Bijective Proofs of Monk's rule for Schubert and Double Schubert Polynomials with Bumpless Pipe Dreams

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Submitted: Jan 27, 2023; Accepted: May 19, 2023; Published: Jul 14, 2023

Abstract

We give bijective proofs of Monk's rule for Schubert and double Schubert polynomials computed with bumpless pipe dreams. In particular, they specialize to bijective proofs of transition and cotransition formulas of Schubert and double Schubert polynomials, which can be used to establish bijections with ordinary pipe dreams.

Mathematics Subject Classifications: 05E05, 05E14

1 Introduction

Bumpless pipe dreams are introduced in the context of back stable Schubert calculus by Lam, Lee, and Shimozono [LLS21]. In that paper, the authors introduced bumpless pipe dream polynomials and proved that they agree with double Schubert polynomials. Subsequently, Weigandt [Wei21] expressed Lascoux's transition formula with bumpless pipe dream polynomials and gave a bijective proof with bumpless pipe dreams. In a recent paper, Knutson [Knu22] gave several proofs of the cotransition formula of double Schubert polynomials, including a combinatorial proof with ordinary pipe dreams. Both transition and cotransition formulas are specializations of (an equivalent formulation of) Monk's rule for double Schubert polynomials, which is an expansion formula of the product of a linear double Schubert polynomial and a double Schubert polynomial. The original Monk's rule is a geometric version for single Schubert polynomials, studied first in [Mon59]. A combinatorial proof of it with ordinary pipe dreams (called RC-graphs there) is given in [BB93]. In this paper, we give a new bijective proof of Monk's rule for single Schubert polynomials with bumpless pipe dreams, and show that a slight modification of the construction gives us a bijective proof of Monk's rule for double Schubert polynomials using decorated bumpless pipe dreams, which are bumpless pipe dreams with a binary label on each blank tile. Combinatorial proofs of Monk's rule for double Schubert polynomials

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 https://doi.org/10.37236/11824

were not known before. We also remark that with the cotransition bijections on bumpless pipe dreams, together with similar known results on ordinary pipe dreams, one can establish bijections between ordinary pipe dreams and bumpless pipe dreams. In [GH23], it is shown that these inductive bijections all agree with a single direct bijection.

Definition 1. A (reduced) **bumpless pipe dream** is a tiling of the $n \times n$ grid with the six kinds of tiles shown below

such that

(a) there are n pipes total,

(b) travelling from south to east, each pipe starts vertically at the south edge of the grid, and ends horizontally at the east edge of the grid, and

(c) no two pipes cross twice.

Condition (c) is the reducedness condition. In this paper we only consider reduced bumpless pipe dreams. For convenience, we call these tiles **r-tile**, **j-tile**, **"+"-tile**, **blank tile, "−"-tile**, and **"|"-tile**. The term "bumpless" comes from the fact that the tiling disallows the **bump tile**  .

We index the tiles in a bumpless pipe dream with *matrix coordinates*. Given a bumpless pipe dream, one can read off a permutation by labeling the pipes from 1 to n along the south edge, follow the pipes from south to east, and read the labels top-down along the east edge. Given a permutation $\pi \in S_n$, we denote the set of bumpless pipe dreams associated to π by $\mathrm{BPD}(\pi)$.

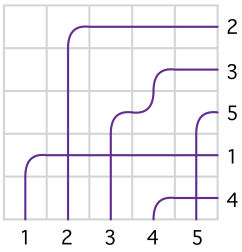

Figure 1: A bumpless pipe dream for $\pi = 23514$

For example, in Figure 1, the j-tile at $(3, 4)$ belongs to pipe $\pi(2) = 3$.

Definition 2. For $D \in \mathrm{BPD}(\pi)$, let $blank(D) \subseteq [n] \times [n]$ denote the set of blank tiles in the bumpless pipe dream D. For $\pi \in S_n$, let

$$\mathfrak{S}_\pi(\mathbf{x}, -\mathbf{y}) := \sum_{D \in \mathrm{BPD}(\pi)} \prod_{(i,j) \in blank(D)} (x_i - y_j) \in \mathbb{Z}[x_1, \cdots, x_n, y_1, \cdots, y_n].$$

Lam, Lee, and Shimozono showed that that $\mathfrak{S}_\pi(\mathbf{x}, -\mathbf{y})$ is the double Schubert polynomial for $\pi \in S_n$ [LLS21, Theorem 5.13]. Setting all the y variables to 0, we get the expression for single Schubert polynomials

$$\mathfrak{S}_\pi(\mathbf{x}) := \sum_{D \in \mathrm{BPD}(\pi)} \prod_{(i,j) \in blank(D)} x_i \in \mathbb{Z}[x_1, \cdots, x_n].$$

2 A Bijective Proof of Monk's Rule with Bumpless Pipe Dreams

Theorem 3 (Monk's rule)**.** *Let $\pi \in S_n$, $1 \leqslant \alpha < n$, such that there exists some $l > \alpha$ such that $\pi\, t_{\alpha,l} \gtrdot \pi$, where $\gtrdot$ denotes the covering relation in Bruhat order, and $t_{a,b}$ denotes transposition of a and b in S_n. Then*

$$\mathfrak{S}_\alpha(\mathbf{x})\mathfrak{S}_\pi(\mathbf{x}) = \sum_{\substack{k \leqslant \alpha < l \\ \pi\, t_{k,l} \gtrdot \pi}} \mathfrak{S}_{\pi\, t_{k,l}}(\mathbf{x}),$$

where $\mathfrak{S}_\alpha := \mathfrak{S}_{s_\alpha}$.

Subtracting $\mathfrak{S}_{\alpha-1}(\mathbf{x})\mathfrak{S}_\pi(\mathbf{x})$ from the equality and rearranging, we get

$$x_\alpha \mathfrak{S}_\pi(\mathbf{x}) + \sum_{\substack{k < \alpha \\ \pi\, t_{k,\alpha} \gtrdot \pi}} \mathfrak{S}_{\pi\, t_{k,\alpha}}(\mathbf{x}) = \sum_{\substack{\alpha < l \\ \pi\, t_{\alpha,l} \gtrdot \pi}} \mathfrak{S}_{\pi\, t_{\alpha,l}}(\mathbf{x}) \tag{1}$$

Remark 4. Note that we can remove the conditions on α and the existence of l such that $\pi\, t_{\alpha,l} \gtrdot \pi$ if we consider $\pi \in S_\infty := \bigcup_n S_n$, and this is how the rule is usually stated. For convenience of our combinatorial proofs we choose to work with the version stated for $\pi \in S_n$, but this does not lose the level of generality.

The goal of this section is to give a bijective proof of formula (1) with bumpless pipe dreams, as stated in the following theorem.

Theorem 5. *Given $\pi \in S_n$ and $1 \leqslant \alpha < n$ such that there exists $l > \alpha$ where $\pi\, t_{\alpha,l} \gtrdot \pi$, there exists a bijection*

$$\Phi_\pi : \mathrm{BPD}(\pi) \sqcup \coprod_{\substack{k < \alpha \\ \pi\, t_{k,\alpha} \gtrdot \pi}} \mathrm{BPD}(\pi\, t_{k,\alpha}) \longrightarrow \coprod_{\substack{\alpha < l \\ \pi\, t_{l,\alpha} \gtrdot \pi}} \mathrm{BPD}(\pi\, t_{l,\alpha}).$$

such that for any $D \in \mathrm{BPD}(\pi)$, the number of blank tiles on each row other than row α is preserved under the map, the number of blank tiles on row α increases by 1, and for any $D \in \bigcup_{\substack{k < \alpha \\ \pi\, t_{k,\alpha} \gtrdot \pi}} \mathrm{BPD}(\pi\, t_{k,\alpha})$ the number of blank tiles on each row is preserved under the map.

We start by preparing a few technical lemmas. In [LLS21, Section 5.2], the authors defined **droop moves** on bumpless pipe dreams (see e.g., Figure 2). We use the same language here. Define an **almost bumpless pipe dream** of π at (i, j) by allowing a

bumpless pipe dream diagram to have exactly one bump tile at position (i, j). (Double crossing of two pipes is still not allowed.) Note that an almost bumpless pipe dream may be created from a bumpless pipe dream by drooping a pipe into an r-tile (or undrooping into a j-tile), or replacing a "+"-tile with a bump tile without creating a double crossing. We also introduce the terminology **r/j-shaped turn** to refer to the corresponding pipe segments in an r/j-tile or bump tile.

Lemma 6. *Let (i, j) be the position of an r-shaped turn of pipe $p = \pi(x)$. If there exists $y > x$ such that $\pi\, t_{x,y} \gtrdot \pi$, then there exist $a, b > 0$ such that $(i, j+b)$ and $(i+a, j)$ are not "+"-tiles. Pick the smallest such possible a, b, then p is allowed to droop into $(i+a, j+b)$* with the possibility of creating a bump in $(i + a, j + b)$ (but not a double crossing).

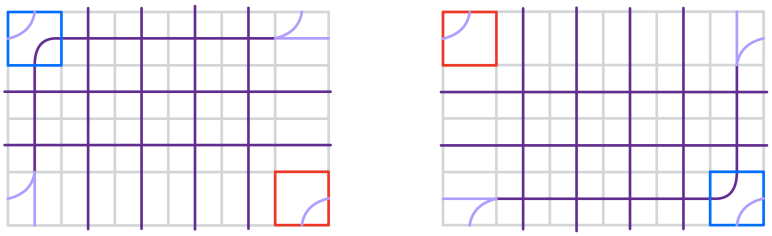

Figure 2: Droop to the closest tile as described in Lemma 6 (lighter color indicates possibilities). Here the top-left tile is at position (i, j) and the bottom-right tile is at position $(i + a, j + b)$.

Proof. Suppose for all $j' > j$, (i, j') is a "+"-tile. Then since all pipes need to exit from the east edge, the only way to fill the region (i', j') with $i' \geqslant i$, $j' \geqslant j$, $(i, j) \neq (i', j')$ is with "+"-tiles. This implies that there is no $y > x$ such that $\pi\, t_{x,y} \gtrdot \pi$. The same reasoning applies if for all $i' > i$, (i', j) is a "+"-tile.

Now pick the smallest a, b as stated in the lemma. Observe that in this case $(i, j + b)$ is either a "−"-tile or a j-tile, $(i + a, j)$ is either a "|"-tile or a j-tile, and all tiles (i', j') with $i < i' < i + a$ and $j < j' < j + b$ must be "+"-tiles. This means that all $(i', j + b)$ for $i < i' < i + a$ must be "−"-tiles, and all $(i + a, j')$ for $j < j' < j + b$ must be "|"-tiles. Therefore, the tile at $(i + a, j + b)$ has a "−"-tile above and a "|"-tile to the left, so it can only be a blank or r-tile. It is then easy to see p may droop into $(i + a, j + b)$, with the possibility of creating a bump but not a double crossing. □

Intuitively, Lemma 6 is about finding the closest tile an r-shaped corner can droop into. It is not hard to see that the droop move described in this lemma has an inverse operation.

Lemma 7. *Let (i, j) be the position of a j-shaped turn of pipe $p = \pi(x)$. Pick the largest $a, b > 0$ such that the tiles on row i strictly between $(i, j - b)$ and (i, j) are "+"-tiles and the tiles on column j strictly between $(i - a, j)$ and (i, j) are "+"-tiles. Then p is allowed to undroop to $(i - a, j - b)$, with the possibility of creating a bump in $(i - a, j - b)$.*

Proof. Note that a, b always exist since bumpless pipe dreams cannot have crosses on the north or west border. The rest of the proof is symmetric to the proof of the second half of Lemma 6. □

Lemma 8. *Suppose $p = \pi(x)$ and $q = \pi(y)$ are two pipes that cross once and bump once, and that the j-shaped corner in the bump tile belongs to p. If we swap the positions of the cross and the bump, then in the new bump tile, the r-shaped turn belongs to p. See Figure 3.*

Proof. Suppose p and q cross at (i, j) before the swap. Consider the pipes travelling from south to east. If the bump is after the cross, we must have $p = \pi(x) > q = \pi(y)$ and $x < y$, namely the "|" in the cross at (i, j) must belong to p. After the swap, p still enters from the bottom of (i, j), and therefore it makes an r-shaped turn. If the bump is before the cross, we must have $q = \pi(y) > p = \pi(x)$ and $y < x$, namely the "−" in the cross at (i, j) must belong to p. After the swap, p still exits from the right, and therefore makes an r-shaped turn at (i, j). □

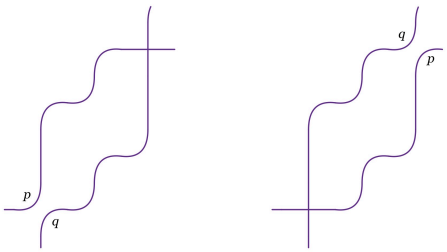

Figure 3: A case of bump-cross swap

Again, we have the opposite version of this statement. We omit the proof.

Lemma 9. *Suppose $p = \pi(x)$ and $q = \pi(y)$ are two pipes that cross once and bump once, and that the r-shaped corner in the bump tile belongs to p. If we swap the positions of the cross and the bump, then in the new bump tile, the j-shaped turn belongs to p.*

Figure 4 shows a walk-through of the algorithm below that inserts a blank tile on row 4 for a bumpless pipe dream. The reader is invited to guess the the algorithm before reading the description. The shaded square in each diagram denotes the r-tile at which a blank tile is about to be inserted, or a bump tile that needs to be resolved.

We now describe an algorithm for inserting a blank tile at position (i, j) where there is an r-tile or, as will be made clear below, to resolve a conflict where there is temporarily a bump tile. Suppose this r-shaped corner belongs to pipe $p = \pi(x)$. Let $(i + a, j + b)$, $a, b > 0$, be the tile southeast to (i, j) such that the tiles on the ith row strictly between (i, j) and $(i, j + b)$ are all "+"-tiles, and the tiles in the jth column between (i, j) and $(i + a, j)$ are all "+"-tiles. By Lemma 6, p may droop into $(i + a, j + b)$ with the possibility of creating a bump. We let the pipe p droop into $(i + a, j + b)$. If $(i + a, j + b)$ used to be a blank tile (now a "j"), we have newly occupied a blank tile on row $i + a$, so we find the

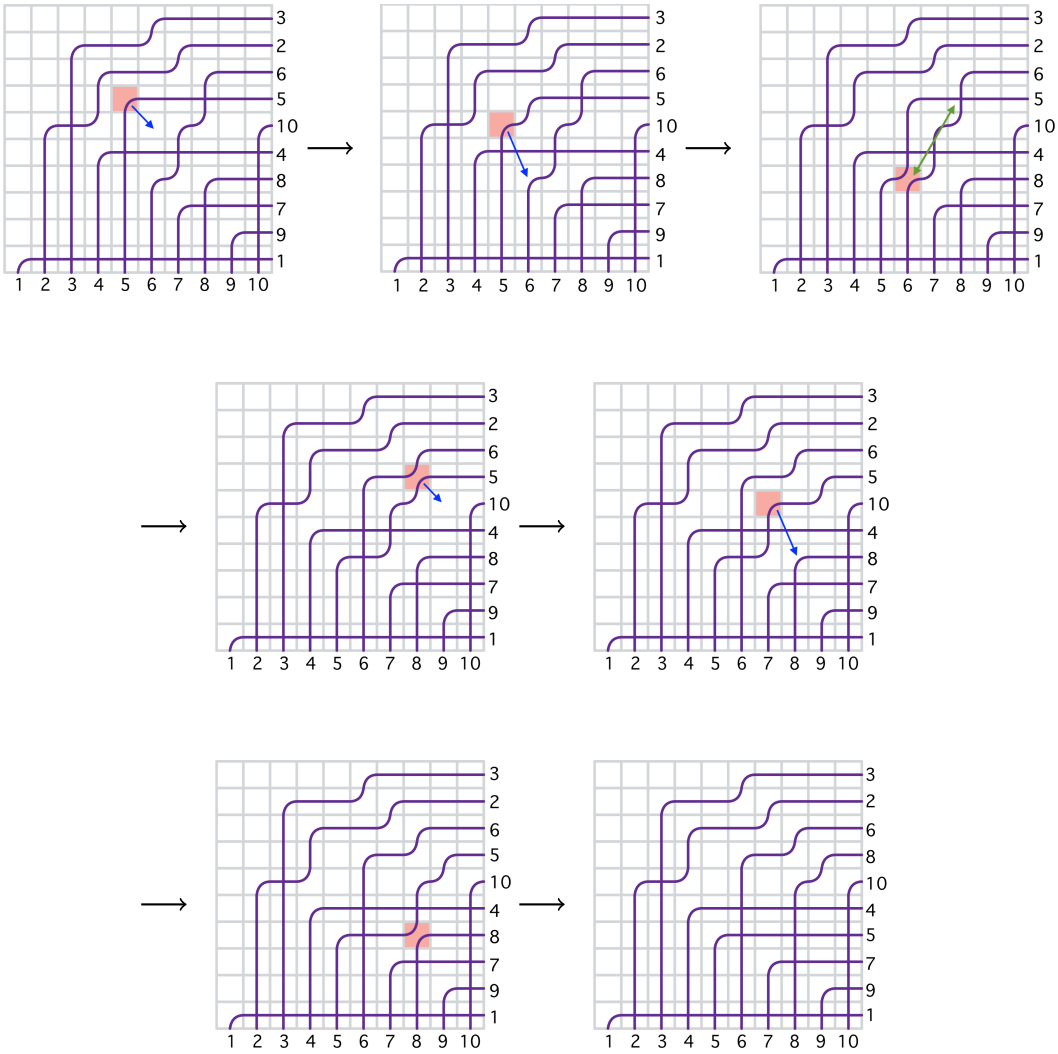

Figure 4: Insertion of a blank tile at $(i, j) = (4, 5)$ for a BPD of $\pi = [3, 2, 6, 5, 10, 4, 8, 7, 9, 1]$

r-tile on row $i + a$ that belongs to p and repeat the same algorithm for inserting a blank tile at an r-tile, as before. (Note that such an r-tile always exists.) The other possibility is that $(i + a, j + b)$ used to be an r-tile (now a bump). Suppose the bump is with pipe $q = \pi(y)$. If $\pi\, t_{x,y} \gtrdot \pi$, or in other words, p and q do not cross, we replace the bump tile with a "+"-tile, and terminate the algorithm. If p and q cross each other, we find the tile (i', j') where the crossing is, replace the existing bump tile in $(i + a, j + b)$ with a cross, and replace the cross in (i', j') with a bump tile. After this, by Lemma 8, the r-shaped turn in (i', j') must belong to p. We resolve this bump by going back to the beginning of the algorithm and repeat the process.

We give the pseudocode of the algorithm below. Let $\mathrm{BPD}'_{i,j}(\pi)$ denote the set of bumpless pipe dreams of π that have an r-tile at (i, j), plus the *almost* bumpless pipe

dreams of π which have exactly one bump at position (i, j). For $D \in \mathrm{BPD}'_{i,j}(\pi)$, let $D(m, n)$ denote the tile in D at position (m, n).

Algorithm 1:

```
1  insert_blank_or_resolve_bump_at_r(D,i,j)
       Data: D ∈ BPD'_{i,j}(π), where the r-shaped turn in D(i,j) belongs to pipe p = π(x),
             which satisfies ∃y > x such that π t_{x,y} ⋗ π.
2      a, b ← 1, 1;
3      while D(i + a, j) = "+" do a ← a + 1;
4      while D(i, j + b) = "+" do b ← b + 1;
5      Droop p into (i + a, j + b);
6      if D(i + a, j + b) = "j" then
7          (i + a, j') ← position of r-tile of p on row i + a;
8          insert_blank_or_resolve_bump_at_r (D, i + a, j')
9      else
10         Let q = π(y) be the pipe that p bumps into at (i + a, j + b);
11         if π t_{x,y} ⋗ π then
12             D(i + a, j + b) ← "+";
13             return D
14         else
15             (i', j') ← position of existing cross of p and q;
16             D(i', j') ← bump tile;
17             D(i + a, j + b) ← "+";
18             insert_blank_or_resolve_bump_at_r (D, i', j')
```

Proposition 10. *The algorithm* insert_blank_or_resolve_bump_at_r *terminates and produces a bumpless pipe dream of $\pi\, t_{x,y}$ for some $y > x$ such that $\pi\, t_{x,y} \gtrdot \pi$.*

Proof. The well-definedness of the algorithm follows from Lemmas 6 and 8, as explained in the construction. For termination, observe that we modify the pipe p either by droop moves or cross-bump-swap moves. The area under the pipe p (as a curve) in the $n \times n$ square strictly decreases after each of these moves. Since the modification to the diagram in each iteration of the function before the final modification at line 12 right before it returns preserves the property that the diagram is a BPD or an almost BPD of π, there is a finite set of possible areas under the pipe p. Therefore, the algorithm must terminate, and by the terminating condition, p must have bumped into q after drooping, therefore occupying the j-shaped corner at this bump. This means $p < q$, so $x < y$. Therefore, it produces a bumpless pipe dream of $\pi\, t_{x,y}$ for some $y > x$ such that $\pi\, t_{x,y} \gtrdot \pi$. □

Now, the algorithm has an opposite version which inserts a blank tile at a position where there is a j-tile, or resolves a conflict where there is a bump tile, in the opposite direction.

Algorithm 2:

```
1  insert_blank_or_resolve_bump_at_j(D,i,j)
       Data: D ∈ BPD'_{i,j}(π), where the j-shaped turn in D(i,j) belongs to pipe p = π(x)
2      a, b ← 1, 1;
3      while D(i − a, j) = "+" do a ← a + 1;
4      while D(i, j − b) = "+" do b ← b + 1;
5      Undroop p into (i − a, j − b);
6      if D(i − a, j − b) = "r" then
7          if ∀j' > j − b, D(i − a, j') does not have a j-tile then
8              return D
9          (i − a, j') ← position of j-tile of p on row i − a;
10         insert_blank_or_resolve_bump_at_j (D, i − a, j')
11     else
12         Let q = π(y) be the pipe that p bumps into at (i − a, j − b);
13         if π t_{y,x} ⋗ π then
14             D(i − a, j − b) ← "+";
15             return D
16         else
17             (i', j') ← position of existing cross of p and q;
18             D(i', j') ← bump tile;
19             D(i + a, j + b) ← "+";
20             insert_blank_or_resolve_bump_at_j (D, i', j')
```

Proposition 11. *The algorithm* insert_blank_or_resolve_bump_at_j *terminates and produces a either a bumpless pipe dream of $\pi\, t_{y,x}$ for some $y < x$, or a bumpless pipe dream of π with one fewer blank tile on row x, compared to the input.*

Proof. By Lemmas 7 and 9, this algorithm is well-defined. By similar reasoning as in Proposition 10, this algorithm terminates.

If it terminates by triggering the condition on line 7, p only turns once on row $i - a$, so this must also be the row in which p exits, which means $i - a = x$. This entire process does not change the permutation, so the output is a bumpless pipe dream of π. The last undrooping step ate a blank tile on row x and did not give it back, so this bumpless pipe dream has one fewer blank tile on row x.

If the algorithm terminates by triggering the condition on line 13, p must have bumped into q after undrooping, therefore occupying the r-shaped corner at this bump. This means $p > q$, and therefore $x > y$. □

We are now ready to describe the bijection Φ_π.

Proof of Theorem 5. Let $D \in \mathrm{BPD}(\pi)$, $p = \pi(\alpha)$. Pipe p exits on row α. Let (α, j) be the position of the r-tile of p on row α. We run the insert_blank_or_resolve_bump_at_r function on (D, α, j). By Proposition 10, the output is a bumpless pipe dream $D' \in \mathrm{BPD}(\pi\, t_{\alpha,l})$ for some $l > \alpha$ and $\pi\, t_{\alpha,l} \gtrdot \pi$. By construction of the algorithm, the number of blank tiles on each row stays the same, except for row α where D' has one more blank tile than D.

Let $D \in \mathrm{BPD}(\pi\, t_{k,\alpha})$ for some $k < \alpha$ such that $\pi\, t_{k,\alpha} \gtrdot \pi$. Let $q = \pi\, t_{k,\alpha}(\alpha)$ and $p = \pi\, t_{k,\alpha}(k)$ be pipes. Since $\pi\, t_{k,\alpha} \gtrdot \pi$, p and q must cross. Let (i, j) be the position of the tile where they cross. Notice that since $q < p$, the "|" segment in this cross must belong to p. We now replace this "+" tile with a bump tile. Notice that by doing so we have uncrossed p and q, therefore creating an almost bumpless pipe dream, $D' \in \mathrm{BPD}'_{i,j}(\pi)$. Also, now $p = \pi(\alpha)$, $q = \pi(k)$, and the r-shaped corner in this new bump tile belongs to pipe p.

We run the function insert_blank_or_resolve_bump_at_r on (D', i, j). Again by Proposition 10, the output is a bumpless pipe dream $D'' \in \mathrm{BPD}(\pi\, t_{\alpha,l})$ for some $l > \alpha$ and $\pi\, t_{\alpha,l} \gtrdot \pi$. The number of blank tiles on each row remains constant during this process.

To go the opposite direction, let $E \in \mathrm{BPD}(\pi\, t_{l,\alpha})$ for some $\alpha < l$ such that $\pi\, t_{\alpha,l} \gtrdot \pi$. Let $q = \pi\, t_{\alpha,l}(\alpha)$ and $p = \pi\, t_{\alpha,l}(l)$ be pipes. Since $\pi\, t_{\alpha,l} \gtrdot \pi$, p and q must cross. Let (i, j) be the position of the tile where they cross. Notice that since $q > p$, the "|" segment in this cross must belong to q. We now replace this "+" tile with a bump tile. Again this uncrosses p and q, creating an almost bumpless pipe dream $E' \in \mathrm{BPD}'_{i,j}(\pi)$, and making $p = \pi(\alpha)$ and $q = \pi(l)$. The j-shaped corner in this new bump tile belongs to p.

We run the function insert_blank_or_resolve_bump_at_j on (E', i, j). By Proposition 11, there are two possible outcomes. The output is either some $E'' \in \mathrm{BPD}(\pi\, t_{k,\alpha})$ for some $k < \alpha$, in which case the number of blank tiles on each row stays invariant, or some $E'' \in \mathrm{BPD}(\pi)$ that occupies one more blank tile on row α as compared to E'.

By the construction of the two algorithms, it is easy to see that the processes described above are inverses of each other, giving a bijection between the two sets of bumpless pipe dreams. □

3 Monk's Rule for Double Schubert Polynomials

The version of Monk's rule for double Schubert polynomials states that

Theorem 12 (Monk's rule for double Schuberts)**.** *Let $\pi \in S_n$, $1 \leqslant \alpha < n$, such that there exists some $l > \alpha$ with $\pi\, t_{\alpha,l} \gtrdot \pi$. Then*

$$\mathfrak{S}_\alpha(\mathbf{x}, -\mathbf{y})\mathfrak{S}_\pi(\mathbf{x}, -\mathbf{y}) = \sum_{\substack{k \leqslant \alpha < l \\ \pi\, t_{k,l} \gtrdot \pi}} \mathfrak{S}_{\pi\, t_{k,l}}(\mathbf{x}, -\mathbf{y}) + \sum_{i=1}^{\alpha} (y_{\pi(i)} - y_i)\mathfrak{S}_\pi(\mathbf{x}, -\mathbf{y}).$$

Computing $\mathfrak{S}_\alpha(\mathbf{x}, -\mathbf{y})\mathfrak{S}_\pi(\mathbf{x}, -\mathbf{y}) - \mathfrak{S}_{\alpha-1}(\mathbf{x}, -\mathbf{y})\mathfrak{S}_\pi(\mathbf{x}, -\mathbf{y})$ and rearranging terms, we find

$$(x_\alpha - y_{\pi(\alpha)})\mathfrak{S}_\pi(\mathbf{x}, -\mathbf{y}) + \sum_{\substack{k < \alpha \\ \pi\, t_{k,\alpha} \gtrdot \pi}} \mathfrak{S}_{\pi\, t_{k,\alpha}}(\mathbf{x}, -\mathbf{y}) = \sum_{\substack{\alpha < l \\ \pi\, t_{\alpha,l} \gtrdot \pi}} \mathfrak{S}_{\pi\, t_{\alpha,l}}(\mathbf{x}, -\mathbf{y}). \tag{2}$$

We give a bijective proof of formula (2) in this section.

We will first need to introduce decorations on blank tiles of bumpless pipe dreams. A **decorated bumpless pipe dream** of π is a bumpless pipe dream together with a decoration on the blank tiles, where each blank tile must be decorated with either an x

or a $-\mathsf{y}$ label. Let $\widetilde{\mathrm{BPD}}(\pi)$ be the set of decorated bumpless pipe dreams of π. In other words,

$$\widetilde{\mathrm{BPD}}(\pi) = \{(D, f) : D \in \mathrm{BPD}(\pi), f : blank(D) \to \{\mathsf{x}, -\mathsf{y}\}\}.$$

Note that $|\widetilde{\mathrm{BPD}}(\pi)| = |\,\mathrm{BPD}(\pi)| \times 2^{|blank(D)|}$ for any $D \in \mathrm{BPD}(\pi)$. Expand the double Schubert polynomial as a sum of monomials, we get the following expression

$$\mathfrak{S}_\pi(\mathbf{x}, -\mathbf{y}) = \sum_{(D,f)\in\widetilde{\mathrm{BPD}}(\pi)} \mathrm{mon}(D, f),$$

where

$$\mathrm{mon}(D, f) = \prod_{\substack{(i,j)\in blank(D)\\ f(i,j)=\mathsf{x}}} x_i \prod_{\substack{(i,j)\in blank(D)\\ f(i,j)=-\mathsf{y}}} (-y_j).$$

Similarly, we define

$$\widetilde{\mathrm{BPD}}'_{i,j}(\pi) := \{(D, f) : D \in \mathrm{BPD}'_{i,j}(\pi), f : blank(D) \to \{\mathsf{x}, -\mathsf{y}\}\}.$$

The combinatorial version of formula (2) is stated as follows.

Theorem 13. *Let $\pi \in S_n$, $1 \leqslant \alpha < n$, such that there exists some $l > \alpha$ with $\pi\, t_{\alpha,l} \gtrdot \pi$. Then there exists a bijection*

$$\widetilde{\Phi}_\pi : (\{\mathsf{x}, -\mathsf{y}\} \times \widetilde{\mathrm{BPD}}(\pi)) \sqcup \coprod_{\substack{k<\alpha\\ \pi\, t_{k,\alpha}\gtrdot\pi}} \widetilde{\mathrm{BPD}}(\pi\, t_{k,\alpha}) \longrightarrow \coprod_{\substack{\alpha<l\\ \pi\, t_{l,\alpha}\gtrdot\pi}} \widetilde{\mathrm{BPD}}(\pi\, t_{l,\alpha}),$$

such that for any $(D, f) \in \widetilde{\mathrm{BPD}}(\pi))$,

$$\mathrm{mon}(\widetilde{\Phi}_\pi(\mathsf{x}, D, f)) = x_\alpha\, \mathrm{mon}(D, f),$$

$$\mathrm{mon}(\widetilde{\Phi}_\pi(-\mathsf{y}, D, f)) = -y_{\pi(\alpha)}\, \mathrm{mon}(D, f),$$

and for any $k < \alpha, \pi\, t_{k,\alpha} \gtrdot \pi$, $(D, f) \in \widetilde{\mathrm{BPD}}(\pi\, t_{k,\alpha})$,

$$\mathrm{mon}(\widetilde{\Phi}_\pi(D, f)) = \mathrm{mon}(D, f).$$

Proof. We will modify the algorithms given in the previous section slightly to get the bijection $\widetilde{\Phi}_\pi$. The algorithms in both directions will now take as input $(D, f) \in \widetilde{\mathrm{BPD}}'_{i,j}(\pi)$, *as well as a label $u \in \{\mathsf{x}, -\mathsf{y}\}$ in the case when $D(i, j)$ is an r-tile in the forward direction, and when $D(i, j)$ is a j-tile in the backward direction.* The outputs will also be decorated bumpless pipe dreams, as well as a label $v \in \{\mathsf{x}, -\mathsf{y}\}$ in certain cases.

In Algorithm 1, before the droop of p on line 5, if $D(i + a, j + b)$ is a blank tile (in which case the condition on line 6 will be true), we remember its label v. If the input $D(i, j) =$ "r", after the droop on line 5, $D(i, j)$ will become a blank tile. We decorate it with the label specified by input. Now instead of always choosing the position of the r-tile

of p on row $i+a$, we check the label v. If $v = \mathtt{x}$, we choose the position of the r-tile of p on row $i+a$ as before, but if $v = -\mathtt{y}$, we choose the position $(i', j+b)$ of the r-tile of p on column $j+b$. Note that this construction guarantees that if the input is (D, f) where $D(i,j) =$ "r" and label u, and the output is (D', f'), then $x_i \operatorname{mon}(D,f) = \operatorname{mon}(D',f')$ if $u = \mathtt{x}$, and $-y_j \operatorname{mon}(D,f) = \operatorname{mon}(D',f')$ if $u = -\mathtt{y}$. The pseudocode for this modification is the following snippet, and we replace lines 5–8 in Algorithm 1 with it.

```
1  if D(i+a, j+b) is blank then
2  |  v ← f(i+a, j+b)
3  Droop p into (i+a, j+b);
4  if D(i, j) is blank then f(i, j) ← u;
5  if D(i+a, j+b) = "j" then
6  |  if v = x then
7  |  |  (i+a, j') ← position of r-tile of p on row i+a;
8  |  |  insert_blank_or_resolve_bump_at_r(D, f, i+a, j', v)
9  |  else
10 |  |  (i', j+b) ← position of r-tile of p on column j+b;
11 |  |  insert_blank_or_resolve_bump_at_r(D, f, i', j+b, v)
```

(To be pedantic, on line 20 of Algorithm 1 we also need to pass the decoration as an argument, and the decoration must also be returned.)

Similarly, we replace lines 5–10 in Algorithm 2 with the following snippet.

```
1  if D(i−a, j−b) is blank then
2  |  v ← f(i−a, j−b)
3  Undroop p into (i−a, j−b);
4  if D(i, j) is blank then f(i, j) ← u;
5  if D(i−a, j−b) = "r" then
6  |  if v = x then
7  |  |  if ∀j' > j−b, D(i−a, j') does not have a j-tile then
8  |  |  |  return D, f, v
9  |  |  (i−a, j') ← position of r-tile of p on row i−a;
10 |  |  insert_blank_or_resolve_bump_at_j(D, f, i−a, j', v)
11 |  else
12 |  |  if ∀i' > i−a, D(i', j−b) does not have a j-tile then
13 |  |  |  return D, f, v
14 |  |  (i', j+b) ← position of r-tile of p on column j+b;
15 |  |  insert_blank_or_resolve_bump_at_j(D, f, i', j−b, v)
```

Note that if the condition on line 12 in the snippet above is triggered, then p only turns once on column $j-b$, which means that $j - b = \pi(x)$. The rest of the analysis for the modified algorithm is the same as before. □

Figure 5 shows an example for the same bumpless pipe dream as Figure 4, but with labelled tiles.

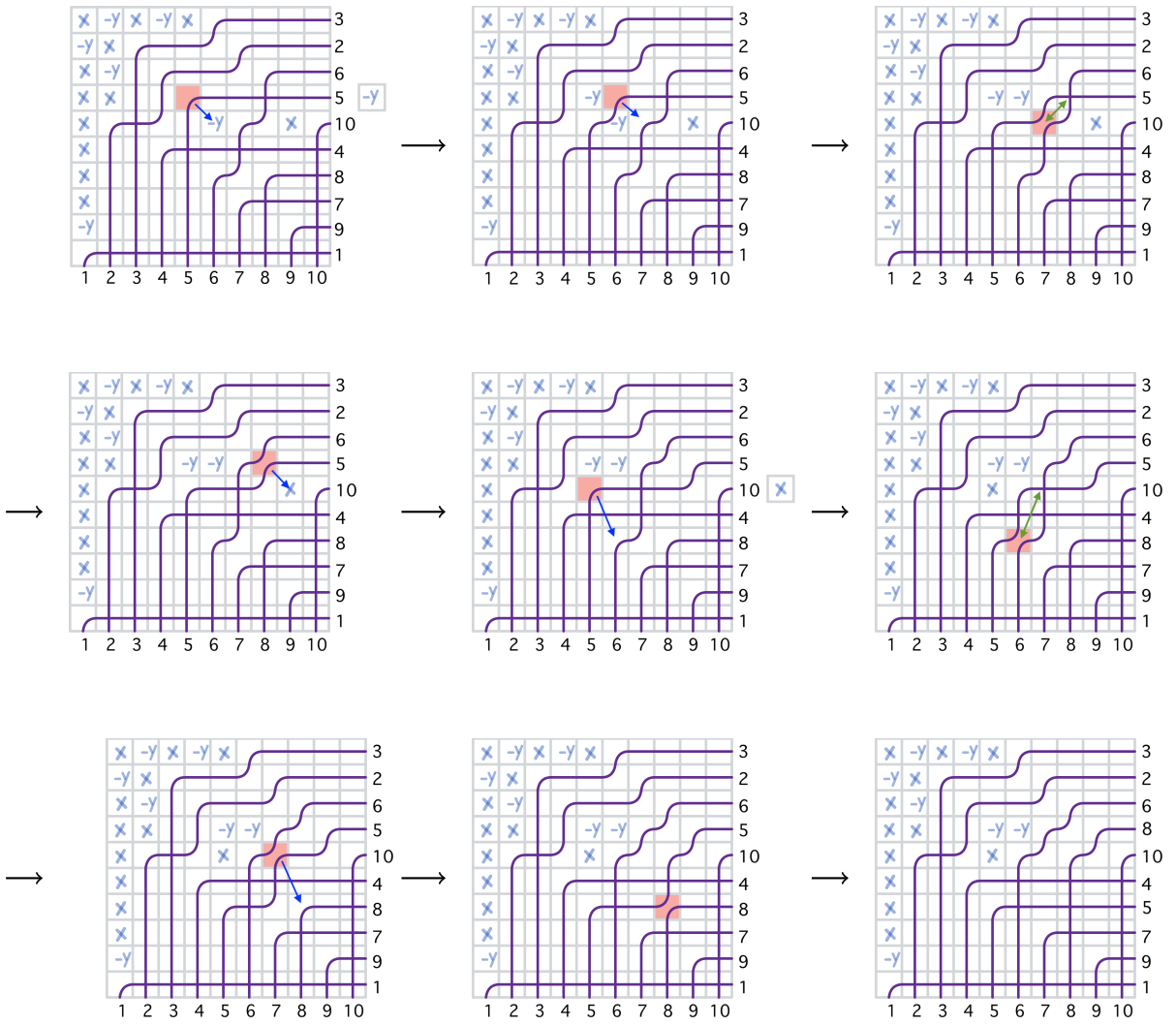

Figure 5: Insertion of a blank tile marked $-\mathsf{y}$ at $(i, j) = (4, 5)$ for a decorated BPD of $\pi = [3, 2, 6, 5, 10, 4, 8, 7, 9, 1]$

4 Transition and Cotransition Formulas

We discuss briefly the implication our results have on the transition and cotransition formulas of (double) Schubert polynomials. Transition and cotransition formulas are specializations of formula (2). If there is a unique $l > \alpha$ such that $\pi\, t_{\alpha,l} \gtrdot \pi$, namely if the right side of formula (2) only has one summand, we get the *transition formula* for $\mathfrak{S}_{\pi\, t_{\alpha,l}}$. In terms of combinatorial bijections, this is the simplest case because only a single droop/undroop move is required to go between the bijection, and each move only modifies four tiles locally. The details of this are given in [Wei21]. Therefore, to establish the transition formula alone for double Schubert polynomials, we do not need to consider decorated bumpless pipe dreams. Billey, Holroyd, and Young gave a bijective proof for transition with ordinary pipe dreams [BHY19]. There, the construction only works for

single Schubert polynomials.

If, on the other hand, there is no $k < \alpha$ such that $\pi\, t_{k,\alpha} \gtrdot \pi$, we get the *cotransition formula*. Unlike the transition formula, if we only work with bumpless pipe dreams without decorations, we can only get the version for single Schubert polynomials. This is analogous to the phenomenon in [BHY19]. On the other hand, in [Knu22] a simple bijective proof of cotransition for double Schubert polynomials is given with ordinary pipe dreams, which only requires changing one tile locally to go between the bijection. As a direct consequence, using the cotransition bijections for ordinary and bumpless pipe dreams, we get a bijection of ordinary and bumpless pipe dreams by reverse induction on the length of the permutation. This idea is similar to the approach in [FGS18] where a shape preserving bijection between reduced word tableaux for a permutation w and Edelman-Greene pipe dreams of w is constructed. In [GH23], a direct bijection between reduced pipe dreams and bumpless pipe dreams is given, and its canonical nature is justified by the fact that all inductive bijections based on the transition and cotrasition formulas agree with the direct bijection.

Acknowledgements

This work was supported by the National Science Foundation DMS-1439786 while the author was in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the Combinatorial Algebraic Geometry program.

The author would like to thank Allen Knutson for helpful discussion and comments on the draft of the paper, and Anna Weigandt for discussion of the transition formula.

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