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Flat Families of Point Schemes for Connected Graded Algebras

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Abstract. We study truncated point schemes of connected graded algebras as families over the parameter space of varying relations for the algebras, proving that the families are flat over the open dense locus where the point schemes achieve the expected (i.e., minimal) dimension.

When the truncated point scheme is zero-dimensional, we obtain its number of points counted with multiplicity via a Chow ring computation. This latter application in particular confirms a conjecture of Brazfield that a generic two-generator two-relation algebra has seventeen truncated point modules of length six.

Introduction

The context for the present note is that of noncommutative projective algebraic geometry, in the sense of studying graded algebras and modules as (analogues of) homogeneous coordinate rings, as exemplified, for instance, by the seminal paper [AS87]. The follow-up work of [ATvdB90; ATvdB91] introduced novel methods of handling the difficulties inherent in working with noncommutative rings by leveraging classical (as opposed to noncommutative) algebraic geometry to probe the nature of the "noncommutative projective schemes" embodied by the rings in question. We recall the relevant setup briefly.

To fix ideas and notation, consider an algebraically closed field $\Bbbk$, an r-dimensional vector space V whose dual is spanned by basis elements x_i, $1 \le i \le r$, and s multilinear (of degree at least two) forms f_j on V. The typical algebra we consider is of the form

$$A = \frac{T(V^*)}{I} = \frac{\Bbbk\langle x_1, \dots, x_r\rangle}{(f_1, \dots, f_s)}$$

(regarding the degree-one generators as elements of the dual V^* is simply a matter of convention).

A *point module* of A is a graded A-module that is cyclic and has Hilbert series $(1-t)^{-1}$. If A is commutative, then these correspond to the closed points of the projective scheme $\operatorname{Proj} A$, justifying the nomenclature. One of the innovations of [ATvdB90] was introducing a scheme Γ whose closed points parameterize the isomorphism classes of point modules over A; this is the so-called *point scheme* of A. The scheme Γ is the inverse limit of the *truncated point schemes* $\{\Gamma_n\}_n$ defined as follows:

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Regard the relations f_j with degrees d_j as elements of the respective tensor powers $(V^*)^{\otimes d_j}$. For every $n \geq 2$, we define

$$\Gamma_n \subseteq \mathbb{P}(V)^n \cong (\mathbb{P}^{r-1})^n$$

to be the zero scheme of the degree-n component I_n of the ideal I generated by the f_j.

The closed points of Γ_n parameterize the isomorphism classes of *truncated point modules* of length $n+1$, defined as cyclic graded A-modules with Hilbert series $1+t+t^2+\cdots+t^n$. If the number n is larger than or equal to the highest degree of the defining relations $f_1,\ldots,f_s$, then Γ_n determines all truncated point schemes with indices larger than n and hence the point scheme Γ.

Truncated point schemes play an important role in the study of three-dimensional AS-regular algebras in [ATvdB90]. A three-dimensional AS-regular algebra A is either of quadratic or cubic type. If A is of quadratic type, then it has three (degree-one) generators and three degree-two relations, as a polynomial ring with three variables does. The inverse system of its truncated point schemes satisfies $\cdots \xrightarrow{\sim} \Gamma_3 \xrightarrow{\sim} \Gamma_2$, and hence the point scheme is canonically isomorphic to Γ_2. Moreover, via the first (or the second) projection π of $\Gamma_2 \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ to $\mathbb{P}^2$, Γ_2 is realized as a graph of an automorphism σ of the scheme-theoretic image $E := \pi(\Gamma_2)$ in $\mathbb{P}^2$. The image E is either a degree-3 divisor or the entire $\mathbb{P}^2$. If A is of cubic type, then it has two generators and two degree-3 relations. It has similar properties, but the truncated point schemes are stable from Γ_3, and it is realized as a graph of an automorphism σ of a bidegree $(2,2)$-divisor or the entire $\mathbb{P}^1 \times \mathbb{P}^1$. A crucial observation made in [ATvdB90] was that every three-dimensional AS-regular algebra can be recovered, up to isomorphism, from a triple consisting of its point scheme Γ (or E), an automorphism of Γ (which is the above σ), and a line bundle on Γ.

We note in passing that in this regard three-dimensional AS-regular algebras behave very differently from four(or higher)-dimensional ones. As observed in [van88] (see also [VvRW98]), generic graded Clifford algebras of global dimension 4, which are 4-generator 6-relation quadratic AS-regular algebras, have 20 point modules. Specific examples of four-dimensional AS-regular algebras with finitely many points appear in numerous sources (e.g., [van88; VvRW98; SV99; SV06; SV07; CS17; CPS19]).

In the present note, we study the behavior of the truncated point schemes Γ_n upon varying the set of relations $\{f_j\}_j$ or the relation space

$$\operatorname{span}\{f_j\}_j \tag{0.1}$$

while keeping the degrees d_j of the f_j fixed. In other words, we regard $\{f_j\}_j$ as a point in the relevant product

$$\mathbb{G} = \prod_{j=1}^{s} \mathbb{P}((V^*)^{\otimes d_j}) \tag{0.2}$$

of projective spaces and study Γ_n as fibers of a family over the latter scheme. For a comparison between $\mathbb{G}$ and the space parameterizing (0.1), see Remark 1.8.

Theorem 1.3 shows that under appropriate bounds on the degrees d_j, the locus $U \subseteq \mathbb{G}$ over which Γ_n has the expected minimal dimension is open and dense in $\mathbb{G}$. Moreover, according to Theorem 1.6, the resulting family is flat over U. This implies that a suite of algebro-geometric invariants we might compute for Γ_n (e.g., the arithmetic genus) stays constant so long as the dimension of Γ_n is that provided by the naive count.

When Γ_n is zero-dimensional, a simple computation in the Chow ring of $\mathbb{P}(V)^n$ returns the number of points of Γ_n counted with multiplicity. We apply this procedure and our main results to algebras of the same "shape" (i.e., having the same number of generators and degrees of relations) as the four-dimensional Artin–Schelter regular algebras listed in [L+07, Proposition 1.4]. There are three types of such algebras, and in each case, we compute the number of points (counted with multiplicity) of Γ_n for the smallest number n such that $\dim(\Gamma_n) = 0$. This includes via Proposition 1.7 a confirmation in Proposition 2.5 of Brazfield's conjecture:

Conjecture 0.1 ([Bra99, Conjecture IV.8.1]). *Let A be a connected graded algebra with two degree-one generators. If the defining ideal of A is generated by a generic cubic and a generic quartic relation, then Γ_5 consists of exactly seventeen distinct points.*

1. Main Results

Fix positive integers r and s. We consider the following family of algebras associated with an s-tuple:

Definition 1.1. For a tuple

$$\mathbf{d} = (d_1 \leq \cdots \leq d_s)$$

with $d_j \geq 2$, an *algebra of type* $(r, \mathbf{d})$ is a connected graded algebra with r degree-one generators and s relations of degrees $d_1, \ldots, d_s$.

We retain the notations in Introduction and focus on Γ_n for $n \geq d_s$, henceforth referred to as the *stable range* for n. Note that for stable n, the scheme Γ_n is defined as the joint zero locus in $\mathbb{P}(V)^n$ of

$$\sum_{j=1}^{s} (n - d_j + 1) \tag{1.1}$$

multilinear equations whose respective degrees are indicated by the summands of (1.1), $n - d_j + 1$ equations of degree d_j.

Definition 1.2. Given r, $\mathbf{d}$, and n as above, the *defect* $\mathrm{df}(r, \mathbf{d}, n)$ attached to this data is the sum (1.1).

Recall that we denote by $\mathbb{G}$ the space (0.2) of relations for type-$(r, \mathbf{d})$ algebras. It is, in other words, the variety of tuples of homogeneous polynomials f_j of prescribed degrees d_j up to scaling. Now define the *universal truncated point scheme* to be the closed subscheme $\mathbb{X}_n$ of $\mathbb{G} \times \mathbb{P}(V)^n$ given by

$$\mathbb{X}_n = \{((f_j)_j, (a_i)_i) \in \mathbb{G} \times \mathbb{P}(V)^n \mid f_j(a_{i+1}, \dots, a_{i+d_j}) = 0 \\ \text{for } 1 \le j \le s \text{ and } 0 \le i \le n - d_j\}.$$

The fiber $(\mathbb{X}_n)_R$ at $R \in \mathbb{G}$ along the projection $\pi : \mathbb{X}_n \to \mathbb{G}$ is the truncated point scheme Γ_n attached to the type-$(r, \mathbf{d})$ algebra $T(V^*)/(R)$.

Our main results are as follows. First, we have the following observation to the effect that Γ_n has "expected dimension" generically.

THEOREM 1.3. *Fix r, $\mathbf{d}$, and n, and suppose the associated defect* df *is $\le n \times (r-1)$.*

(1) *For each $R \in \mathbb{G}$, $\Gamma_n = (\mathbb{X}_n)_R$ is nonempty, and all components have dimension $\ge n(r-1) - \mathrm{df}$.*
(2) *The locus U of $R \in \mathbb{G}$ where all components of Γ_n have dimension $n(r-1) - \mathrm{df}$ is open and dense.*

Proof. We prove the two claims separately.

(1) As observed above, Γ_n is by definition the scheme-theoretic intersection of df hypersurfaces in the $n(r-1)$-dimensional scheme $\mathbb{P}(V)^n$, so the lower bound $n(r-1) - \mathrm{df}$ for the dimensions of the components is a consequence, for instance, of [Har77, Proposition I.7.1]: that result is stated for subschemes of affine space, but our scheme $\mathbb{P}(V)^n$ admits a cover by open patches isomorphic to $\mathbb{A}^n$.

The nonemptiness follows from the following computation in the Chow ring $A^* = A^*(\mathbb{P}(V)^n)$. According to the Künneth theorem for Chow rings (e.g., [Tot14, Propositions 1 and 2]) A^* is isomorphic to the nth tensor power of $A^*(\mathbb{P}(V))$, which is simply $\mathbb{Z}[\varepsilon]/(\varepsilon^r)$ for the class ε of a hyperplane:

$$A^* \cong \bigotimes_{i=1}^{n} \mathbb{Z}[\varepsilon_i]/(\varepsilon_i^r).$$

Now consider the multilinearizations $f_{i,j}$ of f_j on $\mathbb{P}(V)^n$ with $0 \le i \le n - d_j$; Γ_n is the intersection of the respective zero loci $V_{i,j}$ of $f_{i,j}$, represented in the Chow ring by sums of the form

$$\varepsilon_{i+1} + \varepsilon_{i+2} + \cdots + \varepsilon_{i+d_j}. \tag{1.2}$$

The product of elements (1.2) in the Chow ring will be shown to be nonzero in Lemma 1.4. In turn, this then implies that the intersection of the schemes $V_{i,j}$ represented by (1.2) is nonempty.

To verify this last point, recall, for example, from [Ful98, §8.1] that the product

$$\prod_{j=1}^{s} \prod_{i=0}^{n-d_j} [V_{i,j}]$$

can be obtained as the pushforward through

$$\bigcap_{j=1}^{s}\bigcap_{i=0}^{n-d_j} V_{i,j} \to \mathbb{P}(V)^n$$

of an element in the Chow ring of the left-hand intersection (see especially [Ful98, Example 8.1.9]). If this intersection were trivial, then the element in question would vanish, and hence the conclusion.

(2) Since $\mathbb{G}$ is irreducible, it suffices to prove that U is open and nonempty. We relegate the nonemptiness to Lemma 1.5.

By [GW10, Corollary 14.113] the locus of $R \in \mathbb{G}$ where $\Gamma_n = (\mathbb{X}_n)_R$ has dimension at most $n(r-1) - \mathrm{df}$ is open. It follows from (1) that this locus is U. □

LEMMA 1.4. *In the context of Theorem* 1.3,

$$\prod_{j=1}^{s}\Bigg(\prod_{i=0}^{n-d_j}(\varepsilon_{i+1}+\cdots+\varepsilon_{i+d_j})\Bigg)$$

is a nonzero element of the ring $\bigotimes_{i=1}^{n}\mathbb{Z}[\varepsilon_i]/(\varepsilon_i^r)$.

Proof. We will prove the statement for all $\mathbf{d} = (d_1,\dots,d_s)$ and n with the milder restriction $1 \le d_j \le n$ for all j and the same assumption $\mathrm{df} \le n(r-1)$. As before, we may assume that $d_1 \le \cdots \le d_s$ without loss of generality.

If we append $d_{s+1} = n$ at the end of $\mathbf{d}$, then the defect is increased by one, and the element in question is multiplied by $\varepsilon_1 + \cdots + \varepsilon_n$. By applying this operation as many times as necessary we can assume $\mathrm{df} = n(r-1)$. Then the inequality $r-1 \le s$ follows from

$$n(r-1) = \sum_{j=1}^{s}(n-d_j+1) \le \sum_{j=1}^{s} n = ns.$$

The number of j with $d_j = 1$ is at most $r-1$. Indeed, if $1 = d_1 = \cdots = d_{r-1}$, then

$$n(r-1) = \sum_{j=1}^{s}(n-d_j+1) = n(r-1) + \sum_{j=r}^{s}(n-d_j+1),$$

and $n - d_j + 1 \ge 1$. Hence $s = r-1$ in this case.

Now we complete the proof by induction on n. If $n = 1$, then $d_1 = \cdots = d_s = 1$, and hence $s = r-1$. The element in question is $\varepsilon_1^{r-1} \ne 0$.

Let $n \ge 2$. For two elements $P_1, P_2 \in \bigotimes_{i=1}^{n}\mathbb{Z}[\varepsilon_i]/(\varepsilon_i^r) = \mathbb{Z}[\varepsilon_1,\dots,\varepsilon_n]/(\varepsilon_1^r,\dots,\varepsilon_n^r)$, we write $P_1 \le P_2$ if $P_2 - P_1$ is represented by a polynomial whose coefficients are all nonnegative. Then we have

$$\prod_{j\le r-1}\Bigg(\prod_{i=0}^{n-d_j}(\varepsilon_{i+1}+\cdots+\varepsilon_{i+d_j})\Bigg) \ge \prod_{j\le r-1}\Bigg(\varepsilon_1\prod_{i=1}^{n-d_j}(\varepsilon_{i+1}+\cdots+\varepsilon_{i+d_j})\Bigg) \quad \text{and}$$

$$\prod_{j\ge r}\Big(\prod_{i=0}^{n-d_j}(\varepsilon_{i+1}+\cdots+\varepsilon_{i+d_j})\Big)\ge\prod_{j\ge r}\Big(\prod_{i=0}^{n-d_j}(\varepsilon_{i+2}+\cdots+\varepsilon_{i+d_j})\Big).$$

Therefore

$$\begin{aligned}\prod_{j=1}^{s}&\Big(\prod_{i=0}^{n-d_j}(\varepsilon_{i+1}+\cdots+\varepsilon_{i+d_j})\Big)\\ &\ge\varepsilon_1^{r-1}\prod_{j=1}^{s}\Big(\prod_{i=0}^{(n-1)-d'_j}(\varepsilon_{i+2}+\cdots+\varepsilon_{i+1+d'_j})\Big),\end{aligned}\tag{1.3}$$

where $d'_j=d_j$ for $j\le r-1$ and $d'_j=d_j-1$ for $j\ge r$. The right-hand side of (1.3) is of the form $\varepsilon_1^{r-1}P$, where P is the element in question for the tuple $(d'_1,\dots,d'_s)$ in variables $\varepsilon_2,\dots,\varepsilon_n$. Since the defect for this new tuple is

$$\sum_{j=1}^{s}((n-1)-d'_j+1)=\sum_{j=1}^{s}(n-d_j+1)-(r-1)=(n-1)(r-1),$$

the induction hypothesis implies that P is a nonzero element of $\mathbb{Z}[\varepsilon_2,\dots,\varepsilon_n]/(\varepsilon_2^r,\dots,\varepsilon_n^r)$. Therefore both sides of (1.3) are ≥ 0 and nonzero. This completes the proof. $\square$

LEMMA 1.5. *In the context of Theorem* 1.3, *there are choices of relations* f_j, $1\le j\le s$, *for which all components of* Γ_n *achieve the lower dimension bound of* $n(r-1)-\mathrm{df}$.

Proof. Simply select the forms f_j to be of the form

$$f_j=\prod_{i=1}^{d_j}\ell_{i,j}$$

for linear forms $\ell_{i,j}$ on $\mathbb{P}(V)$, chosen so that the zero locus of any r is empty (i.e., the zero loci $Z(\ell_{i,j})$ are in general position in $\mathbb{P}(V)$).

The components of the joint zero locus of the multilinearizations of the f_j are obtained by imposing df linear constraints on the n coordinates of points in $\mathbb{P}(V)^n$, and the fact that such components have the requisite dimension $n(r-1)-\mathrm{df}$ follows from the generic choice of $\ell_{i,j}$. $\square$

Additionally, the following result ensures that when Γ_n has the expected size, various invariants such as multidegrees as subschemes of products of projective spaces, genus, and so on remain constant. The result is analogous to [CS, Theorem 4.4], and its proof is similarly based on [Eis95, Theorem 18.16].

THEOREM 1.6. *The restriction of the family* $\mathbb{X}_n\to\mathbb{G}$ *to the open dense subscheme* $U\subseteq\mathbb{G}$ *from Theorem* 1.3 *is flat.*

Proof. Denote $\mathbb{X}_n$ by $\mathbb{X}$; we indicate restriction of families by subscripts, as in $\mathbb{X}_U$ for the restriction of $\pi : \mathbb{X} \to \mathbb{G}$ to $U \subseteq \mathbb{G}$. We will apply the miracle flatness theorem [Eis95, Theorem 18.16 (b)] (also called the local criterion for flatness) to the following setup.

Let $x \in \mathbb{X}_U$. Then the theorem in question applies to the local rings

$$(S, P) = (\mathcal{O}_{U,\pi(x)}, \mathfrak{m}_{\pi(x)}) \to (\mathcal{O}_{\mathbb{X},x}, \mathfrak{m}_x) = (A, Q).$$

For this, we need

- S to be regular; this is the case since S is the local ring of a point on a product of projective spaces.
- A to be Cohen–Macaulay; this follows from the fact that A is a complete intersection. Indeed, applying [GW10, Proposition 14.107 (1)] to the morphism $\mathbb{X}_U \to U$, we have

  $$\dim \mathbb{X}_U \leq \dim U + n(r-1) - \mathrm{df} = \dim(\mathbb{G} \times \mathbb{P}(V)^n) - \mathrm{df}.$$

  Since $\mathbb{X}_U$ is defined by df relations in the regular scheme $\mathbb{G} \times \mathbb{P}(V)^n$, the local ring $\mathcal{O}_{\mathbb{X},x}$ is a complete intersection.
- The dimension of the fiber A/PA equals the relative dimension $\dim(A) - \dim(S)$; this is simply a paraphrase of the fact that we are restricting to the locus U where π has fibers of the lowest possible expected dimension, that is, $n(r-1) - \mathrm{df}(r, \mathbf{d}, n)$.

This completes the proof. □

The following result will come in handy below, when we examine some examples.

Proposition 1.7. *In the setting of Theorem* 1.6, *suppose furthermore that* $n(r-1) = \mathrm{df}$. *Then the set* $W \subseteq U$ *over which* Γ_n *is reduced is open and dense.*

Proof. The irreducibility of U means that it is sufficient to prove that the set in question is open and nonempty.

Under the present hypotheses, at each point in U the scheme Γ_n is finite, that is, consists of several points, some, perhaps, with multiplicity. The flatness result in Theorem 1.6 ensures that the length $|\Gamma_n|$ is constant throughout U, counting multiplicity; we denote this common number by ℓ.

By the functorial description of the Hilbert scheme of points (e.g., [Har66, p. 15], [Ber12, Definition 2.1], or [TP17, Tag 0B94]), the flat family $\mathbb{X}_U \to U$ entails a map $\phi : U \to \mathrm{Hilb}^{\ell}_{\mathbb{P}(V)^n}$.

The Hilbert scheme contains an open subscheme Hilb° consisting of ℓ-tuples of *distinct* points in $\mathbb{P}(V)^n$ (see [Ber12, Proposition 2.4] and the remarks following it). In conclusion, the openness and nonemptiness of W will follow once we argue that $\phi(U)$ intersects Hilb°, that is, Γ_n is reduced for at least one point of U.

To verify this last claim, note that Γ_n will indeed be reduced for a generic choice of linear forms in the construction used in the proof of Lemma 1.5. □

REMARK 1.8. So far we have been using $\mathbb{G}$, the space of sets of relations, as a parameterizing scheme, but we can also use

$$\mathbb{H} = \prod_{d=2}^{m} \mathrm{Gr}(r_d, (V^*)^{\otimes d}),$$

which, instead, parameterizes linear spans of relations. Here m is the maximum of the degrees of relations, each r_d is the number of relations of degree d, and $\mathrm{Gr}(r_d, (V^*)^{\otimes d})$ denotes the Grassmannian of r_d-dimensional subspaces (of relations) in $(V^*)^{\otimes d}$. However, this makes no essential difference to our main results such as Theorem 1.3 and Proposition 1.7 nor counting points of various truncated point schemes in the next section.

To see this, write

$$\mathbb{G} = \prod_{d=2}^{m} \mathbb{P}((V^*)^{\otimes d})^{r_d}$$

and consider $D = \prod_d D_d$ where D_d is the open dense subset of $\mathbb{P}((V^*)^{\otimes d})^{r_d}$ consisting of tuples that span r_d-dimensional subspaces of $(V^*)^{\otimes d}$. Then there is a canonical surjective morphism $D \to \mathbb{H}$ that sends a set of relations (up to scalar) to its linear span in each degree. Since our earlier results state some properties of Γ_n on an open dense subset of $\mathbb{G}$, it suffices to show that the morphism $D \to \mathbb{H}$ is open. The miracle flatness theorem we used in the proof of Theorem 1.6 implies that it is flat, because the fiber of $W \in \mathrm{Gr}(r_d, (V^*)^{\otimes d})$ under the morphism $D_d \to \mathrm{Gr}(r_d, (V^*)^{\otimes d})$ is $\mathbb{P}(W)^{r_d} \cap D_d$. According to [Har77, Exercise III.9.1], it follows that the morphism $D \to \mathbb{H}$ is open.

REMARK 1.9. Applying Theorem 1.3(2) to $r = 3$, $s = 3$, $d_1 = d_2 = d_3 = 2$, and $n = 3$, we deduce that an algebra with three generators and three quadratic relations has zero-dimensional Γ_3 if the set of relations is taken generically. A three-dimensional AS-regular algebra of quadratic type has the same generator-relation pattern, but its relations are *not* generic because Γ_3 is at least one-dimensional as mentioned in Introduction. Similarly, the set of relations for a three-dimensional AS-regular algebra of cubic type is not generic, either.

On the other hand, [ATvdB90, Theorem 1] claims that the three-dimensional AS-regular algebras are exactly the non-degenerate standard algebras, which in particular means that 3-dimensional AS-regular algebras are generic among the *standard* algebras. Since being standard is not an open condition, this does not contradict the previous paragraph.

2. Examples and Connections to Prior Work

The preceding material ties in with a number of results of similar flavor in the literature, as we now document.

We will focus on algebras with the same generator-relation pattern as the four-dimensional AS-regular ones classified in [L+07, Proposition 1.4]:

- four generators and six relations of degree 2;

- three generators, two degree-two relations, and two degree-three relations;
- two generators and one relation in each degree 3 and 4.

Under the regularity assumptions of [L+07], the Betti numbers of these types of algebras are, respectively,

- 1, 4, 6, 4, 1;
- 1, 3, 4, 3, 1;
- 1, 2, 2, 2, 1.

When applying the contents of Section 1, the relevant critical dimension $n(r-1) - \mathrm{df}(r, \mathbf{d}, n)$ becomes zero for certain n, that is, the Γ_n in question will be nonempty finite (perhaps nonreduced) schemes.

2.1. Four Generators, Six Quadratic Relations (Type 14641)

In this case the results of Section 1 essentially recapture the main result of [van88] to the effect that generically, such algebras have 20 point modules, counted with multiplicity.

In the absence of regularity conditions the scheme Γ_2 will be our stand-in for the scheme of point modules, and hence the n to which Section 1 applies here is 2.

We thus have $r = 4$ and $s = 6$, and the vector space V of the above discussion is dual to the span V^* of linearly independent generators $x_1, \ldots, x_4$. The scheme $\mathbb{G}$ is $\mathbb{P}(V^* \otimes V^*)^6$, all d_j are equal to 2, and the defect is 6.

We then have the following:

Proposition 2.1. *Under the conventions of the present subsection, the scheme Γ_2 is nonempty, and the locus $U \subseteq \mathbb{G}$ where Γ_2 is zero-dimensional is open and dense.*

For relation spaces $R \in U$, Γ_2 consists of twenty points, counted with multiplicity. These points are distinct for $R \in W$ as in Proposition 1.7.

Proof. Everything but the claim about the count of 20 is an immediate application of Theorems 1.3 and 1.6 and Proposition 1.7.

As for the count itself, it follows from the fact that examples with $|\Gamma_2| = 20$ exist, as first constructed in [van88] (see also [SV07; CV15; CS17] and references therein) together with flatness; the latter ensures the constancy of the degree throughout the open parameter family U.

Alternatively, we can avoid having to handle any examples at all by resorting to a Chow ring-based argument: $A^*(\mathbb{P}(V)^2)$ is in this case isomorphic to

$$\mathbb{Z}[\varepsilon_1]/(\varepsilon_1^4) \otimes \mathbb{Z}[\varepsilon_2]/(\varepsilon_2^4),$$

and each bilinearization of a relation cuts out a hypersurface V_i, $1 \le i \le 6$ of class $\varepsilon_1 + \varepsilon_2$. Since $\varepsilon_i^4 = 0$, this then implies that the product of the Chow classes $[V_i]$ is

$$(\varepsilon_1 + \varepsilon_2)^6 = 20\varepsilon_1^3\varepsilon_2^3. \tag{2.1}$$

On the other hand, [Ful98, Example 8.2.1] implies that product (2.1) is the Chow class of the scheme-theoretic intersection $\bigcap_i V_i$. The assumption of the mentioned result is ensured by [Ful98, Example 8.2.7] and the fact that each V_i is Cohen–Macaulay. Since $\varepsilon_1^3\varepsilon_2^3$ is the Chow class of a point, this means that the said intersection consists of 20 points with multiplicity. □

REMARK 2.2. It is the second proof of $|\Gamma_2| = 20$ given above that would presumably be more portable and flexible, as it is available even when Γ_n is not zero-dimensional. We will treat such a case in Proposition 2.6.

REMARK 2.3. The number n such that the critical dimension becomes zero is equal to $\ell - 2$, where ℓ is the Gorenstein parameter of a four-dimensional AS-regular algebra of the same generator-relation pattern. Indeed, in the proof of [L+07, Proposition 1.4], it is observed that the AS-regular algebras considered there have Hilbert series $1/p(t)$, where $p(t)$ has a zero at $t = 1$ with multiplicity ≥ 3. In our terminology, $p(1) = 0$ implies that $s = 2r - 2$ and $p'(1) = 0$ implies that the sum of d_j is $(r - 1)\ell$. Thus the defect is

$$\sum_{j=1}^{s}(n - d_j + 1) = (2n + 2 - \ell)(r - 1),$$

which is equal to $n(r - 1)$ if and only if $n = \ell - 2$.

2.2. *Three Generators, Quadratic and Cubic Relations (Type 13431)*

We now tackle the second bullet point listed at the beginning of the present section, corresponding to three-generator algebras with two quadratic and two cubic relations. We will then study Γ_3 (i.e., here $n = 3$).

PROPOSITION 2.4. *Under the conventions of the present subsection, the scheme Γ_3 is nonempty, and the locus $U \subseteq \mathbb{G}$ where Γ_3 is zero-dimensional is open and dense.*

For relation spaces $R \in U$, Γ_3 consists of 19 points, counted with multiplicity. These points are distinct for $R \in W$ as in Proposition 1.7.

Proof. The proof is entirely parallel to that of Proposition 2.1, only the count requiring modification.

This time the relevant Chow ring is

$$A^*(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \cong \bigotimes_{i=1}^{3} \mathbb{Z}[\varepsilon_i]/(\varepsilon_i^3),$$

and the class of Γ_3 is the coefficient of $\varepsilon_1^2\varepsilon_2^2\varepsilon_3^2$ in

$$(\varepsilon_1 + \varepsilon_2)^2(\varepsilon_2 + \varepsilon_3)^2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2.$$

This is easily seen to be 19 by direct computation. □

2.3. Two Generators, Cubic and Quartic Relations (Type 12221)

This case in fact motivated the present note and corresponds to the third bullet point of the discussion at the start of the present section.

This investigation is a follow-up to [CKS19] (in turn, inspired by [L+07]) and was prompted by our learning belatedly of the thesis [Bra99], where some of the algebras of interest here are studied. Specifically, the following result resolves [Bra99, Conjecture IV.8.1] (i.e., Conjecture 0.1) in the affirmative.

PROPOSITION 2.5. *Under the conventions of the present subsection, the scheme Γ_5 is nonempty, and the locus $U \subseteq \mathbb{G}$ where Γ_5 is zero-dimensional is open and dense.*

For relation spaces $R \in U$, Γ_5 consists of 17 points, counted with multiplicity. The points are distinct for $R \in W$ as in Proposition 1.7.

Proof. Once more, the argument is precisely parallel to those of Propositions 2.1 and 2.4, except for inessential numerical differences in the last portion of the proof.

The Chow ring to consider here is

$$A^*((\mathbb{P}^1)^5) \cong \bigotimes_{i=1}^{5} \mathbb{Z}[\varepsilon_i]/(\varepsilon_i^2),$$

and the sought-after degree is the coefficient of $\prod_i \varepsilon_i$ in

$$(\varepsilon_1+\varepsilon_2+\varepsilon_3)(\varepsilon_2+\varepsilon_3+\varepsilon_4)(\varepsilon_3+\varepsilon_4+\varepsilon_5)(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5);$$

this is indeed 17.

Alternatively, we can repeat the example-based argument at the end of the proof of Proposition 2.1: the family $\mathbb{X}_U \to U$ is flat, and we know that its fiber has degree 17 for at least one point in U via the examples in [Bra99, Chapter V]. Flatness then ensures that the degree is 17 throughout U. □

For the type of algebras of this subsection, we can apply our general result also to $n = 4$. In this case the expected dimension of Γ_4 is one.

Let Y be a closed subscheme of a product $\mathbb{P} = (\mathbb{P}^{r-1})^n$ of projective spaces such that all irreducible components of Y have the same dimension, say d. Let $(b_1, \dots, b_n)$ be a tuple of nonnegative integers whose sum is d. Recall (e.g., [C+20, §2.1]) that the *multidegree* of Y of type $(b_1, \dots, b_n)$ is the number of points (with multiplicities) in

$$Y \cap (L_1 \times \cdots \times L_n)$$

for a generic choice of linear subspaces $\{L_i\}_i$ of $\mathbb{P}^{r-1}$ such that $\dim L_i = r - 1 - b_i$.

When $d = 1$, we can express the multidegrees of Y simply as a sequence of n nonnegative integers

$$|Y \cap H_i|, \quad 1 \le i \le n,$$

where

$$H_i = (\mathbb{P}^{r-1})^{\times(i-1)} \times Z(\ell_i) \times (\mathbb{P}^{r-1})^{\times(n-i)}$$

for generic linear forms ℓ_i.

PROPOSITION 2.6. *Under the conventions of the present subsection, the scheme* Γ_4 *is nonempty, and the locus* $U \subseteq \mathbb{G}$ *where* Γ_4 *is one-dimensional is open and dense.*

For relation spaces $R \in U$, Γ_4 *has multidegrees* $(4, 3, 3, 4)$.

Proof. The proof is similar to that of Proposition 2.1, but now we consider the Chow ring

$$A^*((\mathbb{P}^1)^4) \cong \bigotimes_{i=1}^{4} \mathbb{Z}[\varepsilon_i]/(\varepsilon_i^2)$$

and compute the product

$$(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)(\varepsilon_2 + \varepsilon_3 + \varepsilon_4)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4).$$

The result is

$$4\varepsilon_1\varepsilon_2\varepsilon_3 + 3\varepsilon_1\varepsilon_2\varepsilon_4 + 3\varepsilon_1\varepsilon_3\varepsilon_4 + 4\varepsilon_2\varepsilon_3\varepsilon_4.$$

The same argument as the latter part of the proof of Proposition 2.1 implies that this is the Chow class of Γ_4.

Finally, the last statement follows from the fact that, as explained in [C+20, Remark 2.8], the multidegrees can be read off as the tuple of coefficients of the Chow class. □

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