0\bar{0}i}$) does not respect the lifted B_{n+1} action, and therefore does not give a bi-grading on $H^*\mathcal{Y}_{n+1}^1$.

4 The $d = 3$ case in Type B

We now turn to the space Z_n^3 , which was studied by Feichtner–Ziegler in [24] and Xicoténcatl in [54]; we will first review their work⁷ in Sect. 4.1. Then, we will work to understand the action of B_n on the basis for $H^*Z_n^3$ (Sect. 4.2). The presentation for $H^*Z_n^3$ and its consequences are given in Sect. 4.3.

4.1 Tools from topology

In [24], Feichtner–Ziegler use the space Z_n^d to compute a presentation for the cohomology of the $\mathbb{Z}_2$ orbit configuration space $\text{Conf}_n^{\mathbb{Z}_2}(\mathbb{S}^d)$. Let

$$\begin{aligned} \Pi_{[n]} : Z_{n+1}^d &\longrightarrow Z_n^d \\ (x_1, \dots, x_{n+1}) &\longmapsto (x_1, \dots, x_n) \end{aligned}$$

⁷ In [24], Feichtner–Ziegler give a presentation of $H^*Z_n^d$ for $d \geq 2$. However, their computation of the action of B_n on the generators of $H^*Z_n^d$ has an error [24, Lemma 7(iv)]. Xicoténcatl also gives a presentation of $H^*Z_n^d$, which agrees with our presentation; however his work does not explicitly compute the action of B_n on the generators of $H^*Z_n^d$. We will see in Proposition 4.6 that the B_n -action on $H^*Z_n^d$ is delicate, and so we include all the details of our computations for completeness.

be the map which “forgets” the last coordinate in $\mathcal{Z}_{n+1}^d$, and define

$$\mathcal{Q}_n = \{0, x_1, \varphi(x_1), \dots, x_{n-1}, \varphi(x_{n-1})\},$$

where $\varphi(x_i) = -x_i/|x_i|^2$, as before. This induces a locally trivial fiber bundle

$$\mathbb{R}^d \setminus \mathcal{Q}_n \longrightarrow \mathcal{Z}_{n+1}^d \longrightarrow \mathcal{Z}_n^d. \quad (4.1.1)$$

Feichtner–Ziegler prove that the associated spectral sequence collapses in its second term whenever $d > 2$, and that the corresponding cohomology ring $H^*\mathcal{Z}_n^d$ is torsion-free with Hilbert series⁸

$$\text{Hilb}(H^*\mathcal{Z}_n^d, t) = (1 + t^{d-1})(1 + 3t^{d-1}) \cdots (1 + (2n-1)t^{d-1}). \quad (4.1.2)$$

Using the spectral sequence induced by (4.1.1), Feichtner–Ziegler provide a multiplicative generating set and $\mathbb{Z}$ -module basis for $H^*\mathcal{Z}_n^d$.

Proposition 4.1 ([24]: Prop 8) *A $\mathbb{Z}$ -linear basis for $H^*\mathcal{Z}_n^d$ is given by choosing one element from each set:*

$$\mathbb{Z}\{1, z_1\} \cdot \mathbb{Z}\{1, z_2, z_{12}, z_{1\bar{2}}, \dots\} \cdots \mathbb{Z}\{1, z_n, z_{1n}, z_{1\bar{n}}, \dots, z_{(n-1)n}, z_{(n-1)\bar{n}}\}.$$

A number of relations that hold in $H^*\mathcal{Z}_n^3$ can be established from the above topological framework. Recall that $t_i \in B_n$ is the element sending $i \mapsto -i$ and $j \mapsto j$ for all other j , and $s_i \in B_n$ is the Coxeter generator $(i, i+1)$. Let (i, j) be the transposition swapping i and j .

Proposition 4.2 *The following identities hold in $H^*\mathcal{Z}_n^3$ for distinct $i, j, k \in [n]^\pm$*

- (1) (Feichtner–Ziegler [24, Lemma 7(iii)]): $t_i \cdot z_i = z_{\bar{i}} = -z_i$;
- (2) $(i, j) \cdot z_{ij} = z_{ji} = -z_{ij}$;
- (3) $z_{ij}^2 = z_i^2 = 0$.
- (4) (Feichtner–Ziegler [24, Prop. 11]): $z_{ij}z_{jk} - z_{ik}z_{ij} - z_{ik}z_{jk} = 0$
- (5) (Feichtner–Ziegler [24, Prop. 11]): $z_{ij}z_i - z_{ij}z_j - z_i z_j = 0$.
- (6) $z_j z_{i\bar{j}} - z_{ij} z_{i\bar{j}} - z_j z_{ij} = 0$.

We give a general sketch of each of these relations for intuition.

Proof (1) The first claim in Proposition 4.2 follows because the generator z_i is the image of the (dual) fundamental class induced from the projection map

$$\begin{aligned} \pi_i : \mathcal{Z}_n^d &\longrightarrow S^{d-1} \\ (x_1, \dots, x_n) &\longmapsto \frac{x_i}{|x_i|}. \end{aligned}$$

The action of t_i then restricts to the antipodal action on S^{d-1} .

(2) Analogously, the second claim in Proposition 4.2 can be understood via the projection map

$$\begin{aligned} \pi_{ij} : \mathcal{Z}_n^d &\longrightarrow S^{d-1} \\ (x_1, \dots, x_n) &\longmapsto \frac{x_i - x_j}{|x_i - x_j|}. \end{aligned}$$

The generator z_{ij} is similarly defined as the image of $\pi_{ij}^*([S^{d-1}])$ (where $[S^{d-1}]$ is the dual fundamental class of S^{d-1}). Once again the action of (i, j) can be traced back via π_{ij} , where it restricts to the antipodal map on S^{d-1} .

⁸ They further prove (with a bit more work) that the $d = 2$ case is also torsion free and satisfies (4.1.2).

- (3) Again, the third claim comes from the fact that the generators z_{ij} and z_i are the images of $[S^{d-1}]$, and $[S^{d-1}]^2 = 0$ in $H^*(S^{d-1})$.
- (4) Let $U_{ij}^+ := \{(x_1, \dots, x_n) \in \mathbb{R}^{3n} : x_i = x_j\}$, and consider the complement space $\mathcal{M}\{U_{ij}^+, U_{jk}^+\} \subset \mathbb{R}^{3n}$. There is a natural inclusion

$$\mathcal{Z}_n^3 \hookrightarrow \mathcal{M}\{U_{ij}^+, U_{jk}^+\};$$

the induced map in cohomology must send the relation $z_{ij}z_i - z_{ij}z_j - z_i z_j = 0$ in $H^*(\mathcal{M}\{U_{ij}^+, U_{jk}^+\})$ to 0 in $H^*\mathcal{Z}_n^3$.

- (5) Let $U_i := \{(x_1, \dots, x_n) \in \mathbb{R}^{3n} : x_i = 0\}$, and again consider the complement and corresponding inclusion $\mathcal{Z}_n^3 \hookrightarrow \mathcal{M}\{U_{ij}^+, U_i, U_j\}$. Then (5) holds by the same logic as relation (4).
- (6) Finally, we obtain the relation (6) using a similar argument as the last part of [24, Prop 11.]. It is sufficient to work in the case that $n = 2$. Let

$$U_{12}^- := \{(x_1, x_2 \in (\mathbb{R}^3 \setminus \{0\})^2 : x_1 \neq x_2, x_1 \neq \varphi(x_2)\},$$

and U_1, U_2, U_{12}^+ be as before. Consider the map

$$\begin{aligned} \Phi : \mathcal{M}\{U_1, U_2, U_{12}^+\} &\longrightarrow \mathcal{M}\{U_{12}^+, U_{12}^-, U_2\} \\ (x_1, x_2) &\longmapsto (x_1 - x_2, x_1 - \varphi(x_2)). \end{aligned}$$

Let d_1, d_2 and d_{12} be the respective generators of U_1, U_2, U_{12} . The same argument used in (i) of the proof of [24, Prop. 11] shows that $\Phi^*(d_1) = z_{12}$.

To show that $\Phi^*(d_2) = z_{1\bar{2}}$, note that

$$\Phi \circ s_2(x_1, x_2) = (x_1 - \varphi(x_2), x_1 - x_2) = s_1 \circ \Phi(x_1, x_2).$$

Hence

$$\Phi^*(d_2) = \Phi^* \circ s_1(d_1) = s_2 \circ \Phi^*(d_1) = s_2(z_{12}) = z_{1\bar{2}}.$$

Finally, we show that $\Phi^*(d_{12}) = -c_2$. Let $\sigma : \mathcal{M}\{U_1, U_2, U_{12}^+\} \rightarrow \mathbb{S}^2$ be the projection onto the second coordinate, and then the unit sphere. Thus $\sigma^*(c) = d_2$, where c is the generator of $\mathbb{S}^2$. Let

$$\begin{aligned} \bar{\theta} : \mathcal{M}\{U_1, U_2, U_{12}^+\} &\longrightarrow \mathbb{S}^2 \\ (x_1, x_2) &\longmapsto \frac{1}{|x_1 - x_2|} (x_1 - x_2). \end{aligned}$$

Then

$$\begin{aligned} \bar{\theta} \circ \Phi(x_1, x_2) &= \frac{1}{|\varphi(x_2) - x_2|} (\varphi(x_2) - x_2) \\ &= \frac{1}{|\frac{-x_2}{|x_2|^2} \left(1 + \frac{1}{|x_2|^2}\right)|} \cdot \frac{-x_2}{|x_2|^2} \left(1 + \frac{1}{|x_2|^2}\right) \\ &= \frac{1}{\left(1 + \frac{1}{|x_2|^2}\right)} \cdot \frac{-x_2}{|x_2|^2} \left(1 + \frac{1}{|x_2|^2}\right) \\ &= \frac{-x_2}{|x_2|^2} \end{aligned}$$

Table 6 The action of B_2 on the basis of $H^* \mathcal{Z}_2^3$. Bold indicates an element not in the basis

	s_1	t_2	$s_1 t_2$	$(s_1 t_2)^2 = -1$
1	1	1	1	1
z_1	z_2	z_1	z_2	$-z_1$
z_2	z_1	$-z_2$	$-z_1$	$-z_2$
z_{12}	$-z_{12}$	$z_{1\bar{2}}$	$-z_{\bar{1}2}$	$z_{\bar{1}\bar{2}}$
$z_{1\bar{2}}$	$-z_{\bar{1}2}$	z_{12}	$-z_{12}$	$z_{\bar{1}2}$
$z_1 z_2$	$z_1 z_2$	$-z_1 z_2$	$-z_1 z_2$	$z_1 z_2$
$z_1 z_{12}$	$-z_2 z_{12} = -z_1 z_{12} + z_1 z_2$	$z_1 z_{1\bar{2}}$	$-z_2 z_{\bar{1}2} = -z_1 z_{\bar{1}2} - z_1 z_2$	$-z_1 z_{\bar{1}\bar{2}}$
$z_1 z_{1\bar{2}}$	$-z_2 z_{\bar{1}2} = -z_1 z_{\bar{1}2} - z_1 z_2$	$z_1 z_{12}$	$-z_2 z_{12} = -z_1 z_{12} + z_1 z_2$	$-z_1 z_{\bar{1}2}$

$$= -\sigma(x_1, x_2).$$

Thus $\Phi^* \circ \bar{\theta}^* = -\sigma^*$ because the antipodal map in this context has degree -1 . Noting that $\bar{\theta}^*(c) = z_{12}$ implies that

$$\Phi^*(d_{12}) = \Phi^* \circ \bar{\theta}^*(c) = -\sigma^*(c) = -z_2,$$

and therefore

$$\Phi^*(d_{12}d_1 - d_{12}d_2 - d_1d_2) = -z_2z_{12} + z_2z_{1\bar{2}} - z_{12}z_{1\bar{2}}.$$

□

4.2 Tools from representation theory

The relations in Proposition 4.2 are insufficient to give an algebra presentation for $H^* \mathcal{Z}_n^3$ —that is, they do not give all the relations among the generators of the algebra. In particular, we are missing information about how to rewrite the generators $z_{\bar{i}j}$ and $z_{\bar{i}\bar{j}}$ in terms of the basis discussed in Proposition 4.1. This is relevant in part because we would like to understand the action of B_n on $H^* \mathcal{Z}_n^3$. Consider, for example, what is known based on Proposition 4.2 about the action of B_2 on the basis for $H^* \mathcal{Z}_2^3$, summarized in Table 6. The teal entries indicate elements which are not in the basis.

Thus the next step is to develop tools to complete the presentation of $H^* \mathcal{Z}_n^3$. In order to understand the representations carried by $H^* \mathcal{Z}_n^3$ and $H^* \mathcal{Y}_{n+1}^3$, assume henceforth that both spaces have coefficients in $\mathbb{Q}$. Our subsequent computations will use the representation theory of the hyperoctahedral group, reviewed in § 2.3.3.

4.2.1 A recursion for $H^* \mathcal{Y}_{n+1}^3$

We will now develop a recursion between the B_n -representations carried by $H^* \mathcal{Z}_n^3$ and $H^* \mathcal{Y}_n^3$, in addition to being of independent interest, this recursion will be instrumental to determining the presentation for $H^* \mathcal{Z}_n^3$.

Recall that the two spaces $\mathcal{Z}_n^3$ and $\mathcal{Y}_{n+1}^3$ are homeomorphic, and therefore their cohomologies are isomorphic as B_n -representations. We will use this fact to understand $H^* \mathcal{Y}_{n+1}^3$ as a B_{n+1} -representation. The space $\mathcal{Y}_{n+1}^3$ has a fiber sequence

$$SU_2 \setminus \{\pm p_1, \pm p_2, \dots, \pm p_n\} \longrightarrow \mathcal{Y}_{n+1}^3 \longrightarrow \mathcal{Y}_n^3. \quad (4.2.1)$$

Note that there is a B_n -action on F , which comes from the B_n action on $\{\pm p_1, \dots, \pm p_n\}$. In particular, for $\sigma \in B_n$, the point $(p_1, \dots, p_n, p_{n+1})$ and $(p_{\sigma(1)}, \dots, p_{\sigma(n)}, p_{n+1})$ have homeomorphic fibers in the fiber sequence (4.2.2), and the action of B_n permutes the punctures of F . It follows that B_n acts on each space in (4.2.2), and is equivariant with respect to both maps.

Because $\mathcal{Z}_n^3$ is simply connected and has homology concentrated only in even degrees, the same must be true for $\mathcal{Y}_{n+1}^3$. It follows that the spectral sequence collapses, yielding the isomorphism:

$$H^* \mathcal{Y}_{n+1}^3 \cong_{B_n} H^* \mathcal{Y}_n^3 \otimes H^*(F) \cong_{B_n} H^* \mathcal{Z}_n^3, \quad (4.2.2)$$

where F is the fiber $SU_2 \setminus \{\pm p_1, \pm p_2, \dots, \pm p_n\}$ from (4.2.1).

Since F is connected, $H^0(F) \cong \mathbb{Q}$ and carries the trivial representation. The only other non-trivial degree is $H^2(F)$, which is $2n - 1$ dimensional.

Lemma 4.3 *As a B_n -representation,*

$$H^2(F) \cong_{B_n} \chi^{(n-1), (1)} + \chi^{(n-1), \emptyset}.$$

Proof Since F is a punctured sphere, the generators of the homology can be understood as cycles around these punctures. Let e_i be the cocycle dual to the cycle around the point p_i and $\bar{e}_i$ the cocycle dual to the cycle around its antipodal point $-p_i$. The B_n action on $H^2(F)$ is determined by how B_n permutes these cocycles. It follows that

$$H^2(F) \cong \frac{\mathbb{Q} e_1 \oplus \mathbb{Q} \bar{e}_1 \oplus \dots \oplus \mathbb{Q} e_n \oplus \mathbb{Q} \bar{e}_n}{\mathbb{Q}(e_1 + \bar{e}_1 + \dots + e_n + \bar{e}_n)}. \quad (4.2.3)$$

The denominator carries the trivial representation, which we shall denote by $\mathbb{1}$. To compute the numerator, note that the subgroup fixing e_1 pointwise is $S_1 \times B_{n-1}$, which has index $2n$ in B_n . Hence

$$\mathbb{Q} e_1 \oplus \mathbb{Q} \bar{e}_1 \oplus \dots \oplus \mathbb{Q} e_n \oplus \mathbb{Q} \bar{e}_n \cong_{B_n} \text{Ind}_{S_1 \times B_{n-1}}^{B_n} \mathbb{1} \cong_{B_n} \text{Ind}_{B_1 \times B_{n-1}}^{B_n} \left(\text{Ind}_{S_1 \times B_{n-1}}^{B_1 \times B_{n-1}} \mathbb{1} \right)$$

by transitivity of induction. The inner term on the left-hand-side can be expanded as

$$\text{Ind}_{S_1 \times B_{n-1}}^{B_1 \times B_{n-1}} \mathbb{1} = (\chi^{(1), \emptyset} \oplus \chi^{\emptyset, (1)}) \times \chi^{(n-1), \emptyset} = (\chi^{(1), \emptyset} \times \chi^{(n-1), \emptyset}) \oplus (\chi^{\emptyset, (1)} \times \chi^{(n-1), \emptyset}).$$

Using the Type B branching rules, it follows that

$$\text{Ind}_{B_1 \times B_{n-1}}^{B_n} (\chi^{\emptyset, (1)} \times \chi^{(n-1), \emptyset}) = \chi^{\emptyset, (1)} \cdot \chi^{(n-1), \emptyset} = \chi^{(n-1), (1)},$$

which is an n -dimensional representation of B_n .

The other term is

$$\text{Ind}_{B_1 \times B_{n-1}}^{B_n} (\mathbb{1}_{B_1} \times \mathbb{1}_{B_{n-1}}) = \mathbb{Q}[B_n / (B_1 \times B_{n-1})],$$

which is also n -dimensional and decomposes as $\chi^{(n), \emptyset} \oplus \chi^{(n-1), (1), \emptyset}$. The numerator in (4.2.3) therefore has description

$$\mathbb{Q} e_1 \oplus \mathbb{Q} \bar{e}_1 \oplus \dots \oplus \mathbb{Q} e_n \oplus \mathbb{Q} \bar{e}_n = \chi^{(n), \emptyset} \oplus \chi^{(n-1), (1), \emptyset} \oplus \chi^{(n-1), (1)}$$

and so

$$H^2(F) = \frac{\mathbb{Q} e_1 \oplus \mathbb{Q} \bar{e}_1 \oplus \dots \oplus \mathbb{Q} e_n \oplus \mathbb{Q} \bar{e}_n}{\mathbb{Q}(e_1 + \bar{e}_1 + \dots + e_n + \bar{e}_n)} = \chi^{(n-1), (1), \emptyset} \oplus \chi^{(n-1), (1)}. \quad (4.2.4)$$

□

Plugging the description of $H^*(F)$ into (4.2.2) yields the following recursion.

Corollary 4.4 *There is a isomorphism of B_n -modules*

$$H^{2j} \mathcal{Z}_n^3 \cong_{B_n} H^{2j} \mathcal{Y}_n^3 \oplus \left(H^{2(j-1)} \mathcal{Y}_n^3 \otimes V \right)$$

for $0 \leq j \leq n$, where $V = \chi^{(n-1,1),\emptyset} \oplus \chi^{(n-1),(1)}$.

4.2.2 The case that $n = 2$

Our strategy will be to compute the representation for $H^* \mathcal{Z}_2^3$ using Corollary 4.4, and then use the fact that the cohomology generators z_{ij} , z_i are compatible with projections between configuration spaces to deduce a presentation for $H^* \mathcal{Z}_n^3$. In particular, if

$$\Pi_{[S]} : H^* \mathcal{Z}_n^3 \longrightarrow H^* \mathcal{Z}_{|T|}^3$$

is the map projecting $(x_1, \dots, x_n)$ to the points with index $T \subset [n]$, then Feichtner-Ziegler show in [24, Lemma 6] that in cohomology, $\Pi_{|T|}^*(z_{ij}) = z_{ij}$ for $i, j \in T$.

Part of our upcoming argument will rely on Theorem 5.10, which states that as an ungraded representation,

$$H^* \mathcal{Z}_n^3 \cong_{B_n} \mathbb{Q}[B_n].$$

We will defer the proof⁹ of Theorem 5.10 to Sect. 5. One consequence of this fact is that $H^* \mathcal{Z}_n^3$ will contain exactly one copy of every 1-dimensional representation of B_n .

Our goal in Proposition 4.5 is to determine where each irreducible of $\mathbb{Q}[B_2]$ appears in $H^* \mathcal{Z}_2^3$. Denote by $\rho[j]$ a representation ρ in cohomological degree j .

Proposition 4.5 *As B_2 representations, $H^* \mathcal{Z}_2^3$ decomposes as*

$$\begin{aligned} H^0 \mathcal{Z}_2^3 &= \chi^{(2),\emptyset} \\ H^2 \mathcal{Z}_2^3 &= \chi^{(1,1),\emptyset} + \chi^{\emptyset,(1,1)} + \chi^{(1),(1)} \\ H^4 \mathcal{Z}_2^3 &= \chi^{\emptyset,(2)} + \chi^{(1),(1)}. \end{aligned}$$

Proof By Corollary 4.4, when $n = 2$,

$$V = \chi^{(1,1),\emptyset} \oplus \chi^{(1),(1)}$$

and $H^0 \mathcal{Z}_2^3 = \chi^{(2),\emptyset}$. Since $H^0 \mathcal{Z}_1^3 = \chi^{(1),\emptyset}$, by Theorem 5.10 it follows that $H^2 \mathcal{Z}_1^3 = \chi^{\emptyset,(1)}$. Write the B_2 representation carried by $H^2 \mathcal{Y}_2^3$ as χ ; note that χ is 1-dimensional and must restrict to $\chi^{\emptyset,(1)}$. Inspection of Tables 2 and 3 then shows that χ must be either $\chi^{\emptyset,(1,1)}$ or $\chi^{\emptyset,(2)}$.

Expanding the recursion in Corollary 4.4 gives

$$\begin{aligned} H^* \mathcal{Z}_2^3 &= (\chi^{(2),\emptyset}(0) + (\chi^{(1),(1)} + \chi^{(1,1),\emptyset}[2])) \otimes (\chi^{(2),\emptyset}[0] + \chi[2]) \\ &= \chi^{(2),\emptyset}[0] + (\chi^{(1),(1)} + \chi^{(1,1),\emptyset} + \chi)[2] + (\chi^{(1),(1)} + \chi^{(1,1),\emptyset} \otimes \chi)[4], \end{aligned}$$

where we use the fact that $\chi^{(1),(1)} \otimes \chi = \chi^{(1),(1)}$ no matter which 1-dimensional representation χ is.

⁹ The proof of Theorem 5.10 does not rely on the remaining results in § 4. In particular, all subsequent work in § 4 will focus on computing the presentation of $H^* \mathcal{Z}_n^3$. Theorem 5.10 will be proved using equivariant formality, which we will see follows from the Hilbert series for $H^* \mathcal{Z}_n^3$ given in (4.1.2).

Observe by Proposition 4.2 that $z_1 z_2 \in H^4 \mathcal{Z}_2^3$ is an eigenbasis for $\chi^{\emptyset, (2)}$. Since the total representation of $H^* \mathcal{Z}_2^3$ is the regular representation of B_2 , it follows that $\chi^{\emptyset, (2)}$ must appear in cohomological degree 4 only, and so it must be that

$$\chi^{\emptyset, (2)} = \chi^{(1,1), \emptyset} \otimes \chi.$$

Again using the fact that the total representation in this context is $\mathbb{Q}[B_2]$, one deduces that $\chi = \chi^{\emptyset, (1,1)}$ and therefore

$$H^* \mathcal{Z}_2^3 = \chi^{(2), \emptyset}[0] + (\chi^{(1), (1)} + \chi^{(1,1), \emptyset} + \chi^{\emptyset, (1,1)})[2] + (\chi^{(1), (1)} + \chi^{\emptyset, (2)})[4].$$

□

We will now determine (1) the 1-dimensional eigenspaces of $H^* \mathcal{Z}_2^3$ as B_2 representations and (2) a complete description of the B_2 action on the generators of $H^* \mathcal{Z}_2^3$, which can then be extended to the B_n action on $H^* \mathcal{Z}_n^3$. Recall that since $H^* \mathcal{Z}_n^3 \cong \mathbb{Q}[B_n]$, each 1-dimensional representation of B_n will occur once in $H^* \mathcal{Z}_n^3$.

Proposition 4.6 *Let $V(\chi)$ be an eigenvector in $H^* \mathcal{Z}_2^3$ of a 1-dimensional representation χ of B_2 . The four 1-dimensional representations χ of B_2 occur in $H^* \mathcal{Z}_2^3$ as the span of the following elements:*

$V(\chi)$	Eigenvector
$V(\chi^{(2), \emptyset})$	1
$V(\chi^{(1,1), \emptyset})$	$z_{12} + z_{1\bar{2}} + z_1$
$V(\chi^{\emptyset, (1,1)})$	$z_{12} - z_{1\bar{2}} - z_2$
$V(\chi^{\emptyset, (2)})$	$z_1 z_2$

This in turn determines the following relations:

$$z_{1\bar{2}} = z_{12} + z_1 - z_2 \quad (4.2.5)$$

$$z_{1\bar{2}} = z_{1\bar{2}} + z_1 + z_2. \quad (4.2.6)$$

Proof The copy of $\chi^{(1), (1)}$ in $H^2 \mathcal{Z}_2^3$ has basis z_1 and z_2 . By Proposition 4.5, the remaining representations in $H^2 \mathcal{Z}_2^3$ are $\chi^{(1,1), \emptyset}$ and $\chi^{\emptyset, (1,1)}$. For vectors $\alpha = (\alpha_{12}^+, \alpha_{12}^-, \alpha_1, \alpha_2) \in \mathbb{Q}^4$ and $\beta = (\beta_{12}^+, \beta_{12}^-, \beta_1, \beta_2) \in \mathbb{Q}^4$, write

$$V(\chi^{(1,1), \emptyset}) = \alpha_{12}^+ z_{12} + \alpha_{12}^- z_{1\bar{2}} + \alpha_1 z_1 + \alpha_2 z_2$$

and

$$V(\chi^{\emptyset, (1,1)}) = \beta_{12}^+ z_{12} + \beta_{12}^- z_{1\bar{2}} + \beta_1 z_1 + \beta_2 z_2.$$

Our goal will be to solve for the undetermined coefficients above.

Starting with $V(\chi^{(1,1), \emptyset})$, we have that

$$\begin{aligned} t_2 \cdot V(\chi^{(1,1), \emptyset}) &= \alpha_{12}^+ z_{1\bar{2}} + \alpha_{12}^- z_{12} + \alpha_1 z_1 - \alpha_2 z_2 \\ &= V(\chi^{(1,1), \emptyset}) \\ &= \alpha_{12}^+ z_{12} + \alpha_{12}^- z_{1\bar{2}} + \alpha_1 z_1 + \alpha_2 z_2. \end{aligned}$$

This forces $\alpha_{12}^+ = \alpha_{12}^-$ and $\alpha_2 = 0$. Note that $\alpha_{12}^+ \neq 0$, because otherwise $V(\chi^{(1,1),\emptyset})$ would not be linearly independent with (z_1, z_2) . Thus without loss of generality, scale $\alpha_{12}^+ = \alpha_{12}^- = 1$; write $\alpha_1/\alpha_{12}^+ = \alpha$, so that

$$V(\chi^{(1,1),\emptyset}) = z_{12} + z_{1\bar{2}} + \alpha z_1.$$

Acting by s_1 gives

$$\begin{aligned} s_1 \cdot V(\chi^{(1,1),\emptyset}) &= -z_{12} - z_{1\bar{2}} + \alpha z_2 \\ &= -V(\chi^{(1,1),\emptyset}) \\ &= -z_{12} - z_{1\bar{2}} - \alpha z_1. \end{aligned}$$

This implies that

$$z_{1\bar{2}} = z_{12} + \alpha z_1 + \alpha z_2.$$

By a similar argument,

$$\begin{aligned} t_2 \cdot V(\chi^{\emptyset,(1,1)}) &= \beta_{12}^+ z_{1\bar{2}} + \beta_{12}^- z_{12} + \beta_1 z_1 - \beta_2 z_2 \\ &= -V(\chi^{\emptyset,(1,1)}) \\ &= -\beta_{12}^+ z_{12} - \beta_{12}^- z_{1\bar{2}} - \beta_1 z_1 - \beta_2 z_2. \end{aligned}$$

Hence one can conclude that $\beta_{12}^+ = -\beta_{12}^- \neq 0$ and $\beta_1 = 0$. Again, normalize by $\beta_{12}^+ = 1$ and set $\beta = \beta_2/\beta_{12}^+$; therefore

$$V(\chi^{\emptyset,(1,1)}) = z_{12} - z_{1\bar{2}} + \beta z_2.$$

Acting again by s_1 gives

$$\begin{aligned} s_1 \cdot V(\chi^{\emptyset,(1,1)}) &= -z_{12} + z_{1\bar{2}} + \beta z_1 \\ &= -V(\chi^{\emptyset,(1,1)}) \\ &= -z_{12} + z_{1\bar{2}} - \beta z_2, \end{aligned}$$

which implies that

$$z_{1\bar{2}} = z_{12} - \beta z_1 - \beta z_2. \quad (4.2.7)$$

We therefore conclude that $\alpha = -\beta$, and so

$$V(\chi^{(1,1),\emptyset}) = z_{12} + z_{1\bar{2}} - \beta z_1.$$

Applying w_0 to both expressions gives:

$$\begin{aligned} w_0 \cdot V(\chi^{(1,1),\emptyset}) &= z_{1\bar{2}} + z_{12} + \beta z_1 = V(\chi^{(1,1),\emptyset}) = z_{12} + z_{1\bar{2}} - \beta z_1 \\ w_0 \cdot V(\chi^{\emptyset,(1,1)}) &= z_{1\bar{2}} - z_{12} - \beta z_2 = V(\chi^{\emptyset,(1,1)}) = z_{12} - z_{1\bar{2}} + \beta z_2. \end{aligned}$$

Simplifying and adding the above two expressions gives

$$2z_{1\bar{2}} = 2z_{12} - 2\beta z_1 + 2\beta z_2.$$

Finally, we must show that $\beta = -1$. By Proposition 4.2 (5),

$$z_{12}z_1 - z_{12}z_2 - z_1z_2 = 0$$

holds in $H^* \mathcal{Z}_2^3$. Acting by t_2 gives

$$z_{1\bar{2}}z_1 + z_{1\bar{2}}z_2 + z_1z_2 = 0.$$

One the other hand, acting by $s_1t_2s_1$ gives

$$-z_{1\bar{2}}z_1 - z_{1\bar{2}}z_2 + z_1z_2 = 0.$$

Using (4.2.7), and Propopsition 4.2 (3), we have

$$\begin{aligned} -z_{1\bar{2}}z_1 - z_{1\bar{2}}z_2 + z_1z_2 &= -(z_{1\bar{2}} - \beta z_1 - \beta z_2)z_1 - (z_{1\bar{2}} - \beta z_1 - \beta z_2)z_2 + z_1z_2 \\ &= -z_{1\bar{2}}z_1 - z_{1\bar{2}}z_2 + (1 + 2\beta)z_1z_2. \end{aligned}$$

It follows that

$$\begin{aligned} 0 &= (z_{1\bar{2}}z_1 + z_{1\bar{2}}z_2 + z_1z_2) + (-z_{1\bar{2}}z_1 - z_{1\bar{2}}z_2 + (1 + 2\beta)z_1z_2) \\ &= (2 + 2\beta)z_1z_2. \end{aligned}$$

Since z_1z_2 is a basis element, $\beta = -1$. $\square$

The relations (4.2.5) and (4.2.6) are the final relations needed to determine a presentation for $H^* \mathcal{Z}_n^3$; note that they precisely match the relation (2) in Corollary 3.11.

4.3 Presentation for $H^* \mathcal{Z}_n^3$

We are now ready to give a presentation for $H^* \mathcal{Z}_n^3$. As before, the action of $\sigma \in B_n$ on the variables z_{ij} and z_i for $i, j \in [n]^\pm$ is given by

$$\sigma \cdot \begin{cases} z_{ij} = z_{\sigma(i)\sigma(j)} \\ z_i = z_{\sigma(i)}, \end{cases}$$

with the convention that $\bar{\bar{i}} = i$.

Theorem 4.7 *There is a B_n -equivariant ring isomorphism*

$$H^* \mathcal{Z}_n^3 \cong \mathbb{Z}[z_{ij}, z_i]/\mathcal{J},$$

for $i, j \in [n]^\pm$, where $\mathcal{J}$ is generated by

- $z_{\bar{i}j} = z_{i\bar{j}} + z_i + z_j$ and
- The relations in Proposition 4.2.

Proof Proposition 4.2 and 4.6 show that the relations in $\mathcal{J}$ hold in $H^* \mathcal{Z}_n^3$. (The relation $z_{\bar{i}j} = z_{i\bar{j}} + z_i + z_j$ comes from (4.2.6); note that it also implies $z_{\bar{i}\bar{j}} = z_{ij} + z_i - z_j$ from the action by t_j .)

Hence it is sufficient to prove these are the only relations needed to generate $\mathcal{J}$. To do so we will employ a standard argument, very similar in spirit to the proof of Theorem 3.7, to show that

$$\underbrace{\mathbb{Z}\{1, z_1\}}_{h'_1} \cdot \underbrace{\mathbb{Z}\{1, z_2, z_{12}, z_{1\bar{2}}\}}_{h'_2} \cdots \underbrace{\mathbb{Z}\{1, z_n, z_{1n}, z_{1\bar{n}}, \dots, z_{(n-1)n}, z_{(n-1)\bar{n}}\}}_{h'_n}.$$

forms a Gröbner basis for $H^* \mathcal{Z}_n^3$. We will again use the lexicographic ordering on z_i, z_{ij} for $i, j \in [n]^\pm$ induced by the ordering

$$\bar{0} < 1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}.$$

Table 7 Action of the elements t_i, t_j and $(i, j) \in B_n$ on the generators $z_{ij}^+, z_{ij}^-, z_i \in H^* \mathcal{Z}_n^3$

	t_i	t_j	(i, j)
z_{ij}^+	$z_{ij}^- + z_i + z_j$	z_{ij}^-	$-z_{ij}^+$
z_{ij}^-	$z_{ij}^+ + z_i - z_j$	z_{ij}^+	$-z_{ij}^- - z_i - z_j$
z_i	$-z_i$	z_i	z_j

Here we identify z_i with z_{0i} so that for $i < j$ one has $z_i < z_j < z_{ij} < z_{i\bar{j}} < z_{i\bar{j}}^- < z_{i\bar{j}}^+$. The argument amounts to showing that any monomial with two terms from the same set h_i' can be rewritten in terms of elements lower in the above ordering using the relations in $\mathcal{J}$. Call such monomials *broken-circuits*. The same argument as in Theorem 3.7 shows that these broken-circuits (the underlined terms) can be rewritten using relations in $\mathcal{J}$:

- $0 = z_{ij}z_{jk} - z_{ij}z_{jk} - \underline{z_{jk}z_{ik}}$
- $0 = z_{ij}z_{j\bar{k}} - z_{ij}z_{j\bar{k}} - \underline{z_{j\bar{k}}z_{i\bar{k}}}$
- $0 = z_i z_{ij} - \underline{z_{ij}z_j} - z_i z_j$
- $0 = z_i z_{i\bar{j}} + \underline{z_{i\bar{j}}z_j} + z_i z_j$
- $0 = z_j z_{i\bar{j}} - \underline{z_j z_{ij}} - z_{ij} z_{i\bar{j}}$
- $0 = z_j z_{i\bar{j}} + z_k z_{i\bar{j}} + \underline{z_{i\bar{j}}z_{j\bar{k}}} - z_j z_{ik} - z_k z_{ik} - \underline{z_{j\bar{k}}z_{ik}} - z_{i\bar{j}} z_{ik}$
- $0 = (z_i + z_j + z_{i\bar{j}})z_{jk} - (z_i + z_j + z_{i\bar{j}})(z_i + z_k + z_{i\bar{k}}) - \underline{z_{jk}(z_i + z_k + z_{i\bar{k}})}$

for $i, j, k \in [n]$. Note that each underlined term is the largest with respect to our ordering, which completes the proof. $\square$

The presentation of $H^* \mathcal{Z}_n^3$ in Theorem 4.7 is identical to the presentation of $\mathcal{G} = \text{gr}(H^* \mathcal{Z}_n^1)$ in Corollary 3.11, which gives the following.

Corollary 4.8 *There is an isomorphism of B_n -modules*

$$H^* \mathcal{Z}_n^3 \cong \text{gr}(H^* \mathcal{Z}_n^1) = \mathcal{G}.$$

It turns out that Corollary 4.8 can be *lifted* to the spaces $H^* \mathcal{Y}_{n+1}^3$ and $H^* \mathcal{Y}_{n+1}^1$:

Corollary 4.9 *There is an isomorphism of B_{n+1} -modules*

$$H^* \mathcal{Y}_{n+1}^3 \cong \text{gr}(H^* \mathcal{Y}_{n+1}^1)$$

The proof of Corollary 4.9 requires additional tools that will be discussed in Sect. 5.

Remark 4.10 Theorem 4.7 and its proof reveal an asymmetry in the space $H^* \mathcal{Z}_n^3$; in particular, only the generators $z_{i\bar{j}}, z_{ij}$ and z_i for $1 \leq i < j \leq n$ are needed, but the presentation can be described more compactly (and elegantly) if the generator $z_{i\bar{j}}$ is also included. One alternative way to describe the presentation of $H^* \mathcal{Z}_n^3$ is to set $z_{ij}^+ := z_{ij}$ and $z_{ij}^- := z_{i\bar{j}}$ with the action of B_n given by Table 7.

One could then write the presentation of $H^* \mathcal{Z}_n^3$ as $\mathbb{Z}[z_{ij}^+, z_{ij}^-, z_i]/\mathcal{K}$, where $\mathcal{K}$ is generated by the B_n -image of the relations $(z_{ij}^+)^2 = (z_{i\bar{j}})^2 = (z_i)^2$ and

- (i) $z_{ij}^+ z_{jk}^+ - z_{ij}^+ z_{jk}^+ - z_{jk}^+ z_{ik}^+$
- (ii) $z_i z_{ij}^+ - z_{ij}^+ z_j - z_i z_j$
- (iii) $z_j z_{i\bar{j}}^- - z_j z_{ij}^+ - z_{i\bar{j}}^+ z_{ij}^-$.

This is more or less a rewriting of Theorem 4.7, but has the disadvantage of being less explicit and less obviously related to the presentation of $H^* \mathcal{Z}_n^1$. However, there are some advantages to this perspective, and we will return to it in Sect. 5.

Table 8 Action of the elements t_i, t_j and $(i, j) \in B_n$ on the generators $z_{ij}^+, z_{ij}^-, z_i \in \mathcal{G}$

	t_i	t_j	(i, j)
z_{ij}^+	z_{ij}^-	z_{ij}^-	$-z_{ij}^+$
z_{ij}^-	z_{ij}^+	z_{ij}^+	$-z_{ij}^-$
z_i	$-z_i$	z_i	z_j

Similarly, the presentation of the ring $\mathcal{G}$ can be re-written

$$\mathcal{G} = \mathbb{Z}[z_{ij}^+, z_{ij}^-, z_i] / \mathcal{L}'$$

for $1 \leq i < j \leq n$, where $\mathcal{L}'$ is generated by $(z_{ij}^+)^2 = (z_{ij}^-)^2 = (z_i)^2$ and

- (i) $z_{ij}^+ z_{ij}^-$ (ii) $z_{ij}^+ z_j - z_{ij}^- z_i$ (iii) $z_{ij}^- z_j + z_{ij}^+ z_i$ (iv) $z_{ij}^+ z_{jk}^+ - z_{ij}^+ z_{ik}^+ - z_{jk}^+ z_{ik}^+$,
(v) $z_{ij}^- z_{jk}^+ - z_{ij}^- z_{ik}^- - z_{jk}^+ z_{ik}^-$
(vi) $z_{ij}^- z_{jk}^- - z_{ij}^- z_{ik}^+ - z_{jk}^- z_{ik}^+$ (vii) $z_{ij}^+ z_{jk}^- - z_{ij}^+ z_{ik}^- - z_{jk}^- z_{ik}^-$

Now the action by B_n on the generators (see Table 8) is significantly simpler.

5 T-equivariant cohomology

There are several results whose proofs we have deferred thus far. In this section, we will build the tools to prove these theorems and more deeply understand the connections between the $d = 1$ and $d = 3$ cases for $\mathcal{Z}_n^d$ and $\mathcal{Y}_{n+1}^d$. To do so, we will delve into the world of equivariant cohomology; background material on equivariant cohomology can be found in Sect. 5.1. Experts on the topic can proceed directly to Sect. 5.2.

5.1 Background

Equivariant topology is a powerful tool by which to study a topological space X with a group action by a Lie group T ; see Anderson [3], Proudfoot [41], and Tymoczko [49] for excellent introductions to the topic. Recently in [39], Moseley successfully used equivariant cohomology to study complements of hyperplane and subspace arrangements in real Euclidean space. Our approach will follow Moseley's lead.

Henceforth, the key objects will be:

- T , a Lie group. For us, T will almost exclusively be $U(1)$, the unit circle;
- X , a topological space that T acts on; call X a T -space and let X^T be the space fixed by T . For our purposes X will be an oriented manifold. Call such a manifold a T -manifold, and any submanifold of X that is stable under the action of T a T -submanifold;
- ET , a contractible space which T acts upon freely;
- $X_T := ET \times_T X$, is defined to be the quotient of $ET \times X$ by

$$(t \cdot e, x) \sim (e, t \cdot x)$$

for $t \in T$, $e \in ET$ and $x \in X$. A nice special case is when X is a point; then X_T is simply the classifying space of T , written BT . See example 5.1 for more details.

- $H_T^*(X) := H^*(X_T)$ is the T -equivariant cohomology of X . This is precisely the ordinary cohomology for X_T . Intuitively, if T acts freely on X then $H_T^*(X) = H^*(X/T)$.

- $[Y]_T \in H_T^k(X)$ is the class of a codimension k oriented T -stable submanifold of X .

To make the above more concrete, consider the following example.

Example 5.1 Let X be a single point, denoted pt , and $T = U(1)$. To determine ET , one must find a space that (1) T acts upon freely and (2) is contractible. There is a natural free action of T on $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ by rotation, where $e^{\pi i} \in U(1)$ acts on $(z_0, \dots, z_n)$ by

$$e^{\pi i} \cdot (z_0, \dots, z_n) = (e^{\pi i} z_0, \dots, e^{\pi i} z_n).$$

While $\mathbb{S}^{2n+1}$ satisfies the first requirement, it is not contractible. To correct for this, note that $\mathbb{S}^{2n+1}$ can be successively embedded in $\mathbb{S}^{2n+2\ell+1}$ for $\ell \geq 0$. Following this procedure iteratively allows us to view $\mathbb{S}^{2n+1}$ inside of $\mathbb{S}^\infty$ (defined as the union of $\mathbb{S}^{2n+1}$ for all n) which is a contractible space. Thus $ET = \mathbb{S}^\infty$. Since $X = \text{pt}$, the space pt_T is simply

$$\text{pt}_T = \mathbb{S}^\infty \times_T \text{pt} = \mathbb{S}^\infty / \mathbb{S}^1.$$

In general, the quotient of $\mathbb{S}^{2n+1}$ by the rotation action of $\mathbb{S}^1$ gives complex projective space $\mathbb{C}\mathbb{P}^n$, and in the limit

$$\mathbb{S}^\infty / \mathbb{S}^1 = \mathbb{C}\mathbb{P}^\infty,$$

where $\mathbb{C}\mathbb{P}^\infty$ is infinite complex projective space. Hence $H_T^*(\text{pt}) = H^*(\mathbb{C}\mathbb{P}^\infty)$. The cohomology of $\mathbb{C}\mathbb{P}^\infty$ (with coefficients in $\mathbb{Q}$) is the univariate polynomial ring $\mathbb{Q}[u]$ where $u \in H^2(\mathbb{C}\mathbb{P}^\infty)$.

Here, we understand u as the image of $[0]_T$ under the induced map from the inclusion

$$\begin{aligned} i : \text{pt} &\hookrightarrow \mathbb{C} \\ \text{pt} &\mapsto 0. \end{aligned}$$

Example 5.1 illustrates one of the key differences between ordinary and equivariant cohomology: in the latter, $H^*(\text{pt}) = \mathbb{Q}$ a (relatively) uninteresting ring. By contrast, $H_T^*(\text{pt}) = \mathbb{Q}[u]$ has a richer structure.

Many of the important tools in ordinary cohomology also hold in the equivariant setting, including functoriality, cohomological ring structure, Poincaré duality for smooth orientable spaces, the Leray-Serre spectral sequence, the Künneth formula, and more (see [49]). This means, for starters, that the argument showing that $H^*(\text{pt}) = \mathbb{Q}$ is a module over $H^*(X)$ can be used to show that $H_T^*(\text{pt})$ is a module over $H_T^*(X)$. In the case most relevant to us, where $T = U(1)$, this says that $H_T^*(X)$ is a $\mathbb{Q}[u]$ -module. Remark 5.2 discusses another “standard” fact.

Remark 5.2 One property which we will use repeatedly is that if T -manifolds $Y, Z \subset X$ have empty intersection, then in $H_T^*(X)$, their product $[Y]_T \cdot [Z]_T$ is 0. This is a special case of the more general fact that if Y and Z intersect transversely their product in equivariant cohomology will be an intersection number. The same statement is a standard fact in ordinary cohomology, and can be proved in much the same way for equivariant cohomology. See Dorpalen-Barry–Proudfoot–Wang [20, § 2.5] for a proof of this fact over $\mathbb{Z}$, along with more intuition for the equivariant classes represented by submanifolds.

The space X_T is a fiber bundle over the classifying space $BT = ET/T$:

$$X \longrightarrow X_T \longrightarrow BT. \quad (5.1.1)$$

In principal, one might try to compute the ordinary cohomology of X_T using a Leray spectral sequence. In general this is very difficult, but we will see that in our case the relevant spectral sequence collapses.

In particular, a space X is *equivariantly formal* (with respect to T) if the spectral sequence in (5.1.1) collapses at the E_2 page; this means that $H_T^*(X)$ is a free module over $H_T(\text{pt})$. Importantly, if X and BT have cohomology concentrated only in even degrees, then X is equivariantly formal with respect to T . All of the spaces we consider (e.g. $\mathcal{Z}_n^3$ and $\mathcal{Y}_{n+1}^3$) will satisfy this hypothesis.

If X is equivariantly formal with respect to T , there is an $H_T^*(\text{pt})$ -module isomorphism

$$H_T^*(X) = H^*(X) \otimes H_T^*(\text{pt}).$$

Again, we almost exclusively care about the case that $T = U(1)$; the above then says that

$$H_T^*(X) = H^*(X) \otimes \mathbb{Q}[u].$$

Equivariant formality thus allows us to recover information about $H^*(X)$ from $H_T^*(X)$ and vice-versa. For instance, if X is equivariantly formal, then any $\mathbb{Q}$ -basis for $H^*(X)$ is an $H_T^*(\text{pt})$ -basis for $H_T^*(X)$; if $T = U(1)$ and $v_1, \dots, v_k$ is a basis for $H^*(X)$, then $v_1, \dots, v_k, u$ is a $\mathbb{Q}$ -basis for $H_T^*(X)$.

Example 5.3 Let $T = U(1)$ and $X = \mathbb{R}^3 \setminus \{0\}$. Then T acts on $\mathbb{R}^3 \setminus \{0\}$ by rotation around the x -axis. Let Z^+ and Z^- be the positive and negative x -axes, respectively. Once we fix an orientation of $\mathbb{R}^3 \setminus \{0\}$ and orient Z^+ and Z^- outwards, we obtain two classes in $H_T^2(\mathbb{R}^3 \setminus \{0\})$, namely $[Z^+]_T$ and $[Z^-]_T$.

Consider the map

$$\begin{aligned} f : \mathbb{Q}[x, y]/(xy) &\longrightarrow H_T^*(\mathbb{R}^3 \setminus \{0\}) \\ y &\longmapsto [Z^+]_T \\ x &\longmapsto [Z^-]_T. \end{aligned}$$

By Remark 5.2, since $Z^+ \cap Z^- = \emptyset$, their corresponding classes in equivariant cohomology multiply as

$$[Z^+]_T \cdot [Z^-]_T = 0,$$

and so f is well-defined. In fact, f defines an isomorphism; this argument is written out in detail in [41, Prop 5.22] and [39, Example 2.4].

Since both $\mathbb{R}^3 \setminus \{0\}$ and $BT = \mathbb{C}\mathbb{P}^\infty$ have cohomology concentrated in even degrees, $\mathbb{R}^3 \setminus \{0\}$ is equivariantly formal, and therefore

$$H_T^*(\mathbb{R}^3 \setminus \{0\}) \cong \mathbb{Q}[x, y]/(xy) \cong \mathbb{Q}[u] \otimes H^*(\mathbb{R}^3 \setminus \{0\}).$$

We would like an explicit map between $\mathbb{Q}[x, y]/(xy)$ and $\mathbb{Q}[u] \otimes H^*(\mathbb{R}^3 \setminus \{0\})$. To this end, consider the T -equivariant projection

$$\begin{aligned} \pi : \mathbb{R}^3 \setminus \{0\} &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (y, z), \end{aligned}$$

so that $\pi^{-1}(0) = Z^+ \sqcup -Z^-$ (when accounting for orientation). We are interested in the image of u under the induced maps

$$H_T^*(pt) \xrightarrow{i^*} H_T^*(\mathbb{R}^2) \xrightarrow{\pi^*} H_T^*(\mathbb{R}^3 \setminus \{0\}).$$

The first map sends u to $[0]_T$. By the above discussion, $\pi^*([0]_T) = y - x$, and therefore

$$\pi^*(i^*(u)) = y - x.$$

Note that in $H^*(\mathbb{R}^3 \setminus \{0\})$, the classes $[Z^+]$ and $[Z^-]$ represent the same generator of $H^*(\mathbb{R}^3 \setminus \{0\})$. This gives a (non-canonical) isomorphism:

$$\begin{aligned} f' : \mathbb{Q}[x, y]/(xy) &\longrightarrow \mathbb{Q}[u] \otimes H^*(\mathbb{R}^3 \setminus \{0\}) \\ y - x &\longmapsto u \\ y &\longmapsto [Z^+]. \end{aligned}$$

See [41, Prop 5.22] and [39, Example 2.4] for a more in-depth description of this example.

There are several consequences of equivariant formality proved by Moseley in [39] that will play an important role in our analysis.

Proposition 5.4 (Moseley [39, Prop. 2.5, Prop 2.9]) *Let X be an equivariantly formal T -space. Then there are surjections*

$$\Phi_0 : H_T^*(X) \rightarrow H^*(X)$$

sending u to 0 and

$$\Phi_1 : H_T^*(X) \rightarrow H^*(X^T)$$

sending u to 1, which induce the ring isomorphisms

$$\begin{aligned} H_T^*(X)/\langle u \rangle &\cong H^*(X), \\ H_T^*(X)/\langle u - 1 \rangle &\cong H^*(X^T). \end{aligned}$$

If a group W acts on X and commutes with the action of T , then there is a W -action on $H_T^(X)$ which fixes u , and the maps Φ_0 and Φ_1 are W -equivariant.*

One corollary to Proposition 5.4 is that there is a natural filtration on $H^*(X^T)$ coming from the cohomological grading on $H_T^*(X)$. Note that u has cohomological degree 2 in $H_T^*(X)$.

Definition 5.5 (Equivariant filtration) Suppose X is an equivariantly formal T -space, so that $H_T^*(X)/\langle u - 1 \rangle \cong H^*(X^T)$. The equivariant filtration on $H^*(X^T)$ is then

$$F_0(X) \subset F_1(X) \subset \cdots \subset H^*(X^T),$$

where $F_k(X)$ is the image under Φ_1 of the classes in $H_T^*(X)$ of cohomological degree at most k .

Write the associated graded ring with respect to the equivariant filtration as $\mathrm{gr}_T(H^*(X^T))$.

Corollary 5.6 (Moseley, [39, Cor. 2.7]) *There is an isomorphism*

$$\mathrm{gr}_T(H^*(X^T)) \cong H^*(X).$$

Corollary 5.6 has a similar flavor to Corollaries 4.8 and 4.9. This connection will be formalized in Corollary 5.13 and Theorem 5.14.

It is useful to see Proposition 5.4 and Corollary 5.6 in the context of Example 5.3.

Example 5.7 Continuing Example 5.3, consider first the subspace $\mathbb{R}^3 \setminus \{0\}$ fixed by the rotation action of $T = U(1)$. Since T rotates around the x -axis,

$$(\mathbb{R}^3 \setminus \{0\})^T = \{(x, 0, 0) : x \in \mathbb{R} \setminus \{0\}\} = Z^+ \sqcup Z^-.$$

By Example 5.3 and Proposition 5.4:

$$H_T^*(\mathbb{R}^3 \setminus \{0\})/\langle u \rangle \cong \mathbb{Q}[y]/(y^2) \cong H^*(\mathbb{R}^3 \setminus \{0\}), \quad (5.1.2)$$

$$H_T^*(\mathbb{R}^3 \setminus \{0\})/\langle u - 1 \rangle \cong \mathbb{Q}[y]/(y^2 - y) \cong H^*(Z^+ \sqcup Z^-). \quad (5.1.3)$$

The first isomorphism in (5.1.2) follows because $f' : y - x \mapsto u$ so we identify $y - x = 0$ in $H_T^*(\mathbb{R}^3 \setminus \{0\})/\langle u \rangle$. The first isomorphism in (5.1.3) is because $y - x = 1$ in $H_T^*(\mathbb{R}^3 \setminus \{0\})/\langle u - 1 \rangle$, so

$$0 = xy = (y - 1)y = y^2 - y.$$

The second isomorphisms in both (5.1.2) and (5.1.3) are applications of Proposition 5.4, but can be verified directly by computing $H^*(\mathbb{R}^3 \setminus \{0\})$ and $H^*(Z^+ \sqcup Z^-)$. The former is standard; the latter is concentrated in degree 0, and can be thought of as the $\mathbb{Q}$ -vector space of linear functionals on $Z^+ \sqcup Z^-$. One can then understand y as a single Heaviside function $y : Z^+ \sqcup Z^- \rightarrow \mathbb{Z}$ given by

$$y(p) := \begin{cases} 1 & p \in Z^+ \\ 0 & p \in Z^- \end{cases}.$$

It follows naturally that $y^2 = y$ and that y is the only function needed to generate this space. (This line of reasoning should feel reminiscent of § 3!)

Finally, note that although $\mathbb{Q}[y]/(y^2 - y)$ is not graded, it has an ascending filtration by degree, and the associated graded ring with respect to this filtration is $\mathbb{Q}[y]/(y^2)$. Corollary 5.6 says

$$\mathrm{gr}_T(H_T^* \mathbb{R}^3 \setminus \{0\}) \cong H^*(\mathbb{R}^3 \setminus \{0\}),$$

and so we see that **the two associated graded rings—e.g. coming from the equivariant filtration and the filtration by degree in the Heaviside functions—coincide**. One way of interpreting this is that the presentation of $H^*(Z^+ \sqcup Z^-)$ using Heaviside functions is particularly natural with respect to equivariant cohomology. We will see this phenomena again in the case of $H^* \mathcal{Z}_n^1$ and $H^* \mathcal{Y}_{n+1}^1$.

5.2 Applications to $\mathcal{Z}_n^3$ and $\mathcal{Y}_{n+1}^3$

Our goal is to apply Proposition 5.4 and Corollary 5.6 to $\mathcal{Z}_n^3$ and its lift, $\mathcal{Y}_{n+1}^3$. To do so, we must identify a torus action and prove that it commutes with the respective actions of B_n and B_{n+1} .

Proposition 5.8 *There is an action by the circle group $T = U(1)$ on $\mathcal{Z}_n^3$ and $\mathcal{Y}_{n+1}^3$ which commutes with the actions of B_n and B_{n+1} respectively, and has fixed spaces*

$$\begin{aligned} (\mathcal{Z}_n^3)^{U(1)} &= \mathcal{Z}_n^1 \\ (\mathcal{Y}_{n+1}^3)^{U(1)} &= \mathcal{Y}_{n+1}^1. \end{aligned}$$

Proof Identify $\mathbb{R}^3$ with $\mathbb{C} \oplus \mathbb{R}$, and let $\omega \in U(1)$ act on $(z, x) \in \mathbb{C} \oplus \mathbb{R}$ by

$$\omega \cdot (z, x) = (\omega z, x).$$

Let $|(z, x)|$ be the magnitude of (z, x) , thought of as a vector in $\mathbb{R}^3$. Importantly,

$$|(\omega z, x)|^2 = |(z, x)|^2.$$

This shows that the action by ω commutes with the action by φ :

$$\omega \cdot (\varphi \cdot (z, x)) = \omega \cdot \frac{-(z, x)}{|(z, x)|^2} = \frac{-(\omega z, x)}{|(z, x)|^2} = \varphi(\omega z, x).$$

The S_n -action on $\mathcal{Z}_n^3$ by coordinate permutation commutes with the action by $U(1)$ in a more straightforward way, and hence the action of B_n commutes with the $U(1)$ action. The fixed space of the $U(1)$ action on $\mathbb{C} \oplus \mathbb{R}$ is $\{0\} \oplus \mathbb{R}$, from which it follows that

$$(\mathcal{Z}_n^3)^{U(1)} = \{(p_1, \dots, p_n) \in (\{0\} \oplus (\mathbb{R} \setminus \{0\}))^n \subset (\mathbb{C} \oplus \mathbb{R})^n : p_i \neq p_j, p_i \neq \varphi(p_j)\} \cong \mathcal{Z}_n^1.$$

To study the action of $U(1)$ on $\mathcal{Y}_{n+1}^3$, first embed $U(1)$ into SU_2 via the isomorphism

$$U(1) \cong \left\{ \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix} : \omega = e^{2\pi i \theta} \right\}.$$

From this identification it follows that $U(1)$ acts on SU_2 on the *right* by multiplication, inducing a diagonal action on $\mathcal{Y}_{n+1}^3$. (Recall that SU_2 acts on the left in $\mathcal{Y}_{n+1}^3$). The action by $U(1)$ commutes with the $\mathbb{Z}_2$ antipodal action on $\mathcal{Y}_{n+1}^3$ (e.g. left multiplication by -1) because -1 is in the center of SU_2 . Since $U(1)$ acts diagonally, it must also commute with the action of S_{n+1} on $\mathcal{Y}_{n+1}^3$ by coordinate permutation. Hence the $U(1)$ action commutes with the B_{n+1} -action on $\mathcal{Y}_{n+1}^3$. To determine the fixed space of the $U(1)$ -action, note that $g \in SU_2$ commutes with every element of $U(1)$ if and only if $g \in U(1)$. Thus in $\mathcal{Y}_{n+1}^3$

$$(p_1, \dots, p_n) \sim (p_1 \omega, \dots, p_n \omega)$$

for every $\omega \in U(1)$ if and only if $p_i \in U(1)$ for all $1 \leq i \leq n$, and so

$$(\mathcal{Y}_{n+1}^3)^{U(1)} = \mathcal{Y}_{n+1}^1.$$

□

As a Corollary, we immediately obtain the following:

Corollary 5.9 *The spaces $\mathcal{Z}_n^3$ and $\mathcal{Y}_{n+1}^3$ are equivariantly formal with respect to the action by $T = U(1)$. Hence*

- $H_T^*(\mathcal{Z}_n^3)$ and $H_T^*(\mathcal{Y}_{n+1}^3)$ are $\mathbb{Q}[u]$ modules;
- There is an action of B_n (resp. B_{n+1}) on $H_T^*(\mathcal{Z}_n^3)$ (resp. $H_T^*(\mathcal{Y}_{n+1}^3)$) that fixes u ;
- The maps Φ_0 and Φ_1 as in Proposition 5.4 are surjective and B_n (resp. B_{n+1}) equivariant;
- There are isomorphisms

$$\begin{aligned} \mathrm{gr}_T(H^*(\mathcal{Z}_n^1)) &\cong_{B_n} H^*(\mathcal{Z}_n^3) \\ \mathrm{gr}_T(H^*(\mathcal{Y}_{n+1}^1)) &\cong_{B_{n+1}} H^*(\mathcal{Y}_{n+1}^3), \end{aligned}$$

where here the associated graded ring is with respect to the equivariant filtration.

Proof The equivariant formality follows from the fact that both $\mathcal{Z}_n^3$ and $\mathcal{Y}_{n+1}^3$ have cohomology concentrated in even degrees. The remainder of the statement follows by applying Corollary 5.6. $\square$

The equivariant setting allows us to give a description of the total representations of $H^* \mathcal{Z}_n^3$ and $H^* \mathcal{Y}_{n+1}^3$ by applying Proposition 3.3 and adapting techniques used in [39] and [40].

Theorem 5.10 *As ungraded representations*

$$\begin{aligned} H^* \mathcal{Z}_n^3 &\cong_{B_n} \mathbb{Q}[B_n], \\ H^* \mathcal{Y}_{n+1}^3 &\cong_{B_{n+1}} \operatorname{Ind}_{\langle c \rangle}^{B_{n+1}} \mathbb{1} = \mathbb{Q}[B_{n+1}/\langle c \rangle], \end{aligned}$$

where c is a Coxeter element of B_{n+1} and $\mathbb{1}$ is the trivial representation.

Proof By Proposition 3.3, $\mathcal{Y}_{n+1}^1$ is isomorphic to the B_{n+1} action on the cosets $B_{n+1}/\langle c \rangle$. Since $\mathcal{Y}_{n+1}^1$ is a disjoint union of contractible pieces, $H^0 \mathcal{Y}_{n+1}^1 = H^* \mathcal{Y}_{n+1}^1$ and so passing to cohomology simply gives

$$H^* \mathcal{Y}_{n+1}^1 \cong_{B_{n+1}} \mathbb{Q}[B_{n+1}/\langle c \rangle].$$

Proposition 3.3 then says that the B_n action on the set of connected components of $\mathcal{Y}_{n+1}^1$ is simply transitive. Since $\mathcal{Y}_{n+1}^1$ is B_n -equivariantly homeomorphic to $\mathcal{Z}_n^1$, this implies that B_n also acts simply transitively on the set of connected components of $\mathcal{Z}_n^1$.

Thus in cohomology,

$$H^0 \mathcal{Z}_n^1 = H^* \mathcal{Z}_n^1 = \mathbb{Q}[B_n].$$

Finally, passing to the associated graded ring $\operatorname{gr}_T(H^* \mathcal{Y}_{n+1}^1) = H^* \mathcal{Y}_{n+1}^3$ and $\operatorname{gr}_T(H^* \mathcal{Z}_n^1) = H^* \mathcal{Z}_n^3$ will not change the isomorphism type, so the claim follows. $\square$

5.3 Presentation of equivariant cohomology

Having established some basic properties of $H_T^*(\mathcal{Z}_n^3)$ and $H_T^*(\mathcal{Y}_{n+1}^3)$, we turn to computing their presentations. Our arguments will closely resemble Moseley [39].

Recall that the equivariant formality of $\mathcal{Z}_n^3$ implies that

- (1) $H_T^*(\mathcal{Z}_n^3) \cong H^* \mathcal{Z}_n^3 \otimes \mathbb{Q}[u]$ as a $\mathbb{Q}[u]$ -module, where u is the image of $H_T^2(\operatorname{pt})$ under the map induced from $\mathcal{Z}_n^3 \rightarrow \operatorname{pt}$, and
- (2) Any generating set for $H^* \mathcal{Z}_n^3$ can be lifted to a generating set of $H_T^*(\mathcal{Z}_n^3)$ over $\mathbb{Q}[u]$.

We will take a slight shortcut here and use the smaller generating set for $H^* \mathcal{Z}_n^3$ discussed in Remark 4.10: in particular, pick generators of $H^* \mathcal{Z}_n^3$ as z_{ij}^+ , z_{ij}^- , and z_i for integers $1 \leq i < j \leq n$. Since the maps $\Phi_0 : H_T^*(\mathcal{Z}_n^3) \rightarrow H^* \mathcal{Z}_n^3$ and $\Phi_1 : H_T^*(\mathcal{Z}_n^3) \rightarrow H^* \mathcal{Z}_n^1$ are B_n -equivariant by Corollary 5.9, the B_n action on the generators in $H_T^*(\mathcal{Z}_n^3)$ is inherited from these spaces (see Table 7).

We would first like to relate these generators to T -submanifolds of $\mathcal{Z}_n^3$. Recall the definition of the hyperplanes

$$\begin{aligned} H_{ij} &:= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \neq x_j\} \\ H_i &:= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \neq 0\} \end{aligned}$$

and that the action of t_i on $\mathcal{Z}_n^3$ is

$$t_i \cdot (x_1, \dots, x_i, \dots, x_n) = (x_1, \dots, \varphi(x_i), \dots, x_n).$$

Let $w_i = (x_i, y_i, z_i) \in \mathbb{R}^3 \setminus \{0\}$. For integers $i, j \in [n]$, define the maps from $(\mathbb{R}^3 \setminus \{0\})^n \rightarrow \mathbb{R}^3$ by

$$\begin{aligned}\omega_{ij} &: (w_1, \dots, w_n) \mapsto w_j - w_i, \\ \omega_{i\bar{j}} &: (w_1, \dots, w_n) \mapsto \varphi(w_j) - w_i, \\ \omega_i &: (w_1, \dots, w_n) \mapsto w_i.\end{aligned}$$

These maps are not linear. However, they are T -equivariant and have the property that $(\omega_{ij})^{-1}(\mathbb{R}^3 \setminus \{0\}) = \mathbb{R}^{3n} - (H_{ij} \otimes \mathbb{R}^3)$ (and similarly for ω_i). Note that

$$t_j \cdot (\omega_{ij})^{-1}(\mathbb{R}^3 \setminus \{0\}) = (\omega_{i\bar{j}})^{-1}(\mathbb{R}^3 \setminus \{0\})$$

and that

$$\mathcal{Z}_n^3 = \bigcap_{1 \leq i \neq j \leq n} \left((\omega_{ij})^{-1}(\mathbb{R}^3 \setminus \{0\}) \cap (\omega_{i\bar{j}})^{-1}(\mathbb{R}^3 \setminus \{0\}) \right) \cap \bigcap_{1 \leq i \leq n} \left(\omega_i^{-1}(\mathbb{R}^3 \setminus \{0\}) \right).$$

For ease of notation, write ω_J to refer to an arbitrary ω -map, so ω_J is the place-holder for ω_{ij} for $i, j \in [n]^\pm$ and ω_i for $i \in [n]$. Define

$$\begin{aligned}Y_J^+ &:= \omega_J^{-1}(Z^+) \\ Y_J^- &:= \omega_J^{-1}(Z^-),\end{aligned}$$

where Z^+ (resp. Z^-) is the positive (resp. negative) part of the x -axis in $\mathbb{R}^3 \setminus \{0\}$. This induces classes $\omega_J^*([Y^+]_T)$ and $\omega_J^*([Y^-]_T)$ in $H_T^*(\mathcal{Z}_n^3)$. By Example 5.3, and functoriality,

$$\omega_J^*([Y^-]_T) = \omega_J^*([Y^+]_T - u) = \omega_J^*([Y^+]_T) - u.$$

We are now ready to obtain relations in $H_T^*(\mathcal{Z}_n^3)$. Our primary method will be to show that $Y_J^\pm \cap Y_{J'}^\pm = \emptyset$ for certain $J, J' \subset [n]^\pm$, which by Remark 5.2 will imply that the corresponding classes in equivariant cohomology multiply to 0.

Lemma 5.11 For $1 \leq i < j \leq n$, let

$$\Psi : \mathbb{Q}[z_{ij}^+, z_{ij}^-, z_i, u] \longrightarrow H_T^*(\mathcal{Z}_n^3)$$

be the map sending

$$z_{ij}^+ \mapsto \omega_{ij}^*([Z^+]_T) \quad z_{ij}^- \mapsto \omega_{i\bar{j}}^*([Z^+]_T) \quad z_i \mapsto \omega_i^*([Z^+]_T) \quad u \mapsto u.$$

Then Ψ is surjective and the following relations lie in $\ker(\Psi)$:

$$0 = z_{ij}^+(z_{ij}^+ - u) = z_{ij}^-(z_{ij}^- - u) = z_i(z_i - u) \tag{5.3.1}$$

$$0 = u^{-1} \left(z_{ij}^+ z_{jk}^+ (z_{ik}^+ - u) - (z_{ij}^+ - u)(z_{jk}^+ - u)z_{ik}^+ \right) \tag{5.3.2}$$

$$0 = u^{-1} \left(z_{ij}^+ z_i (z_j - u) - (z_{ij}^+ - u)(z_i - u)z_j \right) \tag{5.3.3}$$

$$0 = u^{-1} \left(z_{ij}^+ z_{i\bar{j}}^-(z_i - u) - (z_{ij}^+ - u)(z_{i\bar{j}}^- - u)z_i \right) \tag{5.3.4}$$

Note that relations (5.3.2)–(5.3.4) are all polynomials (i.e. the expression in the parentheses has a factor of u).

Proof Because $\mathcal{Z}_n^3$ is formal and z_{ij}^+, z_{ij}^- and z_i for $1 \leq i < j \leq n$ is a generating set for $H^*\mathcal{Z}_n^3$, it follows that z_{ij}^+, z_{ij}^-, z_i and u form a generating set for $H_T^*(\mathcal{Z}_n^3)$. Hence Ψ is surjective.

For (5.3.1),

$$\Psi(z_J(z_J - u)) = \omega_J^*([Z^+]_T) \omega_J^*([Z^-]_T) = \omega_J^*(0) = 0$$

because $Z^+ \cap Z^- = \emptyset$.

For $1 \leq i < j < k \leq n$, consider

$$Y_{ij}^+ = \{(w_1, \dots, w_n) : x_i < x_j\},$$

$$Y_{jk}^+ = \{(w_1, \dots, w_n) : x_j < x_k\},$$

$$Y_{ik}^- = \{(w_1, \dots, w_n) : x_k < x_i\}.$$

Then $Y_{ij}^+ \cap Y_{jk}^+ \cap Y_{ik}^- = \emptyset$ since we cannot have $x_i < x_j < x_k < x_i$. This implies that

$$\Psi(z_{ij}^+ z_{jk}^+ (z_{ik}^- - u)) = \omega_{ij}^*([Z^+]_T) \omega_{jk}^*([Z^+]_T) \omega_{ik}^*([Z^-]_T) = 0,$$

and analogously,

$$\Psi((z_{ij}^+ - u)(z_{jk}^+ - u)z_{ik}^+) = \omega_{ij}^*([Z^-]_T) \omega_{jk}^*([Z^-]_T) \omega_{ik}^*([Z^+]_T) = 0.$$

The expansion of the above expression has a factor of u ; removing this factor recovers (5.3.2).

Relations (5.3.3) and (5.3.4) follow similarly. For the former, $Y_{ij}^+ \cap Y_i^+ \cap Y_j^- = \emptyset$ since we cannot have $0 < x_i < x_j < 0$. By the same argument, $Y_{ij}^- \cap Y_i^- \cap Y_j^+ = \emptyset$ and following the logic of (5.3.2), we obtain (5.3.3).

For (5.3.4), elements in $Y_j^+ \cap Y_{ij}^+$ must have $x_i < 0$ since $0 < x_j$ implies $\varphi(x_j) < 0$. From this it follows that $Y_j^+ \cap Y_{ij}^+ \cap Y_{ij}^- = \emptyset$ since we cannot have $0 < x_j < x_i < 0$. Analogously $Y_j^- \cap Y_{ij}^- \cap Y_{ij}^+ = \emptyset$, and (5.3.4) follows. $\square$

We now use these relations to give a presentation for $H_T^*(\mathcal{Z}_n^3)$.

Theorem 5.12 *Let $\mathcal{K}'$ be the ideal generated by B_n -images of the relations in Lemma 5.11. Then*

$$H_T^*(\mathcal{Z}_n^3) = \mathbb{Z}[z_{ij}^+, z_{ij}^-, z_i, u : 1 \leq i < j \leq n] / \mathcal{K}'.$$

Proof Lemma 5.11 shows that Ψ induces a surjective map

$$\overline{\Psi} : \mathbb{Q}[z_J, u] / \mathcal{K}' \rightarrow H_T^*(\mathcal{Z}_n^3).$$

We would like to show $\overline{\Psi}$ is injective as well.

Recall that Φ_0 is the map defined by sending u to 0. By Corollary 5.9, applying Φ_0 to $H_T^*(\mathcal{Z}_n^3)$ gives a surjective, B_n equivariant map to $H^*\mathcal{Z}_n^3$. Applying Φ_0 to $\mathbb{Z}[z_J, u]$ via

$$\Phi_0 : \mathbb{Z}[z_J, u] \longrightarrow \mathbb{Z}[z_J]$$

we see that $\Phi_0(\mathcal{K}') = \mathcal{K}$ from Remark 4.10. (Note that the image of relations (5.3.2), (5.3.3), (5.3.4) under Ψ_0 is the negation of the relations (i), (ii) and (iii) in $H^*\mathcal{Z}_n^3$ given in Remark 4.10.)

It follows that we have a commutative diagram with exact bottom row and surjective columns:

$$\begin{array}{ccccccc} \mathcal{K}' & \longrightarrow & \mathbb{Z}[z_J, u] & \xrightarrow{\Psi} & H_T^*(\mathcal{Z}_n^3) & \longrightarrow & 0 \\ \Phi_0 \downarrow & & \downarrow \Phi_0 & & \downarrow \Phi_0 & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathbb{Z}[z_J] & \longrightarrow & H^*\mathcal{Z}_n^3 \longrightarrow 0. \end{array}$$

The same diagram chase argument used by Moseley in [39, Thm 4.6] then applies to show that $\ker(\Psi) = \mathcal{K}'$. $\square$

One key takeaway from Theorem 5.12 is that (as in Example 5.3) the Heaviside function presentation of $H^*\mathcal{Z}_n^1$ is quite natural with respect to equivariant cohomology.

Corollary 5.13 *The filtration by Heaviside-like functions z_{ij} , z_i for $i, j \in [n]^\pm$ and the equivariant filtration on $H^*\mathcal{Z}_n^1$ coincide. Thus*

$$H_T^*(\mathcal{Z}_n^3)/\langle u \rangle = \mathrm{gr}_T(H^*\mathcal{Z}_n^1) \cong \mathrm{gr}(H^*\mathcal{Z}_n^1) \cong H^*\mathcal{Z}_n^3.$$

Proof This follows from noting that setting $u = 1$ in $H_T^*(\mathcal{Z}_n^3)$ recovers the presentation of $H^*\mathcal{Z}_n^1$ in Corollary 3.11. $\square$

5.4 Lifting to $\mathcal{Y}_{n+1}^3$

We would like make an analogous statement to Corollary 5.13 for $H_T^*(\mathcal{Y}_{n+1}^3)$ by applying two key facts established earlier:

- (1) there is a B_n -equivariant homeomorphism between $\mathcal{Z}_n^1$ and $\mathcal{Y}_{n+1}^1$ and
- (2) $\mathrm{gr}_T(\mathcal{Z}_n^1)$ coincides with $\mathrm{gr}(\mathcal{Z}_n^1)$.

Using a similar argument to [40, Remark 2.9], this is enough to establish the desired connection.

Theorem 5.14 *The filtration by signed cyclic Heaviside functions y_{ijk} for $i, j, k \in [n]_0^\pm$ and the equivariant filtration on $H^*\mathcal{Y}_{n+1}^1$ coincide. Thus*

$$H_T^*(\mathcal{Y}_{n+1}^3)/\langle u \rangle = \mathrm{gr}_T(H^*\mathcal{Y}_{n+1}^1) \cong \mathrm{gr}(H^*\mathcal{Y}_{n+1}^1) \cong H^*\mathcal{Y}_{n+1}^3.$$

Proof The key idea here is to recognize that the homeomorphism between $\mathcal{Z}_n^1$ and $\mathcal{Y}_{n+1}^1$ derived from Proposition 2.10 is a special case ($\ell = 0$) of a more general family of homeomorphisms for $0 \leq \ell \leq n$:

$$\begin{aligned} f_\ell : \mathcal{Y}_{n+1}^1 &\longrightarrow \mathcal{Z}_n^1 \\ (p_0, \dots, p_n) &\longmapsto (p_\ell^{-1} p_0, \dots, p_\ell^{-1} p_{\ell-1}, p_\ell^{-1} p_{\ell+1}, \dots, p_\ell^{-1} p_n). \end{aligned}$$

In cohomology, this induces a family of maps f_ℓ^* sending z_{ij} to $y_{\ell ij}$ and z_i to $y_{\ell \bar{\ell} i}$. By Corollary 5.6, in $\mathcal{Z}_n^1$ the filtration by Heaviside-like functions coincides with the filtration arising from equivariant cohomology. It follows that for a fixed ℓ , the same must be true of $H^*\mathcal{Y}_{n+1}^1$ with respect to the filtration arising from the $y_{\ell ij}$ and the filtration arising from $H_T^*(\mathcal{Y}_{n+1}^3)$. Since the latter filtration is stable under the action of B_{n+1} , it must coincide with the filtration by *all* signed cyclic Heaviside functions (e.g. allowing the indices $i, j, k \in [n]_0^\pm$ to vary.) $\square$

Since we have already obtained a description of $\text{gr}(H^*\mathcal{Y}_{n+1}^1)$ by Corollary 3.8, we obtain a description of $H^*\mathcal{Y}_{n+1}^3$ as a Corollary.

Corollary 5.15 *The ring $H^*\mathcal{Y}_{n+1}^3$ has presentation*

$$H^*\mathcal{Y}_{n+1}^3 \cong \mathbb{Z}[y_{ijk}]/\mathcal{T},$$

for distinct $i, j, k \in [n]_0^\pm$, where $\mathcal{T}$ is generated by the relations

$$\begin{aligned} (i) \quad & y_{ijk}^2, & (ii) \quad & y_{ijk} - y_{ij\ell} + y_{ik\ell} - y_{j\ell k} & (iii) \quad & y_{i\bar{j}k} - y_{i\bar{j}\bar{k}} \\ & & (iv) \quad & y_{ijk} = y_{jik} & (v) \quad & y_{ijk}y_{ik\ell} - y_{ijk}y_{ij\ell} - y_{ik\ell}y_{ij\ell}. \end{aligned}$$

6 Connection to the Mantaci-Reutenauer algebra

Finally, we will complete the Type B story by studying the family of B_n -representations arising in the *Mantaci-Reutenauer algebra* (Definition 6.4), introduced in [35]. Our primary goal is to prove Theorem 6.26 relating the topological spaces from Sect. 3–5 to these representations. In order to do so, we will need to define a family of idempotents in the Mantaci-Reutenauer algebra (§ 6.1) and the representations they generate (§ 6.1.1), as well as decompose the ring $\mathcal{G}$ further by *signed set compositions* (§ 6.1.2).

An *integer composition* of n is a sequence $(a_1, \dots, a_\ell)$ where $a_i \in [n]$ and $|a_1| + \dots + |a_\ell| = n$. Similarly a *signed integer composition* of n will be a sequence $(a_1, \dots, a_\ell)$ where $a_i \in [n]^\pm$ and again $|a_1| + \dots + |a_\ell| = n$.

Definition 6.1 For σ in one-line notation, the *Mantaci-Reutenauer descent set* of σ is

$$\text{MRDes}(\sigma) := \left\{ i \in [n-1] : \begin{array}{l} |\sigma_i| > |\sigma_{i+1}| \text{ and } \sigma_i \text{ and } \sigma_{i+1} \text{ have the same sign or} \\ \sigma_i \text{ and } \sigma_{i+1} \text{ have opposite signs.} \end{array} \right.$$

Note that $\text{MRDes}(\sigma)$ partitions σ into $\ell := |\text{MRDes}(\sigma)| + 1$ many blocks, $b_1, \dots, b_\ell$ of size $m_1, \dots, m_\ell$. By construction every element in b_i will have the same sign; let

$$\text{sgn}(m_i) = \begin{cases} m_i & \text{if } b_i \subset [n] \\ \bar{m}_i & \text{if } b_i \subset [n]^- \end{cases}.$$

Then the *shape* of $\sigma \in B_n$ is the signed integer composition

$$\text{sh}(\sigma) := (\text{sgn}(m_1), \dots, \text{sgn}(m_\ell)).$$

Example 6.2 If $\sigma = (3, 4, \bar{1}, \bar{5}, \bar{2}) \in B_5$, then $\text{MRDes}(\sigma) = \{2, 4\}$, which partitions σ into ordered blocks $(\{3, 4\}, \{\bar{1}, \bar{5}\}, \{\bar{2}\})$. Therefore $\text{sh}(\sigma) = (2, \bar{2}, \bar{1})$.

Remark 6.3 Recall that for $\sigma \in S_n$, the *descent set* of σ is $\text{Des}(\sigma) = \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}$. Because $S_n \leq B_n$, both $\text{MRDes}(\sigma)$ and $\text{sh}(\sigma)$ are well-defined for $\sigma \in S_n$. In particular, when $\sigma \in S_n$, $\text{MRDes}(\sigma) = \text{Des}(\sigma)$ and $\text{sh}(\sigma)$ will be an unsigned integer composition of n .

Definition 6.4 The *Mantaci-Reutenauer algebra* is the algebra $\Sigma'[B_n]$ generated by Y_α in $\mathbb{Q}[B_n]$ where

$$Y_\alpha := \sum_{\substack{\sigma \in B_n \\ \text{sh}(\sigma) = \alpha}} \sigma.$$

The dimension of $\Sigma'[B_n]$ is $2 \cdot 3^{n-1}$; for other bases of $\Sigma'[B_n]$, see [2].

6.1 The Vazirani idempotents

In [51], Vazirani introduces a complete family of orthogonal idempotents for $\Sigma'[B_n]$. These idempotents generalize a family of idempotents $\{e_\lambda\}_{\lambda \vdash n}$ introduced by Garsia–Reutenauer in [26] for $\Sigma[S_n]$; see [22, § 3] for complete details on both definitions.

We will use an equivalent construction of Vazirani's idempotents given by Douglass–Tomlin in [22, Prop. 2.5]. This will require the following objects and maps:

- Let $\mathcal{C}(n)$ and $\mathcal{SC}(n)$ be the set of unsigned and signed integer compositions of n , respectively.
- For $p = (p_1, \dots, p_\ell) \in \mathcal{SC}(n)$, let $|p| := (|p_1|, \dots, |p_\ell|) \in \mathcal{C}(n)$.
- For $p = (p_1, \dots, p_\ell) \in \mathcal{SC}(n)$, define

$$\widehat{p}_i = \sum_{j=1}^i |p_j| \in [n],$$

and the map $\widehat{}$ sending signed compositions to subsets of $[n-1]$:

$$\begin{aligned} \widehat{} : \mathcal{SC}(n) &\longrightarrow 2^{[n-1]} \\ p = (p_1, \dots, p_\ell) &\longmapsto \widehat{p} = (\widehat{p}_1, \dots, \widehat{p}_{\ell-1}). \end{aligned}$$

If we restrict the domain of $\widehat{}$ to ordinary compositions $\mathcal{C}(n)$, then $\widehat{}$ defines a bijection, and allows us to identify the sets

$$\left\{ \sigma \in S_n : \text{sh}(\sigma) = p \in \mathcal{C}(n) \right\} = \left\{ \sigma \in S_n : \text{Des}(\sigma) = \widehat{p} \right\}.$$

- A set composition of $[n]$ is a partition of $[n]$ into disjoint blocks $(b_1, \dots, b_\ell)$. Let $\Lambda_{[n]}$ be the set of such compositions. Define

$$\begin{aligned} \Lambda : \mathcal{SC}(n) &\longrightarrow \Lambda_{[n]} \\ p = (p_1, \dots, p_\ell) &\longmapsto \Lambda(p) = (\Lambda(p_1), \dots, \Lambda(p_\ell)), \end{aligned}$$

where $\Lambda(p_i) := \{\widehat{p}_{i-1} + 1, \dots, \widehat{p}_i\}$. Note that $\widehat{p}$ and $\Lambda(p)$ differ because the former is a subset of $[n-1]$ while the latter is a genuine set composition of $[n]$.

- Recall that a *signed partition* of n is a pair of partitions $\lambda = (\lambda^+, \lambda^-)$ such that λ^+ is a partition of $n_1 \leq n$, λ^- is a partition of $n_2 \leq n$, and $n_1 + n_2 = n$. Let $\ell(\lambda^+)$ (resp. $\ell(\lambda^-)$) be the number of parts in λ^+ (resp. λ^-), and $\ell(\lambda) = \ell(\lambda^+) + \ell(\lambda^-)$.
- For $p \in \mathcal{SC}(n)$, let $\overleftarrow{p}$ be the reordering of the parts of p into a signed partition (λ^+, λ^-) so that λ^+ consists of all positive parts of p in decreasing order and λ^- consists of all negative parts of p in decreasing order.

- For a set $A \subset [n-1]$, define an element in the Type A Descent algebra $\Sigma[S_n]$:

$$X_A := \sum_{\substack{w \in S_n \\ \text{Des}(w) \subset A}} w.$$

Varying over all subsets $A \subset [n-1]$ gives a basis for $\Sigma[S_n]$.

- For $m > 0$, the *Reutenauer idempotent* $r_m \in \Sigma[S_m] \subset \mathbb{Q}[S_n]$ is

$$r_m = \sum_{A \subset [m-1]} \frac{(-1)^{|A|}}{|A|+1} X_A.$$

Note that the definition of r_m can be extended to any ordered subset $J \subset [n]$ by replacing $[m]$ by J , in which case we will write $r_J \in \mathbb{Q}[S_J]$. Furthermore, since $S_n \leq B_n$, one also has $r_J \in \mathbb{Q}[B_n]$.

- For $J \subset [n]$ the element $w_{0,J} \in B_n$ is the product

$$\prod_{i \in J} t_i.$$

In other words, $w_{0,J}$ acts like -1 on J and 1 off J .

- For $J \subset [n]$, define

$$\epsilon_J^\pm := \frac{1}{2}(1 \pm w_{0,J}).$$

Example 6.5 Suppose $p = (2, \bar{2}, 1)$. Then $|p| = (2, 2, 1)$, $\widehat{p} = \{2, 4, 5\}$ and $\Lambda(p) = \{\{1, 2\}, \{3, 4\}, \{5\}\}$. Finally,

$$\overleftarrow{p} = \left(\underbrace{(2, 1)}_{\lambda^+}, \underbrace{(2)}_{\lambda^-} \right),$$

so $\ell(\lambda^+) = 2$, $\ell(\lambda^-) = 1$.

We are at last ready to define the Vazirani idempotents.

Definition 6.6 (Douglass-Tomlin [22, Prop 2.5]). Given $p = (p_1, \dots, p_\ell) \in \mathcal{SC}(n)$, define the element

$$I_p := X_{\widehat{p}} \cdot \epsilon_{\Lambda(p_1)}^{\zeta_1} \cdot r_{\Lambda(p_1)} \cdots \epsilon_{\Lambda(p_\ell)}^{\zeta_\ell} \cdot r_{\Lambda(p_\ell)},$$

where ζ_i is the sign of p_i . Then for each signed partition (λ^+, λ^-) , the *Vazirani idempotent* is

$$\mathfrak{g}_{(\lambda^+, \lambda^-)} := \sum_{\substack{p \in \mathcal{SC}(n) \\ \overleftarrow{p} = (\lambda^+, \lambda^-)}} \frac{1}{\ell(\lambda)!} I_p.$$

The Vazirani idempotents extend the Garsia-Reutenauer idempotents in $\Sigma[S_n]$, which are defined as:

$$\epsilon_\lambda = \frac{1}{\ell(\lambda)!} \sum_{\substack{p \in \mathcal{C}(n) \\ \overleftarrow{p} = \lambda}} X_{\widehat{p}} \cdot r_{\Lambda(p_1)} \cdots r_{\Lambda(p_\ell)}. \quad (6.1.1)$$

In fact, we can make the relationship between the $\mathfrak{g}_{(\lambda^+, \lambda^-)}$ and ϵ_λ precise. In [2], Aguiar-Bergeron-Nyman study the surjection $\tau : B_n \rightarrow S_n$ that forgets the signs of $\sigma \in B_n$. For example, if $\sigma = (\bar{2}, 1, \bar{3})$, then $\tau(\sigma) = (2, 1, 3)$.

Theorem 6.7 (Aguiar–Bergeron–Nyman [2, Prop 7.5]). *The map τ is an algebra homomorphism from*

$$\tau : \mathbb{Q}[B_n] \rightarrow \mathbb{Q}[S_n].$$

When restricted to $\Sigma'[B_n]$, the map τ surjects onto $\Sigma[S_n]$.

This will allow us to precisely relate the $\mathfrak{g}_{(\lambda^+, \lambda^-)}$ to the $\mathfrak{e}_\lambda$; the author is grateful to M. Aguiar for suggesting this line of inquiry.

Proposition 6.8 *For a signed partition (λ^+, λ^-) , one has*

$$\tau(\mathfrak{g}_{(\lambda^+, \lambda^-)}) = \begin{cases} \mathfrak{e}_{\lambda^+} & \lambda^- = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Proof Since τ is an algebra homomorphism, it is enough to consider how τ maps each term in I_p for $\overleftarrow{p} = (\lambda^+, \lambda^-)$. In particular, for any $\Lambda(p_i)$, since $\tau(w_{0, p_i}) = 1$, it follows that

$$\tau(\epsilon_{\Lambda(p_i)}^{\zeta_i}) = \begin{cases} 1 & \zeta_i = + \\ 0 & \zeta_i = -. \end{cases}$$

Thus $\tau(I_p) = 0$ if p has any negative parts, from which it follows that $\tau(\mathfrak{g}_{(\lambda^+, \lambda^-)}) = 0$ if $\lambda^- \neq \emptyset$. On the other hand, if $\overleftarrow{p} = (\lambda^+, \emptyset)$, then $p \in \mathcal{C}(n)$, $|p| = p$ and $\tau(\epsilon_{p_i}^+) = 1$ for all i . Hence $\tau(\mathfrak{g}_{(\lambda^+, \emptyset)}) = \mathfrak{e}_{\lambda^+}$ by (6.1.1). $\square$

In [26], Garsia and Reutenauer show that the definition of the Type A Eulerian idempotents given in (2.1.1) is equivalent to

$$\mathfrak{e}_k = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = k-1}} \mathfrak{e}_\lambda. \quad (6.1.2)$$

Our analog of the Eulerian idempotents will take inspiration from (6.1.2):

Definition 6.9 For $0 \leq k \leq n$, define the idempotent in $\Sigma'[B_n]$

$$\mathfrak{g}_k := \sum_{\substack{\lambda = (\lambda^+, \lambda^-) \\ \ell(\lambda^+) = k}} \mathfrak{g}_{(\lambda^+, \lambda^-)}. \quad (6.1.3)$$

Proposition 6.8 implies the following relationship between the $\mathfrak{g}_k$ and the $\mathfrak{e}_k$.

Corollary 6.10 *The map $\tau : \Sigma'[B_n] \rightarrow \Sigma[S_n]$ sends*

$$\tau(\mathfrak{g}_k) = \begin{cases} 0 & k = 0 \\ \mathfrak{e}_{k-1} & k > 0. \end{cases}$$

6.1.1 Representations generated by $\mathfrak{g}_{(\lambda^+, \lambda^-)}$ and $\mathfrak{g}_k$

The idempotents in Definitions 6.6 and 6.9 generate families of B_n -representations.

Definition 6.11 For any signed partition (λ^+, λ^-) of n , define the B_n -representation

$$G_{(\lambda^+, \lambda^-)} := \mathfrak{g}_{(\lambda^+, \lambda^-)} \mathbb{Q}[B_n].$$

Further, for $0 \leq k \leq n$ define the B_n -representation

$$G_n^{(k)} := \mathfrak{g}_k \mathbb{Q}[B_n].$$

Douglass–Tomlin proved in [22] that the characters of the $G_{(\lambda^+, \lambda^-)}$ could be described as induced representations from one-dimensional characters of certain centralizers. In order to describe and analyze these induced characters, we will temporarily extend scalars from $\mathbb{Q}$ to $\mathbb{C}$.

Recall from Definition 2.18 that conjugacy classes in B_n are determined by cycle type and are parametrized by signed partitions (λ^+, λ^-) . Our first goal is to construct a standard element for each conjugacy class $\mathcal{C}_{(\lambda^+, \lambda^-)}$ of B_n which we will call $\sigma_{(\lambda^+, \lambda^-)}$. We will then express the representation $G_{(\lambda^+, \lambda^-)}$ as an induced representation from the centralizer $Z_{(\lambda^+, \lambda^-)}$ of $\sigma_{(\lambda^+, \lambda^-)}$.

Fix the signed partition $\lambda = (\lambda^+, \lambda^-)$; we first obtain $\sigma_{(\lambda^+, \lambda^-)}$ as follows. As before, let $\widehat{\lambda}_i = \sum_{j=1}^i |\lambda_j|$ and define $\Lambda_i := \Lambda(\lambda_i) := \{\widehat{\lambda}_{i-1} + 1, \widehat{\lambda}_{i-1} + 2, \dots, \widehat{\lambda}_i\}$. If $\lambda_i \in \lambda^+$, define the positive λ_i -cycle

$$c_i := ((\widehat{\lambda}_{i-1} + 1)\widehat{\lambda}_{i-1} \cdots \widehat{\lambda}_i).$$

By construction c_i has order $|\lambda_i|$.

To construct a “standard” negative cycle, recall that for any $J \subset [n]$, the element $w_{0,J} \in B_n$ is the product of t_i for $i \in P$ and that for $j_1, j_2, \dots, j_\ell \in [n]$,

$$(j_1 \cdots j_\ell)^-$$

is the negative cycle in B_n where $j_i \mapsto j_{i+1}$ for $1 \leq i \leq \ell - 1$ and $j_\ell \mapsto \overline{j_1}$. Then for $\lambda_i \in \lambda^-$, define the negative $|\lambda_i|$ cycle

$$d_i := \begin{cases} c_i \cdot w_{0, \Lambda_i} & |\lambda_i| \text{ is odd} \\ ((\widehat{\lambda}_{i-1} + 1)\widehat{\lambda}_{i-1} \cdots \widehat{\lambda}_i)^- & |\lambda_i| \text{ is even.} \end{cases}$$

The cycle d_i has order $2|\lambda_i|$ in B_n . The signed permutation

$$\sigma_{(\lambda^+, \lambda^-)} := c_1 \cdots c_{\ell(\lambda^+)} d_{\ell(\lambda^+)+1} \cdots d_{\ell(\lambda)-1}$$

will be our standard representative of $\mathcal{C}_{(\lambda^+, \lambda^-)}$.

Finally, suppose $\lambda_i = \lambda_{i+1}$ and both λ_i, λ_{i+1} occur in either λ^+ or λ^- . Define δ_i to be a particular choice of permutation swapping the blocks Λ_i and Λ_{i+1} . Specifically,

$$\delta_i : j \mapsto \begin{cases} j & j \notin \Lambda_i \cup \Lambda_{i+1} \\ j + |\lambda_i| & j \in \Lambda_i \\ j - |\lambda_i| & j \in \Lambda_{i+1}. \end{cases}$$

The centralizer $Z_{(\lambda^+, \lambda^-)}$ of $\sigma_{(\lambda^+, \lambda^-)}$ is then generated by

- c_i and w_{0, Λ_i} for every $\lambda_i \in \lambda^+$
- d_i for every $\lambda_i \in \lambda^-$;
- δ_i for every pair $\lambda_i = \lambda_{i+1}$ for $1 \leq i \leq \ell(\lambda) - 1$. (Again, we require that λ_i, λ_{i+1} are both in λ^+ or both in λ^- .)

Example 6.12 Suppose $\lambda = (\lambda^+, \lambda^-) = ((2, 1), (2, 2))$. Then the relevant elements of B_7 (written in one-line notation) are

$$\begin{aligned} c_1 &= (2, 1, 3, 4, 5, 6, 7) \\ w_{0,\{1,2\}} &= (\bar{1}, \bar{2}, 3, 4, 5, 6, 7) \\ c_2 &= (1, 2, 3, 4, 5, 6, 7) \\ w_{0,\{3\}} &= (1, 2, \bar{3}, 4, 5, 6, 7) \\ d_3 &= (1, 2, 3, 5, \bar{4}, 6, 7) \\ d_4 &= (1, 2, 3, 4, 5, 7, \bar{6}) \\ \sigma_{((2,1),(\bar{2},\bar{2}))} &= (2, 1, 3, 5, \bar{4}, 7, \bar{6}) \\ \delta_3 &= (1, 2, 3, 6, 7, 4, 5). \end{aligned}$$

Since c_i , d_i , $w_{0,J}$ and δ_i generate $Z_{(\lambda^+, \lambda^-)}$, any representation of $Z_{(\lambda^+, \lambda^-)}$ is determined by its values on these elements.

Define $\omega_k := e^{2\pi i/k}$. The representation of $Z_{(\lambda^+, \lambda^-)}$ we are interested in is 1-dimensional, that is, a map $Z_{(\lambda^+, \lambda^-)} \rightarrow \mathbb{C}$.

Definition 6.13 Let $\rho_{(\lambda^+, \lambda^-)}$ be the character of $Z_{(\lambda^+, \lambda^-)}$ given by

$$\rho_{(\lambda^+, \lambda^-)}(\sigma) := \begin{cases} \omega_{|\lambda_i|} & \sigma = c_i \text{ for } \lambda_i \in \lambda^+ \\ \omega_{2|\lambda_i|} & \sigma = d_i \text{ for } \lambda_i \in \lambda^- \\ 1 & \sigma = w_{0,\Lambda_i} \text{ for } 1 \leq i \leq \ell(\lambda^+) \\ 1 & \sigma = \delta_i \text{ for } 1 \leq i \leq \ell(\lambda) - 1 \text{ and } \lambda_i = \lambda_{i+1}. \end{cases}$$

The representation $\rho_{(\lambda^+, \lambda^-)}$ is of interest because of the following theorem.

Theorem 6.14 (Douglass–Tomlin [22]) *There is an isomorphism of B_n -representations*

$$G_{(\lambda^+, \lambda^-)} = \text{Ind}_{Z_{(\lambda^+, \lambda^-)}}^{B_n} \rho_{(\lambda^+, \lambda^-)}.$$

Our aim going forward is to connect the representations $G_{(\lambda^+, \lambda^-)}$ to the topological spaces we have been studying.

6.1.2 Decomposition of $\mathcal{G}$ by signed set partitions

Recall from Definition 3.14 that $\mathcal{G}$ is the associated graded ring of $\text{gr}(H^*Z_n^1) \cong H^*Z_n^3$ with respect to the filtration described in Proposition 3.13. We will use the presentation of $\mathcal{G}$ described by Remark 4.10 to show that $\mathcal{G}$ admits an even finer decomposition by *signed set compositions*, defined below.

Definition 6.15 Let $J_1, J_2 \subset [n]$ be any partition of $[n]$. Then $\alpha = (\alpha^+, \alpha^-)$ is a *signed set partition* of $[n]$ if α^+ is a set partition of J_1 and α^- is a set partition of J_2 . Let $\Lambda_{[n]}^\pm$ be the collection of all such signed set partitions of $[n]$.

Note that we allow either α^+ or α^- to be empty. The blocks in α^+ are said to be *positive* while the blocks in α^- are said to be *negative*. Let $\ell(\alpha^+)$ (respectively, $\ell(\alpha^-)$) be the number of parts of α^+ (respectively, α^-).

Write $E = \mathbb{Z}[z_{ij}^+, z_{ij}^-, z_i]$ for $1 \leq i < j \leq n$. In order to decompose E by $\Lambda_{[n]}^\pm$, we introduce the *type map* of a monomial $f \in E$.

Definition 6.16 From f , create a graph $G(f) = (V, E)$, with $V = [n]$ and E described by:

- For every z_{ij}^+ or z_{ij}^- appearing in f , there is an edge between i and j
- For every z_i appearing in f , there is a loop at the vertex i .

Define the $\text{type}_n(f) \in \Lambda_{[n]}^\pm$ to be the signed set partition with blocks comprised of the connected components of $G(f)$; if a connected component contains a loop, it is a negative block, and otherwise it is a positive block.

Example 6.17 Suppose $n = 8$ and $f = (z_{12}^-)^2 z_5 z_{56}^- z_7^3$. Then

$$\text{type}_8((z_{12}^-)^2 \cdot z_5 z_{56}^- \cdot z_7^3) = (\underbrace{(\{1, 2\}, \{3\}, \{4\}, \{8\})}_{(\text{type}_8(f))^+}, \underbrace{(\{5, 6\}, \{7\})}_{(\text{type}_8(f))^-}).$$

The map type_n is well-defined on monomials, but not polynomials. Note that type_n is surjective but not injective. (To see that it is surjective, note that for any $\alpha \in \Lambda_{[n]}^\pm$, one can easily construct an $f \in E$ with $\text{type}(f) = \alpha$.)

Definition 6.18 For a signed set partition $\alpha = (\alpha^+, \alpha^-) \in \Lambda_{[n]}^\pm$, define

$$E_\alpha := \mathbb{Z}[\{\text{monomials } f \in E : \text{type}_n(f) = \alpha\}].$$

Note that by the above description, there is a vector space decomposition

$$E = \bigoplus_{\alpha \in \Lambda_{[n]}^\pm} E_\alpha.$$

We would like to show that this decomposition descends to the quotient $\mathcal{G}$ as well.

Recall that

$$\mathcal{G} = \mathbb{Z}[z_{ij}^+, z_{ij}^-, z_i] / \mathcal{L}'$$

for $1 \leq i < j \leq n$, where $\mathcal{L}'$ is generated by $(z_{ij}^+)^2 = (z_{ij}^-)^2 = (z_i)^2$ and

$$\begin{aligned} & (i) \ z_{ij}^+ z_{ij}^- \quad (ii) \ z_{ij}^+ z_j - z_{ij}^+ z_i \quad (iii) \ z_{ij}^- z_j + z_{ij}^- z_i \quad (iv) \ z_{ij}^+ z_{jk}^+ - z_{ij}^+ z_{ik}^+ - z_{jk}^+ z_{ik}^+, \\ & (v) \ z_{ij}^- z_{jk}^+ - z_{ij}^- z_{ik}^- - z_{jk}^+ z_{ik}^- \quad (vi) \ z_{ij}^- z_{jk}^- - z_{ij}^- z_{ik}^+ - z_{jk}^- z_{ik}^+ \quad (vii) \ z_{ij}^+ z_{jk}^- - z_{ij}^+ z_{ik}^- - z_{jk}^- z_{ik}^-. \end{aligned}$$

Proposition 6.19 With $\mathcal{G}$ as above, there is a vector space decomposition

$$\mathcal{L}' = \bigoplus_{\alpha \in \Lambda_{[n]}^\pm} \mathcal{L}' \cap E_\alpha,$$

inducing the decomposition

$$\mathcal{G} = \bigoplus_{\alpha \in \Lambda_{[n]}^\pm} \mathcal{G}_\alpha,$$

where $\mathcal{G}_\alpha := E_\alpha / (\mathcal{L}' \cap E_\alpha)$.

Proof It is sufficient to check that each relation in $\mathcal{L}'$ has summands in a single E_α for some $\alpha \in \Lambda_{[n]}^\pm$. Relation (i) has type $\alpha^+ = \{\{i, j\}, \{k\}_{k \neq i, j}\}$ and $\alpha^- = \emptyset$. Each summand in relations (ii) and (iii) has type $\alpha^+ = \{\{k\}\}_{k \neq i, j}$ and $\alpha^- = \{i, j\}$, while each summand in relations (iv)—(vii) has type $\alpha^+ = \{i, j, k\}, \{\{\ell\}\}_{\ell \neq i, j, k}$, and $\alpha^- = \emptyset$. $\square$

The decomposition of $\mathcal{G}$ by signed set partitions is also compatible with the bi-grading on $\mathcal{G}$.

Proposition 6.20 For $0 \leq \ell \leq k \leq n$,

$$\mathcal{G}_{k,\ell} = \bigoplus_{\substack{\alpha=(\alpha^+, \alpha^-) \in \Lambda_{[n]}^\pm \\ \ell(\alpha^+) = n-k, \\ \ell(\alpha^-) = \ell}} \mathcal{G}_\alpha$$

Proof Fix $\alpha \in \Lambda_{[n]}^\pm$ with $\ell(\alpha^+) = n - k$ and $\ell(\alpha^-) = \ell$. Consider $f \in \mathcal{G}_\alpha$. Factor f by the blocks of α , and write f_i as the factor of f with support $\alpha_i \in \alpha$. Note that if $\alpha_i \in \alpha^+$, then $f_i \in \mathcal{G}_{|\alpha_i|-1,0}$. Similarly, if $\alpha_j \in \alpha^-$, then $f_j \in \mathcal{G}_{|\alpha_j|,1}$.

Hence

$$\begin{aligned} f &= f_1 \cdots f_{n-k} \cdot f_{n-k+1} \cdots f_{n-k+\ell} \in \underbrace{(\mathcal{G}_{|\alpha_1|-1,0} \cdot \mathcal{G}_{|\alpha_2|-1,0} \cdots \mathcal{G}_{|\alpha_{n-k}|-1,0})}_{\alpha^+} \\ &\quad \cdot \underbrace{(\mathcal{G}_{|\alpha_{n-k+1}|,1} \cdots \mathcal{G}_{|\alpha_{n-k+\ell}|,1})}_{\alpha^-} \\ &\subseteq \mathcal{G}_{|\alpha^+|-(n-k),0} \cdot \mathcal{G}_{|\alpha^-|,\ell} \\ &\subseteq \mathcal{G}_{k,\ell}, \end{aligned}$$

because $|\alpha^+| + |\alpha^-| = n$.

For the other containment, consider any monomial $f \in \mathcal{G}_{k,\ell}$ and let $\alpha = \text{type}_n(f)$. Using the above notation, we have that $f_i \in \mathcal{G}_{|\alpha_i|-1,0}$ if $\alpha_i \in \alpha^+$ and $\mathcal{G}_{|\alpha_i|,1}$ if $\alpha_i \in \alpha^-$. Since $f \in \mathcal{G}_{k,\ell}$, we must therefore have $\ell(\alpha^-) = \ell$, and

$$\begin{aligned} \deg \left(\prod_{\substack{1 \leq i \leq n \\ \alpha_i \in \alpha^+}} f_i \right) &= |\alpha^+| - \ell(\alpha^+), \\ \deg \left(\prod_{\substack{1 \leq j \leq n \\ \alpha_j \in \alpha^-}} f_j \right) &= |\alpha^-|. \end{aligned}$$

Since $\deg(f) = k$,

$$k = |\alpha^+| - \ell(\alpha^+) + |\alpha^-| = n - \ell(\alpha^+),$$

from which it follows that $\ell(\alpha^+) = n - k$. $\square$

6.2 Induced representations

Using the decomposition of $\mathcal{G}$ by $\Lambda_{[n]}^\pm$, we may begin to describe the pieces $\mathcal{G}_\alpha$ as induced representations.

Given $\alpha = (\alpha^+, \alpha^-) = (\alpha_1, \dots, \alpha_\ell) \in \Lambda_{[n]}^\pm$, let $[\alpha] \in SC(n)$ be the signed integer composition

$$[\alpha] = (\#\alpha_1, \dots, \#\alpha_\ell),$$

where $\#\alpha_i \in [n]^\pm$ is the size of the set α_i with $\#\alpha_i \in [n]$ if $\alpha_i \in \alpha^+$ and in $[n]^-$ if $\alpha_i \in \alpha^-$. As before, $\overleftarrow{[\alpha]} = ([\alpha]^+, [\alpha]^-)$ is the signed integer partition obtained by ordering the parts of $[\alpha]$ in decreasing order.

Consider the B_n orbit of $\mathcal{G}_\alpha$, which will be indexed by signed partitions (λ^+, λ^-) :

$$\mathcal{G}_{(\lambda^+, \lambda^-)} := \bigoplus_{\substack{\alpha \in \Lambda_{[n]}^\pm \\ \overleftarrow{[\alpha]} = (\lambda^+, \lambda^-)}} \mathcal{G}_\alpha.$$

The first step is to compute the characters of $\mathcal{G}_{n,1} = \mathcal{G}_{(0,(n))}$ and $\mathcal{G}_{n-1,0} = \mathcal{G}_{((n),0)}$ which will form a base-case for the other pieces of $\mathcal{G}_{(\lambda^+, \lambda^-)}$.

Theorem 6.21 *As B_n -representations*

$$\mathcal{G}_{n,1} = \text{Ind}_{Z_{(\emptyset,(n))}}^{B_n} \rho_{(\emptyset,(n))} = G_{(\emptyset,(n))}, \quad \mathcal{G}_{n-1,0} = \text{Ind}_{Z_{((n),\emptyset)}}^{B_n} \rho_{((n),\emptyset)} = G_{((n),\emptyset)}.$$

To prove Theorem 6.21, we will first prove that $\mathcal{G}_{n,1}$ is isomorphic to a different induced representation, and then appeal to Lemma 6.22. Both the proof of Theorem 6.21 and Lemma 6.22 will use techniques developed by Berget [10] and in the case of Lemma 6.22, by Douglass–Tomlin [22].

Lemma 6.22 *Let η be the n -cycle $(12 \cdots n) \in S_n$. Then as B_n -representations,*

$$\text{Ind}_{\langle \eta, -1 \rangle}^{B_n} \chi = \text{Ind}_{Z_{(\emptyset,(n))}}^{B_n} \rho_{(\emptyset,(n))}$$

where $\chi(\eta) = \omega_n$ and $\chi(-1) = -1$.

Proof Note that $Z_{(\emptyset,(n))}$ is the cyclic group generated by a Coxeter element c of B_n and

$$\rho_{(\emptyset,(n))}(c) = \omega_{2n}.$$

When n is odd, we use similar methods to Berget in [10, Corollary 9.2]. In this case, $-\eta$ is a Coxeter element with eigenvalue $-e^{2\pi i/n}$, implying that $\langle \eta, -1 \rangle = Z_{(\emptyset,(n))}$. Because n is odd, $-e^{2\pi i/n}$ has order $2n$ and is in fact a primitive $2n$ -th root of unity. Thus the representation χ of $\langle \eta, -1 \rangle$ coincides with the representation $\rho_{(\emptyset,(n))}$ of $Z_{(\emptyset,(n))}$.

When n is even, we use a result by Douglass–Tomlin [22, Prop 4.1], which states that for even n ,

$$\text{Ind}_{\langle \eta, -1 \rangle}^{B_n} \chi = \text{Ind}_{Z_{(\emptyset,(n))}}^{B_n} \rho_{(\emptyset,(n))}.$$

Note that unlike the odd case, the above is a statement about the induced representations rather than the representations χ and $\rho_{(\emptyset,(n))}$. $\square$

We may now prove Theorem 6.21.

Proof of Theorem 6.21 Define $v := z_1 z_{12}^+ z_{23}^+ \cdots z_{(n-1)n}^+ \in \mathcal{G}_{n,1}$, and let V_n be the S_n -module generated by v (cf. proof of Berget [10, Thm. 9.1, Cor. 9.2].) We claim that as an S_n -representation, $V_n \cong_{S_n} \text{Lie}_n$, the multilinear component of the free Lie algebra on n generators (see Example 2.6). This follows from comparing the presentation in Remark 4.10 with the presentation of Lie_n discussed in Example 2.6. In particular, sending v to $z_{12} z_{23} \cdots z_{(n-1)n} \in H^{(d-1)(n-1)} \text{Conf}_n(\mathbb{R}^d)$ (for $d \geq 3$, odd) induces an S_n equivariant isomorphism since $z_i z_{ij}^+ = z_j z_{ij}^+$ in V_n (see Remark 4.10, relation (ii) of $\mathcal{L}'$). Thus $V_n \cong \text{Ind}_{\langle w \rangle}^{S_n} e^{2\pi i/n}$, where η is an n -cycle in S_n .

Note that $w_0 = -1$ acts on z_{ij}^+ trivially in $\mathcal{G}$, and $w_0 z_i = -z_i$. Hence for any monomial in $\mathcal{G}_{n,1}$, w_0 acts as -1 . Because w_0 is central, it follows that the $\langle S_n, w_0 \rangle$ module generated by v is isomorphic to

$$\text{Ind}_{\langle \eta, w_0 \rangle}^{\langle S_n, w_0 \rangle} \chi,$$

where $\chi(\eta) = e^{2\pi i/n}$ and $\chi(w_0) = -1$. Write this module as V'_n , and note that

$$\dim(V'_n) = \dim(V_n) = (n-1)!$$

Now consider the B_n -module generated by v . Inspection shows that one obtains all of $\mathcal{G}_{n,1}$ by acting on v by B_n . Hence there is a surjection of B_n -modules:

$$\text{Ind}_{\langle S_n, w_0 \rangle}^{B_n} V'_n \rightarrow \mathcal{G}_{n,1},$$

where

$$\dim\left(\text{Ind}_{\langle S_n, w_0 \rangle}^{B_n} V'_n\right) = \frac{n!2^n}{2n!} \dim(V'_n) = 2^{n-1}(n-1)! = \dim(\mathcal{G}_{n,1}).$$

The dimension count of $\mathcal{G}_{n,1}$ follows from picking a cyclic ordering on the set $[n]$ which will determine the indices of the $n-1$ generators $z_{i_1 i_2}^{\zeta_1} z_{i_2 i_3}^{\zeta_2} \cdots z_{i_{n-1} i_n}^{\zeta_{n-1}}$, and then choosing $\zeta_j \in \{+, -\}$ for $0 \leq j \leq n-1$. (The choice of z_i is irrelevant by the relations in $\mathcal{G}$.)

Thus this surjection is in fact an isomorphism. By transitivity of induction,

$$\begin{aligned} \text{Ind}_{\langle \eta, -1 \rangle}^{B_n} \chi &= \text{Ind}_{\langle S_n, -1 \rangle}^{B_n} \text{Ind}_{\langle \eta, -1 \rangle}^{\langle S_n, -1 \rangle} (\chi) \\ &= \text{Ind}_{\langle S_n, -1 \rangle}^{B_n} (V'_n) \\ &= \mathcal{G}_{n,1}. \end{aligned}$$

The claim then follows by Lemma 6.22.

An identical argument (without the need for Lemma 6.22) shows the claim for $\mathcal{G}_{n-1,0}$, where now -1 acts as trivially on every monomial in $\mathcal{G}_{n-1,0}$. $\square$

Remark 6.23 As suggested by the proof of Theorem 6.21, the space $\mathcal{G}_{n,1}$ seems to be a good candidate for a Type B analog of Lie_n in the sense that

- $\mathcal{G}_{n,1}$ can be described as an induced representation from a Coxeter element of B_n and
- One can show that as a B_{n-1} -representation, $\mathcal{G}_{n,1}$ restricts to the regular representation $\mathbb{Q}[B_{n-1}]$.

A natural next step would be to extend some of the other properties of Lie_n in Type A ; see for instance [1].

Remark 6.24 In the case that n is odd, the B_n -representation carried by $\mathcal{G}_{n,1}$ coincides with a representation studied by Berget in [10] coming from the internal zonotopal algebra of the Type B hyperplane arrangement. In the case that n is even, the B_n -module structure is almost the same, except that the element w_0 acts trivially in Berget's representation while it acts as -1 on $\mathcal{G}_{n,1}$.

We are almost ready to prove the main result of this Sect. (Theorem 6.26), but need one last lemma. For (λ^+, λ^-) , write

$$\lambda^+ = (1^{m_1}, \dots, n^{m_n}) \tag{6.2.1}$$

$$\lambda^- = (1^{q_1}, \dots, n^{q_n}), \tag{6.2.2}$$

where $m_i, q_j \geq 0$. Then fix $\alpha \in \Lambda_{[n]}^\pm$ so that $[\alpha] = \lambda = (\lambda^+, \lambda^-)$, and

$$\alpha_i = \{\widehat{\lambda_{i-1}} + 1, \widehat{\lambda_{i-1}}, \dots, \widehat{\lambda_i}\}.$$

Consider H_α , the set-wise stabilizer of $\mathcal{G}_\alpha$. One can write explicitly the isomorphism type of H_α :

$$H_\alpha = \prod_{\substack{1 \leq i \leq n \\ m_i \neq 0}} B_i \wr S_{m_i} \times \prod_{\substack{1 \leq j \leq n \\ q_j \neq 0}} B_j \wr S_{q_j}.$$

By construction, $Z_{(\lambda^+, \lambda^-)}$ is a subgroup of H_α .

Lemma 6.25 *For $\alpha \in \Lambda_{[n]}^\pm$ as described above, there is an H_α -isomorphism*

$$\mathcal{G}_\alpha = \text{Ind}_{Z_{(\lambda^+, \lambda^-)}}^{H_\alpha} \varphi_{(\lambda^+, \lambda^-)}.$$

Proof Recall that in the character description of $\varphi_{(\lambda^+, \lambda^-)}$, we have $\varphi_{(\lambda^+, \lambda^-)}(\delta_i) = 1$, where δ_i swaps blocks of the same size that are both in the positive (resp. negative) part of α . Hence in considering the induced representation of $\varphi_{(\lambda^+, \lambda^-)}$ from $Z_{(\lambda^+, \lambda^-)}$ to H_α , it is enough to understand the representation on the distinct, commuting factors of H_α coming from each α_i .

Similarly, the action by δ_i on $\mathcal{G}_\alpha$ is trivial, so again to understand the H_α -representation on $\mathcal{G}_\alpha$ it is sufficient to describe the distinct commuting factors of H_α .

Write $a_i := \#\alpha_i$. In the case of $\text{Ind}_{Z_{(\lambda^+, \lambda^-)}}^{H_\alpha} \varphi_{(\lambda^+, \lambda^-)}$, the B_{a_i} -representation coming from the block α_i in α^+ is precisely

$$\text{Ind}_{Z_{((a_i), \emptyset)}}^{B_{a_i}} \varphi_{((a_i), \emptyset)}$$

and for the block of size α_j in α^- the corresponding B_{a_j} -representation is

$$\text{Ind}_{Z_{(\emptyset, (a_j))}}^{B_{a_j}} \varphi_{(\emptyset, (a_j))}.$$

By Theorem 6.21,

$$\begin{aligned} \mathcal{G}_{a_i-1,0} &= \text{Ind}_{Z_{(a_i), \emptyset}}^{B_{a_i}} \varphi_{((a_i), \emptyset)} \\ \mathcal{G}_{a_j,1} &= \text{Ind}_{Z_{(\emptyset, (a_j))}}^{B_{a_j}} \varphi_{(\emptyset, (a_j))}. \end{aligned}$$

Hence the representations agree on each commuting factor of H_α , and therefore must agree on all of H_α . $\square$

Theorem 6.26 *As B_n -representations,*

$$\mathcal{G}_{(\lambda^+, \lambda^-)} = \text{Ind}_{Z_{(\lambda^+, \lambda^-)}}^{B_n} \varphi_{(\lambda^+, \lambda^-)},$$

and therefore

$$\mathcal{G}_{(\lambda^+, \lambda^-)} \cong G_{(\lambda^+, \lambda^-)}.$$

Proof By construction,

$$\text{Ind}_{H_\alpha}^{B_n} \mathcal{G}_\alpha = \mathcal{G}_{(\lambda^+, \lambda^-)}.$$

By transitivity of induction and Lemma 6.25,

$$\mathrm{Ind}_{Z(\lambda^+, \lambda^-)}^{B_n} \varphi_{(\lambda^+, \lambda^-)} = \mathrm{Ind}_{H_\alpha}^{B_n} \left(\mathrm{Ind}_{Z_\lambda}^{H_\alpha} \varphi_{(\lambda^+, \lambda^-)} \right) = \mathrm{Ind}_{H_\alpha}^{B_n} \mathcal{G}_\alpha = \mathcal{G}_{(\lambda^+, \lambda^-)}.$$

□

6.3 Consequences

Theorem 6.26 has several implications for the representations generated by the idempotents $\mathfrak{g}_k$,

$$G_n^{(k)} = \mathfrak{g}_k \mathbb{Q}[B_n].$$

Theorem 6.27 *There is a B_n -representation isomorphism*

$$G_n^{(n-k)} \cong \mathcal{G}_k = H^{2k} \mathcal{Z}_n^3.$$

Proof Note that

$$\mathfrak{g}_{n-k} = \sum_{\substack{(\lambda^+, \lambda^-) \\ \ell(\lambda^+) = n-k}} \mathfrak{g}_{(\lambda^+, \lambda^-)}$$

and

$$\mathcal{G}_k = \bigoplus_{\ell=0}^k \mathcal{G}_{k,\ell} = \bigoplus_{\substack{(\alpha^+, \alpha^-) \in \Lambda_{[n]}^\pm \\ \ell(\alpha^+) = n-k}} \mathcal{G}_\alpha.$$

The claim follows by applying Theorem 6.26. □

As a further consequence, this implies that the $G_n^{(n-k)}$ have a lift to B_{n+1} .

Theorem 6.28 *The representations $G_n^{(k)}$ lift to B_{n+1} , where they are described by $H^{2k} \mathcal{Y}_{n+1}^3$.*

Proof By Corollary 4.8, $\mathcal{G}_k \cong H^{2k} \mathcal{Z}_n^3$, which lifts to $H^{2k} \mathcal{Y}_{n+1}^3$. □

6.4 Further questions

We conclude with some lingering questions that our investigation has brought to light:

- (1) In Type A, the Whitehouse representations have the form $\mathbb{Q}[S_{n+1}] f_{n+1}^{(k)}$ for certain idempotents $f_{n+1}^{(k)} \in \mathbb{Q}[S_{n+1}]$. Is there a family of idempotents in $\mathbb{Q}[B_{n+1}]$ that generate representations isomorphic to the graded pieces of $H^* \mathcal{Y}_{n+1}^3$?
- (2) The Type A Eulerian idempotents generate a commutative subalgebra of $\Sigma[S_n]$ spanned by elements with the same descent number (e.g. descent set size). The idempotents $\mathfrak{g}_k$ also generate a commutative subalgebra of $\Sigma'[B_n]$. Is there a combinatorial description of this subalgebra?
- (3) The Whitehouse representation $F_{n+1}^{(0)}$ has other combinatorial-topological interpretations, for instance related to the homology of the space of trees by work of Robinson–Whitehouse [43], as well as the homology of subposets of the partition lattice by work of Sundaram [47]. Do the Type B lifts have analogous interpretations?
- (4) Can the results of this paper be extended to the complex reflection groups $G(r, 1, n) \cong \mathbb{Z}_r \wr S_n$?

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