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Residual torsion-free nilpotence, biorderability and pretzel knots

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The residual torsion-free nilpotence of the commutator subgroup of a knot group has played a key role in studying the biorderability of knot groups. A technique developed by Mayland (1975) provides a sufficient condition for the commutator subgroup of a knot group to be residually torsion-free nilpotent using work of Baumslag (1967, 1969). We apply Mayland's technique to several genus one pretzel knots and a family of pretzel knots with arbitrarily high genus. As a result, we obtain a large number of new examples of knots with biorderable knot groups. These are the first examples of biorderable knot groups for knots which are not fibered or alternating.

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1 Introduction

Let J be a knot in S^3. The *knot exterior* of J is $M_J := S^3 - \nu(J)$, where $\nu(J)$ is the interior of a tubular neighborhood of J, and the *knot group* of J is $\pi_1(M_J)$. Denote the Alexander polynomial of J by Δ_J.

A group Γ is *nilpotent* if its lower central series terminates (is trivial) after finitely many steps. In other words, for some nonnegative integer n,

$$\Gamma_0 \rhd \Gamma_1 \rhd \cdots \rhd \Gamma_n = 1,$$

where $\Gamma_0 = \Gamma$ and $\Gamma_{i+1} = [\Gamma_i, \Gamma]$ for each $i = 0, \dots, n-1$. A group Γ is *residually torsion-free nilpotent* if, for every nontrivial element $x \in \Gamma$, there is a normal subgroup $N \lhd \Gamma$ such that $x \notin N$ and G/N is a torsion-free nilpotent group. We are concerned with when the commutator subgroup of a knot group is residually torsion-free nilpotent, which has applications to ribbon concordance (see Gordon [15]) and the biorderability of the knot group; see Linnell, Rhemtulla and Rolfsen [25].

Several knots are known to have groups with residually torsion-free nilpotent commutator subgroups. The commutator subgroup of fibered knot groups are finitely generated free groups, which are residually torsion-free nilpotent; see Magnus [27]. Work of Mayland and Murasugi [30] shows that the knot groups of pseudoalternating knots, whose Alexander polynomials have a prime power leading coefficient, have residually torsion-free nilpotent commutator subgroups; pseudoalternating knots are defined in Section 3. The knot groups of two-bridge knots have residually torsion-free nilpotent commutator subgroups; see Johnson [20].

There is also the following obstruction to a knot group having residually torsion-free nilpotent commutator subgroup:

Proposition 1.1 *If J is a knot in S^3 with trivial Alexander polynomial, then the commutator subgroup of $\pi_1(M_J)$ cannot be residually torsion-free nilpotent.*

Proof Let G be the commutator subgroup of $\pi_1(M_J)$. Let M^∞ be the infinite cyclic cover of M_J, the covering space of M_J corresponding to G so that $\pi_1(M^\infty) = G$; see Rolfsen [36, Chapter 7] for details. Then

$$H_1(M^\infty, \mathbb{Z}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[t, t^{-1}]/\langle a_i(t)\rangle,$$

where $a_1(t), \dots, a_n(t)$ are polynomials such that

$$\prod_{i=1}^{n} a_i(t) = \Delta_J(t).$$

Since the Alexander polynomial of J is trivial $G/[G,G] \cong H_1(M^\infty, \mathbb{Z}) = 1$, so $G = [G,G]$. It follows that every term of the lower central series of G is isomorphic to G. Suppose $N \lhd G$ is a proper normal subgroup of G. For each term of the lower central series of G/N,

$$(G/N)_i \cong G_i/N \cong G/N \neq 1,$$

so G/N cannot be nilpotent. Thus, G is not residually torsion-free nilpotent. □

Given the integers $k_1, k_2, \dots, k_n$, define $P(k_1, k_2, \dots, k_n)$ to be the *pretzel knot* represented in the diagram in Figure 1. Mayland [29] describes a technique to examine the commutator subgroup of the group of a knot bounding an unknotted minimal genus Seifert surface; see Section 2. In fact, this is the technique Mayland and Murasugi used to prove their result for pseudoalternating knots [30]. Applying Seifert's algorithm to the diagram in Figure 1 yields an unknotted minimal genus Seifert surface (see Gabai [12]) making pretzel knots ideal candidates for Mayland's technique.

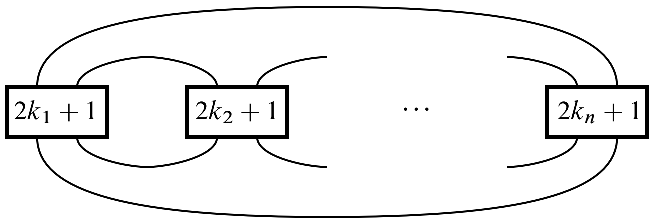

Figure 1: A pretzel knot diagram. The integers in the boxes indicate the number of right-hand half-twist when positive and left-hand half-twist when negative.

Let J be the $P(2p+1, 2q+1, 2r+1)$ pretzel knot for some integers p, q and r. J is a two-bridge knot (possibly trivial) precisely when at least one of p, q and r is equal to 0 or -1 (see Kawauchi [23, Chapter 2]) so for our purposes, we can assume that none of p, q and r are 0 or -1. Permuting the parameters $2p+1$, $2q+1$ and $2r+1$ yields the same (unoriented) knot. Also, $P(-2p-1, -2q-1, -2r-1)$ and $P(2p+1, 2q+1, 2r+1)$ are mirrors of each other. Since $\pi_1(M_J)$ is invariant of reversing orientation and mirroring, we can assume that $1 \le q \le r$.

Theorem 1.2 *Given integers p, q and r with $1 \le q \le r$ and $p \ne 0$ or -1, let J be the $P(2p+1, 2q+1, 2r+1)$ pretzel knot with Alexander polynomial Δ_J whose leading coefficient is a prime power. The commutator subgroup of $\pi_1(M_J)$ is residually torsion-free nilpotent if*

- $p \ge 1$,
- *J is $P(2p+1, 3, 2r+1)$,*
- *J is $P(-3, 2q+1, 2r+1)$ and J is not $P(-3, 5, 5)$, $P(-3, 5, 7)$, $P(-3, 5, 9)$, $P(-3, 5, 11)$ or $P(-3, 7, 7)$, or*
- *J is $P(-5, 2q+1, 2r+1)$ and J is not*
  - *$P(-5, 7, R)$ when R is 11, 13, 15, 17, 19, 21, 23 or 25,*
  - *$P(-5, 9, R)$ when R is 9, 11, 13, 15 or 17, or*
  - *$P(-5, 11, R)$ when R is 11 or 13.*

Remark 1.3 Proposition 1.1 is the only known obstruction to the commutator subgroup of a genus one pretzel knot group being residually torsion-free nilpotent, so the exceptional cases in Theorem 1.2 with nontrivial Alexander polynomial remain unresolved and cannot be resolved with the technique used in this paper.

If $p \leq -2$ and $1 \leq q \leq r$, then $P(2p+1, 2q+1, 2r+1)$ is not a pseudoalternating knot; see Proposition 3.1. Therefore, all of the examples from Theorem 1.2 where $p < -1$ are new examples of knots with residually torsion-free nilpotent commutator subgroups.

In addition, we also obtain pretzel knots of arbitrarily high genus whose groups have residually torsion-free nilpotent commutator subgroups. However, we were not able to determine whether or not these knots are pseudoalternating so it is possible this result follows from Mayland and Murasugi's work.

Theorem 1.4 *If J is a $P(3, -3, \ldots, 3, -3, 2r+1)$ pretzel knot for some integer r, then the commutator subgroup of $\pi_1(M_J)$ is residually torsion-free nilpotent.*

1.1 Possible generalizations

The techniques used here have a few limitations. First, while our method can be applied to many families of genus one pretzel knots on a case by case basis, this method does not lend itself well to generalizing to all genus one pretzel knots since many of the details depend on the arithmetic properties of p, q and r. Secondly, Mayland's method requires a couple conditions (an unknotted Seifert surface satisfying the free factor property and an Alexander polynomial with prime power leading coefficient) which may not be necessary for a knot group to have residually torsion-free nilpotent commutator subgroup. Nevertheless, we make the following prediction for genus one pretzel knots.

Conjecture 1.5 If J is a genus one pretzel knot then the commutator subgroup of $\pi_1(M_J)$ is residually torsion-free nilpotent if and only if the Alexander polynomial of J is nontrivial.

1.2 Application to biorderability

A group is said to be *biorderable* if there exists a total order of the group's elements, invariant under both left and right multiplication. Chiswell, Glass and Wilson proved the following fact, using work of Linnell, Rhemtulla and Rolfsen [25], and it has been instrumental in determining the biorderability of several knot groups; see Clay, Desmarais and Naylor [8], Johnson [20] and Perron and Rolfsen [35].

Theorem 1.6 [7, Theorem B] *Let J be a knot in S^3. If $\pi_1(M_J)$ has residually torsion-free nilpotent commutator subgroup and all the roots of Δ_J are real and positive then $\pi_1(M_J)$ is biorderable.*

Furthermore, Ito obtained the following obstruction to a knot group being biorderable when the knot is rationally homologically fibered; see Section 2 for the definition of rationally homologically fibered.

Theorem 1.7 [18, Theorem 2] *Let J be a rationally homologically fibered knot. If $\pi_1(M_J)$ is biorderable then Δ_J has at least one real positive root.*

The Alexander polynomial of the pretzel knot $P(2p+1, 2q+1, 2r+1)$ has the form

$$\Delta_J(t) = Nt^2 + (1-2N)t + N,$$

where

$$N = \det \begin{pmatrix} p+q+1 & -q-1 \\ -q & q+r+1 \end{pmatrix}. \tag{1-1}$$

See Section 3 for details. Note that Δ_J has two positive real roots when $N < 0$ and two nonreal roots when $N > 0$. If $N = 0$, then $\Delta_J(t) = 1$. Therefore, we have the following proposition:

Proposition 1.8 *Let J be the $P(2p+1, 2q+1, 2r+1)$ pretzel knot, and let N be defined as in* (1-1). *If the commutator subgroup of $\pi_1(M_J)$ is residually torsion-free nilpotent and $N < 0$, then $\pi_1(M_J)$ is biorderable. If $N > 0$, then $\pi_1(M_J)$ is never biorderable, regardless of whether or not the commutator subgroup of $\pi_1(M_J)$ is residually torsion-free nilpotent.*

Applying Proposition 1.8 to the results in Theorem 1.2 yields the following corollary.

Corollary 1.9 *Given integers p, g and r with $1 \le q \le r$ and $p \ne 0$ or -1, let J be the $P(2p+1, 2q+1, 2r+1)$ pretzel knot with Alexander polynomial Δ_J.*

(1) *$\pi_1(M_J)$ is biorderable if*
   - *J is $P(-3, 3, 2r+1)$,*
   - *J is $P(-5, 3, 2r+1)$ and $r+4$ is a prime power, or*
   - *J is $P(-5, 7, 7)$ or $P(-5, 7, 9)$.*

(2) *$\pi_1(M_J)$ is not biorderable if*
   - *$p \ge 1$,*
   - *J is $P(-3, 5, 2r+1)$ with $r > 3$,*
   - *J is $P(-3, 2q+1, 2r+1)$ with $q \ge 2$,*
   - *J is $P(-5, 7, 2r+1)$ with $r \ge 9$,*
   - *J is $P(-5, 9, 2r+1)$ with $r \ge 6$, or*
   - *J is $P(-5, 2q+1, 2r+1)$ with $q \ge 5$.*

We also have the following corollary to Theorem 1.4.

Corollary 1.10 *If J is the $P(3,-3,\dots,3,-3,2r+1)$ pretzel knot for some integer r, then $\pi_1(M_J)$ is biorderable.*

Details of the proof of Corollary 1.10 are provided in Section 4.

1.3 A possible connection of biorderability to branched covers

Given a knot J in S^3, let $\Sigma_n(J)$ be the *n–fold cyclic cover of S^3 branched over J*; see Rolfsen [36, Chapter 10] for the definition and construction of a cyclic branched cover. Part of the motivation for studying the biorderability of pretzel knots is to investigate the following questions.

Question 1.11 Do there exist knots with $\pi_1(M_J)$ biorderable and $\pi_1(\Sigma_n(J))$ left-orderable for some n?

Question 1.12 Does $\pi_1(M_J)$ not being biorderable imply that $\pi_1(\Sigma_n(J))$ is left-orderable for some n?

Question 1.11 is resolved here.

Theorem 1.13 *For each integer $q \geq 3$, let J_q be the $P(1-2q, 2q+1, 4q-3)$ pretzel knot. When $q-1$ is a prime power, $\pi_1(M_{J_q})$ is biorderable and $\pi_1(\Sigma_2(J_q))$ is left-orderable.*

Remark 1.14 Question 1.11 is still unanswered for fibered knots and alternating knots.

Question 1.12 remains unresolved as of the writing of this paper. However, some important remarks can be made about this question.

Suppose J is a pretzel knot $P(2p+1, 2q+1, 2r+1)$ with $1 \leq q \leq r$. When $p \geq 1$, the signature of J is nonzero which likely means that $\pi_1(\Sigma_n(J))$ is left-orderable for n sufficiently large; see Gordon [16, Corollary 1.2 and Question 1.3].

Suppose $p < -1$. By the Montesinos trick [31], the double branched cover of J is the Seifert fibered space

$$\Sigma_2(J) = M\left(0; -1, \frac{-2p-2}{-2p-1}, \frac{1}{2q+1}, \frac{1}{2r+1}\right).$$

By work of Eisenbud, Hirsch and Neumann [10], Lisca and Stipsicz [26], Jankins and Neumann [19], Naimi [34] and Boyer, Rolfsen and Wiest [4], $\Sigma_2(J)$ is left-orderable if and only if there are positive integers a and m such that the triple

$((-2p-2)/(-2p-1), 1/(2q+1), 1/(2r+1))$ is less than some permutation of the triple $(a/m, (m-a)/m, 1/m)$. This happens precisely when $1 < -p \leq q$. In this case, $m = 2q$ and $a = 2q-1$. Therefore, we can state the following proposition.

Proposition 1.15 *Suppose J is the $P(2p+1, 2q+1, 2r+1)$ pretzel knot with $p < -1$ and $1 \leq q \leq r$. Then $\pi_1(\Sigma_2(J))$ is left-orderable if and only if $-p \leq q$.*

Thus, if $p < -1$ and the double branched cover of J does not have left-orderable fundamental group, then $q < -p$ so N as defined in (1-1) is negative. Therefore, if Conjecture 1.5 is true, $\pi_1(M_J)$ would be biorderable when $q < -p$ by Proposition 1.8. In particular, if Conjecture 1.5 is true, it's not likely that any nonalternating genus one pretzel knot would be a counterexamples for Question 1.12.

There is some evidence that genus one pretzel knots with no left-orderable cyclic branched covers do exists. It is conjectured (see Boyer, Gordon and Watson [3]) that given a prime orientable closed rational homology sphere Y, $\pi_1(Y)$ is not left-orderable if and only if Y is an L-space, and Issa and Turner show that the cyclic branched covers of the $P(-3, 3, 2r+1)$ pretzel knots are all L-spaces; see [17].

Outline

In Section 2, we review how Mayland's technique [29] can be used to analyze when the commutator subgroup of a knot group is residually torsion-free nilpotent. In Section 3, we apply this technique to genus one pretzel knots and prove Theorems 1.2 and 1.13. In Section 4, we prove Theorem 1.4. Appendix A contains the proofs of some key lemmas. We also provide a chart of our results in Appendix B.

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2 Preliminaries on Mayland's technique

Mayland used a description of the commutator subgroup of a knot group to investigate when they are residual finite [29]. In this section, we show how Mayland's technique can be used to find a sufficient condition for the commutator subgroup of a knot group to be residually torsion-free nilpotent.

2.1 Mayland's technique

Let J be a knot in S^3 and suppose J bounds a minimal genus Seifert surface S such that S is *unknotted*; in other words, $\pi_1(S^3 \backslash S)$ is a free group. Let $\widehat{S} = M_J \cap S$. Let G be the commutator subgroup of $\pi_1(M_J)$.

Let U be the image of a bicollar embedding $\widehat{S} \times [-1,1] \hookrightarrow M_J$ where $\widehat{S}$ is the image of $\widehat{S} \times \{0\}$, and let $M_S = M_J \backslash \widehat{S}$. Denote the images of $\widehat{S} \times (0,1]$ and $\widehat{S} \times [-1,0)$ in M_S by U^+ and U^-, respectively. Let $X = \pi_1(M_S)$, which is a free group of rank $2g$ where g is the genus of J. Consider the inclusion maps $i^+ \colon U^+ \to M_S$ and $i^- \colon U^- \to M_S$. Let H be the image of the induced map $i^+_* \colon \pi_1(U^+) \to \pi_1(M_S)$ and K be the image of $i^-_* \colon \pi_1(U^-) \to \pi_1(M_S)$.

For each integer n, let X_n be a copy of X, $H_n \subset X_n$ be a copy of H, and $K_n \subset X_n$ be a copy of K. The fundamental groups of U, U^+ and U^- are canonically isomorphic, and since S has minimal genus, i^+_* and i^-_* are injective. Therefore, H_n and K_{n+1} are identified with a rank $2g$ free group F. By Brown and Crowell [5, Theorem 2.1], G is an amalgamated free product of the form

$$G \cong \cdots *_F X_{-2} *_F X_{-1} *_F X_0 *_F X_1 *_F X_2 *_F \cdots. \tag{2-1}$$

Baumslag provides the following sufficient condition for a group to be residually torsion-free nilpotent when G is an ascending chain of *parafree* subgroups; see [1; 2] for a definition and discussion of parafree groups.

Proposition 2.1 [2, Proposition 2.1(i)] *Suppose G is a group which is the union of an ascending chain of subgroups*

$$G_0 < G_1 < G_2 < \cdots < G_n < \cdots < G = \bigcup_{n=1}^{\infty} G_n.$$

Suppose each G_n is parafree of the same rank. If, for each nonnegative integer n, $|G_{n+1} : G_n[G_{n+1}, G_{n+1}]|$ is finite, then G is residually torsion-free nilpotent.

For each nonnegative integer m, define Z^m as follows:

$$Z^m := X_{-m} *_F X_{1-m} *_F \cdots *_F X_{m-1} *_F X_m. \tag{2-2}$$

The direct limit of the Z^m is isomorphic to G. Furthermore, since i^+_* and i^-_* are injective, the natural inclusion $Z^m \hookrightarrow Z^{m+1}$ is an embedding, so G is an ascending chain of subgroups

$$Z^0 < Z^1 < Z^2 < \cdots < Z^m < \cdots < G = \bigcup_{m=1}^{\infty} Z^m.$$

A subgroup A of a free group B is a *free factor* if $B = A * D$ for some subgroup D of B. It immediately follows that A is a free factor of B if and only if every (equivalently, at least one) free basis of A extends to a free basis of B. A theorem of Mayland provides sufficient conditions for each Z^m to be parafree.

Proposition 2.2 [29, Theorem 3.2] *If H and K are free factors of $H[X,X]$ and $K[X,X]$, respectively, and $|X : H[X,X]| = |X : K[X,X]| = p^l$ for some prime p and nonnegative integer l, then for every nonnegative m, Z^m is parafree of rank $2g$.*

The knot J is *rationally homologically fibered* if the induced map on homology, $i_h^+ \colon H_1(U^+;\mathbb{Q}) \to H_1(M_S;\mathbb{Q})$ (or equivalently $i_h^- \colon H_1(U^-;\mathbb{Q}) \to H_1(M_S;\mathbb{Q})$), is an isomorphism. Let S_+ be a Seifert matrix representing i_h^+ such that $S_- := S_+^T$ is a Seifert matrix representing i_h^-. S_+ is also a presentation matrix for the abelian group $X/H[X,X]$. Similarly, S_- is a presentation matrix for $X/K[X,X]$. Thus,

$$\frac{X}{H[X,X]} \cong \frac{X}{K[X,X]}. \tag{2-3}$$

Denote the standard form of the Alexander polynomial of J by Δ_J. For some nonnegative integer k,

$$t^k \Delta_J(t) = \det(tS_+ - S_+^T) = d_0 + d_1 t + \cdots + d_{2g} t^{2g}.$$

It is a well-known fact that $d_i = d_{2g-i}$; see [33, Chapter 6].

Proposition 2.3 *Suppose J is a knot in S^3. The following statements are equivalent:*

(a) *J is rationally homologically fibered.*

(b) *$|X : H[X,X]|$ is finite.*

(c) *$|X : K[X,X]|$ is finite.*

(d) *$\deg \Delta_J = 2g$.*

Proof The equivalence of (b) and (c) follows from (2-3).

Since S_+ is a presentation matrix for $X/H[X,X]$, we have that $|X : H[X,X]|$ is finite if and only if $|\det(S_+)| \neq 0$. It follows that (a) and (b) are equivalent.

Since $d_{2g} = d_0 = \det(S_+)$, we have $\deg \Delta_J = 2g$ if and only if $\det(S_+) \neq 0$, so (a) and (d) are equivalent. □

Proposition 2.4 *When J is rationally homologically fibered,*

$$|X : H[X,X]| = |X : K[X,X]| = |\Delta_J(0)|.$$

Proof When J is rationally homologically fibered

$$|X : H[X, X]| = |\det(S_+)| = |\Delta_J(0)|,$$

so the proposition follows from (2-3). □

For each nonnegative m,

$$\frac{Z^{m+1}}{Z^m[Z^{m+1}, Z^{m+1}]} \cong \frac{X}{H[X, X]} \times \frac{X}{K[X, X]}.$$

So, when J is rationally homologically fibered,

$$|Z^{m+1} : Z^m[Z^{m+1}, Z^{m+1}]| = |X : H[X, X]||K : H[X, X]| = \Delta_J(0)^2 \tag{2-4}$$

by Proposition 2.4.

The Seifert surface S is said to *satisfy the free factor property* if H and K are free factors of $H[X, X]$ and $K[X, X]$, respectively. Note that this property is independent of the orientation of S. A sufficient condition for the residual torsion-free nilpotence of G can be summarized as follows.

Proposition 2.5 *Suppose J is a rationally homologically fibered knot in S^3 with unknotted minimum genus Seifert surface S. If S satisfies the free factor property and $|\Delta_J(0)|$ is a prime power, then the commutator subgroup G is residually torsion-free nilpotent.*

Proof Suppose J is a rationally homologically fibered with unknotted minimum genus Seifert surface S satisfying the free factor property, and suppose $|\Delta_J(0)|$ is a prime power.

Define Z^m for each nonnegative integer m as in (2-2). By Proposition 2.4, $|X : H[X, X]|$ and $|K : H[X, X]|$ are prime powers since J is rationally homologically fibered. Thus, by Proposition 2.2, each Z^m is parafree of rank twice the genus of J.

By (2-4), $|Z^{m+1} : Z^m[Z^{m+1}, Z^{m+1}]| = \Delta_J(0)^2$, so $|Z^{m+1} : Z^m[Z^{m+1}, Z^{m+1}]|$ is finite. Therefore, by Proposition 2.1, G is residually torsion-free nilpotent. □

2.2 Pseudoalternating knots

A *special alternating diagram* is an alternating link diagram in which all of the crossings have the same sign. Any link with such a diagram is called a *special alternating link*. The Seifert surface described by performing Seifert's algorithm on a special alternating diagram is a *primitive flat surface*. A *generalized flat surface* is any surface which can be obtained by combining some number of primitive flat surfaces by Murasugi sums. See Gabai [11] for a definition and exposition of Murasugi sums. A link which bounds a generalized flat surface is a *pseudoalternating link*. Alternating links are pseudoalternating links. However, all torus links, many of which are not alternating, are also pseudoalternating links.

Pseudoalternating knots are rationally homologically fibered and bound surfaces satisfying the free factor condition [30, Theorem 2.5]. Therefore, the knot group of a pseudoalternating knot, whose Alexander polynomial has a prime power leading coefficient, has residually torsion-free nilpotent commutator subgroup.

3 Genus one pretzel knots

Let J be the $P(2p+1,2q+1,2r+1)$ pretzel knot for some integers p, q and r with $1 \le q \le r$ and $p \ne -1$ or 0. Let S be the unknotted genus one surface depicted in Figure 2, which we refer to as the *standard Seifert surface* of J. For the genus one pretzel knots which are not two-bridge knots, the standard Seifert surface is the unique Seifert surface of minimal genus, up to isotopy [13].

In this section, we analyze when S satisfies the free factor property. When $p > 0$, $P(2p+1,2q+1,2r+1)$ is an alternating knot, and thus $P(2p+1,2q+1,2r+1)$ is pseudoalternating. However, this is not true when $p \le -2$.

Proposition 3.1 *When $1 \le q \le r$ and $p \le -2$, the pretzel knot $P(2p+1,2q+1,2r+1)$ is not a pseudoalternating knot.*

Proof Suppose $P(2p+1,2q+1,2r+1)$ is pseudoalternating. When $1 \le q \le r$ and $p \le -2$, the diagram in Figure 1 has a minimal number of crossings [24, Theorem 10].

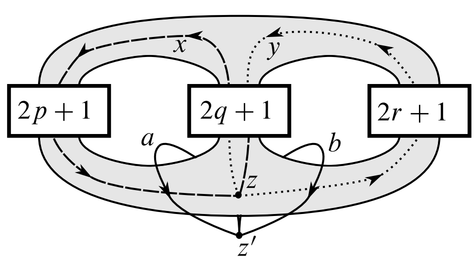

Figure 2: The Seifert surface S of $P(2p+1,2q+1,2r+1)$.

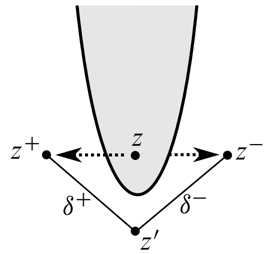

Figure 3: Isotopy of basepoints.

Since this diagram is not alternating, $P(2p+1, 2q+1, 2r+1)$ cannot be alternating by a theorem of Kauffman, Murasugi and Thistlethwaite [21; 22; 32; 38]. In particular, $P(2p+1, 2q+1, 2r+1)$ is not special alternating. Thus, $P(2p+1, 2q+1, 2r+1)$ must be the boundary of a surface S which is the Murasugi sum of two generalized flat surfaces, S_1 and S_2, which are not disks.

By Gabai [11], S must be a minimal genus Seifert surface, so $\chi(S) = -1$. Analyzing the effect of a Murasugi sum on the Euler characteristic yields

$$-1 = \chi(S) = \chi(S_1) + \chi(S_2) - 1.$$

Since S_1 and S_2 are not disks, neither S_1 nor S_2 has positive Euler characteristic. It follows that $\chi(S_1) = \chi(S_2) = 0$, so S_1 and S_2 are both annuli.

The boundary of a Murasugi sum of two annuli is a double twist knot which is alternating. Thus $P(2p+1, 2q+1, 2r+1)$ is alternating, which is a contradiction. □

Since J is pseudoalternating when $p \geq 0$, we will only need to focus on the case when p is negative.

3.1 Mayland's technique for genus one pretzel knots

Define M_J, M_S, X, H and K as in Section 2. Here we offer a concrete description of the maps on fundamental groups i_*^+ and i_*^- for genus one pretzel knots. This is the same description used by Crowell and Trotter in [9]. Choose a basepoint z on the lower part of S, and let x and y be the classes generating $\pi_1(S, z)$ represented by the loops indicated in Figure 2. Let z^+ and z^- be push-offs of z of each side of S. Let z' be the basepoint of M_S obtained by shifting z tangentially along S through ∂S. Let δ^+ and δ^- be arcs connecting z' to z^+ and z^-, respectively; see Figure 3. Finally, let a and b be the indicated classes generating $\pi_1(M_S, z')$.

By slightly isotoping elements of $\pi_1(S,z)$ off of S, $\pi_1(U^+,z^+)$ and $\pi_1(U^-,z^-)$ are canonically identified to $\pi_1(S,z)$, which is a rank two free group F generated by x and y. The group $X := \pi_1(M_S,z')$ is a rank two free group generated by a and b. The map $i_*^+ \colon F \to X$ takes a class $[\gamma]$ in $\pi_1(U^+,z^+) = F$ to the class $[\delta^+ * \gamma * (-\delta^+)]$ in $\pi_1(M_S,z') = X$. Likewise, the map $i_*^- \colon F \to X$ takes $[\gamma]$ to $[\delta^- * \gamma * (-\delta^-)]$.

With these choices, we define the elements

$$\begin{aligned} \alpha_H &:= i_*^+(x) = (b^{-1}a)^{q+1}a^p, \quad & \alpha_K &:= i_*^-(x) = (ab^{-1})^q a^{p+1}, \\ \beta_H &:= i_*^+(y) = b^{r+1}(a^{-1}b)^q, \quad & \beta_K &:= i_*^-(y) = b^r(ba^{-1})^{q+1}, \end{aligned} \tag{3-1}$$

so that

$$H = \langle \{\alpha_H, \beta_H\} \rangle \quad \text{and} \quad K = \langle \{\alpha_K, \beta_K\} \rangle.$$

Thus, the Seifert matrices for i_*^+ and i_*^- are

$$S_+ = \begin{pmatrix} p+q+1 & -q-1 \\ -q & q+r+1 \end{pmatrix} \quad \text{and} \quad S_- = \begin{pmatrix} p+q+1 & -q \\ -q-1 & q+r+1 \end{pmatrix}. \tag{3-2}$$

Let $N = \det S_+ = \det S_-$. Up to multiplication by a signed power of t, the Alexander polynomial of J is

$$\Delta_J(t) = Nt^2 + (1-2N)t + N.$$

When $N \neq 0$, J is rationally homologically fibered by Proposition 2.3. Simply considering the integer N can provide useful information.

Proposition 3.2 *When $N = 0$, G is not residually torsion-free nilpotent.*

Proof When $N = 0$ we have $\Delta_J(t) = 1$, so G cannot be residually nilpotent by Proposition 1.1. □

Proposition 3.3 *If $|N| = 1$, then the standard Seifert surface S does not satisfy the free factor property.*

Proof Let S be the standard Seifert surface of J, and define X, H, and K as in Section 2. Each of these are rank two free groups. Suppose S satisfies the free factor property.

When $|N| = 1$ we have that $X/H[X,X] \cong X/K[X,X] \cong 1$ by Proposition 2.4, so $X = H[X,X] = K[X,X]$. Since H is a free factor of $H[X,X]$ and both are rank two free groups, $H = H[X,X] = X$. Similarly, since K is a free factor of $K[X,X]$ and both are rank two free groups, $K = X$. This implies that i_*^+ and i_*^- are isomorphisms. Thus, $\pi_1(M_J)$ is an extension of $\mathbb{Z}$ described by the short exact sequence

$$1 \to X \to \pi_1(M_J) \to \mathbb{Z} \to 1.$$

The Stallings fibration theorem implies that J is a genus one fibered knot [37]. However, the only genus one fibered knots are the trefoil and the figure eight knot [6; 14], which is a contradiction since we are assuming J is not a two-bridge knot. □

In light of Proposition 2.5, to prove the commutator subgroup of $\pi_1(M_J)$ is residually torsion-free nilpotent, it is sufficient to show S satisfies the free factor property.

3.2 Outline of the procedure

In each case we use the same basic procedure, outlined below, to analyze whether or not S satisfies the free factor property.

(1) Find a presentation matrix for $X/H[X,X]$ of the form

$$\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} u & 0 \\ v & w \end{pmatrix}$$

using row operations. Note, u and w can always be made positive. Thus, $X/H[X,X]$ is isomorphic to $(\mathbb{Z}/u\mathbb{Z}) \times (\mathbb{Z}/w\mathbb{Z})$. The $\mathbb{Z}/u\mathbb{Z}$ factor is generated by the class of a, and the $\mathbb{Z}/w\mathbb{Z}$ factor is generated by the class of b.

(2) Since $X/H[X,X]$ is abelian, the set $\mathcal{C}$ is a set of coset representatives of $H[X,X]$:

$$\mathcal{C} = \{a^k b^l \mid 0 \le k < u,\ 0 \le l < w\}.$$

Given $x \in X$, denote by $\bar{x}$ the coset representative of x in $\mathcal{C}$. Define

$$x_{c,x} := cx(\overline{cx})^{-1},$$

where $c \in \mathcal{C}$ and $x \in \{a,b\}$. From this we find the following free basis for $H[X,X]$ using the Reidemeister–Schreier method:

$$\mathcal{B} = \{x_{c,x} \mid c \in \mathcal{C},\ x \in \{a,b\},\ x_{c,x} \ne 1\}.$$

See [28] for details.

(3) Use the Reidemeister–Schreier rewriting process to rewrite the generating set of H from (3-1). A word $\alpha \in H$, where $\alpha = \alpha_1^{s_1} \dots \alpha_k^{s_k}$ with $\alpha_i \in \{a,b\}$ and $s_i = \pm 1$, can be rewritten as

$$\alpha = x_{c_1,\alpha_1}^{s_1} \dots x_{c_k,\alpha_k}^{s_k},$$

where

$$c_i = \begin{cases} \overline{\alpha_1 \dots \alpha_{i-1}} & \text{when } s_i = 1, \\ \overline{\alpha_1 \dots \alpha_i} & \text{when } s_i = -1. \end{cases}$$

(4) Determine if the generating set of H can be extended to a free basis of $H[X,X]$.

(5) Repeat this procedure for K.

When the free bases of H and K can be extended to free bases of $H[X,X]$ and $K[X,X]$, respectively, S satisfies the free factor property. If the chosen basis of either H or K fails to extend, then S cannot satisfy the free factor property.

3.3 Knots whose standard Seifert surface satisfies the free factor property

Lemma 3.4 *If J is $P(-5,7,7)$ or $P(-5,7,9)$ then S satisfies the free factor property.*

Proof Suppose J is $P(-5,7,7)$. From (3-1),

$$\alpha_H = (b^{-1}a)^4a^{-3}, \quad \alpha_K = (ab^{-1})^3a^{-2},$$
$$\beta_H = b^4(a^{-1}b)^3, \quad \beta_K = b^3(ba^{-1})^4.$$

The abelian group $X/H[X,X]$ has presentation matrix

$$\begin{pmatrix} 1 & -4 \\ -3 & 7 \end{pmatrix},$$

which becomes

$$\begin{pmatrix} 1 & 1 \\ 0 & 5 \end{pmatrix}$$

after row operations.

From this we get $\mathcal{C} = \{1, b, b^2, b^3, b^4\}$ as a set of coset representatives of $H[X,X]$. We apply Reidemeister–Schreier to obtain the following free basis of $H[X,X]$:

$$\mathcal{B} = \{ab, ba, b^2ab^{-1}, b^3ab^{-2}, b^4ab^{-3}, b^5\}.$$

Label the basis elements as follows: $x_k := b^kab^{1-k}$ for $0 \le k \le 4$ and $x_5 := b^5$.

Now we can rewrite α_H and β_H in terms of $\mathcal{B}$, obtaining

$$\begin{aligned} \alpha_H &= (b^{-5})(b^4ab^{-3})(b^2ab^{-1})(ab)(b^{-5})(b^3a^{-1}b^{-4})(b^5)(b^{-1}a^{-1}) \\ &= x_5^{-1}x_4x_2x_0x_5^{-1}x_4^{-1}x_5x_0^{-1} \end{aligned}$$

and

$$\beta_H = (b^5)(b^{-1}a^{-1})(ba^{-1}b^{-2})(b^3a^{-1}b^{-4})(b^5) = x_5x_0^{-1}x_2^{-1}x_4^{-1}x_5.$$

Thus

$$\alpha_H = \beta_H^{-1}x_4^{-1}x_5x_0^{-1},$$

so

$$x_4 = x_5x_0^{-1}\alpha_H^{-1}\beta_H^{-1}$$

and

$$x_2 = \beta_H\alpha_H x_0\beta_H^{-1}x_5x_0^{-1}.$$

Therefore, the set

$$\{\alpha_H, \beta_H, x_0, x_1, x_3, x_5\}$$

is a generating set of six elements for $H[X,X]$, and thus is a free basis. It follows that

$$H[X,X] = H * \{x_0, x_1, x_3, x_5\},$$

so H is a free factor of $H[X,X]$.

After row reductions, $X/K[X,X]$ has presentation matrix

$$\begin{pmatrix} 1 & -3 \\ 0 & 5 \end{pmatrix}.$$

From this we get a free basis of $K[X,X]$:

$$x_k := \begin{cases} b^k a b^{-3-k} & \text{for } 0 \le k \le 1, \\ b^k a b^{2-k} & \text{for } 2 \le k \le 4, \\ b^5 & \text{for } k = 5. \end{cases}$$

Rewriting α_K and β_K, we get

$$\alpha_K = (ab^{-3})(b^2a)(b^{-5})(b^4ab^{-2})(ba^{-1}b^{-3})(b^3a^{-1}) = x_0x_2x_5^{-1}x_4x_3^{-1}x_0^{-1}$$

and

$$\beta_K = (b^4a^{-1}b^{-1})(b^2a^{-1}b^{-4})(b^5)(a^{-1}b^{-2})(b^3a^{-1}) = x_1^{-1}x_4^{-1}x_5x_2^{-1}x_0^{-1}.$$

Thus

$$x_4 = x_5x_2^{-1}x_0^{-1}\beta_K^{-1}x_1^{-1}$$

and

$$x_3 = x_0^{-1}\alpha_K^{-1}\beta_K^{-1}x_1^{-1}.$$

Therefore, the set

$$\{\alpha_K, \beta_K, x_0, x_1, x_2, x_5\}$$

is a free basis of $K[X,X]$ so K is a free factor of $K[X,X]$. Therefore, S satisfies the free factor property.

Suppose J is $P(-5,7,9)$. $X/H[X,X]$ has presentation matrix

$$\begin{pmatrix} 1 & -4 \\ -3 & 8 \end{pmatrix},$$

which becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

after row operations.

By applying Reidemeister–Schreier, we obtain the free basis $\{x_0, x_1, x_2, x_3, x_4\}$, where $x_i = b^iab^{-i}$ for $i = 0, \dots, 3$ and $x_4 = b^4$. Then

$$\alpha_H = (b^{-1}a)^4a^{-3} = (b^{-4})(b^3ab^{-3})(b^2ab^{-2})(bab^{-1})(a^{-1})(a^{-1}) = x_4^{-1}x_3x_2x_1x_0^{-2}$$

and

$$\begin{aligned}\beta_H &= b^5(a^{-1}b)^3\\ &= (b^{-4})(ba^{-1}b^{-1})(b^2a^{-1}b^{-2})(b^3a^{-1}b^{-3})(b^4)\\ &= x_4x_1^{-1}x_2^{-1}x_3^{-1}x_4.\end{aligned}$$

Thus

$$x_4 = \beta_H\alpha_H x_0^2$$

and

$$x_3 = x_4\alpha_H x_0^2x_1^{-1}x_2^{-1}.$$

Therefore, the set

$$\{\alpha_H, \beta_H, x_0, x_1, x_2\}$$

is a free basis of $H[X,X]$, so H is a free factor of $H[X,X]$.

A similar argument shows K is a free factor of $K[X,X]$. Therefore, S satisfies the free factor property. □

Lemma 3.5 *If J is a $P(-3,3,2r+1)$ pretzel knot then S satisfies the free factor property.*

Proof From (3-1),

$$\begin{aligned}\alpha_H &= b^{-1}ab^{-1}a^{-1}, &\quad \alpha_K &= ab^{-1}a^{-1},\\ \beta_H &= b^{r+1}a^{-1}b, &\quad \beta_K &= b^{r+1}a^{-1}ba^{-1}.\end{aligned}$$

The abelian group $X/H[X,X]$ has presentation matrix

$$\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$$

when r is even and

$$\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}$$

when r is odd.

Using $\mathcal{C} = \{1, b\}$ as a set of coset representatives, we apply Reidemeister–Schreier to obtain $\mathcal{B} = \{x_0, x_1, x_2\}$, a free basis of $H[X,X]$.

When r is even

$$x_0 = a, x_1 = bab^{-1} \quad\text{and}\quad x_2 = b^2,$$

so

$$\alpha_H = (b^{-2})(bab^{-1})(a^{-1}) = x_2^{-1}x_1x_0^{-1}$$

and

$$\beta_H = (b^{2k})(ba^{-1}b^{-1})(b^2) = x_2^kx_1^{-1}x_2,$$

where $r = 2k$.

When r is odd
$$x_0 = ab^{-1}, \quad x_1 = ba \quad \text{and} \quad x_2 = b^2,$$
so
$$\alpha_H = (b^{-2})(ba)(b^{-2})(ba^{-1}) = x_2^{-1}x_1x_2^{-1}x_0^{-1}$$
and
$$\beta_H = (b^{2k+2})(a^{-1}b^{-1})(b^2) = x_2^{k+1}x_1^{-1}x_2,$$
where $r = 2k+1$.

In either case, the set $\{\alpha_H, \beta_H, x_2\}$ is a free basis of $H[X,X]$ so H is a free factor of $H[X,X]$.

$X/K[X,X]$ has presentation matrix
$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$
Using $\mathcal{C} = \{1, a\}$ as a set of coset representatives, we get the free basis of $K[X,X]$, $\mathcal{B} = \{x_0, x_1, x_2\}$, where
$$x_0 = a^2, \quad x_1 = b \quad \text{and} \quad x_2 = aba^{-1}.$$
Thus,
$$\alpha_K = x_2^{-1} \quad \text{and} \quad \beta_K = x_1^{r+1}x_0^{-1}x_2.$$
The set $\{\alpha_K, \beta_K, x_1\}$ is a free basis of $K[X,X]$ so K is a free factor of $K[X,X]$. Therefore, S satisfies the free factor property. □

The proofs of the following results can be found in Appendix A.

Lemma 3.6 *If J is a $P(2p+1, 3, 2r+1)$ pretzel knot with $p < -2$ then S satisfies the free factor property.*

Lemma 3.7 *Suppose J is $P(-3, 2q+1, 2r+1)$ and one of the following conditions holds*:

(1) $q = 2$ *and* $r \geq 6$,

(2) $q = 3$ *and* $r \geq 4$,

(3) $q > 3$.

Then S satisfies the free factor property.

Lemma 3.8 *Suppose J is $P(-5, 2q+1, 2r+1)$ and one of the following conditions holds*:

(1) $q = 3$ *and* $r \geq 13$,

(2) $q = 4$ *and* $r \geq 9$,

(3) $q = 5$ and $r \geq 7$,

(4) $q > 5$.

Then S satisfies the free factor property.

3.4 Proof of Theorem 1.13

For each integer $q \geq 3$, let J_q be the pretzel knot $P(1-2q, 2q+1, 4q-3)$, so $p = -q$ and $r = 2q-2$.

Lemma 3.9 *For all $q \geq 3$, the standard Seifert surface S of J_q satisfies the free factor property.*

Proof The knot J_3 is $P(-5, 7, 9)$. Thus, for J_3, S satisfies the free factor property by Lemma 3.4.

Assume $q \geq 4$. Define X, H and K as above. After row reductions, $X/H[X, X]$ has presentation matrix

$$\begin{pmatrix} 1 & -(q+1) \\ 0 & -N \end{pmatrix},$$

where $N = -(q-1)^2$.

Let $C = -N = (q-1)^2$. Using Reidemeister–Schreier, we obtain the basis

$$\{ab^{-q-1}, bab^{-q-2}, \dots, b^{C-q-2}ab^{1-C}, b^{C-q-1}a, b^{C-q}ab^{-1}, \dots, b^{C-1}ab^{-q}, b^C\}.$$

To simplify computations we modify this basis by multiplying some of the elements by b^{-C} on the right, and obtain a free basis $\mathcal{B} = \{x_0, \dots, x_C\}$ of $H[X, X]$, where $x_k = b^k ab^{-q-1-k}$ for $k = 0, \dots, C-1$ and $x_C = b^C$.

We can rewrite α_H and β_H as

$$\begin{aligned} \alpha_H &= (b^{-1}a)^{q+1}a^{-q} \\ &= x_C^{-1}x_{C-1}x_C(x_{q-1}\cdots x_{i(q-2)-1}\cdots x_{q(q-2)-1})x_C x_{q-2}x_{q-3}^{-1}x_C^{-1} \\ &\qquad (x_{(q-3)(q+1)}^{-1}x_{(q-4)(q+1)}^{-1}\cdots x_{(q-i)(q+1)}^{-1}\cdots x_0^{-1}) \end{aligned}$$

and

$$\beta_H = b^{2q-1}(a^{-1}b)^q = x_{q-2}^{-1}x_C^{-1}(x_{q(q-2)-1}^{-1}\cdots x_{q(q-i)-1}^{-1}\cdots x_{q-1}^{-1})x_C^{-1}x_{C-1}^{-1}x_C.$$

Since $q \geq 4$, the generator x_0 appears once in the expression for α_H and does not appear in the expression for β_H. Also, since $q-2 < C-1$ and $qk-1 < C-1$ for all $k = 1, \dots, q-2$, x_{C-1} only appears once in the expression for β_H.

Thus x_{C-1} is a product of $\beta_H, x_1, \dots, x_{C-2}, x_C$ and x_0 is a product of $\alpha_H, x_1, \dots, x_C$. Therefore, the set $\{\alpha_H, \beta_H, x_1, \dots, x_{C-2}, x_C\}$ is a free basis of $H[X, X]$, so H is a free factor of $H[X, X]$.

After row reductions, $X/K[X, X]$ has presentation matrix

$$\begin{pmatrix} 1 & -q \\ 0 & C \end{pmatrix}.$$

We obtain a free basis $\mathcal{B} = \{x_0, \dots, x_C\}$ of $K[X, X]$, where $x_k = b^k a b^{-(q+k)}$ for $k = 0, \dots, C-1$ and $x_C = b^C$.

We can rewrite α_K and β_K as

$$\begin{aligned}\alpha_K &= (ab^{-1})^q a^{-q+1} \\ &= (x_0 x_{q-1} x_{2(q-1)} \cdots x_{(q-2)(q-1)}) x_C x_0 x_C^{-1} (x_{q(q-2)}^{-1} x_{q(q-3)}^{-1} \cdots x_0^{-1})\end{aligned}$$

and

$$\beta_K = b^{2q-2}(ba^{-1})^{q+1} = x_{q-1}^{-1} x_0^{-1} x_C^{-1} (x_{(q-2)(q-1)}^{-1} x_{(q-3)(q-1)}^{-1} \cdots x_0^{-1}.$$

The generator x_q appears once in the expression for α_K and does not appear in the expression for β_K. Also, x_C only appears once in the expression for β_K. Therefore, the set $\{\alpha_K, \beta_K, x_0, \dots, x_{q-1}, x_{q+1}, \dots, x_{C-1}\}$ is a free basis of $K[X, X]$, so K is a free factor of $K[X, X]$. Hence, S satisfies the free factor property. □

Proof of Theorem 1.13 By Lemma 3.9, J_q has a Seifert surface satisfying the free factor property. The Alexander polynomial of J_q is $Nt^2 + (1-2N)t + N$ where $N = -(q-1)^2$, so J_q is rationally homologically fibered and Δ_{J_q} has two positive real roots.

When $q-1$ is a prime power, $|\Delta_{J_q}(0)| = (q-1)^2$ is also a prime power. Therefore, when $q-1$ is a prime power, $\pi_1(M_{J_q})$ has residually torsion-free nilpotent commutator subgroup by Proposition 2.5, and $\pi_1(M_{J_q})$ is biorderable by Proposition 1.8. Since $p = -q$, we have that $\Sigma_2(J_q)$ is left-orderable by Proposition 1.15 for all $q \geq 3$. □

3.5 Knots where the standard Seifert surface does not satisfy the free factor property

Lemma 3.10 *If J is $P(1-2q, 2q+1, 2q^2+1)$ or $P(1-2q, 2q+1, 2q^2-3)$ then S does not satisfy the free factor property.*

Proof If J is $P(1-2q, 2q+1, 2q^2+1)$ then $p = -q$ and $r = q^2$, and if J is $P(1-2q, 2q+1, 2q^2-3)$ then $p = -q$ and $r = q^2-2$. In both cases $|N| = 1$, so by Proposition 3.3 S does not satisfy the free factor property. □

Lemma 3.11 *Suppose J is one of*

- $P(-3,5,11)$,
- $P(-3,7,7)$,
- $P(-5,7,R)$ *for* $R=11,13,21,23$ *or* 25,
- $P(-5,9,R)$ *for* $R=9,13,15$ *or* 17,
- $P(-5,11,11)$, *or*
- $P(-5,11,13)$.

Then S, the standard Seifert surface of J, does not satisfy the free factor property.

Proof If J is $P(-3,5,11)$, then $X/H[X,X]$ has presentation matrix

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}.$$

We have the free basis $\mathcal{B}=\{ab^{-1}, bab^{-2}, b^2\}$ of $H[X,X]$. Then let $x_0=ab^{-1}$, $x_1=bab^{-2}$ and $x_2=b^2$, so

$$\beta_H=b^6(a^{-1}b)^2=x_2^2x_1^{-2}x_2.$$

Let

$$\Gamma:=\frac{H[X,X]}{\langle\beta_H^{H[X,X]}\rangle}\cong\langle x_0,x_1,x_2 : x_2^3x_1^{-2}\rangle,$$

where $\langle\beta_H^{H[X,X]}\rangle$ is the normal closure of β_H in $H[X,X]$. Suppose $\{\alpha_H,\beta_H\}$ could be extended to a basis of $H[X,X]$. Then Γ is a free group and Γ has a subgroup isomorphic to $E:=\langle x_1,x_2 : x_2^3x_1^{-2}\rangle$. The abelianization of E is $\mathbb{Z}$, but E is not abelian since x_1 and x_2 do not commute. Thus E is not free, and Γ cannot be free either, which is a contradiction.

Therefore H is not a free factor of $H[X,X]$, and S does not satisfy the free factor property.

If J is $P(-5,7,25)$, then $H[X,X]$ has a free basis $x_0=a$, $x_1=bab^{-1}$, $x_2=b^2ab^{-2}$, $x_3=b^3ab^{-3}$ and $x_4=b^4$. Under this basis

$$\beta_H\alpha_H=b^{12}a^{-2}=x_4^3x_0^{-2}.$$

We can extend $\{\alpha_H,\beta_H\}$ to a free basis of $H[X,X]$ if and only if $\{\alpha_H,\beta_H\alpha_H\}$ can be extended to a free basis. However, an argument similar to the previous case shows that $\beta_H\alpha_H$ cannot be extended to a basis of $H[X,X]$.

Therefore, H is not a free factor of $H[X,X]$, and S does not satisfy the free factor property.

If J is $P(-5,7,13)$ or $P(-5,7,21)$, then $H[X,X]$ has free basis $x_0=a$, $x_1=bab^{-1}$, and $x_2=b^2$. Thus, the set $\{x_0,x_1,x_2^{-1}x_1x_0\}$ is also a free basis of $H[X,X]$. Denote $x_2^{-1}x_1x_0$ by y.

Using the basis $\{x_0,x_1,y\}$,

$$\alpha_H=(b^{-1}a)^4a^{-3}=y^2x_0^{-3}.$$

An argument similar to the previous cases shows that α_H cannot be extended to a basis of $H[X,X]$. Therefore H is not a free factor of $H[X,X]$, and S does not satisfy the free factor property.

The proofs of the other cases are similar to the cases above. Here we provide the elements obstructing the free factor property.

When J is $P(-3,7,7)$,

$$\beta_H=b^4(a^{-1}b)^3=x_2x_1^{-3}x_2,$$

where $x_0=ab^{-1}$, $x_1=bab^{-2}$ and $x_2=b^2$.

When J is $P(-5,7,11)$,

$$\beta_H=b^6(a^{-1}b)^3=x_3x_2^{-3}x_3,$$

where $x_0=ab^{-1}$, $x_1=bab^{-2}$, $x_2=b^2ab^{-3}$ and $x_3=b^3$.

When J is $P(-5,7,23)$,

$$\beta_H=b^12(a^{-1}b)^3=x_3^3x_2^{-3}x_3,$$

where $x_0=ab^{-1}$, $x_1=bab^{-2}$, $x_2=b^2ab^{-3}$ and $x_3=b^3$.

When J is $P(-5,9,9)$,

$$\beta_H=b^5(a^{-1}b)^4=x_0^5(x_2^{-1}x_1x_0)^2,$$

where $x_0=b$, $x_1=aba^{-1}$ and $x_2=a^2$.

When J is $P(-5,9,13)$,

$$\beta_H=b^7(a^{-1}b)^4=x_0^7(x_2^{-1}x_1x_0)^2,$$

where $x_0=b$, $x_1=aba^{-1}$ and $x_2=a^2$.

When J is $P(-5,9,15)$,

$$\beta_K\alpha_K=b^8a^{-3}=x_4^2x_0^{-3},$$

where $x_0=a$, $x_1=bab^{-1}$, $x_2=b^2ab^{-2}$, $x_3=b^3ab^{-3}$ and $x_4=b^4$.

When J is $P(-5,9,17)$,

$$\beta_H = b^9(a^{-1}b)^4 = (x_0x_6^{-1}x_4x_2)^3(x_6^{-1}x_5x_2)^2,$$

where $x_0 = ba^2$, $x_1 = aba$, $x_2 = a^2b$, $x_3 = a^3ba^{-1}$, $x_4 = a^4ba^{-2}$, $x_5 = a^5ba^{-3}$ and $x_6 = a^6$.

When J is $P(-5,11,11)$,

$$\beta_H = b^6(a^{-1}b)^5 = x_3x_2^{-5}x_3,$$

where $x_0 = ab^{-1}$, $x_1 = bab^{-2}$, $x_2 = b^2ab^{-3}$ and $x_3 = b^3$.

When J is $P(-5,11,13)$,

$$\beta_K = b^6(a^{-1}b)^6 = (x_0x_3x_6)^3(x_0x_2x_4x_6)^2,$$

where $x_0 = ba^{-3}$, $x_1 = aba^{-4}$, $x_2 = a^2ba^{-5}$, $x_3 = a^3ba^{-6}$, $x_4 = a^4ba^{-7}$, $x_5 = a^5ba^{-8}$ and $x_6 = a^6$. □

3.6 Proof of Theorem 1.2

Lemma 3.12 *If J is $P(-3,5,7)$, $P(-5,7,17)$ or $P(-5,9,11)$ then $\pi_1(M_J)$ does not have a residually torsion-free nilpotent commutator subgroup.*

Proof For each of these knots $N=0$, so this follows from Proposition 3.2. □

Proof of Theorem 1.2 When $p \geq 1$, S is pseudoalternating so S satisfies the free factor property [30]. Therefore, when $p \geq 1$, the knot group of $P(2p+1, 2q+1, 2r+1)$ has residually torsion-free nilpotent commutator subgroups when $|\Delta_J(0)|$ is a prime power.

The other positive results follow from applying Proposition 2.5 to Lemmas 3.4, 3.5, 3.6, 3.7, 3.8, and 3.12. □

4 Higher genus pretzel knots

In this section we prove Theorem 1.4, which presents a family of pretzel knots with arbitrarily high genus whose groups have residually torsion-free nilpotent commutator subgroups.

Let k be a positive integer, and let r be any integer. Suppose J is the $(2k+1)$–parameter pretzel knot $P(3,-3,\dots,3,-3,2r+1)$ with genus k Seifert surface S as shown in Figure 4. Define X, H and K as in Section 2.

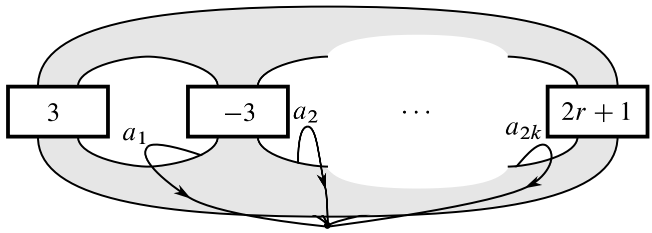

Figure 4: Seifert surface for higher genus pretzel knots.

Proof of Theorem 1.4 X is a free group of rank $2k$ with generating set $\{a_1, \dots, a_{2k}\}$ as shown in Figure 4. By choosing a suitable free basis for $\pi_1(S)$, the subgroup H has the free basis

$$\begin{aligned}
\alpha_1 &= (a_1^{-1}a_2)a_1,\\
\alpha_2 &= (a_3^{-1}a_2)^2(a_2^{-1}a_1)^2,\\
&\;\vdots\\
\alpha_{2i-1} &= (a_{2i-1}^{-1}a_{2i})(a_{2i-2}^{-1}a_{2i-1}),\\
\alpha_{2i} &= (a_{2i+1}^{-1}a_{2i})^2(a_{2i}^{-1}a_{2i-1})^2,\\
&\;\vdots\\
\alpha_{2k-1} &= (a_{2k-1}^{-1}a_{2k})(a_{2k-2}^{-1}a_{2k-1}),\\
\alpha_{2k} &= a_{2k}^{r+1}(a_{2k}^{-1}a_{2k-1})^2.
\end{aligned}$$

$X/H[X,X]$ has the presentation matrix

$$\begin{pmatrix}
0 & 1 & & & & & & \\
2 & 0 & -2 & & & & & \\
 & -1 & 0 & 1 & & & & \\
 & & 2 & 0 & -2 & & & \\
 & & & \ddots & \ddots & \ddots & & \\
 & & & & -1 & 0 & 1 \\
 & & & & & 2 & r-1
\end{pmatrix}, \tag{4-1}$$

which after row operations becomes

$$\begin{pmatrix}
0 & 1 & & & & & \\
2 & 0 & 0 & & & & \\
 & 0 & 0 & 1 & & & \\
 & & 2 & 0 & 0 & & \\
 & & & \ddots & \ddots & \ddots & \\
 & & & & 0 & 0 & 1 \\
 & & & & & 2 & 0
\end{pmatrix}.$$

It follows that

$$\frac{X}{H[X,X]} \cong \bigoplus_{j=1}^{k} (\mathbb{Z}/2\mathbb{Z}),$$

where the j^{th} $\mathbb{Z}/2\mathbb{Z}$ factor is generated by the class of a_{2j} in $X/H[X,X]$, and when i is odd the class of a_i is trivial.

Define

$$a_{\boldsymbol{\sigma}} := a_1^{\sigma_1} a_3^{\sigma_2} \dots a_{2k-1}^{\sigma_k},$$

where $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_k) \in \{0,1\}^k$. $H[X,X]$ is an index 2^k subgroup of X, so the rank of $H[X,X]$ is $2^k + 1$.

The following set is a set of coset representatives of $H[X,X]$:

$$\mathcal{C} = \{a_{\boldsymbol{\sigma}} \mid \boldsymbol{\sigma} \in \{0,1\}^k\}.$$

From $\mathcal{C}$, we find a free basis $\mathcal{B}$ of elements of the form $x_{k,\boldsymbol{\sigma}} := a_{\boldsymbol{\sigma}} a_k \overline{a_{\boldsymbol{\sigma}} a_k}^{-1}$.

We point out a few important examples of basis elements. For i odd,

$$a_i^2 = a_i a_i \overline{a_i a_i}^{-1} \in \mathcal{B}.$$

For i even,

$$a_i = 1 a_i \overline{1 a_i}^{-1} \in \mathcal{B}.$$

For i odd and j even,

$$a_i a_j a_i^{-1} = a_i a_j \overline{a_i a_j}^{-1} \in \mathcal{B}.$$

Using the basis $\mathcal{B}$ rewrite the α_i as

$$\begin{aligned}
\alpha_1 &= (a_1^{-2})(a_1 a_2 a_1^{-1})(a_1^2), \\
\alpha_2 &= (a_3^{-2})(a_3 a_2 a_3^{-1})(a_1 a_2^{-1} a_1^{-1})(a_1^2), \\
&\ \vdots \\
\alpha_{2i-1} &= (a_{2i-1}^{-2})(a_{2i-1} a_{2i} a_{2i-1}^{-1})(a_{2i-1} a_{2i-2}^{-1} a_{2i-1}^{-1})(a_{2i-1}^2), \\
\alpha_{2i} &= (a_{2i+1}^{-2})(a_{2i+1} a_{2i} a_{2i+1}^{-1})(a_{2i-1} a_{2i}^{-1} a_{2i-1}^{-1})(a_{2i-1}^2), \\
&\ \vdots \\
\alpha_{2k-1} &= (a_{2k-1}^{-2})(a_{2k-1} a_{2k} a_{2k-1}^{-1})(a_{2k-1} a_{2k-2}^{-1} a_{2k-1}^{-1})(a_{2k-1}^2), \\
\alpha_{2k} &= a_{2k}^r (a_{2k-1} a_{2k}^{-1} a_{2k-1}^{-1})(a_{2k-1}^2),
\end{aligned}$$

which can be extended to the free basis $\mathcal{B}'$ of $H[X,X]$

$$\mathcal{B}' = (\mathcal{B} - (\mathcal{B}_1 \cup \mathcal{B}_2 \cup \{a_{2k-1}^2\})) \cup \{\alpha_1, \dots, \alpha_{2k}\},$$

where

$$\mathcal{B}_1 = \{a_1a_2a_1^{-1}, a_3a_4a_3^{-1}, \dots, a_{2k-1}a_{2k}a_{2k-1}^{-1}\}$$

and

$$\mathcal{B}_2 = \{a_3a_2a_3^{-1}, a_5a_4a_5^{-1}, \dots, a_{2k-1}a_{2k-2}a_{2k-1}^{-1}\}.$$

Thus, H is a free factor of $H[X,X]$.

A similar argument shows K is a free factor of $K[X,X]$. Thus, S satisfies the free factor property.

From (4-1) we compute $|X : H[X,X]| = 2^k$, so by Proposition 2.3 J is rationally homologically fibered. Thus, S is an unknotted minimal genus Seifert surface, and J is rationally homologically fibered. It follows from Proposition 2.5 that the commutator subgroup of J is residually torsion-free nilpotent. □

Proof of Corollary 1.10 From the Seifert matrix (4-1), we compute the Alexander polynomial

$$\Delta_J(t) = (t-2)^k(2t-1)^k.$$

It follows from Theorems 1.4 and 1.6 that $\pi_1(M_J)$ is biorderable. □

Appendix A Proofs of lemmas

In this appendix, we present the proofs of Lemmas 3.6, 3.7 and 3.8. Let J be a pretzel knot $P(2p+1, 2q+1, 2r+1)$ with $1 \le q \le r$. Define the Seifert surface S and the groups $X \cong \langle a, b\rangle$, $H \cong \langle \alpha_H, \beta_H\rangle$ and $K \cong \langle \alpha_K, \beta_K\rangle$ as in Section 3.

A.1 Proof of Lemma 3.6

Lemma 3.6 *If J is a $P(2p+1, 3, 2r+1)$ pretzel knot with $p < -2$, then S satisfies the free factor property.*

Proof From (3-1),

$$\alpha_H = b^{-1}ab^{-1}a^{p+1}, \quad \alpha_K = ab^{-1}a^{p+1},$$
$$\beta_H = b^{r+1}a^{-1}b, \qquad \beta_K = b^{r+1}a^{-1}ba^{-1}.$$

The abelian group $X/H[X,X]$ has a presentation matrix

$$\begin{pmatrix} 1 & -r-2 \\ 0 & -N \end{pmatrix},$$

where $N = pr + 2p + 2r + 2 = (p+2)(r+2) - 2$, which is negative since $p \le -2$.

Let $C = -N$. Using $\mathcal{C} = \{1, b, \ldots, b^C\}$ as a set of coset representatives, we apply Reidemeister–Schreier to obtain a free basis of $H[X, X]$. Modifying this basis, we get

$$\mathcal{B} = \{x_0, \ldots, x_C\},$$

where $x_k := b^k a b^{-r-2-k}$ when $0 \le k \le C-1$ and $x_C := b^C$.

Using the rewriting process, we have that

$$\alpha_H = x_C^{-1} x_{C-1} (x_{C-r-4}^{-1} x_{C-2r-6}^{-1} \cdots x_{C-i(r+2)-2}^{-1} \cdots x_{r+2}^{-1} x_0^{-1})$$

and

$$\beta_H = x_C^{-1} x_{C-1}^{-1} x_C.$$

(Note that $C > r+2$ since $p < -2$, so x_{r+2} is defined.) We can extend $\{\alpha_H, \beta_H\}$ to the set $\{\alpha_H, \beta_H, x_1, \ldots, x_{C-2}, x_C\}$, which is a free basis of $H[X, X]$, so H is a free factor of $H[X, X]$.

$X/K[X, X]$ has a presentation matrix

$$\begin{pmatrix} -N & 0 \\ -p-2 & 1 \end{pmatrix}.$$

Let $l = -p-2$ so $C = l(r+2)+2$. Note that l is a positive integer. We obtain a free basis of $K[X, X]$

$$\mathcal{B} = \{x_0, \ldots, x_C\},$$

where $x_k := a^k b a^{l-k}$ when $0 \le k \le C-1$ and $x_C := a^C$.

Using the rewriting process,

$$\alpha_K = x_{l+1}^{-1}$$

and

$$\beta_K = x_0 x_C^{-1} x_{l(r+1)+2} x_{lr+2} x_{l(r-1)+2} \cdots x_{2l+2} x_{l+1}.$$

The set $\{\alpha_K, \beta_K, x_1, \ldots, x_l, x_{l+2}, \ldots, x_C\}$ is a free basis of $K[X, X]$ so K is a free factor of $K[X, X]$. Thus, S satisfies the free factor property. □

A.2 Proof of Lemma 3.7

Lemma 3.7 *Suppose J is $P(-3, 2q+1, 2r+1)$ and one of the following conditions holds*:

(1) $q = 2$ *and* $r \ge 6$,

(2) $q = 3$ *and* $r \ge 4$,

(3) $q > 3$.

Then S satisfies the free factor property.

Proof This lemma is shown by applying the outline from Section 2 to two cases. First, we address the case when $q = 2$ and $r \geq 6$, then we show the lemma is true when $q \geq 3$, $r \geq 4$ and $q \leq r$.

Case $q = 2$ and $r \geq 6$ $X/H[X,X]$ has a presentation matrix

$$\begin{pmatrix} 1 & -3 \\ 0 & N \end{pmatrix},$$

where $N = r - 3$.

$H[X,X]$ has free basis $x_k = b^k a b^{-k-3}$ for $k = 0, \dots, N-1$, and $x_N = b^N$. Under this basis

$$\alpha_H = (b^{-1}a)^2 b^{-1} a^{-1} = x_N^{-1} x_{N-1} x_N x_1 x_0^{-1}$$

and

$$\beta_H = b^{r+1}(ba^{-1})^2 = x_N x_1^{-1} x_N x_{N-1}^{-1} x_N.$$

Since $r \geq 6$, we have $N \geq 3$, so $x_{N-1} \neq x_1$. Thus, the set $\{\alpha_H, \beta_H, x_2, \dots, x_N\}$ is a free basis of $H[X,X]$ so H is a free factor of $H[X,X]$.

$X/K[X,X]$ has a presentation matrix

$$\begin{pmatrix} 1 & -2 \\ 0 & N \end{pmatrix},$$

where $N = r - 3$.

$K[X,X]$ has free basis $x_k = b^k a b^{-k-2}$ for $k = 0, \dots, N-1$, and $x_N = b^N$. Under this basis

$$\alpha_K = (ab^{-1})^2 a^{-1} = x_0 x_1 x_0^{-1}$$

and

$$\beta_K = b^{r+1} a^{-1} (ba^{-1})^2 = x_N x_2^{-1} x_1^{-1} x_0^{-1}.$$

The set $\{\alpha_K, \beta_K, x_0, x_3, \dots, x_N\}$ is a free basis of $K[X,X]$ so K is a free factor of $K[X,X]$.

Case $q \geq 3$ and $r \geq 4$ $X/H[X,X]$ has a presentation matrix

$$\begin{pmatrix} 1 & -r \\ 0 & N \end{pmatrix},$$

where $N = qr - q - r - 1 = (q-1)(r-1) - 2$. Note that since $q \geq 3$ and $r \geq 4$, $N > r - 2 > 1$.

We then obtain a free basis $x_k = b^k a b^{-r-k}$ for $k = 0, \dots, N-1$ and $x_N = b^N$. Under this basis

$$\alpha_H = (b^{-1}a)^{q+1}a^{-3} = x_N^{-1} x_{N-1} x_N x_{r-2} x_{2r-3} \cdots x_{N-r+2} x_N x_1 x_0^{-1}$$

and

$$\beta_H = b^{r+1(a^{-1}b)^q} = x_1^{-1} x_N^{-1} x_{N-r+2}^{-1} x_{N-2r+3}^{-1} \cdots x_{r-2} x_{N-1}.$$

Since $N > r-2 > 1$, the set $\{\alpha_H, \beta_H, x_2, \dots, x_N\}$ is a free basis of $H[X,X]$ so H is a free factor of $H[X,X]$.

For K, we begin by substituting $a = a_* b_*$ and $b = b_*$ so that

$$\alpha_K = a_*^q b_*^{-1} a_*^{-1} \quad \text{and} \quad \beta_K = b_*^r a_*^{-q-1}.$$

$X/K[X,X]$ has a presentation matrix

$$\begin{pmatrix} N & 0 \\ 1-q & 1 \end{pmatrix},$$

where $N = qr - q - r - 1$.

Under the basis $x_k = a_* b_* a_*^{1-q-k}$ for $k = 0, \dots, N-1$ and $x_N = a_*^N$,

$$\alpha_K = x_1^{-1}$$

and

$$\beta_K = x_0 x_{q-1} \cdots x_{(q-1)(r-1)} x_N.$$

Similarly to H, K is a free factor of $K[X,X]$. Therefore, S satisfies the free factor property. □

A.3 Proof of Lemma 3.8

Lemma 3.8 *Suppose J is $P(-5, 2q+1, 2r+1)$ and one of the following conditions holds*:

(1) $q = 3$ *and* $r \geq 13$,

(2) $q = 4$ *and* $r \geq 9$,

(3) $q = 5$ *and* $r \geq 7$,

(4) $q > 5$.

Then S satisfies the free factor property.

This lemma is shown by applying the outline from Section 2 to several cases.

Lemma A.1 *If J is $P(-5,7,2r+1)$ with $r \geq 13$, then S satisfies the free factor property.*

Proof In this case, $q=3$ and $N=r-8$. $X/H[X,X]$ has a presentation matrix
$$\begin{pmatrix} 1 & -4 \\ 0 & N \end{pmatrix}.$$
We use the free basis, $x_k = b^k a b^{-4-k}$ for $k=0,\dots,N-1$ and $x_N = b^N$.

When $r=13$,
$$\alpha_H = x_5^{-1} x_4 x_5 x_2 x_5 x_0 x_5^{-1} x_4^{-1} x_0^{-1}$$
and
$$\beta_H = x_5^2 x_0^{-1} x_5^{-1} x_2^{-1} x_5^{-1} x_4^{-1} x_5,$$
so
$$\beta_H \alpha_H = x_5 x_4^{-1} x_0^{-1}.$$
The set $\{\alpha_H, \beta_H, x_1, x_3, x_4, x_5\}$ is a free basis of $H[X,X]$ so H is a free factor of $H[X,X]$.

When $r \geq 14$,
$$\alpha_H = x_N^{-1} x_{N-1} x_N x_2 x_5 x_4^{-1} x_0^{-1}$$
and
$$\beta_H = x_N x_5^{-1} x_2^{-1} x_N^{-1} x_{N-1}^{-1} x_N.$$
The set $\{\alpha_H, \beta_H, x_1, x_3, \dots, x_N\}$ is a free basis of $H[X,X]$ so H is a free factor of $H[X,X]$.

$X/K[X,X]$ has a presentation matrix
$$\begin{pmatrix} 1 & -3 \\ 0 & N \end{pmatrix}.$$
We use the free basis $x_k = b^k a b^{-3-k}$ for $k=0,\dots,N-1$ and $x_N = b^N$.

Using this basis,
$$\alpha_K = x_0 x_2 x_4 x_3^{-1} x_0^{-1}.$$
When $r=13$ or $r=14$,
$$\beta_K = x_N^2 x_{6-N}^{-1} x_N^{-1} x_4^{-1} x_2^{-1} x_0^{-1},$$
and, when $r \geq 15$,
$$\beta_K = x_N x_6^{-1} x_4^{-1} x_2^{-1} x_0^{-1}.$$
In both cases, the set $\{\alpha_K, \beta_K, x_0, x_1, x_4, \dots, x_N\}$ is a free basis of $K[X,X]$ so K is a free factor of $K[X,X]$. □

Lemma A.2 *If J is $P(-5,9,2r+1)$ with $r \geq 9$, then S satisfies the free factor property.*

Proof In this case, $q=4$ and $N=2r-10$. $X/H[X,X]$ has a presentation matrix

$$\begin{pmatrix} N & 0 \\ -2 & 1 \end{pmatrix}$$

after the substitutions $a=b_*^2a_*$ and $b=b_*$. We use the free basis $x_k=a_*^k b_* a_*^{-2-k}$ for $k=0,\dots,N-1$ and $x_N=a_*^N$.

When $r=9$,

$$\alpha_H = x_0x_3x_6x_8x_1x_2^{-1}x_8^{-1}x_7^{-1}x_5^{-1}x_2^{-1}x_0^{-1}$$

and

$$\beta_H = (x_0x_2x_4x_6x_8)^2x_0x_2x_1^{-1}x_8^{-1}x_6^{-1}x_3^{-1}x_0^{-1}.$$

The set $\{\alpha_H,\beta_H,x_0,x_1,x_2,x_4,x_6,x_7,x_8\}$ is a free basis of $H[X,X]$ so H is a free factor of $H[X,X]$.

When $r=10$,

$$\alpha_H = x_0x_3x_6x_9x_{10}x_0^{-1}x_{10}^{-1}x_7^{-1}x_5^{-1}x_2^{-1}x_0^{-1}$$

and

$$\beta_H = (x_0x_2x_4x_6x_8x_{10})^2x_0x_{10}^{-1}x_9^{-1}x_6^{-1}x_3^{-1}x_0^{-1}.$$

When $r \geq 11$,

$$\alpha_H = x_0x_3x_6x_9x_{10}^{-1}x_7^{-1}x_5^{-1}x_2^{-1}x_0^{-1}$$

and

$$\beta_H = x_0x_2\cdots x_{N-2}x_Nx_0x_2x_4x_6x_8x_{10}x_9^{-1}x_6^{-1}x_3^{-1}x_0^{-1}.$$

In both cases, the set $\{\alpha_H,\beta_H,x_0,\dots,x_6,x_8,x_{10},\dots,x_N\}$ is a free basis of $H[X,X]$.

$X/K[X,X]$ has a presentation matrix

$$\begin{pmatrix} 1 & 3-r \\ 0 & N \end{pmatrix}.$$

We use the free basis $x_k=b^kab^{3-r-k}$ for $k=0,\dots,N-1$ and $x_N=b^N$. Using this basis,

$$\alpha_K = x_0x_{r-4}x_Nx_2x_{r-2}x_{r-3}^{-1}x_0^{-1}$$

and

$$\beta_K = x_4^{-1}x_N^{-1}x_{r-2}^{-1}x_2^{-1}x_N^{-1}x_{r-4}^{-1}x_0^{-1}.$$

Since $r \geq 9$,

$$N = r-8+r-2 > r-2 > r-3 > r-4 > 0,$$

so the generators x_{r-2}, x_{r-3} and x_{r-4} are valid generators.

The set $\{\alpha_K, \beta_K, x_1, \dots, x_{r-4}, x_{r-2}, \dots, x_N\}$ is a free basis of $K[X,X]$ so K is a free factor of $K[X,X]$. □

Lemma A.3 *If J is $P(-5,11,2r+1)$ with $r \geq 7$, then S satisfies the free factor property.*

Proof In this case, $q=5$ and $N=3r-12$. $X/H[X,X]$ has a presentation matrix

$$\begin{pmatrix} 1 & r-6 \\ 0 & N \end{pmatrix}.$$

We use the free basis $x_k = b^k a b^{r-6-k}$ for $k = 0, \dots, N-1$ and $x_N = b^N$.

Using this basis,

$$\alpha_H = x_{2r-6}^{-1} x_N x_0^{-1}.$$

When $r=7$,

$$\beta_H = x_9 x_0^{-1} x_2^{-1} x_4^{-1} x_6^{-1} x_8^{-1} x_9,$$

and, when $r \geq 8$,

$$\beta_H = x_{2r-5}^{-1} x_N x_2^{-1} x_{r-3}^{-1} x_{2r-8}^{-1} x_{3r-13}^{-1} x_N.$$

Note that, when $r \geq 8$,

$$N > 3r-13 > 0,$$
$$N = r-7+2r-5 > 2r-5 > 2r-6 > 2r-8 > 0,$$

and

$$N = 2r-9+r-3 > r-3 > 0,$$

so the generators x_{3r-13}, x_{2r-5}, x_{2r-6}, x_{2r-8} and x_{r-3} are valid generators.

In both cases, the set $\{\alpha_H, \beta_H, x_1, x_3, \dots, x_N\}$ is a free basis of $H[X,X]$, so H is a free factor of $H[X,X]$.

After making the substitutions $a = b_*^2 a_*$ and $b = b_*$, $X/K[X,X]$ has a presentation matrix

$$\begin{pmatrix} N & 0 \\ 3 & 1 \end{pmatrix}.$$

We use the free basis $x_k = a_*^k b_* a_*^{3-k}$ for $k = 0, \dots, N-1$ and $x_N = b_*^N$. Using this basis,

$$\alpha_K = x_1 x_N^{-1} x_{N-1} x_{N-3} x_{N-5} x_{N-4}^{-1} x_{N-2}^{-1} x_N x_1^{-1}$$

and

$$\beta_K = x_0 x_N^{-1} x_{N-3} x_{N-6} \cdots x_3 x_0 x_N^{-1} x_{N-3} x_{N-6} x_{N-7}^{-1} x_{N-5}^{-1} x_{N-3}^{-1} x_{N-1}^{-1} x_N x_1^{-1}.$$

Since $r \geq 7$, we have $N \geq 9$, so all the generators used are valid generators.

The set $\{\alpha_K, \beta_K, x_0, \dots, x_{N-8}, x_{N-6}, x_{N-5}, x_{N-4}, x_{N-3}, x_{N-1}, x_N\}$ is a free basis of $K[X, X]$, so K is a free factor of $K[X, X]$. □

Lemma A.4 *If J is $P(-5, 13, 13)$ or $P(-5, 13, 15)$, then S satisfies the free factor property.*

Proof If J is $P(-5, 13, 13)$, then $p = -3$ and $q = r = 6$. $X/H[X, X]$ has presentation matrix

$$\begin{pmatrix} 10 & 0 \\ -2 & 1 \end{pmatrix}.$$

We use the free basis $x_k := a^k b a^{-2-k}$ for $k = 0, \dots, 9$ and $x_{10} := a^{10}$.

Using this basis,

$$\alpha_H = x_{10}^{-1} x_8^{-1} x_7^{-1} x_6^{-1} x_5^{-1} x_4^{-1} x_3^{-1} x_2^{-1}$$

and

$$\beta_H = x_0 x_2 x_4 x_6 x_8 x_{10} x_0 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_{10},$$

so

$$\beta_H \alpha_H = x_0 x_2 x_4 x_6 x_8 x_{10} x_0.$$

The set $\{\alpha_H, \beta_H, x_0, x_1, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ is a free basis of $H[X, X]$, so H is a free factor of $H[X, X]$.

$X/K[X, X]$ has presentation matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 10 \end{pmatrix}.$$

We use the free basis $x_k := a^k b a^{1-k}$ for $k = 0, \dots, 9$ and $x_{10} := a^{10}$.

Using this basis,

$$\alpha_K = x_0 x_{10}^{-1} x_8 x_6 x_4 x_2 x_0 x_{10}^{-1} x_9^{-1} x_{10} x_0^{-1}$$

and

$$\beta_K = x_8^{-1} x_{10} x_0^{-1} x_2^{-1} x_4^{-1} x_6^{-1} x_8^{-1} x_{10} x_0^{-1}.$$

The set $\{\alpha_K, \beta_K, x_0, x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}\}$ is a free basis of $K[X,X]$ so K is a free factor of $K[X,X]$.

If J is $P(-5,13,15)$, then $p=-3$, $q=6$ and $r=7$. After making the substitutions $a=b_*^2 a_*$ and $b=b_*$, $X/H[X,X]$ has presentation matrix

$$\begin{pmatrix} 14 & 0 \\ 4 & 1 \end{pmatrix}.$$

We use the free basis $x_k := a_*^k b_* a_*^{4-k}$ for $k=0,\dots,13$ and $x_{14} := a_*^{14}$.

Using this basis,

$$\alpha_H = x_0 x_{14}^{-1} x_{11} x_8 x_5 x_2 x_{14}^{-1} x_{13} x_{14} x_0^{-1} x_3^{-1} x_7^{-1} x_{10}^{-1} x_{14} x_0^{-1}$$

and

$$\beta_H = x_0 x_{14}^{-1} x_{10} x_6 x_2 x_{14}^{-1} x_{12} x_8 x_4 x_0 x_{14}^{-1} x_{13}^{-1} x_{14} x_2^{-1} x_5^{-1} x_8^{-1} x_{11}^{-1} x_{14} x_0^{-1}.$$

The set $\{\alpha_H, \beta_H, x_0, \dots, x_5, x_8, \dots, x_{14}\}$ is a free basis of $H[X,X]$ so H is a free factor of $H[X,X]$.

$X/K[X,X]$ has presentation matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 14 \end{pmatrix}.$$

We use the free basis $x_k := b^k a b^{2-k}$ for $k=0,\dots,13$ and $x_{14} := b^{14}$.

Using this basis,

$$\alpha_K = x_0 x_{14}^{-1} x_{11} x_8 x_5 x_2 x_{14}^{-1} x_{13} x_{10} x_{14} x_0^{-1}$$

and

$$\beta_K = x_{10}^{-1} x_{13}^{-1} x_{14} x_2^{-1} x_5^{-1} x_8^{-1} x_{11}^{-1} x_{14} x_0^{-1},$$

so

$$\beta_H \alpha_H = x_{14} x_0^{-1}.$$

The set $\{\alpha_K, \beta_K, x_1, x_3, \dots, x_{14}\}$ is a free basis of $K[X,X]$, so K is a free factor of $K[X,X]$. □

Lemma A.5 *If J is $P(-5, 2q+1, 2r+1)$ with q even, $q \geq 6$ and $r \geq 8$, then H is a free factor of $H[X, X]$.*

Proof Let c be the integer such that $q = 2c$. $X/H[X, X]$ has a presentation matrix

$$\begin{pmatrix} 2 & -r \\ 0 & w \end{pmatrix},$$

where $w = cr - 2c - r - 1$ and $N = 2w$.

We have the set of coset representatives

$$\mathcal{C} = \{1, b, b^2, \dots, b^{w-1}, a, ab, ab^2, \dots, ab^{w-1}\}.$$

We apply Reidemeister–Schreier to find a free basis of $H[X, X]$. In the following computations we assume that the coset representative $\overline{a^2}$ of a^2 is b^r. For this to be correct, it must be true that $r < w$, which we verify here.

Since $q \geq 6$, we have $c \geq 2$, and $r \geq 8$, so

$$w = cr - 2c - r - 1 = (c-3)(r-2) + (r-7) + r > r.$$

We apply Reidemeister–Schreier to find $x_{c,x} = cx(\overline{cx})^{-1}$ for each $c \in \mathcal{C}$ and $x \in \{a, b\}$:

$$\begin{aligned}
x_{b^i,a} &= b^i a(\overline{b^i a})^{-1} &&= \begin{cases} b^i a b^{-i} a^{-1} & \text{if } 0 < i \leq w-1, \\ 1 & \text{if } i = 0, \end{cases} \\
x_{b^i,b} &= b^{i+1}(\overline{b^{i+1}})^{-1} &&= \begin{cases} 1 & \text{if } 0 \leq i < w-1, \\ b^w & \text{if } i = w-1, \end{cases} \\
x_{ab^i,a} &= ab^i a(\overline{ab^i a})^{-1} &&= \begin{cases} ab^i ab^{-i-r} & \text{if } 0 \leq i < w-r, \\ ab^i ab^{w-i-r} & \text{if } w-r \leq i \leq w-1, \end{cases} \\
x_{ab^i,b} &= ab^{i+1}(\overline{ab^{i+1}})^{-1} &&= \begin{cases} 1 & \text{if } 0 \leq i < w-1, \\ ab^w a^{-1} & \text{if } i = w-1. \end{cases}
\end{aligned}$$

The nontrivial elements $x_{c,x}$ form a basis $\{x_1, \dots, x_w, y_0, \dots, y_w\}$, where

$$x_i = \begin{cases} b^i a b^{-i} a^{-1} & \text{if } 1 \leq i \leq w-1, \\ b^w & \text{if } i = w, \end{cases}$$

and

$$y_i = \begin{cases} ab^i ab^{-i-r} & \text{if } 0 \leq i < w-r, \\ ab^i ab^{w-i-r} & \text{if } w-r \leq i < w, \\ ab^w a^{-1} & \text{if } i = w. \end{cases}$$

Using this basis,

$$\beta_H \alpha_H = y_0^{-1}$$

and

$$\beta_H = y_1^{-1} x_2^{-1} \prod_{i=1}^{c-2} (y_{\delta(i)+1}^{-1} x_{\delta(i)+2}^{-1}) y_{w-2}^{-1} x_{w-1}^{-1} x_w,$$

where

$$\delta(i) = w - i(r-2).$$

We claim that $\delta(i) \neq 0$ for all i. Since $w = (c-1)(r-2) - 3$

$$\delta(i) = w - i(r-2) = (r-2)(c-i-1) - 3,$$

so if $\delta(i) = 0$ then $(r-2)(c-i-1) = 3$. However, since $r \geq 8$, we have that $r-2$ does not divide 3.

Thus, y_1 only appears once in β_H so the set $\{\beta_H \alpha_H, \beta_H, x_1, \dots, x_w, y_2, \dots, y_w\}$ is a free basis of $H[X,X]$. Since $\{\beta_H \alpha_H, \beta_H\}$ is a free basis of H, H is a free factor of $H[X,X]$. □

Lemma A.6 *If J is $P(-5, 2q+1, 2r+1)$ with q odd and $q \geq 7$, then H is a free factor of $H[X,X]$.*

Proof Let c be the integer such that $q = 2c+1$. $X/H[X,X]$ has a presentation matrix

$$\begin{pmatrix} 1 & v \\ 0 & N \end{pmatrix},$$

where $v = cr - 2c - r - 2$ and $N = 2cr - 4c - r - 4 = 2v + r$.

We use the free basis $x_k = b^k a b^{v-k}$ for $k = 0, \dots, N-1$ and $x_N = b^N$. Using this basis,

$$\beta_H \alpha_H = x_{v+r}^{-1} x_N x_0^{-1}$$

and

$$\beta_H = x_{v+r+1}^{-1} \prod_{i=0}^{2c-1} y_i,$$

where

$$y_i = \begin{cases} x_{\epsilon(i)}^{-1} & \text{if } \epsilon(i) < N - v - 1, \\ x_{\epsilon(i)}^{-1} x_N & \text{if } \epsilon(i) \geq N - v - 1, \end{cases}$$

and

$$\epsilon(i) = 2 + i(v+1) \bmod N.$$

Since $q \geq 7$, we have $c \geq 3$, and, since $r \geq 7$,

$$v = cr - 2c - r - 2 = (c-2)(r-2) + r - 6 > 1.$$

This means that

$$N = 2v + r > v + r + 1 > v + r > 0,$$

so x_{v+r} and x_{v+r+1} are valid generators.

We claim that $\epsilon(i)$ is distinct for each $i = 0, \dots, 2c-1$. Suppose that $\epsilon(i) = \epsilon(j)$ for some i and j. Then $(j-i)(v+1)$ is a multiple of N. In particular, N divides $(j-i)\gcd(N, v+1)$. Applying the Euclidean algorithm to N and $v+1$, we have

$$N = 2(v+1) + r - 2$$

and

$$v + 1 = (c-1)(r-2) - 3,$$

so

$$\gcd(N, v+1) = \gcd(r-2, 3) \le 3.$$

The maximum value of $j - i$ is $2c - 1$. It follows that

$$N \le 3(2c-1).$$

However, since $c \ge 3$ and $r \ge 7$,

$$N = 2cr - 4c - r - 4 = (2c-1)(r-4) + 4c - 8 \ge 3(2c-1) + 4 > 3(2c-1),$$

which is a contradiction.

Thus $x_{\epsilon(0)} = x_2$ only appears once in β_H so the set $\{\beta_H \alpha_H, \beta_H, x_1, x_3, \dots, x_N\}$ is a free basis of $H[X, X]$. Therefore, H is a free factor of $H[X, X]$. □

Lemma A.7 *If J is $P(-5, 2q+1, 2r+1)$ with $q \equiv 0 \bmod 3$, $q \ge 6$ and $r \ge 8$, then K is a free factor of $K[X, X]$.*

Proof Let c be the integer such that $q = 3c$. $X/K[X, X]$ has a presentation matrix

$$\begin{pmatrix} 1 & v \\ 0 & N \end{pmatrix},$$

where $v = cr - 2c - r - 1$ and $N = 3cr - 6c - 2r - 2 = 3v + r + 1$.

We use the free basis $x_k = b^k a b^{v-k}$ for $k = 0, \dots, N-1$ and $x_N = b^N$. Using this basis,

$$\beta_K \alpha_K = x_{v+r+1}^{-1} x_{2v+r+1}^{-1} x_N x_0^{-1}$$

and

$$\beta_K = x_{v+r+1}^{-1} x_{2v+r+2}^{-1} \prod_{i=0}^{3c-2} y_i,$$

where

$$y_i = \begin{cases} x_{\zeta(i)}^{-1} & \text{if } \zeta(i) < N - v - 1, \\ x_{\zeta(i)}^{-1} x_N & \text{if } \zeta(i) \geq N - v - 1, \end{cases}$$

and

$$\zeta(i) = 1 + i(v+1) \bmod N.$$

Since $q \geq 6$, we have $c \geq 2$, and, since $r \geq 8$,

$$v = cr - 2c - r - 1 = (c-1)(r-5) + 3c - 6 > 1.$$

This means that

$$N = 3v + r + 1 > 2v + r + 2 > 2v + r + 1 > v + r + 1 > 0,$$

so x_{v+r+1}, x_{2v+r+1} and x_{2v+r+2} are valid generators.

Suppose that $\zeta(i) = \zeta(j)$ for some i and j. Then N divides $(j-i)\gcd(N, v+1)$. Applying the Euclidean algorithm to N and $v+1$, we have

$$N = 3(v+1) + r - 2$$

and

$$v + 1 = (c-1)(r-2) - 2,$$

so

$$N \leq 2(3c-2).$$

However, since $c \geq 2$ and $r \geq 8$,

$$N = 3cr - 6c - 2r - 2 = (3c-2)(r-4) + 6c - 10 > 2(3c-2),$$

so $\zeta(i)$ is distinct for each $i = 0, \dots, 3c-2$. Thus $x_{\zeta(0)} = x_1$ only appears once in β_K so the set $\{\beta_K \alpha_K, \beta_K, x_2, \dots, x_N\}$ is a free basis of $K[X, X]$. Therefore, K is a free factor of $K[X, X]$. □

Lemma A.8 *If J is $P(-5, 2q+1, 2r+1)$ with $q \equiv 1 \bmod 3$ and $q \geq 7$, then K is a free factor of $K[X, X]$.*

Proof Let c be the integer such that $q = 3c + 1$. $X/K[X, X]$ has a presentation matrix

$$\begin{pmatrix} 1 & -v \\ 0 & N \end{pmatrix},$$

where $v = cr - 2c - 1$ and $N = 3cr - 6c - r - 4 = 3v - r - 1$.

We use the free basis $x_k = b^k a b^{-v-k}$ for $k = 0, \dots, N-1$ and $x_N = b^N$. Using this basis,

$$\beta_K \alpha_K = x_N^{-1} x_{2v}^{-1} x_v^{-1} x_0^{-1}$$

and

$$\beta_K = x_N^{-1} x_{2v}^{-1} x_{v+1}^{-1} \prod_{i=0}^{3c-1} y_i,$$

where

$$y_i = \begin{cases} x_{\eta(i)}^{-1} & \text{if } \eta(i) < N - v + 1, \\ x_{x_N^{-1}\eta(i)}^{-1} & \text{if } \eta(i) \geq N - v + 1, \end{cases}$$

and

$$\eta(i) = 2 - i(v-1) \bmod N.$$

Since $q \geq 7$, we have $c \geq 2$, and, since $r \geq 7$,

$$v = cr - 2c - 1 = (c-1)(r-8) + 4c - 8 + r + 1 > r + 1.$$

This means that

$$N = 3v - r - 1 > 2v > v + 1 > v > 0,$$

so x_v, x_{v+1} and x_{2v} are valid generators.

Suppose that $\eta(i) = \eta(j)$ for some i and j. Then N divides $(j-i)\gcd(N, v-1)$. Applying the Euclidean algorithm to N and $v-1$, we have

$$N = 3(v-1) - (r-2)$$

and

$$v - 1 = c(r-2) - 2,$$

so

$$N \leq 2(3c-1).$$

However, since $c \geq 2$ and $r \geq 7$,

$$N = 3cr - 6c - r - 4 = (3c-1)(r-4) + 6c - 8 > 2(3c-1),$$

so $\eta(i)$ is distinct for each $i = 0, \dots, 3c-2$. Thus $x_{\eta(0)} = x_2$ only appears once in β_K so the set $\{\beta_K \alpha_K, \beta_K, x_1, x_3, \dots, x_N\}$ is a free basis of $K[X, X]$. Therefore, K is a free factor of $K[X, X]$. □

Lemma A.9 *If J is $P(-5, 2q+1, 2r+1)$ with $q \equiv 2 \bmod 3$ and $q \geq 8$, then K is a free factor of $K[X, X]$.*

Proof Let c be the integer such that $q = 3c + 2$. $X/K[X, X]$ has a presentation matrix

$$\begin{pmatrix} 3 & -(r+1) \\ 0 & w \end{pmatrix},$$

where $w = cr - 2c - 2$ and $N = 3w$.

We have the set of coset representatives

$$\mathcal{C} = \{1, b, b^2, \dots, b^{w-1}, a, ab, \dots, ab^{w-1}, a^2, a^2b, \dots, a^2b^{w-1}\}.$$

We apply Reidemeister–Schreier to find a free basis of $K[X, X]$.

Since $q \geq 6$ we have $c \geq 2$, and $r \geq 8$ so $r + 1 < w$:

$$w = cr - 2c - 2 = (c-2)(r-2) + (r-7) + r + 1 > r + 1.$$

Thus, the coset representative, $\overline{a^3}$ is b^{r+1}.

We apply Reidemeister–Schreier to find a basis $\{x_1, \dots, x_w, y_1, \dots, y_w, z_0, \dots, z_w\}$, where

$$x_i = \begin{cases} b^i a b^{-i} a^{-1} & \text{if } 1 \leq i \leq w-1, \\ b^w & \text{if } i = w, \end{cases}$$

$$y_i = \begin{cases} a b^i a b^{-i} a^{-2} & \text{if } 1 \leq i \leq w-1, \\ a b^w a^{-1} & \text{if } i = w, \end{cases}$$

$$z_i = \begin{cases} a^2 b^i a b^{-i-r-1} & \text{if } 0 \leq i < w-r-1, \\ a^2 b^i a b^{w-i-r-1} & \text{if } w-r-1 \leq i < w, \\ a^2 b^w a^{-2} & \text{if } i = w. \end{cases}$$

Using this basis,

$$\beta_K \alpha_K = z_0^{-1}$$

and

$$\beta_K = z_0^{-1} y_1^{-1} x_2^{-1} \prod_{i=1}^{c-1} (z_{\delta(i)}^{-1} y_{\delta(i)+1}^{-1} x_{\delta(i)+2}^{-1}) z_{w-2}^{-1} y_{w-1}^{-1} y_w,$$

where

$$\delta(i) = w - i(r-2).$$

Since $w = c(r-2) - 2$,

$$\delta(i) - 1 = w - i(r-2) - 1 = (r-2)(c-i) - 3,$$

so if $\delta(i) = 1$ then $(r-2)(c-i) = 3$. However, $r-2$ does not divide 3 since $r \geq 7$, so $\delta(i)$ is never 1, so y_1 only appears once in β_K.

Therefore, the set $\{\beta_K \alpha_K, \beta_K, x_1, \dots, x_w, y_2, \dots, y_w, z_1, \dots, z_w\}$ is a free basis of $K[X, X]$. Since $\{\beta_K \alpha_K, \beta_K\}$ is a free basis of K, K is a free factor of $K[X, X]$. □

Appendix B Chart of results

Table 1 summarizes the results we've found for the pretzel knots $P(-3, Q, R)$ and $P(-5, Q, R)$ where $Q = 2q + 1$ and $R = 2r + 1$. The shapes around the cells in each chart indicate whether or not the knot's standard Seifert surface S satisfies the

values of Q \ values of R	3	5	7	9	11	13	15
3	(−2)	(−2)	(−2)	(−2)	(−2)	(−2)	(−2)
5		[−1]	0	[1]	[2]	(3)	(4)
7			[2]	(4)	(6)	(8)	(10)
9				(7)	(10)	(13)	(16)
11					(14)	(18)	(22)

values of Q \ values of R	3	5	7	9	11	13	15	17	19	21	23	25	27
3	(−5)	(−6)	(−7)	(−8)	(−9)	(−10)	(−11)	(−12)	(−13)	(−14)	(−15)	(−16)	(−17)
5		(−6)	(−6)	(−6)	(−6)	(−6)	(−6)	(−6)	(−6)	(−6)	(−6)	(−6)	(−6)
7			(−5)	(−4)	[−3]	[−2]	[−1]	0	[1]	[2]	[3]	[4]	(5)
9				[−2]	0	[2]	[4]	[6]	(8)	(10)	(12)	(14)	(16)
11					[3]	[6]	(9)	(12)	(15)	(18)	(21)	(24)	(27)
13						(10)	(14)	(18)	(22)	(26)	(30)	(34)	(38)
15							(19)	(24)	(29)	(34)	(39)	(44)	(49)

Table 1: The results for some $P(-3, Q, R)$ (top) and $P(-5, Q, R)$ (bottom) pretzel knots where $Q = 2q + 1$ and $R = 2r + 1$. The integer in each cell is the value of N. Each cell is in a circle if the knot's standard Seifert surface satisfies the free factor property, and in a square if the knot's standard Seifert surface does not satisfy the free factor property.

free factor property. Cells of knots with trivial Alexander polynomial have no shapes. The integer in each cell is the value of $N = \det(S_+) = \det(S_-)$, which is also the leading coefficient of the Alexander polynomial. If a pretzel knot's cell is contained in a circle and N is a prime power, then the knot group has residually torsion-free nilpotent commutator subgroup. If in addition $N < 0$, then the knot group is biorderable.

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