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Elliptic R-matrices and Feigin and Odesskii's elliptic algebras

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Abstract

The algebras $Q_{n,k}(E, \tau)$ introduced by Feigin and Odesskii as generalizations of the 4-dimensional Sklyanin algebras form a family of quadratic algebras parametrized by coprime integers $n > k \geq 1$, a complex elliptic curve E, and a point $\tau \in E$. The main result in this paper is that $Q_{n,k}(E, \tau)$ has the same Hilbert series as the polynomial ring on n variables when τ is not a torsion point. We also show that $Q_{n,k}(E, \tau)$ is a Koszul algebra, hence of global dimension n when τ is not a torsion point, and, for all but countably many τ, $Q_{n,k}(E, \tau)$ is Artin–Schelter regular. The proofs use the fact that the space of quadratic relations defining $Q_{n,k}(E, \tau)$ is the image of an operator $R_\tau(\tau)$ that belongs to a family of operators $R_\tau(z) : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$, $z \in \mathbb{C}$, that (we will show) satisfy the quantum Yang–Baxter equation with spectral parameter.

Keywords Elliptic algebra · Quantum Yang–Baxter equation · Sklyanin algebra · Koszul algebra · Artin–Schelter regular algebra

Mathematics Subject Classification Primary 14A22; Secondary 16S38 · 16W50 · 17B37 · 14H52

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1 Introduction

Until the end of Proposition 1.3, E is a complex elliptic curve, τ is a point on E, $n > k \geq 1$ are relatively prime integers, and $Q_{n,k}(E,\tau)$ denotes the elliptic algebra that was defined by Feigin and Odesskii in their 1989 papers [25] and [37]. This and our earlier papers [16–18] are the first steps in a project to develop a "geometric representation theory" for the $Q_{n,k}(E,\tau)$'s. This paper establishes some fundamental algebraic properties of $Q_{n,k}(E,\tau)$ (Theorems 1.1 and 1.2 and Proposition 1.3 below).

The algebras $Q_{n,1}(E,\tau)$ are called Sklyanin algebras in honor of Sklyanin's discovery of $Q_{4,1}(E,\tau)$.

It is often useful to think of the $Q_{n,k}(E,\tau)$'s as generalizations of enveloping algebras, $U(\mathfrak{g})$, of finite dimensional semisimple Lie algebras and their quantizations $U_q(\mathfrak{g})$.

1.1 Geometric representation theory and elliptic algebras

Geometric representation theory is one of the major mathematical developments of the past half-century. It emerged in the context of Lie theory but its development has required and stimulated connections and tools that play a role in many other areas. Remarkably, the algebraic varieties that appear in "classical" geometric representation theory[1] over $\mathbb{C}$ are almost always *rational* varieties.[2] (The only exceptions we know are some Hessenberg varieties: see [21, Rmk. 4.2].) We do not know any meta-theorem that explains this phenomenon, but its first manifestation is Chevalley's theorem proving that every connected linear algebraic group over an algebraically closed field of characteristic zero is rational [12]. Over the past 35 years evidence has accumulated that there should be a "geometric representation theory" for elliptic algebras in which the relevant geometric objects are no longer rational varieties but elliptic curves, powers of elliptic curves, symmetric powers of elliptic curves, higher secants and secant varieties to elliptic curves, and mixtures of such things. One reason this might not be surprising is that many rational affine varieties can be usefully thought of as degenerations of such non-rational varieties. Likewise some of the algebras that appear in the context of quantum groups are degenerations of elliptic algebras, and some representations of quantum groups are degenerations of representations of elliptic algebras.[3] See, for example, Cherednik's papers [13–15] and, more recently, [20] which shows that some of the representation theory of $U_q(\mathfrak{sl}_2)$ is a "degenerate" version of the representation theory of $Q_{4,1}(E,\tau)$. We note, too, the similarity between the results

[1] By "classical" we mean the representation theory of the enveloping algebra $U(\mathfrak{g})$ and its quantization $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a finite dimensional Lie algebra over $\mathbb{C}$.

[2] An irreducible algebraic variety X over an algebraically closed field $\Bbbk$ is rational if $\Bbbk(X)$, its field of rational functions, is a purely transcendental extension $\Bbbk(x_1,\ldots,x_n)$ of $\Bbbk$; in more geometric terms, there is a non-empty open set $U \subseteq X$ and an open set $V \subseteq \mathbb{A}^n_{x_1,\ldots,x_n}$ such that $U \cong V$.

[3] This is related to the fact that Belavin's elliptic solutions to the quantum Yang–Baxter equation with spectral parameter degenerate to trigonometric and rational solutions.

about finite dimensional representations of $Q_{4,1}(E,\tau)$ in [49, 52] and [57, 58] and results about the representation theory of $U_q(\mathfrak{sl}_2)$.[4]

One way in which the $Q_{n,k}(E,\tau)$'s differ from enveloping algebras is that no information can be gleaned from induced representations of subalgebras; $Q_{n,k}(E,\tau)$ does not appear to have any useful subalgebras that one can induce from; this seems to be related to the fact that $Q_{n,k}(E,\tau)$ does not possess anything like a PBW basis (but see [55]); indeed, one can hardly do any hand calculations in $Q_{n,k}(E,\tau)$. Nevertheless, there is a rich theory of "linear modules" for elliptic algebras and these are good replacements for induced modules of the form $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda$ (where $\mathfrak{p}$ is a Lie subalgebra of a Lie algebra of $\mathfrak{g}$ and $\mathbb{C}_\lambda$ is a 1-dimensional representation of $\mathfrak{p}$). Evidence supporting the claims in the previous sentence can be found in [31, Thm. 2.2], [55, Thm. 1.4], [53, §5] and [20, §1B]. Feigin and Odesskii were the first to recognize that linear modules for $Q_{n,k}(E,\tau)$, and the homomorphisms between them, are related to the higher secants to E embedded as a degree n normal curve in $\mathbb{P}^{n-1}$.

The simplest linear modules are the point modules: a point module is a cyclic graded left $Q_{n,k}(E,\tau)$-module having Hilbert series $(1-t)^{-1}$. Point modules played a central role in [2, 3]. When $k=1$, the isomorphism classes of point modules are parametrized by E for generic τ, except when $n=4$ in which case there are four additional points. The survey article [49] describes the beautiful interaction between linear modules for $Q_{4,1}(E,\tau)$ and the geometry associated to E embedded as a quartic normal curve in $\mathbb{P}^3$.

For all k, Feigin and Odesskii showed there is a certain variety, $X_{n/k} \subseteq \mathbb{P}^{n-1}$, called the characteristic variety (see [16] for its definition), that parametrizes an important subset of the point modules (in many cases this might be all the point modules but we don't know this yet). In [16], it is shown that $X_{n/k} \cong E^g/\Sigma_{n/k}$, the quotient of a certain power E^g by the action of a subgroup of the symmetric group of order $(g+1)!$. In [18] it is shown that for some (n,k) there is a fully faithful embedding of $\mathrm{Qcoh}(X_{n/k})$ into a certain quotient category of graded $Q_{n.k}(E,\tau)$-modules. This is strikingly different from what happens for $U(\mathfrak{g})$ or $U_q(\mathfrak{g})$.

Despite the differences some of the themes in the representation theory of $U(\mathfrak{g})$ appear in the context of the $Q_{n,k}(E,\tau)$'s: for example, Van den Bergh's remarkable paper [56] establishes a "translation principle" for $Q_{4,1}(E,\tau)$, which is expressed in terms of an equivalence between certain categories of representations having "different central characters". It seems likely that there will be translation principles for other elliptic algebras. For example, when n is even Feigin and Odesskii [37, §3, Rmk. 2] surmise that $Q_{n,1}(E,\tau)$ has two linearly independent central elements of degree $\frac{1}{2}n$ that "correspond" to the Poisson central elements that appear in [40, Thm. A] and there might be a translation principle relating certain categories of modules that are annihilated by different linear combinations of those central elements.

Since $Q_{n,k}(E,0)=\mathbb{C}[x_0,\ldots,x_{n-1}]$ is a polynomial ring on n variables, there is a Poisson structure $\{x_i,x_j\} := \lim_{\tau\to 0}\frac{[x_i,x_j]}{\tau}$ on $Q_{n,k}(E,0)$ and, because $\deg\{x_i,x_j\}=2$, this induces a Poisson structure on $\mathbb{P}^{n-1}=\mathrm{Proj}(Q_{n,k}(E,0))$, which is commonly denoted $q_{n,k}$ and called the Feigin–Odesskii bracket [25, 37]. It is conjectured that $q_{n,k}$

4 Odesskii and Feigin [38] examine finite dimensional representations of $Q_{n,k}(E,\tau)$ when τ has finite order.

coincides with a "natural" Poisson structure discovered in 1998 by Feigin–Odesskii [26] and Polishchuk [39], on $\mathbb{P}(\mathrm{Ext}^1_E(\mathcal{V}_{n,k}, \mathcal{O}_E))$ where $\mathcal{V}_{n,k}$ is a stable bundle on E of rank k and degree n. Hua and Polishchuk [27, Thm. 5.2] showed that the conjecture holds when $k = 1$.[5] The symplectic leaves for $q_{n,k}$ are known only when $k = 1$ [19]. The symplectic leaves for $q_{n,1}$ can be described in terms of the higher secant varieties to E embedded in $\mathbb{P}^{n-1}$ as a degree-n elliptic normal curve: see [19, §1] for a precise description.[6,7] Since linear modules for $Q_{n,k}(E, \tau)$ seem to be analogous to Verma modules, and linear modules for $Q_{n,1}(E, \tau)$ (for the most part) correspond to higher-dimensional secants to E, the fact that the symplectic leaves for $q_{n,1}$ are related to the secants suggests, again, that the relation between the representation theory of $Q_{n,k}(E, \tau)$ and the geometry of $E \subseteq \mathbb{P}^{n-1}$ is analogous to the relation between the representation theory of $U(\mathfrak{g})$ and the geometry associated to the action of the adjoint group on $\mathfrak{g}^*$.

When $k > 1$, very little is known about $q_{n,k}$. It is possible that the symplectic leaves for $q_{n,k}$ are related to the geometry of the higher secant varieties for the characteristic variety, $X_{n/k}$, embedded in $\mathbb{P}^{n-1} = \mathbb{P}(\mathrm{Ext}^1_E(\mathcal{V}_{n,k}, \mathcal{O}_E))$. When $\mathbb{P}(\mathrm{Ext}^1_E(\mathcal{V}_{n,k}, \mathcal{O}_E))$ is interpreted as a moduli space for certain stable bundles on E of degree n and rank $k+1$ it has a "natural" Poisson structure [39], which is expected to coincide with $q_{n,k}$. For more about $q_{n,k}$, see [27–29] and [42–44]. It would be good to know whether each point on $X_{n/k}$ is a symplectic leaf for $q_{n,k}$.

Feigin and Odesskii's explicit construction of certain linear modules, and the results about linear modules for $Q_{n,1}(E, \tau)$ due to Staniszkis [53] and Tate–Van den Bergh [55], provide evidence that the representation theory of $Q_{n,k}(E, \tau)$ is related to the symplectic leaves for $q_{n,k}$, and their Lagrangian subvarieties, in "the same way" as the representation theory of a finite dimensional semisimple Lie algebra $\mathfrak{g}$ is related to the symplectic leaves (= the coadjoint orbits) in $\mathfrak{g}^*$ for the natural Poisson bracket on the symmetric algebra $S(\mathfrak{g})$ (see [30] for a nice survey of the role codajoint orbits play in representation theory).

The Poincaré–Birkhoff–Witt theorem allows one to use filtered-graded methods to show that if $\mathfrak{g}$ is a finite dimensional Lie algebra, then $U(\mathfrak{g})$ is a noetherian domain whose global dimension and Gelfand–Kirillov dimension equal $\dim(\mathfrak{g})$. We do not know if $Q_{n,k}(E, \tau)$ is a noetherian domain, though it is when $k = 1$ [55]. Having finite global dimension is a rather weak property for non-commutative rings so one often seeks to establish additional homological properties that are consequences of finite global dimension in the commutative case. It is known that $U(\mathfrak{g})$ has essentially all the homological properties the polynomial ring has: $U(\mathfrak{g})$ is Cohen–Macaulay in the sense that if M is a non-zero finitely generated left $U(\mathfrak{g})$-module, then $\mathrm{GKdim}(M)+j(M) = \dim(\mathfrak{g})$ where $j(M) = \min\{j \mid \mathrm{Ext}^j_{U(\mathfrak{g})}(M, U(\mathfrak{g})) \neq 0\}$ (see [9, Ch. 2, Thm. 7.1] and [32]); $U(\mathfrak{g})$ has the Auslander property, meaning that if M is as before, then $\mathrm{Ext}^i_{U(\mathfrak{g})}(N, U(\mathfrak{g})) = 0$ for all submodules $N \subseteq \mathrm{Ext}^j_{U(\mathfrak{g})}(M, U(\mathfrak{g}))$ when $i < j$

5 The Poisson structure $q_{n,k}$ is analogous to the Poisson structure $\{x, y\} := [x, y]$ on the symmetric algebra $S(\mathfrak{g})$.

6 [40, Thm. A] shows that certain Poisson central elements for $q_{n,1}$ are related to a higher secant variety.

7 The symplectic leaves for the Poisson structure on the affine variety $\mathbb{C}^n = \mathrm{Spec}(Q_{n,k}(E, 0))$ have been studied by Feigin and Odesskii in [25, 26] and [35, 36], for example.

[7, 10]. The algebras $Q_{n,1}(E,\tau)$ also have all these properties, though the methods required to establish them are very different [3, 51, 55].

Amongst other things, the results in this paper, in particular Theorem 1.1(1) and Theorem 1.2, determine the size (i.e., the Gelfand–Kirillov dimension) of $Q_{n,k}(E,\tau)$ and some of its fundamental homological properties. Roughly, these results say that like $U(\mathfrak{g})$, $Q_{n,k}(E,\tau)$ shares some of the homological properties of the polynomial ring on n variables.

1.2 The algebras $Q_{n,k}(E,\tau)$ are deformations of polynomial rings

It has long been expected that the $Q_{n,k}(E,\tau)$'s have the same size as the polynomial ring on n variables. More formally, for fixed (E,n,k), it was expected that the $Q_{n,k}(E,\tau)$'s form a flat family of graded algebras that are deformations of polynomial rings—the algebra $Q_{n,k}(E,0)$ is a polynomial ring on n variables. To prove that the $Q_{n,k}(E,\tau)$'s form a flat family of graded algebras that are deformations of polynomial rings one must show that the homogeneous components of $Q_{n,k}(E,\tau)$ have the same dimension as those of the polynomial ring on n variables. When $k=1$, this was proved by Tate and Van den Bergh [55] over 20 years ago. One of the main results in this paper is that this is true for *all* k provided τ is not a torsion point on E (Theorem 1.1).

The fundamental homological properties of $Q_{n,1}(E,\tau)$ were worked out by Tate and Van den Bergh [55]: they are Artin–Schelter regular, Auslander–Gorenstein, and Cohen–Macaulay. The starting point for [55] is the geometric description of the quadratic relations for $Q_{n,1}(E,\tau)$ in [55, §4.1], which is inspired by [37, §2]; the relation between these two geometric descriptions of the relations is explained in [17, §3.2]. Because there is not yet a similar geometric description of the relations for $Q_{n,k}(E,\tau)$ when $k>1$ we need a new method to understand $Q_{n,k}(E,\tau)$. The starting point for the results in this paper is the fact that the quadratic defining relations for $Q_{n,k}(E,\tau)$ can be defined in terms of an elliptic solution of the quantum Yang–Baxter equation. Although we focus on $Q_{n,k}(E,\tau)$, the techniques we develop in this paper should be useful for other algebras.

The $Q_{n,k}(E,\tau)$'s are graded $\mathbb{C}$-algebras generated by n degree-one elements. The Hilbert series of $Q_{n,k}(E,\tau)$ is the formal power series $\sum_{i=0}^{\infty}\dim(Q_{n,k}(E,\tau)_i)t^i$. The quadratic dual of $Q_{n,k}(E,\tau)$ is denoted by $Q_{n,k}(E,\tau)^!$.

The main results in this paper are as follows. (The notation is explained after their statement.)

Theorem 1.1 *Assume $\tau\in E$ is not a torsion point.*

(1) (Theorem 6.12) The Hilbert series of $Q_{n,k}(E,\tau)$ is the same as that of the polynomial ring on n variables placed in degree one, namely $(1-t)^{-n}$.
(2) (Theorem 7.7) The Hilbert series of $Q_{n,k}(E,\tau)^!$ is the same as that of the exterior algebra on n variables placed in degree one, namely $(1+t)^n$.
(3) (Theorems 9.17 and 10.1) $Q_{n,k}(E,\tau)$ is a Koszul algebra whose global dimension is n.

Theorem 1.2 (Theorem 10.4) *$Q_{n,k}(E,\tau)$ is Artin–Schelter regular for all but countably many $\tau \in E$.*

Proposition 1.3 (Proposition 10.3) *$Q_{n,k}(E,\tau)^!$ is a Frobenius algebra for all but finitely many $\tau \in E$.*

Although $Q_{n,k}(E,0)$ is a polynomial ring, most $Q_{n,k}(E,\tau)$'s are not commutative. We observed in [17, §4.2.1] that there are some values of $\tau \in \mathbb{C} - \Lambda$ for which $Q_{n,k}(E,\tau)$ is commutative (in fact, a polynomial ring); for example, $Q_{4,1}(E,\tau)$ is a polynomial ring when $\tau \in \frac{1}{2}\Lambda$.

The algebras $Q_{n,k}(E,\tau)$ depend on a pair of relatively prime integers $n > k \geq 1$, a point $\tau \in \mathbb{C}$, and a complex elliptic curve $E := \mathbb{C}/\Lambda$ where $\Lambda := \mathbb{Z} + \mathbb{Z}\eta$ is the lattice spanned by 1 and a point η lying in the upper half plane. Fix a vector space $V \cong \mathbb{C}^n$ with basis $x_0, \ldots, x_{n-1}$ indexed by the cyclic group $\mathbb{Z}_n$. We fix this notation for the rest of the paper.

The algebra $Q_{n,k}(E,\tau)$ is defined to be the quotient of the tensor algebra TV modulo the ideal generated by the subspace $\mathrm{rel}_{n,k}(E,\tau) \subseteq V^{\otimes 2}$ spanned by the n^2 elements

$$r_{ij} := \sum_{r \in \mathbb{Z}_n} \frac{\theta_{j-i+(k-1)r}(0)}{\theta_{j-i-r}(-\tau)\theta_{kr}(\tau)}\, x_{j-r} \otimes x_{i+r} \tag{1.1}$$

where the indices i and j belong to $\mathbb{Z}_n = \mathbb{Z}/n$ and $\theta_0(z), \ldots, \theta_{n-1}(z)$ are certain theta functions of order n (defined in [17, Prop. 2.6] and (2.1) below), indexed by $\mathbb{Z}$, that are quasi-periodic with respect to Λ.

If $\tau \in \frac{1}{n}\Lambda$, then $\theta_{kr}(\tau) = 0$ for some r so the relations do not make sense. Nevertheless, we can extend the definition of $Q_{n,k}(E,\tau)$ to all $\tau \in \mathbb{C}$ (see Sect. 5.4.1 and [17, §3.3]).

Up to isomorphism, $Q_{n,k}(E,\tau)$ depends only on the image of τ in E so we often regard τ as a point in E and call it a torsion point if $m\tau = 0$ in E for some integer $m \geq 1$.

1.3 The algebra $Q_{n,k}(E,\tau)$ can be defined in terms of Belavin's elliptic solutions to the quantum Yang–Baxter equation

Belavin's solution [8] to the quantum Yang–Baxter equation with spectral parameter (see (QYBE1) in Sect. 2.2 below) is the linear operator

$$S_k(z) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$$

defined in (3.6) below. As we will now explain,

$$\text{the space of relations for } Q_{n,k}(E,\tau) = \text{the image of } P \circ S_k(-n\tau) \tag{1.2}$$

where P is the linear operator $v \otimes v' \mapsto v' \otimes v$ on $V^{\otimes 2}$. The fact that $S_k(z)$ satisfies (QYBE1) seems to account for the rich structure of $Q_{n,k}(E,\tau)$. In particular, the proofs of the main results in this paper use this fact repeatedly.[8]

[8] Surprisingly, the results in our earlier papers about $Q_{n,k}(E,\tau)$ do not use this fact in an explicit way.

1.3.1 The linear operator $R_\tau(z)$

Fix (n,k) and $\tau \in \mathbb{C} - \frac{1}{n}\Lambda$. We define the linear operator

$$R_{n,k,\tau}(z) \;=\; R_\tau(z) \;=\; R(z) : V^{\otimes 2} \to V^{\otimes 2}$$

by the formula

$$R(z)(x_i \otimes x_j) \;:=\; \frac{\theta_0(-z)\cdots\theta_{n-1}(-z)}{\theta_1(0)\cdots\theta_{n-1}(0)} \sum_{r\in\mathbb{Z}_n} \frac{\theta_{j-i+r(k-1)}(-z+\tau)}{\theta_{j-i-r}(-z)\theta_{kr}(\tau)}\, x_{j-r} \otimes x_{i+r} \tag{1.3}$$

for all $(i,j) \in \mathbb{Z}_n^2$. Since $\tau \notin \frac{1}{n}\Lambda$, the $\theta_{kr}(\tau)$ term in the denominator is never zero; the term $\theta_{j-i-r}(-z)$ in the denominator always cancels with a factor in the numerator before the Σ sign; hence $z \mapsto R_\tau(z)$ is a well defined holomorphic function $\mathbb{C} \to \mathrm{End}_{\mathbb{C}}(V^{\otimes 2})$. By [17, Lem. 3.13], the function $\tau \mapsto R_\tau(\tau)$, initially defined on $\mathbb{C} - \frac{1}{n}\Lambda$, extends in a unique way to a holomorphic function $\mathbb{C} \to \mathrm{End}_{\mathbb{C}}(V^{\otimes 2})$; from now on $R_\tau(\tau)$, or just $R(\tau)$, denotes this extension.[9]

Comparing the formula for $R_\tau(z)$ with the defining relations for $Q_{n,k}(E,\tau)$ in (1.1), one sees that

$$\mathrm{rel}_{n,k}(E,\tau) \;=\; \mathrm{span}\{r_{ij} \mid i,j \in \mathbb{Z}_n\} \;=\; \text{the image of } R_\tau(\tau). \tag{1.4}$$

In Propositions 3.4 and 3.5 we show that

$$S_k(-nz) \;=\; ne(\tfrac{1}{2}n(n+1)z)\, PR_{n,k,\tau}(z). \tag{1.5}$$

This equality implies the equality in (1.2). The operator $S_k(z)$ is defined in terms of certain theta functions with characteristics, i.e., the functions $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}$ defined in Sect. 2.5, whereas $R(z)$ is defined in terms the θ_α's defined in (2.1); the relation between the two types of theta functions is given in Lemma 2.9. A version of (1.5) must have been known to Feigin and Odesskii, but we could not find it in the literature so we have proved it here.[10]

1.3.2. It has been known since the 1980's that $S_k(z)$ satisfies (QYBE1) (see the discussion after Theorem 3.1). Hence $PS_k(z)$, and therefore $R(z)$, satisfies (QYBE2). We record this fact in Theorem 3.7.

[9] In this paper we need an improved version of [17, Lem. 3.13]: Lemma 5.1 below shows that for each $m \in \mathbb{Z}$ and each $\zeta \in \frac{1}{n}\Lambda$ there is a holomorphic function $\mathbb{C} \to \mathrm{End}_{\mathbb{C}}(V^{\otimes 2})$, $\tau \mapsto R_{n,k,\tau}(m\tau+\zeta)$.

[10] In [37, Rmk. 4, §1], Feigin and Odesskii say there is a close connection between the $Q_{n,k}(E,\tau)$'s and Belavin's elliptic solutions to the QYBE. They do not specify the connection but refer the reader to [15]; although [15, §4] concerns an algebra $\mathcal{R}^d_\eta$ that is defined in terms of $S_k(z)$, [15] does not refer to $R(z)$. The algebras $\mathcal{R}^d_\eta$ in [15, §4] are generated by n^2d elements whereas $Q_{n,k}(E,\tau)$ is generated by n elements. At the end of the introduction to [37] is an equality $A^{(d)} = Q_{n^2d,nd-1}(E,\tau)$. The algebra $A^{(d)}$ is not defined (perhaps it is $\mathcal{R}^d_\eta$) and there is no explanation of the equality.

It is easy to see that $R_\tau(0) = I \otimes I$ where I denotes the identity operator on V (Proposition 2.4). As a consequence of this and the fact that $R(z)$ satisfies (QYBE2), for each $z \in \mathbb{C}$, $R_\tau(z)$ is either an isomorphism or satisfies $R_\tau(z)R_\tau(-z) = 0$ (Lemma 2.3). Corollary 5.9 shows that $R_\tau(z)$ is a non-isomorphism if and only if $z \in \pm\tau + \frac{1}{n}\Lambda$. That result also shows that rank $R_\tau(\tau+\zeta) = \binom{n}{2}$ for all $\zeta \in \frac{1}{n}\Lambda$.[11] It follows that

$$\text{the image of } R_\tau(z) \;=\; \text{the kernel of } R_\tau(-z)$$

for all $z \in \pm\tau + \frac{1}{n}\Lambda$. The next result follows from these remarks and the calculation before Theorem 7.4.

Theorem 1.4 (Theorem 7.4) *The quadratic dual $Q_{n,k}(E,\tau)^!$ is isomorphic to the tensor algebra TV modulo the ideal generated by the kernel of the operator $R_{n,n-k,\tau}(\tau) : V^{\otimes 2} \to V^{\otimes 2}$.*

1.3.3. The fact that $Q_{n,k}(E,0)$ is a polynomial ring on n variables (see [17, Prop. 5.1] for a proof) is related to the fact that $\lim_{\tau\to 0} R_\tau(\tau)$ is the anti-symmetrization operator $v \otimes v' \mapsto v \otimes v' - v' \otimes v$. The fact that $Q_{n,k}(E,0)^!$ is an exterior algebra on n variables is related to the fact that $\lim_{\tau\to 0} R_\tau(-\tau)$ is the symmetrization operator $v \otimes v' \mapsto v \otimes v' + v' \otimes v$. These are special cases of Proposition 5.2 which shows that $\lim_{\tau\to 0} R_\tau(m\tau)$ is the skew-symmetrization operator $v \otimes v' \to v \otimes v' - mv' \otimes v$ for all $m \in \mathbb{Z}$. This observation is used in an essential way in the proof of Theorem 1.1(1): it is used to show that the space of degree-d relations for $Q_{n,k}(E,\tau)$ is the kernel of a certain operator $F_d(-\tau) : V^{\otimes d} \to V^{\otimes d}$. Like $R_\tau(\tau)$, $F_d(-\tau)$ belongs to a family of operators $F_d(z)$, $z \in \mathbb{C}$, and Proposition 6.4 shows that the limits of $F_d(-\tau)$ and $F_d(\tau)$ as $\tau \to 0$ are the symmetrization and anti-symmetrization operators on $V^{\otimes d}$, respectively. This gives a heuristic explanation as to why we might expect that $Q_{n,k}(E,\tau)$ and $Q_{n,k}(E,\tau)^!$ should be deformations of the polynomial and exterior algebras, respectively.

1.4 Methods

The methods in this paper might be useful in other situations so we say a little about them. For the purposes of the discussion we write $A(\tau) = Q_{n,k}(E,\tau)$. Thus, $A(0)$ is the polynomial ring $SV = \mathbb{C}[x_0, \ldots, x_{n-1}]$.

The main results in this paper are of the following form: $A(\tau)$ has property $\mathbf{P}(\tau)$, where $\mathbf{P}(0)$ is a property of $A(0)$. In all cases of interest $\mathbf{P}(\tau)$ can be formulated as a statement that a certain subspace $S(\tau) \subseteq V^{\otimes d}$ has the same dimension as $S(0)$.

We realize $S(\tau)$ as the image or kernel of a linear operator $P(\tau') : V^{\otimes d} \to V^{\otimes d}$, where τ' is usually an integer multiple of τ, and reduce the question of interest to a question about the rank of $P(\tau')$. In all cases of interest, $P(\tau')$ belongs to a family of linear operators $P(z) : V^{\otimes d} \to V^{\otimes d}$, $z \in \mathbb{C}$, whose matrix entries (with respect to some, hence every, basis) are theta functions with respect to Λ having the same quasi-periodicity properties. We call such a $P(z)$ a theta operator (see Sect. 4.2). The

[11] This implies that the dimension of $\mathrm{rel}_{n,k}(E,\tau)$ is $\binom{n}{2}$, which is the first step toward proving Theorem 1.1(1).

determinant $\det P(z)$ is then a theta function, of order r say, and Lemma 2.5 tells us that $\det P(z)$ has r zeros (counted with multiplicity) in a fundamental parallelogram (and also tells us the sum of those zeros). In other words, if $\operatorname{mult}_p(\det P(z))$ denotes the multiplicity of p as a zero of $\det P(z)$, $\sum_p \operatorname{mult}_p(\det P(z)) = r$ where the sum is taken over all points in a fundamental parallelogram. By Lemma 4.1, $\operatorname{mult}_p(\det P(z)) \geq \dim(\ker P(p))$.

Often, we are able to narrow down the possibilities for the zeros of $\det P(z)$ to a finite number of $\frac{1}{n}\Lambda$-cosets of the form $m\tau + \frac{1}{n}\Lambda$; see Propositions 5.8 and 9.8 for example. We then obtain for "enough" of those m's a result of the form $\dim(\ker P(m\tau)) \geq$ some number, d_m say. It then follows that

$$r = \sum_p \operatorname{mult}_p(\det P(z)) \geq \sum_p \dim(\ker P(p)) \geq \sum_m d_m.$$

If the right-most sum equals r, then these inequalities are equalities and we conclude that we have found all the zeros of $\det P(z)$ and their individual multiplicities. In particular, we now know $\dim(\ker P(\tau'))$.

Among the operators playing the role of $P(z)$ are:

- $R(z)$ in Sect. 5 where we show that $R(z)$ is not an isomorphism if and only if $z \in \pm\tau + \frac{1}{n}\Lambda$ and that rank $R(\tau) = \binom{n}{2}$; the dimension of the space of quadratic relations for $Q_{n,k}(E,\tau)$ is therefore the same as for SV;
- $F_d(z)$ and $G_\tau(z)$ in Sect. 6 where we show that the dimension of the space of degree-d relations for $Q_{n,k}(E,\tau)$, which is the kernel of $F_d(-\tau)$, is the same as for SV;
- $H_\tau(z)$ in Sect. 9 where we prove that a certain lattice of subspaces of $V^{\otimes d}$ is distributive by showing that certain elements of it have the same dimension as their counterparts for SV.

The operators $G_\tau(z)$ and $H_\tau(z)$ are not defined on all of $V^{\otimes d}$.

1.5 Contents of this paper

The main result in Sect. 3 is a proof of (1.5) then, as a consequence of that and the fact that $S_k(z)$, which is defined in (3.6), satisfies (QYBE1), we conclude that $R_{n,k,\tau}(z)$ satisfies (QYBE2).

Section 4 establishes some general results about a holomorphic linear operator $A(z)$ on a finite-dimensional vector space and relates the location and multiplicities of the zeros of $\det A(z)$ to the dimension of the kernel of $A(z)$. These results are used in Sects. 5, 6 and 9. We also introduce the notion of a theta operator in this section.

Section 5 takes the first step toward showing that $Q_{n,k}(E,\tau)$ has the same Hilbert series as the polynomial ring $\mathbb{C}[x_0,\ldots,x_{n-1}]$ by showing that the dimension of $\mathrm{rel}_{n,k}(E,\tau)$ is $\binom{n}{2}$. This is not straightforward. We must understand the kernel and image of $\lim_{\tau\to 0} R(\pm\tau+\zeta)$ when $\zeta \in \frac{1}{n}\Lambda$. To do this we show that $\det R(z)$ is a theta function with respect to $\frac{1}{n}\Lambda$; we also need to know the location and multiplicities of the zeros of $\det R(z)$. Odesskii already knew this but he did not prove the formula for

$\det R(z)$ in his survey [36] so we do that in Proposition 5.8 (and in doing so make a small correction to his formula); that, and a proof that the dimension of $\mathrm{rel}_{n,k}(E,\tau)$ is $\binom{n}{2}$, are the main results in Sect. 5.

In Sect. 6 we show that the Hilbert series of $Q_{n,k}(E,\tau)$ is $(1-t)^{-n}$ for all $\tau \in (\mathbb{C} - \bigcup_{m\geq 1} \frac{1}{m}\Lambda) \cup \frac{1}{n}\Lambda$. The method is a jazzed up version of the method in Sect. 5. The space of degree-d relations for $Q_{n,k}(E,\tau)$ is realized as the kernel of the linear operator $F_d(-\tau) : V^{\otimes d} \to V^{\otimes d}$, and the proof of the Hilbert series result requires a careful analysis of the theta operator $F_d(z)$ that is similar in spirit to some of the arguments in Sect. 5. The argument we use to prove the Hilbert series result bears no resemblance to earlier arguments showing that the Hilbert series of $Q_{n,1}(E,\tau)$ is $(1-t)^{-n}$. We make some further remarks about this in Sect. 6.4.3.

In Sect. 7 we show that the Hilbert series of $Q_{n,k}(E,\tau)^!$ is $(1+t)^n$ for all $\tau \in (\mathbb{C} - \bigcup_{m=1}^{n+1} \frac{1}{mn}\Lambda) \cup \frac{1}{n}\Lambda$. The methods there resemble those in Sect. 6 but now the space of degree-d relations for (a quotient of TV that is isomorphic to) $Q_{n,k}(E,\tau)^!$ is realized as the kernel of $F_d(\tau)$.

Since the space of degree-d relations for $Q_{n,k}(E,\tau)$ is the kernel of $F_d(-\tau)$, there is a canonical graded vector space isomorphism $Q_{n,k}(E,\tau) \cong \bigoplus_{d=0}^{\infty} \mathrm{im}\, F_d(-\tau)$. The multiplication on $Q_{n,k}(E,\tau)$ can therefore be transferred to this subspace of TV in a canonical way. Section 8 gives an explicit description of this multiplication via the operators $M_{a,b}$ defined there. This multiplication is analogous to the shuffle product on the subspace of the tensor algebra consisting of the symmetric tensors.

In Sect. 9, we show $Q_{n,k}(E,\tau)$ is a Koszul algebra for all $\tau \in (\mathbb{C} - \bigcup_{m\geq 1} \frac{1}{m}\Lambda) \cup \frac{1}{n}\Lambda$ by verifying the "distributive lattice" criterion. The operators $M_{a,b}$ defined in Sect. 8, and others derived from them, play a crucial role. Once more, we use the methods described in Sect. 1.4.

In Sect. 10 we show that, for fixed n, k, and E, $Q_{n,k}(E,\tau)$ is an Artin–Schelter regular algebra for all but countably many τ.

2 Preliminaries

Whenever possible, the notation in this paper is the same as that in our earlier papers [16–18]. (We will advise the reader to consult those papers when necessary.) For example, we always use the notation

$$e(z) = e^{2\pi i z}.$$

We introduced the notation (n, k, E), η, Λ, $\tau \in \mathbb{C}$, and $\mathrm{rel}_{n,k}(E,\tau)$ in Sect. 1.2. This notation will be fixed throughout the paper. The space of theta functions $\Theta_n(\Lambda)$ and its distinguished basis θ_α, $\alpha \in \mathbb{Z}_n$, are defined in [17, §2]. The basic properties of the θ_α's are recorded in [17, Prop. 2.6], so we pause here briefly only to recall the definition:

$$\theta_\alpha(z) \;=\; \theta_\alpha(z \,|\, \eta) \;:=\; e\Big(\alpha z + \tfrac{\alpha}{2n} + \tfrac{\alpha(\alpha-n)}{2n}\eta\Big) \prod_{m=0}^{n-1} \theta\big(z + \tfrac{m}{n} + \tfrac{\alpha}{n}\eta\big), \tag{2.1}$$

where θ is the order-1 theta function

$$\theta(z) \;=\; \theta(z \,|\, \eta) \;:=\; \sum_{n\in\mathbb{Z}} (-1)^n e\big(nz + \tfrac{1}{2}n(n-1)\eta\big).$$

An explicit formula for $\theta_\alpha(z)$ as an infinite exponential sum can be obtained by combining Lemma 2.9 with the definition of the function $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z \,|\, \eta)$ at the start of Sect. 2.5.

2.1 Notation for linear operators

Always, V denotes a complex vector space of dimension n with basis x_i, $i \in \mathbb{Z}_n$.

We will write I for the identity operator on V.

If $A : V \to V$ is a linear operator and $1 \le i \le d$, we write A_i for the operator $I^{\otimes(i-1)} \otimes A \otimes I^{\otimes(d-i)}$ on $V^{\otimes d}$.

If $A : V^{\otimes 2} \to V^{\otimes 2}$ is a linear operator and $1 \le i \le d-1$, we write $A_{i,i+1}$ for the operator $I^{\otimes(i-1)} \otimes A \otimes I^{\otimes(d-i-1)}$ on $V^{\otimes d}$.

Given integers $0 \le p \le d$ and a linear operator $A : V^{\otimes p} \to V^{\otimes p}$, we write A^L (resp., A^R) for the operator $A \otimes I^{\otimes(d-p)}$ (resp., $I^{\otimes(d-p)} \otimes A$) on $V^{\otimes d}$ given by A acting on the left-most (resp., right-most) p tensorands of $V^{\otimes d}$. For a family of linear operators $A(z_1, \ldots, z_p)$, we write $A^L(z_1, \ldots, z_p)$ for $A(z_1, \ldots, z_p)^L$. We also write $A^R(z_1, \ldots, z_p) := A(z_1, \ldots, z_p)^R$.

Various linear operators of the form $A(z_1, \ldots, z_p)$ will be evaluated when several of its arguments are the same. If $i \le j$ and $z_i = \cdots = z_j = \nu$ we write $A(z_1, \ldots, z_{i-1}, \nu^{j-i+1}, z_{j+1}, \ldots, z_p)$ for $A(z_1, \ldots, z_p)$.

2.2 The quantum Yang–Baxter equation with spectral parameter

The material in this subsection is standard.

Let $A \in \operatorname{End}_{\mathbb{C}}(V \otimes V)$. We define linear operators $A_{12}, A_{23}, A_{13} \in \operatorname{End}(V^{\otimes 3})$ by $A_{12} := A \otimes I$, $A_{23} := I \otimes A$, where I is the identity operator on V, and A_{13} acts as the identity on the middle V and as A does on the first and third factors of $V \otimes V \otimes V$.

A family of linear operators $R(z) \in \operatorname{End}(V \otimes V)$, parametrized by $z \in \mathbb{C}$, satisfies the first quantum Yang–Baxter equation if

$$R(u)_{12} R(u+v)_{13} R(v)_{23} \;=\; R(v)_{23} R(u+v)_{13} R(u)_{12} \tag{QYBE1}$$

for all $u, v \in \mathbb{C}$. We say that $R(z)$ satisfies the second quantum Yang–Baxter equation if

$$R(u)_{12} R(u+v)_{23} R(v)_{12} \;=\; R(v)_{23} R(u+v)_{12} R(u)_{23} \tag{QYBE2}$$

for all $u, v \in \mathbb{C}$. The family of operators $R(z)$ satisfying (QYBE1) or (QYBE2) is called an R-matrix.

Let $P \in \text{End}(V \otimes V)$ be the linear map $P(x \otimes y) = y \otimes x$. If $A \in \text{End}(V \otimes V)$ we define

$$A' := PA \quad \text{and} \quad A'' := AP.$$

Multiplication by P provides a bijection between solutions to (QYBE1) and (QYBE2).

Proposition 2.1 *A family of operators $R(z)$ satisfies (QYBE1) if and only if $R(z)'$ satisfies (QYBE2) if and only if $R(z)''$ satisfies (QYBE2).*

This is an immediate consequence of the following routine lemma.

Lemma 2.2 *Let $A, B, C \in \text{End}(V \otimes V)$. Then $A_{12}B_{13}C_{23} = C_{23}B_{13}A_{12}$ if and only if $A'_{12}B'_{23}C'_{12} = C'_{23}B'_{12}A'_{23}$ if and only if $A''_{12}B''_{23}C''_{12} = C''_{23}B''_{12}A''_{23}$.*

The next result plays a crucial role in Sect. 5.3.

Lemma 2.3 *Let $R(z)$, $z \in \mathbb{C}$, be a family of operators satisfying (QYBE2). If $R(0) = I \otimes I$, then there are scalars $c(z) \in \mathbb{C}$ such that*

$$R(z)R(-z) = R(-z)R(z) = c(z)I \otimes I.$$

In particular, if $R(z)$ is not an isomorphism, then $R(z)R(-z) = 0 = R(-z)R(z)$.

Proof Since $R(0) = I \otimes I$, substituting $u = -v = z$ in (QYBE2) yields

$$R(z)_{12}R(-z)_{12} = R(-z)_{23}R(z)_{23}. \tag{2.2}$$

We use the fixed basis $\{x_i\}_i$ for V. Applying both sides of (2.2) to $x_i \otimes x_j \otimes x_k$ yields

$$R(z)R(-z)(x_i \otimes x_j) \otimes x_k = x_i \otimes R(-z)R(z)(x_j \otimes x_k).$$

Hence there is a linear map $F(u) : V \to V$ such that

$$R(z)R(-z) \otimes I = I \otimes F(z) \otimes I = I \otimes R(-z)R(z),$$

which implies $R(z)R(-z) = I \otimes F(z)$ and $R(-z)R(z) = F(z) \otimes I$. The same argument for $-z$ implies $R(-z)R(z) = I \otimes F(-z)$ and $R(z)R(-z) = F(-z) \otimes I$. Hence there is $c(z) \in \mathbb{C}$ such that

$$I \otimes F(z) = F(-z) \otimes I = c(z)I \otimes I.$$

Therefore $F(z) = c(z)I$. $\square$

Proposition 2.4 *If $R_\tau(z)$ is the operator defined in (1.3), then*

(1) $R_\tau(0) = I \otimes I$ *and*
(2) $R_\tau(\tau)R_\tau(-\tau) = 0 = R_\tau(-\tau)R_\tau(\tau)$.

Proof (1) Since the zeros of $\theta_\alpha(z)$ are the points in $-\frac{\alpha}{n}\eta+\mathbb{Z}\eta+\frac{1}{n}\mathbb{Z}$ ([17, Prop. 2.6(6)]), the $\theta_0(0)$ term appearing before the Σ sign in the expression for $R_\tau(0)$ annihilates all the terms after the Σ sign except the $r = j - i$ summand whose denominator, $\theta_{j-i-r}(0)$, cancels the $\theta_0(0)$ term. Hence

$$R_\tau(0)(x_i \otimes x_j) \;=\; x_i \otimes x_j$$

for all $i, j \in \mathbb{Z}_n$.

(2) Theorem 3.7 below shows that the $R_\tau(z)$ defined in (1.3) satisfies (QYBE2) so, since $R_\tau(0) = I \otimes I$, the conclusion of Lemma 2.3 applies to $R_\tau(z)$.

Since $Q_{n,k}(E, \tau)$ has an infinite-dimensional cyclic module, namely a point module (see [16, §1.4]), $\mathrm{rel}_{n,k}(E, \tau) \neq V^{\otimes 2}$. But $\mathrm{rel}_{n,k}(E, \tau)$ is the image of $R_\tau(\tau)$ so the result follows from Lemma 2.3. □

2.2.1 R-matrices in arbitrary algebras

It will be convenient to generalize the setup for R-matrices and the quantum Yang–Baxter equation. Instead of operators $R(z)$ in $\mathrm{End}(V \otimes V)$ we can take elements $R(z) \in S \otimes_Z S$ where S is a $\mathbb{C}$-algebra and $Z \subseteq S$ is a central subalgebra. There are obvious definitions of $R(z)_{ij} \in S \otimes_Z S \otimes_Z S$ for $(ij) \in \{(12), (13), (23)\}$. The equations (QYBE1) and (QYBE2) then acquire the obvious meanings. If V is a left S-module, then the various $R(z)_{ij}$ act on $V^{\otimes 3}$. If S is a finite-dimensional $\mathbb{C}$-algebra we can speak of holomorphic or meromorphic $R(z)$.

Section 3 uses this idea with $S = \mathbb{C}\Gamma$ for a finite group Γ and $Z = \mathbb{C}\Delta$ for a central subgroup $\Delta < \Gamma$.

2.3 Theta functions in one variable

We make frequent use of the following result. A proof of it appears in the appendix to [17].

Lemma 2.5 *Assume $\Lambda = \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$ is a lattice in $\mathbb{C}$ such that $\mathrm{Im}(\eta_2/\eta_1) > 0$, and suppose f is a non-constant holomorphic function on $\mathbb{C}$. If there are constants $a, b, c, d \in \mathbb{C}$ such that*

$$\begin{aligned} f(z+\eta_1) \;&=\; e^{-2\pi i(az+b)} f(z) \quad \textit{and} \\ f(z+\eta_2) \;&=\; e^{-2\pi i(cz+d)} f(z), \end{aligned}$$

then

(1) *$c\eta_1 - a\eta_2 \in \mathbb{Z}_{\geq 0}$, and*
(2) *f has $c\eta_1 - a\eta_2$ zeros (counted with multiplicity) in every fundamental parallelogram for Λ, and*
(3) *the sum of those zeros is $\frac{1}{2}(c\eta_1^2 - a\eta_2^2) + (c - a)\eta_1\eta_2 + b\eta_2 - d\eta_1$ modulo Λ.*

2.4 Transformation properties of $R_\tau(z)$

Let $S, T, N \in \mathrm{GL}(V)$, and $P \in \mathrm{GL}(V^{\otimes 2})$, be the automorphisms

$$S \cdot x_\alpha = e\big(\tfrac{\alpha}{n}\big)x_\alpha, \quad T \cdot x_\alpha = x_{\alpha+1}, \quad N \cdot x_\alpha = x_{-\alpha}, \quad P(u \otimes v) := v \otimes u.$$

The group generated by S and T is the Heisenberg group H_n of order n^3, and V is an irreducible representation of H_n (see (3.3)). At [37, §1, Rmk. 2], Odesskii and Feigin observed that S and T extend to automorphisms of $Q_{n,k}(E,\tau)$ (see [17, Prop. 3.23] for the details). We will often use the projective representation of $\frac{1}{n}\Lambda/\Lambda$ on V given by

$$\tfrac{a}{n} + \tfrac{b}{n}\eta \mapsto T^b S^{ka}.$$

Let

$$b(z) := e\big(-nz + \tau + \tfrac{1}{2} - \tfrac{n+1}{2}\eta\big). \tag{2.3}$$

Proposition 2.6 *Let $k' \in \mathbb{Z}$ be the unique integer such that $n > k' \geq 1$ and $kk' = 1$ in $\mathbb{Z}_n$. Then*

$$\begin{aligned}
R_\tau\big(z + \tfrac{1}{n}\big) &= (-1)^{n-1}(I \otimes S^{-k})\, R_\tau(z)\, (S^k \otimes I), && (2.4)\\
R_\tau\big(z + \tfrac{1}{n}\eta\big) &= b(z)(I \otimes T^{-1})\, R_\tau(z)\, (T \otimes I), && (2.5)\\
R_\tau(-z) &= e(n^2 z) P\, R_{-\tau}(z)\, P, && (2.6)\\
R_\tau(-z) &= e(n^2 z)(N \otimes N)\, R_{-\tau}(z)\, (N \otimes N), && (2.7)\\
R_{\tau+\frac{1}{n}}(z) &= (S \otimes I)\, R_\tau(z)\, (S^{-1} \otimes I), && (2.8)\\
R_{\tau+\frac{1}{n}\eta}(z) &= e(z)(I \otimes T^{-k'})\, R_\tau(z)\, (I \otimes T^{k'}). && (2.9)
\end{aligned}$$

Furthermore,

(1) $\operatorname{rank} R_\tau(z+\zeta) = \operatorname{rank} R_\tau(z)$ *for all* $\zeta \in \frac{1}{n}\Lambda$*;*
(2) $R_\tau(z)R_\tau(-z) = 0 = R_\tau(-z)R_\tau(z)$ *for all* $z \in \pm\tau + \frac{1}{n}\Lambda$*;*[12]
(3) $S \otimes S$ *and* $T \otimes T$ *commute with* $R_\tau(z)$*.*[13]

Proof We will use the notation $D := \theta_1(0)\cdots\theta_{n-1}(0)$.
Proof of (2.4). Since $\theta_\alpha\left(z + \frac{1}{n}\right) = e\left(\frac{\alpha}{n}\right)\theta_\alpha(z)$, by [17, Prop. 2.6(3)], and

$$e(-\tfrac{kr}{n})x_{j-r} \otimes x_{i+r} = e(\tfrac{ki}{n})(I \otimes S^{-k})(x_{j-r} \otimes x_{i+r}),$$

12 It follows from Corollary 5.9 that $R(z)R(-z) = 0 = R(-z)R(z)$ if and only if $z \in \pm\tau + \frac{1}{n}\Lambda$, and that $R(z)$ is an isomorphism if $z \notin \pm\tau + \frac{1}{n}\Lambda$.

13 This implies that S and T extend to automorphisms of $Q_{n,k}(E,\tau)$ (cf., [17, Prop. 3.23]).

we see that $R_\tau(z+\frac{1}{n})(x_i \otimes x_j)$ equals

$$\begin{aligned}
&\frac{1}{D}\left(\prod_{\alpha\in\mathbb{Z}_n}\theta_\alpha(-z-\tfrac{1}{n})\right)\sum_{r\in\mathbb{Z}_n}\frac{\theta_{j-i+r(k-1)}(-z-\frac{1}{n}+\tau)}{\theta_{j-i-r}(-z-\frac{1}{n})\theta_{kr}(\tau)}\,x_{j-r}\otimes x_{i+r}\\
&\quad= e\big(-\tfrac{n-1}{2}\big)\frac{1}{D}\left(\prod_{\alpha\in\mathbb{Z}_n}\theta_\alpha(-z)\right)\sum_{r\in\mathbb{Z}_n}e\big(-\tfrac{kr}{n}\big)\frac{\theta_{j-i+r(k-1)}(-z+\tau)}{\theta_{j-i-r}(-z)\theta_{kr}(\tau)}\,x_{j-r}\otimes x_{i+r}\\
&\quad= (-1)^{n-1}e(\tfrac{ki}{n})\,\frac{1}{D}\left(\prod_{\alpha\in\mathbb{Z}_n}\theta_\alpha(-z)\right)\\
&\qquad\cdot\sum_{r\in\mathbb{Z}_n}\frac{\theta_{j-i+r(k-1)}(-z+\tau)}{\theta_{j-i-r}(-z)\theta_{kr}(\tau)}\,(I\otimes S^{-k})(x_{j-r}\otimes x_{i+r})\\
&\quad= (-1)^{n-1}e(\tfrac{ki}{n})\,(I\otimes S^{-k})R_\tau(z)(x_i\otimes x_j)\\
&\quad= (-1)^{n-1}\,(I\otimes S^{-k})R_\tau(z)(S^k\otimes I)(x_i\otimes x_j).
\end{aligned}$$

Proof of (2.5). By [17, Prop. 2.6(4)], $\theta_\alpha\big(z-\frac{1}{n}\eta\big)=e\big(z+\frac{1}{2n}-\frac{n+1}{2n}\eta\big)\,\theta_{\alpha-1}(z)$.
Therefore $R_\tau(z+\frac{1}{n}\eta)(x_i\otimes x_j)$ equals

$$\begin{aligned}
&\frac{1}{D}\left(\prod_{\alpha\in\mathbb{Z}_n}\theta_\alpha(-z-\tfrac{1}{n}\eta)\right)\sum_{r\in\mathbb{Z}_n}\frac{\theta_{j-i+r(k-1)}(-z-\frac{1}{n}\eta+\tau)}{\theta_{j-i-r}(-z-\frac{1}{n}\eta)\theta_{kr}(\tau)}\,x_{j-r}\otimes x_{i+r}\\
&\quad= e\Big(-nz+\tfrac{1}{2}-\tfrac{n+1}{2}\eta\Big)\,\frac{1}{D}\left(\prod_{\alpha\in\mathbb{Z}_n}\theta_{\alpha-1}(-z)\right)\\
&\qquad\cdot\sum_{r\in\mathbb{Z}_n}e(\tau)\frac{\theta_{j-i+r(k-1)-1}(-z+\tau)}{\theta_{j-i-r-1}(-z)\theta_{kr}(\tau)}\,x_{j-r}\otimes x_{i+r}\\
&\quad= b(z)\,\frac{1}{D}\left(\prod_{\alpha\in\mathbb{Z}_n}\theta_\alpha(-z)\right)\sum_{r\in\mathbb{Z}_n}\frac{\theta_{j-i+r(k-1)-1}(-z+\tau)}{\theta_{j-i-r-1}(-z)\theta_{kr}(\tau)}\,(I\otimes T^{-1})(x_{j-r}\otimes x_{i+1+r})\\
&\quad= b(z)\,(I\otimes T^{-1})R_\tau(z)(T\otimes I)(x_i\otimes x_j).
\end{aligned}$$

Proof of (2.6). Since $\theta_\alpha(-z)=-e\big(-nz+\frac{\alpha}{n}\big)\theta_{-\alpha}(z)$ by [17, Prop. 2.6(5)],

$$\begin{aligned}
R_\tau(-z)(x_i\otimes x_j) &= \frac{\theta_0(z)\cdots\theta_{n-1}(z)}{\theta_1(0)\cdots\theta_{n-1}(0)}\sum_{r\in\mathbb{Z}_n}\frac{\theta_{j-i+r(k-1)}(z+\tau)}{\theta_{j-i-r}(z)\theta_{kr}(\tau)}\,x_{j-r}\otimes x_{i+r}\\
&= (-1)^n e(n^2z+\tfrac{n-1}{2})\frac{\theta_0(-z)\cdots\theta_{n-1}(-z)}{\theta_1(0)\cdots\theta_{n-1}(0)}\\
&\qquad\cdot\sum_{r\in\mathbb{Z}_n}(-1)\frac{\theta_{i-j-r(k-1)}(-z-\tau)}{\theta_{i-j+r}(-z)\theta_{-kr}(-\tau)}\,x_{j-r}\otimes x_{i+r}\\
&= e(n^2z)\frac{\theta_0(-z)\cdots\theta_{n-1}(-z)}{\theta_1(0)\cdots\theta_{n-1}(0)}\sum_{r\in\mathbb{Z}_n}\frac{\theta_{i-j+r(k-1)}(-z-\tau)}{\theta_{i-j-r}(-z)\theta_{kr}(-\tau)}\,P(x_{i-r}\otimes x_{j+r})\\
&= e(n^2z)P\,R_{-\tau}(z)\,P(x_i\otimes x_j).
\end{aligned}$$

Proof of (2.7). This follows from (2.6) and

$$\begin{aligned} PR_\tau(z)P(x_i \otimes x_j) &= \frac{\theta_0(-z)\cdots\theta_{n-1}(-z)}{\theta_1(0)\cdots\theta_{n-1}(0)} \sum_{r\in\mathbb{Z}_n} \frac{\theta_{i-j+r(k-1)}(-z+\tau)}{\theta_{i-j-r}(-z)\theta_{kr}(\tau)}\, x_{j+r}\otimes x_{i-r} \\ &= (N\otimes N)R_\tau(z)(N\otimes N)(x_i\otimes x_j). \end{aligned}$$

Proof of (2.8). Since $\theta_\alpha\left(z+\frac{1}{n}\right) = e\left(\frac{\alpha}{n}\right)\theta_\alpha(z)$, $R_{\tau+\frac{1}{n}}(z)(x_i\otimes x_j)$ equals

$$\begin{aligned} &\frac{1}{D}\left(\prod_{\alpha\in\mathbb{Z}_n}\theta_\alpha(-z)\right)\sum_{r\in\mathbb{Z}_n}\frac{\theta_{j-i+r(k-1)}(-z+\tau+\frac{1}{n})}{\theta_{j-i-r}(-z)\theta_{kr}(\tau+\frac{1}{n})}\, x_{j-r}\otimes x_{i+r} \\ &= \frac{1}{D}\left(\prod_{\alpha\in\mathbb{Z}_n}\theta_\alpha(-z)\right)\sum_{r\in\mathbb{Z}_n} e\left(\tfrac{j-i-r}{n}\right)\frac{\theta_{j-i+r(k-1)}(-z+\tau)}{\theta_{j-i-r}(-z)\theta_{kr}(\tau)}\, x_{j-r}\otimes x_{i+r} \\ &= e\left(\tfrac{-i}{n}\right)\frac{1}{D}\left(\prod_{\alpha\in\mathbb{Z}_n}\theta_\alpha(-z)\right)\sum_{r\in\mathbb{Z}_n}\frac{\theta_{j-i+r(k-1)}(-z+\tau)}{\theta_{j-i-r}(-z)\theta_{kr}(\tau)}\,(S\otimes I)(x_{j-r}\otimes x_{i+r}) \\ &= (S\otimes I)R_\tau(z)(S^{-1}\otimes I)(x_i\otimes x_j). \end{aligned}$$

Proof of (2.9). Since $\theta_\alpha\left(z+\frac{1}{n}\eta\right) = e\left(-z-\frac{1}{2n}+\frac{n-1}{2n}\eta\right)\theta_{\alpha+1}(z)$, $R_{\tau+\frac{1}{n}\eta}(z)(x_i\otimes x_j)$ equals

$$\begin{aligned} &\frac{1}{D}\left(\prod_{\alpha\in\mathbb{Z}_n}\theta_\alpha(-z)\right)\sum_{r\in\mathbb{Z}_n}\frac{\theta_{j-i+r(k-1)}(-z+\tau+\frac{1}{n}\eta)}{\theta_{j-i-r}(-z)\theta_{kr}(\tau+\frac{1}{n}\eta)}\, x_{j-r}\otimes x_{i+r} \\ &= e(z)\frac{1}{D}\left(\prod_{\alpha\in\mathbb{Z}_n}\theta_\alpha(-z)\right)\sum_{r\in\mathbb{Z}_n}\frac{\theta_{j-i+r(k-1)+1}(-z+\tau)}{\theta_{j-i-r}(-z)\theta_{kr+1}(\tau)}\, x_{j-r}\otimes x_{i+r} \\ &= e(z)\frac{1}{D}\left(\prod_{\alpha\in\mathbb{Z}_n}\theta_\alpha(-z)\right)\sum_{s\in\mathbb{Z}_n}\frac{\theta_{j-i+k'+s(k-1)}(-z+\tau)}{\theta_{j-i+k'-s}(-z)\theta_{ks}(\tau)}\, x_{j+k'-s}\otimes x_{i-k'+s} \\ &\quad (\text{where } s=r+k') \\ &= e(z)\frac{1}{D}\left(\prod_{\alpha\in\mathbb{Z}_n}\theta_\alpha(-z)\right)\sum_{s\in\mathbb{Z}_n}\frac{\theta_{j-i+k'+s(k-1)}(-z+\tau)}{\theta_{j-i+k'-s}(-z)\theta_{ks}(\tau)}\,(I\otimes T^{-k'})(x_{j+k'-s}\otimes x_{i+s}) \\ &= e(z)(I\otimes T^{-k'})\,R_\tau(z)\,(x_i\otimes x_{j+k'}) \\ &= e(z)(I\otimes T^{-k'})\,R_\tau(z)\,(I\otimes T^{k'})(x_i\otimes x_j). \end{aligned}$$

(1) This is an immediate consequence of (2.4) and (2.5).

(2) We observed in Proposition 2.4 that $R_\tau(\tau)$ is not an isomorphism. Hence $R_\tau(\tau+\zeta)$ is not an isomorphism either. The result now follows from Lemma 2.3 and Proposition 2.4(1).

(3) To show that $[R_\tau(z), S\otimes S]=0$ we make $V^{\otimes 2}$ a $\mathbb{Z}_n$-graded vector space by setting $\deg(x_\alpha\otimes x_\beta):=\alpha+\beta$. Since the action of $R_\tau(z)$ preserves degree and the homogeneous components are $S^{\otimes 2}$-eigenspaces, the actions of $R_\tau(z)$ and $S\otimes S$ on $V^{\otimes 2}$ commute with each other.

Write $c_{i,j,r}$ for the coefficient of $x_{j-r}\otimes x_{i+r}$ in $R_\tau(z)(x_i\otimes x_j)$. Since $c_{i+1,j+1,r}=c_{i,j,r}$,

$$\begin{aligned} R_\tau(z)(T\otimes T)(x_i\otimes x_j) &= R_\tau(z)(x_{i+1}\otimes x_{j+1}) \\ &= \sum_r c_{i+1,j+1,r}x_{j+1-r}\otimes x_{i+1+r} \\ &= (T\otimes T)\Big(\sum_r c_{i,j,r}\,x_{j-r}\otimes x_{i+r}\Big). \end{aligned}$$

Hence $R_\tau(z)(T\otimes T)=(T\otimes T)R_\tau(z)$, as claimed. □

An induction argument using (2.4) and (2.5) proves the following.

Corollary 2.7 *If $a,b\in\mathbb{Z}$ and $\zeta=\frac{a}{n}+\frac{b}{n}\eta$, then*

$$R_\tau(z+\zeta) \;=\; f(z,\zeta,\tau)(I\otimes T^bS^{ka})^{-1}R_\tau(z)(T^bS^{ka}\otimes I)$$

where $f(z,\zeta,\tau)=e(-bnz)e\big(b\tau+\frac{b+a(n-1)}{2}-\frac{b(n+b)}{2}\eta\big)$.

2.5 Theta functions with characteristics

The Jacobi theta function with respect to Λ is the holomorphic function

$$\vartheta(z\,|\,\eta) \;:=\; \sum_{m\in\mathbb{Z}} e\Big(mz+\frac{1}{2}m^2\eta\Big).$$

Clearly, $\vartheta(z+1\,|\,\eta)=\vartheta(z\,|\,\eta)$ and $\vartheta(z+\eta\,|\,\eta)=e(-z-\frac{1}{2}\eta)\,\vartheta(z\,|\,\eta)$.

For real numbers a and b the theta function with characteristics a and b is

$$\begin{aligned} \vartheta\begin{bmatrix} a\\ b\end{bmatrix}(z\,|\,\eta) &:= e\Big(a(z+b)+\frac{1}{2}a^2\eta\Big)\vartheta(z+a\eta+b\,|\,\eta) \\ &= \sum_{m\in\mathbb{Z}} e\Big((a+m)(z+b)+\frac{1}{2}(a+m)^2\eta\Big). \end{aligned}$$

This is the same as the definition at [46, (2.5)]. In [34, p. 10] and [54, (3.1)], $\vartheta\begin{bmatrix} a\\ b\end{bmatrix}(z\,|\,\eta)$ is denoted by $\vartheta_{a,b}(z,\eta)$. The papers [46] and [54] play a role in Sect. 3.

It is easy to see that

$$\vartheta\begin{bmatrix} a+1 \\ b \end{bmatrix}(z\,|\,\eta) \;=\; \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z\,|\,\eta) \quad\text{and}\quad \vartheta\begin{bmatrix} a \\ b+1 \end{bmatrix}(z\,|\,\eta) \;=\; e(a)\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z\,|\,\eta). \tag{2.10}$$

Since $\vartheta(z\,|\,\eta)=0$ if and only if $z\in\frac{1}{2}(1+\eta)+\Lambda$ by Lemma 2.5, $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z\,|\,\eta)=0$ if and only if

$$z \;\in\; \tfrac{1}{2}(1+\eta)\;-\;(a\eta+b)\;+\;\Lambda.$$

Proposition 2.8 *If $s,t\in\mathbb{Z}$, then* $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z+s\eta+t\,|\,\eta) \;=\; e\big(at-s(z+b)-\frac{1}{2}s^2\eta\big)\,\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z\,|\,\eta)$.

Proof We observed above that $\vartheta(z+1\,|\,\eta)=\vartheta(z\,|\,\eta)$ and $\vartheta(z+\eta\,|\,\eta)=e(-z-\frac{1}{2}\eta)\,\vartheta(z\,|\,\eta)$. An induction argument shows that $\vartheta(z+s\eta\,|\,\eta)=e(-sz-\frac{1}{2}s^2\eta)\vartheta(z\,|\,\eta)$ for all integers s and it follows from this that $\vartheta(z+s\eta+t\,|\,\eta)=e(-sz-\frac{1}{2}s^2\eta)\vartheta(z\,|\,\eta)$ for all integers s and t. Hence

$$\begin{aligned}
\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z+s\eta+t\,|\,\eta) &= e\big(a(z+s\eta+t+b)+\frac{1}{2}a^2\eta\big)\vartheta(z+s\eta+t+a\eta+b\,|\,\eta)\\
&= e\big(a(z+s\eta+t+b)+\frac{1}{2}a^2\eta\big)\\
&\qquad\cdot e(-s(z+a\eta+b)-\tfrac{1}{2}s^2\eta)\vartheta(z+a\eta+b\,|\,\eta)\\
&= e(a(s\eta+t))e(-s(z+a\eta+b)-\tfrac{1}{2}s^2\eta)\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z\,|\,\eta)\\
&= e\big(at-s(z+b)-\tfrac{1}{2}s^2\eta\big)\,\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z\,|\,\eta)
\end{aligned}$$

as claimed. □

The functions $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}$ are related to the θ_α's defined in [17, Prop. 2.6] in the following way.

Lemma 2.9 *There is a non-zero constant $c\in\mathbb{C}$, independent of α and z, such that*

$$\vartheta\begin{bmatrix} \frac{\alpha}{n}+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z\,|\,n\eta) \;=\; c^{-1}\,e(-\tfrac{1}{2}z)\,\theta_\alpha\big(\tfrac{z}{n}\,\big|\,\eta\big)$$

for all $\alpha\in\mathbb{Z}$ and all $z\in\mathbb{C}$.

Proof Since the functions $\theta_\alpha(z)$, $\alpha\in\mathbb{Z}$, are characterized up to a common non-zero scalar multiple by their quasi-periodicity properties

$$\theta_\alpha\big(z+\tfrac{1}{n}\big) \;=\; e\big(\tfrac{\alpha}{n}\big)\theta_\alpha(z) \quad\text{and}\quad \theta_\alpha\big(z+\tfrac{1}{n}\eta\big) \;=\; e\big(-z-\tfrac{1}{2n}+\tfrac{n-1}{2n}\eta\big)\theta_{\alpha+1}(z)$$

it suffices to show that the functions $e(\frac{1}{2}nz)\vartheta\begin{bmatrix}\frac{\alpha}{n}+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(nz\,|\,n\eta)$ have the same quasi-periodicity properties. The first equality in (2.10) implies that $e(\frac{1}{2}nz)\vartheta\begin{bmatrix}\frac{\alpha}{n}+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(nz\,|\,n\eta)$ depends only on the image of α in $\mathbb{Z}_n$. By [34, p. 10],

$$\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(z+1\,|\,\eta)=e(a)\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(z\,|\,\eta)\quad\text{and}\quad \vartheta\begin{bmatrix}a\\ b\end{bmatrix}(z+\tfrac{1}{n}\eta\,|\,\eta)$$
$$=e(-\tfrac{1}{n}z-\tfrac{b}{n}-\tfrac{1}{2n^2}\eta)\vartheta\begin{bmatrix}a+\frac{1}{n}\\ b\end{bmatrix}(z\,|\,\eta).$$

Therefore

$$\begin{aligned}e\big(\tfrac{1}{2}n(z+\tfrac{1}{n})\big)\vartheta\begin{bmatrix}\frac{\alpha}{n}+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\big(n(z+\tfrac{1}{n})\,|\,n\eta\big) &= e\big(\tfrac{1}{2}\big)e\big(\tfrac{1}{2}nz\big)e\big(\tfrac{\alpha}{n}+\tfrac{1}{2}\big)\vartheta\begin{bmatrix}\frac{\alpha}{n}+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(nz\,|\,n\eta)\\ &= e\big(\tfrac{\alpha}{n}\big)e\big(\tfrac{1}{2}nz\big)\vartheta\begin{bmatrix}\frac{\alpha}{n}+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(nz\,|\,n\eta)\end{aligned}$$

and

$$\begin{aligned}&e\big(\tfrac{1}{2}n(z+\tfrac{1}{n}\eta)\big)\vartheta\begin{bmatrix}\frac{\alpha}{n}+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\big(n(z+\tfrac{1}{n}\eta)\,|\,n\eta\big)\\ &\quad= e\big(\tfrac{1}{2}\eta\big)e\big(\tfrac{1}{2}nz\big)e\big(-z-\tfrac{1}{2n}-\tfrac{1}{2n}\eta\big)\vartheta\begin{bmatrix}\frac{\alpha+1}{n}+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(nz\,|\,n\eta)\\ &\quad= e\big(-z-\tfrac{1}{2n}+\tfrac{n-1}{2n}\eta\big)e\big(\tfrac{1}{2}nz\big)\vartheta\begin{bmatrix}\frac{\alpha+1}{n}+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(nz\,|\,n\eta).\end{aligned}$$

Thus, $e(\frac{1}{2}nz)\vartheta\begin{bmatrix}\frac{\alpha}{n}+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(nz\,|\,n\eta)$ has the same quasi-periodicity properties as $\theta_\alpha(z)$. □

3 Elliptic solutions to the quantum Yang–Baxter equation

In this section we assume $\tau\notin\frac{1}{n}\Lambda$ and set

$$\xi\;:=\;\tau\;+\;\tfrac{1}{2}(1+\eta).$$

Notice that $\xi\notin\frac{1}{2}(1+\eta)+\frac{1}{n}\Lambda$.

We will use the proof of Theorem 3.1 in [46] to show that $R(z)$ satisfies (QYBE2), i.e., to prove Theorem 3.7 below.[14]

[14] The $\sum$ symbol in [46, (3.11)] should be $\prod$, and the symbol γ_0 in that equation denotes a non-zero scalar.

Following [46, (3.2)], for each $(a, b) \in \mathbb{Z}^2$, we define

$$w_{(a,b)}(z) \;:=\; \frac{\vartheta\begin{bmatrix} a/n \\ b/n \end{bmatrix}(z+\xi \mid \eta)}{\vartheta\begin{bmatrix} a/n \\ b/n \end{bmatrix}(\xi \mid \eta)}. \tag{3.1}$$

Since $\xi \notin \frac{1}{2}(1+\eta)+\frac{1}{n}\Lambda$, the denominator of $w_{(a,b)}(z)$ is non-zero whence $w_{(a,b)}(z)$ is a holomorphic function of z. It follows from (2.10) that $w_{(a,b)}(z)$ depends only on the images of a and b in $\mathbb{Z}_n$. Thus, if $p = (a, b) \in \mathbb{Z}_n^2$, there is a well-defined holomorphic function $w_p(z)$.

Theorem 3.1 [54, Thm. 4.4] *For $p = (a, b) \in \mathbb{Z}_n^2$, let $I_p : V \to V$ be the linear map $I_p(x_i) = \omega^{ib} x_{i-a}$, where $\omega := e(\frac{1}{n})$. The operator*

$$S(z) \;:=\; \sum_{p \in \mathbb{Z}_n^2} w_p(z) I_p \otimes I_p^{-1} \tag{3.2}$$

satisfies (QYBE1).

We will refer to $S(z)$ as Belavin's elliptic solution to the QYBE.

For $n = 2$, $S(z)$ was discovered by R. Baxter who also proved Theorem 3.1 [4–6]. Theorem 3.1 was formulated and conjectured to be true for all n by Belavin [8], and was subsequently proved by Cherednik [13], Chudnovsky and Chudnovsky [11], and by Tracy [54].

We need a slightly more elaborate version of Theorem 3.1. In [46, 54], and in the other papers showing that $S(z)$ satisfies (QYBE1), the operators I_p are defined after first realizing V as an irreducible representation of the Heisenberg group

$$H_n \;:=\; \big\langle \gamma, \chi, \epsilon \mid \gamma^n = \chi^n = \epsilon^n = 1,\ [\gamma, \epsilon] = [\chi, \epsilon] = 1,\ [\gamma, \chi] = \epsilon \big\rangle. \tag{3.3}$$

of order n^3. The representation on V is via operators $\gamma \mapsto g \in \operatorname{End}(V)$ and $\chi \mapsto h \in \operatorname{End}(V)$ where $g \cdot x_i := \omega^i x_i$ and $h \cdot x_i := x_{i-1}$ and $\omega = e\big(\frac{1}{n}\big)$; the central element $\epsilon \in H_n$ now acts as multiplication by ω^{-1}, and we have $I_{(a,b)} = h^a g^b$.

We can now apply the discussion in Sect. 2.2.1 to the group algebra $S := \mathbb{C}H_n$ with $Z := \mathbb{C}\langle\epsilon\rangle$, the group algebra of the center $\langle\epsilon\rangle < H_n$.

Theorem 3.2 *For $p = (a, b) \in \mathbb{Z}_n^2$, let $J_p \in \mathbb{C}H_n$ be the element $\chi^a \gamma^b$. The family of operators*

$$S(z) \;:=\; \sum_{p \in \mathbb{Z}_n^2} w_p(z) J_p \otimes J_p^{-1} \quad \in \quad \mathbb{C}H_n \otimes_{\mathbb{C}\langle\epsilon\rangle} \mathbb{C}H_n \tag{3.4}$$

satisfies (QYBE1).

Proof This is essentially what the proof of [54, Thm. 4.4] shows; at no point does that proof use the specific realization of h and g as operators on V, beyond the fact that their commutator is a root of unity of order (dividing) n. One can therefore replace those operators with their abstract versions in (3.3), and ω with the generator ϵ of the center of H_n. $\square$

As a consequence, we have the following generalization of Theorem 3.1.

Corollary 3.3 *Let $\phi : H_n \to \mathrm{End}(V)$ be a representation such that ϵ acts on V as a scalar multiplication and define $I^{\phi}_{(a,b)} := \phi(J_{(a,b)}) = \phi(\chi^a\gamma^b)$. The operator*

$$S^{\phi}(z) := \sum_{p\in\mathbb{Z}_n^2} w_p(z) I^{\phi}_p \otimes (I^{\phi}_p)^{-1} \tag{3.5}$$

satisfies (QYBE1).

Recall that $P \in \mathrm{End}(V\otimes V)$ is defined by $P(x\otimes y) = y\otimes x$.

Proposition 3.4 *Let k' be the unique integer such that $kk' = 1$ in $\mathbb{Z}_n$ and $n > k' \geq 1$, and define*

$$S_k(z) := \sum_{(a,b)\in\mathbb{Z}_n^2} w_{(a,b)}(z) I_{(-k'a,b)} \otimes I^{-1}_{(-k'a,b)} \tag{3.6}$$

where $I_{(-k'a,b)} : V \to V$ is the operator $x_i \mapsto \omega^{ib}x_{i+k'a}$.[15] *Then*

$$S_k(-nz) = ne(\tfrac{1}{2}n(n+1)z)\, PR_{n,k,\tau}(z). \tag{3.7}$$

Proof We first prove the result for $k = -1$. When $k = -1$, $S_k(z)$ is the operator $S(z)$ in (3.2). The coefficient of $x_{i+r}\otimes x_{j-r}$ in $S(z)(x_i\otimes x_j)$ is

$$S(z)^{i,j}_{i+r,j-r} := \sum_{b\in\mathbb{Z}_n} w_{(-r,b)}(z)\,\omega^{-b(j-i-r)}.$$

This is the function $S^{r,j-i-r}(z, w, \ldots)$ in [46, (3.4)] (after replacing their a, b, α, and τ in [46, (3.4)] by our r, $j-i-r$, b, and η, respectively). If, in [46, (3.3)], we replace their τ and w by our η and $n\tau$, respectively, then their η becomes our ξ. The second variable w in $S^{r,j-i-r}(z, w, \ldots)$ becomes $n\tau$. We now have

$$\begin{aligned} S(z)^{i,j}_{i+r,j-r} &= f(z)\,\frac{\vartheta\begin{bmatrix}\frac{j-i-2r}{n}+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z+n\tau\,|\,n\eta)}{\vartheta\begin{bmatrix}-\frac{r}{n}+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(n\tau\,|\,n\eta)\cdot\vartheta\begin{bmatrix}\frac{j-i-r}{n}+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z\,|\,n\eta)} && \text{by [46, (3.10)]}\\ &= f(z)\,\frac{c^{-1}e\big(-\frac{1}{2}(z+n\tau)\big)\theta_{j-i-2r}(\frac{z}{n}+\tau)}{c^{-1}e\big(-\frac{1}{2}n\tau\big)\theta_{-r}(\tau)\cdot c^{-1}e\big(-\frac{1}{2}z\big)\theta_{j-i-r}(\frac{z}{n})} && \text{by Lemma 2.9}\\ &= cf(z)\,\frac{\theta_{j-i-2r}(\frac{z}{n}+\tau)}{\theta_{-r}(\tau)\theta_{j-i-r}(\frac{z}{n})} \end{aligned}$$

[15] The operator $S_k(z)$ is defined in the same way as $S(z)$ after replacing the generator h by the new generator $h^{-k'}$.

where c is the constant in Lemma 2.9 and

$$f(z) = ne\big(-\tfrac{1}{2}z\big)\, \vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z\,|\,n\eta) \prod_{\alpha=1}^{n-1} \left(\frac{\vartheta\begin{bmatrix} \frac{\alpha}{n}+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z\,|\,n\eta)}{\vartheta\begin{bmatrix} \frac{\alpha}{n}+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(0\,|\,n\eta)} \right) \qquad \text{by [46, (3.10)]}$$

$$= ne\big(-\tfrac{1}{2}z\big)c^{-1}e(-\tfrac{1}{2}z)\theta_0(\tfrac{z}{n}) \prod_{\alpha=1}^{n-1} \left(\frac{e(-\frac{1}{2}z)\theta_\alpha(\frac{z}{n})}{\theta_\alpha(0)} \right) \qquad \text{by Lemma 2.9.}$$

Therefore

$$\begin{aligned} S(-nz)^{i,j}_{i+r,j-r} &= c\, f(-nz)\, \frac{\theta_{j-i-2r}(-z+\tau)}{\theta_{-r}(\tau)\theta_{j-i-r}(-z)} \\ &= n\, e(\tfrac{1}{2}n(n+1)z)\, \frac{\theta_0(-z)\cdots\theta_{n-1}(-z)}{\theta_1(0)\cdots\theta_{n-1}(0)}\, \frac{\theta_{j-i-2r}(-z+\tau)}{\theta_{-r}(\tau)\theta_{j-i-r}(-z)}. \end{aligned}$$

The last expression is $ne(\frac{1}{2}n(n+1)z)$ times the coefficient of $x_{j-r}\otimes x_{i+r}$ in $R(z)(x_i \otimes x_j)$ when $k=-1$ (see (1.3)). Thus, the proposition is true for $k=-1$.

We now address the general case.

The coefficient of $x_{i+r}\otimes x_{j-r}$ in $S_k(-nz)(x_i\otimes x_j)$ is

$$S_k(-nz)^{i,j}_{i+r,j-r} := \sum_{b\in\mathbb{Z}_n} w_{(kr,b)}(-nz)\, \omega^{-b(j-i-r)}.$$

A suitable adjustment to the arguments in [46, §3] shows that

$$\begin{aligned} S_k(-nz)^{i,j}_{i+r,j-r} &= f(-nz)\, \frac{\vartheta\begin{bmatrix} \frac{j-i+r(k-1)}{n}+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(-nz+n\tau\,|\,n\eta)}{\vartheta\begin{bmatrix} \frac{kr}{n}+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(n\tau\,|\,n\eta)\cdot \vartheta\begin{bmatrix} \frac{j-i-r}{n}+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(-nz\,|\,n\eta)} \\ &= cf(-nz)\, \frac{\theta_{j-i+r(k-1)}(-z+\tau)}{\theta_{kr}(\tau)\theta_{j-i-r}(-z)} \\ &= n\, e\big(\tfrac{1}{2}n(n+1)z\big)\, \theta_0(-z) \left(\prod_{\alpha=1}^{n-1} \frac{\theta_\alpha(-z)}{\theta_\alpha(0)} \right) \frac{\theta_{j-i+r(k-1)}(-z+\tau)}{\theta_{kr}(\tau)\theta_{j-i-r}(-z)} \end{aligned}$$

where c is the constant in Lemma 2.9. Comparing this with the definition of $R(z)$ in (1.3) completes the proof. □

We now give another proof of Proposition 3.4 that does not rely on the calculations in [46].

Proposition 3.5 *Let k' and $S_k(z)$ be as in Proposition 3.4. Then*

$$S_k(-nz) = n\, e\big(\tfrac{1}{2}n(n+1)z\big)\, PR_{n,k,\tau}(z). \tag{3.8}$$

Proof[16] When the operators on the left- and right-hand sides of (3.8) are evaluated at $x_i \otimes x_j$ the result is a linear combination of $x_{i+r} \otimes x_{j-r}$, $r \in \mathbb{Z}_n$. Thus, to prove the proposition it suffices to show that the coefficients of all $x_{i+r} \otimes x_{j-r}$ in these evaluations are the same (for all i, j, r). This is what we will prove.

For the remainder of the proof we fix i, j, r and set $s := j - i - r$.

The coefficient of $x_{i+r} \otimes x_{j-r}$ in $S(-nz)(x_i \otimes x_j)$ is

$$F(z) \;:=\; \sum_{b \in \mathbb{Z}_n} w_{(kr,b)}(-nz)\,\omega^{-bs}.$$

The coefficient of $x_{i+r} \otimes x_{j-r}$ in $ne\big(\frac{1}{2}n(n+1)z\big)PR_{n,k,\tau}(z)(x_i \otimes x_j)$ is

$$G(z) \;:=\; n\,e\big(\tfrac{1}{2}n(n+1)z\big)\,\frac{\theta_0(-z)\cdots\theta_{n-1}(-z)}{\theta_1(0)\cdots\theta_{n-1}(0)}\,\frac{\theta_{s+kr}(-z+\tau)}{\theta_s(-z)\theta_{kr}(\tau)}.$$

We must show that $F(z) = G(z)$. To do this we will show that $F(z)$ and $G(z)$ have the same quasi-periodicity properties (with respect to the lattice $\frac{1}{n}\mathbb{Z} + \mathbb{Z}\eta$), the same zeros, and that $F\big(\frac{s}{n}\eta\big) = G\big(\frac{s}{n}\eta\big)$. It follows from the first two of these facts that $F(z)$ and $G(z)$ are scalar multiples of each other, and it then follows from the equality that this scalar is 1.

Quasi-periodicity properties of $G(z)$: Since $\theta_\alpha(z + \frac{1}{n}) = e\big(\frac{\alpha}{n}\big)\theta_\alpha(z)$,

$$G\big(z + \tfrac{1}{n}\big) \;=\; e\big(-\tfrac{kr}{n}\big)\,G(z).$$

Since $\theta_\alpha(-z-\eta) = e\big(n(-z-\eta) - \frac{1}{2}\big)\theta_\alpha(-z)$,

$$\prod_{\substack{\alpha=0\\ \alpha\neq s}}^{n-1} \theta_\alpha(-z-\eta) \;=\; e\big((n-1)n(-z-\eta) - \tfrac{n-1}{2}\big)\prod_{\substack{\alpha=0\\ \alpha\neq s}}^{n-1}\theta_\alpha(-z)$$

and

$$\theta_{s+kr}(-z-\eta+\tau) \;=\; e\big(n(-z-\eta+\tau) - \tfrac{1}{2}\big)\theta_{s+kr}(-z+\tau).$$

Therefore

$$\begin{aligned} G(z+\eta) \;&=\; e\big(\tfrac{1}{2}n(n+1)\eta\big)\,e\big(n^2(-z-\eta) + n\tau - \tfrac{n}{2}\big)\,G(z)\\ &=\; e\big(-n^2 z - \tfrac{1}{2}n^2\eta + \tfrac{n}{2}(1+\eta) + n\tau\big)\,G(z). \end{aligned}$$

Computation of $G\big(\frac{s}{n}\eta\big)$: By [17, Prop. 2.6(7)],

$$\theta_\alpha(z - \tfrac{s}{n}\eta) \;=\; e\big(sz + \tfrac{s}{2n} - \tfrac{sn+s^2}{2n}\eta\big)\theta_{\alpha-s}(z),$$

16 When $k = n-1$, $S_k(z)$ equals the operator $S(z)$ in Eq. (3.1) of Richey and Tracy's paper [46]. Some of the calculations in this proof are similar to those that produce Eqs. (3.3)–(3.12) in [46].

whence

$$\frac{\theta_{s+kr}(-\frac{s}{n}\eta+\tau)}{\theta_{kr}(\tau)} \;=\; e\big(s\tau+\tfrac{s}{2n}-\tfrac{sn+s^2}{2n}\eta\big)$$

and

$$\theta_\alpha(-\tfrac{s}{n}\eta) \;=\; e\big(\tfrac{s}{2n}-\tfrac{sn+s^2}{2n}\eta\big)\theta_{\alpha-s}(0).$$

The singularity of the function

$$\frac{\theta_0(-z)\cdots\theta_{n-1}(-z)}{\theta_s(-z)}$$

at $z=\frac{s}{n}\eta$ is removable and the value of the associated holomorphic function at $z=\frac{s}{n}\eta$ is

$$\begin{aligned}\prod_{\substack{\alpha=0\\ \alpha\neq s}}^{n-1}\theta_\alpha\big(-\tfrac{s}{n}\eta\big) \;&=\; \prod_{\substack{\alpha=0\\ \alpha\neq s}}^{n-1} e\big(\tfrac{s}{2n}-\tfrac{sn+s^2}{2n}\eta\big)\theta_{\alpha-s}(0).\\ &=\; e\big((n-1)\big(\tfrac{s}{2n}-\tfrac{sn+s^2}{2n}\eta\big)\big)\prod_{\alpha=1}^{n-1}\theta_\alpha(0).\end{aligned}$$

Therefore

$$\begin{aligned}G\big(\tfrac{s}{n}\eta\big) \;&=\; n\, e\big(\tfrac{1}{2}(n+1)s\eta\big)\, e\big((n-1)\big(\tfrac{s}{2n}-\tfrac{sn+s^2}{2n}\eta\big)\big)\, e(s\tau+\tfrac{s}{2n}-\tfrac{sn+s^2}{2n}\eta)\\ &=\; n\, e\big(\tfrac{s}{2}(\eta+1)-\tfrac{s^2}{2}\,\eta+s\tau\big).\end{aligned}$$

The zeros of $G(z)$: Since $\theta_\alpha(z)$ has zeros at points in $-\frac{\alpha}{n}\eta+\frac{1}{n}\mathbb{Z}+\mathbb{Z}\eta$, $G(z)$ has zeros at the points in the set

$$\big\{\tau+\tfrac{s+kr}{n}\eta\big\}\;\cup\;\big\{0,\tfrac{1}{n}\eta,\dots,\tfrac{n-1}{n}\eta\big\}\;-\;\big\{\tfrac{s}{n}\eta\big\}.$$

Quasi-periodicity properties of $F(z)$: Since $\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z+1\mid\eta)=e(a)\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z\mid\eta)$,

$$\begin{aligned}F(z+\tfrac{1}{n}) \;&=\; \sum_{b\in\mathbb{Z}_n} w_{(kr,b)}(-nz-1)\,\omega^{-bs}\\ &=\; \sum_{b\in\mathbb{Z}_n}\frac{\vartheta\begin{bmatrix}kr/n\\b/n\end{bmatrix}(-nz-1+\xi\mid\eta)}{\vartheta\begin{bmatrix}kr/n\\b/n\end{bmatrix}(\xi\mid\eta)}\,\omega^{-bs}\\ &=\; \sum_{b\in\mathbb{Z}_n} e\big(-\tfrac{kr}{n}\big)\frac{\vartheta\begin{bmatrix}kr/n\\b/n\end{bmatrix}(-nz+\xi\mid\eta)}{\vartheta\begin{bmatrix}kr/n\\b/n\end{bmatrix}(\xi\mid\eta)}\,\omega^{-bs}\end{aligned}$$

$$= \omega^{-kr} F(z).$$

If $p, q \in \mathbb{Z}$, then $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(z + p\eta + q \mid \eta) = e\big(aq - p(z+b) - \frac{p^2}{2}\eta\big)\, \vartheta \begin{bmatrix} a \\ b \end{bmatrix}(z \mid \eta)$ by Proposition 2.8, so

$$\begin{aligned}
F(z + \tfrac{m}{n}\eta) &= \sum_{b \in \mathbb{Z}_n} w_{(kr,b)}(-nz - m\eta)\, \omega^{-bs} \\
&= \sum_{b \in \mathbb{Z}_n} \frac{\vartheta \begin{bmatrix} kr/n \\ b/n \end{bmatrix}(-nz - m\eta + \xi \mid \eta)}{\vartheta \begin{bmatrix} kr/n \\ b/n \end{bmatrix}(\xi \mid \eta)}\, \omega^{-bs} \\
&= \sum_{b \in \mathbb{Z}_n} e\big(m(-nz + \xi + b/n) - \tfrac{m^2}{2}\eta\big) \frac{\vartheta \begin{bmatrix} kr/n \\ b/n \end{bmatrix}(-nz + \xi \mid \eta)}{\vartheta \begin{bmatrix} kr/n \\ b/n \end{bmatrix}(\xi \mid \eta)}\, \omega^{-bs} \\
&= e\big(-nmz + m\xi - \tfrac{m^2}{2}\eta\big) \sum_{b \in \mathbb{Z}_n} e\big(\tfrac{bm}{n}\big) \frac{\vartheta \begin{bmatrix} kr/n \\ b/n \end{bmatrix}(-nz + \xi \mid \eta)}{\vartheta \begin{bmatrix} kr/n \\ b/n \end{bmatrix}(\xi \mid \eta)}\, \omega^{-bs} \\
&= e\big(-nmz + m\xi - \tfrac{m^2}{2}\eta\big) \sum_{b \in \mathbb{Z}_n} \frac{\vartheta \begin{bmatrix} kr/n \\ b/n \end{bmatrix}(-nz + \xi \mid \eta)}{\vartheta \begin{bmatrix} kr/n \\ b/n \end{bmatrix}(\xi \mid \eta)}\, \omega^{-b(s-m)}.
\end{aligned}$$

Setting $m = n$, we see that

$$F(z + \eta) = e\big(-n^2 z + n\tau + \tfrac{n}{2}(1+\eta) - \tfrac{n^2}{2}\eta\big) F(z).$$

Thus, $F(z)$ and $G(z)$ have the same quasi-periodicity properties with respect to $\frac{1}{n}\mathbb{Z} + \mathbb{Z}\eta$.

The zeros of $F(z)$: It follows from the formulas for $F(z + \frac{1}{n})$ and $F(z+\eta)$ that $F(z)$ has n zeros[17] in each fundamental parallelogram for $\frac{1}{n}\mathbb{Z} + \mathbb{Z}\eta$, and the sum of these zeros is $\tau + \frac{kr}{n}\eta + \frac{1}{2}(n+1)\eta$ modulo $\frac{1}{n}\mathbb{Z} + \mathbb{Z}\eta$.

[17] Since $F(z + \frac{1}{n}) = e(-\frac{kr}{n})F(z)$ we may apply Lemma 2.5 to $F(z)$ with $\eta_1 = \frac{1}{n}$, $\eta_2 = \eta$, $a = 0$, $b = \frac{kr}{n}$, $c = n^2$, and $d = n(-\tau - \frac{1}{2}(1+\eta) + \frac{n}{2}\eta)$. Thus, $c\eta_1 - a\eta_2 = n$ and

$$\begin{aligned}
\tfrac{1}{2}(c\eta_1^2 - a\eta_2^2) + (c-a)\eta_1\eta_2 + b\eta_2 - d\eta_1 &= \tfrac{1}{2} + n\eta + \tfrac{kr}{n}\eta + \tau + \tfrac{1}{2}(1+\eta) - \tfrac{n}{2}\eta \\
&= \tau + \tfrac{kr}{n}\eta + \tfrac{1}{2}(n+1)\eta \quad \text{modulo } \tfrac{1}{n}\mathbb{Z} + \mathbb{Z}\eta.
\end{aligned}$$

Setting $z = 0$ above, we see that

$$F(\tfrac{m}{n}\eta) = e\big(m\xi - \tfrac{m^2}{2}\eta\big) \sum_{b\in\mathbb{Z}_n} \frac{\vartheta\begin{bmatrix} kr/n \\ b/n \end{bmatrix}(\xi \mid \eta)}{\vartheta\begin{bmatrix} kr/n \\ b/n \end{bmatrix}(\xi \mid \eta)}\, \omega^{-b(s-m)}$$

which is zero when $m \neq s$ in $\mathbb{Z}_n$. Thus, $F(z)$ vanishes at the $n-1$ points in the set

$$\big\{0, \tfrac{1}{n}\eta, \ldots, \tfrac{n-1}{n}\eta\big\} - \big\{\tfrac{s}{n}\eta\big\}. \tag{3.9}$$

These points belong to a single fundamental parallelogram for $\frac{1}{n}\mathbb{Z} + \mathbb{Z}\eta$ and their sum is $\frac{1}{2}(n-1)\eta - \frac{s}{n}\eta$. Hence there is another zero at

$$\tau + \tfrac{kr}{n}\eta + \tfrac{1}{2}(n+1)\eta - \tfrac{1}{2}(n-1)\eta + \tfrac{s}{n}\eta = \tau + \tfrac{s+kr}{n}\eta \quad \text{modulo } \tfrac{1}{n}\mathbb{Z} + \mathbb{Z}\eta.$$

Comparison of $F(z)$ and $G(z)$: Thus $F(z)$ and $G(z)$ have the same zeros. They also have the same quasi-periodicity properties with respect to $\frac{1}{n}\mathbb{Z} + \mathbb{Z}\eta$ so their ratio is a doubly periodic meromorphic function without zeros or poles, and therefore a constant. However, the formula for $F(\frac{s}{n}\eta)$ above gives

$$F(\tfrac{s}{n}\eta) = n\, e\big(s\xi - \tfrac{s^2}{2}\eta\big) = n\, e\big(s\tau + \tfrac{s}{2}(1+\eta) - \tfrac{s^2}{2}\eta\big)$$

which equals $G(\frac{s}{n}\eta)$ so that constant is 1. The proof is complete. $\square$

Corollary 3.6 *$Q_{n,k}(E,\tau)^{\mathrm{op}}$ is the quotient of TV by the ideal generated by the image of $S_k(-n\tau)$.*

Theorem 3.7 *The family of operators $R_\tau(z) : V^{\otimes 2} \to V^{\otimes 2}$ in (1.3) satisfies (QYBE2).*

Proof Since $S_k(z) = S^\phi(z)$ where

$$\phi : H_n \to \mathrm{End}(V)$$

is the representation

$$\chi \mapsto h^{-k'}, \ \gamma \mapsto g,$$

and ϵ acts as multiplication by $\omega^{k'}$, Corollary 3.3 tells us that $S_k(z)$ satisfies (QYBE1) and hence so does $S_k(-nz)$. It now follows from (3.7) that $PR(z)$ satisfies (QYBE1) and therefore $R(z)$ satisfies (QYBE2) by Proposition 2.1. $\square$

4 Families of linear operators

In Sects. 6, 7 and 9, we need to determine the zeros, and their multiplicities, of the determinants of certain linear operators $G_\tau(z)$, $G_\tau^+(z)$ and $H_\tau(z)$, on $V^{\otimes d}$. These operators, which are analytic functions of z, are compositions of operators of the form $I^{\otimes i-1} \otimes R(w) \otimes I^{d-i+1}$ for various w's and i's.

4.1 Dimensions of kernels and the multiplicity of the zeros of the determinant

The multiplicity of a point $p \in \mathbb{C}$ as a zero of a meromorphic function $f(z)$ is denoted

$$\mathrm{mult}_p\, f(z).$$

If $f(z)$ is identically zero in a neighborhood of p we set $\mathrm{mult}_p\, f(z) = \infty$.

The next result is used in Sects. 6, 7 and 9.

Lemma 4.1 *Let V be a finite-dimensional complex vector space and $A : D \to \mathrm{End}(V)$ a holomorphic map defined on a domain $D \subseteq \mathbb{C}$. For all $p \in D$,*

$$\mathrm{mult}_p(\det A(z)) \;\geq\; \mathrm{nullity}\, A(p).$$

Proof This is trivial if $\det A(z)$ is identically zero in a neighborhood of p, so we assume that p is an isolated zero of $\det A(z)$. Let $e_1, \ldots, e_\ell$ be an ordered basis for $\ker A(p)$, and extend it to an ordered basis $e_1, \ldots, e_\ell, \ldots$ for V. With respect to this basis, the entries of the matrix $A(z)$ are holomorphic functions whose first ℓ columns are divisible by $z - p$ (in the ring of functions holomorphic in a neighborhood of p). Hence $\det A(z)$ is divisible by $(z - p)^\ell$, finishing the proof. □

In Sect. 9, we need a stronger version of Lemma 4.1. First, some terminology. If $A : D \to \mathrm{End}(V)$ is a holomorphic map as above and p is a fixed point in D, we define

$$A_m(z) \;:=\; \frac{A(z)}{(z - p)^m}$$

with the convention that $A_{-1}(z) \equiv 0$.

Definition 4.2 The singularity partition of $A(z)$ at a point $p \in D$ is the tuple $\sigma_p(A) := (\lambda_0 \geq \lambda_1 \geq \cdots)$ of non-negative integers defined by

$$\lambda_m \;:=\; \text{the dimension of the kernel of } A_m(p)\big|_{\ker A_{m-1}(p)}.$$

The size of the singularity partition is the number $|\sigma_p(A)| := \sum_i \lambda_i$.

Remark 4.3 Since $\lambda_m = 0$ for $m \gg 0$, we ignore those zeros and regard the partition as a *finite* tuple.

The next result, which improves on Lemma 4.1, is used in the proof of Proposition 9.15.

Lemma 4.4 *Let V be a finite-dimensional complex vector space and $A : D \to \mathrm{End}(V)$ a holomorphic map for a domain $D \subseteq \mathbb{C}$. For all $p \in D$,*

$$\mathrm{mult}_p(\det A(z)) \;\geq\; |\sigma_p(A)|.$$

Proof Without loss of generality, we assume that $p = 0$. A direct sum decomposition $V = \ker A(0) \oplus V_0$ leads to a decomposition

$$A_0 = A : D \to \mathrm{Hom}(\ker A(0), V) \oplus \mathrm{Hom}(V_0, V) \tag{4.1}$$

whose left-hand component is a multiple of z, contributing z^{λ_0} to $\det A(z)$ where

$$\sigma_0(A) = (\lambda_0 \geq \lambda_1 \geq \cdots).$$

Dividing the left hand component of (4.1) by z and using a splitting $\ker A_0(0) = \ker A_1(0) \oplus V_1$, we obtain

$$A_1 : D \to \mathrm{Hom}(\ker A_1(0), V) \oplus \mathrm{Hom}(V_1, V),$$

with the left-hand component once more a multiple of z contributing z^{λ_1} to the determinant. Simply repeat the procedure until the singularity partition has been exhausted, noting that if at any point any of the functions

$$A_m(0)\big|_{\ker A_{m-1}(0)}$$

vanish identically then $\det A(z)$ does too, making the statement trivial. □

4.2 Theta operators

Assume $\Lambda = \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$ is a lattice in $\mathbb{C}$ such that $\mathrm{Im}(\eta_2/\eta_1) > 0$. A theta function of order r with respect to Λ is a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ satisfying the quasi-periodicity conditions $f(z+\eta_1) = e(-az-b)f(z)$ and $f(z+\eta_2) = e(-cz-d)f(z)$ in Lemma 2.5 for some constants a, b, c, d such that $c\eta_1 - a\eta_2 = r$. If f is not the zero function it has r zeros in every fundamental parallelogram for Λ. For example, the functions that belong to the space $\Theta_{r,c}(\Lambda)$, defined in [17, §2.1], are theta functions of order r. In particular, θ_α is a theta function of order n with respect to Λ and $\{\theta_0, \ldots, \theta_{n-1}\}$ is a basis for $\Theta_n(\Lambda) = \Theta_{n,\frac{n-1}{2}}(\Lambda)$ [17, Prop. 2.6].

Definition 4.5 A holomorphic map $A : \mathbb{C} \to \mathrm{End}(V)$ is a theta operator of order r with respect to Λ if $A(z+\eta_1) = e(-az-b)A(z)$ and $A(z+\eta_2) = e(-cz-d)A(z)$ for some constants a, b, c, d such that $c\eta_1 - a\eta_2 = r$. Equivalently, if $\langle v^*, A(-)v\rangle$ is a theta functions of order r having the same quasi-periodicity properties for all $v \in V$ and all $v^* \in V^*$, where V^* is the dual vector space of V. Equivalently, the matrix entries for $A(z)$ with respect to any basis for V are theta functions of order r having the same quasi-periodicity properties.

For example, $R(z)$ is a theta operator of order n^2 with respect to Λ because its matrix entries belong to $\Theta_{n^2, n\tau-n^2\eta}(\Lambda)$ (see [17, §2.1.2]).

If $A_i(z)$, $i = 1, 2$, are theta operators whose matrix entries belong to $\Theta_{r_i,c_i}(\Lambda)$, then $A_1(z)A_2(z)$ is a theta operator of order $r_1 + r_2$ because its matrix entries belong to $\Theta_{r_1+r_2, c_1+c_2}(\Lambda)$.

If $A(z)$ is a theta operator whose matrix entries belong to $\Theta_{r,c}(\Lambda)$ and $d \in \mathbb{C}$, then $A(z+d)$ is a theta operator of order r because its matrix entries belong to $\Theta_{r,c-rd}(\Lambda)$.

4.2.1 "Determinants"

We often encounter theta operators that preserve a fixed subspace $W \subseteq V$ or, more generally, map it to a fixed subspace $W' \subseteq V$ of the same dimension. If $A(z)(W) \subseteq W'$ for all $z \in \mathbb{C}$ and $\dim W = m = \dim W'$, then A induces a holomorphic function

$$\det_{W \to W'}(A(z)) := \bigwedge^m A(z) : \bigwedge^m W \longrightarrow \bigwedge^m W' \tag{4.2}$$

that is well-defined up to a non-zero scalar multiple (depending on a choice of bases for W and W'). We will often be interested in the location and multiplicities of the zeros of this function. That data does not depend on the choice of bases. These remarks, and the next result, apply to the theta operators $G_\tau(z)$ defined in Sect. 6.4.1 and $H_\tau(z)$ defined in Proposition 9.6.

Proposition 4.6 *Assume W and W' are subspaces of V of the same dimension. If $A : \mathbb{C} \to \mathrm{End}(V)$ is a theta operator of order N with respect to Λ such that $A(z)(W) \subseteq W'$ for all z, then* $\det_{W \to W'}(A(z))$

(1) is a theta function of order $N \dim W$ and
(2) has $N \dim W$ zeros in every fundamental parallelogram for Λ if it is not identically zero.

In particular, $\det A(z)$ *is a theta function of order $N \dim V$ with respect to Λ.*

Proof (1) Composing A with an automorphism of V that maps W' isomorphically onto W, we may as well assume $W = W'$, whence $\det_{W \to W}(A(z))$ becomes the usual determinant of $A(z)|_W$.

Choose an ordered basis for W and extend it to one for V. The operators $A(z)$ then have the shape

$$\begin{pmatrix} A_{11}(z) & A_{12}(z) \\ 0 & A_{22}(z) \end{pmatrix},$$

and $\det(A(z)|_W) = \det A_{11}(z)$. Since the summands in the usual expression for $\det A_{11}(z)$ are products of $\dim W$ theta functions of order N having the same quasi-periodicity properties, those summands, and therefore their sum, are theta functions of order $N \times \dim W$.

(2) A non-zero theta function of order r has r zeros in a fundamental parallelogram (Lemma 2.5). □

4.3 Families of kernels and images

Let $\mathrm{Grass}(d, W)$ denote the Grassmannian of d-dimensional subspaces of a finite-dimensional $\mathbb{C}$-vector space W.

In this subsection we consider algebraic or analytic morphisms f_i from a complex variety (algebraic or analytic,[18] though we typically specialize to the algebraic case) to either

- the Grassmannian $\mathrm{Grass}(d, W)$, or
- the space of linear maps $\mathrm{Hom}(W, W')$ for finite-dimensional $\mathbb{C}$-vector spaces W and W'.

We will be interested in the families of intersections (or sums) of $f_i(y)$ in the first case or $\ker f_i(y)$ (or $\mathrm{im}\, f_i(y)$) in the second.

Proposition 4.7 [47, Prop. 13.4] *Let $f : Y \to \mathrm{Hom}(W, W')$ be a morphism of algebraic or analytic varieties and write $r := \max\{\mathrm{rank}\, f(y) \mid y \in Y\}$.*

(1) The set $U := \{y \in Y \mid \mathrm{rank}\, f(y) = r\}$ is an open dense subset of Y.
(2) The map $\ker f : U \to \mathrm{Grass}(\dim W - r, W)$, $u \mapsto \ker f(u)$, is a morphism.
(3) The map $\mathrm{im}\, f : U \to \mathrm{Grass}(r, W')$, $u \mapsto \mathrm{im}\, f(u)$, is a morphism.

Proof As we said above, we focus on the algebraic situation.

(1) Since Y is a variety, and therefore irreducible, density follows from openness and non-emptiness. The latter holds by construction (since the maximal rank is, of course, achieved somewhere), so it remains to argue that $U \subseteq Y$ is open. This is clear from the fact that the condition rank $< r$ is expressible as a collection of algebraic equations (the vanishing of $r \times r$ minors).

Parts (2) and (3) are proved in [17, Prop. 3.17]. □

To further strengthen the connection between families of subspaces and families of operators, we have a kind of converse to parts (2) and (3) of Proposition 4.7.

If $F : X \to \mathrm{Hom}(W, W')$ is a function we write $\ker F$ and $\mathrm{im}\, F$ for the functions $x \mapsto \ker F(x)$ and $x \mapsto \mathrm{im}\, F(x)$, respectively.

Lemma 4.8 *Let $f : Y \to \mathrm{Grass}(d, W)$ be a morphism.*

(1) Let W' be a fixed vector space of dimension $\geq \dim W - d$. Then Y can be covered with open subvarieties U for which there are morphisms

$$F_U : U \to \mathrm{Hom}(W, W')$$

such that $f|_U = \ker F_U$.

(2) Let W'' be a fixed vector space of dimension $\geq d$. Then Y can be covered with open subvarieties U for which there are morphisms

$$G_U : U \to \mathrm{Hom}(W'', W)$$

such that $f|_U = \mathrm{im}\, G_U$.

[18] We adopt the following convention: a (complex) *algebraic variety* is a scheme over $\mathbb{C}$ that is reduced, irreducible, separated, and of finite type. An *analytic variety* is a (Hausdorff) analytic space that is reduced and irreducible.

Proof We prove (1); the dual argument shows (2) (using possibly different U's). Cover the Grassmannian $G := \mathrm{Grass}(d, W)$ with the affine open sets U_Γ described in [23, §3.2.2], consisting of the d-subspaces of W that intersect a fixed $(\dim W - d)$-dimensional subspace Γ trivially. Pulling these back to Y, we may as well assume the image of f lies entirely within a single open patch $U_\Gamma \subset G$ for a fixed Γ; i.e., we are now assuming that $f(y) \cap \Gamma = \{0\}$ for all $y \in Y$.

Now Γ is naturally isomorphic to $W/f(y)$ via the quotient map $W \to W/f(y)$ so, for each $y \in Y$, define $F(y) \in \mathrm{Hom}(W, W')$ to be the composition

$$W \to W/f(y) \cong \Gamma \to W'$$

where $\Gamma \to W'$ is some fixed embedding. This is the desired morphism $F : Y \to \mathrm{Hom}(W, W')$. □

Proposition 4.9 [47, Prop. 13.5] *Let f_i, $1 \le i \le r$, be morphisms $Y \to \mathrm{Grass}(d, W)$. Then,*

(1) The sets

$$U := \left\{ y \in Y \,\middle|\, \bigcap_i f_i(y) \textit{ has minimal dimension } e \right\} \quad \textit{and}$$
$$U' := \left\{ y \in Y \,\middle|\, \sum_i f_i(y) \textit{ has maximal dimension } e' \right\}$$

are open dense subsets of Y.

(2) The maps

$$\bigcap_i f_i : U \to \mathrm{Grass}(e, W), \quad y \mapsto \bigcap_i f_i(y), \quad \textit{and}$$
$$\sum_i f_i : U' \to \mathrm{Grass}(e', W), \quad y \mapsto \sum_i f_i(y),$$

are morphisms.

Proof Both (1) and (2) are true if they are true locally so, after Lemma 4.8, we can assume that there are morphisms $F_i : Y \to \mathrm{Hom}(W, W')$ and $G_i : Y \to \mathrm{Hom}(W'', W)$ such that $\ker F_i = f_i = \mathrm{im}\, G_i$. Now

$$\bigcap_i f_i = \ker\left(F_1 \oplus \cdots \oplus F_r : Y \to \mathrm{Hom}(W, W'^{\oplus r})\right),$$

and

$$\sum_i f_i = \mathrm{im}\left((G_1, \ldots, G_r) : Y \to \mathrm{Hom}(W''^{\oplus r}, W)\right),$$

so (1) and (2) follow from Proposition 4.7. □

4.3.1 Generically large and generically small functions

Let X be a topological space. We say that a function $f : X \to \mathbb{R}$ is generically large (resp., generically small) if $f^{-1}(c)$ is an open dense set for some $c \in \mathbb{R}$ and $f(x) < c$ (resp., $f(x) > c$) for all $x \notin f^{-1}(c)$.

In the setting of Proposition 4.7, the function rank $f(x)$ on X is generically large (*images are generically large*) and nullity $f(x)$ is generically small (*kernels are generically small*).

We also say that *sums are generically large* and *intersections are generically small* based on the facts in Proposition 4.9.

4.3.2. We will apply these ideas to situations where we have two functions $f_1, f_2 : X \to \mathbb{R}$ with the following properties: f_1 is generically small (e.g., nullity); f_2 is generically large (e.g., rank); $f_1(x) \leq f_2(x)$ for all x; $f_1(x) = f_2(x)$ on an open dense subset of X. It follows that $f_1(x) = f_2(x)$ for all x.

5 The determinant of $R_\tau(z)$ and the space of quadratic relations for $Q_{n,k}(E, \tau)$

In Sect. 6.4 we will show, for all $\tau \in (\mathbb{C} - \bigcup_{m \geq 1} \frac{1}{m}\Lambda) \cup \frac{1}{n}\Lambda$, that the Hilbert series for $Q_{n,k}(E, \tau)$ is the same as that for the polynomial ring $\mathbb{C}[x_0, \ldots, x_{n-1}]$. In this section we prove that, for all $\tau \in \mathbb{C} - (\frac{1}{2n}\Lambda - \frac{1}{n}\Lambda)$, the degree-two components of $Q_{n,k}(E, \tau)$ and $\mathbb{C}[x_0, \ldots, x_{n-1}]$ have the same dimension, namely $\binom{n+1}{2}$. Since $\mathrm{rel}_{n,k}(E, \tau)$ is the image of $R_\tau(\tau)$ (see Sect. 5.4.1), it suffices to show that the nullity of $R_\tau(\tau)$ is $\binom{n+1}{2}$. That is what we will do.

Since we showed that $Q_{n,k}(E, \tau)$ has the same Hilbert series as the polynomial ring on n variables when $\tau \in \frac{1}{n}\Lambda$ in [17, §5], we only have to prove the result for $\tau \in \mathbb{C} - \frac{1}{2n}\Lambda$.

5.1 The limit of $R_\tau(m\tau + \zeta)$ as $\tau \to 0$

As a function of z, $R_\tau(z)$ is not defined when $\tau \in \frac{1}{n}\Lambda$ (because some $\theta_{kr}(\tau)$ will then be 0). Nevertheless, as we observed in [17, §3.3.2], the holomorphic function $\tau \mapsto R_\tau(\tau)$ on $\mathbb{C} - \frac{1}{n}\Lambda$ extends in a unique way to a holomorphic function on $\mathbb{C}$. We need a slightly more general result here.

Lemma 5.1 *Fix $\zeta \in \frac{1}{n}\Lambda$ and $m \in \mathbb{Z}$. As a function of τ, the operator $R_\tau(m\tau + \zeta)$ is holomorphic on $\mathbb{C} - \frac{1}{n}\Lambda$, and its singularities at $\frac{1}{n}\Lambda$ are removable; i.e., $R_\tau(m\tau + \zeta)$ extends in a unique way to a holomorphic function of τ on the entire complex plane.*

Proof By definition, $R_\tau(m\tau + \zeta)(x_i \otimes x_j)$ is

$$\frac{1}{\theta_1(0)\cdots\theta_{n-1}(0)}\left(\prod_{i\in\mathbb{Z}_n}\theta_i(-m\tau-\zeta)\right)\sum_{r\in\mathbb{Z}_n}\frac{\theta_{j-i+r(k-1)}((1-m)\tau-\zeta)}{\theta_{j-i-r}(-m\tau-\zeta)\theta_{kr}(\tau)}x_{j-r}\otimes x_{i+r}. \tag{5.1}$$

Suppose both theta functions in the denominator of a summand are zero at τ. Then $\tau = -\frac{kr}{n}$ and $-m\tau - \zeta = -\frac{j-i-r}{n}$ modulo $\frac{1}{n}\mathbb{Z} + \mathbb{Z}\eta$, so $(1-m)\tau - \zeta = -\frac{j-i+(k-1)}{n}$ modulo $\frac{1}{n}\mathbb{Z} + \mathbb{Z}\eta$; thus the numerator in the same summand is also zero; each summand therefore has at most a pole of order one at τ and, since $m\tau + \zeta \in \frac{1}{n}\Lambda$, such a pole is canceled out by the order-one zero in the term before the Σ sign. □

Proposition 5.2 *For all $m \in \mathbb{Z}$,*

$$\lim_{\tau \to 0} R_\tau(m\tau) \;=\; \mathrm{sym}_m \tag{5.2}$$

where $\mathrm{sym}_m : V^{\otimes 2} \to V^{\otimes 2}$ *is the skew-symmetrization operator*

$$\mathrm{sym}_m(v \otimes v') \;:=\; v \otimes v' - m v' \otimes v.$$

Proof The limit as $\tau \to 0$ of $R_\tau(m\tau)$ is the same as the limit as $\tau \to 0$ of the operator

$$x_i \otimes x_j \;\mapsto\; \theta_0(-m\tau) \sum_{r \in \mathbb{Z}_n} \frac{\theta_{j-i+r(k-1)}((1-m)\tau)}{\theta_{j-i-r}(-m\tau)\theta_{kr}(\tau)} x_{j-r} \otimes x_{i+r}.$$

As $\tau \to 0$, multiplication by $\theta_0(-m\tau)$ annihilates those terms in the sum $\sum_{r \in \mathbb{Z}_n}$ whose denominators do not vanish at 0 so only the $r = 0$ and $r = j - i$ terms contribute to $\lim_{\tau \to 0} R_\tau(m\tau)$. Hence $\lim_{\tau \to 0} R_\tau(m\tau) = \lim_{\tau \to 0} X_\tau(m\tau)$ where

$$\begin{aligned} X_\tau(m\tau)(x_i \otimes x_j) := \theta_0(-m\tau) \\ \cdot \left(\frac{\theta_{k(j-i)}((1-m)\tau)}{\theta_0(-m\tau)\theta_{k(j-i)}(\tau)} x_i \otimes x_j + \frac{\theta_{j-i}((1-m)\tau)}{\theta_{j-i}(-m\tau)\theta_0(\tau)} x_j \otimes x_i \right) \end{aligned} \tag{5.3}$$

for $i \neq j$ and

$$X_\tau(m\tau)(x_i \otimes x_i) \;:=\; \theta_0(-m\tau) \cdot \frac{\theta_0((1-m)\tau)}{\theta_0(\tau)\theta_0(-m\tau)} x_i \otimes x_i. \tag{5.4}$$

Assume $i \neq j$. The two $\theta_{k(j-i)}(\,\cdot\,)$ factors in the left-hand term of (5.3) cancel out as $\tau \to 0$ because both converge to $\theta_{k(j-i)}(0)$ which is non-zero; the two $\theta_0(-m\tau)$ terms also cancel out so the first term on the right-hand side of (5.3) converges to $x_i \otimes x_j$. Since $\theta_0(\tau)$ vanishes at $\tau = 0$ with multiplicity 1,

$$\theta_0(\tau) = a_1\tau + a_2\tau^2 + \cdots$$

with $a_1 \neq 0$. Hence the ratio $\frac{\theta_0(-m\tau)}{\theta_0(\tau)}$ converges to $-m$. The two $\theta_{j-i}(\,\cdot\,)$ factors in the right-hand term of (5.3) cancel out as $\tau \to 0$ because both converge to $\theta_{j-i}(0)$ which is non-zero. The second term on the right-hand side of (5.3) therefore converges to $-mx_j \otimes x_i$. Thus, $X_\tau(m\tau)(x_i \otimes x_j)$ converges to $x_i \otimes x_j - mx_j \otimes x_i$ as $\tau \to 0$.

Assume $i = j$. Similar analysis shows that $X_\tau(m\tau)(x_i \otimes x_i)$ converges to $(1-m)x_i \otimes x_i$ as $\tau \to 0$.

Combining the cases $i \neq j$ and $i = j$ gives the uniform result

$$X_\tau(m\tau)(x_i \otimes x_j) \longrightarrow x_i \otimes x_j - m x_j \otimes x_i$$

as $\tau \to 0$. The proof is now complete. □

For each $\zeta \in \frac{1}{n}\Lambda$, we define

$$R_+(\zeta) := \lim_{\tau \to 0} R_\tau(\tau + \zeta) \quad \text{and} \quad R_-(\zeta) := \lim_{\tau \to 0} R_\tau(-\tau + \zeta). \tag{5.5}$$

Corollary 5.3 *When $\zeta = 0$, the operators in (5.5) are $R_\pm(0)(x_i \otimes x_j) = x_i \otimes x_j \mp x_j \otimes x_i$.*

Proof Apply Proposition 5.2 with $m = 1$ for $R_+(0)$ and with $m = -1$ for $R_-(0)$. □

Lemma 5.4 *For all $\zeta \in \frac{1}{n}\Lambda$,*

$$\begin{aligned}
\ker R_+(\zeta) &= \operatorname{im} R_-(-\zeta), \\
\operatorname{im} R_+(\zeta) &= \ker R_-(-\zeta), \\
\operatorname{nullity} R_+(\zeta) &= \tbinom{n+1}{2}, \\
\operatorname{nullity} R_-(\zeta) &= \tbinom{n}{2}.
\end{aligned}$$

Proof By Proposition 5.2,

$$R_+(0)(x_i \otimes x_j) = x_i \otimes x_j - x_j \otimes x_i \quad \text{and} \quad R_-(0)(x_i \otimes x_j) = x_i \otimes x_j + x_j \otimes x_i.$$

Therefore $\operatorname{im} R_+(0) = \ker R_-(0)$, $\operatorname{im} R_-(0) = \ker R_+(0)$, $\operatorname{nullity} R_+(0) = \binom{n+1}{2}$, and $\operatorname{nullity} R_-(0) = \binom{n}{2}$. Thus, the lemma is true when $\zeta = 0$.

We now consider an arbitrary $\zeta = \frac{a}{n} + \frac{b}{n}\eta$. The argument in the next two paragraphs will show that $R_+(\zeta)R_-(-\zeta) = R_-(-\zeta)R_+(\zeta) = 0$.

Define $C := T^b S^{ka}$, $C' := T^{-b} S^{-ka}$, $A := C \otimes I$, $A' = C' \otimes I$, $B := I \otimes C^{-1}$, and $B' := I \otimes C'^{-1}$. By Corollary 2.7,

$$\begin{aligned}
R_+(\zeta) &= \lim_{\tau \to 0} f(\tau, \zeta, \tau) B R_\tau(\tau) A \;=\; f(0, \zeta, 0) B \big(\lim_{\tau \to 0} R_\tau(\tau)\big) A \\
&= f(0, \zeta, 0) B R_+(0) A
\end{aligned}$$

and

$$\begin{aligned}
R_-(-\zeta) &= \lim_{\tau \to 0} f(-\tau, -\zeta, \tau) B' R_\tau(-\tau) A' \;=\; f(0, -\zeta, 0) B' \big(\lim_{\tau \to 0} R_\tau(-\tau)\big) A' \\
&= f(0, -\zeta, 0) B' R_-(0) A'
\end{aligned}$$

where $f(z,\zeta,\tau)$ does not vanish at any point in $\mathbb{C}\times\frac{1}{n}\Lambda\times\mathbb{C}$. To show that $R_+(\zeta)R_-(-\zeta)=0$ it suffices to show that $R_+(0)AB'R_-(0)=0$; i.e., that $R_+(0)(C\otimes C'^{-1})R_-(0)=0$.

Let $\varepsilon=e\big(\frac{1}{n}\big)$. Since $ST=\varepsilon TS$, $C'^{-1}=S^{ka}T^b=\varepsilon^{kab}T^bS^{ka}=\varepsilon^{kab}C$. Therefore

$$\begin{aligned}R_+(0)(C\otimes C'^{-1})R_-(0)&=R_+(0)(I\otimes\varepsilon^{kab})(C\otimes C)R_-(0)\\&=(C\otimes C)R_+(0)(I\otimes\varepsilon^{kab})R_-(0);\end{aligned}$$

we used the fact that $S^{\otimes 2}$ and $T^{\otimes 2}$ commute with $R_\tau(z)$ and hence with limits of $R_\tau(m\tau+\zeta)$. Since $I\otimes\varepsilon^{kab}$ is a scalar multiple of the identity, it also commutes with $R_\pm(0)$. Hence $R_+(0)(C\otimes C'^{-1})R_-(0)=0$. This completes the proof that $R_+(\zeta)R_-(-\zeta)=0$. A similar argument shows that $R_-(-\zeta)R_+(\zeta)=0$.

Since A and B are invertible operators and $f(0,\zeta,0)$ is a non-zero scalar, rank $R_+(\zeta)=$ rank $R_+(0)$. Similarly, rank $R_-(\zeta)=$ rank $R_-(0)$. The lemma therefore holds for all $\zeta\in\frac{1}{n}\Lambda$. □

5.2 The ranks of $R_\tau(\tau)$ and $R_\tau(-\tau)$

Lemma 5.5 *Assume $\tau\in\mathbb{C}-\frac{1}{n}\Lambda$. For all $\zeta\in\frac{1}{n}\Lambda$,*

$$\operatorname{im}R_\tau(\tau+\zeta)\subseteq\ker R_\tau(-\tau-\zeta),\quad \operatorname{im}R_\tau(-\tau-\zeta)\subseteq\ker R_\tau(\tau+\zeta),\tag{5.6}$$

$$\operatorname{nullity}R_\tau(\tau+\zeta)\ \ge\ \binom{n+1}{2},\quad \textit{and}\quad \operatorname{nullity}R_\tau(-\tau-\zeta))\ \ge\ \binom{n}{2}.\tag{5.7}$$

Proof (5.6) is an immediate consequence of Proposition 2.6(2).

By Lemma 5.1, $R_\tau(\tau+\zeta)$ and $R_\tau(-\tau-\zeta)$ extend to holomorphic functions of τ on the whole complex plane. By Lemma 5.4,

$$\operatorname{nullity}\big(\lim_{\tau\to 0}R_\tau(\tau+\zeta)\big)\ =\ \binom{n+1}{2}\ =\ \operatorname{rank}\big(\lim_{\tau\to 0}R_\tau(-\tau-\zeta)\big).$$

Since nullity is generically small and rank is generically large it follows that

$$\operatorname{nullity}R_\tau(\tau+\zeta)\ \le\ \binom{n+1}{2}\ \le\ \operatorname{rank}R_\tau(-\tau-\zeta)$$

for generic τ. But $\operatorname{im}R_\tau(-\tau-\zeta)\subseteq\ker R_\tau(\tau+\zeta)$ so

$$\operatorname{nullity}R_\tau(\tau+\zeta)\ =\ \binom{n+1}{2}\ =\ \operatorname{rank}R_\tau(-\tau-\zeta)$$

for generic τ. However, nullity is generically small and rank is generically large so

$$\operatorname{nullity}R_\tau(\tau+\zeta)\ \ge\ \binom{n+1}{2}\ \ge\ \operatorname{rank}R_\tau(-\tau-\zeta)$$

for *all* $\tau \in \mathbb{C}$. This proves the first inequality in (5.7). A similar argument proves the second.[19] □

5.3 The determinant of $R_\tau(z)$

Since $R(z)$ is a theta operator of order n^2 with respect to Λ, $\det R(z)$ is a theta function of order n^4 with respect to Λ (Proposition 4.6). The next result improves on this.

Proposition 5.6 *If $\tau \notin \frac{1}{n}\Lambda$, then $\det R_\tau(z)$ is a theta function of order n^2 with respect to $\frac{1}{n}\Lambda$,*

(1) $\det R_\tau\big(z+\frac{1}{n}\big) = \det R_\tau(z)$,
(2) $\det R_\tau\big(z+\frac{1}{n}\eta\big) = b(z)^{n^2}\det R_\tau(z)$, *and*
(3) $\det R_\tau(z)$ *has n^2 zeros in every fundamental parallelogram for $\frac{1}{n}\Lambda$.*

Proof By (2.4) and (2.5), $R_\tau\big(z+\frac{1}{n}\big) = (-1)^{n-1}(I\otimes S^{-k})\,R_\tau(z)\,(S^k\otimes I)$ and $R_\tau\big(z+\frac{1}{n}\eta\big) = b(z)(I\otimes T^{-1})\,R_\tau(z)\,(T\otimes I)$. Since $\dim(V^{\otimes 2}) = n^2$ implies $\det(-I\otimes I)^{n-1} = 1$, it follows that $\det R_\tau\big(z+\frac{1}{n}\big) = \det R_\tau(z)$ and $\det R_\tau\big(z+\frac{1}{n}\eta\big) = b(z)^{n^2}\det R_\tau(z)$.

(3) We note that $b(z)^{n^2} = e(-n^3z - B)$ as functions of z for a suitable $B \in \mathbb{C}$. Applying Lemma 2.5 to the function $f(z) = \det R(z)$, with $\eta_1 = \frac{1}{n}$, $\eta_2 = \frac{1}{n}\eta$, $a = b = 0$, $c = n^3$, and $d = B$, we see that the number of zeros (counted with multiplicity) in each fundamental parallelogram for $\frac{1}{n}\Lambda$ is $c\eta_1 - a\eta_2 = n^3 \times \frac{1}{n} = n^2$. □

Theorem 5.7 *If $\tau \notin \frac{1}{2n}\Lambda$, then*

(1) $R_\tau(z)$ *is an isomorphism if and only if $z \notin \pm\tau + \frac{1}{n}\Lambda$;*
(2) $\operatorname{im} R_\tau(\tau+\zeta) = \ker R_\tau(-\tau-\zeta)$ *and* $\operatorname{im} R_\tau(-\tau-\zeta) = \ker R_\tau(\tau+\zeta)$ *for all $\zeta \in \frac{1}{n}\Lambda$;*
(3) if $p \in \mathbb{C}$, then

$$\operatorname{mult}_p(\det R_\tau(z)) \;=\; \operatorname{nullity} R_\tau(p) \;=\; \begin{cases} \binom{n+1}{2} & \textit{if} \quad p \in \tau + \frac{1}{n}\Lambda, \\ \binom{n}{2} & \textit{if} \quad p \in -\tau + \frac{1}{n}\Lambda, \\ 0 & \textit{otherwise;} \end{cases}$$

If $\tau \in \frac{1}{2n}\Lambda - \frac{1}{n}\Lambda$, then $\operatorname{nullity} R_\tau(\tau) = \operatorname{nullity} R_\tau(-\tau) \ge \binom{n+1}{2}$.

Proof If $\tau \in \mathbb{C} - \frac{1}{2n}\Lambda$, then τ and $-\tau$ are distinct points modulo $\frac{1}{n}\Lambda$.
Lemmas 5.5 and 4.1 imply that

$$\operatorname{mult}_\tau(\det R_\tau(z)) \;\ge\; \operatorname{nullity} R_\tau(\tau) \;\ge\; \binom{n+1}{2}$$

19 Alternatively, the second inequality in (5.7) follows from the first because

$$\operatorname{nullity} R_\tau(-\tau-\zeta) \;=\; \dim V^{\otimes 2} - \operatorname{rank} R_\tau(-\tau-\zeta) \;\ge\; n^2 - \binom{n+1}{2} \;=\; \binom{n}{2}.$$

and

$$\mathrm{mult}_{-\tau}(\det R_\tau(z)) \;\geq\; \text{nullity}\, R_\tau(-\tau) \;\geq\; \binom{n}{2}.$$

But $\det R_\tau(z)$ has exactly $n^2 = \binom{n+1}{2} + \binom{n}{2}$ zeros in every fundamental parallelogram for $\frac{1}{n}\Lambda$ so the displayed inequalities are equalities, which then implies (1) and (2). Since the rank of $R_\tau(z)$ is the same as that of $R_\tau(z+\zeta)$ (Proposition 2.6(1)), part (3) follows from (2) and the inclusions in (5.6).

Finally, if $\tau \in \frac{1}{2n}\Lambda - \frac{1}{n}\Lambda$, then $-\tau = \tau + \zeta$ for some $\zeta \in \frac{1}{n}\Lambda$ so $R_\tau(\tau)$ and $R_\tau(-\tau)$ have the same nullity by Proposition 2.6(1), and this is $\geq \binom{n+1}{2}$ by (5.7). □

Proposition 5.8 *For all* $\tau \in \mathbb{C} - \frac{1}{n}\Lambda$,

$$\det R_\tau(z) \;=\; \left(\prod_{\alpha\in\mathbb{Z}_n} \frac{\theta_\alpha(-z-\tau)}{\theta_\alpha(-\tau)}\right)^{\frac{n(n-1)}{2}} \left(\prod_{\alpha\in\mathbb{Z}_n} \frac{\theta_\alpha(-z+\tau)}{\theta_\alpha(\tau)}\right)^{\frac{n(n+1)}{2}}. \tag{5.8}$$

In particular, $\det R_\tau(z)$ *does not depend on* k.

Proof We first prove this under the assumption that $\tau \notin \frac{1}{2n}\Lambda$.

Let $D(z)$ denote the function on the right-hand side of (5.8). Both $\det R(z)$ and $D(z)$ are holomorphic functions of z. It follows from Proposition 5.6 and [17, Prop. 2.6] that $\det R(z)$ and $D(z)$ have the same quasi-periodicity properties with respect to the lattice $\frac{1}{n}\Lambda$. It follows from Theorem 5.7(3) and [17, Prop. 2.6] that $\det R(z)$ and $D(z)$ have the same zeros with the same multiplicities; the ratio $(\det R(z))/D(z)$ is therefore a meromorphic function on the elliptic curve $\mathbb{C}/\frac{1}{n}\Lambda$ with neither zeros nor poles, and therefore a constant. Since $R(0) = I \otimes I$ by Proposition 2.4, $\det R(0) = 1 = D(0)$. So the constant is 1.

The result is therefore true when $\tau \notin \frac{1}{2n}\Lambda$. By continuity, it also holds when $\tau \in \frac{1}{2n}\Lambda - \frac{1}{n}\Lambda$. □

Corollary 5.9 *Let* $\tau \in \mathbb{C} - \frac{1}{n}\Lambda$. *Then* $R_\tau(z)$ *is an isomorphism if and only if* $z \notin \pm\tau + \frac{1}{n}\Lambda$.

Proof Proposition 5.8 tells us that $\det R_\tau(z) = 0$ if and only if $z \in \pm\tau + \frac{1}{n}\Lambda$. Thus the result follows. □

5.4 The space of quadratic relations for $Q_{n,k}(E,\tau)$ has dimension $\binom{n}{2}$

This subsection completes the proof that the dimension of $\mathrm{rel}_{n,k}(E,\tau)$ is the same as that of the space of quadratic relations for the polynomial ring on n variables when $\tau \notin (\frac{1}{2n}\Lambda - \frac{1}{n}\Lambda)$.

5.4.1 Definition of $\mathrm{rel}_{n,k}(E, \tau)$ for all $\tau \in \mathbb{C}$

Since Lemma 5.1 ensures that $R_\tau(\tau)$ extends in a unique way to a holomorphic function on $\mathbb{C}$, we can now define, for all $\tau \in \mathbb{C}$,

$$\begin{aligned} \mathrm{rel}_{n,k}(E,\tau) &:= \mathrm{im}\, R_\tau(\tau), \\ Q_{n,k}(E,\tau) &:= \frac{TV}{(\mathrm{rel}_{n,k}(E,\tau))} \end{aligned}$$

where, if $\tau \in \frac{1}{n}\Lambda$, then $R_\tau(\tau)$ is not defined so it should be regarded as the limit $R_+(\tau)$ in (5.5).[20]

5.4.2 Remark

In their first paper [37], Odesskii and Feigin define $\mathrm{rel}_{n,k}(E,\tau)$ to be $\mathrm{im}\, R_\tau(\tau)$; however, in his survey [36, p. 1145], Odesskii defines $\mathrm{rel}_{n,k}(E,\tau)$ to be $\ker R_\tau(-\tau)$. By Theorem 5.7(2), $\mathrm{im}\, R_\tau(\tau) = \ker R_\tau(-\tau)$ if $\tau \notin \frac{1}{2n}\Lambda$ (we do not know whether this equality holds when $\tau \in \frac{1}{2n}\Lambda - \frac{1}{n}\Lambda$).

Theorem 5.10 *For all $\tau \in \mathbb{C} - (\frac{1}{2n}\Lambda - \frac{1}{n}\Lambda)$, $\dim \mathrm{rel}_{n,k}(E,\tau) = \binom{n}{2}$.*

Proof We proved this in [17, §5] for $\tau \in \frac{1}{n}\Lambda$. Suppose $\tau \in \mathbb{C} - \frac{1}{2n}\Lambda$. Then $\mathrm{rel}_{n,k}(E,\tau)$ is the image of $R_\tau(\tau)$ and, by Theorem 5.7(3),

$$\mathrm{rank}\, R_\tau(\tau) = \dim V^{\otimes 2} - \mathrm{nullity}\, R_\tau(\tau) = n^2 - \binom{n+1}{2} = \binom{n}{2}.$$

□

5.5 Some twists of $Q_{n,k}(E,\tau)$

By Proposition 2.6(1), $\mathrm{rank}\, R_\tau(\tau+\zeta) = \mathrm{rank}\, R_\tau(\tau)$ for all $\zeta \in \frac{1}{n}\Lambda$. Since $Q_{n,k}(E,\tau) = TV/(\mathrm{im}\, R_\tau(\tau))$ it is reasonable to ask whether the algebras $TV/(\mathrm{im}\, R_\tau(\tau+\zeta))$ are, perhaps, "new" elliptic algebras.

Proposition 5.11 *Assume $a, b \in \mathbb{Z}$. Let $\tau \in \mathbb{C}$, $\zeta = \frac{a}{n} + \frac{b}{n}\eta$, and $\phi := S^{-ka}T^{-b} \in$ $\mathrm{GL}(V)$.*

(1) The map ϕ extends in a unique way to an algebra automorphism of TV that descends to an automorphism of $Q_{n,k}(E,\tau)$ (that we also denote by ϕ) and

$$\frac{TV}{(\mathrm{im}\, R_\tau(\tau+\zeta))} \cong Q_{n,k}(E,\tau)^\phi$$

where $Q_{n,k}(E,\tau)^\phi$ is the twist of $Q_{n,k}(E,\tau)$ in the sense of [17, §4.1].

[20] There are some other ways to extend the definition of $\mathrm{rel}_{n,k}(E,\tau)$ to all $\tau \in \mathbb{C}$; see [17, §3.3] for more discussion.

(2) *The categories of* $\mathbb{Z}$*-graded left modules over* $TV/(\operatorname{im} R_\tau(\tau+\zeta))$ *and* $Q_{n,k}(E,\tau)$ *are equivalent.*
(3) *If* $k+1$ *is a unit in* $\mathbb{Z}_n$*, then* $TV/(\operatorname{im} R_\tau(\tau+\zeta))$ *is isomorphic to* $Q_{n,k}(E,\tau+\zeta')$ *where* $\zeta' = -\frac{kc}{n} - \frac{kd}{n}\eta$ *and* c *and* d *are arbitrary integers such that* $(k+1)c = a$ *and* $(k+1)d = b$ *in* $\mathbb{Z}_n$*.*

Proof We use the convention that, for $\tau \in \frac{1}{n}\Lambda$, $R_\tau(\tau)$ and $R_\tau(\tau+\zeta)$ mean $R_+(\tau)$ and $R_+(\tau+\zeta)$, respectively.

(1) Let $\mathfrak{a}$ denote the ideal in TV generated by $\operatorname{im} R_\tau(\tau)$; thus $Q_{n,k}(E,\tau) = TV/\mathfrak{a}$. Certainly ϕ extends in a unique way to an algebra automorphism of TV that we continue to denote by ϕ. By Proposition 2.6(3), the operator $\phi\otimes\phi$ on $V\otimes V$ commutes with $R_\tau(\tau)$ so $\operatorname{im} R_\tau(\tau)$ is stable under the action of $\phi\otimes\phi$. The automorphism ϕ of TV therefore preserves $\mathfrak{a}$, i.e., $\phi(\mathfrak{a}) = \mathfrak{a}$, and so descends to an automorphism of $Q_{n,k}(E,\tau)$ (that we continue to denote by ϕ). We define the algebra $Q_{n,k}(E,\tau)^\phi$ as in [17, §4].

By [17, Lem. 4.1], $(TV/\mathfrak{a})^\phi \cong TV/\phi'(\mathfrak{a})$ where $\phi'(\mathfrak{a})$ denotes the image of $\mathfrak{a}$ under the action of the linear map ϕ' that is $I\otimes\phi\otimes\cdots\otimes\phi^{d-1}$ on each $V^{\otimes d}$. Since our $\mathfrak{a}$ is generated by its degree-two component, $\mathfrak{a}_2$, $\phi'(\mathfrak{a})$ is generated by $(I\otimes\phi)(\mathfrak{a}_2)$. But

$$\begin{aligned}(I\otimes\phi)(\mathfrak{a}_2) &= (I\otimes\phi)R_\tau(\tau)(V^{\otimes 2})\\ &= (I\otimes\phi)R_\tau(\tau)(\phi^{-1}\otimes I)(V^{\otimes 2})\\ &= \operatorname{im} R_\tau(\tau+\zeta) \qquad \text{by Corollary 2.7.}\end{aligned}$$

Hence

$$\frac{TV}{(\operatorname{im} R_\tau(\tau+\zeta))} = \frac{TV}{(I\otimes\phi)(\mathfrak{a}_2)} = \frac{TV}{\phi'(\mathfrak{a})} \cong Q_{n,k}(E,\tau)^\phi.$$

(2) This is an immediate consequence of [3, Cor. 8.5].

(3) Assume $k+1$ is a unit in $\mathbb{Z}_n$ (equivalently, $k'+1$ is a unit in $\mathbb{Z}_n$).

Let $c,d \in \mathbb{Z}$ be such that $c = (k+1)^{-1}a$ and $d = (k+1)^{-1}b$ in $\mathbb{Z}_n$. Define $\sigma := S^{-kc}T^{-d}$. Then $\phi = \sigma^{k+1}$, and if $\zeta' = -\frac{kc}{n} - \frac{kd}{n}$,

$$Q_{n,k}(E,\tau)^\phi = Q_{n,k}(E,\tau)^{\sigma^{k+1}} \cong Q_{n,k}(E,\tau+\zeta')$$

by [17, Thm. 4.3]. □

5.6 The relation between $R_\tau(z)$ and Odesskii's R-matrix

Odesskii defines a family of operators that we will denote by $R^{\mathrm{Od}}(z)$ at [36, p. 1145].[21] The relation between the two operators is

[21] What we are calling $R^{\mathrm{Od}}(z)$ is obtained from Odesskii's formula for $R_{n,k}(\mathcal{E},\eta)(u-v)$ by identifying $x_\alpha(u)$ and $x_\alpha(v)$ with x_α and setting $v=0$ and $u=z$.

$$R_\tau(z) \;=\; \left(\prod_{\alpha\in\mathbb{Z}_n} \frac{\theta_\alpha(-z+\tau)}{\theta_\alpha(\tau)}\right) R^{\mathrm{Od}}(z). \tag{5.9}$$

We chose to work with $R_\tau(z)$ rather than $R^{\mathrm{Od}}(z)$ for several reasons. For example, the $\theta_\alpha(-z+\tau)$ terms in (5.9) cancel the poles that occur in the formula for $R^{\mathrm{Od}}(z)$ when $z \in \tau + \frac{1}{n}\Lambda$. Those poles mean that $R^{\mathrm{Od}}(z)$ does not satisfy (QYBE2) for all values of u and v. Nevertheless, away from those poles, $R^{\mathrm{Od}}(z)$ satisfies (QYBE2) if and only if $R_\tau(z)$ does. Our other reasons for preferring $R_\tau(z)$ included the following: we wanted $R_\tau(0)$ to be the identity operator (Proposition 2.4), and also wanted the equality $\lim_{\tau\to 0} R_\tau(m\tau) = \mathrm{sym}_m$ in Proposition 5.2 and the equalities in Proposition 6.4; other choices of $R_\tau(z)$ would only give those equalities up to a non-zero scalar multiple.

Proposition 5.12 *The determinant of the operator $R^{\mathrm{Od}}(z)$ defined at* [36, p. 1145] *is*

$$(-1)^{\frac{n^2(n-1)}{2}} e\left(\frac{n^3(n-1)}{2}\,\tau\right)\left(\frac{\theta_0(-z-\tau)\cdots\theta_{n-1}(-z-\tau)}{\theta_0(-z+\tau)\cdots\theta_{n-1}(-z+\tau)}\right)^{\frac{n(n-1)}{2}}. \tag{5.10}$$

Proof It follows from (5.8) that

$$\det R^{\mathrm{Od}}(z) = \left(\prod_{\alpha\in\mathbb{Z}_n} \frac{\theta_\alpha(\tau)}{\theta_\alpha(-z+\tau)}\right)^{n^2} \left(\prod_{\alpha\in\mathbb{Z}_n} \frac{\theta_\alpha(-z-\tau)}{\theta_\alpha(-\tau)}\right)^{\frac{n(n-1)}{2}} \cdot \left(\prod_{\alpha\in\mathbb{Z}_n} \frac{\theta_\alpha(-z+\tau)}{\theta_\alpha(\tau)}\right)^{\frac{n(n+1)}{2}}.$$

We leave to the reader the pleasant task of showing this equals the expression in (5.10). □

The formula for $\det R^{\mathrm{Od}}(z)$ at [36, p. 1145] omits the term $(-1)^{\frac{n^2(n-1)}{2}} e\left(\frac{n^3(n-1)}{2}\,\tau\right)$.

6 The Hilbert series of $Q_{n,k}(E,\tau)$

This section shows $Q_{n,k}(E,\tau)$ has the same Hilbert series as the polynomial ring SV for all $\tau \in (\mathbb{C} - \bigcup_{m\ge 1} \frac{1}{m}\Lambda) \cup \frac{1}{n}\Lambda$.

6.1 Introduction

Section 5 showed that, for all $\tau \in \mathbb{C} - (\frac{1}{2n}\Lambda - \frac{1}{n}\Lambda)$, the degree-two component of $Q_{n,k}(E,\tau)$ has the "right" dimension, namely $\binom{n+1}{2}$, by showing in Theorem 5.7(2) that $Q_{n,k}(E,\tau)$'s space of quadratic relations, which is, by definition, the image of $R_\tau(\tau)$, is equal to the kernel of $R_\tau(-\tau)$. A similar idea is used in this section: we show, for all $\tau \in \mathbb{C} - \bigcup_{m=1}^{d} \frac{1}{mn}\Lambda$, that the space of degree-d relations for $Q_{n,k}(E,\tau)$, which is $\sum_{s+t+2=d} V^{\otimes s} \otimes \operatorname{im} R_\tau(\tau) \otimes V^{\otimes t}$, is the kernel of the operator $F_d(-\tau) : V^{\otimes d} \to V^{\otimes d}$ defined in (6.2) below.

The analogy with the polynomial ring SV is helpful. The space of degree-d relations for SV, i.e., the kernel of the natural map $V^{\otimes d} \to S^d V$, is the kernel of the symmetrization operator $\sum_{\sigma \in S_d} \sigma$ acting in the natural way on $V^{\otimes d}$. Proposition 6.4 shows that $\lim_{\tau \to 0} F_d(-\tau)$ is a non-zero scalar multiple of $\sum_{\sigma \in S_d} \sigma$.

6.2 The linear operators T_d, $S_{d\to 1}$, $S^{\mathrm{rev}}_{d\to 1}$, F_d, etc., on $V^{\otimes d}$

The results in this subsection apply to any family of linear operators $R(z) : V^{\otimes 2} \to V^{\otimes 2}$, $z \in \mathbb{C}$, satisfying (QYBE2), which is the parametrized braid relation

$$R(u)_{12} R(u+v)_{23} R(v)_{12} \;=\; R(v)_{23} R(u+v)_{12} R(u)_{23}$$

for all $u, v \in \mathbb{C}$.

If $t_p, \ldots, t_q \in \mathbb{C}$ we will write $\Sigma_p^q := t_p + \cdots + t_q$. Let i, j, and d be positive integers with $i \le j \le d$. We will use the following operators on $V^{\otimes d}$:

$$\begin{aligned}
S_{i\to j}(t_i, \ldots, t_{j-1}) &:= R\big(\Sigma_i^{j-1}\big)_{i,i+1} \cdots R\big(\Sigma_q^{j-1}\big)_{q,q+1} \cdots R\big(t_{j-1}\big)_{j-1,j},\\
S_{j\to i}(t_{j-1}, \ldots, t_i) &:= R\big(\Sigma_i^{j-1}\big)_{j-1,j} \cdots R\big(\Sigma_i^{q}\big)_{q,q+1} \cdots R\big(t_i\big)_{i,i+1},\\
S^{\mathrm{rev}}_{i\to j}(t_i, \ldots, t_{j-1}) &:= R\big(t_i\big)_{i,i+1} \cdots R\big(\Sigma_i^{q}\big)_{q,q+1} \cdots R\big(\Sigma_i^{j-1}\big)_{j-1,j},\\
S^{\mathrm{rev}}_{j\to i}(t_{j-1}, \ldots, t_i) &:= R\big(t_{j-1}\big)_{j-1,j} \cdots R\big(\Sigma_q^{j-1}\big)_{q,q+1} \cdots R\big(\Sigma_i^{j-1}\big)_{i,i+1},
\end{aligned}$$

with the convention that these are the identity operators when $i = j$. For example,

$$S_{1\to 4}(t_1, t_2, t_3) \;=\; R(t_1 + t_2 + t_3)_{1,2} R(t_2 + t_3)_{2,3} R(t_3)_{3,4}.$$

Each of these is a theta operator (Definition 4.5) of order $(i-j)n^2$ with respect to Λ. We also define

$$T_d(z_1, \ldots, z_{d-1}) \;:=\; S_{2\to 1}(z_1) S_{3\to 1}(z_1, z_2) \cdots S_{d\to 1}(z_1, \ldots, z_{d-1}). \tag{6.1}$$

and

$$F_d(z) \;:=\; T_d(z, \ldots, z). \tag{6.2}$$

When $d = 0, 1$ we declare that these operators are the identity.

The choice of labeling for the arguments in the S-operators allows the elegant factorizations

$$\begin{aligned}
S_{i\to k}(t_i, \ldots, t_{k-1}) &= S_{i\to j}(t_i, \ldots, t_{j-2}, \Sigma_{j-1}^{k-1}) S_{j\to k}(t_j, \ldots, t_{k-1}),\\
S_{k\to i}(t_{k-1}, \ldots, t_i) &= S_{k\to j}\big(t_{k-1}, \ldots, t_{j+1}, \Sigma_i^{j}\big) S_{j\to i}\big(t_{j-1}, \ldots, t_i\big),
\end{aligned}$$

$$
\begin{aligned}
S^{\mathrm{rev}}_{i\to k}(t_i,\ldots,t_{k-1}) &= S^{\mathrm{rev}}_{i\to j}(t_i,\ldots,t_{j-1})S^{\mathrm{rev}}_{j\to k}(\Sigma_i^j, t_{j+1},\ldots,t_{k-1}),\\
S^{\mathrm{rev}}_{k\to i}(t_{k-1},\ldots,t_i) &= S^{\mathrm{rev}}_{k\to j}(t_{k-1},\ldots,t_j)S^{\mathrm{rev}}_{j\to i}\big(\Sigma_{j-1}^{k-1}, t_{j-2},\ldots,t_i\big),
\end{aligned}
$$

when $1\le i\le j\le k$. These factorizations make no use of the Yang–Baxter equation. We also note that

$$
S_{i\to j}(t_i,\ldots,t_{j-1}) = S^{\mathrm{rev}}_{i\to j}\big(\Sigma_i^{j-1}, -t_i,\ldots,-t_{j-2}\big), \tag{6.3}
$$

$$
S_{j\to i}(t_{j-1},\ldots,t_i) = S^{\mathrm{rev}}_{j\to i}\big(\Sigma_i^{j-1}, -t_{j-1},\ldots,-t_{i+1}\big). \tag{6.4}
$$

These equalities make no use of the Yang–Baxter equation either.

Recall the notation in Sect. 2.1: T^L_{d-1} (resp., T^R_{d-1}) denotes T_{d-1} applied to the left-most (resp., right-most) $d-1$ tensorands of $V^{\otimes d}$. The identities in the next result will be used repeatedly in subsequent sections.

Lemma 6.1 *For all $d\ge 2$ and all $z_1,\ldots,z_{d-1}\in\mathbb{C}$,*

$$
\begin{aligned}
T_d(z_1,\ldots,z_{d-1}) &= T^L_{d-1}(z_1,\ldots,z_{d-2})S_{d\to 1}(z_1,\ldots,z_{d-1})\\
&= T^R_{d-1}(z_2,\ldots,z_{d-1})S_{1\to d}(z_{d-1},\ldots,z_1)\\
&= S^{\mathrm{rev}}_{1\to d}(z_1,\ldots,z_{d-1})T^L_{d-1}(z_2,\ldots,z_{d-1})\\
&= S^{\mathrm{rev}}_{d\to 1}(z_{d-1},\ldots,z_1)T^R_{d-1}(z_1,\ldots,z_{d-2}).
\end{aligned}
$$

Proof (1) The equality $T_d(z_1,\ldots,z_{d-1}) = T^L_{d-1}(\cdots)S_{d\to 1}(\cdots)$ follows at once from the definition of $T_d(z_1,\ldots,z_{d-1})$.

(2) We will now show that $T_d(z_1,\ldots,z_{d-1})S^{\mathrm{rev}}_{1\to d}(z_1,\ldots,z_{d-1})T^L_{d-1}(z_2,\ldots,z_{d-1})$. We first replace each factor $S_{i\to 1}(z_1,\ldots,z_{i-1})$ in (6.1) by

$$
R(z_1+\cdots+z_{i-1})_{i-1,i}S_{i-1\to 1}(z_2,\ldots,z_{i-1}).
$$

Since $S_{i-1\to 1}(z_2,\ldots,z_{i-1})$ acts on the left-most $i-1$ tensorands of $V^{\otimes d}$, it commutes with $R(z_1+\cdots+z_{j-1})_{j-1,j}$ whenever $i<j$. Therefore

$$
\begin{aligned}
T_d(z_1,\ldots,z_{d-1}) &= R(z_1)_{12}\cdot R(z_1+z_2)_{23}S_{2\to 1}(z_2)\\
&\qquad\cdots R(z_1+\cdots+z_{d-1})_{d-1,d}S_{d-1\to 1}(z_2,\ldots,z_{d-1})\\
&= R(z_1)_{12}\cdots R(z_1+\cdots+z_{d-1})_{d-1,d}\\
&\qquad\cdot S_{2\to 1}(z_2)\cdots S_{d-1\to 1}(z_2,\ldots,z_{d-1})\\
&= S^{\mathrm{rev}}_{1\to d}(z_1,\ldots,z_{d-1})T^L_{d-1}(z_2,\ldots,z_{d-1}).
\end{aligned}
$$

(3) We now prove $T_d(z_1,\ldots,z_{d-1}) = T^R_{d-1}(\cdots)S_{1\to d}(\cdots)$ by induction on d. The case $d=2$ is trivial, so we assume the equality holds for all integers $\le d$ and prove it for $d+1$.

By (1) and the induction hypothesis,

$$
T_{d+1}(z_1,\ldots,z_d) = T^L_d(z_1,\ldots,z_{d-1})S_{d+1\to 1}(z_1,\ldots,z_d)
$$

$$= T^M_{d-1}(z_2, \dots, z_{d-1}) S_{1\to d}(z_{d-1}, \dots, z_1) S_{d+1\to 1}(z_1, \dots, z_d) \tag{6.5}$$

where T^M_{d-1} denotes T_{d-1} applied to the middle $d-1$ tensorands of $V^{\otimes(d+1)}$.

The product $S_{1\to d}(z_{d-1}, \dots, z_1) S_{d+1\to 1}(z_1, \dots, z_d)$ equals

$$\begin{aligned} &S_{1\to d}(z_{d-1}, \dots, z_1) R(z_1 + \dots + z_d)_{d,d+1} S_{d\to 1}(z_2, \dots, z_d) \\ &= S_{1\to d-1}(z_{d-1}, \dots, z_3, z_1 + z_2)\, R(z_1)_{d-1,d}\, R\left(\sum_{i=1}^d z_i\right)_{d,d+1} R\left(\sum_{i=2}^d z_i\right)_{d-1,d} \\ &\quad \cdot S_{d-1\to 1}(z_3, \dots, z_d). \end{aligned} \tag{6.6}$$

By (QYBE2), the product of the three R's in the middle equals

$$R(z_2 + \dots + z_d)_{d,d+1} R(z_1 + \dots + z_d)_{d-1,d} R(z_1)_{d,d+1}. \tag{6.7}$$

Since $R_{d,d+1}$ commutes with $S_{1\to d-1}$ and $S_{d-1\to 1}$, (6.6) equals

$$\begin{aligned} &R(z_2 + \dots + z_d)_{d,d+1} S_{1\to d-1}(z_{d-1}, \dots, z_3, z_1 + z_2) R(z_1 + \dots + z_d)_{d-1,d} \\ &\cdot S_{d-1\to 1}(z_3, \dots, z_d) R(z_1)_{d,d+1}. \end{aligned}$$

The product of the three factors in the middle has the same form as the second line of (6.6), so we can repeat the procedure. Eventually we see that (6.6) equals $S_{d+1\to 2}(z_2, \dots, z_d) S_{1\to d+1}(z_d, \dots, z_1)$. Hence (6.5) equals

$$\begin{aligned} &T^M_{d-1}(z_2, \dots, z_{d-1}) S_{d+1\to 2}(z_2, \dots, z_d) S_{1\to d+1}(z_d, \dots, z_1) \\ &\quad = T^R_d(z_2, \dots, z_d) S_{1\to d+1}(z_d, \dots, z_1) \end{aligned}$$

as desired.

(4) Mimic the argument in (2) to show $T^R_{d-1}(\cdots) S_{1\to d}(\cdots) = S^{\mathrm{rev}}_{d\to 1}(\cdots) T^R_{d-1}(\cdots)$. ◻

Lemma 6.2 *For every* $1 \le i \le d-1$,

$$T_d(z_1, \dots, z_{d-1}) = R(z_i)_{i,i+1} Q_i = Q'_i R(z_{d-i})_{i,i+1}$$

where Q_i *and* Q'_i *are products of terms* $R(w)_{j,j+1}$ *for various integers* $1 \le j \le d-1$ *and partial sums* w *of* $z_1, \dots, z_{d-1}$.

Proof We argue by induction on d. The case $d = 2$ is trivial. Assuming the claim up to and including $d-1$, the first equality follows from the first and second identities in Lemma 6.1, and second equality follows from the third and fourth identities in Lemma 6.1. ◻

Proposition 6.3 *Assume* $1 \le i \le d-1$.

(1) $F_d(-\tau) = Q R(-\tau)_{i,i+1} = R(-\tau)_{i,i+1} Q'$ *for some* Q *and* Q' *that are products of terms* $R(m\tau)_{j,j+1}$ *for various integers* $1 \le j \le d-1$ *and* m.

(2) $F_d(\tau) = QR(\tau)_{i,i+1} = R(\tau)_{i,i+1}Q'$ *for some* Q *and* Q' *that are products of terms* $R(m\tau)_{j,j+1}$ *for various integers* $1 \le j \le d-1$ *and* m.

(3) If $R(\tau)R(-\tau) = 0$, *then*

$$\operatorname{im} F_d(-\tau) \subseteq \bigcap_{s+t+2=d} V^{\otimes s} \otimes \ker R(\tau) \otimes V^{\otimes t}.$$

(4) If $R(-\tau)R(\tau) = 0$, *then*

$$\ker F_d(-\tau) \supseteq \sum_{s+t+2=d} V^{\otimes s} \otimes \operatorname{im} R(\tau) \otimes V^{\otimes t}.$$

Proof Both (1) and (2) are special cases of Lemma 6.2.

(3) Since $F_d(-\tau) = R(-\tau)_{i,i+1}Q'$, $R(\tau)_{i,i+1}F_d(-\tau) = 0$; i.e., $\operatorname{im} F_d(-\tau) \subseteq \ker R(\tau)_{i,i+1}$. But

$$\ker R(\tau)_{i,i+1} = V^{\otimes(i-1)} \otimes \ker R(\tau) \otimes V^{\otimes(d-i-1)}$$

so the result follows.

(4) Since $F_d(-\tau) = QR(-\tau)_{i,i+1}$, $F_d(-\tau)R(\tau)_{i,i+1} = 0$; i.e., $\ker F_d(-\tau) \supseteq \operatorname{im} R(\tau)_{i,i+1}$. □

In proving that $Q_{n,k}(E,\tau)$ has the "right" Hilbert series we will show that the inclusions in parts (3) and (4) of Proposition 6.3 are equalities when $R(z)$ is the operator in (1.3).

6.3 The limit $F_d(\pm\tau)$ as $\tau \to 0$

To determine the Hilbert series of $Q_{n,k}(E,\tau)$ and $Q_{n,k}(E,\tau)^!$ we must understand the limits of $F_d(\pm\tau)$ as $\tau \to 0$.

Proposition 6.4 *If* d *is an integer* ≥ 2, *then*

$$\lim_{\tau\to 0} F_d(-\tau) = \prod_{m=1}^{d-1} m! \cdot \sum_{\sigma \in S_d} \sigma \quad \textit{and}$$
$$\lim_{\tau\to 0} F_d(\tau) = \prod_{m=1}^{d-1} m! \cdot \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma)\sigma$$

where the symmetric group S_d *acts on* $V^{\otimes d}$ *by permuting tensorands.*

Proof We prove the proposition for $F_d(-\tau)$. The argument for $F_d(\tau)$ is virtually identical.

We argue by induction on d. Since $F_2(z) = R(z)$, the $d = 2$ case is a consequence of Corollary 5.3. Assume $d \ge 3$. Since $F_d(-\tau) = T_d(-\tau, \ldots, -\tau)$, it follows from

Lemma 6.1 and Proposition 5.2 that

$$\begin{aligned}\lim_{\tau\to 0} F_d(-\tau) &= \lim_{\tau\to 0} R(-\tau)_{12}\cdots R(-(d-1)\tau)_{d-1,d}F^L_{d-1}(-\tau)\\ &= \mathrm{sym}^{1,2}_{-1}\cdots \mathrm{sym}^{d-1,d}_{-(d-1)}\cdot \lim_{\tau\to 0} F^L_{d-1}(-\tau)\end{aligned} \tag{6.8}$$

where the superscripts on the operators sym_m indicate which tensorands of $V^{\otimes d}$ they apply to. By the induction hypothesis, the factor $\lim_{\tau\to 0} F^L_{d-1}(-\tau)$ in (6.8) is the map

$$v_1\cdots v_{d-1}v_d \;\mapsto\; \prod_{m=1}^{d-2} m!\cdot \sum_{\sigma\in S_{d-1}} v_{\sigma(1)}\cdots v_{\sigma(d-1)}v_d \tag{6.9}$$

where we have suppressed tensor symbols for readability.

Claim: the product of the sym factors in (6.8) sends the sum part of (6.9) to

$$(d-1)!\sum_{\sigma\in S_d} v_{\sigma(1)}\cdots v_{\sigma(d)}.$$

To see this, we will count, for every t and every $\sigma'\in S_{d-1}$, the number of times the term

$$v_{\sigma'(1)}\cdots v_{\sigma'(t-1)}\, v_d\, v_{\sigma'(t)}\cdots v_{\sigma'(d-1)} \tag{6.10}$$

appears in

$$\mathrm{sym}^{1,2}_{-1}\cdots \mathrm{sym}^{d-1,d}_{-(d-1)}\left(\sum_{\sigma\in S_{d-1}} v_{\sigma(1)}\cdots v_{\sigma(d-1)}v_d\right). \tag{6.11}$$

If we write $\mathrm{sym}_{-m} = I + mP$ using $I = \mathrm{id}_V$ and the flip $P\in \mathrm{End}(V\otimes V)$, then $\mathrm{sym}^{m,m+1}_{-m} = I^{\otimes d} + mP_{m,m+1}$, where $P_{m,m+1}$ interchanges the m^{th} and $(m+1)^{\text{th}}$ tensorands in $V^{\otimes d}$. Since v_d starts out on the extreme right in (6.11) and crosses leftward past $d-t$ tensorands to reach its position in (6.10), the latter must be the result of applying the summands

$$(d-1)P_{d-1,d},\ (d-2)P_{d-2,d-1},\ \ldots,\ tP_{t,t+1}$$

of the rightmost $(d-t)$ sym operators in (6.11). These yield a factor of

$$(d-1)\cdots(t+1)t \;=\; \frac{(d-1)!}{(t-1)!} \tag{6.12}$$

in front of (6.10), and we must show that the remaining sym operators

$$\mathrm{sym}^{1,2}_{-1}\ldots \mathrm{sym}^{t-1,t}_{-(t-1)} \tag{6.13}$$

contribute the missing $(t-1)!$ factor to produce the requisite $(d-1)!$. Only the $I^{\otimes d}$ term in

$$\mathrm{sym}^{t-1,t}_{-(t-1)} \;=\; I^{\otimes d} \;+\; (t-1)P_{t-1,t}$$

can contribute a (6.10) term because $P_{t-1,t}$ would slide v_d further to the left. On the other hand, each of the product of the remaining sym factors in the product (6.13), namely

$$\mathrm{sym}_{-1}^{1,2} \cdots \mathrm{sym}_{-(t-2)}^{t-2,t-1},$$

contributes terms of the form (6.10) via both its $I^{\otimes d}$ and P terms, whence each

$$\mathrm{sym}_{-j}^{j,j+1} \;=\; I^{\otimes d} + jP_{j,j+1}, \quad 1 \le j \le t-2,$$

supplements (6.12) with an additional factor of $1 + j$ scaling the (6.10) term. The overall coefficient of (6.10) is therefore

$$\frac{(d-1)!}{(t-1)!} \cdot (t-1) \cdots 2 \;=\; (d-1)!$$

as claimed. □

6.4 Computation of the Hilbert series for $Q_{n,k}(E, \tau)$

Theorem 6.5 *Let $d \ge 2$. Assume $\tau \in \mathbb{C} - \bigcup_{m=1}^{d} \frac{1}{mn}\Lambda$.*

(1) We have

$$\ker F_d(-\tau) \;=\; \sum_{s+t+2=d} V^{\otimes s} \otimes \operatorname{im} R_\tau(\tau) \otimes V^{\otimes t} \tag{6.14}$$

and its dimension is the same as the dimension of the space of degree-d relations of a polynomial algebra in n variables, namely $n^d - \binom{n+d-1}{d}$.

(2) We have

$$\operatorname{im} F_d(-\tau) \;=\; \bigcap_{s+t+2=d} V^{\otimes s} \otimes \ker R_\tau(\tau) \otimes V^{\otimes t} \tag{6.15}$$

and its dimension is the same as the dimension of the degree-d component of a polynomial algebra in n variables, namely $\binom{n+d-1}{d}$.

We assume that $\tau \in \mathbb{C} - \bigcup_{m=1}^{d} \frac{1}{mn}\Lambda$ until the end of the proof, i.e., $\pm\tau, \pm 2\tau, \ldots, \pm d\tau \notin \frac{1}{n}\Lambda$.

When $d = 2$, Theorem 6.5 follows from Theorem 5.7 since $F_2(\tau) = R(\tau)$. We now argue by induction on d.

6.4.1 The operators $G_\tau(z)$

Taking $(z_1, \ldots, z_{d-1}) = (z, -\tau, \ldots, -\tau)$ in Lemma 6.1 we see that

$$\begin{aligned} T_d(z, -\tau, \ldots, -\tau) \;&=\; S_{1\to d}^{\mathrm{rev}}(z, -\tau, \ldots, -\tau) T_{d-1}^{L}(-\tau, \ldots, -\tau) \\ &=\; T_{d-1}^{R}(-\tau, \ldots, -\tau) S_{1\to d}(-\tau, , \ldots, -\tau, z) \end{aligned}$$

which implies that the operator

$$S^{\mathrm{rev}}_{1\to d}(z,-\tau,\ldots,-\tau) \;=\; R(z)_{12}R(z-\tau)_{23}\cdots R(z-(d-2)\tau)_{d-1,d} \tag{6.16}$$

on $V^{\otimes d}$ restricts to a linear map

$$G_\tau(z) : \operatorname{im} F_{d-1}(-\tau)\otimes V \;\longrightarrow\; V\otimes \operatorname{im} F_{d-1}(-\tau). \tag{6.17}$$

Since $T_d(z,-\tau,\ldots,-\tau) = S^{\mathrm{rev}}_{1\to d}(z,-\tau,\ldots,-\tau)T^L_{d-1}(-\tau,\ldots,-\tau)$,

$$\operatorname{im} G_\tau(z) \;=\; \operatorname{im} T_d(z,-\tau,\ldots,-\tau). \tag{6.18}$$

In particular, $\operatorname{im} G_\tau(-\tau) = \operatorname{im} F_d(-\tau)$.

Since $R_\tau(z)$ (resp., $\det R_\tau(z)$) is a theta operator (resp., function) of order n^2 with respect to Λ (resp., $\frac{1}{n}\Lambda$), the operator (resp., determinant of the operator) in (6.16) is a theta operator (resp., function) of order $(d-1)n^2$ with respect to Λ (resp., $\frac{1}{n}\Lambda$).

6.4.2 The "determinant" of $G_\tau(z)$

By the induction hypothesis, rank $F_{d-1}(-\tau) = \binom{n+d-2}{d-1}$ so

$$\dim(\operatorname{im} F_{d-1}(-\tau)\otimes V) \;=\; \dim(V\otimes \operatorname{im} F_{d-1}(-\tau)) \;=\; n\binom{n+d-2}{d-1}.$$

We fix arbitrary bases for the subspaces $\operatorname{im} F^L_{d-1}(-\tau)$ and $\operatorname{im} F^R_{d-1}(-\tau)$ of $V^{\otimes d}$ and write $\det G_\tau(z)$ for the determinant of the matrix for $G_\tau(z)$ with respect to those bases; although $\det G_\tau(z)$ depends on the choice of bases, the location and multiplicities of its zeros do not (see Sect. 4.2.1).

Proposition 6.6 $\det G_\tau(z)$ *is a theta function with respect to* $\frac{1}{n}\Lambda$ *and has exactly* $(d-1)n\binom{n+d-2}{d-1}$ *zeros in every fundamental parallelogram for* $\frac{1}{n}\Lambda$.

Proof Let W and W' denote the domain and codomain in (6.17). By the induction hypothesis,

$$\dim W \;=\; \dim W' \;=\; n\binom{n+d-2}{d-1}.$$

As remarked above, the operator in (6.16) is a theta operator of order $(d-1)n^2$ with respect to Λ. It now follows from Proposition 4.6 applied to $A(z) = G_\tau(z)$ that $\det G_\tau(z)$ is a theta function of order $(d-1)n^3\binom{n+d-2}{d-1}$ with respect to Λ. However, because the determinant of the operator in (6.16) is a theta function with respect to $\frac{1}{n}\Lambda$ so is $\det G_\tau(z)$, and it has exactly $(d-1)n\binom{n+d-2}{d-1}$ zeros in every fundamental parallelogram for $\frac{1}{n}\Lambda$. (Note that $\det G_\tau(z)$ does not vanish identically because each factor in (6.16) is an isomorphism for all but finitely many z.) □

Lemma 6.7 *For all* $m = 1,\ldots,d-2$, $G_\tau(m\tau) = 0$.

Proof By definition, $G_\tau(m\tau)$ is the restriction of $R(m\tau)_{12}R((m-1)\tau)_{23}\dots R((m-d+2)\tau)_{d-1,d}$ to im $F_{d-1}(-\tau)\otimes V$ so, to prove the lemma, it suffices to show that

$$R(m\tau)_{12}R((m-1)\tau)_{23}\cdots R((m-d+2)\tau)_{d-1,d}\cdot(F_{d-1}(-\tau)\otimes I)=0. \quad (6.19)$$

Assume $1\le m\le d-3$. The product $R(\tau)_{m,m+1}R(0)_{m+1,m+2}R(-\tau)_{m+2,m+3}$ is a factor of the operator in (6.19). But $R(0)_{m+1,m+2}$ is the identity so

$$\begin{aligned} R(\tau)_{m,m+1}R(0)_{m+1,m+2}R(-\tau)_{m+2,m+3} &= R(0)_{m+1,m+2}R(\tau)_{m,m+1}R(-\tau)_{m+2,m+3} \\ &= R(0)_{m+1,m+2}R(-\tau)_{m+2,m+3}R(\tau)_{m,m+1}. \end{aligned}$$

In fact, $R(\tau)_{m,m+1}$ commutes with all the R-factors to the right of it in (6.19) so (6.19) can be written as $Q''R(\tau)_{m,m+1}\cdot(F_{d-1}(-\tau)\otimes I)$ for some Q''. However, by Proposition 6.3, there is a factorization of the form $F_{d-1}(-\tau)=R(-\tau)_{m,m+1}Q'$ and hence a factorization $F_{d-1}(-\tau)\otimes I=R(-\tau)_{m,m+1}(Q'\otimes I)$. The product in (6.19) is therefore of the form

$$Q''R(\tau)_{m,m+1}\cdot R(-\tau)_{m,m+1}(Q'\otimes I),$$

and this product is zero (Lemma 5.5 tells us that $R(\tau)_{i,i+1}R(-\tau)_{i,i+1}=0$ for all i).

Assume $m=d-2$. The left-hand side of (6.19) is now

$$R((d-2)\tau)_{12}R((d-3)\tau)_{23}\cdots R(\tau)_{d-2,d-1}R(0)_{d-1,d}\cdot(F_{d-1}(-\tau)\otimes I)$$

which equals $Q''R(\tau)_{d-2,d-1}\cdot(F_{d-1}(-\tau)\otimes I)$ for some Q''. However, by Proposition 6.3, $F_{d-1}(-\tau)=R(-\tau)_{d-2,d-1}Q$ for some Q so, as before, (6.19) is zero. ◻

Lemma 6.8 nullity $G_\tau((d-1)\tau)\ge\binom{n+d-1}{d}$.

Proof When $z=(d-1)\tau$, the right-most factor in (6.16) is $R(\tau)_{d-1,d}$ so

$$\begin{aligned} \ker G_\tau((d-1)\tau) &\supseteq \ker R(\tau)_{d-1,d}\cap(\operatorname{im} F_{d-1}(-\tau)\otimes V) \\ &= \bigcap_{s+t+2=d} V^{\otimes s}\otimes\ker R_\tau(\tau)\otimes V^{\otimes t} \\ &\supseteq \operatorname{im} F_d(-\tau) \end{aligned}$$

where the equality comes from the induction hypothesis (6.15) applied to im $F_{d-1}(-\tau)$. The operator $F_d(-\tau)$ can be extended to all τ and its rank is generically large. Thus the desired inequality follows from the fact that $\operatorname{rank}(\lim_{\tau\to0}F_d(-\tau))=\binom{n+d-1}{d}$, which is a consequence of Proposition 6.4. ◻

Lemma 6.9 nullity $G_\tau(-\tau)\ge n\binom{n+d-2}{d-1}-\binom{n+d-1}{d}$.

Proof By Proposition 6.3,

$$\sum_{s+t+2=d} V^{\otimes s}\otimes\operatorname{im} R(\tau)\otimes V^{\otimes t}\subseteq\ker F_d(-\tau). \quad (6.20)$$

Extending the function $R_\tau(\tau)$ to all $\tau \in \mathbb{C}$, the left-hand side of (6.20) makes sense for all τ and its dimension is generically large by Propositions 4.7 and 4.9. When $\tau = 0$ the left-hand side of (6.20) is the space of degree-d relations for the polynomial ring SV by Corollary 5.3, so its dimension is $n^d - \binom{n+d-1}{d}$. Hence $\dim(\ker F_d(-\tau)) \geq n^d - \binom{n+d-1}{d}$ for generic τ; but the dimension of this kernel is generically small by Proposition 4.7, so this inequality holds for all τ. Therefore

$$\begin{aligned} \dim(\ker G_\tau(-\tau)) &= \dim(\operatorname{im} F_{d-1}(-\tau) \otimes V) - \dim(\operatorname{im} G_\tau(-\tau)) \\ &= n \dim(\operatorname{im} F_{d-1}(-\tau)) - \dim(\operatorname{im} F_d(-\tau)) \\ &\geq n\binom{n+d-2}{d-1} - \binom{n+d-1}{d} \end{aligned}$$

where the inequality comes from the induction hypothesis $\dim(\ker F_{d-1}(-\tau)) = n^{d-1} - \binom{n+d-2}{d-1}$. □

Lemma 6.10 *For all $\zeta \in \frac{1}{n}\Lambda$, $G_\tau(z+\zeta)$ and $G_\tau(z)$ have the same nullity.*

Proof Assume $\zeta = \frac{a}{n} + \frac{b}{n}\eta$ where $a, b \in \mathbb{Z}$, and let $C = T^b S^{ka} : V \to V$. In this proof we use the notation $C_i := I^{\otimes(i-1)} \otimes C \otimes I^{\otimes(d-i)}$.

By Corollary 2.7, $R_\tau(z+\zeta) = f(z,\zeta,\tau)C_2^{-1}R_\tau(z)C_1$ where $f(z,\zeta,\tau)$ is a nowhere vanishing function.

By definition, $G_\tau(z)$ is the restriction of $S^{\text{rev}}_{1\to d}(z, -\tau, \ldots, -\tau)$ to the image of $F^L_{d-1}(-\tau)$. Therefore $G_\tau(z+\zeta)$ is the restriction of

$$\begin{aligned} &R(z+\zeta)_{12} R(z-\tau+\zeta)_{23} \ldots R(z-(d-2)\tau+\zeta)_{d-1,d} \\ &\quad = g(z,\zeta,\tau) C_2^{-1} R(z)_{12} C_1 \cdot C_3^{-1} R(z-\tau)_{23} C_2 \cdots C_d^{-1} R(z-(d-2)\tau)_{d-1,d} C_{d-1} \\ &\quad = g(z,\zeta,\tau)(C_2 C_3 \cdots C_d)^{-1} R(z)_{12}(z-\tau)_{23} \cdots R(z-(d-2)\tau)_{d-1,d}(C_1 \cdots C_{d-1}) \\ &\quad = g(z,\zeta,\tau)(C_2 C_3 \cdots C_d)^{-1}\, S^{\text{rev}}_{1\to d}(z, -\tau, \ldots, -\tau)\,(C_1 \cdots C_{d-1}) \end{aligned}$$

to the image of $F^L_{d-1}(-\tau)$, where the function $g(z,\zeta,\tau)$ is a product of various $f(\cdot,\cdot,\cdot)$'s and therefore never vanishes. It follows that $G_\tau(z+\zeta)$ and the restriction of $S^{\text{rev}}_{1\to d}(z,-\tau,\ldots,-\tau)\,(C_1 \cdots C_{d-1})$ to the image of $F^L_{d-1}(-\tau)$ have the same nullity. Equivalently, the nullity of $G_\tau(z+\zeta)$ equals the nullity of the restriction of $S^{\text{rev}}_{1\to d}(z,-\tau,\ldots,-\tau)$ to the image of $C_1 \cdots C_{d-1} F^L_{d-1}(-\tau)$.

By Proposition 2.6(3), $C^{\otimes d-1}$ commutes with $R(z)_{i,i+1}$ for all z and all $1 \leq i \leq d-2$, and therefore with $F_{d-1}(z)$. The image of $C_1 \cdots C_{d-1} F^L_{d-1}(-\tau)$ is therefore the same as the image of $F^L_{d-1}(-\tau)C_1 \cdots C_{d-1}$ which is, since C is an automorphism of V, the same as the image of $F^L_{d-1}(-\tau)$. Thus $G_\tau(z+\zeta)$ has the same nullity as $G_\tau(z)$. □

By Lemma 6.10, the results in Lemmas 6.7 to 6.9 hold when $\zeta \in \frac{1}{n}\Lambda$ is added to the input of $G_\tau(z)$. Therefore

$$
\begin{aligned}
&\text{nullity}\, G_\tau(z)\\
&\geq \begin{cases}
\binom{n+d-1}{d} & \text{if} \quad z \in (d-1)\tau + \frac{1}{n}\Lambda,\\
n\binom{n+d-2}{d-1} & \text{if} \quad z \in m\tau + \frac{1}{n}\Lambda \quad \text{for some} \quad m = 1, \ldots, d-2,\\
n\binom{n+d-2}{d-1} - \binom{n+d-1}{d} & \text{if} \quad z \in -\tau + \frac{1}{n}\Lambda,\\
0 & \text{otherwise.}
\end{cases}
\end{aligned} \tag{6.21}
$$

Lemma 6.11 *Equality holds in* (6.21). *Consequently,*

$$
\begin{aligned}
&\text{rank}\, T_d(z, -\tau, \ldots, -\tau)\\
&= \begin{cases}
n\binom{n+d-2}{d-1} - \binom{n+d-1}{d} & \textit{if} \quad z \in (d-1)\tau + \frac{1}{n}\Lambda,\\
0 & \textit{if} \quad z \in m\tau + \frac{1}{n}\Lambda \quad \textit{for some} \quad m = 1, \ldots, d-2,\\
\binom{n+d-1}{d} & \textit{if} \quad z \in -\tau + \frac{1}{n}\Lambda,\\
n\binom{n+d-2}{d-1} & \textit{otherwise.}
\end{cases}
\end{aligned}
$$

Proof By Lemma 4.1, the sum of the dimensions of $\ker G_\tau(z)$, as z runs over all zeros of $\det G_\tau(z)$ in a fundamental parallelogram for Λ, does not exceed the number of zeros of $\det G_\tau(z)$ in that parallelogram, which is $(d-1)n^3\binom{n+d-2}{d-1}$ by Proposition 6.6. By the assumption $\tau \in \mathbb{C} - \bigcup_{m=1}^{d} \frac{1}{mn}\Lambda$, the cosets $m\tau + \frac{1}{n}\Lambda$, $-1 \leq m \leq d-1$, are pairwise disjoint. Since the sum of the three non-zero numbers appearing on the right-hand side of (6.21) is

$$
\binom{n+d-1}{d} + n\binom{n+d-2}{d-1}\cdot(d-2) + \left(n\binom{n+d-2}{d-1} - \binom{n+d-1}{d}\right) = (d-1)n\binom{n+d-2}{d-1}
$$

and the fundamental parallelogram for Λ contains exactly n^2 points in $\frac{1}{n}\Lambda$, every inequality in (6.21) is an equality. Therefore

$$
\begin{aligned}
\dim(\operatorname{im} T_d(z, -\tau, \ldots, -\tau)) &= \dim(\operatorname{im} G_\tau(z)) \qquad \text{by (6.18)}\\
&= \dim(\operatorname{im} F_{d-1}(-\tau) \otimes V) \; - \; \dim(\ker G_\tau(z))\\
&= n\binom{n+d-2}{d-1} \; - \; \dim(\ker G_\tau(z))
\end{aligned}
$$

where the last equality follows from the induction hypothesis. □

Proof of Theorem 6.5(2) We observed in the proof of Lemma 6.8 that

$$
\ker G_\tau((d-1)\tau) \supseteq \bigcap_{s+t+2=d} V^{\otimes s} \otimes \ker R_\tau(\tau) \otimes V^{\otimes t} \supseteq \operatorname{im} F_d(-\tau).
$$

But $\dim(\ker G_\tau((d-1)\tau)) = \binom{n+d-1}{d}$ and $\dim(\operatorname{im} F_d(-\tau)) = \binom{n+d-1}{d}$ by Lemma 6.11 since $F_d(-\tau) = T_d(-\tau, \ldots, -\tau)$, so these three subspaces of $V^{\otimes d}$ coincide. □

Proof of Theorem 6.5(1) We must show that the inclusion

$$\sum_{s+t+2=d} V^{\otimes s} \otimes \operatorname{im} R(\tau) \otimes V^{\otimes t} \subseteq \ker F_d(-\tau) \tag{6.22}$$

(in Proposition 6.3(4)) is an equality. We will do this by showing that the dimension of the left-hand side is $\geq$ the dimension of the right-hand side.

By the induction hypothesis, the left-hand side of (6.22) equals $\ker F^R_{d-1}(-\tau) + \operatorname{im} R(\tau)_{12}$ and its dimension is

$$\begin{aligned} &\dim(\ker F^R_{d-1}(-\tau)) + \dim(\operatorname{im} R(\tau)_{12}) - \dim(\ker F^R_{d-1}(-\tau) \cap \operatorname{im} R(\tau)_{12}) \\ &= n\big(n^{d-1} - \tbinom{n+d-2}{d-1}\big) + n^{d-2}\tbinom{n}{2} - \dim(\ker F^R_{d-1}(-\tau) \cap \operatorname{im} R(\tau)_{12}), \end{aligned}$$

so it suffices to show that

$$\begin{aligned} &n\big(n^{d-1} - \tbinom{n+d-2}{d-1}\big) + n^{d-2}\tbinom{n}{2} - \dim(\ker F^R_{d-1}(-\tau) \cap \operatorname{im} R(\tau)_{12}) \\ &\geq \dim(\ker F_d(-\tau)) = n^d - \dim(\operatorname{im} F_d(-\tau)) = n^d - \tbinom{n+d-1}{d} \end{aligned}$$

where the last equality appears in the proof of (2). Equivalently, it suffices to show that

$$\begin{aligned} \dim(\ker F^R_{d-1}(-\tau) \cap \operatorname{im} R(\tau)_{12}) &\leq n\big(n^{d-1} - \tbinom{n+d-2}{d-1}\big) + n^{d-2}\tbinom{n}{2} - \big(n^d - \tbinom{n+d-1}{d}\big) \\ &= -n\tbinom{n+d-2}{d-1} + n^{d-2}\tbinom{n}{2} + \tbinom{n+d-1}{d}. \end{aligned}$$

Since

$$\begin{aligned} \dim(\ker F^R_{d-1}(-\tau) \cap \operatorname{im} R(\tau)_{12}) &= \dim(\ker F^R_{d-1}(-\tau)R(\tau)_{12}) - \dim(\ker R(\tau)_{12}) \\ &= \dim(\ker F^R_{d-1}(-\tau)R(\tau)_{12}) - \big(n^d - n^{d-2}\tbinom{n}{2}\big), \end{aligned}$$

it suffices to show that

$$\begin{aligned} \dim(\ker F^R_{d-1}(-\tau)R(\tau)_{12}) &\leq -n\tbinom{n+d-2}{d-1} + n^{d-2}\tbinom{n}{2} + \tbinom{n+d-1}{d} + \big(n^d - n^{d-2}\tbinom{n}{2}\big) \\ &= n^d - n\tbinom{n+d-2}{d-1} + \tbinom{n+d-1}{d}. \end{aligned}$$

By Lemma 6.1,

$$T_d((d-1)\tau, -\tau, \ldots, -\tau) = F^R_{d-1}(-\tau)R(\tau)_{12}R(2\tau)_{23} \cdots R((d-1)\tau)_{d-1,d}$$

so

$$\operatorname{im} F^R_{d-1}(-\tau)R(\tau)_{12} \supseteq \operatorname{im} T_d((d-1)\tau, -\tau, \ldots, -\tau)$$

and

$$\dim(\ker F^R_{d-1}(-\tau)R(\tau)_{12}) \leq \dim(\ker T_d((d-1)\tau, -\tau, \ldots, -\tau))$$

$$= n^d - n\binom{n+d-2}{d-1} + \binom{n+d-1}{d}$$

where the equality comes from Lemma 6.11. The proof is now complete. □

Theorem 6.12 *For all* $\tau \in (\mathbb{C} - \bigcup_{m\geq 1} \frac{1}{m}\Lambda) \cup \frac{1}{n}\Lambda$, *the Hilbert series of* $Q_{n,k}(E,\tau)$ *is the same as that of the polynomial ring on n variables.*

Proof We proved this in [17, §5] for $\tau \in \frac{1}{n}\Lambda$, so we assume that $\tau \in \mathbb{C} - \bigcup_{m\geq 1} \frac{1}{m}\Lambda$. By Theorem 6.5(1), the space of degree-d relations for $Q_{n,k}(E,\tau)$ has the same dimension as that for a polynomial algebra in n variables. Hence the result follows. □

6.4.3 Remarks

Our proof that $Q_{n,k}(E,\tau)$ has the "right" Hilbert series has nothing in common with earlier proofs that $Q_{n,1}(E,\tau)$ has the "right" Hilbert series. The proofs for $n = 3$ and $n = 4$, and $k = 1$, by Artin–Tate–Van den Bergh [2] and Smith–Stafford [51], respectively, relied on the following facts, none of which is guaranteed to hold for other $Q_{n,k}(E,\tau)$'s: (1) $Q_{3,1}(E,\tau)$ has a central element of degree 3 ([1, p. 211], [22, Thm. 4.4])[22] and the quotient by it is a twisted homogeneous coordinate ring for E; $Q_{4,1}(E,\tau)$ has a regular sequence consisting of two degree-2 central elements ([48, Thm. 2], [22, Thm. 6.5]) and the quotient by them is a twisted homogeneous coordinate ring for E;[23] (2) the Riemann–Roch theorem for curves allows one to determine the Hilbert series of these two twisted homogeneous coordinate rings; (3) a tricky induction argument then allows one to "climb up the regular sequence" to show that the dimension of the degree-i component $Q_{n,1}(E,\tau)_i$ (for $n = 3, 4$) is "right" and, simultaneously, that the central elements form a regular sequence with respect to homogeneous elements of degree $< i$. Tate–Van den Bergh [55] proved that $Q_{n,1}(E,\tau)$ has the "right" Hilbert series when $n \geq 5$. Their argument relies on modules of I-type (a notion they introduce) and a geometric definition of the defining relations [55, (4.2)] (at the end of their Sect. 1, they suggest that $Q_{n,k}(E,\tau)$ might be amenable to their techniques).

It would be good to know whether the methods in this paper apply to other graded algebras whose defining relations are the image of a specialization of a family of operators $R(z)$ satisfying the QYBE.

7 The Hilbert series for $Q_{n,k}(E,\tau)^!$

The argument in this section showing that the Hilbert series of $Q_{n,k}(E,\tau)^!$ is $(1+t)^n$ is modeled on the argument used in Sect. 6 to show that the Hilbert series of $Q_{n,k}(E,\tau)$ is $(1-t)^{-n}$.

22 Artin–Schelter's proof is "by computer". De Laet's is "by algebra".

23 These facts are analogues of the fact that when E is embedded in $\mathbb{P}^2$ or $\mathbb{P}^3$ as an elliptic normal curve of degree 3, or 4, respectively, it is a complete intersection. However, when E is embedded in $\mathbb{P}^{n-1}$ as an elliptic normal curve of degree $n \geq 5$ it is not a complete intersection.

7.1 The algebras $S_{n,k}(E,\tau)$

By definition, $Q_{n,k}(E,\tau) = TV/(\operatorname{im} R_\tau(\tau))$ with the convention that $R_\tau(\tau)$ is replaced by $R_+(\tau)$ in (5.5) when $\tau \in \frac{1}{n}\Lambda$. We now define

$$S_{n,k}(E,\tau) := \frac{TV}{(\ker R_\tau(\tau))}$$

with the same convention. Theorem 7.4 shows that $Q_{n,k}(E,\tau)^! \cong S_{n,n-k}(E,\tau)$.

By [17, Prop. 3.22], the automorphism $N(x_\alpha) = x_{-\alpha}$ of V extends to algebra isomorphisms $Q_{n,k}(E,\tau) \to Q_{n,k}(E,-\tau)$ and $Q_{n,k}(E,\tau) \to Q_{n,k}(E,\tau)^{\mathrm{op}}$. There is a similar result for $S_{n,k}(E,\tau)$.

Proposition 7.1 *Let $N \in \mathrm{GL}(V)$ be the map $N(x_\alpha) = x_{-\alpha}$. For all $\tau \in \mathbb{C}$, N extends to algebra isomorphisms $S_{n,k}(E,\tau) \to S_{n,k}(E,-\tau)$ and $S_{n,k}(E,\tau) \to S_{n,k}(E,\tau)^{\mathrm{op}}$. In particular,*

$$S_{n,k}(E,\tau) \cong S_{n,k}(E,\tau)^{\mathrm{op}} = S_{n,k}(E,-\tau).$$

Proof It is clear that

$$S_{n,k}(E,\tau)^{\mathrm{op}} = \frac{TV}{(\ker R_\tau(\tau)P)}$$

where P is the operator $P(u\otimes v) = v\otimes u$. By (2.6), $R_\tau(\tau) = e(-n^2\tau)PR_{-\tau}(-\tau)P$ for all $\tau \in \mathbb{C} - \frac{1}{n}\Lambda$, and thus for all $\tau \in \mathbb{C}$ (we define $R_\tau(\tau)$ and $R_{-\tau}(-\tau)$ as limits when $\tau \in \frac{1}{n}\Lambda$). Hence $\ker R_\tau(\tau)P = \ker R_{-\tau}(-\tau)$. Hence $S_{n,k}(E,\tau)^{\mathrm{op}} = S_{n,k}(E,-\tau)$.

By (2.7),

$$(N\otimes N)R_\tau(\tau) = e(-n^2\tau)R_{-\tau}(-\tau)\,(N\otimes N)$$

for all $\tau \in \mathbb{C}$. It follows that $\ker R_{-\tau}(-\tau)\,(N\otimes N) = \ker R_\tau(\tau)$.

Thus, $N\otimes N$ is an automorphism of $V^{\otimes 2}$ that sends $\ker R_\tau(\tau)$, the space of quadratic relations for $S_{n,k}(E,\tau)$, to $\ker R_{-\tau}(-\tau)$, the space of quadratic relations for $S_{n,k}(E,-\tau)$. Therefore N induces an isomorphism $S_{n,k}(E,\tau) \to S_{n,k}(E,-\tau)$. $\square$

7.2 The quadratic dual of $Q_{n,k}(E,\tau)$

Let $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{C}$ and $\langle\cdot,\cdot\rangle : V^{\otimes 2}\times V^{\otimes 2} \to \mathbb{C}$ be the non-degenerate symmetric bilinear forms $\langle x_i, x_j\rangle = \delta_{ij}$ and $\langle x_i\otimes x_k, x_j\otimes x_\ell\rangle = \delta_{ij}\delta_{k\ell}$. The maps

$$\begin{aligned} V &\to V^*, \quad v\mapsto \langle v,\cdot\rangle, \\ V\otimes V &\to (V\otimes V)^*, \quad u\otimes v\mapsto \langle u\otimes v,\cdot\rangle, \\ V^*\otimes V^* &\to (V\otimes V)^*, \quad \langle u,\cdot\rangle\otimes\langle v,\cdot\rangle \mapsto \langle u\otimes v,\cdot\rangle, \end{aligned}$$

are isomorphisms. We will treat them as identifications. The third isomorphism is the composition of $V^*\otimes V^* \to V\otimes V$, induced by the first, and the second. We also define the isomorphism $V^{\otimes d} \to (V^*)^{\otimes d}$ for each $d\geq 3$ in the same way as $d=2$ and identify TV with TV^*.

If R is a subspace of $V \otimes V$, then the quadratic dual $A = TV/(R)$ is defined to be $A^! := TV^*/(R^\perp)$, where $R^\perp$ is the annihilator with respect to the form $\langle \cdot, \cdot \rangle : V^{\otimes 2} \times V^{\otimes 2} \to \mathbb{C}$ and is regarded as a subspace of $V^* \otimes V^*$.

Given a linear operator $R : V^{\otimes 2} \to V^{\otimes 2}$, we denote by $R^\bullet : V^{\otimes 2} \to V^{\otimes 2}$ the unique linear map such that $\langle R^\bullet x, y\rangle = \langle x, Ry\rangle$ for all $x, y \in V^{\otimes 2}$.

Lemma 7.2 *If V is a finite-dimensional vector space and $R^* : (V \otimes V)^* \to (V \otimes V)^*$ is the dual of a linear map $R : V^{\otimes 2} \to V^{\otimes 2}$, then*

(1) $(\operatorname{im} R)^\perp = \ker R^*$ *and*
(2) $(\ker R)^\perp = \operatorname{im} R^*$.

With the conventions stated just before this lemma,

$$\left(\frac{TV}{(\operatorname{im} R)}\right)^! = \frac{TV}{(\ker R^\bullet)}.$$

Proof Parts (1) and (2) are basic linear algebra. The displayed equality follows because the left-hand side of it equals $TV^*/(\ker R^*)$ which we are identifying with the right-hand side (by convention). □

Lemma 7.3 *For all $\tau \in \mathbb{C} - \frac{1}{n}\Lambda$ and $z \in \mathbb{C}$,*

$$R_{n,k,\tau}(z)^\bullet = e(-n^2 z) R_{n,n-k,-\tau}(-z).$$

Proof Let $\mathcal{B} = \{x_i \otimes x_j \mid i, j \in \mathbb{Z}_n\}$. This is a basis for $V^{\otimes 2}$. We must show, for all $p, q, s, t \in \mathbb{Z}_n$, that

$$\langle x_p \otimes x_q, R_{n,k,\tau}(z)(x_s \otimes x_t)\rangle = e(-n^2 z)\langle R_{n,n-k,-\tau}(-z)(x_p \otimes x_q), x_s \otimes x_t\rangle. \quad (7.1)$$

or, equivalently, that the coefficient of $x_p \otimes x_q$ in $R_{n,k,\tau}(z)(x_s \otimes x_t)$ with respect to $\mathcal{B}$ equals $e(-n^2 z)$ times the coefficient of $x_s \otimes x_t$ in $R_{n,n-k,-\tau}(-z)(x_p \otimes x_q)$ with respect to $\mathcal{B}$.

If $p + q \neq s + t$ then both sides of (7.1) are zero so we assume $p + q = s + t$ for the remainder of the proof.

By (2.7), the right-hand side of (7.1) is equal to

$$\langle R_{n,n-k,\tau}(z)(x_q \otimes x_p), x_t \otimes x_s\rangle = \frac{\theta_0(-z)\cdots\theta_{n-1}(-z)}{\theta_1(0)\cdots\theta_{n-1}(0)} \frac{\theta_{p-q+r(-k-1)}(-z+\tau)}{\theta_{p-q-r}(-z)\theta_{-kr}(\tau)}$$

where $r \in \mathbb{Z}_n$ is determined by $p - r = t$ or, equivalently, by $q + r = s$. Since $p - q = t - s + 2r$,

$$\langle R_{n,n-k,\tau}(z)(x_q \otimes x_p), x_t \otimes x_s\rangle = \frac{\theta_0(-z)\cdots\theta_{n-1}(-z)}{\theta_1(0)\cdots\theta_{n-1}(0)} \frac{\theta_{t-s+(-r)(k-1)}(-z+\tau)}{\theta_{t-s-(-r)}(-z)\theta_{k(-r)}(\tau)},$$

which equals the left-hand side of (7.1). □

Theorem 7.4 *For all $\tau \in \mathbb{C}$,*

$$Q_{n,k}(E,\tau)^! \cong S_{n,n-k}(E,\tau). \tag{7.2}$$

Proof Lemma 7.3 implies that $R_{n,k,\tau}(\tau)^\bullet = e(-n^2\tau)R_{n,n-k,-\tau}(-\tau)$ for all $\tau \in \mathbb{C} - \frac{1}{n}\Lambda$, and thus for all $\tau \in \mathbb{C}$ if we define both sides as limits when $\tau \in \frac{1}{n}\Lambda$. Therefore, by Lemma 7.2,

$$Q_{n,k}(E,\tau)^! = \frac{TV}{(\ker R_{n,k,\tau}(\tau)^\bullet)} = \frac{TV}{(\ker R_{n,n-k,-\tau}(-\tau))} = S_{n,n-k}(E,-\tau).$$

The desired isomorphism now follows from Proposition 7.1. □

Corollary 7.5 *For all $\tau \in \mathbb{C}$, $S_{n,1}(E,\tau) \cong \Lambda V$, the exterior algebra on V.*

Proof By Theorem 7.4, $S_{n,1}(E,\tau)$ is isomorphic to the quadratic dual of $Q_{n,n-1}(E,\tau)$. However, $Q_{n,n-1}(E,\tau) \cong SV$ by [17, Prop. 5.5] so the result follows from the well-known fact that the quadratic dual of the polynomial algebra SV is the exterior algebra ΛV^*. □

7.3 Computation of the Hilbert series for $Q_{n,k}(E,\tau)^!$

In this subsection we prove Theorem 7.7. After Theorem 7.4 it suffices to show that the Hilbert series of $S_{n,k}(E,\tau)$ is $(1+t)^n$ for all k and all $\tau \in (\mathbb{C} - \bigcup_{m=1}^{n+1} \frac{1}{mn}\Lambda) \cup \frac{1}{n}\Lambda$. The arguments we use to show this are similar to those in Sect. 6.4 with the essential change that images and kernels of $R_\tau(\tau)$ are replaced by images and kernels of $R_\tau(-\tau)$.

We adopt the convention that $\binom{n}{e} = 0$ when $e > n$.

Theorem 7.6 *Let $d \geq 2$. Assume $\tau \in \mathbb{C} - \bigcup_{m=1}^{d} \frac{1}{mn}\Lambda$.*

(1) We have

$$\ker F_d(\tau) = \sum_{s+t+2=d} V^{\otimes s} \otimes \operatorname{im} R_\tau(-\tau) \otimes V^{\otimes t} \tag{7.3}$$

and its dimension is the same as the dimension of the space of degree-d relations for the exterior algebra in n variables, namely $n^d - \binom{n}{d}$.

(2) We have

$$\operatorname{im} F_d(\tau) = \bigcap_{s+t+2=d} V^{\otimes s} \otimes \ker R_\tau(-\tau) \otimes V^{\otimes t} \tag{7.4}$$

and its dimension is the same as the dimension of the degree-d component of an exterior algebra in n variables, namely $\binom{n}{d}$.

Proof The proof is like that for Theorem 6.5 with some natural changes. The binomial coefficients $\binom{n+e-1}{e}$ are replaced by $\binom{n}{e}$. The operator

$$G_\tau(z) : \operatorname{im} F_{d-1}(-\tau) \otimes V \longrightarrow V \otimes \operatorname{im} F_{d-1}(-\tau)$$

that is the restriction of $S^{\mathrm{rev}}_{1\to d}(z,-\tau,\ldots,-\tau)$ is replaced by

$$G^+_\tau(z): V\otimes \operatorname{im} F_{d-1}(\tau) \longrightarrow \operatorname{im} F_{d-1}(\tau)\otimes V$$

which is the restriction of $S^{\mathrm{rev}}_{d\to 1}(z,\tau,\ldots,\tau)$. The result in Proposition 6.6 showing that $\det G_\tau(z)$ has exactly $(d-1)n^3\binom{n+d-2}{d-1}$ zeros in a fundamental parallelogram for Λ is replaced by the result that $\det G^+_\tau(z)$ has exactly $(d-1)n^3\binom{n}{d-1}$ zeros in a fundamental parallelogram for Λ. The analogue of (6.21) is now

$$\dim(\ker G^+_\tau(z)) \ \ge\ \begin{cases} \binom{n}{d} & \text{if } \ z\in -(d-1)\tau+\frac{1}{n}\Lambda,\\ n\binom{n}{d-1} & \text{if } \ z\in -m\tau+\frac{1}{n}\Lambda \ \text{ for some } \ m=1,\ldots,d-2,\\ n\binom{n}{d-1}-\binom{n}{d} & \text{if } \ z\in \tau+\frac{1}{n}\Lambda,\\ 0 & \text{otherwise.}\end{cases} \tag{7.5}$$

(The hypothesis that $\tau\notin\bigcup_{m=1}^{d}\frac{1}{mn}\Lambda$ ensures that the four cases in (7.5) are pairwise disjoint.) After these changes, the argument then proceeds as before since

$$\operatorname{rank}\Big(\lim_{\tau\to 0} F_d(\tau)\Big) \ =\ \binom{n}{d}$$

by Proposition 6.4. □

In analogy with Lemma 6.11,

$$\begin{aligned}&\dim(\operatorname{im} T_d(\tau,\ldots,\tau,z))\\ &\quad=\begin{cases} n\binom{n}{d-1}-\binom{n}{d} & \text{if } z\in -(d-1)\tau+\frac{1}{n}\Lambda,\\ 0 & \text{if } \ z\in -m\tau+\frac{1}{n}\Lambda \ \text{ for some } \ m=1,\ldots,d-2,\\ \binom{n}{d} & \text{if } \ z\in \tau+\frac{1}{n}\Lambda,\\ n\binom{n}{d-1} & \text{otherwise.}\end{cases}\end{aligned} \tag{7.6}$$

Theorem 7.7 *If $\tau\in(\mathbb{C}-\bigcup_{m=1}^{n+1}\frac{1}{mn}\Lambda)\cup\frac{1}{n}\Lambda$, then $Q_{n,k}(E,\tau)^!$ has the same Hilbert series as the exterior algebra on n variables.*

Proof If $\tau\in\frac{1}{n}\Lambda$, then $Q_{n,k}(E,\tau)$ is a twist of the polynomial ring on n variables [17, Cor. 5.2] so its category of graded modules is equivalent to the category of graded modules over that polynomial ring. But the Koszulity of a finitely generated connected graded algebra generated in degree one depends only on its category of graded modules (see the argument preceding [59, Prop. 5.7]). Since the polynomial ring is a Koszul algebra so is every twist of it. In particular, $Q_{n,k}(E,\tau)$ is a Koszul algebra. The Hilbert series for Q and $Q^!$ therefore satisfy the functional equation $H_{Q^!}(t)H_Q(-t)=1$. The Hilbert series for $Q_{n,k}(E,\tau)^!$ is therefore $(1+t)^n$.

For the rest of the proof we assume that $\tau\notin\bigcup_{m=1}^{n+1}\frac{1}{mn}\Lambda$.

Since $\tau\notin\frac{1}{2n}\Lambda$, $\operatorname{im} R_\tau(-\tau)=\ker R_\tau(\tau)$ by Theorem 5.7(2). By Theorem 7.6(1), the degree-d part of $S_{n,k}(E,\tau)$ has the same dimension as the degree-d part of the exterior algebra on n variables for all $0 \le d \le n+1$. In particular, $S_{n,k}(E,\tau)_{n+1} = 0$; since $S_{n,k}(E,\tau)$ is generated in degree-one, $S_{n,k}(E,\tau)_d = 0$ for all $d \ge n+1$. Thus $S_{n,k}(E,\tau)$ has the same Hilbert series as the exterior algebra on n variables for all k. The result now follows from Theorem 7.4. □

8 Multiplication in $Q_{n,k}(E,\tau)$

Let $\mathrm{Sym}^d V$ (resp., $\mathrm{Alt}^d V$) denote the subspace of $V^{\otimes d}$ consisting of the symmetric (resp., anti-symmetric) tensors. The restriction to $\mathrm{Sym}^d V$ of the natural map from the tensor algebra TV to the symmetric algebra $SV := TV/(\mathrm{Alt}^2 V)$ is an isomorphism onto its image, $S^d V$, the degree-d component of SV. The multiplication on SV can therefore be transferred in a canonical way to a multiplication on $\mathrm{Sym}\, V := \bigoplus_{d\ge 0} \mathrm{Sym}^d V$. The induced multiplication is called the shuffle product.

In a similar way, the equality in (6.14) leads to a canonical isomorphism from $Q_{n,k}(E,\tau)$ to the subspace of TV that is the direct sum of the images of the operators $F_d(-\tau)$ which are, by Proposition 6.4, elliptic analogues of the symmetrization operators. Following this line of reasoning, the multiplication on $Q_{n,k}(E,\tau)$ can be transferred in a canonical way to this graded subspace of TV.

In this section we make this multiplication explicit in terms of certain operators, those in Proposition 8.6(1), that should be thought of as elliptic analogues of the shuffle operators.

8.1 The operators $M_{b,a} : V^{\otimes a} \otimes V^{\otimes b} \to V^{\otimes(a+b)}$

At first sight, the calculations in this section might appear mysterious. They have been guided by a desire to find an elliptic analogue of the equality (8.1) which says that the product on $\mathrm{Sym}\, V$ induced by the usual product on SV *is* the shuffle product. We need some notation to explain this.

Let $a, b \in \mathbb{Z}_{\ge 0}$. Let S_{a+b} denote the group of permutations of $\{1,\dots,a+b\}$. Define

$$
\begin{aligned}
S_{a|b} &:= \{\sigma \in S_{a+b} \,|\, \sigma(1) < \cdots < \sigma(a) \text{ and } \sigma(a+1) < \cdots < \sigma(a+b)\},\\
S_{a|\circ} &:= \{\sigma \in S_{a+b} \,|\, \sigma(i) = i \text{ for all } i \ge a+1\},\\
S_{\circ|b} &:= \{\sigma \in S_{a+b} \,|\, \sigma(i) = i \text{ for all } i \le a\}.
\end{aligned}
$$

Elements in $S_{a|b}$ are called shuffles. If $\sigma \in S_{a+b}$, then there are unique elements $\omega \in S_{a|b}$, $\alpha \in S_{a|\circ}$, $\beta \in S_{\circ|b}$ such that $\sigma = \omega\alpha\beta$. Hence, in the group algebra $\mathbb{C}S_{a+b}$, we have

$$
\Bigg(\sum_{\omega \in S_{a|b}} \omega\Bigg)\Bigg(\sum_{\alpha \in S_{a|\circ}} \alpha\Bigg)\Bigg(\sum_{\beta \in S_{\circ|b}} \beta\Bigg) = \sum_{\sigma \in S_{a+b}} \sigma. \tag{8.1}
$$

The shuffle product $u \otimes v \mapsto u * v$ on $V^{\otimes(a+b)}$, and its restriction $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V \to \mathrm{Sym}^{a+b} V$, is given by $\frac{a!b!}{(a+b)!}$ times the left-most term in (8.1).

The second equality in Lemma 8.5, which is one of the main results in this section, namely

$$M_{b,a}(-\tau)\cdot\big(F_a(-\tau)\otimes F_b(-\tau)\big) \;=\; F_{a+b}(-\tau),$$

is analogous to (8.1) (with some factorial terms thrown in). By Proposition 6.4, $\lim_{\tau\to 0} F_a(-\tau) = \prod_{m=1}^{a-1} m!\cdot\sum_{\sigma\in S_a}\sigma$; i.e., $F_a(-\tau)$ is analogous to the the middle term on the left-hand side of (8.1). We now introduce the operator $M_{b,a}(-\tau)$ that will be analogous to the left-most term in (8.1).

Definition 8.1 Let $a, b \in \mathbb{Z}_{\ge 0}$. Define the operator

$$M_{a,b}(z; x_1,\dots,x_{a-1}; y_1,\dots,y_{b-1}) : V^{\otimes(a+b)} \to V^{\otimes(a+b)}$$

to be

$$\begin{matrix} R(z)_{a,a+1} & R(z+y_1)_{a+1,a+2} & \cdots & R(z+\sum_k y_k)_{a+b-1,a+b} \\ R(z+x_1)_{a-1,a} & R(z+x_1+y_1)_{a,a+1} & \cdots & R(z+x_1+\sum_k y_k)_{a+b-2,a+b-1} \\ \vdots & \vdots & \ddots & \vdots \\ R(z+\sum_j x_j)_{12} & R(z+\sum_j x_j+y_1)_{23} & \cdots & R(z+\sum_j x_j+\sum_k y_k)_{b,b+1} \end{matrix}$$

interpreted as either

(1) the downward product of the rightward products along rows, or
(2) the rightward product of the downward products along columns.

If $a = 0$ or $b = 0$, we regard the operator as the identity.

For example, $M_{2,3}(z; x; y_1, y_2)$ is

$$\begin{aligned} &R(z)_{23}R(z+y_1)_{34}R(z+y_1+y_2)_{45}R(z+x)_{12}R(z+x+y_1)_{23}R(z+x+y_1+y_2)_{34} \\ &\quad = R(z)_{23}R(z+x)_{12}R(z+y_1)_{34}R(z+x+y_1)_{23} \\ &\qquad\cdot R(z+y_1+y_2)_{45}R(z+x+y_1+y_2)_{34}. \end{aligned}$$

Let $\mathsf{x} := (x_1,\dots,x_{a-1})$ and $\mathsf{y} := (y_1,\dots,y_{b-1})$. The interpretations in (1) and (2) yield

$$M_{a,b}(z;\mathsf{x};\mathsf{y}) = S^{\mathrm{rev}}_{a\to a+b}(z,\mathsf{y})S^{\mathrm{rev}}_{a-1\to a+b-1}(z+x_1,\mathsf{y})\cdots S^{\mathrm{rev}}_{1\to b+1}(z+\sum_j x_j,\mathsf{y}) \tag{8.2}$$

$$= S^{\mathrm{rev}}_{a+1\to 1}(z,\mathsf{x})S^{\mathrm{rev}}_{a+2\to 2}(z+y_1,\mathsf{x})\cdots S^{\mathrm{rev}}_{a+b\to b}(z+\textstyle\sum_k y_k,\mathsf{x}) \tag{8.3}$$

respectively. We leave the reader to verify that the procedures in (1) and (2) produce the same result (this does not involve using the Yang–Baxter equation). The first step in verifying this is to notice that if one starts with the product produced by (1), then the factors $R(z+x_1+\cdots+x_j)_{a-j,a-j+1}$ coming from the left-most column commute with all the entries in the array that appear to the northeast of that factor; thus the product produced by (1) is equal to $S^{\text{rev}}_{a+1\to 1}(z, \mathsf{x})$ times

$$S^{\text{rev}}_{a+1\to a+b}(z+y_1,\dots,y_{b-1})S^{\text{rev}}_{a\to a+b-1}(z+x_1+y_1,y_2,\dots,y_{b-1})$$
$$\cdots S^{\text{rev}}_{2\to b+1}\left(z+\sum_j x_j+y_1,y_2,\dots,y_{b-1}\right).$$

One then treats this product in the same way, and so on.

We write $M_{a,b}(z)$ for $M_{a,b}(z;\mathsf{x};\mathsf{y})$ if $z=x_1=\cdots=x_{a-1}=y_1=\cdots=y_{b-1}$.

Lemma 8.2 *Let* $\mathsf{x}=(x_1,\dots,x_{a-1})$ *and* $\mathsf{y}=(y_1,\dots,y_{b-1})$. *As operators on* $V^{\otimes(a+b)}$,

$$T^L_a(\mathsf{x})M_{a,b}(z+\textstyle\sum_j x_j;-\mathsf{x};\mathsf{y}) = M_{a,b}(z;\mathsf{x}^{\text{rev}};\mathsf{y})T^R_a(\mathsf{x}) \tag{8.4}$$

and

$$T^R_b(\mathsf{y})M_{a,b}(z+\textstyle\sum_k y_k;\mathsf{x}^{\text{rev}};-\mathsf{y}^{\text{rev}}) = M_{a,b}(z;\mathsf{x}^{\text{rev}};\mathsf{y})T^L_b(\mathsf{y}) \tag{8.5}$$

where $\mathsf{x}^{\text{rev}}:=(x_{a-1},\dots,x_1)$ *and* $\mathsf{y}^{\text{rev}}:=(y_{b-1},\dots,y_1)$.

Proof By (8.3) and (6.4),

$$\begin{aligned}
&M_{a,b}\left(z+\sum_j x_j;-\mathsf{x};\mathsf{y}\right)\\
&= S^{\text{rev}}_{a+1\to 1}\left(z+\sum_j x_j,-\mathsf{x}\right)S^{\text{rev}}_{a+2\to 2}\left(z+\sum_j x_j+y_1,-\mathsf{x}\right)\cdots\\
&\quad\cdot S^{\text{rev}}_{a+b\to b}\left(z+\sum_j x_j+\sum_k y_k,-\mathsf{x}\right)\\
&= S_{a+1\to 1}(\mathsf{x},z)S_{a+2\to 2}(\mathsf{x},z+y_1)\cdots S_{a+b\to b}\left(\mathsf{x},z+\sum_k y_k\right).
\end{aligned}$$

By Lemma 6.1,

$$T^L_{d-1}(z_1,\dots,z_{d-2})S_{d\to 1}(z_1,\dots,z_{d-1}) = S^{\text{rev}}_{d\to 1}(z_{d-1},\dots,z_1)T^R_{d-1}(z_1,\dots,z_{d-2}). \tag{8.6}$$

Hence $T^L_a(\mathsf{x})M_{a,b}(z+\sum_j x_j;-\mathsf{x};\mathsf{y})$ equals

$$
\begin{aligned}
&T_a^L(\mathsf{x})S_{a+1\to 1}(\mathsf{x},z)S_{a+2\to 2}(\mathsf{x},z+y_1)S_{a+3\to 3}(\mathsf{x},z+y_1+y_2)\cdots\\
&\quad = S^{\mathrm{rev}}_{a+1\to 1}(z,\mathsf{x}^{\mathrm{rev}})(I\otimes T_a(\mathsf{x}))^L S_{a+2\to 2}(\mathsf{x},z+y_1)S_{a+3\to 3}(\mathsf{x},z+y_1+y_2)\cdots\\
&\quad = S^{\mathrm{rev}}_{a+1\to 1}(z,\mathsf{x}^{\mathrm{rev}})S^{\mathrm{rev}}_{a+1\to 1}(z+y_1,\mathsf{x}^{\mathrm{rev}})(I^{\otimes 2}\otimes T_a(\mathsf{x}))^L S_{a+3\to 3}(\mathsf{x},z+y_1+y_2)\cdots
\end{aligned}
$$

where the last equality is obtained by applying (8.6) after observing that the previous $(I\otimes T_a(\mathsf{x}))^L$ is of the form $T_a^L(\mathsf{x})$ with respect to $S_{a+2\to 2}(\mathsf{x},z+y_1)$. Repeating this procedure we eventually see that $T_a^L(\mathsf{x})M_{a,b}(z+\sum_j x_j;-\mathsf{x};\mathsf{y})$ equals

$$
S^{\mathrm{rev}}_{a+1\to 1}(z,\mathsf{x}^{\mathrm{rev}})S^{\mathrm{rev}}_{a+2\to 2}(z+y_1,\mathsf{x}^{\mathrm{rev}})\cdots S^{\mathrm{rev}}_{a+b\to b}(z+\textstyle\sum_k y_k,\mathsf{x}^{\mathrm{rev}})T_a^R(\mathsf{x})
$$

which is $M_{a,b}(z;\mathsf{x}^{\mathrm{rev}};\mathsf{y})T_a^R(\mathsf{x})$.

A similar argument proves (8.5). □

Lemma 8.3 *For positive integers a, b,*

$$
\begin{aligned}
&T_a^L(z_1,\ldots,z_{a-1})S_{a+1\to 1}(z_1,\ldots,z_a)S_{a+2\to 2}(z_1,\ldots,z_{a-1},z_a+z_{a+1})\cdots\\
&\quad\cdots S_{a+b\to b}(z_1,\ldots,z_{a-1},z_a+\cdots+z_{a+b-1})T_b^L(z_{a+1},\ldots,z_{a+b-1})\\
&\quad = T_{a+b}(z_1,\ldots,z_{a+b-1}).
\end{aligned}
\tag{8.7}
$$

Proof Since $T_{d-1}^L(z_1,\ldots,z_{d-2})S_{d\to 1}(z_1,\ldots,z_{d-1})=T_d(z_1,\ldots,z_{d-1})$ (by Lemma 6.1), the product of the two left-most factors on the left-hand side of (8.7) equals

$$
T_{a+1}^L(z_1,\ldots,z_a).
$$

By definition (6.1), the right-most factor $T_b^L(z_{a+1},\ldots,z_{a+b-1})$ on the left-hand side is

$$
T_b^L(z_{a+1},\ldots,z_{a+b-1}) = S_{2\to 1}(z_{a+1})S_{3\to 1}(z_{a+1},z_{a+2})\ldots S_{b\to 1}(z_{a+1},\ldots,z_{a+b-1}).
\tag{8.8}
$$

Each of the $b-1$ resulting $S_{j\to 1}$ factors commutes with all $S_{a+k\to k}$ factors ending the left-hand side of (8.7) for $k>j$. Implementing this commutation for each of the S factors in (8.8) means attaching $S_{j\to 1}$ in (8.8) to $S_{a+j\to j}$ in (8.7) to produce

$$
\begin{aligned}
&S_{a+j\to j}(z_1,\ldots,z_{a-1},z_a+\cdots+z_{a+j-1})S_{j\to 1}(z_{a+1},\ldots,z_{a+j-1})\\
&\quad = S_{a+j\to 1}(z_1,\ldots,z_{a+j-1}).
\end{aligned}
$$

Multiplying these by the $T_{a+1}^L(z_1,\ldots,z_a)$ we already have and applying Lemma 6.1 successively now yields the right-hand side $T_{a+b}(z_1,\ldots,z_{a+b-1})$ of (8.7), as claimed. □

Lemma 8.4 *If* $\mathsf{x}=(x_1,\ldots,x_{a-1})$ *and* $\mathsf{y}=(y_1,\ldots,y_{b-1})$*, then*

$$
T_{a+b}(\mathsf{x},z,\mathsf{y}) \;=\; M_{a,b}\big(z;\mathsf{x}^{\mathrm{rev}};\mathsf{y}\big)\cdot T_a^R(\mathsf{x})\cdot T_b^L(\mathsf{y}).
$$

Proof Lemma 8.3 can be re-stated as

$$T_{a+b}(\mathsf{x}, z, \mathsf{y}) \;=\; T_a^L(\mathsf{x}) \cdot M_{a,b}\big(z + \textstyle\sum_j x_j; -\mathsf{x}; \mathsf{y}\big) \cdot T_b^L(\mathsf{y}).$$

So the result follows from (8.4). □

8.2 Multiplication in $Q_{n,k}(E, \tau)$

Assume $\tau \in \mathbb{C} - \bigcup_{m\geq 1} \frac{1}{m}\Lambda$. By (6.14),

$$Q_{n,k}(E, \tau)_d \;=\; \frac{V^{\otimes d}}{\ker F_d(-\tau)},$$

which is canonically isomorphic to $\operatorname{im} F_d(-\tau)$. Thus

$$Q_{n,k}(E, \tau) \;\cong\; \bigoplus_{d\geq 0} \operatorname{im} F_d(-\tau)$$

as graded vector spaces, so the multiplication on $Q_{n,k}(E, \tau)$ induces a multiplication on the right-hand space making it a graded $\mathbb{C}$-algebra. Proposition 8.6 describes the induced multiplication.

Lemma 8.5 *With the notation above,*

$$\begin{aligned} M_{b,a}(\tau) \cdot (F_a(\tau) \otimes F_b(\tau)) \;&=\; F_{a+b}(\tau), \\ M_{b,a}(-\tau) \cdot (F_a(-\tau) \otimes F_b(-\tau)) \;&=\; F_{a+b}(-\tau). \end{aligned}$$

Proof By Lemma 8.4,

$$\begin{aligned} F_{a+b}(\pm\tau) \;&=\; M_{b,a}(\pm\tau; (\pm\tau)^{b-1}; (\pm\tau)^{a-1}) T_b^R((\pm\tau)^{b-1}) T_a^L((\pm\tau)^{a-1}) \\ &=\; M_{b,a}(\pm\tau) F_b^R(\pm\tau) F_a^L(\pm\tau) \\ &=\; M_{b,a}(\pm\tau) \cdot (F_a(\pm\tau) \otimes F_b(\pm\tau)) \end{aligned}$$

as desired. □

Proposition 8.6 *Let* $\tau \in \mathbb{C} - \bigcup_{m\geq 1} \frac{1}{m}\Lambda$.

(1) Define a bilinear multiplication on $A := \bigoplus_{d\geq 0} \operatorname{im} F_d(-\tau)$ *by the maps*

$$\operatorname{im} F_a(-\tau) \otimes \operatorname{im} F_b(-\tau) \;\longrightarrow\; \operatorname{im} F_{a+b}(-\tau)$$

induced from $M_{b,a}(-\tau)$ *for all* $a, b \geq 0$. *Then* A *is a graded algebra isomorphic to* $Q_{n,k}(E, \tau)$.

(2) Similarly, the maps induced from $M_{b,a}(\tau)$ *make* $\bigoplus_{a+b\geq 0} \operatorname{im} F_{a+b}(\tau)$ *a graded algebra isomorphic to the quadratic dual* $Q_{n,k}(E, \tau)^!$.

Proof Write Q for $Q_{n,k}(E,\tau)$ and Q_i for its degree-i component. Consider the diagram

$$\begin{tikzcd} V^{\otimes a}\otimes V^{\otimes b} \arrow[r, two heads] \arrow[d, equal] & Q_a\otimes Q_b \arrow[r, "\sim"] \arrow[d] & A_a\otimes A_b \arrow[r, hook] & V^{\otimes a}\otimes V^{\otimes b} \arrow[d, "M_{b,a}(-\tau)"] \\ V^{\otimes(a+b)} \arrow[r, two heads] & Q_{a+b} \arrow[r, "\sim"] & A_{a+b} \arrow[r, hook] & V^{\otimes(a+b)} \end{tikzcd}$$

where the first and second rows are factorizations of $F_a(-\tau)\otimes F_b(-\tau)$ and $F_{a+b}(-\tau)$, respectively. The left-hand square commutes since Q is defined as a quotient of the tensor algebra, and the commutativity of the outer square follows from Lemma 8.5. Thus the right-hand square also commutes, whence $M_{b,a}(-\tau)$ induces a map $A_a\otimes A_b\to A_{a+b}$, which is equal to the one induced from the multiplication of Q. This proves the first statement.

If we replace all $-\tau$'s in the above argument by τ we get a proof of the second statement using (7.4). $\square$

9 Koszulity of $Q_{n,k}(E,\tau)$

Throughout this section, we assume that $\tau\in\mathbb{C}-\bigcup_{m\geq 1}\frac{1}{m}\Lambda$ so we can apply Theorems 6.5 and 7.6.

Let $\mathsf{Lat}(V^{\otimes d})$ denote the lattice of subspaces of $V^{\otimes d}$.

We will use the following result to show that $Q_{n,k}(E,\tau)$ is a Koszul algebra.

Lemma 9.1 (Backelin) [41, Thm. 2.4.1] *Let $\tau\in\mathbb{C}$. $Q_{n,k}(E,\tau)$ is a Koszul algebra if and only if the sublattice of $\mathsf{Lat}(V^{\otimes d})$ generated by*

$$W_i := V^{\otimes(i-1)}\otimes \mathrm{rel}_{n,k}(E,\tau)\otimes V^{\otimes(d-i-1)},\quad i=1,\ldots,d-1, \tag{9.1}$$

is distributive for all $d\geq 2$.

9.1 Distributive lattices

Recall that a lattice $(\mathcal{L},\vee,\wedge)$ is distributive if

$$x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)\quad\text{and} \tag{9.2}$$
$$x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z) \tag{9.3}$$

for all $x,y,z\in\mathcal{L}$. Condition (9.2) holds for all x,y,z if only if (9.3) holds for all x,y,z.

A lattice $\mathcal{L}$ is modular if (9.2) (or equivalently, (9.3)) holds for all triples (x,y,z) satisfying $x\geq z$. As explained in [41, Lem. 1.6.1], if $\mathcal{L}$ is modular, then (9.2) and (9.3) are equivalent for *each* triple (x,y,z) and these conditions are invariant under permutations of x, y and z. If those equivalent conditions hold, we say that the triple (x,y,z) is distributive.

We write $\mathsf{Lat}(d)$ for the sublattice of $\mathsf{Lat}(V^{\otimes d})$ generated by $W_1, \dots, W_{d-1}$. Like the lattice of subspaces of any vector space, $\mathsf{Lat}(V^{\otimes d})$ is modular, and so is $\mathsf{Lat}(d)$.

We say that $X \in \mathsf{Lat}(d)$ has **classical dimension** if it has the same dimension as its counterpart for the polynomial ring SV. This terminology is not really for X, an element of $\mathsf{Lat}(d)$, but rather for an expression of X using join and meet.

Since $\dim \mathrm{rel}_{n,k}(E, \tau) = \binom{n}{2} = \dim(\mathrm{Alt}^2 V)$, every W_i has classical dimension; its classical counterpart is $V^{\otimes(i-1)} \otimes \mathrm{Alt}^2 V \otimes V^{\otimes(d-i-1)}$. The subspaces

$$\Sigma_s := \sum_{i=1}^{s} W_i, \quad \text{and} \quad I_t := \bigcap_{j=d-t}^{d-1} W_j$$

also have classical dimension for all s and t by Theorems 6.5 and 7.6. It follows that $\Sigma_s \cap I_t$ has classical dimension if and only if $\Sigma_s + I_t$ does.

Because $\mathsf{Lat}(V^{\otimes d})$ is modular, the second half of [41, Thm. 1.6.3] tells us the following.

Proposition 9.2 *Let $d \geq 3$ and let W_i, $1 \leq i \leq d-1$, be the subspaces of $V^{\otimes d}$ defined in (9.1). If $\mathsf{Lat}(2), \dots, \mathsf{Lat}(d-1)$ are distributive and, for $1 \leq \ell \leq d-1$, the triple*

$$\left(\sum_{i=1}^{\ell-1} W_i,\ W_\ell,\ \bigcap_{j=\ell+1}^{d-1} W_j \right) \tag{9.4}$$

is distributive, then $\mathsf{Lat}(d)$ is distributive and $Q_{n,k}(E, \tau)$ is a Koszul algebra.

We will prove that $\mathsf{Lat}(d)$ is distributive by induction on d.

Lemma 9.3 *Fix $d \geq 3$. Assume $\mathsf{Lat}(2), \dots, \mathsf{Lat}(d-1)$ are distributive. If $\Sigma_i \cap I_{d-i-1}$ has classical dimension for all $i = 0, \dots, d-1$, then $\mathsf{Lat}(d)$ is distributive.*

Proof It suffices to show that $(\Sigma_{\ell-1}, W_\ell, I_{d-\ell-1})$ is a distributive triple for all integers ℓ in $[1, d-1]$.

Fix ℓ and write $r := d - \ell - 1$.

Since $\Sigma_{\ell-1} + W_\ell = \Sigma_\ell$ and $W_\ell \cap I_r = I_{r+1}$, the distributivity condition

$$\Sigma_{\ell-1} + (W_\ell \cap I_r) = (\Sigma_{\ell-1} + W_\ell) \cap (\Sigma_{\ell-1} + I_r).$$

is equivalent to the condition

$$\Sigma_{\ell-1} + I_{r+1} = \Sigma_\ell \cap (\Sigma_{\ell-1} + I_r). \tag{9.5}$$

The two terms on the right-hand side of (9.5) have classical dimensions:

- Σ_ℓ does by Theorem 6.5;
- $\Sigma_{\ell-1} + I_r$ does by Theorem 6.5 and the observation that $\Sigma_{\ell-1} = X \otimes V^{\otimes d-\ell}$ and $I_r = V^{\otimes \ell} \otimes Y$ where

$$X = \Sigma_{\ell-1} \subseteq V^{\otimes \ell}, \quad \text{and} \quad Y = I_r \subseteq V^{\otimes(d-\ell)}$$

whence $\dim(\Sigma_{\ell-1} \cap I_r) = \dim(X \otimes Y) = \dim X \cdot \dim Y =$ the classical dimension.

Since Σ_ℓ and I_r have classical dimension, and $\Sigma_\ell \cap I_r$ has classical dimension by assumption, $\Sigma_\ell + I_r$ also has classical dimension. But this is true for all ℓ so the left-hand side of (9.5) has classical dimension. So does the right-hand side because

$$\begin{aligned}\dim(\Sigma_\ell \cap (\Sigma_{\ell-1} + I_r)) &= \dim \Sigma_\ell + \dim(\Sigma_{\ell-1} + I_r) - \dim(\Sigma_\ell + (\Sigma_{\ell-1} + I_r)) \\ &= \dim \Sigma_\ell + \dim(\Sigma_{\ell-1} + I_r) - \dim(\Sigma_\ell + I_r).\end{aligned}$$

Thus the left- and right-hand sides of (9.5) have classical dimensions. However,

$$\Sigma_{\ell-1} + I_{r+1} \subseteq \Sigma_\ell \cap (\Sigma_{\ell-1} + I_r)$$

and this inclusion is an equality in the case of the polynomial ring so both the left- and right-hand sides have the same dimension for the polynomial ring *and* in the present situation too because they have classical dimensions (by hypothesis). Hence this inclusion is an equality in our case too; i.e., (9.5) holds, and the proof is complete. □

Thus, to show that $Q_{n,k}(E, \tau)$ is a Koszul algebra it suffices to show that $\Sigma_\ell \cap I_r$ (with $r = d - \ell - 1$) has classical dimension for all $d \geq 3$ and $0 \leq \ell \leq d - 1$. We will achieve this goal in Proposition 9.16.

Let $r = d - \ell - 1$. If $\ell \in \{0, 1, d - 1\}$, then $\Sigma_\ell \cap I_r$ has classical dimension so we can assume that $2 \leq \ell \leq d - 2$ (i.e., $1 \leq r \leq d - 3$), but for now we also allow the case $\ell = 1$ (i.e., $r = d - 2$) to show some necessary results for induction and exclude this case later.

To show that $\Sigma_\ell \cap I_r$ has classical dimension we first convert the problem into a question about the rank of the operator $F^L_{\ell+1}(-\tau)F^R_{r+1}(\tau) : V^{\otimes d} \to V^{\otimes d}$.

Lemma 9.4 *Let $F_p(z)$ be the operator on $V^{\otimes p}$ defined in (6.2). We have*

$$\dim(\Sigma_\ell \cap I_r) = \dim\Big(\ker F^L_{\ell+1}(-\tau)F^R_{r+1}(\tau)\Big) - n^\ell \left(n^{r+1} - \tbinom{n}{r+1}\right).$$

Proof Since $\Sigma_\ell \cap I_r = \sum_{i=1}^{\ell} W_i \cap \bigcap_{j=\ell+1}^{d-1} W_j$ (by definition),

$$\Sigma_\ell \cap I_r = \ker\big(F^L_{\ell+1}(-\tau) : V^{\otimes d} \to V^{\otimes d}\big) \cap \operatorname{im}\big(F^R_{r+1}(\tau) : V^{\otimes d} \to V^{\otimes d}\big)$$

by Theorems 6.5 and 7.6. Therefore

$$\dim(\Sigma_\ell \cap I_r) = \dim\Big(\ker F^L_{\ell+1}(-\tau)F^R_{r+1}(\tau)\Big) - \dim\Big(\ker F^R_{r+1}(\tau)\Big).$$

But $\dim\big(\ker F^R_{r+1}(\tau)\big) = n^\ell \left(n^{r+1} - \binom{n}{r+1}\right)$ by Theorem 7.6, so the proof is complete. □

9.1.1 Notation for the classical case

For the symmetric algebra, SV, the analogue of W_i is

$$\Lambda_{i,i+1} \;:=\; V^{\otimes(i-1)} \otimes \mathrm{Alt}^2 V \otimes V^{\otimes(d-i-1)}$$

The classical analogue of $\Sigma_\ell \cap I_r$ is therefore the space

$$W^{\ell+1,r+1} \;:=\; \sum_{i=1}^{\ell} \Lambda_{i,i+1} \cap \bigcap_{i=\ell+1}^{d-1} \Lambda_{i,i+1}.$$

9.2 The operators $T_{\ell,r}(z)$ and $H_\tau(z)$ on $V^{\otimes d}$

When $\ell \geq 2$, we will use the notation

$$T_{\ell,r}(z) \;:=\; T_d(\tau^r, -(r+1)\tau, (-\tau)^{\ell-2}, z).$$

When $\ell = 1$, we only define

$$T_{1,r}(-\tau) \;:=\; T_d(\tau^r, -(r+1)\tau).$$

Lemma 9.5 *With the above notation,*

$$T_{\ell,r}(z) \;=\; M_{r+1,\ell}(-(r+1)\tau;\, \tau^r;\, (-\tau)^{\ell-2}, z) \cdot T^L_\ell((-\tau)^{\ell-2}, z) \cdot T^R_{r+1}(\tau^r). \quad (9.6)$$

Proof This follows from Lemma 8.4. □

We now define $H_\tau(z)$.

Proposition 9.6 *If $\ell \geq 2$, then the theta operator*

$$S^{\mathrm{rev}}_{d\to 1}(z,\ (-\tau)^{\ell-2},\ -(r+1)\tau,\ \tau^r) \quad (9.7)$$

on $V^{\otimes d}$ restricts to a theta operator

$$H_\tau(z) : V \otimes \operatorname{im} T_{\ell-1,r}(-\tau) \;\longrightarrow\; \operatorname{im} T_{\ell-1,r}(-\tau) \otimes V$$

and $\operatorname{im} H_\tau(z) = \operatorname{im} T_{\ell,r}(z)$.

Proof By Lemma 6.1,

$$\begin{aligned} T_{\ell,r}(z) \;&=\; S^{\mathrm{rev}}_{d\to 1}(z,\ (-\tau)^{\ell-2},\ -(r+1)\tau,\ \tau^r) \cdot T^R_{d-1}(\tau^r,\ -(r+1)\tau, (-\tau)^{\ell-2}) \\ &=\; T^L_{d-1}(\tau^r,\ -(r+1)\tau,\ (-\tau)^{\ell-2}) \cdot S_{d\to 1}(\tau^r,\ -(r+1)\tau,\ (-\tau)^{\ell-2},\ z). \end{aligned} \quad (9.8)$$

In other words,

$$\begin{aligned} T_{\ell,r}(z) &= S^{\text{rev}}_{d\to 1}(z,\ (-\tau)^{\ell-2},\ -(r+1)\tau,\ \tau^r)\ \cdot\ T^R_{\ell-1,r}(-\tau) \\ &= T^L_{\ell-1,r}(-\tau)\ \cdot\ S_{d\to 1}(\tau^r,\ -(r+1)\tau,\ (-\tau)^{\ell-2},\ z) \end{aligned} \tag{9.9}$$

The results follow because $\operatorname{im} T^R_{\ell-1,r}(-\tau) = V \otimes \operatorname{im} T_{\ell-1,r}(-\tau)$ and $\operatorname{im} T^L_{\ell-1,r}(-\tau) = \operatorname{im} T_{\ell-1,r}(-\tau) \otimes V$. □

Lemma 9.7 *Let $1 \le \ell \le d-2$. Then $\Sigma_\ell \cap I_r$ has classical dimension if and only if*

$$\dim\bigl(\operatorname{im} T_{\ell,r}(-\tau)\bigr) = n^\ell \tbinom{n}{r+1} - \dim W^{\ell+1,r+1}. \tag{9.10}$$

Proof Setting $z = -\tau$ and using (8.2) to factor the $M_{r+1,\ell}$ term in (9.6) as $A \cdot S^{\text{rev}}_{1\to\ell+1}((-\tau)^\ell)$, we have

$$\begin{aligned} T_{\ell,r}(-\tau) &= A \cdot S^{\text{rev}}_{1\to\ell+1}((-\tau)^\ell) T^L_\ell((-\tau)^{\ell-1}) T^R_{r+1}(\tau^r) \\ &= A \cdot T^L_{\ell+1}((-\tau)^\ell) T^R_{r+1}(\tau^r) \qquad \text{by Lemma 6.1} \\ &= A \cdot F^L_{\ell+1}(-\tau) F^R_{r+1}(\tau). \end{aligned}$$

Since the R's appearing in A belong to $\{R(-2\tau), \ldots, R(-(d-1)\tau)\}$, the assumption $\tau \in \mathbb{C} - \bigcup_{m\ge 1} \frac{1}{m}\Lambda$ implies that A is an isomorphism. Hence, by Lemma 9.4,

$$\begin{aligned} \dim(\Sigma_\ell \cap I_r) &= \dim\Bigl(\ker F^L_{\ell+1}(-\tau) F^R_{r+1}(\tau)\Bigr) - n^\ell \left(n^{r+1} - \tbinom{n}{r+1}\right) \\ &= \dim V^{\otimes d} - \dim(\operatorname{im} T_{\ell,r}(-\tau)) - n^\ell \left(n^{r+1} - \tbinom{n}{r+1}\right) \\ &= n^\ell \tbinom{n}{r+1} - \dim(\operatorname{im} T_{\ell,r}(-\tau)). \end{aligned}$$

Since $W^{\ell+1,r+1}$ is the classical analogue of $\Sigma_\ell \cap I_r$, the result follows. □

9.2.1 The induction hypothesis

We will prove that (9.10) holds by induction on d. Thus, we assume (9.10) is true for $d-1$ or fewer tensorands. If $\ell = 1$, then (9.10) follows from Lemma 9.7 since $\Sigma_1 \cap I_r = I_{r+1}$ has classical dimension. So we also assume $2 \le \ell \le d-2$, i.e., $1 \le r \le d-3$. The induction hypothesis implies that

$$\dim\bigl(\text{the domain of } H_\tau(z)\bigr) = n^\ell \tbinom{n}{r+1} - n \dim W^{\ell,r+1}. \tag{9.11}$$

The function $\det H_\tau(z)$ in the next result is only defined up to a non-zero scalar multiple (see Sect. 4.2.1).

Proposition 9.8 *The function* $\det H_\tau(z)$ *is a theta function with respect to $\frac{1}{n}\Lambda$ having*

$$(d-1)\dim\bigl(V \otimes \operatorname{im} T_{\ell-1,r}(-\tau)\bigr) = (d-1)\Bigl(n^\ell \tbinom{n}{r+1} - n \dim W^{\ell,r+1}\Bigr) \tag{9.12}$$

zeros in each fundamental parallelogram for $\frac{1}{n}\Lambda$, *all of which belong to*

$$\{-\tau, 0, \tau, \ldots, (d-1)\tau\} + \tfrac{1}{n}\Lambda. \tag{9.13}$$

Proof Since $R(z)$ is a theta operator of order n^2 with respect to Λ, $S^{\mathrm{rev}}_{1\to d}(z,\ldots)$ in (9.7) is a theta operator of order $(d-1)n^2$ with respect to Λ. Since $H_\tau(z)$ is the restriction of $S^{\mathrm{rev}}_{1\to d}(z,\ldots)$ to $\operatorname{im} T^R_{\ell-1,r}(-\tau)$, Proposition 4.6 tells us that $\det H_\tau(z)$ has

$$(d-1)n^2 \dim\bigl(\operatorname{im} T^R_{\ell-1,r}(-\tau)\bigr)$$

zeros in every fundamental parallelogram for Λ. However, since $\det R(z)$ is a theta function with respect to $\frac{1}{n}\Lambda$, so is $\det H_\tau(z)$. Hence $\det H_\tau(z)$ has $(d-1)\dim\bigl(\operatorname{im} T^R_{\ell-1,r}(-\tau)\bigr)$ zeros in every fundamental parallelogram for $\frac{1}{n}\Lambda$.

By the induction hypothesis, (9.10) tells us that $\dim(\operatorname{im} T_{\ell-1,r}(-\tau)) = n^{\ell-1}\binom{n}{r+1} - \dim W^{\ell,r+1}$. The equality in (9.12) now follows once we observe that $\det H_\tau(z)$ is not identically zero: it isn't because the factors of $S^{\mathrm{rev}}_{1\to d}(z,\ldots)$ are invertible for all but finitely many z's.

Since $H_\tau(z)$ is the restriction of a product of terms of the form $R(z-m\tau)_{i,i+1}$ for various i's and $m = 0,\ldots,d-2$, the zeros of $\det H_\tau(z)$ belong to

$$\{z \mid \det R(z-m\tau) = 0 \text{ for some } m \in [0, d-2]\}.$$

But $\det R(z) = 0$ if and only if $z \in \pm\tau + \frac{1}{n}\Lambda$, so this set is $\{-\tau, 0, \tau, \ldots, (d-1)\tau\} + \frac{1}{n}\Lambda$. □

We now examine $\operatorname{mult}_p(\det H_\tau(z))$ for the p's in (9.13). In truth, we will only examine $\operatorname{mult}_p(\det H_\tau(z))$ when $p \in \{-\tau, 0, \tau, \ldots, (d-1)\tau\}$ and then apply Lemma 9.14.

As the next result shows, $H_\tau(p) = 0$ for some of these p's.

Lemma 9.9 *If* $m \in \mathbb{Z} \cap [1, d-3]$, *then* $\dim(\ker H_\tau(m\tau)) \geq n^\ell\binom{n}{r+1} - n\dim W^{\ell,r+1}$.

Proof By (9.11), the inequality is equivalent to $H_\tau(m\tau) = 0$.

Since $\operatorname{im} H_\tau(m\tau) = \operatorname{im} T_{\ell,r}(m\tau)$ by Proposition 9.6, $H_\tau(m\tau) = 0$ if $T_{\ell,r}(m\tau) = 0$. Thus we will prove the lemma by showing that $T_{\ell,r}(m\tau) = 0$ for the m's in $[1, d-3]$. We split the proof into two parts.

(1) Assume $1 \leq m \leq \ell - 2$. (This case is vacuous if $\ell = 2$ so we assume $\ell \geq 3$.) By Lemma 9.5, $T^L_\ell((-\tau)^{\ell-2}, m\tau)$ is a factor of $T_{\ell,r}(m\tau)$ so it suffices to show that $T_\ell((-\tau)^{\ell-2}, m\tau) = 0$. By Lemma 6.1,

$$T_\ell((-\tau)^{\ell-2}, m\tau) \;=\; S^{\mathrm{rev}}_{\ell\to 1}(m\tau, (-\tau)^{\ell-2}) \cdot T^R_{\ell-1}((-\tau)^{\ell-2}).$$

But

$$S^{\mathrm{rev}}_{\ell\to 1}(m\tau, (-\tau)^{\ell-2}) \;=\; R(m\tau)_{\ell-1,\ell} R((m-1)\tau)_{\ell-2,\ell-1} \cdots R((m-\ell+2)\tau)_{12}$$

and this has two consecutive factors of the form

$$R(\tau)_{j,j+1}\, R(0)_{j-1,j}$$

for some $j \geq 2$. The $R(\tau)_{j,j+1}$ commutes past the $R(0) = I \otimes I$ term, and also commutes past the other factors to its right, to ultimately annihilate the term $T^R_{\ell-1}((-\tau)^{\ell-2}) = R(-\tau)_{j,j+1}Q'$ (where the latter equality uses Proposition 6.3(1)).

(2) Assume $\ell - 1 \leq m \leq d-3$. By (8.3), $M_{r+1,\ell}(-(r+1)\tau;\, \tau^r;\, (-\tau)^{\ell-2}, m\tau)$ is right-divisible by

$$\begin{aligned} &S^{\mathrm{rev}}_{d\to\ell}((m-d+2)\tau, \tau^r) \\ &\quad = R((m-d+2)\tau)_{d-1,d}R((m-d+3)\tau)_{d-2,d-1}\cdots R((m-\ell+1)\tau)_{\ell,\ell+1}, \end{aligned}$$

and $m-d+2 \leq -1 \leq 0 \leq m-\ell+1$. Hence it has two consecutive factors of the form $R(-\tau)_{j,j+1}R(0)_{j-1,j}$ for some $j \geq \ell+1$. Since $R(0) = I \otimes I$, $R(-\tau)_{j,j+1}$ commutes with all factors to the right of it in $S^{\mathrm{rev}}_{d\to\ell}((m-d+2)\tau, \tau^r)$. After moving it all the way to the right in the expression (9.6), we conclude that $T_{\ell,r}(m\tau)$ is right-divisible by $R(-\tau)_{j,j+1}T^R_{r+1}(\tau^r)$. However, by Proposition 6.3(2), $T^R_{r+1}(\tau^r) = R(\tau)_{j,j+1}Q$ for some Q so $R(-\tau)_{j,j+1}T^R_{r+1}(\tau^r) = 0$. □

Lemma 9.10 *If $T^R_{\ell-1,r}(-\tau)$ denotes $I \otimes T_{\ell-1,r}(-\tau)$ acting on $V^{\otimes d}$, then*

$$\operatorname{rank} T^R_{\ell-1,r}(-\tau)S_{1\to\ell}((-\tau)^{\ell-1}) \;=\; \tbinom{n+\ell-1}{\ell}\tbinom{n}{r+1}.$$

Proof As operators on $V^{\otimes(d-1)}$,

$$\begin{aligned} T_{\ell-1,r}(-\tau) \;&=\; M_{r+1,\ell-1}(-(r+1)\tau;\, \tau^r;\, (-\tau)^{\ell-2}) \cdot T^L_{\ell-1}((-\tau)^{\ell-2}) \cdot T^R_{r+1}(\tau^r) \\ &=\; M^R_{r,\ell-1}(-(r+1)\tau;\, \tau^{r-1};\, (-\tau)^{\ell-2}) \cdot S^{\mathrm{rev}}_{1\to\ell}((-\tau)^{\ell-1}) \cdot T^L_{\ell-1}((-\tau)^{\ell-2}) \cdot T^R_{r+1}(\tau^r) \\ &=\; M^R_{r,\ell-1}(-(r+1)\tau;\, \tau^{r-1};\, (-\tau)^{\ell-2}) \cdot T^L_{\ell}((-\tau)^{\ell-1}) \cdot T^R_{r+1}(\tau^r) \end{aligned}$$

where $M^R_{r,\ell-1}(\cdots)$ is acting on the $(d-2)$ right-most tensorands of $V^{\otimes(d-1)}$; the second equality comes from (8.2), and the third from Lemma 6.1. We now view this as an equality of operators on $V^{\otimes d} = V \otimes V^{\otimes(d-1)}$ by considering each of the four operators in it as acting on the right-most $(d-1)$ tensorands; i.e., we replace each operator by ($I \otimes$ itself). But $I \otimes T_{\ell-1,r}(-\tau) = T^R_{\ell-1,r}(-\tau)$, so the equality implies

$$\begin{aligned} T^R_{\ell-1,r}(-\tau) \cdot S_{1\to\ell}((-\tau)^{\ell-1}) = M^R_{r,\ell-1}(\cdots) \cdot (I \otimes T^L_{\ell}((-\tau)^{\ell-1})) \cdot T^R_{r+1}(\tau^r) \\ \cdot S_{1\to\ell}((-\tau)^{\ell-1}). \end{aligned}$$

By the assumption $\tau \in \mathbb{C} - \bigcup_{m\geq 1} \frac{1}{m}\Lambda$, all the R's appearing in the term $M^R_{r,\ell-1}(\cdots)$ are isomorphisms. The rank of $T^R_{\ell-1,r}(-\tau)S_{1\to\ell}((-\tau)^{\ell-1})$ therefore equals that of

$$\big(I \otimes T^L_{\ell}((-\tau)^{\ell-1})\big) \cdot T^R_{r+1}(\tau^r) \cdot S_{1\to\ell}((-\tau)^{\ell-1}). \tag{9.14}$$

The left-most ℓ tensorands of $V^{\otimes d}$ are "disjoint" from the right-most $(r+1)$ tensorands so

$$(9.14) = \big(I \otimes T^L_\ell((-\tau)^{\ell-1})\big) \cdot S_{1\to\ell}((-\tau)^{\ell-1}) \cdot T^R_{r+1}(\tau^r).$$

But $S_{1\to\ell}(z_1,\dots,z_{\ell-1}) = S_{1\to\ell+1}(z_1,\dots,z_{\ell-1},0)$ and

$$T^R_\ell(z_2,\dots,z_\ell)\cdot S_{1\to\ell+1}(z_\ell,\dots,z_1) = S^{\mathrm{rev}}_{1\to\ell+1}(z_1,\dots,z_\ell)\cdot T^L_\ell(z_2,\dots,z_\ell)$$

by Lemma 6.1, so, as operators on $V^{\otimes(\ell+1)}$,

$$\begin{aligned}(9.14) &= \big(I \otimes T^L_\ell((-\tau)^{\ell-1})\big) \cdot S_{1\to\ell}((-\tau)^{\ell-1}) \cdot T^R_{r+1}(\tau^r)\\ &= \big(I \otimes T^L_\ell((-\tau)^{\ell-1})\big) \cdot S_{1\to\ell+1}((-\tau)^{\ell-1},0) \cdot T^R_{r+1}(\tau^r)\\ &= S^{\mathrm{rev}}_{1\to\ell+1}(0,(-\tau)^{\ell-1}) \cdot T^L_\ell((-\tau)^{\ell-1}) \cdot T^R_{r+1}(\tau^r).\end{aligned}$$

By Lemma 6.11, $G_\tau(0)$ is an isomorphism. But $G_\tau(0)$ is the restriction of $S^{\mathrm{rev}}_{1\to\ell+1}(0,(-\tau)^{\ell-1})$ to the image of $T^L_\ell((-\tau)^{\ell-1})$ so $S^{\mathrm{rev}}_{1\to\ell+1}(0,(-\tau)^{\ell-1})$ acts injectively on the image of $T^L_\ell((-\tau)^{\ell-1})$, whence

$$\operatorname{rank}(9.14) = \operatorname{rank} T^L_\ell((-\tau)^{\ell-1})T^R_{r+1}(\tau^r).$$

As operators on $V^{\otimes d} = V^{\otimes \ell}\otimes V^{\otimes(r+1)}$, $T^L_\ell((-\tau)^{\ell-1})T^R_{r+1}(\tau^r) = T_\ell((-\tau)^{\ell-1}) \otimes T_{r+1}(\tau^r)$ so

$$\begin{aligned}\operatorname{rank} T^L_\ell((-\tau)^{\ell-1})T^R_{r+1}(\tau^r) &= \operatorname{rank} T^L_\ell((-\tau)^{\ell-1}) \cdot \operatorname{rank} T^R_{r+1}(\tau^r)\\ &= \operatorname{rank} F_\ell(-\tau)\cdot \operatorname{rank} F_{r+1}(\tau)\end{aligned}$$

which is $\binom{n+\ell-1}{\ell}\binom{n}{r+1}$ by Theorems 6.5 and 7.6. □

9.2.2 Terminology

A family of linear operators $A(z)$ has a zero of multiplicity m at a point $p \in \mathbb{C}$ if $A(z) = (z-p)^m B(z)$ where $B(z)$ is an operator such that $\det B(z)$ has neither a zero nor a pole at p.

Lemma 9.11 *The restriction of $H_\tau(z)$ to $\operatorname{im} T^R_{\ell-1,r}(-\tau)S_{1\to\ell}((-\tau)^{\ell-1})$ has a zero of multiplicity ≥ 2 at $z = (\ell-1)\tau$.*

Proof Since $H_\tau(z) : \operatorname{im} T^R_{\ell-1,r}(-\tau) \to \operatorname{im} T^L_{\ell-1,r}(-\tau)$ is the restriction of $S^{\mathrm{rev}}_{d\to 1}(z,\ (-\tau)^{\ell-2},\ -(r+1)\tau,\ \tau^r)$, it suffices to prove that

$$S^{\mathrm{rev}}_{d\to 1}(z,(-\tau)^{\ell-2},-(r+1)\tau,\tau^r)\cdot T^R_{\ell-1,r}(-\tau)\cdot S_{1\to\ell}((-\tau)^{\ell-1}) \tag{9.15}$$

has a zero of multiplicity ≥ 2 on its domain $V^{\otimes d}$. Since

$$
\begin{aligned}
&S^{\mathrm{rev}}_{d\to 1}(z, (-\tau)^{\ell-2}, -(r+1)\tau, \tau^r) \cdot T^R_{\ell-1,r}(-\tau) \\
&\quad = T_d(\tau^r, -(r+1)\tau, (-\tau)^{\ell-2}, z) \\
&\quad = M_{r+1,\ell}(-(r+1)\tau; \tau^r; (-\tau)^{\ell-2}, z) \cdot T^R_{r+1}(\tau^r) \cdot T^L_{\ell}((-\tau)^{\ell-2}, z)
\end{aligned}
$$

by (9.9) and Lemma 8.4, the operator in (9.15) is the composition of

$$
M_{r+1,\ell}(-(r+1)\tau; \tau^r; (-\tau)^{\ell-2}, z) \cdot T^R_{r+1}(\tau^r) \tag{9.16}
$$

and

$$
T^L_{\ell}((-\tau)^{\ell-2}, z) \cdot S_{1\to\ell}((-\tau)^{\ell-1}). \tag{9.17}
$$

We will show that each of these operators is zero at $z = (\ell-1)\tau$.

First,

$$
\begin{aligned}
(9.16) &= M^L_{r+1,\ell-1}(-(r+1)\tau; \tau^r; (-\tau)^{\ell-2}) \cdot S^{\mathrm{rev}}_{d\to\ell}(z-(\ell+r-1)\tau, \tau^r) \cdot T^R_{r+1}(\tau^r) \\
&= M^L_{r+1,\ell-1}(-(r+1)\tau; \tau^r; (-\tau)^{\ell-2}) \cdot T^R_{r+2}(\tau^r, z-(\ell+r-1)\tau)
\end{aligned}
$$

where the first and second equalities follow from (8.3) and Lemma 6.1, respectively. When $z = (\ell-1)\tau$, the right-most factor is $T^R_{r+2}(\tau^r, -r\tau)$ which $= 0$ by (7.6). Hence (9.16)$= 0$.

When $z = (\ell-1)\tau$,

$$
\begin{aligned}
(9.17) &= T^L_{\ell-1}((-\tau)^{\ell-2}) \cdot S_{\ell\to 1}((-\tau)^{\ell-2}, (\ell-1)\tau) \cdot S_{1\to\ell}((-\tau)^{\ell-1}) \qquad \text{by Lemma 6.1} \\
&= T^L_{\ell-1}((-\tau)^{\ell-2}) \cdot R(\tau)_{\ell-1,\ell} \cdots R((\ell-1)\tau)_{12} \cdot R(-(\ell-1)\tau)_{12} \cdots R(-\tau)_{\ell-1,\ell}.
\end{aligned}
$$

By Lemma 2.3, for all u the operator $R(u)R(-u)$ is a scalar multiple of the identity so we can rearrange the terms in this product to obtain a factor of the form $R(\tau)_{\ell-1,\ell}R(-\tau)_{\ell-1,\ell}$ which $= 0$. □

Lemma 9.12 $\dim(\ker H_\tau(-\tau)) \geq \dim W^{\ell+1,r+1} - n \dim W^{\ell,r+1}$.

Proof Since $\mathrm{im}\, H_\tau(m\tau) = \mathrm{im}\, T_{\ell,r}(m\tau)$ by Proposition 9.6,

$$
\begin{aligned}
\dim(\ker H_\tau(z)) + \dim(\mathrm{im}\, T_{\ell,r}(z)) &= \dim(\mathrm{im}\, T^R_{\ell-1,r}(-\tau)) \\
&= n^\ell \tbinom{n}{r+1} - n \dim W^{\ell,r+1}
\end{aligned}
$$

where the second equality follows from the induction hypothesis. Thus, to prove the lemma we must show that

$$
n^\ell \tbinom{n}{r+1} - \dim(\mathrm{im}\, T_{\ell,r}(-\tau)) \geq \dim W^{\ell+1,r+1}.
$$

However,

$$
\dim(\ker T_{\ell,r}(-\tau)) \geq \dim(\ker F^L_{\ell+1}(-\tau) F^R_{r+1}(\tau)) \quad \text{because, as observed in Lemma 9.7,}
$$

$$\begin{aligned}
T_{\ell,r}(-\tau) &= A \cdot F^L_{\ell+1}(-\tau) F^R_{r+1}(\tau) \text{ for some } A \\
&= \dim \left(\ker F^L_{\ell+1}(-\tau) \cap \operatorname{im} F^R_{r+1}(\tau)\right) + \dim(\ker F^R_{r+1}(\tau)) \\
&= \dim \left(\ker F^L_{\ell+1}(-\tau) \cap \operatorname{im} F^R_{r+1}(\tau)\right) + n^d - n^\ell \tbinom{n}{r+1} \quad \text{by Theorem 7.6} \\
&= \dim \left(\sum_{i=1}^{\ell} W_i \cap \bigcap_{j=\ell+1}^{d-1} W_j \right) + n^d - n^\ell \tbinom{n}{r+1} \quad \text{by Theorems 6.5 and 7.6} \\
&= \dim\left(\Sigma_\ell \cap I_r\right) + n^d - n^\ell \tbinom{n}{r+1}.
\end{aligned}$$

It follows that $n^d - \dim(\operatorname{im} T_{\ell,r}(-\tau)) \geq \dim\left(\Sigma_\ell \cap I_r\right) + n^d - n^\ell \binom{n}{r+1}$; i.e.,

$$n^\ell \tbinom{n}{r+1} - \dim(\operatorname{im} T_{\ell,r}(-\tau)) \geq \dim\left(\Sigma_\ell \cap I_r\right).$$

Thus, the proof will be complete once we show that

$$\dim\left(\Sigma_\ell \cap I_r\right) \geq \dim W^{\ell+1,r+1}$$

(note that the right-hand side is the classical analogue of the left-hand side). Since

$$\Sigma_\ell \cap I_r \supseteq \sum_{i=1}^{\ell} \left(W_i \cap \bigcap_{j=\ell+1}^{d-1} W_j \right),$$

with equality when $\mathsf{Lat}(d)$ is distributive, we consider the expression on the right. The term inside the parentheses has classical dimension by Theorem 7.6 and hence the right hand sum has generically large dimension by Proposition 4.9. By Corollary 5.3 the sum on the right has classical dimension in the limit as $\tau \to 0$, so

$$\dim\left(\Sigma_\ell \cap I_r\right) \geq \dim \sum_{i=1}^{\ell} \left(W_i \cap \bigcap_{j=\ell+1}^{d-1} W_j \right) \geq \dim W^{\ell+1,r+1}$$

on a dense set of τ's. But $\dim\left(\Sigma_\ell \cap I_r\right)$ is generically small because $\dim(\ker F^L_{\ell+1}(-\tau) F^R_{r+1}(\tau))$ is, so $\dim\left(\Sigma_\ell \cap I_r\right) \geq \dim W^{\ell+1,r+1}$ for all τ. □

Lemma 9.13 *We have*

$$\dim(\ker H_\tau((d-1)\tau)) \geq \dim W^{\ell,r+2} + n^\ell \tbinom{n}{r+1} - n^{\ell-1} \tbinom{n}{r+2} - n \dim W^{\ell,r+1}. \tag{9.18}$$

Proof As observed in the proof of Lemma 9.12,

$$\dim(\ker H_\tau(z)) + \dim(\operatorname{im} T_{\ell,r}(z)) = n^\ell \tbinom{n}{r+1} - n \dim W^{\ell,r+1}.$$

Thus, to prove the lemma we must show that

$$n^\ell\binom{n}{r+1} - n\dim W^{\ell,r+1} - \dim(\operatorname{im} T_{\ell,r}((d-1)\tau)) \geq \text{the right-hand side of (9.18)}$$

or, equivalently, that

$$\dim(\ker T_{\ell,r}((d-1)\tau)) \geq n^d - n^{\ell-1}\binom{n}{r+2} + \dim W^{\ell,r+2}. \tag{9.19}$$

We now consider $\dim\ker T_{\ell,r}((d-1)\tau)$. We have

$$\begin{aligned}
&T_{\ell,r}((d-1)\tau)\\
&\quad = M_{r+1,\ell}(-(r+1)\tau;\tau^r;(-\tau)^{\ell-2},(d-1)\tau)\\
&\qquad \cdot \big(T_\ell((-\tau)^{\ell-2},(d-1)\tau)\otimes T_{r+1}(\tau^r)\big) \qquad \text{by (9-6)}\\
&\quad = B\cdot S^{\text{rev}}_{d\to\ell}(\tau,\tau^r)\cdot\big(T_\ell((-\tau)^{\ell-2},(d-1)\tau)\otimes T_{r+1}(\tau^r)\big) \qquad \text{by (8-3)}
\end{aligned}$$

where

$$\begin{aligned}
B &= S^{\text{rev}}_{r+2\to 1}(-(r+1)\tau,\tau^r)S^{\text{rev}}_{r+3\to 2}(-(r+2)\tau,\tau^r)\cdots S^{\text{rev}}_{d-1\to\ell-1}(-(d-2)\tau,\tau^r)\\
&= S^{\text{rev}}_{r+1\to d-1}(-(r+1)\tau,(-\tau)^{\ell-2})S^{\text{rev}}_{r\to d-2}(-r\tau,(-\tau)^{\ell-2})\cdots S^{\text{rev}}_{1\to\ell}((-\tau)^{\ell-1}),
\end{aligned}$$

the equality being essentially the same as (8.2) = (8.3). We write $B = C\cdot S^{\text{rev}}_{1\to\ell}((-\tau)^{\ell-1})$.

By Lemma 6.1, $T_\ell((-\tau)^{\ell-2},z) = T^L_{\ell-1}((-\tau)^{\ell-2})\cdot S_{\ell\to 1}((-\tau)^{\ell-2},z)$. Hence $\dim\ker T_{\ell,r}((d-1)\tau)$ is $\geq$ the dimension of the kernel of

$$\begin{aligned}
&B\cdot S^{\text{rev}}_{d\to\ell}(\tau,\tau^r)\cdot\big(T_{\ell-1}((-\tau)^{\ell-2})\otimes I\otimes T_{r+1}(\tau^r)\big)\\
&\quad = B\cdot\big(I^{\otimes(\ell-1)}\otimes S^{\text{rev}}_{r+2\to 1}(\tau^{r+1})\big)\cdot\big(T_{\ell-1}((-\tau)^{\ell-2})\otimes I\otimes T_{r+1}(\tau^r)\big)\\
&\quad = B\cdot\big(T_{\ell-1}((-\tau)^{\ell-2})\otimes S^{\text{rev}}_{r+2\to 1}(\tau^{r+1})T^R_{r+1}(\tau^r)\big)\\
&\quad = B\cdot\big(T_{\ell-1}((-\tau)^{\ell-2})\otimes T_{r+2}(\tau^{r+1})\big) \qquad \text{by Lemma 6.1}\\
&\quad = B\cdot T^L_{\ell-1}((-\tau)^{\ell-2})\cdot T^R_{r+2}(\tau^{r+1})\\
&\quad = C\cdot S^{\text{rev}}_{1\to\ell}((-\tau)^{\ell-1})\cdot T^L_{\ell-1}((-\tau)^{\ell-2})\cdot T^R_{r+2}(\tau^{r+1})\\
&\quad = C\cdot T^L_\ell((-\tau)^{\ell-1})\cdot T^R_{r+2}(\tau^{r+1}) \qquad \text{by Lemma 6.1.}
\end{aligned}$$

In particular,

$$\begin{aligned}
&\dim(\ker T_{\ell,r}((d-1)\tau)) \geq \dim\big(\ker F^L_\ell(-\tau)F^R_{r+2}(\tau)\big)\\
&\quad = \dim\big(\ker F^L_\ell(-\tau)\cap\operatorname{im} F^R_{r+2}(\tau)\big) + \dim(\ker F^R_{r+2}(\tau))\\
&\quad = \dim(\Sigma_{\ell-1}\cap I_{r+1}) + \dim(\ker F^R_{r+2}(\tau)) \qquad \text{by Theorems 6.5 and 7.6}\\
&\quad \geq \dim W^{\ell,r+2} + \dim(\ker F^R_{r+2}(\tau)) \quad \text{as in the proof of Lemma 9.12}\\
&\quad = \dim W^{\ell,r+2} + n^d - n^{\ell-1}\binom{n}{r+2} \quad \text{by Theorem 7.6.}
\end{aligned}$$

Thus, the inequality in (9.19) holds and the proof is complete. □

Lemma 9.14 *For all $\zeta \in \frac{1}{n}\Lambda$, $H_\tau(z+\zeta)$ and $H_\tau(z)$ have the same nullity.*

Proof The proof resembles that of Lemma 6.10; actually, it's a little simpler because of some cancellation. As in that proof, we write $\zeta = \frac{a}{n} + \frac{b}{n}\eta$ with $a, b \in \mathbb{Z}$ and set $C := T^b S^{ka}$.

By Corollary 2.7, $R_\tau(z+\zeta) = f(z,\zeta,\tau)C_2^{-1}R_\tau(z)C_1$.

By definition, $H_\tau(z)$ is the restriction of $S^{\mathrm{rev}}_{d\to 1}(z,\ldots) = R(z+*)_{d-1,d}\cdots R(z+*)_{12}$, where the $*$'s represent some terms that play no role in the calculations below, to the image of $T^R_{\ell-1,r}(-\tau)$. Hence $H_\tau(z+\zeta)$ is the restriction of

$$\begin{aligned}
&R(z+\zeta+*)_{d-1,d}R(z+\zeta+*)_{d-2,d-1}\cdots R(z+\zeta+*)_{12}\\
&\quad = g(z,\zeta,\tau)C_d^{-1}R(z+*)_{d-1,d}C_{d-1}\cdot C_{d-1}^{-1}R(z+*)_{d-2,d-1}C_{d-2}\\
&\qquad \cdots C_2^{-1}R(z+*)_{12}C_1\\
&\quad = g(z,\zeta,\tau)C_d^{-1}\,R(z+*)_{d-1,d}R(z+*)_{d-2,d-1}\cdots R(z+*)_{12}C_1\\
&\quad = g(z,\zeta,\tau)C_d^{-1}\,S^{\mathrm{rev}}_{d\to 1}(z,\ldots)\,C_1
\end{aligned}$$

to the image of $T^R_{\ell-1,r}(-\tau)$, where the function $g(z,\zeta,\tau)$ is a product of various $f(\cdot,\cdot,\cdot)$'s and therefore never vanishes.

Thus, the nullity of $H_\tau(z+\zeta)$ is the same as the nullity of the restriction of $S^{\mathrm{rev}}_{1\to d}(z,\ldots)C_1$ to $\operatorname{im} T^R_{\ell-1,r}(-\tau) = V\otimes \operatorname{im} T_{\ell-1,r}(-\tau)$. But C_1 is an automorphism so the nullity of $H_\tau(z+\zeta)$ equals the nullity of the restriction of $S^{\mathrm{rev}}_{1\to d}(z,\ldots)$ to $V\otimes \operatorname{im} T_{\ell-1,r}(-\tau)$; i.e., it equals the nullity of $H_\tau(z)$. □

Lemmas 9.9, 9.12 and 9.13, which also hold when $\zeta \in \frac{1}{n}\Lambda$ is added to the input of $H_\tau(z)$, tell us that

$$\begin{aligned}
&\dim(\ker H_\tau(p))\\
&\quad \ge \begin{cases}
\dim W^{\ell,r+2} + n^\ell\binom{n}{r+1} - n^{\ell-1}\binom{n}{r+2} & \\
\quad -n\dim W^{\ell,r+1} & p\in (d-1)\tau+\frac{1}{n}\Lambda,\\
n^\ell\binom{n}{r+1} - n\dim W^{\ell,r+1} & p\in m\tau+\frac{1}{n}\Lambda \text{ and}\\
 & \ m\in\{1,\ldots,d-3\},\\
\dim W^{\ell+1,r+1} - n\dim W^{\ell,r+1} & p\in -\tau+\frac{1}{n}\Lambda,\\
0 & \text{otherwise.}
\end{cases}
\end{aligned} \tag{9.20}$$

Further taking into account the multiplicity-two result in Lemma 9.11 for $p = (\ell-1)\tau$, and for $p = (\ell-1)\tau+\zeta$ with $\zeta\in\frac{1}{n}\Lambda$, the following inequalities for the singularity partitions of H_τ (Definition 4.2) hold:

$$|\sigma_p(H_\tau)| \geq \begin{cases} \dim W^{\ell,r+2} + n^\ell\binom{n}{r+1} - n^{\ell-1}\binom{n}{r+2} \\ \quad -n\dim W^{\ell,r+1} & p \in (d-1)\tau + \frac{1}{n}\Lambda, \\ n^\ell\binom{n}{r+1} - n\dim W^{\ell,r+1} & p \in m\tau + \frac{1}{n}\Lambda \text{ and} \\ & \; m \in \{1,\ldots,d-3\} - \{\ell-1\}, \\ n^\ell\binom{n}{r+1} - n\dim W^{\ell,r+1} + \binom{n+\ell-1}{\ell}\binom{n}{r+1} & p \in (\ell-1)\tau + \frac{1}{n}\Lambda, \\ \dim W^{\ell+1,r+1} - n\dim W^{\ell,r+1} & p \in -\tau + \frac{1}{n}\Lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{9.21}$$

Proposition 9.15 *All inequalities in (9.21) are equalities. Furthermore, if $p \notin (\ell-1)\tau + \frac{1}{n}\Lambda$, all the inequalities in (9.20) are equalities.*

Proof Let P be a fundamental parallelogram with respect to Λ.

We will show that

$$\dim W^{\ell,r+2} + \dim W^{\ell+1,r+1} \;=\; n^\ell\binom{n}{r+1} + n^{\ell-1}\binom{n}{r+2} - \binom{n+\ell-1}{\ell}\binom{n}{r+1} \tag{9.22}$$

in the last paragraph of this proof. For now, assume that (9.22) is true. With that assumption,

$$\begin{aligned} \sum_{p\in P}|\sigma_p(H_\tau)| \;&\geq\; (d-1)n^2\left(n^\ell\binom{n}{r+1} \;-\; n\dim W^{\ell,r+1}\right) \\ &= \text{the number of zeros } \det H_\tau(z) \text{ has in } P \quad \text{by Proposition 9.8} \\ &= \sum_{p\in P}\mathrm{mult}_p(\det H_\tau(z)) \\ &\geq \sum_{p\in P}|\sigma_p(H_\tau)| \quad \text{by Lemma 4.4.} \end{aligned} \tag{9.23}$$

Hence the two inequalities in (9.23) are equalities. It follows that the inequalities in (9.21) are equalities, as claimed.

The second sentence in the proposition follows since the only points p where we have to consider zeros of multiplicity ≥ 2 are those $p \in (\ell-1)\tau + \frac{1}{n}\Lambda$.

We will now prove (9.22). Recall the notation in Sect. 9.1.1, and consider

$$W^{\ell+1,r+1} \;=\; \sum_{i=1}^{\ell}\Lambda_{i,i+1} \cap \bigcap_{i=\ell+1}^{d-1}\Lambda_{i,i+1} \;=\; (X+Y)\cap Z$$

where

$$X = \sum_{i=1}^{\ell-1}\Lambda_{i,i+1}, \quad Y = \Lambda_{\ell,\ell+1}, \quad Z = \bigcap_{i=\ell+1}^{d-1}\Lambda_{i,i+1}.$$

The lattice generated by the $\Lambda_{i,i+1}$'s is distributive so $(X+Y)\cap Z = X\cap Z + Y\cap Z$. Hence

$$\dim W^{\ell+1,r+1} = \dim(X\cap Z) + \dim(Y\cap Z) - \dim(X\cap Y\cap Z). \tag{9.24}$$

But $X\cap Y\cap Z = W^{\ell,r+2}$, $\dim(X\cap Z) = \left(n^\ell - \binom{n+\ell-1}{\ell}\right)\binom{n}{r+1}$, and $\dim(Y\cap Z) = n^{\ell-1}\binom{n}{r+2}$; substituting these into (9.24) yields (9.22). □

Proposition 9.16 $\Sigma_\ell \cap I_r$ *has classical dimension.*

Proof We have

$$\begin{aligned}
&\dim(\mathrm{im} T_{\ell,r}(-\tau))\\
&\quad = n^\ell\binom{n}{r+1} - n\dim W^{\ell,r+1} - \dim(\ker H_\tau(-\tau)) \quad \text{by the proof of Lemma 9.12}\\
&\quad = n^\ell\binom{n}{r+1} - n\dim W^{\ell,r+1} - \dim W^{\ell+1,r+1}\\
&\qquad + n\dim W^{\ell,r+1} \quad \text{by Proposition 9.15}\\
&\quad = n^\ell\binom{n}{r+1} - \dim W^{\ell+1,r+1}.
\end{aligned}$$

The result now follows from Lemma 9.7. □

Theorem 9.17 *For all* $\tau \in (\mathbb{C} - \bigcup_{m\geq 1}\frac{1}{m}\Lambda)\cup\frac{1}{n}\Lambda$, $Q_{n,k}(E,\tau)$ *is a Koszul algebra.*

Proof We observed this in the proof of Theorem 7.7 for $\tau \in \frac{1}{n}\Lambda$. The result for $\tau \in \mathbb{C} - \bigcup_{m\geq 1}\frac{1}{m}\Lambda$ follows from the arguments in this section. □

10 Artin–Schelter regularity of $Q_{n,k}(E,\tau)$

In this section we show, for all but countably many τ, that $Q_{n,k}(E,\tau)$ is an Artin–Schelter regular algebra in the sense of [1]. Suppose $\tau \in (\mathbb{C} - \bigcup_{m\geq 1}\frac{1}{m}\Lambda)\cup\frac{1}{n}\Lambda$. Since $Q := Q_{n,k}(E,\tau)$ has finite Gelfand–Kirillov dimension, it is Artin–Schelter regular of dimension n if the global dimension of Q is n and

$$\mathrm{Ext}^i_Q(\mathbb{C},Q) \cong \begin{cases}\mathbb{C} & \text{if } \ i=n,\\ 0 & \text{if } \ i\neq n.\end{cases}$$

The next result provides a partial confirmation of Artin–Schelter regularity.

Theorem 10.1 *For all* $\tau \in (\mathbb{C} - \bigcup_{m\geq 1}\frac{1}{m}\Lambda)\cup\frac{1}{n}\Lambda$, *the global dimension of* $Q_{n,k}(E,\tau)$ *is* n.

Proof Let A be a connected graded algebra over a field $\Bbbk$. It is well known that the global dimension of A is the largest integer d such that $\mathrm{Ext}^d_A(\Bbbk,\Bbbk)\neq 0$ (see, e.g., [45, Prop. 3.18] and [24, Thm. 11]). If A is a Koszul algebra, then $\mathrm{Ext}^d_A(\Bbbk,\Bbbk)\cong A^!_d$ so its global dimension is the largest integer d such that $A^!_d\neq 0$. Since $Q_{n,k}(E,\tau)^!$ is a Koszul algebra with Hilbert series $(1+t)^n$, the global dimension of $Q_{n,k}(E,\tau)$ is n. □

We will use the following result with $A = Q_{n,k}(E,\tau)$; see, e.g., [50, Prop. 5.10] or [33, Thm. 1.9].

Theorem 10.2 *Let A be a connected graded Koszul algebra over a field $\Bbbk$. If its global dimension is finite, then A is Artin–Schelter regular if and only if $A^!$ is Frobenius.*

A finite-dimensional $\Bbbk$-algebra S is Frobenius if it is isomorphic as a left S-module to its dual $S^* := \operatorname{Hom}_{\Bbbk}(S, \Bbbk)$ equipped with the left module structure resulting from right multiplication. By [50, Lem. 3.2], $S := S_{n,k}(E,\tau)$ is Frobenius if and only if the multiplication maps

$$S_i \times S_{n-i} \longrightarrow S_n \cong \Bbbk$$

are non-degenerate bilinear forms for all $i = 0, \dots, n$; this happens if and only if S_n is the socle of S as a left (or right) S-module.

Proposition 10.3 *For each (n, k, E), $S_{n,k}(E,\tau)$ is a Frobenius algebra for all but finitely many $\tau \in E$.*

Proof In this proof, τ denotes a complex number.

Let $\mathcal{F} = \bigcup_{m=1}^{n+1} E[mn]$. Assume $\tau + \Lambda \in E - \mathcal{F}$; i.e., $\tau \in \mathbb{C} - \bigcup_{m=1}^{n+1} \frac{1}{mn}\Lambda$.

Let $S := S_{n,k}(E,\tau)$. By Theorem 7.6, $S_n \cong \mathbb{C}$ and $S_{n+1} = 0$. Since S is generated in degree one, $S_d = 0$ for all $d \geq n+1$. In particular, S is finite-dimensional.

By Proposition 8.6(2), S is Frobenius if and only if the bilinear maps

$$\operatorname{im} F_i(\tau) \times \operatorname{im} F_{n-i}(\tau) \longrightarrow \operatorname{im} F_i(\tau) \otimes \operatorname{im} F_{n-i}(\tau) \xrightarrow{M_{n-i,i}(\tau)} \operatorname{im} F_n(\tau) \cong \mathbb{C}$$

are non-degenerate for all $i = 0, \dots, n$. Since $M_{n-i,i}(\tau) \cdot (F_i(\tau) \otimes F_{n-i}(\tau)) = F_n(\tau)$ (Lemma 8.5), this happens if and only if, for all $i = 0, \dots, n$, the rank of the bilinear map

$$V^{\otimes i} \times V^{\otimes (n-i)} \longrightarrow V^{\otimes i} \otimes V^{\otimes (n-i)} \xrightarrow{F_n(\tau)} \operatorname{im} F_n(\tau) \cong \mathbb{C} \tag{10.1}$$

equals $\dim(\operatorname{im} F_i(\tau)) = \dim(\operatorname{im} F_{n-i}(\tau)) = \binom{n}{i}$. Clearly, the rank can be no larger than this.

Fix $x \in V^{\otimes n}$ such that $F_n(0)(x) \neq 0$. There is a Zariski-open dense subset $U \subseteq E$ such that $F_n(\tau)(x) \neq 0$ for all $\tau + \Lambda \in U$. Assume $\tau + \Lambda \in U$.

Let $\{v_j\}$ and $\{w_k\}$ be bases for $V^{\otimes i}$ and $V^{\otimes (n-i)}$, respectively. The rank of the bilinear map in (10.1) is the rank of the matrix $(c_{jk}(\tau))_{j,k}$ where

$$F_n(\tau)(v_j \otimes w_k) = c_{jk}(\tau) \cdot F_n(\tau)(x).$$

Since $F_n(\tau)$ is a theta operator, $c_{jk}(\tau)$ is an elliptic function, and so are all minors of $(c_{jk}(\tau))_{j,k}$. So the rank of $(c_{jk}(\tau))_{j,k}$ is generically large. Since the rank attains the maximal value $\binom{n}{i}$ at $\tau = 0$, it equals $\binom{n}{i}$ for all $\tau + \Lambda$ belonging to a dense open subset $U_i \subseteq U$. Since $U_0 \cap \cdots \cap U_n = E - \mathcal{F}'$ for some finite subset $\mathcal{F}' \subseteq E$, we see that $S_{n,k}(E,\tau)$ is Frobenius for all $\tau + \Lambda \in E - (\mathcal{F} \cup \mathcal{F}')$. ◻

Suppose $\tau \in \mathbb{C} - \bigcup_{m=1}^{n+1} \frac{1}{mn}\Lambda$. Since the space of degree-d relations for $S_{n,k}(E,\tau)$ is

$$\ker F_d(\tau) = \sum_{s+t+2=d} V^{\otimes s} \otimes \operatorname{im} R_\tau(-\tau) \otimes V^{\otimes t}$$

(Theorem 7.6), the Frobenius property for $S_{n,k}(E,\tau)$ can be reduced to a statement about the kernel of the operators $F_d(\tau)$ (defined in (6.2)) and $G_\tau^+(\tau)$ (defined in the proof of Theorem 7.6). The algebra $S_{n,k}(E,\tau)$ is Frobenius if and only if the following statements (and their left-right symmetric versions which we do not state) are true for all $d = 0, \ldots, n-1$:

the largest subspace $W \subseteq V^{\otimes d}$ such that $V \otimes W \subseteq \ker F_{d+1}(\tau)$ is $W = \ker F_d(\tau)$

or, equivalently,

if $V \otimes \{w\}$ is in the kernel of $G_\tau^+(\tau) : V \otimes \operatorname{im} F_d(\tau) \longrightarrow \operatorname{im} F_d(\tau) \otimes V$, then $w = 0$.

Theorem 10.4 *Let $\tau \in \mathbb{C}$ and fix (n,k,E).*

(1) $Q_{n,k}(E,\tau)$ is Artin–Schelter regular of dimension n for all but countably many τ.
(2) If $Q_{n,k}(E,\tau)$ is a Koszul algebra for all τ, then it is Artin–Schelter regular of dimension n for all but finitely many $\tau + \Lambda$.

Proof (1) The algebra $Q_{n,k}(E,\tau)$ is Artin–Schelter regular of dimension n if the following three statements are true: (a) it is a Koszul algebra; (b) the Hilbert series of $Q_{n,k}(E,\tau)^!$ is $(1+t)^n$; (c) $Q_{n,k}(E,\tau)^!$ is a Frobenius algebra. By Theorem 7.7 and Theorem 10.3, there is a finite set $\mathcal{F} \subseteq E$ such that (b) and (c) are true. By Theorem 9.17, (a) is true for all but countably many cosets $\tau + \Lambda$ and hence for all but countably many τ. Thus (a), (b), and (c) are simultaneously true for all but countably many τ.

(2) The argument follows that in (1). The only difference is that we are now assuming that (a) is true for all τ. Thus, since (b) and (c) are true for all but finitely many $\tau + \Lambda$, $Q_{n,k}(E,\tau)$ is Artin–Schelter regular of dimension n for all but finitely many $\tau + \Lambda$. □

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