

LOCAL WELL-POSEDNESS FOR THE BOLTZMANN EQUATION WITH VERY SOFT POTENTIAL AND POLYNOMIALLY DECAYING INITIAL DATA*

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Abstract. In this paper, we address the local well-posedness of the spatially inhomogeneous noncutoff Boltzmann equation when the initial data decays polynomially in the velocity variable. We consider the case of very soft potentials $\gamma + 2s < 0$. Our main result completes the picture for local well-posedness in this decay class by removing the restriction $\gamma + 2s > -3/2$ of previous works. Our approach is entirely based on the Carleman decomposition of the collision operator into a lower order term and an integro-differential operator similar to the fractional Laplacian. Interestingly, this yields a very short proof of local well-posedness when $\gamma \in (-3, 0]$ and $s \in (0, 1/2)$ in a weighted C^1 space that we include as well.

Key words. Boltzmann equation, local well-posedness, Carleman decomposition, slow decay, inhomogeneous, very soft potential

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1. Introduction. The Boltzmann equation is a kinetic equation arising in statistical physics. Its solution $f(t, x, v) \geq 0$ models the density of particles of a diffuse gas at time $t \in [0, T]$, at location $x \in \mathbb{T}^3$, and with velocity $v \in \mathbb{R}^3$. Roughly, each particle travels with a fixed velocity until a collision, at which time it takes on a new velocity chosen in a way compatible with physical laws. In this article, we focus on the noncutoff version of (1.1) that includes the physically realistic singularity at grazing collisions. The equation reads

$$(1.1) \quad \begin{cases} (\partial_t + v \cdot \nabla_x) f = Q(f, f) & \text{in } [0, T] \times \mathbb{T}^3 \times \mathbb{R}^3, \\ f(0, \cdot, \cdot) = f_{\text{in}} \geq 0 & \text{in } \mathbb{T}^3 \times \mathbb{R}^3. \end{cases}$$

The collision operator Q is defined by

$$Q(f, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) (f(v'_*) f(v') - f(v_*) f(v)) \, d\sigma \, dv_*,$$

where v and v_* are precollisional velocities and v' and v'_* are postcollisional velocities related by

$$v' = \frac{v + v_*}{2} + \sigma \frac{|v - v_*|}{2} \quad \text{and} \quad v'_* = \frac{v + v_*}{2} - \sigma \frac{|v - v_*|}{2}$$

and the collision kernel B is given by

$$B(v - v_*, \sigma) = |v - v_*|^\gamma \theta^{-2-2s} \tilde{b}(\cos \theta), \quad \text{where } \cos \theta = \sigma \cdot \frac{v - v_*}{|v - v_*|}, \gamma \in (-3, 1], s \in (0, 1),$$

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and $\tilde{b}$ is a positive bounded function. In this work, we are mostly interested in the regime of very soft potentials, that is, when $\gamma + 2s < 0$.

There are several active research directions regarding the well-posedness of the Boltzmann equation: global well-posedness in the spatially homogeneous setting (that is, x -independent), global well-posedness and regularity of weak solutions, global well-posedness and convergence of close-to-equilibrium solutions, and local well-posedness with large initial data. Here, we are interested in the local well-posedness of (1.1) with large initial data and, as such, leave it to other references to detail the extensive history of research into the first three categories (see, e.g., [3, 6, 9, 10, 12, 13, 16, 17, 18, 19, 20, 21, 22, 23, 25, 29, 36, 39, 40]).

Alexandre, Morimoto, Ukai, Xu, and Yang, often referred to by the acronym AMUXY, made the first serious progress on the local well-posedness theory for the (noncutoff) Boltzmann equation. In particular, in a sequence of seminal works, by deriving new estimates on the collision operator Q , they were able to establish local well-posedness under the condition that $e^{\alpha|v|^2} f_{\text{in}}$ is bounded in certain Sobolev-based spaces [2, 4, 5, 7, 8]. We note that the Gaussian decay plays a large role in their analysis to compensate for moment loss.

The first results weakening the Gaussian-decay condition on the initial data are due to Morimoto and Yang [37]. They established local well-posedness in an H^6 -based space under the assumptions that $\gamma \in (-3/2, 0]$ and $s \in (0, 1/2)$. This was later extended by Henderson, Snelson, and Tarfulea [28], who showed local well-posedness in an H^5 -based space under the assumption $s \in (0, 1)$ and $\max\{-3, -\frac{3}{2} - 2s\} < \gamma < 0$. Our goal, in the present work, is to remove the restriction $\gamma + 2s > -3/2$. In general, the larger $\gamma + 2s$ is, the more the decay of f at $|v| = +\infty$ is the issue, and the smaller (more negative) $\gamma + 2s$ is, the more regularity is the issue.

Our interest in establishing local well-posedness with initial data that is merely polynomially decaying is due to its relationship to the recent conditional regularity program initiated by Silvestre [38] and continued in collaboration with Imbert and Mouhot [30, 31, 32, 33, 34, 35] (see also a recent work of Alonso [11]). The goal of the program is to understand the regularity theory for the Boltzmann equation conditional to the mass, energy, and entropy densities

$$\begin{aligned} M(t, x) &= \int f(t, x, v) dv, & E(t, x) &= \int f(t, x, v) |v|^2 dv, \\ \text{and } H(t, x) &= \int f(t, x, v) \log f(t, x, v) dv, \end{aligned}$$

satisfying, uniformly in (t, x) ,

$$(1.2) \quad M, E, H \leq C \quad \text{and} \quad \frac{1}{C} \leq M \quad \text{for all } (t, x),$$

where C is a positive constant. When f is x -independent, it is well-known that these conditions are always satisfied. To date, Imbert, Mouhot, and Silvestre have developed a Harnack inequality and Schauder estimates, obtained a sharp lower bound on the tail behavior of f , and proved a propagation of polynomial upper bounds of f result, all of which depended only on the bounds in (1.2).

An upshot of the program of Imbert, Mouhot, and Silvestre is that, roughly, when a suitable local well-posedness result exists, solutions may be continued as long as (1.2) holds (see [32, section 1.1.2] and [28, Corollary 1.2]). In particular, the local well-posedness result must allow for polynomially decaying initial data as that type of

decay can be propagated forward in time depending only on the constant C in (1.2). As no such propagation-of-decay result exists for Gaussian decay, the classical results of AMUXY cannot be used. It is for this reason that it is important to develop the local well-posedness theory when f_{in} decays only polynomially.

Our main theorem removes the restriction of previous results [28, 37] that $\gamma + 2s > -3/2$, thereby completing the picture for local well-posedness with polynomially decaying initial data when $\gamma \in (-3, 0)$ and $s \in (0, 1)$. In order to state our result, we define the following two spaces: given $k, n, m \geq 0$ and $T > 0$, let

$$(1.3) \quad X^{k,n,m} = H^{k,n}(\mathbb{T}^3 \times \mathbb{R}^3) \cap L^{\infty,m}(\mathbb{T}^3 \times \mathbb{R}^3) \quad \text{and} \quad Y_T^{k,n,m} = L^\infty([0, T]; X^{k,n,m}).$$

For any $p \geq 1$, we use $L^{p,n}$ to refer to the space of functions g such that $\langle v \rangle^n g \in L^p$, where $\langle v \rangle^2 = 1 + |v|^2$. The weighted Sobolev space $H^{k,n}$ is defined analogously.

THEOREM 1.1. *Assume that $\gamma + 2s < 0$, $k \geq 5$, $n > 3/2$, and $m > M = M(k, n, \gamma, s)$ sufficiently large. Suppose $0 \leq f_{\text{in}} \in X^{k,n,m}$. Then there exists a time $T > 0$, depending only on $\|f_{\text{in}}\|_{X^{k,n,m}}$ as well as n, k, m, γ, s , and $\tilde{b}$, and a unique solution $f \in Y_T^{k,n,m} \cap C([0, T]; H_{x,v}^{k,n})$ of (1.1) such that $f \geq 0$ and*

$$\|f\|_{Y_T^{k,n,m}} \lesssim \|f_{\text{in}}\|_{X^{k,n,m}}.$$

As discussed above, an important motivation of Theorem 1.1 is to extend the continuation criterion for the Boltzmann equation to the very soft potentials range. While such a result does not directly follow from Theorem 1.1 and the Imbert–Mouhot–Silvestre regularity program (as these results only deal with the regime $\gamma + 2s \in [0, 2]$), it is likely a straightforward exercise after adding an additional assumption on the $L_{t,x}^\infty L_v^p$ norm of f to (1.2). Indeed, this has already been accomplished for the closely related Landau equation in [26] in the analogous parameter regime. An upshot of such a continuation criterion, were it established, is the ability to construct classical solutions from rough initial data as accomplished for the Landau equation [27]. These will be the subject of a future work.

As is typical for nonlinear equations, the main step in the proof of local well-posedness of (1.1) is to establish a priori estimates on solutions. In particular, this requires obtaining bounds on the collision operator Q as a bilinear form from and to various Banach spaces.

In order to explain the strategy and difficulties in obtaining such estimates, we discuss the restriction $\gamma + 2s > -3/2$ in [28]. This is inherited in the application of the estimates on the collision operator developed in [4]. For certain key estimates, AMUXY use Fourier analysis, which is most suited L^2 -based spaces. However, a major lesson from [38] is that one can, very roughly, think of Q as having the decomposition

$$(1.4) \quad Q(g, f) \sim \left(\int g(w) |v - w|^{\gamma+2s} dw \right) (-\Delta)^s f + \left(\int g(w) |v - w|^\gamma dw \right) f,$$

and one sees, after applying the Cauchy–Schwarz inequality, that the coefficient in front of $(-\Delta)^s$ is bounded using the (weighted) L^2 -norm of g only when $\gamma + 2s > -3/2$.

In view of the above, it is required to develop new estimates on the collision operator in spaces that are not L^2 -based. Our approach is to take advantage of the Carleman decomposition (see (2.1)), which views Q as the sum of an integro-differential operator similar to the fractional Laplacian and a lower order term. As this is a real space-based approach, it is possible, through intricate analysis, to obtain

estimates on $Q(g, f)$ in various spaces depending on both L^2 - and L^∞ -based norms. This allows us to circumvent the issues encountered in previous works.

Curiously, this approach leads to an extremely simple proof of local well-posedness in a weighted $C^{0,1}$ space when $s \in (0, 1/2)$ as a priori estimates arise directly from the comparison principle and a naïve choice of supersolution. The reason for this is as follows. First, the intuition in (1.4) roughly gives us that $\|f(t)\|_{L^\infty, m}$ satisfies

$$\frac{d}{dt} \|f(t)\|_{L^\infty, m} \lesssim \|f(t)\|_{L^\infty, m}^2$$

and, hence, must be bounded by the solution φ of $\dot{\varphi} = C\varphi^2$. This is made rigorous by showing, via straightforward computations (see [28, Proposition 3.2]), that $\bar{f}(t, x, v) = \varphi(t)\langle v \rangle^{-m}$ is a supersolution of

$$\bar{f}_t + v \cdot \nabla_x \bar{f} = Q(f, \bar{f}).$$

Hence, f is bounded in L^∞, m . A straightforward attempt to repeat this for the L^∞, m norm of ∂f is complicated by the fact that ∂f solves an equation whose right hand side is $Q(f, \partial f) + Q(\partial f, f)$. While $Q(f, \partial f)$ will have a “good” sign at a maximum, $Q(\partial f, f)$ will not; however, from (1.4), we see that

$$Q(\partial f, f) \sim \left(\int \partial f(w) |v - w|^{\gamma+2s} dw \right) (-\Delta)^s f \lesssim \|\partial f\|_{L^\infty, m} \|(-\Delta)^s f\|_{L^\infty, m}.$$

Fortunately, when $s \in (0, 1/2)$, the last term is lower order:

$$\|(-\Delta)^s f\|_{L^\infty, m} \lesssim \|f\|_{L^\infty, m} + \|\partial f\|_{L^\infty, m}.$$

Hence, roughly,

$$\frac{d}{dt} \|\partial f(t)\|_{L^\infty, m} \lesssim \|\partial f(t)\|_{L^\infty, m} (\|\partial f(t)\|_{L^\infty, m} + \|f(t)\|_{L^\infty, m}).$$

The previous argument can be repeated here to obtain a bound on $\|\partial f\|_{L^\infty, m}$.

We now state the result. We first define the C^k analogue of the spaces X and Y_T (1.3):

$$(1.5) \quad \begin{aligned} \tilde{X}^{k, m_0, m_1} &= \{f : \langle v \rangle^{m_0} \nabla^\ell f \in C(\mathbb{T}^3 \times \mathbb{R}^3)^{6^\ell} \text{ for } 0 \leq \ell \leq k-1, \langle v \rangle^{m_1} \nabla^k f \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)^{6^k}\} \\ \text{and } \tilde{Y}_T^{k, m_0, m_1} &= L^\infty([0, T]; \tilde{X}^{k, m_0, m_1}). \end{aligned}$$

THEOREM 1.2. *Let $k \geq 1$, $\gamma \in (-3, 0]$, $s \in (0, 1/2)$, $m_1 > 3 + \gamma + 2s$, and m_0 be sufficiently large depending only on k , γ , s , and m_1 . Let the initial data $0 \leq f_{\text{in}} \in \tilde{X}^{k, m_0, m_1}$. Then there exists a time $T > 0$, depending only on $\|f_{\text{in}}\|_{\tilde{X}^{k, m_0, m_1}}$, γ , s , m_0 , m_1 , and $\tilde{b}$, and a unique solution $f \geq 0$ of (1.1) such that*

$$f \in \tilde{Y}_T^{k, m_0, m_1} \text{ and } \|f\|_{\tilde{Y}_T^{k, m_0, m_1}} \lesssim \|f_{\text{in}}\|_{\tilde{X}^{k, m_0, m_1}}.$$

It is easy to establish that f is a classical solution with the following heuristic argument. Recalling (1.4) and that $s < 1/2$, it is easy to see that the continuity of $Q(f, f)$ is controlled by the $\tilde{Y}^{1, m_0, m_1}$ -norm of f . Its continuity and (1.1) yield the continuity of $(\partial_t + v \cdot \nabla_x)f$. Hence, all terms in (1.1) “make sense” pointwise and are continuous; that is, f is a classical solution.

We also note that simply by differentiating the equation, we can obtain further time regularity when $k > 1$. This is not the main interest of the statement above so we omit it.

The significance of Theorem 1.2, besides having such a short proof, is that it improves on [28, 37] in two major ways. First, it increases the range of possible γ : [37] requires $\gamma \in (-3/2, 0]$ and [28] requires $\gamma \in (-3/2 - 2s, 0)$. Second, it weakens conditions on the initial regularity: [37] works in an H^6 -based space and [28] works in an H^5 -based space, both of which embed in C^1 . Note that it also reduces the regularity required of the initial data in comparison to Theorem 1.1. On the other hand, like [37] but unlike [28] and Theorem 1.1, it only applies to $s \in (0, 1/2)$.

1.1. Notation. We use the notation $A \lesssim B$ if there is a constant C such that $A \leq CB$. In general, the constant C may depend on $\gamma, s, n, m, k, m_0, m_1$, and $\tilde{b}$. Additionally, if an assumption for an estimate involves a requirement such as $\alpha > \beta$, then the constant C may depend on $\alpha - \beta$. We use $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. Occasionally, it will be necessary to include a constant, in which case we use C to represent such a constant and this constant C may change line-by-line.

Any integral whose domain of integration in v is not specified is understood to be an integral over $\mathbb{R}^3$ and any integral whose domain of integration in x is not specified is understood to be an integral over $\mathbb{T}^3$. For example, for any measurable φ and any measurable sets $\Omega_x \subset \mathbb{T}^3$ and $\Omega_v \subset \mathbb{R}^3$, we have

$$\iint_{\Omega_v} \varphi(x, v) dv dx = \int_{\mathbb{T}^3} \int_{\Omega_v} \varphi(x, v) dv dx \quad \text{and} \quad \int_{\Omega_x} \int \varphi(x, v) dv dx = \int_{\Omega_x} \int_{\mathbb{R}^3} \varphi(x, v) dv dx.$$

Similarly, we often suppress the domain in Lebesgue, Sobolev, and Hölder spaces when it is clear, writing, e.g., $f \in L^{\infty, m}$ instead of $f \in L^{\infty, m}(\mathbb{R}^3)$ if has already been established that $f : \mathbb{R}^3 \rightarrow \mathbb{R}$.

We use B_R to mean a ball of radius R around the origin. Whenever the ball is not centered at the origin, we denote the center v_0 as $B_R(v_0)$.

Finally, when stating estimates on the collision operator $Q(g, f)$, we often omit the assumptions on the involved functions g and f . In these cases, the estimate holds whenever the right hand side is finite.

1.2. Outline. The rest of the paper is organized as follows. In section 2, we consider bounds on the collision operator. In particular, we recall useful known results, prove some easy extensions of them, and state our main new estimates. Then, in section 3, we prove the existence and uniqueness of solutions using the bounds from section 2. Afterward, in section 4, we prove the estimates on the collision operator Q . Finally, in section 5, we give a simple proof of local well-posedness for the case of $s \in (0, 1/2)$ and $\gamma \in (-3, 0]$.

2. Estimates on the collision operator. In this section, we state the key estimates on the collision operator Q that we use in our proof of well-posedness. We begin with a brief overview of the Carleman decomposition allowing us to use ideas from the study of integro-differential operators. Then we state known estimates and their easy extensions. Finally, we state new estimates whose proof, contained in section 4, makes up the bulk of this manuscript.

2.1. Carleman decomposition. A key tool in our analysis is the Carleman decomposition [14, 15] that views the Boltzmann collision operator Q as the sum of a nonlocal diffusion operator locally similar to $-(\Delta)^s$ and a lower order reaction term.

This decomposition is well-known; see [1] for an early discussion of it and [38, sections 4 and 5] for the presentation used here. Indeed,

$$\begin{aligned} Q(g, f) &= Q_s(g, f) + Q_{ns}(g, f), \\ (2.1) \quad Q_s(g, f) &= \int (f(v') - f(v)) K_g(v, v') dv', \\ Q_{ns}(g, f) &= c_b(S_\gamma * g)f, \end{aligned}$$

where $S_\gamma(v) = |v|^\gamma$, $c_b > 0$ is a fixed constant, and K_g satisfies, for any $g \geq 0$ and any $v, v' \in \mathbb{R}^3$,

$$\begin{aligned} (2.2) \quad K_g(v, v') &\approx \frac{1}{|v - v'|^{3+2s}} \int_{w \in v + (v' - v)^\perp} g(w) |v - w|^{\gamma+2s+1} dw \\ \text{and} \quad K_g(v, v + v') &= K_g(v, v - v'). \end{aligned}$$

We refer to Q_s as the “singular” part and Q_{ns} as the “nonsingular” part.

Actually, to be fully rigorous, Q_s should be defined using a principal value. We abuse notation and suppress this as all our estimates occur over symmetric domains near the base point v and are, thus, compatible with the limit involved in the principal value.

2.2. Previously established estimates and easy extensions. In this section, we state various estimates on the collision operator that are well-known or are simple extensions of previous results.

LEMMA 2.1 (estimates of the kernel K_g). *For all $r > 0$ and $v \in \mathbb{R}^3$,*

$$\begin{aligned} (i) \quad &\int_{B_{2r}(v) \setminus B_r(v)} K_g(v', v) dv', \int_{B_{2r}(v) \setminus B_r(v)} K_g(v, v') dv' \lesssim r^{-2s} \int |g(z)| |z - v|^{\gamma+2s} dz. \\ (ii) \quad &\left| \int [K_g(v, v') - K_g(v', v)] dv' \right| \lesssim \int |g(z)| |z - v|^\gamma dz. \\ (iii) \quad &\int_{B_r(v)} (v' - v) K_g(v, v') dv' = 0. \\ (iv) \quad &\left| \int_{B_r(v)} (v' - v) K_g(v, v') dv \right| \lesssim \int |g(z)| |z - v'|^{1+\gamma} dz. \end{aligned}$$

Lemma 2.1 follows from [34, Lemmas 3.4, 3.5, 3.6, and 3.7]. The following lemma can be regarded as a slight generalization of [31, Proposition 2.1].

LEMMA 2.2. *For $0 < s < 1$, $\alpha > 2s$, $r > 0$, and $g : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ there holds*

$$\int_{B_r(v')} K_g(v', v) |v - v'|^\alpha dv, \int_{B_r(v')} K_g(v, v') |v - v'|^\alpha dv \lesssim r^{\alpha-2s} \int |g(w)| |v - w|^{\gamma+2s} dw.$$

Proof. The proofs of both inequalities are similar, so we show only the latter. Assume without loss of generality that $g \geq 0$. We proceed with a simple annular decomposition paired with the existing estimate Lemma 2.1(i). Indeed, letting $A_k = B_{2^{-k}r}(v') \setminus B_{2^{-k-1}r}(v')$, we have

$$\begin{aligned} \int_{B_r(v')} K_g(v, v') |v - v'|^\alpha dv &= \sum_{k=0}^{\infty} \int_{A_k} K_g(v, v') |v - v'|^\alpha dv \leq \sum_{k=0}^{\infty} 2^{-\alpha k} r^\alpha \int_{A_k} K_g(v, v') dv \\ &\leq \sum_{k=0}^{\infty} 2^{-\alpha k} r^\alpha \int_{B_{2^{-k-1}r}^c(v')} K_g(v, v') dv \lesssim \sum_{k=0}^{\infty} 2^{-k(\alpha-2s)} r^{\alpha-2s} \int g(w) |v' - w|^{\gamma+2s} dw. \end{aligned}$$

The claim then follows due to the fact that the sum over k is finite. $\square$

The next lemma concerns bounds on K_g via $\|g\|_{L^\infty, m}$.

LEMMA 2.3. *Fix any $m > 3 + \gamma + 2s$, $g \in L^{\infty, m}$, and $v, v' \in \mathbb{R}^3$. Then*

$$|K_g(v, v')| \lesssim \frac{1}{|v - v'|^{3+2s}} \|g\|_{L^{\infty, m}} \langle v \rangle^{\gamma+2s+1}.$$

We omit the proof as this is obvious from (2.2) and a straightforward parametrization of the two-dimensional hyperplane that is the domain of integration.

On the other hand, under a smallness condition on v' , we can establish a refined estimate involving the decay of g . To our knowledge this was first observed in [28, equation (4.39)] but not stated as a standalone lemma or given a proof in complete generality. As such, we include it here.

LEMMA 2.4. *Fix any $m > 3 + \gamma + 2s$, $\theta \in (0, 1)$, $g \in L^{\infty, m}$, and $v, v' \in \mathbb{R}^3$ with $(1 - \theta)|v| \geq |v'|$. Then*

$$|K_g(v, v')| \lesssim \frac{1}{|v - v'|^{3+2s}} \|g\|_{L^{\infty, m}} \langle v \rangle^{\gamma+2s+3-m}.$$

As the proof of Lemma 2.4 is longer than the others of this subsection, we include it in section 4; however, it is simply a more careful writing of the ideas in the proof of [28, equation (4.39)].

The next lemma provides estimates for the nonsingular part Q_{ns} . Recall (2.2). Then, we have the following estimates.

LEMMA 2.5. *Suppose that $f, g : \mathbb{T}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$. Then, for any $\epsilon > 0$ and $n \geq 0$,*

$$\|Q_{\text{ns}}(g, f)\|_{L^{2, n}} \lesssim \begin{cases} \|g\|_{L^{\infty, 3+\gamma+\epsilon}} \|f\|_{L^{2, n}}, \\ \|g\|_{L^{2, n}} \|f\|_{L^{\infty, n+\epsilon+3/2+\gamma+(3/2-n)_+}}. \end{cases}$$

Remark 1. The first inequality in Lemma 2.5 is obvious by writing $Q_{\text{ns}}(g, f) = (S * g)f$ and bounding $S * g$ in L^∞ using the weighted L^∞ norm of g . The second inequality can be easily proved by using our weighted Young's inequality Lemma 4.2. As this proof is straightforward from the statement of Lemma 4.2, we omit the details.

We also require the following from [28, Lemma 2.6].

LEMMA 2.6 (interpolation lemma). *If $n, m \geq 0$, $k' \in (0, k)$, and $l < (m - \frac{3}{2})(1 - \frac{k'}{k}) + n \frac{k'}{k}$, then*

$$\|f\|_{H^{k', l}} \lesssim \|f\|_{L^{\infty, m}}^{1-\frac{k'}{k}} \|f\|_{H^{k, n}}^{\frac{k'}{k}}.$$

2.3. New estimates. We now state new estimates on the collision operator that are crucial to allowing us to extend well-posedness to the full range of soft potentials. The prior similar work [28] relied heavily on [28, Theorem 2.4, Propositions 2.5 and 3.1]. The first two come directly from [4, Propositions 2.9 and 2.8], respectively. Each result, unfortunately, requires $\gamma + 2s > -\frac{3}{2}$. Thus, these are not applicable in our setting, and the main issue of the present work is to obtain suitable replacements, which we state here.

The first is a commutator estimate (cf. [4, Proposition 2.8], [28, Proposition 2.5]).

PROPOSITION 2.7 (commutator estimate). *For any $\epsilon > 0$, $\gamma \in (-3, 0]$, $\mu \in ((1 - 2s)_+, 2 - 2s)$, $m > \max\{3 + \gamma + 2s, \ell + \gamma + \frac{3}{2}\}$, $\ell > \frac{3}{2}$, and $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$, we have*

$$\|\langle v \rangle^\ell Q_s(g, f) - Q_s(g, \langle v \rangle^\ell f)\|_{L^2} \lesssim (\|f\|_{L^{2, \ell+3/2+\epsilon}} + \|f\|_{H^{2s-1+\mu, \mu+\ell+\gamma+2s}}) \|g\|_{L^{\infty, m}}.$$

The next estimates concern $Q_s(g, f)$ and involve only the v -variable.

PROPOSITION 2.8. *For any $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$, $\gamma \in (-3, 0]$, $\gamma + 2s \leq 0$, and $\epsilon > 0$, the following hold:*

(i) *If $\theta \in (0, 2s)$, then*

$$\int Q_s(g, f) h dv \lesssim \|g\|_{L^\infty, 3+\gamma+2s+\epsilon} \|f\|_{H^{2s-\theta}} \|h\|_{H^\theta}.$$

(ii) *If $\theta > 0$, then*

$$\|Q_s(g, f)\|_{L^2} \lesssim \|g\|_{L^\infty, 3+\gamma+2s+\epsilon} \|f\|_{H^{2s+\theta}}.$$

(iii) *If $n > 3/2 + \gamma + 2s$, $m > 3/2 + \gamma + (3/2 - n)_+$, and $\alpha > 2s$, then*

$$\|Q_s(g, f)\|_{L^{2,n}} \lesssim \|g\|_{L^{2,n}} \left(\|f\|_{L^\infty, m} + \|\langle v \rangle^{n+5/2+(3/2-n)_++\alpha+\gamma+\epsilon} f\|_{C_v^\alpha} \right).$$

(iv) *If $n \geq 0$, $m > n + 6 + \gamma + 2s$, and $g \geq 0$, we have*

$$\int \langle v \rangle^{2n} f Q_s(g, f) dv \lesssim \|g\|_{L^\infty, m} \|f\|_{L^{2,n}}^2.$$

The first two parts above, (i) and (ii), rely heavily on the work in [34]; however, that reference is focused on local estimates and, as such, is not concerned with understanding the dependence on weights. Combined they are a replacement for [28, Theorem 2.4] (see also [4, Proposition 2.9]). The second two parts above, (iii) and (iv), are new. They are replacements for [28, Proposition 3.1(i) and (iii)], respectively.

During the submission of this paper, the anonymous referee made us aware of an earlier result of Desvillettes and Mouhot that is similar to Proposition 2.8(iv) [18, Corollary 2.2] but proved in the case $\gamma \in [0, 1]$ (recall that we consider $\gamma < 0$ here). While the proof is not immediately valid for $\gamma < 0$, it is likely that one can modify it to obtain a result similar to Proposition 2.8(iv) when $\gamma < 0$ that would suffice for our purposes. We note that the proof of [18, Corollary 2.2] differs significantly from the proof of Proposition 2.8(iv) as it is not based on the Carleman decomposition.

We make two brief remarks. First, the result (i) is a slight generalization of the results in [34] as it allows us to choose θ in (i). Second, the result (ii) almost certainly holds without $\theta = 0$; however, as this is not needed for our purposes and the current statement is easy to derive from [34], we are content to use (ii) as is.

The final estimate makes use of the symmetry properties of Q_s in order to avoid having more than one full derivative “land” on f . This is crucial in case two of the proof of the main a priori estimate Proposition 3.1. It is a replacement for [28, Proposition 3.1(iv)].

PROPOSITION 2.9. *Suppose that $f, g : \mathbb{T}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$. If $\gamma \in (-3, 0]$, $\epsilon > 0$, $\mu \in ((1-2s)_+, 2-2s)$, $\kappa \in (s, \min\{2s, 1\})$, $n > \frac{3}{2}$, and $m > \max\{3+\gamma+2s, n+\gamma+\frac{3}{2}\}$, then*

$$\left| \int \langle v \rangle^{2n} Q_s(g, f) \partial f dv dx \right| \lesssim \|g\|_{L^\infty, m} (\|f\|_{L^{2, n+3/2+\epsilon}} + \|f\|_{H^{2s-1+\mu, \mu+n+\gamma+2s}}) \|f\|_{H^{1, n}} \\ + \|\partial g\|_{H^{3/2+(2s-1/2)_++\epsilon, 3+\gamma+2s+\epsilon}} \|f\|_{H^{1, n}}^2 + \|g\|_{C^{\kappa, 3+\epsilon}} \|f\|_{H^{s, n+3/2+\epsilon+(\gamma+2s+1)_+}} \|f\|_{H^{1, n}},$$

where $\partial = \partial_{x_i}$ or ∂_{v_i} for some $i \in \{1, 2, 3\}$.

Recall that we prove the above estimates in section 4.

3. Existence and uniqueness of solutions: Theorem 1.1. In this section, we prove Theorem 1.1. The majority of the work is in the proof of existence and our approach for this follows [28] closely. Indeed, the construction procedure is similar, relying on exhibiting a solution to a suitably regularized and linearized problem. We then use compactness to deregularize and a fixed point argument to pass from the linearized problem to the nonlinear one. The main novelty to the current work as compared to [28] is in the establishment of a priori estimates in $Y_T^{k,n,m}$ of the regularized and linearized problem. When possible, we omit details that are unchanged from [28].

3.1. Proof of existence in Theorem 1.1. First, we define a smooth cutoff function $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ with $0 \leq \psi \leq 1$, $\int \psi(v) dv = 1$,

$$\psi = 1 \quad \text{on } B_{1/2} \quad \text{and} \quad \psi = 0 \quad \text{on } B_1^c.$$

Next, for any $\phi : \mathbb{T}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ and $\epsilon > 0$, we define

$$\phi^\epsilon(x, v) = \frac{1}{\epsilon^6} \int \psi\left(\frac{x-y}{\epsilon}\right) \psi\left(\frac{v-w}{\epsilon}\right) \phi(y, w) dy dw.$$

Then we define the regularized collision operator; for any $\delta > 0$ and $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$,

$$Q_{\epsilon,\delta}(g(x, \cdot), f(x, \cdot))(v) = \psi(\delta v) Q(g^\epsilon(x, \cdot), \psi(\delta \cdot) f(x, \cdot)).$$

Finally, for any $\sigma \in [0, 1]$, we define the differential operator

$$(3.1) \quad \mathcal{L}_{\sigma,\epsilon,\delta}(f) = \partial_t f + \sigma \psi(\delta v) v \cdot \nabla_x f - (\epsilon + (1 - \sigma)) \Delta_{x,v} f - \sigma Q_{\epsilon,\delta}(g, f).$$

The intuition for the above regularizations and cutoffs is given in [28, section 3].

We now establish a priori estimates that hold for both the full equation and the regularized one above. This is done in the following proposition.

PROPOSITION 3.1. *Suppose that $T > 0, k \geq 5, n \geq 3/2 + (\gamma + 2s)_+, \sigma \in [0, 1], \epsilon, \delta \geq 0$, and $m \geq 0$. Supposing that $R, f \in Y_T^{k,n,m}$*

$$(3.2) \quad \begin{cases} \mathcal{L}_{\sigma,\epsilon,\delta} f = R & \text{in } (0, T) \times \mathbb{T}^3 \times \mathbb{R}^3, \\ f(0, \cdot, \cdot) = f_{in} & \text{in } \mathbb{T}^3 \times \mathbb{R}^3. \end{cases}$$

For any $\mu > 0$, if $\delta = 0$ and $m \geq 3/2 + \mu$ or if $\delta > 0$, then

$$(3.3) \quad \|f\|_{L^{\infty,m}} \leq \exp \left\{ C \int_0^T \|g(t)\|_{L^{\infty, \max\{m, 3/2+\mu\}}} dt \right\} \left(\|f_{in}\|_{L^{\infty,m}} + \int_0^T \|R(t)\|_{L^{\infty,m}} dt \right).$$

If $\delta = 0$ and m is sufficiently large depending on k, n, γ , and s , then

$$(3.4) \quad \|f\|_{L^{\infty,m}([0,T]; H_{x,v}^{k,n})} \leq \exp \left\{ C \int_0^T \|g(t)\|_{X_{x,v}^{k,n,m}} dt \right\} \left((1+T) \|f_{in}\|_{X_{x,v}^{k,n,m}} + \int_0^T \|R(t)\|_{X_{x,v}^{k,n,m}} dt \right).$$

Now, we prove Proposition 3.1. The proof follows that of [28, Proposition 3.1] with small changes due to the new estimates on the collision operator necessary in our setting.

Proof. The argument of (3.3) goes exactly as that in [28, Proposition 3.1] and hence we omit the proof here. Now we focus on proving (3.4). First, we let $\alpha, \beta \in \mathbb{N}_0^3$ be any multi-indices such that $|\alpha| + |\beta| = k$. Then, differentiating (3.1), multiplying the resulting equation by $\langle v \rangle^{2n} \partial_x^\alpha \partial_v^\beta f$, and integrating in x and v , we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\langle v \rangle^n \partial_x^\alpha \partial_v^\beta f|^2 dx dv \\ &= -\sigma \int \left(\sum_{i=1}^3 \beta_i \partial_{x_i} \partial_x^\alpha \partial_v^{\beta-e_i} f \right) \langle v \rangle^{2n} \partial_x^\alpha \partial_v^\beta f dx dv \\ & \quad + \sigma \sum_{\substack{\alpha' + \alpha'' = \alpha \\ \beta' + \beta'' = \beta}} C_{\alpha', \beta', \alpha'', \beta''} \int Q(\partial_{x'}^{\alpha'} \partial_v^{\beta'} g, \partial_{x'}^{\alpha''} \partial_v^{\beta''} f) \langle v \rangle^{2n} \partial_x^\alpha \partial_v^\beta f dx dv \\ & \quad - (\epsilon + 1 - \sigma) \int |\nabla_{x,v} \partial_x^\alpha \partial_v^\beta f|^2 + \int \partial_x^\alpha \partial_v^\beta R \langle v \rangle^{2n} \partial_x^\alpha \partial_v^\beta f dx dv \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

for some constants $C_{\alpha', \beta', \alpha'', \beta''} > 0$ depending only on the subscripted quantities.

We see that I_1 is bounded by $\|f\|_{H^{k,n}}^2$, I_4 is bounded by $\|R\|_{H^{k,n}}^2 + \|f\|_{H^{k,n}}^2$, and I_3 has a good sign. Thus, our focus is primarily on I_2 , the term involving the collision operator Q . We argue case by case depending on the size of $|\alpha''| + |\beta''|$ in order to establish that

$$(3.5) \quad I_2 \lesssim \|g\|_{X^{k,n,m}} \|f\|_{X^{k,n,m}}^2.$$

The proof of (3.5) is postponed momentarily while we show how to conclude. Indeed, assuming (3.5) is proved, we arrive at

$$(3.6) \quad \frac{1}{2} \frac{d}{dt} \int |\langle v \rangle^n \partial_x^\alpha \partial_v^\beta f|^2 dx dv \lesssim (\|g\|_{X^{k,n,m}} + 1) \|f\|_{X^{k,n,m}}^2 + \|R\|_{H^{k,n}}^2.$$

Recalling the definition of $X^{k,n,m}$ in (1.3) and using (3.3), we find

$$\begin{aligned} (3.7) \quad & \frac{1}{2} \frac{d}{dt} \|f\|_{H^{k,n}}^2 \lesssim (\|g(t)\|_{X^{k,n,m}} + 1) \|f\|_{H^{k,n}}^2 \\ & + (\|g(t)\|_{X^{k,n,m}} + 1) \exp \left\{ C \int_0^t \|g(t)\|_{X^{k,n,m}} dt \right\} (\|f_{\text{in}}\|_{L^\infty, m}^2 + \|R\|_{X^{k,n,m}}^2). \end{aligned}$$

Therefore, we conclude the proof of (3.4) by applying Grönwall's inequality.

We now establish (3.5). For notational ease, we set

$$(3.8) \quad F = \partial_x^{\alpha''} \partial_v^{\beta''} f, \quad G = \partial_x^{\alpha'} \partial_v^{\beta'} g.$$

Thus, we are estimating terms of the form

$$\begin{aligned} (3.9) \quad & \int \langle v \rangle^{2n} Q(G, F) \partial_x^{\alpha'} \partial_v^{\beta'} F dv dx \\ &= \int \langle v \rangle^{2n} Q_s(G, F) \partial_x^{\alpha'} \partial_v^{\beta'} F dv dx + \int \langle v \rangle^{2n} Q_{\text{ns}}(G, F) \partial_x^{\alpha'} \partial_v^{\beta'} F dv dx. \end{aligned}$$

Case one: $|\alpha''| + |\beta''| = k$, i.e., $\alpha'' = \alpha$, $\beta'' = \beta$, and in the form of $\int \langle v \rangle^{2n} Q(g, F) F$. We estimate the Q_s term first. We proceed by using Proposition 2.8(iv), up to increasing m if necessary,

$$\begin{aligned} \int \langle v \rangle^{2n} Q_s(g, F) F \, dv dx &\lesssim \int \|F\|_{L_v^{2,n}}^2 \|g\|_{L_v^{\infty,m}} \, dx \lesssim \|F\|_{L^{2,n}}^2 \|g\|_{L^{\infty,m}} \\ &\lesssim \|f\|_{H^{k,n}}^2 \|g\|_{L^{\infty,m}} \lesssim \|f\|_{X^{k,n,m}}^2 \|g\|_{X^{k,n,m}}, \end{aligned}$$

as desired.

Furthermore, for $\int \langle v \rangle^{2n} Q_{ns}(g, F) F \, dv dx$, we recall (2.1) and apply Lemma 2.5 to find

$$\begin{aligned} \int \langle v \rangle^{2n} Q_{ns}(g, F) F \, dv dx &\approx \int \langle v \rangle^{2n} (S_\gamma * g) F^2 \, dv dx \lesssim \int \|g\|_{L_v^{\infty,n}} \|F\|_{L_v^{2,n}}^2 \, dx \\ &\lesssim \|g\|_{L^{\infty,m}} \|F\|_{L^{2,n}} \lesssim \|g\|_{L^{\infty,m}} \|f\|_{H^{k,n}}^2 \leq \|g\|_{X^{k,n,m}} \|f\|_{X^{k,n,m}}^2. \end{aligned}$$

This concludes the proof of (3.5) in case one.

Case two: $|\alpha''| + |\beta''| = k - 1$, and in the form, $\int \langle v \rangle^{2n} Q(\partial g, F) \partial F$. Here we denote derivative operator $\partial = \partial_x^{\alpha'} \partial_v^{\beta'}$ as $|\alpha'| + |\beta'| = 1$.

We first estimate the Q_s portion. Fix $\epsilon \in (0, \min\{s, 1-s\})$. Let $\mu = (1-2s)_+ + \epsilon$, $\kappa = s + \epsilon$, and $\tilde{m} = \epsilon + \max\{3 + \gamma + 2s, n + \gamma + 3/2\}$. We then directly apply Proposition 2.9 to find

$$\begin{aligned} &\left| \int \langle v \rangle^{2n} Q_s(\partial g, F) \partial F \, dv dx \right| \\ &\lesssim \|\partial g\|_{L^{\infty,\tilde{m}}} \|F\|_{H^{2s-1+\mu,\mu+n+3/2}} \|F\|_{H^{1,n}} \\ &\quad + \|\partial^2 g\|_{H^{3-s,3+\epsilon}} \|F\|_{H^{1,n}}^2 + \|\partial g\|_{C^{\kappa,3+\epsilon}} \|F\|_{H^{s,n+5/2+\epsilon}} \|F\|_{H^{1,n}} \\ &\lesssim \|\partial g\|_{L^{\infty,\tilde{m}}} \|f\|_{H^{k-2(1-s)+\mu,\mu+n+3/2}} \|f\|_{H^{k,n}} \\ &\quad + \|\partial^2 g\|_{H^{3-s,3+\epsilon}} \|f\|_{H^{k,n}}^2 + \|\partial g\|_{C^{\kappa,3+\epsilon}} \|f\|_{H^{k-(1-s),n+5/2+\epsilon}} \|f\|_{H^{k,n}}. \end{aligned}$$

Applying the Sobolev embedding theorem on terms involving g and then Lemma 2.6 (up to increasing m if necessary) yields

$$\begin{aligned} &\left| \int \langle v \rangle^{2n} Q_s(\partial g, F) \partial F \, dv dx \right| \\ &\lesssim \|g\|_{H^{4+\epsilon,\tilde{m}}} \|f\|_{H^{k-(2-2s)+\mu,\mu+n+3/2}} \|f\|_{H^{k,n}} + \|g\|_{H^{5-s,3+\epsilon}} \|f\|_{H^{k,n}}^2 \\ &\quad + \|g\|_{H^{4+\kappa,3+\epsilon}} \|f\|_{H^{k-(1-s),n+5/2+\epsilon}} \|f\|_{H^{k,n}} \lesssim \|g\|_{X^{k,n,m}} \|f\|_{X^{k,n,m}}^2. \end{aligned}$$

The estimate of the nonsingular part Q_{ns} is the same as in the previous case and is thus omitted.

Case three: $|\alpha''| + |\beta''| = k - 2$ and $|\alpha'| + |\beta'| = 2$. First, we estimate the Q_s term. We see

$$\begin{aligned} &\int \langle v \rangle^{2n} Q_s(G, F) \partial_x^\alpha \partial_v^\beta f \, dv dx \leq \|Q_s(G, F)\|_{L^{2,n}} \|f\|_{H^{k,n}} \\ (3.10) \quad &\leq (\|Q_s(G, \langle v \rangle^n F) - \langle v \rangle^n Q_s(G, F)\|_{L^2} + \|Q_s(G, \langle v \rangle^n F)\|_{L^2}) \|f\|_{H^{k,n}} \\ &=: (B_1 + B_2) \|f\|_{H^{k,n}}. \end{aligned}$$

We estimate B_2 first. Fix any $\theta \in (0, \min\{\frac{2-2s}{3}, \frac{3}{4}\})$ and let $p = 3/(4\theta)$ and $q = \frac{3}{3-4\theta}$. Then we apply Proposition 2.8(ii), Hölder's inequality, and the Sobolev embedding theorem to find

$$\begin{aligned} B_2 &\lesssim \left(\int \|G\|_{L_v^{\infty,3}}^2 \|F\|_{H_v^{2s+\theta,n}}^2 dx \right)^{1/2} \lesssim \|G\|_{L_x^{2p} L_v^{\infty,3}} \|F\|_{L_x^{2q} H_v^{2s+\theta,n}} \\ &\lesssim \|G\|_{H_x^{3/2-2\theta} H_v^{3/2+\theta,3}} \|F\|_{H_x^{2\theta} H_v^{2s+\theta,n}} \lesssim \|G\|_{H^{3-\theta,3}} \|F\|_{H^{2s+3\theta,n}} \lesssim \|g\|_{X^{k,n,m}} \|f\|_{X^{k,n,m}}. \end{aligned}$$

The last inequality follows by our choice of θ .

For B_1 , for any $\mu \in ((1-2s)_+, 2-2s)$ and $\tilde{m} = 1 + \max\{3, n + \gamma + 3/2\}$, we appeal to our commutator estimate Proposition 2.7, the Cauchy-Schwarz inequality, and the Sobolev embedding theorem to obtain

$$\begin{aligned} B_1 &\lesssim \left(\int (\|F\|_{L_v^{2,n+2}} + \|F\|_{H_v^{2s-1+\mu,\mu+n}})^2 \|G\|_{L_v^{\infty,\tilde{m}}}^2 dx \right)^{1/2} \\ &\lesssim \|G\|_{L_x^4 L_v^{\infty,\tilde{m}}} \left(\|F\|_{L_x^4 L_v^{2,n+2}} + \|F\|_{L_x^4 H_v^{2s-1+\mu,\mu+n}} \right) \\ &\lesssim \|G\|_{H^{5/2,\tilde{m}}} \left(\|F\|_{H^{3/4,n+2}} + \|F\|_{H_v^{2s-1/4+\mu,\mu+n}} \right) \\ &\lesssim \|g\|_{H^{9/2,\tilde{m}}} (\|f\|_{H^{k-5/4,n+2}} + \|f\|_{H^{k+2s-9/4+\mu,\mu+n}}). \end{aligned}$$

Notice that $2s - 9/4 + \mu < 0$ as $\mu < 2 - 2s$. With this, observe that all three norms above involve regularity of order strictly less than k . Hence, assuming m is sufficiently large, the interpolation lemma Lemma 2.6 yields

$$B_1 \lesssim \|g\|_{X^{k,n,m}} \|f\|_{X^{k,n,m}}.$$

This concludes the estimates for the singular part.

For the nonsingular part, we apply Lemma 2.5 to find

$$\begin{aligned} &\int \langle v \rangle^{2n} Q_{\text{ns}}(G, F) \partial_x^\alpha \partial_v^\beta f dv dx \\ &\lesssim \|f\|_{H^{k,n}} \|Q_{\text{ns}}(G, F)\|_{L^{2,n}} \lesssim \|f\|_{H^{k,n}} \|G\|_{L^{\infty,3+\gamma+\epsilon}} \|F\|_{L^{2,n}}. \end{aligned}$$

Using the Sobolev embedding theorem and Lemma 2.6, we obtain the desired estimate

$$\int \langle v \rangle^{2n} Q_{\text{ns}}(G, F) \partial_x^\alpha \partial_v^\beta f dv dx \lesssim \|g\|_{X^{k,n,m}} \|f\|_{X^{k,n,m}}^2.$$

This concludes the proof of (3.5) in this case.

Case four: $|\alpha''| + |\beta''| = k - 3$ and $|\alpha'| + |\beta'| = 3$. The proof of (3.5) in this case is exactly as in case three, except with the choices

$$\theta \in \left(0, \min\left\{\frac{1}{2}, \frac{5-4s}{6}\right\}\right), \quad p = \frac{3}{1+4\theta}, \quad \text{and} \quad q = \frac{3}{2-4\theta}$$

in the estimate of Q_s . As such, we omit the proof.

Case five: $|\alpha''| + |\beta''| = k - 4$ and $|\alpha'| + |\beta'| = 4$. We begin with the singular term:

$$\int \langle v \rangle^{2n} Q_s(G, F) \partial_x^\alpha \partial_v^\beta f dv dx \lesssim \|\partial_x^\alpha \partial_v^\beta f\|_{L^{2,n}} \|Q_s(G, F)\|_{L^{2,n}} \leq \|f\|_{H^{k,n}} \|Q_s(G, F)\|_{L^{2,n}}.$$

It is clear that we need only bound the last term above. Recalling Proposition 2.8(iii), we have, for any $\mu \in ((2s-1)_+, 1)$,

$$\|Q_s(G, F)\|_{L_v^{2,n}} \lesssim \left(\|F\|_{L_v^{\infty,3}} + \|\langle v \rangle^{n+5/2+\mu} F\|_{C_v^{1+\mu}} \right) \|G\|_{L_v^{2,n}}.$$

Applying the Sobolev embedding theorem with $\tilde{m}$ sufficiently large (depending only on n), we obtain, for $\epsilon = (1-\mu)/4$,

$$\|Q_s(G, F)\|_{L_v^{2,n}} \lesssim \|F\|_{H_v^{5/2+\mu+\epsilon, \tilde{m}}} \|G\|_{L_v^{2,n}}.$$

Using Hölder's inequality and the Sobolev embedding theorem yields

$$\begin{aligned} \|Q_s(G, F)\|_{L^{2,n}}^2 &\lesssim \int \|F\|_{H_v^{5/2+\mu+\epsilon, \tilde{m}}}^2 \|G\|_{L_v^{2,n}}^2 dx \leq \|F\|_{L_x^3 H_v^{5/2+\mu+\epsilon, \tilde{m}}}^2 \|G\|_{L_x^6 L_v^{2,n}}^2 \\ &\lesssim \|F\|_{H_x^{1/2} H_v^{5/2+\mu+\epsilon, \tilde{m}}}^2 \|G\|_{H^{1,n}}^2 \leq \|g\|_{H^{k,n}}^2 \|f\|_{H^{k-1+\mu+\epsilon, \tilde{m}}}^2. \end{aligned}$$

To control the last term, we use the interpolation lemma, Lemma 2.6, and that, by construction, $\mu + \epsilon < 1$, to find

$$\|f\|_{H^{k-1+\mu, \tilde{m}}} \lesssim \|f\|_{X^{k,n,m}}$$

as long as m is sufficiently large depending only on n and s . This concludes the proof of the bound of the singular term.

We now consider the nonsingular part. As above, it is enough to bound $\|Q_{\text{ns}}(G, F)\|_{L^{2,n}}$. To this end, applying Lemma 2.5 yields

$$\begin{aligned} \|Q_{\text{ns}}(G, F)\|_{L^{2,n}}^2 &\lesssim \left(\int \|F\|_{L^{n+3}}^2 \|G\|_{L_v^{2,n}}^2 dx \right)^{1/2} \\ &\lesssim \|F\|_{L^{n+3}}^2 \|G\|_{L^{2,n}}^2 \leq \|F\|_{L^{n+3}}^2 \|g\|_{X^{k,n,m}}^2. \end{aligned}$$

Thus, we need only bound the norm of F on the right hand side. By the Sobolev embedding theorem and the interpolation lemma Lemma 2.6, we find

$$\|F\|_{L^{n+3}} \lesssim \|F\|_{H^{7/2+1/4, n+3}} \lesssim \|f\|_{H^{k-1/4, n+3}} \lesssim \|f\|_{X^{k,n,m}},$$

as long as m is sufficiently large. This concludes the proof of (3.5) in case five.

Case six: $|\alpha''| + |\beta''| \leq k-5$ and $|\alpha'| + |\beta'| \geq 5$. We begin by bounding the term with Q_s . As above, it is enough to bound $Q_s(G, F)$ in $L^{2,n}$. First, by Proposition 2.8(iii) with $\mu \in ((2s-1)_+, 1)$, we find

$$\|Q_s(G, F)\|_{L^{2,n}}^2 \lesssim \int \|G\|_{L_v^{2,n}}^2 \left(\|F\|_{L_v^{\infty, m}}^2 + \|\langle v \rangle^{n+5/2+\mu} F\|_{C^{1+\mu}}^2 \right) dx.$$

Applying the Sobolev embedding theorem and letting $\epsilon = (1-\mu)/2$, we obtain

$$\begin{aligned} \|Q_s(G, F)\|_{L^{2,n}}^2 &\lesssim \int \|G\|_{L_v^{2,n}}^2 \|F\|_{H_v^{5/2+\mu, \tilde{m}}}^2 dx \leq \|G\|_{L^{2,n}}^2 \|F\|_{L_x^\infty H_v^{5/2+\mu, \tilde{m}}}^2 \\ &\lesssim \|G\|_{L^{2,n}}^2 \|F\|_{H^{4+\mu+\epsilon, \tilde{m}}}^2 \lesssim \|g\|_{H^{k,n}}^2 \|f\|_{H^{k-1+\mu+\epsilon, \tilde{m}}}^2, \end{aligned}$$

where $\tilde{m}$ is a constant depending only on n . The proof concludes as in the previous case by using the fact that $k-1+\mu+\epsilon < k$ and Lemma 2.6.

The estimate of the nonsingular part Q_{ns} is the same as in the previous case and is thus omitted. This concludes the proof of (3.5) in case six and, thus, all cases. $\square$

Having established the bounds above, we now construct a solution.

PROPOSITION 3.2 (construction of solution in the linear equation). *Fix $T > 0$, a function $g \in Y_T^{k,n,m}$, and the initial data $0 \leq f_{\text{in}} \in X^{k,n,m}$. Then there exists $f \in Y_T^{k,n,m}$ such that*

$$(3.11) \quad f_t + v \cdot \nabla_x f = Q(g, f)$$

and $f(0, \cdot, \cdot) = f_{\text{in}}$. Moreover, $f \geq 0$.

Proof. The proof of [28, Proposition 3.3] can be adapted verbatim as it requires only the established bounds in [28, Proposition 3.2] (the analogue of our Proposition 3.1). The proof is composed of three steps: (1) due to the Laplacian in $\mathcal{L}_{\sigma,\epsilon,\delta}$, apply the Schauder estimates to establish boundedness of a linear operator involving of $\mathcal{L}_{\sigma,\epsilon,\delta}$; (2) apply the method of continuity to construct the solution of $\mathcal{L}_{\sigma,\epsilon,\delta} f = 0$ using the bounds from the previous step; and (3) use the a priori estimates from Proposition 3.1 to deregularize. Due to its similarity to [28, Proposition 3.3], we omit the details. $\square$

Proof of existence in Theorem 1.1. The analogous existence result [28, Theorem 1.1] proceeds by constructing a sequence f_i solving

$$(3.12) \quad (\partial_t + v \cdot \nabla_x) f_i = Q(f_{i-1}, f_i),$$

establishing the boundedness of this sequence inductively, showing that $f_i - f_{i-1}$ tends to zero, and then taking the limit $i \rightarrow \infty$. These steps can be repeated verbatim in our setting. Indeed, the existence of each f_i follows from Proposition 3.2. The boundedness of f_i and the smallness of $f_i - f_{i-1}$ in [28, Theorem 1.1] rely only on the bilinearity of Q , the a priori estimates of [28, Proposition 3.2], and a Grönwall argument. As Proposition 3.1 provides *exactly* the same a priori estimates as [28, Proposition 3.2], the convergence of f_i to some f can be repeated exactly as the convergence argument in [28, Theorem 1.1] (f solves (1.1) due to (3.12)). As such, we omit the details, and the proof is concluded. $\square$

3.2. Proof of uniqueness in Theorem 1.1. We now finish the proof of Theorem 1.1 by establishing uniqueness.

Proof of uniqueness in Theorem 1.1. Consider any two solutions f and g of (1.1) with $f(0, \cdot, \cdot) = g(0, \cdot, \cdot) = f_{\text{in}}$ and set $h = f - g$. We have

$$(3.13) \quad h_t + v \cdot \nabla_x h = Q(f, h) + Q(h, g).$$

Then, we multiply (3.13) by $\langle v \rangle^{2n} h$ and integrate with respect to v and x , yielding

$$(3.14) \quad \frac{1}{2} \frac{d}{dt} \|h\|_{L^{2,n}}^2 = \int \langle v \rangle^{2n} Q(f, h) h \, dv dx + \int \langle v \rangle^{2n} Q(h, g) h \, dv dx = I_1 + I_2,$$

where

$$(3.15) \quad I_1 = \int \langle v \rangle^{2n} Q_s(f, h) h \, dv dx + \int \langle v \rangle^{2n} Q_{\text{ns}}(f, h) h \, dv dx = I_{11} + I_{12}$$

and

$$(3.16) \quad I_2 = \int \langle v \rangle^{2n} Q_s(h, g) h \, dv dx + \int \langle v \rangle^{2n} Q_{\text{ns}}(h, g) h \, dv dx = I_{21} + I_{22}.$$

For I_{11} , Proposition 2.8(iv) yields, for $\tilde{m}$ sufficiently large,

$$(3.17) \quad I_{11} \lesssim \int \|h\|_{L_v^{2,n}}^2 \|f\|_{L_v^{\infty,\tilde{m}}} dx \lesssim \|h\|_{L_x^2 L_v^{2,n}}^2 \|f\|_{L_x^\infty L_v^{\infty,\tilde{m}}} \lesssim \|h\|_{L^{2,n}}^2 \|f\|_{X^{k,n,m}}.$$

For I_{12} , we apply Lemma 2.5 to obtain

$$(3.18) \quad I_{12} \lesssim \|h\|_{L^{2,n}}^2 \|f\|_{L^{\infty,3+\gamma+\epsilon}} \lesssim \|h\|_{L^{2,n}}^2 \|f\|_{X^{k,n,m}}.$$

For I_{21} , fix $\alpha \in (2s, 2)$, $\epsilon = (2 - \alpha)/2$, and $\tilde{m}$ to be sufficiently large and apply Proposition 2.8(iii) to find

$$(3.19) \quad \begin{aligned} I_{21} &\lesssim \int \|h\|_{L_v^{2,n}}^2 (\|g\|_{L_v^{\infty,m}} + \|\langle v \rangle^{\tilde{m}} g\|_{C_v^\alpha}) dx \lesssim \|h\|_{L^{2,n}}^2 \|g\|_{L_x^\infty H_v^{3/2+\alpha,\tilde{m}}} \\ &\lesssim \|h\|_{L^{2,n}}^2 \|g\|_{H^{3+\alpha+\epsilon,\tilde{m}}} \lesssim \|h\|_{L^{2,n}}^2 \|g\|_{X^{k,n,m}}. \end{aligned}$$

For I_{22} , apply Lemma 2.5 to find

$$(3.20) \quad I_{22} \lesssim \|h\|_{L^{2,n}}^2 \|g\|_{L^{\infty,n+3}} \lesssim \|h\|_{L^{2,n}}^2 \|g\|_{X^{k,n,m}}.$$

Combining the estimates of I_{11} , I_{12} , I_{21} , and I_{22} , that is, (3.17)–(3.20), and recalling that $\|f\|_{X^{k,n,m}}, \|g\|_{X^{k,n,m}} \lesssim 1$, we find

$$\frac{d}{dt} \|h(t)\|_{L^{2,n}}^2 \lesssim \|h(t)\|_{L^{2,n}}^2.$$

The Grönwall inequality and the fact that $h(0, \cdot, \cdot) = 0$ imply that $h = 0$. We deduce that $f = g$, concluding the proof. $\square$

4. Proof of the estimates on the collision operator Q .

4.1. Proof of the refined estimate on K_g Lemma 2.4.

Proof. We first show that $|v + w| \approx |v| + |w|$ for any $w \in (v - v')^\perp$. The “ $\lesssim$ ” inequality is clear, so we show the other inequality:

$$\begin{aligned} |v + w|^2 &= |v|^2 + 2v \cdot w + |w|^2 = |v|^2 + 2v' \cdot w + |w|^2 \\ &\geq |v|^2 - \frac{1}{1-\theta} |v'|^2 - (1-\theta) |w|^2 + |w|^2 \\ &\geq |v|^2 - (1-\theta) |v|^2 - (1-\theta) |w|^2 + |w|^2 = \theta |v|^2 + \theta |w|^2. \end{aligned}$$

In the second equality, we used that $(v - v') \cdot w = 0$, in the first inequality, we used Young’s inequality, and in the second inequality, we used the hypothesis that $(1 - \theta)|v| \geq |v'|$.

Recalling (2.2) and changing variables, we have

$$|v - v'|^{3+2s} K_g(v, v') \approx \int_{v+(v'-v)^\perp} g(w) |v - w|^{\gamma+2s+1} dw = \int_{(v'-v)^\perp} g(v + w) |w|^{\gamma+2s+1} dw.$$

Clearly, it is enough to simply bound the integral on the right hand side. Using that $|v + w| \approx |v| + |w|$, as established above, we see that

$$\begin{aligned}
 (4.1) \quad & \left| \int_{(v'-v)^\perp} g(v+w) |w|^{\gamma+2s+1} dw \right| \lesssim \|g\|_{L^\infty, m} \int_{(v'-v)^\perp} \frac{|v|^{\gamma+2s+1}}{\langle v+w \rangle^m} dw \\
 & \lesssim \|g\|_{L^\infty, m} \int_{(v'-v)^\perp} \frac{|w|^{\gamma+2s+1}}{\langle v \rangle^m + \langle w \rangle^m} dw \\
 & = \|g\|_{L^\infty, m} \int_{(v'-v)^\perp \cap B_{\langle v \rangle}} \frac{|w|^{\gamma+2s+1}}{\langle v \rangle^m + \langle w \rangle^m} dw + \|g\|_{L^\infty, m} \int_{(v'-v)^\perp \cap B_{\langle v \rangle}^c} \frac{|w|^{\gamma+2s+1}}{\langle v \rangle^m + \langle w \rangle^m} dw \\
 & = \|g\|_{L^\infty, m} (I_1 + I_2).
 \end{aligned}$$

For I_1 , that is, $w \in B_{\langle v \rangle}$, we use the fact that

$$(4.2) \quad \frac{|w|^{\gamma+2s+1}}{\langle v \rangle^m + \langle w \rangle^m} \lesssim |w|^{\gamma+2s+1} \langle v \rangle^{-m}.$$

Thus, we see (recall we are integrating over a subset of a two-dimensional hyperplane)

$$(4.3) \quad I_1 \lesssim \langle v \rangle^{-m} \int_{(v'-v)^\perp \cap B_{\langle v \rangle}} |w|^{\gamma+2s+1} dw \lesssim \langle v \rangle^{\gamma+2s+3-m}.$$

For I_2 , that is, $w \in B_{\langle v \rangle}^c$, we have

$$(4.4) \quad \frac{|w|^{\gamma+2s+1}}{\langle v \rangle^m + \langle w \rangle^m} \lesssim \frac{|w|^{\gamma+2s+1}}{\langle w \rangle^m} \lesssim \langle w \rangle^{\gamma+2s+1-m}.$$

Therefore, we get (again, recall, we are integrating over a subset of a two-dimensional hyperplane)

$$(4.5) \quad I_2 \lesssim \int_{(v'-v)^\perp \cap B_{\langle v \rangle}^c} \langle w \rangle^{\gamma+2s+1-m} dw \lesssim \langle v \rangle^{\gamma+2s+3-m}.$$

Combining (4.1), (4.3), and (4.5), we obtain the desired inequality, concluding the proof. $\square$

4.2. Commutator estimate: Proof of Proposition 2.7. Before beginning, we require a helper lemma concerning the weighted Sobolev norms. While this result is somewhat elementary, we do not know of a reference.

LEMMA 4.1. *For $\tilde{s} \in (0, 1)$, $R > 0$, $\ell \geq 0$, and $\mathcal{D} = \{(v, v') \in \mathbb{R}^6 : |v - v'| \leq \langle v \rangle / R\}$, we have, for any $f \in H^{\tilde{s}, \ell}(\mathbb{R}^3)$,*

$$(4.6) \quad \int_{\mathcal{D}} \langle v \rangle^\ell \frac{|f(v) - f(v')|^2}{|v - v'|^{3+2\tilde{s}}} dv' dv \lesssim \|f\|_{H^{\tilde{s}, \ell}}^2.$$

Before beginning we remark briefly about the content of Lemma 4.1. Recall that $\|f\|_{\dot{H}^{\tilde{s}}} = \int \frac{|f(v) - f(v')|^2}{|v - v'|^{3+2\tilde{s}}} dv' dv$ and, hence,

$$\|f\|_{H^{\tilde{s}, \ell}}^2 = \|\langle v \rangle^\ell f\|_{\dot{H}^{\tilde{s}}}^2 = \|\langle v \rangle^\ell f\|_{L^2}^2 + \int \frac{|\langle v \rangle^\ell f(v) - \langle v' \rangle^\ell f(v')|^2}{|v - v'|^{3+2\tilde{s}}} dv dv'.$$

The difference between the quantity above and the left hand side of (4.6) is now clear.

Proof. To begin, we use the triangle inequality and that $(a+b)^2 \leq 2a^2 + 2b^2$ to find

$$\begin{aligned} \int_{\mathcal{D}} \langle v \rangle^\ell \frac{|f(v) - f(v')|^2}{|v - v'|^{3+2\bar{s}}} dv' dv &\lesssim \int_{\mathcal{D}} \frac{|\langle v \rangle^\ell f(v) - \langle v' \rangle^\ell f(v')|^2 + |\langle v \rangle^\ell - \langle v' \rangle^\ell|^2 |f(v')|^2}{|v - v'|^{3+2\bar{s}}} dv' dv \\ &= \int_{\mathcal{D}} \frac{|\langle v \rangle^\ell f(v) - \langle v' \rangle^\ell f(v')|^2}{|v - v'|^{3+2\bar{s}}} dv' dv + \int_{\mathcal{D}} \frac{|\langle v \rangle^\ell - \langle v' \rangle^\ell|^2 |f(v')|^2}{|v - v'|^{3+2\bar{s}}} dv' dv. \end{aligned}$$

The first term above is clearly bounded above by $\|f\|_{\dot{H}^{\bar{s}, \ell}}$ simply by enlarging the domain of integration. Hence, we consider only the second term.

For $(v, v') \in \mathcal{D}$, we find, via Taylor's theorem, that $|\langle v \rangle^\ell - \langle v' \rangle^\ell|^2 \lesssim \langle v \rangle^{2\ell-2} |v - v'|^2$. Thus,

$$\int_{\mathcal{D}} \frac{|\langle v \rangle^\ell - \langle v' \rangle^\ell|^2 |f(v')|^2}{|v - v'|^{3+2\bar{s}}} dv' dv \lesssim \int_{\mathcal{D}} \frac{\langle v \rangle^{2(\ell-1)} |f(v')|^2}{|v - v'|^{1+2\bar{s}}} dv' dv.$$

Next, clearly there exists $\tilde{R} > 0$ depending only on R such that $\mathcal{D} \subset \{(v, v') \in \mathbb{R}^6 : |v - v'| \leq \langle v' \rangle / \tilde{R}\}$. Additionally, $(v, v') \in \mathcal{D}$ implies that $\langle v \rangle \approx \langle v' \rangle$. These two facts yield

$$\begin{aligned} \int_{\mathcal{D}} \frac{|\langle v \rangle^\ell - \langle v' \rangle^\ell|^2 |f(v')|^2}{|v - v'|^{3+2\bar{s}}} dv' dv &\lesssim \int \langle v' \rangle^{2(\ell-1)} |f(v')|^2 \int_{B_{\langle v' \rangle / \tilde{R}}(v')} \frac{1}{|v - v'|^{1+2\bar{s}}} dv dv' \\ &\lesssim \int \langle v' \rangle^{2(\ell-1)} |f(v')|^2 \langle v' \rangle^{2-2\bar{s}} dv' \lesssim \int \langle v' \rangle^{2\ell} |f(v')|^2 dv' = \|f\|_{L^{2, \ell}}, \end{aligned}$$

which concludes the proof. $\square$

Proof of Proposition 2.7. We prove this using the characterization of the L^2 -norm via duality; that is, fix any $h \in L^2(\mathbb{R}^3)$ and we estimate

$$\int h (\langle v \rangle^\ell Q_s(g, f) - Q_s(g, \langle v \rangle^\ell f)) dv.$$

For any v , let $R_v = \langle v \rangle / 10$ and denote the diagonal strip

$$(4.7) \quad \mathcal{D} = \{(v, v') : |v - v'| < R_v\}.$$

Recalling (2.1), we rewrite the quantity of interest as

$$\begin{aligned} (4.8) \quad \int h (\langle v \rangle^\ell Q_s(g, f) - Q_s(g, \langle v \rangle^\ell f)) dv &= \int K_g(v, v') h(v) (\langle v \rangle^\ell f(v') - \langle v' \rangle^\ell f(v')) dv' dv \\ &= \int_{\mathcal{D}} K_g(v, v') h(v) (\langle v \rangle^\ell f(v') - \langle v' \rangle^\ell f(v')) dv' dv \\ &\quad + \int_{\mathcal{D}^c} K_g(v, v') h(v) (\langle v \rangle^\ell f(v') - \langle v' \rangle^\ell f(v')) dv' dv = I_1 + I_2. \end{aligned}$$

We estimate each of I_1 and I_2 in turn.

Step one: Bounding I_2 . We further decompose I_2 as

$$(4.9) \quad I_2 = \int_{\mathcal{D}^c} K_g(v, v') h(v) \langle v \rangle^\ell f(v') dv' dv - \int_{\mathcal{D}^c} K_g(v, v') h(v) \langle v' \rangle^\ell f(v') dv' dv = I_{21} - I_{22}.$$

We first consider I_{22} . Applying Lemma 2.3, we find

$$(4.10) \quad |I_{22}| \lesssim \|g\|_{L^\infty, m} \int_{\mathcal{D}^c} \frac{|h(v)|}{|v - v'|^{3+2s}} \langle v \rangle^{\gamma+2s+1} \langle v' \rangle^\ell |f(v')| dv' dv.$$

Rewriting the limits of integration, using that $|v - v'| \gtrsim \langle v \rangle$, and applying Cauchy–Schwarz in v' yields

$$\begin{aligned} \int_{\mathcal{D}^c} \frac{|h(v)|}{|v - v'|^{3+2s}} \langle v \rangle^{\gamma+2s+1} \langle v' \rangle^\ell |f(v')| dv' dv &= \int \langle v \rangle^{\gamma+2s+1} |h(v)| \left(\int_{B_{Rv}^c(v)} \frac{\langle v' \rangle^\ell |f(v')|}{|v - v'|^{3+2s}} dv' \right) dv \\ &\lesssim \int \langle v \rangle^{\gamma-2} h(v) \left(\int \langle v' \rangle^{-3-2\epsilon} dv' \right)^{1/2} \left(\int \langle v' \rangle^{\ell+3+2\epsilon} f^2(v') dv' \right)^{1/2} dv. \end{aligned}$$

Noticing that the integral involving f is a weighted L^2 norm of f , the middle integral is finite, and combining this with (4.10), we obtain

$$(4.11) \quad |I_{22}| \lesssim \|g\|_{L^\infty, m} \|f\|_{L^{2, \ell+1}} \int \langle v \rangle^{\gamma-\frac{3}{2}} h(v) dv = \|g\|_{L^\infty, m} \|f\|_{L^{2, \ell+3/2+\epsilon}} \|h\|_{L^2}.$$

We now consider I_{21} . Here, we split the integral as follows:

$$\begin{aligned} I_{21} &= \int_{\mathcal{D}^c \cap \{|v'| \geq |v|/10\}} K_g(v, v') h(v) \langle v \rangle^\ell f(v') dv' dv \\ &\quad + \int_{\mathcal{D}^c \cap \{|v'| \leq |v|/10\}} K_g(v, v') h(v) \langle v \rangle^\ell f(v') dv' dv \\ &= I_{211} + I_{212}. \end{aligned}$$

The estimate of I_{211} reduces to the estimate I_{22} :

$$\begin{aligned} |I_{211}| &\leq \int_{\mathcal{D}^c \cap \{|v'| \geq |v|/10\}} |K_g(v, v') h(v) \langle v \rangle^\ell f(v')| dv' dv \\ &\lesssim \int_{\mathcal{D}^c \cap \{|v'| \geq |v|/10\}} |K_g(v, v') h(v) \langle v' \rangle^\ell f(v')| dv' dv \\ &\leq \int_{\mathcal{D}^c} |K_g(v, v') h(v) \langle v' \rangle^\ell f(v')| dv' dv. \end{aligned}$$

The last term above is exactly the term we estimate in (4.10); hence,

$$(4.12) \quad |I_{211}| \lesssim \|g\|_{L^\infty, m} \|f\|_{L^{2, \ell+3/2+\epsilon}} \|h\|_{L^2}.$$

Turning to I_{212} , we get

$$(4.13) \quad |I_{212}| \lesssim \|g\|_{L^\infty, m} \int_{\mathcal{D}^c \cap \{|v'| \leq |v|/10\}} \frac{h(v) \langle v \rangle^{\ell+\gamma+2s+3-m} f(v')}{|v - v'|^{3+2s}} dv' dv.$$

After applying Cauchy–Schwarz to the integral in v' , a direct computation using that $\ell > 3/2$ yields

$$\begin{aligned} (4.14) \quad &\int_{\mathcal{D}^c \cap \{|v'| \leq |v|/10\}} \frac{h(v) \langle v \rangle^{\ell+\gamma+2s+3-m} f(v')}{|v - v'|^{3+2s}} dv' dv \\ &\leq \int \langle v \rangle^{\ell+\gamma+2s+3-m} h(v) \left(\int_{B_{Rv}(v) \cap B_{|v|/10}} \frac{\langle v' \rangle^{-2\ell} dv'}{|v - v'|^{6+4s}} \right)^{1/2} \left(\int \langle v' \rangle^{2\ell} f^2(v') dv' \right)^{1/2} dv \\ &\lesssim \|f\|_{L^{2, \ell}} \int \langle v \rangle^{\ell+\gamma-m} h(v) dv. \end{aligned}$$

Using that $m > \ell + \gamma + \frac{3}{2}$, we conclude from (4.13) and (4.14) that

$$(4.15) \quad |I_{212}| \lesssim \|g\|_{L^\infty, m} \|f\|_{L^{2, \ell}} \|h\|_{L^{1, \ell + \gamma - m}} \lesssim \|g\|_{L^\infty, m} \|f\|_{L^{2, \ell}} \|h\|_{L^2}.$$

Combining (4.11), (4.12), and (4.15) we deduce that

$$(4.16) \quad |I_2| \lesssim \|f\|_{L^{2, \ell + 1}} \|g\|_{L^\infty, m} \|h\|_{L^2}.$$

This concludes step one.

Step two: Bounding I_1 . For notational convenience, let $W_\ell(v) = \langle v \rangle^\ell$. For any function ψ and any velocities v and v' , let $\delta\psi = \psi(v) - \psi(v')$ (we suppress the dependence on v and v' as no confusion will arise). Then, we rewrite I_1 as

$$I_1 = \int_{\mathcal{D}} K_g(v, v') h(v) \delta f \delta W_\ell dv' dv + \int_{\mathcal{D}} K_g(v, v') h(v) f(v) \delta W_\ell dv' dv = I_{11} + I_{12}.$$

For I_{11} , we see, by the definition of the kernel K_g as in equation (2.2) and get that

$$(4.17) \quad I_{11}^2 \lesssim \|g\|_{L^\infty, m}^2 \left(\int_{\mathcal{D}} \frac{\langle v \rangle^{\gamma + 2s + 1}}{|v - v'|^{3 + 2s}} |h(v)| |\delta f| |\delta W_\ell| dv' dv \right)^2.$$

Next, applying the Cauchy-Schwarz inequality yields

$$(4.18) \quad \begin{aligned} & \left(\int_{\mathcal{D}} \frac{\langle v \rangle^{\gamma + 2s + 1}}{|v - v'|^{3 + 2s}} |h(v)| |\delta f| |\delta W_\ell| dv' dv \right)^2 \\ & \leq \left(\int_{\mathcal{D}} \frac{h(v)^2}{\langle v \rangle^{2\mu}} |v - v'|^{-3 + 2\mu} dv' dv \right) \left(\int_{\mathcal{D}} \langle v \rangle^{2(\mu + \gamma + 2s + 1)} \frac{(\delta f)^2 (\delta W_\ell)^2}{|v - v'|^{3 + 4s + 2\mu}} dv' dv \right). \end{aligned}$$

We first consider the integral involving h . Recalling the definition of $\mathcal{D}$ (4.7), we find

$$(4.19) \quad \begin{aligned} \int_{\mathcal{D}} h(v)^2 |v - v'|^{-3 + 2\mu} \langle v \rangle^{-2\mu} dv' dv & \leq \int h(v)^2 \langle v \rangle^{-2\mu} \int_{B_{R_v}(v)} |v - v'|^{-3 + 2\mu} dv' dv \\ & \lesssim \int h(v)^2 dv = \|h\|_{L^2}^2. \end{aligned}$$

Next, we consider the second integral in (4.18). Recall that $|v - v'| < R_v$ by the definition of $\mathcal{D}$, (4.7). Hence, by Taylor's theorem, we have

$$|\delta W_\ell|^2 \lesssim |v' - v|^2 \langle v \rangle^{2\ell - 2}.$$

Using this and Lemma 4.1, we find

$$(4.20) \quad \int_{\mathcal{D}} \frac{\langle v \rangle^{2(\mu + \gamma + 2s + 1)} (\delta f)^2 (\delta W_\ell)^2}{|v - v'|^{3 + 4s + 2\mu}} dv' dv \lesssim \int_{\mathcal{D}} \frac{\langle v \rangle^{2(\mu + \gamma + 2s + \ell)} (\delta f)^2}{|v - v'|^{3 + 2(2s - 1 + \mu)}} dv' dv \lesssim \|f\|_{H^{2s - 1 + \mu, \mu + \ell}}^2.$$

We conclude by combining (4.17)–(4.20) to obtain

$$(4.21) \quad |I_{11}| \lesssim \|g\|_{L^\infty, m} \|h\|_{L^2} \|f\|_{H^{2s - 1 + \mu, \mu + \ell + \gamma + 2s}}.$$

We consider now I_{12} . Using a second order Taylor expansion of $W_\ell(v) = \langle v \rangle^\ell$, we see that

$$\begin{aligned} I_{12} &= \int_{\mathcal{D}} K_g(v, v') h(v) f(v) \delta W_\ell dv' dv \\ &= \int h(v) f(v) \int_{B_{R_v}(v)} K_g(v, v') \left((D_v W_\ell)|_v(v-v') + \frac{1}{2} (v-v') \cdot (D_v^2 W_\ell)|_{\xi_{v,v'}}(v-v') \right) dv' dv \\ &= I_{121} + I_{122}, \end{aligned}$$

where $\xi_{v,v'} = tv' + (1-t)v$ for some $t \in [0, 1]$. For I_{121} , we use Lemma 2.1(iii) to obtain

$$(4.22) \quad I_{121} = 0.$$

For I_{122} , we use that $|(D_v^2 W_\ell)|_{\xi_{v,v'}}| \lesssim \langle v \rangle^{\ell-2}$, due to the fact that $v' \in B_{R_v}(v)$, in order to find

$$|I_{122}| \lesssim \int_{\mathbb{R}^3} |h(v) f(v)| \int_{B_{R_v}(v)} |K_g(v, v')| \langle v \rangle^{\ell-2} |v - v'|^2 dv' dv.$$

Thus, we have by appealing to Lemma 2.2

$$(4.23) \quad |I_{122}| \lesssim \|g\|_{L^\infty, m} \int_{\mathbb{R}^3} |h(v) f(v)| \langle v \rangle^{\ell-2+\gamma+2s} dv \lesssim \|g\|_{L^\infty, m} \|f\|_{L^{2, \ell-2+\gamma+2s}} \|h\|_{L^2}.$$

Combining (4.22) and (4.23) and the fact that $\ell + 3/2 + \epsilon > \ell - 2 + \gamma + 2s$, we find

$$(4.24) \quad |I_{12}| \lesssim \|g\|_{L^\infty, m} \|f\|_{L^{2, \ell+1}} \|h\|_{L^2}.$$

Thus, by (4.21) and (4.24),

$$|I_1| \lesssim (\|f\|_{L^{2, \ell+3/2+\epsilon}} + \|f\|_{H^{2s-1+\mu, \mu+\ell+\gamma+2s}}) \|g\|_{L^\infty, m} \|h\|_{L^2}.$$

This concludes step two and, thus, the proof. $\square$

4.3. Collection of Q_s estimates: Proof of Proposition 2.8(i)–(iv).

4.3.1. Proof of Proposition 2.8(i).

Proof. Let

$$(4.25) \quad \hat{K}_g(v, v') = \frac{1}{\|g\|_{L^\infty, 3+\gamma+2s+\epsilon}} K_g(v, v'),$$

and we have that $\hat{K}_g$ satisfies the conditions (4.2), (4.3), and (4.4) in [34, section 4] uniformly in v . This allows us to apply their general estimates, which we do now. For clarity, we adopt their notation as closely as possible.

Let $\hat{L}_g$ be the operator defined by replacing the kernel K_g with $\hat{K}_g$ in Q_s , and let $\hat{L}_g^t$ be its transpose. Letting Δ_i be the Littlewood–Paley projectors as in [34, Proof of Theorem 4.1] and using [34, Theorems 4.3 and 4.6] yields, for any θ ,

$$\|\hat{L}_g \Delta_i f\|_{L^2} \lesssim 2^{i\theta} \|\Delta_i f\|_{H^{2s-\theta}} \quad \text{and} \quad \|\hat{L}_g^t \Delta_i h\|_{L^2} \lesssim 2^{i(2s-\theta)} \|\Delta_i h\|_{H^\theta}.$$

Also, recall that $\|\Delta_i \phi\|_{H^\theta} \approx 2^{i\theta} \|\Delta_i \phi\|_{L^2}$ for any θ , i , and ϕ .

Using all estimates above for any fixed $\theta \in (0, 2s)$ yields

$$\begin{aligned} \frac{1}{\|g\|_{L^\infty, 3+\gamma+2s+\epsilon}} \int Q_s(g, f) h dv &= \frac{1}{\|g\|_{L^\infty, 3+\gamma+2s+\epsilon}} \sum_{i,j} \int Q_s(g, \Delta_i f) \Delta_j h dv \\ &= \sum_{i \leq j} \int (\hat{L}_g \Delta_i f) \Delta_j h dv + \sum_{\theta i > (2s-\theta)j} \int \Delta_i f (\hat{L}_g^t \Delta_j h) dv \\ &\lesssim \sum_{\theta i \leq (2s-\theta)j} 2^{\theta i - (2s-\theta)j} \|\Delta_i f\|_{H^{2s-\theta}} \|\Delta_j g\|_{H^\theta} + \sum_{\theta i > (2s-\theta)j} 2^{-\theta i + (2s-\theta)j} \|\Delta_i f\|_{H^{2s-\theta}} \|\Delta_j g\|_{H^\theta} \\ &= \sum_{i,j} 2^{-|\theta i - (2s-\theta)j|} \|\Delta_i f\|_{H^{2s-\theta}} \|\Delta_j g\|_{H^\theta} \\ &\leq \left(\sum_{i,j} 2^{-|\theta i - (2s-\theta)j|} \|\Delta_i f\|_{H^{2s-\theta}}^2 \right)^{1/2} \left(\sum_{i,j} 2^{-|\theta i - (2s-\theta)j|} \|\Delta_j g\|_{H^\theta}^2 \right)^{1/2} \lesssim \|f\|_{H^{2s-\theta}} \|g\|_{H^\theta}. \end{aligned}$$

In the last inequality, we sum first over j , using that $\theta, 2s - \theta > 0$ by assumption, and then recalling that $\sum_i \|\Delta_i f\|_{H^{2s-\theta}}^2 \approx \|f\|_{H^{2s-\theta}}^2$ (and similarly for g). $\square$

4.3.2. Proof of Proposition 2.8(ii).

Proof. We adopt the notation and setting of the proof of Proposition 2.8(i). Then

$$\begin{aligned} \frac{1}{\|g\|_{L^\infty, 3+\gamma+2s+\epsilon}^{1/2}} \|Q_s(g, f)\|_{L^2} &= \|\hat{L}_g f\|_{L^2} \leq \sum_{i=0}^{\infty} \|\hat{L}_g \Delta_i f\|_{L^2} \lesssim \sum_{i=0}^{\infty} 2^{-i\theta} \|\Delta_i f\|_{H^{2s+\theta}} \\ &\lesssim \left(\sum_{i=0}^{\infty} \|\Delta_i f\|_{H^{2s+\theta}}^2 \right)^{1/2} \approx \|f\|_{H^{2s+\theta}}. \end{aligned}$$

4.3.3. Proof of Proposition 2.8(iii). In order to establish part (iii) of Proposition 2.8, we require an analogue of Young's convolution inequality in the setting of the weighted Lebesgue spaces in order to handle terms of the form $\int g(w) |v - w|^{\gamma+2s} dw$. These have been well-studied and are understood in some generality (see, e.g., [24]). However, for the convenience of the reader and because we can get a slightly sharper estimate (due to the specific form considered here), we include the proof.

LEMMA 4.2 (weighted Young's inequality). *Suppose that $n > 3/2 + \eta$, $-3 < \eta < 0$, and $\ell > 3/2 + \eta + (3/2 - n)_+$. If $g \in L^{2,n}$, then*

$$(4.26) \quad \int \langle v \rangle^{-2\ell} \left(\int g(\tilde{v}) |v - \tilde{v}|^\eta d\tilde{v} \right)^2 dv \lesssim \|g\|_{L^{2,n}}^2.$$

Proof. For succinctness, we let $A(v) = |v|^\eta$ and, without loss of generality, we assume that $g \geq 0$. First, we decompose the integral on the left hand side yielding

$$\begin{aligned} \int \langle v \rangle^{-2\ell} (g * A)^2 dv &\leq \int \langle v \rangle^{-2\ell} \left(\int_{B_{R_v}(v)} g(v') A(v - v') dv' \right)^2 dv \\ &\quad + \int \langle v \rangle^{-2\ell} \left(\int_{B_{R_v}^c(v)} g(v') A(v - v') dv' \right)^2 dv = I_1 + I_2. \end{aligned}$$

For I_1 , we use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} I_1 &\leq \int \langle v \rangle^{-2\ell} \left(\int_{B_{|v|/10}(v)} g(v')^2 A(v - v') dv' \right) \left(\int_{B_{|v|/10}(v)} A(v - v') dv' \right) dv \\ &\lesssim \int \langle v \rangle^{-2\ell+3+\eta} \left(\int_{B_{|v|/10}(v)} g(v')^2 A(v - v') dv' \right) dv. \end{aligned}$$

For $v' \in B_{|v|/10}(v)$, we have $\langle v' \rangle \approx \langle v \rangle$ and $v \in B_{|v'|/2}(v')$. Therefore,

$$\begin{aligned} I_1 &\lesssim \int |g(v')|^2 \int_{B_{|v'|/2}(v')} \langle v \rangle^{-2\ell+3+\eta} A(v-v') dv dv' \\ &\lesssim \int |g(v')|^2 \langle v' \rangle^{-2\ell+3+\eta} \int_{B_{|v'|/2}(v')} A(v-v') dv dv' \lesssim \int g(v')^2 \langle v' \rangle^{-2\ell+2(3+\eta)} dv' \lesssim \|g\|_{L^{2,n}}^2, \end{aligned}$$

where we used that $-\ell + (3+\eta) \leq n$. For I_2 , we apply the Cauchy–Schwarz inequality to find

$$\begin{aligned} I_2 &\leq \int \langle v \rangle^{-2\ell} \left(\int_{B_{|v|/10}^c(v)} \langle v' \rangle^{2n} |g(v')|^2 dv' \right) \left(\int_{B_{|v|/10}^c(v)} \langle v' \rangle^{-2n} |v-v'|^{2\eta} dv' \right) dv \\ &\lesssim \|g\|_{L^{2,n}}^2 \int \langle v \rangle^{-2\ell} \left(\int_{B_{|v|/10}^c(v)} \langle v' \rangle^{-2n} |v-v'|^{2\eta} dv' \right) dv \lesssim \|g\|_{L^{2,n}}^2 \int \langle v \rangle^{-2\ell+(3-2n)_++2\eta} dv. \end{aligned}$$

We conclude by using the conditions on n and ℓ . These were also used in the last inequality. Combining the estimates of I_1 and I_2 finishes the proof. $\square$

We are now able to prove Proposition 2.8(iii).

Proof of Proposition 2.8(iii). The proof is somewhat close to that of [28, Proposition 3.1(i)], so we omit details where steps are similar. We may, without loss of generality, assume that $\alpha \in (0, 1) \cup (1, 2)$. If not, we may simply take $\alpha' < \alpha$ such that $\alpha' \in (0, 1) \cup (1, 2)$ and use that $C^\alpha \hookrightarrow C^{\alpha'}$. Finally, the proof is simpler when $\alpha < 1$; hence, we consider only the case $\alpha \in (1, 2)$.

We begin with an annular decomposition: let $A_k(v) = B_{2^k|v|}(v) \setminus B_{2^{k-1}|v|}(v)$ and write

$$(4.27) \quad Q_s(g, f) = \sum_{k \in \mathbb{Z}} \int_{A_k(v)} K_g(v, v') (f(v') - f(v)) dv'.$$

Let $\bar{\mu} = n + 5/2 + (3/2 - n)_+ + \alpha + \gamma + \epsilon$.

Step one: Estimating the sum for any $k \leq 1$. By using a Taylor expansion, we see

$$f(v') - f(v) = ((Df)(\xi_{v',v}) - (Df)(v)) \cdot (v' - v) + (Df)(v) \cdot (v' - v),$$

where $\xi_{v,v'} = tv' + (1-t)v$ for some $t \in [0, 1]$. Thus, by Lemma 2.1(i) and (iii),

$$\left| \int_{A_k(v)} K_g(v, v') (f(v') - f(v)) dv' \right| \lesssim \langle v \rangle^{-\bar{\mu}} (2^k |v|)^{\alpha-2s} \| \langle \cdot \rangle^{\bar{\mu}} Df \|_{C^\alpha} \int |g(v')| |v - v'|^{\gamma+2s} dv'.$$

Recalling that $\alpha - 2s > 0$, by assumption, we have that $|v|^{\alpha-2s} \leq \langle v \rangle^{\alpha-2s}$. Hence,

$$\begin{aligned} &\int \left(\int_{A_k(v)} \langle v \rangle^n K_g(v, v') (f(v') - f(v)) dv' \right)^2 dv \\ &\lesssim 2^{2k(1+\alpha-2s)} \| \langle \cdot \rangle^{\bar{\mu}} Df \|_{C^\alpha}^2 \int \langle v \rangle^{-2(\bar{\mu}-n-1-\alpha+2s)} \left(\int |g(v')| |v - v'|^{\gamma+2s} dv' \right)^2 dv. \end{aligned}$$

We are now in a position to apply the weighted Young's convolution inequality Lemma 4.2. Indeed, by construction, $\ell := \bar{\mu} - n - 1 - \alpha + 2s$ and n satisfy the conditions of Lemma 4.2 so that

$$\int \left(\int_{A_k(v)} \langle v \rangle^n K_g(v, v') (f(v') - f(v)) dv' \right)^2 dv \lesssim 2^{2k(1+\alpha-2s)} \| \langle \cdot \rangle^{\bar{\mu}} Df \|_{C^\alpha}^2 \|g\|_{L^{2,n}}^2.$$

Step two: Estimating the sum for $k \geq 0$ when $|v'| \geq \langle v \rangle/2$. By Lemma 2.1(i),

$$\begin{aligned} \left| \int_{A_k \setminus B_{\langle v \rangle/2}} K_g(v, v')(f(v') - f(v)) dv' \right| &\lesssim \langle v \rangle^{-m} \|f\|_{L^\infty, m} \int_{A_k \setminus B_{\langle v \rangle/2}} |K_g(v, v')| dv' \\ &\lesssim \langle v \rangle^m \|f\|_{L^\infty, m} (2^k \langle v \rangle)^{-2s} \left(\int |g(v')| |v - v'|^{\gamma+2s} dv' \right). \end{aligned}$$

Then, similarly to step one, we apply Lemma 4.2 to obtain

$$\begin{aligned} &\int \langle v \rangle^{2n} \left| \sum_{k \geq 0} \int_{A_k \setminus B_{\langle v \rangle/2}} K_g(v, v')(f(v') - f(v)) dv' \right|^2 dv \\ &\lesssim \|f\|_{L^\infty, m}^2 \left(\sum_{k \geq 0} 2^{-2ks} \right)^2 \int \langle v \rangle^{2(n-m-2s)} \left(\int |g(v')| |v - v'|^{\gamma+2s} dv' \right)^2 dv \lesssim \|f\|_{L^\infty, m}^2 \|g\|_{L^{2, n}}^2, \end{aligned}$$

where we used $n > 3/2 + \gamma + 2s$ and $m > 3/2 + \gamma + (3/2 - n)_+$.

Step three: Estimating the sum for $k \geq 0$ when $|v| \leq 10$ and $|v'| \leq \langle v \rangle/2$. This is similar to step one. The benefit is we are integrating over a compact set in v . As such, we omit the proof and simply state that

$$\begin{aligned} &\int_{B_{10}} \left(\sum_{k \geq 0} \int_{A_k \cap B_{\langle v \rangle/2}} \langle v \rangle^n K_g(v, v')(f(v') - f(v)) dv' \right)^2 dv \\ &= \int_{B_{10}} \left(\int_{B_{\langle v \rangle/2} \setminus B_{|v|/2}} \langle v \rangle^n K_g(v, v')(f(v') - f(v)) dv' \right)^2 dv \lesssim \|Df\|_{C^\alpha}^2 \|g\|_{L^{2, n}}^2. \end{aligned}$$

Hence, we proved step three.

Step four: Estimating the sum for $k \geq 0$ when $|v| \geq 10$ and $|v'| \leq \langle v \rangle/2$. For any $|v| \geq 10$,

$$\begin{aligned} &\left| \sum_{k \geq 0} \int_{A_k \cap B_{\langle v \rangle/2}} K_g(v, v')(f(v') - f(v)) dv' \right| \\ &\lesssim \int_{B_{\langle v \rangle/2}} K_{|g|}(v, v') |f(v')| dv' + \int_{B_{\langle v \rangle/2}} K_{|g|}(v, v') |f(v)| dv' = I_1 + I_2. \end{aligned}$$

For I_2 , we notice that $B_{\langle v \rangle/2} \subseteq (B_{2\langle v \rangle}(v) \setminus B_{\langle v \rangle/4}(v))$ due to the fact that $|v| \geq 10$. Then by Lemma 2.1(i), we have

$$\int_{B_{\langle v \rangle/2}} K_{|g|}(v, v') dv' \lesssim \int_{B_{2\langle v \rangle}(v) \setminus B_{\langle v \rangle/4}(v)} K_{|g|}(v, v') dv' \lesssim \langle v \rangle^{-2s} \int |g(v')| |v - v'|^{\gamma+2s} dv'.$$

Applying the weighted Young's inequality Lemma 4.2 yields

$$\begin{aligned} &\int \langle v \rangle^{2n} I_2^2 dv \lesssim \int \langle v \rangle^{2(n-2s)} |f(v)|^2 \left(\int |g(v')| |v - v'|^{\gamma+2s} dv' \right)^2 dv \\ &\lesssim \|f\|_{L^\infty, m}^2 \int \langle v \rangle^{-2m+2n-4s} \left(\int |g(v')| |v - v'|^{\gamma+2s} dv' \right)^2 dv \lesssim \|f\|_{L^\infty, m}^2 \|g\|_{L^{2, n}}^2, \end{aligned}$$

as desired. For I_1 , the proof is omitted as it is exactly as in [28, Proposition 3.1(i)]. This finishes the proof. $\square$

4.3.4. Proof of Proposition 2.8(iv).

Proof. Without loss of generality, we assume that $f, g \geq 0$. Let $F = \langle v \rangle^n f(v)$. We see

$$\int \langle v \rangle^{2n} Q_s(g, f) f \, dv = \int F Q_s(g, F) \, dv + \int F [\langle v \rangle^n Q_s(g, f) - Q_s(g, F)] \, dv = I_1 + I_2.$$

We further decompose I_1 into three parts:

$$\begin{aligned} I_1 &= - \int [F(v) - F(v')]^2 K_g(v, v') \, dv' \, dv + \int \int [K_g(v, v') - K_g(v', v)] F(v) F(v') \, dv' \, dv \\ &\quad - \int [K_g(v, v') - K_g(v', v)] F(v')^2 \, dv' \, dv = I_{11} + I_{12} + I_{13}. \end{aligned}$$

The first term, I_{11} , has a good sign (and is used for cancellation below). The integrand in I_{12} is antisymmetric with respect to the “pre-post change of variables” $(v, v') \mapsto (v', v)$, so $I_{12} = 0$. To estimate I_{13} , we use Lemma 2.1(ii). Hence, we find

$$|I_{12}| \lesssim \int F(v') \int g(z) |z - v'|^{\gamma+2s} \, dz \, dv' \lesssim \|g\|_{L^{\infty, m}} \|F\|_{L^2} = \|g\|_{L^{\infty, m}} \|f\|_{L^{2, n}}.$$

Here we used that $m > 3 + \gamma + 2s$ and $\gamma + 2s \leq 0$. This concludes the bound on I_1 .

For I_2 , we apply Young’s inequality to find

$$\begin{aligned} I_2 &= \int F(v) f(v') K_g(v, v') (\langle v \rangle^n - \langle v' \rangle^n) \, dv' \, dv \\ &= \int (F(v) - F(v')) f(v') K_g(v, v') (\langle v \rangle^n - \langle v' \rangle^n) \, dv' \, dv \\ &\quad + \int F(v') f(v') K_g(v, v') (\langle v \rangle^n - \langle v' \rangle^n) \, dv' \, dv \\ &\leq -\frac{1}{2} I_{11} + \frac{1}{2} \int f^2(v') K_g(v, v') (\langle v \rangle^n - \langle v' \rangle^n)^2 \, dv' \, dv \\ &\quad + \int F(v') f(v') K_g(v, v') (\langle v \rangle^n - \langle v' \rangle^n) \, dv' \, dv. \end{aligned}$$

Define the last two integrals to be I_{21} and I_{22} . The argument for I_{21} is similar to and easier than I_{22} ; hence, we omit it.

We now bound I_{22} . To do so, we split the integral into domains of integration $\mathcal{D}$, $\mathcal{D}^c \cap \{|v| \leq 10|v'|\}$, and $\mathcal{D}^c \cap \{|v| \geq 10|v'|\}$, where $\mathcal{D} = \{(v, v') : 10|v - v'| \leq \min\{\langle v \rangle, \langle v' \rangle\}\}$. We denote the resulting integrals I_{221} , I_{222} , and I_{223} , respectively.

Considering I_{221} first, we use a Taylor expansion, Lemmas 2.1(iv) and 2.2, and the fact that $\langle v \rangle \approx \langle v' \rangle$ to find, for ξ between v and v' ,

$$\begin{aligned} |I_{221}| &\leq \int \frac{F(v')^2}{\langle v' \rangle} \int_{B_{\langle v' \rangle/2}(v')} K_g(v, v') \left[(v - v') \cdot v' n \langle v' \rangle^{n-2} + \frac{n \langle \xi \rangle^{n-2}}{2} (v - v') \right. \\ &\quad \left. \cdot \left(\text{Id} + \frac{\xi \otimes \xi}{|\xi|^2} \right) (v - v') \right] \, dv \, dv' \end{aligned}$$

$$\begin{aligned}
 & \lesssim \int \frac{F(v')^2}{\langle v' \rangle} \left| \int_{B_{\langle v' \rangle/2}(v')} K_g(v, v')(v - v') dv \right| dv' \\
 & \quad + \int \frac{F(v')^2}{\langle v' \rangle^2} \int_{B_{\langle v' \rangle/2}(v')} K_g(v, v') |v - v'|^2 dv dv' \\
 & \lesssim \int \frac{F(v')^2}{\langle v' \rangle} \int g(w) |v' - w|^{1+\gamma} dw dv' \\
 & \quad + \int \frac{F(v')^2}{\langle v' \rangle^{2s}} \int g(w) |v' - w|^{\gamma+2s} dw dv' \lesssim \|g\|_{L^\infty, m} \|f\|_{L^{2, n}}^2.
 \end{aligned}$$

Above, we used that $m > 3 + \gamma + 2s$.

Next we consider I_{222} . In this case $\langle v \rangle \lesssim \langle v' \rangle$; hence, using that Lemma 2.1(i) and that $m > 3 + \gamma + 2s$ yields

$$\begin{aligned}
 |I_{222}| & \lesssim \int F(v')^2 \int_{B_{\langle v' \rangle/2}^c(v')} K_g(v, v') dv dv' \\
 & \lesssim \int \frac{F(v')^2}{\langle v' \rangle^{2s}} \int g(w) |v' - w|^{\gamma+2s} dw dv' \lesssim \|g\|_{L^\infty, m} \|f\|_{L^{2, n}}^2.
 \end{aligned}$$

Finally, we handle I_{223} . Indeed, we use that $\langle v' \rangle \lesssim \langle v \rangle$, the definition of $\mathcal{D}$, and Lemma 2.4 to get

$$\begin{aligned}
 & \left| \int_{\mathcal{D}^c \cap \{|v| \geq 10|v'|\}} F(v') f(v') K_g(v, v') (\langle v \rangle^n - \langle v' \rangle^n) dv' dv \right| \\
 & \lesssim \int_{\mathcal{D}^c \cap \{|v| \geq 10|v'|\}} F(v') f(v') K_g(v, v') \langle v \rangle^n dv' dv \\
 & \lesssim \int_{\mathcal{D}^c \cap \{|v| \geq 10|v'|\}} F(v') f(v') \frac{|v - v'|^{3+2s}}{\langle v' \rangle^{3+2s}} K_g(v, v') \langle v \rangle^n dv' dv \\
 & \lesssim \|g\|_{L^\infty, m} \int F(v') f(v') \int_{\{10|v'| \leq |v|\}} \langle v \rangle^{-m+3+\gamma+2s+n} dv dv' \lesssim \|g\|_{L^\infty, m} \|f\|_{L^{2, n}}^2.
 \end{aligned}$$

In the last inequality, we used that $m > n + 6 + \gamma + 2s$. This concludes the proof. $\square$

4.4. Proof of Proposition 2.9. In order to prove Proposition 2.9, we first state a useful estimate that follows from work in [34].

LEMMA 4.3. *For any measurable g , if $\gamma + 2s \leq 0$ and $\epsilon > 0$, then*

$$(4.28) \quad \left| \int K_g(f' - f)^2 dv' dv \right| \lesssim \|g\|_{L^\infty, 3+\gamma+2s+\epsilon} \|f\|_{H^s}^2.$$

Proof. Recall that $\hat{K}_g$, defined in (4.25), satisfies the conditions (4.2), (4.3), and (4.4) in [34, section 4] uniformly in v . Thus, applying [34, Lemma 4.2], we find

$$\begin{aligned}
 & \left| \int K_g(f' - f)^2 dv' dv \right| \\
 & = \|g\|_{L^\infty, 3+\gamma+2s+\epsilon} \left| \int \hat{K}_g(v, v') (f' - f)^2 dv' dv \right| \lesssim \|g\|_{L^\infty, 3+\gamma+2s+\epsilon} \|f\|_{H^s}^2,
 \end{aligned}$$

which concludes the proof. $\square$

Now we prove Proposition 2.9.

Proof of Proposition 2.9. We consider only the case $\partial = \partial_{v_i}$ for $i \in \{1, 2, 3\}$. The case when $\partial = \partial_{x_i}$ is similar and simpler as it commutes with $\langle v \rangle^{2n}$. First, let $F = \langle v \rangle^n f$. Then

$$\begin{aligned} \int \langle v \rangle^{2n} Q_s(g, f) \partial f \, dv dx &= \int [\langle v \rangle^n Q_s(g, f) - Q_s(g, \langle v \rangle^n f)] \langle v \rangle^n \partial f \, dv dx \\ &\quad - \int Q_s(g, F) f n v_i \langle v \rangle^{n-2} \, dv dx \\ &\quad + \int Q_s(g, F) \partial F \, dv dx = I_1 + I_2 + I_3. \end{aligned}$$

For I_1 , we apply the commutator estimate Proposition 2.7 to get

$$\begin{aligned} |I_1| &\lesssim \int (\|f\|_{L_v^{2, n+3/2+\epsilon}} + \|f\|_{H_v^{2s-1+\mu, \mu+n+\gamma+2s}}) \|g\|_{L_v^{\infty, m}} \|\partial f\|_{L_v^{2, n}} \, dx \\ (4.29) \quad &\lesssim \int (\|f\|_{L_v^{2, n+3/2+\epsilon}} + \|f\|_{H_v^{2s-1+\mu, \mu+n+\gamma+2s}}) \|g\|_{L_v^{\infty, m}} \|f\|_{H_v^{1, n}} \, dx \\ &\lesssim \|g\|_{L^{\infty, m}} (\|f\|_{L^{2, n+3/2+\epsilon}} + \|f\|_{H^{2s-1+\mu, \mu+n+\gamma+2s}}) \|f\|_{H^{1, n}}. \end{aligned}$$

To estimate I_2 , we apply Proposition 2.8(i) with $\theta = 1$ if $s > 1/2$ or $\theta = 2s - 1 + \mu$ if $s \leq 1/2$ to find

$$\begin{aligned} |I_2| &\lesssim \int \|g\|_{L_v^{\infty, m}} \|F\|_{H_v^\theta} \|f n v_i \langle v \rangle^{n-2}\|_{H_v^{2s-\theta}} \, dx \\ (4.30) \quad &\lesssim \int \|g\|_{L_v^{\infty, m}} \|f\|_{H_v^{\theta, n}} \|f\|_{H_v^{2s-\theta, n-1}} \, dx \lesssim \|g\|_{L^{\infty, m}} \|f\|_{H^{\theta, n}} \|f\|_{H^{2s-\theta, n-1}}. \end{aligned}$$

Using the choice of θ , the right hand side above is less than or equal to (up to a constant) the right hand side of (4.29).

We decompose I_3 into two parts:

$$\begin{aligned} I_3 &= \int Q_s(g, F) \partial F \, dv dx \\ &= \int K_g(F' - F)(\partial F - (\partial F)') \, dv' dv dx \\ &\quad + \int (K_g - K'_g)(F' - F) \partial F \, dv' dv dx = I_{31} + I_{32}. \end{aligned}$$

For I_{31} , we manipulate by integration-by-parts and apply Lemma 4.3 to find

$$\begin{aligned} |I_{31}| &= \left| \int K_g(\partial + \partial')(F' - F)^2 \, dv' dv dx \right| = \left| \int (\partial + \partial') K_g(F' - F)^2 \, dv' dv dx \right| \\ &= \left| \int K_{\partial g}(F' - F)^2 \, dv' dv dx \right| \lesssim \int \|\partial g\|_{L_v^{\infty, 3+\gamma+2s+\epsilon}} \|F\|_{H_v^s}^2 \, dx \\ &= \int \|\partial g\|_{L_v^{\infty, 3+\gamma+2s+\epsilon}} \|f\|_{H_v^s}^2 \, dx. \end{aligned}$$

Fix the conjugate exponents $p = 3/2(1 - s)$ and $q = 3/(2s + 1)$. Applying Hölder's inequality and the Sobolev embedding theorem yields

$$\begin{aligned} |I_{31}| &\lesssim \|\partial g\|_{L_x^p L_v^{\infty, 3+\gamma+2s+\epsilon}} \|f\|_{L_x^{2q} H_v^s}^2 \\ &\lesssim \|\partial g\|_{H_x^{(2s-1/2)+} H_v^{3/2+\epsilon, 3+\gamma+2s+\epsilon}} \|f\|_{H_x^{1-s} H_v^s}^2 \\ &\lesssim \|\partial g\|_{H^{3/2+\epsilon+(2s-1/2)+, 3+\gamma+2s+\epsilon}} \|f\|_{H^{1, n}}^2. \end{aligned}$$

The term I_{32} is considered in [28, Proposition 3.1(iv), estimate of I_2]. A close inspection of the proof shows that it applies in our setting. Hence, for simplicity, we cite directly that, for any $\mu \in (s, \min\{2s, 1\})$,

$$|I_{32}| \lesssim \|g\|_{C^{\mu, 3+\epsilon}} \|f\|_{H^{s, n+3/2+\epsilon+(\gamma+2s+1)_+}} \|f\|_{H^{1, n}}.$$

Combining the above estimates of I_{31} and I_{32} together yields

$$(4.31) \quad |I_3| \lesssim \|\partial g\|_{H^{3-s, 3}} \|f\|_{H^{1, n}}^2 + \|g\|_{C^{\mu, 3+\epsilon}} \|f\|_{H^{s, n+3/2+\epsilon+(\gamma+2s+1)_+}} \|f\|_{H^{1, n}}.$$

The proof is finished after combining (4.29), (4.30), and (4.31). $\square$

5. A simple proof of local well-posedness when $0 < s < 1/2$: **Theorem 1.2.** Here we provide a short proof of local well-posedness when $s \in (0, 1/2)$, taken as a standing assumption throughout the section even when not explicitly stated. As many of the technical details are exactly the same as in the proof of Theorem 1.1, we only outline the main points. As the proof is the same for $k > 1$, we show only the $k = 1$ case. Thus, we simplify the notation using $\tilde{X}^{m_0, m_1}$ in place of $\tilde{X}^{1, m_0, m_0}$ (the definition of $\tilde{X}^{k, m_0, m_1}$ is given in (1.5)).

The first step is to obtain a weighted C^1 estimate of Q_s .

LEMMA 5.1. *Let $m_1 > 3 + \gamma + 2s$ and m_0 sufficiently large depending only on m_1 , s , and γ . The following inequality holds:*

$$\|Q_s(g, f)\|_{L^\infty, m_1} \lesssim \|g\|_{L^\infty, m_1} (\|f\|_{L^\infty, m_0} + \|\nabla_v f\|_{L^\infty, m_1}).$$

Proof. Let $\mu = 1$ if $\gamma \leq -1$ and $\mu = \frac{-\gamma-2s}{1-2s}$ otherwise. Fix $r = \langle v \rangle^\mu / 2$. We first decompose the integral into two parts:

$$|Q_s(g, f)\langle v \rangle^{m_1}| = \int |\langle v \rangle^{m_1} (f(v') - f(v)) K_g(v, v')| dv' \leq I_1 + I_2,$$

where I_1 and I_2 are the integrals over $B_r(v)$ and $B_r(v)^c$, respectively. Applying Lemma 2.2 and using that if $\xi \in B_r(v)$, then $\langle \xi \rangle \approx \langle v \rangle$, we bound I_1 as

$$(5.1) \quad I_1 \lesssim \|\nabla_v f\|_{L^\infty, m_1} \int_{B_r(v)} |v - v'| K_{|g|}(v, v') dv' \lesssim \|\nabla_v f\|_{L^\infty, m_1} r^{1-2s} \int |g(w)| |v - w|^{\gamma+2s} dw.$$

We are finished after bounding the integral by $\langle v \rangle^{\gamma+2s} \|g\|_{L^\infty, m_0}$ and using the definition of r .

The first step to handle I_2 is to split it into the parts containing $f(v)$ and $f(v')$ via the triangle inequality. Call these integrals I_{21} and I_{22} , respectively. Using item (i) again, we see that

$$I_{21} = \int_{B_r(v)^c} \langle v \rangle^{m_1} |f(v)| K_{|g|}(v, v') dv' \lesssim \|f\|_{L^\infty, m_0} \langle v \rangle^{-m_0} r^{-2s} \int g(z) |v - z|^{\gamma+2s} dv'.$$

Bounding the last integral using $\|g\|_{L^\infty, m_1}$ and using the definition of r finishes the estimate of I_{21} .

The last integral, that of I_{22} , requires further decomposition into I_{221} and I_{222} over the domains $B_r(v)^c \cap B_{\langle v \rangle/2}^c$ and $B_r(v)^c \cap B_{\langle v \rangle/2}$. The former is easy to handle using

$$|f(v')| \leq \|f\|_{L^\infty, m_0} \langle v' \rangle^{-m_0} \lesssim \|f\|_{L^\infty, m_0} \langle v \rangle^{-m_0},$$

where we used that $\langle v' \rangle \gtrsim \langle v \rangle$. The rest of the bound follows exactly as for I_{21} .

As for I_{222} , notice that for such v' , $|v - v'| \approx \langle v \rangle$. We use this, along with Lemma 2.4, to find

$$(5.2) \quad \begin{aligned} I_{222} &\lesssim \|g\|_{L^\infty, m_1} \int_{B_r(v)^c \cap B_{\langle v \rangle/2}} \frac{\langle v \rangle^{m_1} f(v')}{\langle v \rangle^{3+2s}} \langle v \rangle^{3+\gamma+2s-m_1} dv' \\ &\lesssim \|g\|_{L^\infty, m_1} \int_{B_r(v)^c \cap B_{\langle v \rangle/2}} \langle v \rangle^\gamma f(v') dv' \lesssim \|g\|_{L^\infty, m_1} \|f\|_{L^\infty, m_0}. \end{aligned}$$

Combining this with the above estimates finishes the proof. $\square$

Next, we give the key estimate for constructing a solution. To that end, we present a proposition that plays the role of Proposition 3.1 above. Recall the space $\tilde{Y}^{m_0, m_1}$ from (1.5).

PROPOSITION 5.2 (propagation of the weighted C^1 bounds). *Fix any $m_1 > 3 + \gamma + 2s$ and m_0 sufficiently large depending only on m_1 , γ , and s . Suppose that $f_{\text{in}} \in \tilde{X}^{m_0, m_1}$ and $g, R \in \tilde{Y}_T^{m_0, m_1}$. If f solves (3.2), then there is a constant $C > 0$ depending only on m , s , and γ such that*

$$\|f\|_{\tilde{Y}^{m_0, m_1}} \lesssim \exp \left\{ C \int_0^T \|g(t)\|_{\tilde{X}^{m_0, m_1}} dt \right\} \left(\|f_{\text{in}}\|_{\tilde{X}^{m_0, m_1}} + T \|R\|_{\tilde{Y}_T^{m_0, m_1}} \right).$$

Proof. First notice that the proof of the bound

$$(5.3) \quad \|f\|_{L^\infty([0, T]; L^\infty, m_0)} \lesssim e^{C \int_0^T \|g(t)\|_{L^\infty, m_0} dt} \left(\|f_{\text{in}}\|_{\tilde{X}^{m_0, m_1}} + \int_0^T \|R(t)\|_{\tilde{X}^{m_0, m_1}} dt \right)$$

is exactly the same as the (brief) proof in [28, Proposition 3.1] and, hence, is omitted here. We note that it is a simpler version of the proof of the bounds on the derivatives that follows.

We now focus instead on bounding $\nabla_{x,v} f$. Fix $\phi(t)$ to be an increasing function to be determined such that $\phi(0) = \|\nabla_{x,v} f_{\text{in}}\|_{L^\infty, m_1}$, and let $F(t, x, v) = \phi(t) \langle v \rangle^{-m_1}$. Clearly we have that

$$(5.4) \quad F(0, x, v) > \max \{ |\partial_{x_i} f_{\text{in}}(x, v)|, |\partial_{v_i} f(x, v)| : i \in \{1, 2, 3\} \} \quad \text{for all } (x, v).$$

Let t_0 be the first time that the above inequality is violated. If t_0 does not exist, we are finished. Hence, we argue by contradiction assuming that there exists $t_0 \in [0, T]$. Without loss of generality,¹ we may assume that there exists $(x_0, v_0) \in \mathbb{T}^3 \times \mathbb{R}^3$ such that equality above holds in (5.4) at the point (t_0, x_0, v_0) . Assume momentarily that

$$(5.5) \quad F(t_0, x_0, v_0) = \partial_{x_1} f(t_0, x_0, v_0).$$

The cases where $i = 2, 3$ are clearly analogous, as is the case when a negative sign appears in the equality (i.e., $F = -\partial_{x_1} f$). The case when the derivative is in the v variable is slightly more complicated as new terms arise, but these new terms can be handled in a straightforward way.

¹Indeed, the only technical issue here is if the inequality is violated at $|v| = \infty$. One may sidestep this by simply including a cutoff as a multiplicative factor of the initial data and of R . It then follows from standard facts about the heat equation that f and its derivatives decay as a Gaussian at high velocities. The cutoff can be removed by a limiting procedure.

Since $F - \partial_{x_1} f \geq 0$ on $[0, t_0] \times \mathbb{T}^3 \times \mathbb{R}^3$, we find
 (5.6)
 $0 \geq \partial_t(F - \partial_{x_1} f) + v \cdot \nabla_x(F - \partial_{x_1} f) - (\epsilon + (1 - \sigma))\Delta_{x,v}(F - \partial_{x_1} f) - \sigma Q_{\epsilon,\delta}(g, F - \partial_{x_1} f).$

We use this to derive a contradiction.

On the one hand, an explicit computation for F , along with [28, Proposition 3.1(v)], yields

(5.7)
 $\partial_t F + v \cdot \nabla_x F - (\epsilon + (1 - \sigma))\Delta_{x,v} F - \sigma Q_{\epsilon,\delta}(g, F) \geq \phi' \langle v_0 \rangle^{-m_1} - C\phi(1 + \|g\|_{L^\infty, m_1}) \langle v_0 \rangle^{-m_1},$

where we used that $m_1 > 3 + \gamma + 2s$, a condition of the quoted result.

On the other hand, using Lemma 5.1, we find

(5.8)
 $\begin{aligned} \partial_t \partial_{x_1} f + v \cdot \nabla_x \partial_{x_1} f - (\epsilon + (1 - \sigma))\Delta_{x,v} \partial_{x_1} f - \sigma Q_{\epsilon,\delta}(g, \partial_{x_1} f) &= \sigma Q_{\epsilon,\delta}(\partial_{x_1} g, f) + \partial_{x_1} R \\ &\lesssim (\|g(t_0)\|_{\tilde{X}^{m_0, m_1}} (\|f\|_{L^\infty, m_0} + \|\nabla f\|_{L^\infty, m_1}) + \|R(t_0)\|_{\tilde{X}^{m_0, m_1}}) \langle v_0 \rangle^{-m_1} \\ &\leq (\|g(t_0)\|_{\tilde{X}^{m_0, m_1}} (\|f\|_{L^\infty, m_0} + \phi) + \|R(t_0)\|_{\tilde{X}^{m_0, m_1}}) \langle v_0 \rangle^{-m_1}. \end{aligned}$

Using (5.3), it is clear from (5.7) and (5.8) that we can choose ϕ to obtain a contradiction in (5.6). This yields a contradiction. Hence (5.4) always holds, finishing the proof. $\square$

As usual, once a priori estimates are established, the construction of a solution follows easily. In fact, in this case, the solution can be constructed exactly as in [28]. Indeed, one can use the method of continuity as well as a smoothing argument in order to establish the existence of solutions to the linear problem. After this, an iteration yields a solution to the nonlinear problem. As it is exactly the same as in [28], we omit the details.

For uniqueness, one can actually simply use an L^2 -based argument. Indeed, a quick check of the arguments in subsection 3.2 reveals that they can be adapted in a straightforward way to use only the Y^{m_0, m_1} norms of two potential solutions f and g . Actually, the proof is *easier* in this case as there is no need to use the Sobolev embedding theorem.

The above concludes the proof of Theorem 1.2.

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