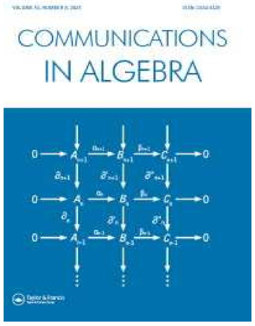

Taylor & Francis
Taylor & Francis Group

Communications in Algebra

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lagb20

Normal subgroups and relative centers of linearly reductive quantum groups

Alexandru Chirvasitu

To cite this article: Alexandru Chirvasitu (2023) Normal subgroups and relative centers of linearly reductive quantum groups, Communications in Algebra, 51:8, 3592-3605, DOI: 10.1080/00927872.2023.2186702

To link to this article: https://doi.org/10.1080/00927872.2023.2186702

Published online: 11 Mar 2023.

COMMUNICATIONS IN ALGEBRA®
2023, VOL. 51, NO. 8, 3592–3605
https://doi.org/10.1080/00927872.2023.2186702

Normal subgroups and relative centers of linearly reductive quantum groups

Alexandru Chirvasitu

Department of Mathematics, University at Buffalo, Buffalo, New York, USA

ABSTRACT

We prove a number of structural and representation-theoretic results on linearly reductive quantum groups, i.e., objects dual to those of cosemisimple Hopf algebras: (a) a closed normal quantum subgroup is automatically linearly reductive if its squared antipode leaves invariant each simple subcoalgebra of the underlying Hopf algebra; (b) for a normal embedding $\mathbb{H} \trianglelefteq \mathbb{G}$ there is a Clifford-style correspondence between two equivalence relations on irreducible $\mathbb{G}$- and, respectively, $\mathbb{H}$-representations; and (c) given an embedding $\mathbb{H} \le \mathbb{G}$ of linearly reductive quantum groups, the Pontryagin dual of the relative center $Z(\mathbb{G}) \cap \mathbb{H}$ can be described by generators and relations, with one generator g_V for each irreducible $\mathbb{G}$-representation V and one relation $g_U = g_V g_W$ whenever U and $V \otimes W$ are not disjoint over $\mathbb{H}$.

This latter center-reconstruction result generalizes and recovers Müger's compact-group analogue and the author's quantum-group version of that earlier result by setting $\mathbb{H} = \mathbb{G}$.

ARTICLE HISTORY
Received 22 December 2021
Revised 18 February 2023
Communicated by Scott Chapman

KEYWORDS
Antipode; center; comodule; cosemisimple Hopf algebra; cotensor; linearly reductive; Quantum group

2020 MATHEMATICS SUBJECT CLASSIFICATION
16T05; 20G42; 16T20

1. Introduction

The quantum groups in the title are as in [25, Section 1.2]: objects $\mathbb{G}$ dual to corresponding Hopf algebras $\mathcal{O}(\mathbb{G})$, with the latter regarded as the algebra of regular functions on (the otherwise non-existent) linear algebraic quantum group $\mathbb{G}$. Borrowing standard linear-algebraic-group terminology (e.g. [23, Chapter 1, Section 1, Definition 1.4]), the linear reductivity condition then simply means that the Hopf algebra $\mathcal{O}(\mathbb{G})$ is cosemisimple.

The unifying thread through the material below is the concept of a (closed) normal quantum subgroup. In the present non-commutative setting normality can be defined in a number of ways that are frequently equivalent [34, Theorem 2.7]. We settle here on the concept introduced in [25, Section 1.5] (and recalled in Definition 3.1): a quotient Hopf algebra

$$\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$$

dual to a closed quantum subgroup $\mathbb{H} \le \mathbb{G}$ is normal if that quotient is an $\mathcal{O}(\mathbb{G})$-comodule under both adjoint coactions $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{G})^{\otimes 2}$:

$$x \mapsto x_2 \otimes S(x_1)x_3 \quad \text{and} \quad x \mapsto x_1 S(x_3) \otimes x_2$$

One piece of motivation for the material is the observation (cf. Remark 3.9) that classically, normal closed subgroups of linearly reductive algebraic groups are again linearly reductive. The non-commutative version of this remark, appearing as Theorem 3.2, can be phrased (in somewhat weakened but briefer form) as follows.

Theorem 1.1. *A normal quantum subgroup $\mathbb{H} \trianglelefteq \mathbb{G}$ of a linearly reductive quantum group is again linearly reductive, provided the squared antipode of $\mathcal{O}(\mathbb{H})$ leaves invariant all simple subcoalgebras of the latter.*

CONTACT Alexandru Chirvasitu achirvas@buffalo.edu Department of Mathematics, University at Buffalo, Buffalo, NY 14260-2900, USA.

In particular, this recovers the classical version: in that case the squared antipode is trivial.

Keeping with the theme of what is (or isn't) afforded by normality, another motivating strand is that of *Clifford theory* (so named for [12], where the relevant machinery was introduced). This is a suite of results relating the irreducible representations of a (finite, compact, etc.) group and those of a normal subgroup via induction/restriction functors; the reader can find a brief illuminating summary in [7, Section 2] (in the context of finite groups).

Hopf-algebra analogues (both purely algebraic and analytic) abound. Not coming close to doing the literature justice, we will point to a selection: [6, 30, 33, 36, 37], say, and the references therein. [13, Section 5, especially Theorem 5.4] provides a version for *compact quantum groups* [38], which are (dual to) cosemisimple complex Hopf $*$-algebras with positive Haar integral (the *CQG algebras* of [14, Definition 2.2]); they thus fit within the confines of the present paper.

The following result paraphrases and summarizes Theorems 4.4, 4.5 and Proposition 4.6. To make sense of it:

- In the language of Section 4, the surjection $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$ of Theorem 1.2 is $H \to B$.
- As explained in Section 2, for a quantum group $\mathbb{G}$ the symbol $\widehat{\mathbb{G}}$ denotes its category of irreducible representations (i.e. simple right $\mathcal{O}(\mathbb{G})$-comodules).
- $\mathrm{Ind}_{\mathbb{H}}^{\mathbb{G}}$ and $\mathrm{Res}_{\mathbb{H}}^{\mathbb{G}}$ denote the induction and restriction functors respectively, as discussed in Section 2.1.

Theorem 1.2. *Let $\mathbb{H} \trianglelefteq \mathbb{G}$ be a normal embedding of linearly reductive quantum groups, and consider the binary relation $\sim$ on $\widehat{\mathbb{G}} \times \widehat{\mathbb{H}}$ defined by*

$$\widehat{\mathbb{G}} \ni V \sim W \in \widehat{\mathbb{H}} \Leftrightarrow \hom_{\mathbb{H}}\left(\mathrm{Res}_{\mathbb{H}}^{\mathbb{G}} V, W\right) \neq 0 \Leftrightarrow \hom_{\mathbb{G}}\left(V, \mathrm{Ind}_{\mathbb{H}}^{\mathbb{G}} W\right) \neq 0.$$

The following statements hold.

(a) The left-hand slices

$$\mathrm{slice}_W := \{V \in \widehat{\mathbb{G}} \mid V \sim W\},\ W \in \widehat{\mathbb{H}}$$

of $\sim$ are the classes of an equivalence relation $\sim_{\mathbb{G}}$, given by

$$V \sim_{\mathbb{G}} V' \Leftrightarrow \mathrm{Res}_{\mathbb{H}}^{\mathbb{G}} V \text{ and } \mathrm{Res}_{\mathbb{H}}^{\mathbb{G}} V' \text{ have the same simple constituents.}$$

(b) The right-hand slices

$${}_V\mathrm{slice} := \{W \in \widehat{\mathbb{H}} \mid V \sim W\},\ V \in \widehat{\mathbb{G}}$$

are the finite classes of an equivalence relation.

A third branch of the present discussion has to do with the *relative centers* of the title: having defined the center $Z(\mathbb{G})$ of a linearly reductive quantum group (Definition 5.3), and given a closed linearly reductive quantum subgroup $\mathbb{H} \le \mathbb{G}$, one can then make sense of the relative center $Z(\mathbb{G}, \mathbb{H})$ as the *intersection* $\mathbb{H} \cap Z(\mathbb{G})$; see Definition 5.4.

Though not immediately obvious, it follows from [11, Section 3] (cited more precisely in the text below) that for embeddings $\mathbb{H}, \mathbb{K} \le \mathbb{G}$ of linearly reductive quantum groups, operations such as the intersection $\mathbb{H} \cap \mathbb{K}$ and the quantum subgroup $\mathbb{H}\mathbb{K}$ generated by the two are well defined and behave as usual when $\mathbb{K}$, say, is normal (hence the relevance of normality, again).

The initial spark of motivation for Section 5 was provided by the main result of [22] (Theorem 3.1 therein), reconstructing the center of a compact group $\mathbb{G}$ as a universal grading group for the category of $\mathbb{G}$-representations. This generalizes to linearly reductive *quantum* groups [8, Proposition 2.9], and, as it turns out, goes through in the relative setting; per Theorem 5.5:

Theorem 1.3. *Let $\mathbb{H} \le \mathbb{G}$ be an embedding of linearly reductive quantum groups, and define the relative chain group $C(\mathbb{G}, \mathbb{H})$ by generators g_V, $V \in \widehat{\mathbb{G}}$ and relations $g_U = g_V g_W$ whenever U and $V \otimes W$ have common simple constituents over $\mathbb{H}$.*

Then, the map

$$C(\mathbb{G},\mathbb{H}) \ni g_V \mapsto W \in \widehat{Z(\mathbb{G},\mathbb{H})} \quad \textit{where} \quad \mathrm{Res}^{\mathbb{G}}_{Z(\mathbb{G},\mathbb{H})} \cong \textit{sum of copies of } W$$

is a group isomorphism.

Or, in words: mapping g_V to the "central character" of V restricted to $Z(\mathbb{G},\mathbb{H})$ gives an isomorphism $C(\mathbb{G},\mathbb{H}) \cong \widehat{Z(\mathbb{G},\mathbb{H})}$. The "plain" (non-relative) version [8, Proposition 2.9] (and hence also its classical compact-group counterpart [22, Theorem 3.1]) are recovered by setting $\mathbb{H} = \mathbb{G}$.

Although strictly speaking outside the scope of the present paper, some further remarks, suggestive of an intriguing connection to semisimple-Lie-group representation theory, will perhaps serve to further motivate the relative chain groups discussed in Theorem 1.3.

Definition 5.1 was inspired by the study of plain (non-relative) chain groups of connected, semisimple Lie groups $\mathbb{G}$ with finite center, studied in [10, Section 4]; specifically, the problem of whether

$$\hom_{\mathbb{H}}(\sigma'', \sigma \otimes \sigma') \neq 0, \quad \sigma,\ \sigma',\ \sigma'' \in \widehat{\mathbb{M}} \tag{1.1}$$

for a compact-group embedding $\mathbb{H} \le \mathbb{M}$ arises naturally while studying the direct-integral decomposition of a tensor product of two *principal-series* representations of such a Lie group $\mathbb{G}$. To summarize, consider the setup of [20] (to which we also refer, along with its own references, for background on the following).

- a connected, semisimple Lie group $\mathbb{G}$ with finite center, with its *Iwasawa decomposition*

  $$\mathbb{G} = \mathbb{K}\mathbb{A}\mathbb{N}$$

  ($\mathbb{K} \le \mathbb{G}$ maximal compact, $\mathbb{A}$ abelian and simply-connected, $\mathbb{N}$ nilpotent and simply-connected);
- the corresponding decomposition

  $$\mathbb{P} = \mathbb{M}\mathbb{A}\mathbb{N}$$

  of a minimal parabolic subgroup, with $\mathbb{M} \le \mathbb{K}$ commuting with $\mathbb{A}$;
- the resulting *principal-series* unitary representations

  $$\pi_{\sigma,\nu} := \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\sigma \otimes \nu \otimes \mathrm{triv}),$$

  where $\sigma \in \widehat{\mathbb{M}}$ and $\nu \in \widehat{\mathbb{A}}$ unitary irreducible representations over those groups.

One is then interested in which $\pi_{\sigma'',\nu''}$ are *weakly contained* [3, Definition F.1.1] in tensor products $\pi_{\sigma,\nu} \otimes \pi_{\sigma',\nu'}$ (i.e. feature in a direct-integral decomposition of the latter); we write

$$\pi_{\sigma'',\nu''} \preceq \pi_{\sigma,\nu} \otimes \pi_{\sigma',\nu'}.$$

It turns out that in the cases worked out in the literature there is a closed subgroup $\mathbb{H} \le \mathbb{M}$ that determines this weak containment via (1.1). Examples:

- When the (connected, etc.) Lie group $\mathbb{G}$ is *complex*, one can simply take $\mathbb{H} = \mathbb{Z}(\mathbb{G})$ (the center of $\mathbb{G}$, which is always automatically contained in $\mathbb{M}$). This follows, for instance, from [35, Theorem 3.5.5] in conjunction with [20, Theorems 1 and 2].
- For $\mathbb{G} = \mathrm{SL}(n,\mathbb{R})$, $n \ge 2$ one can again set $\mathbb{H} = Z(\mathbb{G})$: [27, Section 4] for $n = 2$ and [20, p.210, Theorem] for the rest.
- Finally, for *real-rank-one* $\mathbb{G}$ the main result of [20], Theorem 16 of that paper, provides such an $\mathbb{H} \le \mathbb{M}$ (denoted there by $\mathbb{M}_0$; it is in general non-central, and in fact not even normal).

The phenomenon presumably merits some attention in its own right.

2. Preliminaries

Everything in sight (algebras, coalgebras, etc.) will be linear over a fixed algebraically closed field k. We assume some background on coalgebras and Hopf algebras, as covered by any number of good sources such as [1, 21, 26, 31].

Notation 2.1. A number of notational conventions will be in place throughout.

- Δ, ε and S denote, respectively, coproducts, counits and antipodes. They will occasionally be decorated with letters indicating which coalgebra, Hopf algebra, etc. they are attached to; S_H, for instance, is the antipode of the Hopf algebra H.
- We use an un-parenthesized version of *Heyneman-Sweedler notation* ([21, Notation 1.4.2] or [26, Section 2.1]):
$$\Delta(c) = c_1 \otimes c_2,\ ((\Delta \otimes \mathrm{id}) \circ \Delta)(c) = c_1 \otimes c_2 \otimes c_3$$
and so on for coproducts and
$$c \mapsto c_0 \otimes c_1, \quad c \mapsto c_{-1} \otimes c_0$$
for right and left comodule structures respectively.
- $\mathcal{O}(\mathbb{G})$, $\mathcal{O}(\mathbb{H})$, and so on denote Hopf algebras over a fixed algebraically closed field k; they are to be thought of as algebras of representative functions on linear algebraic *quantum groups* $\mathbb{G}$, H, etc.
- An *embedding* $\mathbb{H} \le \mathbb{G}$ of quantum groups means a Hopf algebra surjection $\mathcal{O}(\mathbb{G}) \twoheadrightarrow \mathcal{O}(\mathbb{H})$ and more generally, a morphism $\mathbb{H} \to \mathbb{G}$ is one of Hopf algebras in the opposite direction $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$.
- Categories of (co)modules are denoted by $\mathcal{M}$, decorated with the symbol depicting the (co)algebra, with the left/right position of the decoration matching the chirality of the (co)module structure. Examples: ${}_A\mathcal{M}$ means left A-modules, $\mathcal{M}^C$ denotes right C-comodules, etc. Comodule structures are right unless specified otherwise.
- These conventions extend to *relative Hopf modules* ([21, Section 8.5] or [26, Section 9.2]): if, say, A is a right comodule algebra [21, Definition 4.1.2] over a Hopf algebra H with structure
$$A \ni a \mapsto a_0 \otimes a_1 \in A \otimes H$$
then $\mathcal{M}_A^H$ denotes the category of right A-modules internal to $\mathcal{M}^H$; that is, right A-modules M that are also right H-comodules via
$$m \mapsto m_0 \otimes m_1$$
such that
$$(ma)_0 \otimes (ma)_1 = m_0a_0 \otimes m_1a_1.$$
There are analogues $\mathcal{M}_H^C$, say, for right H-module coalgebras C, left- or half-left-handed versions thereof, and so on.
- An additional 'f' adornment on one of the above-mentioned categories means *finite-dimensional* (co)modules: $\mathcal{M}_f^C$ is the category of finite-dimensional right C-comodules, for instance.
- Reprising a convention common in the operator-algebra literature (e.g. [15, Section 2.3.2, Section 18.1.1]), $\widehat{C}$ denotes the isomorphism classes of simple and hence finite-dimensional [21, Theorem 5.1.1] (right, unless specified otherwise) C-comodules and $\widehat{\mathbb{G}} = \widehat{\mathcal{O}(\mathbb{G})}$.

  The purely-algebraic and operator-algebraic notations converge when $\mathbb{G}$ is compact and $\mathcal{O}(\mathbb{G})$ denotes the Hopf algebra of representative functions on $\mathbb{G}$: $\widehat{\mathbb{G}}$ as defined above can then be identified with the set of isomorphism classes of irreducible unitary $\mathbb{G}$-representations.
- In the same spirit, it will also occasionally be convenient to write
$$\mathrm{Rep}(\mathbb{G}) := \mathcal{M}^{\mathcal{O}(\mathbb{G})}.$$

The linear algebraic quantum groups $\mathbb{G}$ in the sequel will frequently be *linearly reductive*, in the sense that the Hopf algebra $\mathcal{O}(\mathbb{G})$ is *cosemisimple* [21, Section 2.4]: $\mathrm{Rep}(\mathbb{G})$ is a semisimple category, i.e. every comodule is a direct sum of simple subcomodules. Equivalently ([21, Definition 2.4.1]), $\mathcal{O}(\mathbb{G})$ is a direct sum of simple subcoalgebras.

Cosemisimple Hopf algebras H are equipped with unique unital *integrals* $\int : H \to k$ [21, Theorem 2.4.6] and hence have bijective antipodes (by [16, Corollary 5.4.6], say); more is true, though. Still assuming H cosemisimple, for a simple comodule $V \in \widehat{H}$ the canonical coalgebra morphism
$$\mathrm{End}(V)^* \cong V^* \otimes V \to H$$

(conceptually dual to the analogous map $A \to \mathrm{End}(V)$ giving V a module structure over an algebra A) is one-to-one and gives the direct-sum decomposition

$$H = \bigoplus_{V \in \widehat{H}} (V^* \otimes V) = \bigoplus_{V \in \widehat{H}} C_V \tag{2.1}$$

into simple subcoalgebras $C_V := V^* \otimes V$ (the *Peter-Weyl* decomposition, in compact-group parlance: [14, Definition 2.2], [17, Theorem 27.40], etc.) that makes H cosemisimple to begin with. With this in place, not only is the antipode $S := S_H$ bijective but in fact its square leaves every C_V, $V \in \widehat{H}$ invariant and acts as an automorphism thereon [16, Theorem 7.3.7].

We refer to $C_V = V^* \otimes V$ as the *coefficient coalgebra* of the simple H-comodule V. This is the coalgebra associated to V in [16, Proposition 2.5.3], and is the smallest subcoalgebra $C \le H$ for which the comodule structure

$$V \to V \otimes H$$

factors through $V \otimes C$.

2.1. Restriction, induction and the like

Given a coalgebra morphism $C \to D$, the *cotensor product* ([21, Definition 8.4.2] or [5, Section 10]) $-\,\square_D\, C$ is right adjoint to the natural "scalar corestriction" functor $\mathcal{M}^C \to \mathcal{M}^D$:

$$\begin{tikzcd} \mathcal{M}^C \arrow[r, bend left, "\mathrm{cores}"] \arrow[r, phantom, "\perp"] & \mathcal{M}^D \arrow[l, bend left, "-\square_D C"] \end{tikzcd} \tag{2.2}$$

the central symbol indicating that the top functor is the left adjoint. When $\mathbb{H} \le \mathbb{G}$ is, say, an inclusion of compact groups and $C \to D$ the corresponding surjection $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$ of algebras of representative functions, the cotensor functor

$$-\,\square_{\mathcal{O}(\mathbb{H})}\, \mathcal{O}(\mathbb{G}) : \mathrm{Rep}(\mathbb{H}) \to \mathrm{Rep}(\mathbb{G})$$

is naturally isomorphic with the usual *induction* $\mathrm{Ind}_H^{\mathbb{G}}$ [28, p. 82]. For that reason we repurpose this same notation for the general setting of quantum-group inclusions, writing

$$\mathrm{Ind}_{\mathbb{H}}^{\mathbb{G}} := -\,\square_{\mathcal{O}(\mathbb{H})}\, \mathcal{O}(\mathbb{G}) : \mathrm{Rep}(\mathbb{H}) \to \mathrm{Rep}(\mathbb{G})$$

for any quantum-group inclusion $\mathbb{H} \le \mathbb{G}$; for consistency, we also occasionally also denote the rightward functor in (2.2) by

$$\mathrm{Res}_{\mathbb{H}}^{\mathbb{G}} : \mathrm{Rep}(\mathbb{G}) \to \mathrm{Rep}(\mathbb{H}).$$

3. Normal subgroups and automatic reductivity

Consider a quantum group embedding $\mathbb{H} \le \mathbb{G}$, expressed as a surjective Hopf-algebra morphism $\pi : \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$. As is customary in the literature on quantum homogeneous spaces (e.g. [34, proof of Theorem 2.7]), we write

$$\mathcal{O}(\mathbb{G}/\mathbb{H}) := \{x \in \mathcal{O}(\mathbb{G}) \mid (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\}$$
$$\mathcal{O}(\mathbb{H}\backslash\mathbb{G}) := \{x \in \mathcal{O}(\mathbb{G}) \mid (\pi \otimes \mathrm{id})\Delta(x) = 1 \otimes x\}.$$

According to [2, Definition 1.1.5] a quantum subgroup $\mathbb{H} \le \mathbb{G}$ would be termed *normal* provided the two quantum homogeneous spaces $\mathcal{O}(\mathbb{G}/\mathbb{H})$ and $\mathcal{O}(\mathbb{H}\backslash\mathbb{G})$ coincide. This will not quite do for our purposes (see Example 3.8), so instead we follow [25, Section 1.5] (also, say, [34, Definition 2.6], relying on the same source) in the following

Definition 3.1. The quantum subgroup $\mathbb{H} \leq \mathbb{G}$ cast as the surjection $\pi : \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$

- *left-normal* if π is a morphism of left $\mathcal{O}(\mathbb{G})$-comodules under the left adjoint coaction

$$\mathrm{ad}_l := \mathrm{ad}_{l,\mathbb{G}} : x \mapsto x_1 S(x_3) \otimes x_2.$$

- *right-normal* if similarly, π is a morphism of right $\mathcal{O}(\mathbb{G})$-comodules under the right adjoint coaction

$$\mathrm{ad}_r := \mathrm{ad}_{r,\mathbb{G}} : x \mapsto x_2 \otimes S(x_1) x_3. \tag{3.1}$$

- *normal* if it is both left- and right-normal.

The following result is essentially a tautology in the framework of [11, Section 1.2], but only because in that paper the definition of a normal quantum subgroup is more restrictive (see [11, Definition 1.2.3], which makes an additional (co)flatness requirement).

Theorem 3.2. *Let $\mathbb{H} \leq \mathbb{G}$ be a left- or right-normal quantum subgroup of a linearly reductive group such that S^2 leaves invariant every simple subcoalgebra of $\mathcal{O}(\mathbb{H})$.*

$\mathbb{H}$ is then linearly reductive and normal.

Remark 3.3. The condition that S^2 leave invariant the simple subcoalgebras is certainly necessary for cosemisimplicity [16, Theorem 7.3.7], but I do not know if it is redundant as a hypothesis in the context of Theorem 3.2.

In particular, the squared-antipode condition of Theorem 3.2 is automatic when $S^2 = \mathrm{id}$ (i.e. when $\mathcal{O}(\mathbb{G})$, or $\mathbb{G}$, is *involutory* or *involutive* [26, Definition 7.1.12]). We thus have

Corollary 3.4. *Left- or right-normal quantum subgroups of involutive linearly reductive quantum groups are normal and linearly reductive.*

The proof of Theorem 3.2 requires some preparation. First, a simple remark for future reference.

Lemma 3.5. *Let $\pi : H \to K$ be a surjective morphism of Hopf algebras with H cosemisimple. K then has bijective antipode, and hence π intertwines antipode inverses.*

Proof. That a morphism of bialgebras intertwines antipodes or antipode inverses as soon as these exist is well known, so we focus on the claim that S_K is bijective.

By the very definition of cosemisimplicity H is the direct sum of its simple (hence finite-dimensional [21, Theorem 5.1.1]) subcoalgebras $C_i \leq H$. The assumption is that π is a morphism of Hopf algebras, so the antipode $S := S_H$ restricts to maps

$$S : \ker(\pi|_{C_i}) \to \ker(\pi|_{S(C_i)}), \tag{3.2}$$

injective because S is bijective. On the other hand though, for cosemisimple Hopf algebras the squared antipode leaves every subcoalgebra invariant [16, Theorem 7.3.7], so

$$S^2 : \ker(\pi|_{C_i}) \to \ker(\pi|_{S^2(C_i)}) = \ker(\pi|_{C_i}),$$

being a one-to-one endomorphism of a finite-dimensional vector space, must be bijective. Since that map decomposes as (3.2) followed by its (similarly one-to-one) analogue defined on $S(C_i)$, (3.2) itself must be bijective, and hence the inverse antipode S^{-1} leaves $\ker(\pi)$ invariant. This, in essence, was the claim. $\square$

The conclusion of Lemma 3.5 is by no means true of arbitrary bijective-antipode Hopf algebras H:

Example 3.6. [29, Theorem 3.2] gives an example of a Hopf algebra H with bijective antipode and a Hopf ideal $I \trianglelefteq H$ that is not invariant under the inverse antipode. In other words, even though H has bijective antipode, the quotient Hopf algebra $H \to H/I$ does not.

Proof of Theorem 3.2. The proof proceeds gradually.

Step 1: normality. According to Lemma 3.5 the antipode $S := S_{\mathcal{O}(\mathbb{G})}$ and its inverse both leave the kernel $\mathcal{K}$ of the surjection

$$\pi : \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$$

invariant, so $S(\mathcal{K}) = \mathcal{K}$. The fact that left- and right-normality are equivalent now follows from [25, Proposition 1.5.1].

Step 2: The homogeneous spaces $\mathbb{G}/\mathbb{H}$ and $\mathbb{H}\backslash\mathbb{G}$ coincide. This means that

$$\mathcal{O}(\mathbb{H}\backslash\mathbb{G}) = \mathcal{O}(\mathbb{G}/\mathbb{H}) =: A, \tag{3.3}$$

and follows from [2, Lemma 1.1.7].

Step 3: Reduction to trivial $\mathbb{G}/\mathbb{H}$. The subspace $A \le \mathcal{O}(\mathbb{G})$ of (3.3) is in fact a Hopf subalgebra [2, Lemma 1.1.4]. A is also invariant under the right adjoint action

$$\mathcal{O}(\mathbb{G}) \otimes \mathcal{O}(\mathbb{G}) \ni x \otimes y \mapsto S(y_1)xy_2 \in \mathcal{O}(\mathbb{G})$$

([11, Lemma 1.20]), so by [2, Lemma 1.1.11] the left ideal

$$\mathcal{O}(\mathbb{G})A^- \le \mathcal{O}(\mathbb{G}) \text{ where } A^- := \ker(\varepsilon|_A)$$

is bilateral. The quotient $\mathcal{O}(\mathbb{G})/\mathcal{O}(\mathbb{G})A^-$ must then be a *cosemisimple* quotient Hopf algebra [9, Theorem 2.5] $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{K})$, and we have an exact sequence

$$\begin{tikzcd}[row sep=small] & \mathcal{O}(\mathbb{G}/\mathbb{K}) \arrow[r] \arrow[d, equal] & \mathcal{O}(\mathbb{G}) \arrow[r] & \mathcal{O}(\mathbb{K}) \arrow[dr] & \\ k \arrow[ur] & \mathcal{O}(\mathbb{G}/\mathbb{H}) & & & k \end{tikzcd}$$

of quantum groups in the sense of [2, Section 1.2], with everything in sight cosemisimple. Since furthermore A^- is annihilated by the original surjection $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$, $\mathbb{H}$ can be thought of as a quantum subgroup of $\mathbb{K}$ (rather than $\mathbb{G}$):

$$\mathcal{O}(\mathbb{K}) \to \mathcal{O}(\mathbb{H}).$$

I now claim that the corresponding homogeneous space is trivial:

$$\mathcal{O}(\mathbb{K}/\mathbb{H}) = \mathcal{O}(\mathbb{H}\backslash\mathbb{K}) = k. \tag{3.4}$$

To see this, consider a simple representation $V \in \widehat{\mathbb{K}}$ that contains invariant vectors over $\mathbb{H}$. Because $\mathcal{O}(\mathbb{K})$ is cosemisimple, V is a subcomodule (rather than just a subquotient) of a simple comodule $W \in \widehat{\mathbb{G}}$, and it follows that

$$W|_{\mathbb{H}} \ge V|_{\mathbb{H}}$$

contains invariant vectors. The fact that (3.3) is a Hopf subalgebra means that it is precisely

$$\bigoplus_U C_U, \; U \in \widehat{\mathbb{G}} \text{ and } U|_{\mathbb{H}} \text{ has invariant vectors,}$$

so $C_W \le A$ and the restriction $W|_{\mathbb{K}}$ decomposes completely as a sum of copies of the trivial comodule k. But then $V \le W|_{\mathbb{K}}$ itself must be trivial, proving the claim (3.4). Now simply switch the notation back to $\mathbb{G} := \mathbb{K}$ to conclude Step 3:

$$\mathcal{O}(\mathbb{G}/\mathbb{H}) = \mathcal{O}(\mathbb{H}\backslash\mathbb{G}) = k. \tag{3.5}$$

This latter condition simply means that for an $\mathcal{O}(\mathbb{G})$-comodule V its $\mathbb{G}$- and $\mathbb{H}$-invariants coincide:

$$\hom_{\mathbb{G}}(k, V) = \hom_{\mathbb{H}}(k, V).$$

Equivalently, since

$$\hom_{\mathbb{G}}(V, W) = \hom_{\mathbb{G}}(k, W \otimes V^*),$$

this simply means that the restriction functor

$$\mathrm{Rep}(\mathbb{G}) \ni V \mapsto V|_{\mathbb{H}} \in \mathrm{Rep}(\mathbb{H}) \tag{3.6}$$

is full (for both left and right comodules, but here we focus on the latter).

Step 4: Wrapping up. Because the restriction functor (3.6) is full, simple, non-isomorphic $\mathbb{G}$-representations that remain simple over $\mathbb{H}$ also remain non-isomorphic.

Now, assuming $\mathbb{H} \le \mathbb{G}$ is not an isomorphism (or there would be nothing to prove), some irreducible $V \in \widehat{\mathbb{G}}$ must become reducible over $\mathbb{H}$. There are two possibilities to consider:

(a) All simple subquotients of the reducible representation $V|_{\mathbb{H}}$ are isomorphic. We then have (in $\mathrm{Rep}(\mathbb{H})$) a surjection of V onto a simple quotient thereof, which then embeds into V again. All in all this gives a non-scalar endomorphism of V over $\mathbb{H}$, contradicting the fullness of the restriction functor (3.6).

(b) V acquires at least two non-isomorphic simple subquotients V_i, $i = 1, 2$ over $\mathbb{H}$. Then, the image of the coefficient coalgebra $C_V = V^* \otimes V$ of (2.1) through $\pi : \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$ will contain both

$$C_{V_i} = V_i^* \otimes V_i \le \mathcal{O}(\mathbb{H}),\ i = 1, 2$$

as (simple) subcoalgebras.

The requirement that $S^2(C_{V_i}) = C_{V_i}$ means that the simple comodules V_i are isomorphic to their respective double duals V_i^{**} (as $\mathcal{O}(\mathbb{H})$-comodules, not just vector spaces). But then

$$C_{V_i} = V_i^* \otimes V_i \cong V_i^* \otimes V_i^{**}$$

contains an $\mathbb{H}$-invariant vector, namely the image of the *coevaluation* [19, Definition 9.3.1]

$$\mathrm{coev}_{V_i^*} : k \to V_i^* \otimes V_i^{**}.$$

It follows that the space of $\mathbb{H}$-invariants of the $\mathcal{O}(\mathbb{G})$-comodule $\pi(C_V)$ is at least 2-dimensional, whereas that of $\mathbb{G}$-invariants is at most 1-dimensional (because the same holds true of $C_V = V^* \otimes V$). This contradicts the fullness of (3.6) and hence our assumption that $\mathbb{H} \le \mathbb{G}$ is *not* an isomorphism.

The proof of the theorem is now complete. □

Remark 3.7. Left and right normality are proven equivalent to an alternative notion ([34, Definition 2.3]) in [34, Theorem 2.7] in the context of *CQG algebras*, i.e. complex cosemisimple Hopf $*$-algebras with positive unital integral (this characterization is equivalent to [14, Definition 2.2]).

The substance of Theorem 3.2, however, is the cosemisimplicity claim; this is of no concern in the CQG-algebra case, as a Hopf $*$-algebra that is a quotient of a CQG algebra is automatically again CQG (as follows, for instance, from [14, Proposition 2.4]), and hence cosemisimple.

Example 3.8. The weaker requirement that $\mathcal{O}(\mathbb{G}/\mathbb{H}) = \mathcal{O}(\mathbb{H}\backslash\mathbb{G})$ for normality would render Theorem 3.2 false.

Let $\mathbb{G}$ be a semisimple complex algebraic group and $\mathbb{B} \le \mathbb{G}$ a *Borel subgroup* [18, Part II, Section 1.8]. The restriction functor

$$\mathrm{Res} : \mathrm{Rep}(\mathbb{G}) \to \mathrm{Rep}(\mathbb{B})$$

is full [18, Part II, Corollary 4.7], so in particular

$$\mathcal{O}(\mathbb{G}/\mathbb{B}) = \hom_{\mathbb{B}}(\mathrm{triv}, \mathcal{O}(\mathbb{G})) = \hom_{\mathbb{G}}(\mathrm{triv}, \mathcal{O}(\mathbb{G})) = \mathbb{C}$$

and similarly for $\mathcal{O}(\mathbb{B}\backslash\mathbb{G})$. This means that $\mathcal{O}(\mathbb{G}/\mathbb{B}) = \mathcal{O}(\mathbb{B}\backslash\mathbb{G})$, but $\mathbb{B}$ is nevertheless not reductive.

Remark 3.9. The classical (as opposed to quantum) analogue of Theorem 3.2 admits an alternative, more direct proof relying on the structure of reductive groups:

- In characteristic zero linear reductivity is equivalent (by [24, p.88 (2)], for instance) to plain reductivity [4, Section 11.21], i.e. the condition that the *unipotent radical* $\mathcal{R}_u(\mathbb{G})$ of $\mathbb{G}$ (the largest normal connected unipotent subgroup) be trivial.

  Assuming $\mathbb{G}$ is reductive, for any normal $\mathbb{K} \trianglelefteq \mathbb{G}$ the corresponding unipotent radical $\mathcal{R}_u(\mathbb{K})$ is characteristic in N and hence normal in $\mathbb{G}$, meaning that

  $$\mathcal{R}_u(\mathbb{K}) \le \mathcal{R}_u(\mathbb{G}) = \{1\}$$

  and hence N is again reductive (so linearly reductive, in characteristic zero).
- On the other hand, in positive characteristic p [24, p.88 (1)] says that the linearly reductive groups $\mathbb{G}$ are precisely those fitting into an exact sequence

  $$\{1\} \to \mathbb{K} \to \mathbb{G} \to \mathbb{G}/\mathbb{K} \to \{1\}$$

  with $\mathbb{K}$ a closed subgroup of a torus and $\mathbb{G}/\mathbb{K}$ finite of order coprime to p. Clearly then, normal subgroups of $\mathbb{G}$ have the same structure.

4. Clifford theory

We work with an exact sequence (4.1)

$$k \to A \to H \to B \to k \tag{4.1}$$

of cosemisimple Hopf algebras in the sense of [2, p. 23]. Note that we additionally know that H is left and right coflat over B (simply because the latter is cosemisimple) and left and right faithfully flat over A (by [9, Theorem 2.1]).

We will make frequent use of [32, Theorem 1], to the effect that

$$\begin{tikzcd} \mathcal{M}_A^H \arrow[r, bend left, "M\mapsto M/MA^-"] & \mathcal{M}^B \arrow[l, bend left, "N\otimes A \leftarrow\!\shortmid N"] \end{tikzcd} \tag{4.2}$$

is an equivalence, where the $-$ superscript denotes kernels of counits.

Upon identifying $\mathcal{M}^B$ with $\mathcal{M}_A^H$ via (4.2), the adjunction

$$\begin{tikzcd} \mathcal{M}^H \arrow[r, bend left, "\text{corestrict}"] & \mathcal{M}^B \arrow[l, bend left, "-\square_B H"] \end{tikzcd} \tag{4.3}$$

becomes

$$\begin{tikzcd} \mathcal{M}^H \arrow[r, bend left, "-\otimes A"] & \mathcal{M}_A^H. \arrow[l, bend left, "\text{forget}"] \end{tikzcd} \tag{4.4}$$

We will freely switch points of view between the two perspectives provided by (**??**). Consider the following binary relation $\sim_B$ on $\widehat{B}$.

Definition 4.1. For $V, W \in \widehat{B}$, $V \sim_B W$ provided there is a simple H-comodule U such that V and W are both constituents of the corestriction of U to B.

Similarly, we will study the following relation on $\widehat{H}$:

Definition 4.2. For $V, W \in \widehat{H}$ we set $V \sim_H W$ provided $\hom^B(V, W) \neq 0$.

Remark 4.3. In other words, $\sim_H$ signifies the fact that the corestrictions of V and W to $\mathcal{M}^B$ have common simple constituents.

Our first observation is that $\sim_H$ is an equivalence relation, and provides an alternate characterization for it.

Theorem 4.4. *$\sim_H$ is an equivalence relation on $\widehat{H}$, and moreover, for $V, W \in \widehat{H}$ the following conditions are equivalent*

(1) $V \sim_H W$;

(2) as B-comodules, V and W have the same simple constituents;

(3) V embeds into $W \otimes A \in \mathcal{M}^H$.

Proof. Note first that (2) clearly defines an equivalence relation on $\widehat{H}$, so the first statement of the theorem will be a consequence of

$$(1) \Leftrightarrow (2) \Leftrightarrow (3).$$

We prove the latter result in stages.

(1) $\Leftrightarrow$ (3)**.** By definition, $V \sim_H W$ if and only if

$$\hom^B(V, W) \neq 0.$$

Via (4.2) and the hom-tensor adjunction (4.4), this hom space can be identified with

$$\hom_A^H(V \otimes A, W \otimes A) \cong \hom^H(V, W \otimes A). \tag{4.5}$$

The simplicity of $V \in \widehat{H}$ now implies that every nonzero element of the right hand side of (4.5) is an embedding, hence finishing the proof of the equivalence of (1) and (3).

(1) $\Leftrightarrow$ (2)**.** Let us denote by $\mathrm{const}(\bullet)$ the set of simple constituents of a B-comodule $\bullet$.

By definition $V \sim_H W$ means that *some* of the simple constituents of V and W as objects in $\mathcal{M}^B$ coincide, so (2) is clearly stronger than (1). Conversely, note that by the equivalence (1) $\Rightarrow$ (3) proven above, whenever $V \sim_H W$ we have

$$\mathrm{const}(V) \subseteq \mathrm{const}(W \otimes A), \tag{4.6}$$

where the respective objects are regarded as B-comodules via the corestriction functor $\mathcal{M}^H \to \mathcal{M}^B$.

In turn however, given that $A \in \mathcal{M}^H$ breaks up as a sum of copies of k in $\mathcal{M}^B$ (because of the exactness of (4.1)), the right hand side of (4.6) is simply $\mathrm{const}(W)$. All in all, we have

$$V \sim_H W \Rightarrow \mathrm{const}(V) \subseteq \mathrm{const}(W).$$

This together with the symmetry of $\sim_H$ (obvious by definition from the semisimplicity of $\mathcal{M}^B$) finishes the proof of (1) $\Rightarrow$ (2) and of the theorem. □

Theorem 4.5. *$\sim_B$ is an equivalence relation on $\widehat{B}$ with finite classes.*

Proof. We know from Theorem 4.4 that as U ranges over $\widehat{H}$, the sets $\mathrm{const}(U)$ of constituents of $U \in \mathcal{M}^B$ partition $\widehat{B}$, thus defining an equivalence relation on the latter set.

The definition of $\sim_B$ ensures that $V \sim_B W$ if and only if V and W fall in the same set $\mathrm{const}(U)$, and hence $\sim_B$ coincides with the equivalence relation from the previous paragraph.

Finally, the statement on finiteness of classes is implicit in their description given above: an equivalence class is the set of simple constituents of a simple H-comodule U viewed as a B-comodule, and it must be finite because $\dim(U)$ is. □

Theorems 4.4 and 4.5 establish a connection between the equivalence relations $\sim_H$ and $\sim_B$ on $\widehat{H}$ and $\widehat{B}$ respectively. We record it below.

Before getting to the statement, recall the notation $\mathrm{const}(\bullet) \subseteq \widehat{B}$ for the set of simple summands of an object $\bullet \in \mathcal{M}^B$. With that in mind, we have the following immediate consequence of Theorems 4.4 and 4.5.

Proposition 4.6. *The range of the map*

$$\widehat{H} \to \textit{ finite subsets of } \widehat{B}$$

sending $V \in \widehat{H}$ to $\mathrm{const}(V)$ consists of the equivalence classes of $\sim_B$, and its fibers are the classes of $\sim_H$.

5. Relative chain groups and centers

Definition 5.1. Let $\mathbb{H} \le \mathbb{G}$ be an inclusion of linearly reductive quantum groups. The *(relative) chain group* $C(\mathbb{G}, \mathbb{H})$ is defined by

- generators g_V for simple comodules $V \in \widehat{\mathbb{G}}$;
- relations

$$\hom_{\mathbb{H}}(U, V \otimes W) \neq 0 \Rightarrow g_U = g_V g_W; \tag{5.1}$$

  that is, one such relation whenever the restrictions of U and $V \otimes W$ to $\mathbb{H}$ have non-trivial common summands (i.e. U and $V \otimes W$ are not *disjoint* over $\mathbb{H}$).

We write $C(\mathbb{G}) := C(\mathbb{G}, \mathbb{G})$.

Remark 5.2. For chained inclusions $\mathbb{K} \le \mathbb{H} \le \mathbb{G}$ we have a map $C(\mathbb{G}, \mathbb{K}) \to C(\mathbb{G}, \mathbb{H})$ sending the class of $V \in \widehat{\mathbb{G}}$ in the domain to the class of the selfsame V in the codomain. This is easily seen to be well-defined and a group morphism.

Recall [8, Definition 2.10].

Definition 5.3. Let $\mathbb{G}$ be a linearly reductive quantum group. Its *center* $Z(\mathbb{G}) \le \mathbb{G}$ is the quantum subgroup dual to the largest Hopf algebra quotient

$$\pi : \mathcal{O}(\mathbb{G}) \to \mathcal{O}(Z(\mathbb{G}))$$

that is central in the sense of [8, Definition 2.1]:

$$\pi(x_1) \otimes x_2 = \pi(x_2) \otimes x_1 \in \mathcal{O}(Z(\mathbb{G})) \otimes \mathcal{O}(\mathbb{G}),\ \forall x \in \mathcal{O}(\mathbb{G}).$$

The relative version of this construction, alluded to in the title, is as follows.

Definition 5.4. Let $\mathbb{H} \le \mathbb{G}$ be an embedding of linearly reductive quantum groups. The corresponding *relative center* $Z(\mathbb{G}, \mathbb{H})$ is the *intersection* $Z(\mathbb{G}) \cap \mathbb{H}$ denoted by $Z(\mathbb{G}) \wedge \mathbb{H}$ in [11, Definition 1.15].

This is a quantum subgroup of both $\mathbb{H}$ and $Z(\mathbb{G})$ (and hence also of $\mathbb{G}$), and is automatically linearly reductive by [11, Proposition 3.1].

Each irreducible $\mathbb{G}$-representation breaks up as a sum of mutually isomorphic (one-dimensional) representations over the center $Z(\mathbb{G})$, and hence gets assigned an element of $\widehat{Z(\mathbb{G})}$: its *central character*. Two such irreducible representations that are not disjoint over $\mathbb{H}$ must have corresponding central characters agreeing on

$$Z(\mathbb{G},\mathbb{H}) := Z(\mathbb{G}) \cap \mathbb{H}$$

(the *relative center* associated to the inclusion $\mathbb{H} \le \mathbb{G}$), so we have a canonical morphism

$$\mathrm{CAN} : C(\mathbb{G},\mathbb{H}) \to \widehat{Z(\mathbb{G},\mathbb{H})} \tag{5.2}$$

Theorem 5.5. *For any embedding $\mathbb{H} \le \mathbb{G}$ of linearly reductive quantum groups (5.2) is an isomorphism.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} & C(\mathbb{G},\mathbb{H}) \arrow[dr, "\mathrm{CAN}"] & \\ C(\mathbb{G}) \arrow[ur] \arrow[dr, "\mathrm{CAN}"'] \arrow[rr, phantom, "\cong"] & & \widehat{Z(\mathbb{G},\mathbb{H})} \\ & \widehat{Z(\mathbb{G})} \arrow[ur] & \end{tikzcd} \tag{5.3}$$

where

- the upper left-hand morphism is an instance of the maps noted in Remark 5.2;
- the bottom right-hand map is the (plain) group surjection dual to the quantum-group inclusion $Z(\mathbb{G}) \cap \mathbb{H} \le Z(\mathbb{G})$;
- and the fact that the bottom left-hand map is an isomorphism is a paraphrase of [8, Proposition 2.9] in conjunction with [8, Definition 2.10].

The surjectivity of the bottom composition entails that of (5.2), so it remains to show that the latter is one-to-one.

Let $V \in \widehat{\mathbb{G}}$ be a simple comodule where $Z(\mathbb{G},\mathbb{H})$ operates with trivial character, i.e. one whose class in $C(\mathbb{G},\mathbb{H})$ is annihilated by (5.2). We can then form the quantum subgroup

$$Z(\mathbb{G})\mathbb{H} := Z(\mathbb{G}) \vee \mathbb{H} \le \mathbb{G}$$

generated by $Z(\mathbb{G})$ and $\mathbb{H}$ as in [11, Definition 1.15] (the '$\vee$' notation is used there; we suppress the symbol here for brevity), which then satisfies, according to [11, Theorem 3.4], a quantum-flavored isomorphism theorem:

$$\mathbb{H}/Z(\mathbb{G},\mathbb{H}) \xrightarrow{\cong} Z(\mathbb{G})\mathbb{H}/Z(\mathbb{G})$$

via the canonical map induced from $\mathbb{H} \to Z(\mathbb{G})\mathbb{H}$. Since V (or rather its restriction $V|_{\mathbb{H}}$) is a representation of the former group because $Z(\mathbb{G},\mathbb{H})$ operates trivially, it lifts to a $Z(\mathbb{G})\mathbb{H}$-representation with $Z(\mathbb{G})$ acting trivially. In summary:

The restriction $V|_{\mathbb{H}}$ extends to a $Z(\mathbb{G})\mathbb{H}$-representation W with trivial $Z(\mathbb{G})$-action.

But then the induced representation $\mathrm{Ind}_{Z(\mathbb{G})\mathbb{H}}^{\mathbb{G}} W$ again has trivial central character, and hence so do all of its simple summands V_1. The adjunction (2.2) yields

$$\hom_{Z(\mathbb{G})\mathbb{H}}(V_1|_{Z(\mathbb{G})\mathbb{H}}, W) \cong \hom_{\mathbb{G}}\left(V_1, \mathrm{Ind}_{Z(\mathbb{G})\mathbb{H}}^{\mathbb{G}} W\right) \ne \{0\},$$

meaning that V_1 fails to be disjoint from W over $Z(\mathbb{G})\mathbb{H}$ and hence also from

$$V|_{\mathbb{H}} = W|_{\mathbb{H}} \text{ over } \mathbb{H}.$$

To conclude, observe that

- (5.2) agrees on V and V_1 due to the noted non-disjointness

$$\hom_{\mathbb{H}}(V_1, V) \neq 0;$$

- while the bottom left-hand map $\textsc{can} : C(\mathbb{G}) \to \widehat{Z(\mathbb{G})}$ of (5.3) annihilates V_1 because the latter has trivial central character;
- and hence the top right-hand $\textsc{can}$ map in (5.3) must also annihilate V.

This being the desired conclusion, we are done. □

Acknowledgments

I am also very grateful for the anonymous referee's comments on an earlier version of the paper.

Funding

This work is partially supported by NSF grant DMS-2001128.

References

[1] Abe, E. (1980). *Hopf Algebras*, volume 74 of Cambridge Tracts in Mathematics (Translated from the Japanese by Hisae Kinoshita and Hiroko Tanaka). Cambridge-New York: Cambridge University Press.
[2] Andruskiewitsch, N., Devoto, J. (1995). Extensions of Hopf algebras. *Algebra i Analiz* 7(1):22–61.
[3] Bekka, B., de la Harpe, P., Valette, A. (2008). *Kazhdan's Property (T)*, volume 11 of New Mathematical Monographs. Cambridge: Cambridge University Press.
[4] Borel, A. (1991). *Linear Algebraic Groups*, volume 126 of Graduate Texts in Mathematics, 2nd ed. New York: Springer-Verlag.
[5] Brzezinski, T., Wisbauer, R. (2003). *Corings and Comodules*, volume 309 of London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press.
[6] Burciu, S. (2011). Clifford theory for cocentral extensions. *Isr. J. Math.* 181:111–123.
[7] Ceccherini-Silberstein, T., Scarabotti, F., Tolli, F. (2009). Clifford theory and applications. *J. Math. Sci.* 156:29–43. Functional analysis.
[8] Chirvasitu, A. (2014). Centers, cocenters and simple quantum groups. *J. Pure Appl. Algebra* 218(8):1418–1430.
[9] Chirvasitu, A. (2014). Cosemisimple Hopf algebras are faithfully flat over Hopf subalgebras. *Algebra Number Theory* 8(5):1179–1199.
[10] Chirvasitu, A. (2021). Chain-center duality for locally compact groups. http://arxiv.org/abs/2109.08116v2.
[11] Chirvasitu, A. Hoche, S. O., Kasprzak, P. (2017). Fundamental isomorphism theorems for quantum groups. *Expo. Math.* 35(4):390–442.
[12] Clifford, A. H. (1937). Representations induced in an invariant subgroup. *Ann. Math. (2)* 38(3):533–550.
[13] De Commer, K., Kasprzak, P., Skalski, A., Soł tan, P. M. (2018). Quantum actions on discrete quantum spaces and a generalization of Clifford's theory of representations. *Isr. J. Math.* 226(1):475–503.
[14] Dijkhuizen, M. S., Koornwinder, T. H. (1994). CQG algebras: A direct algebraic approach to compact quantum groups. *Lett. Math. Phys.* 32(4):315–330.
[15] Dixmier, J. (1977). *C^*-algebras*. North-Holland Mathematical Library, Vol. 15 (Translated from the French by Francis Jellett). Amsterdam-New York-Oxford: North-Holland Publishing Co.
[16] Dăscălescu, S. (2001). Constantin Năstăsescu, and Şerban Raianu. In: *Hopf Algebras*, volume 235 of Monographs and Textbooks in Pure and Applied Mathematics. New York: Marcel Dekker, Inc. An introduction.
[17] Hewitt, E., Ross, K. A. (1970). *Abstract Harmonic Analysis. Vol. II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups*. Die Grundlehren der mathematischen Wissenschaften, Band 152. New York-Berlin: Springer-Verlag.
[18] Jantzen, J. C. (2003). *Representations of Algebraic Groups*, volume 107 of Mathematical Surveys and Monographs, 2nd ed. Providence, RI: American Mathematical Society.
[19] Majid, S. (1995). *Foundations of Quantum Group Theory*. Cambridge: Cambridge University Press.
[20] Martin, R. P. (1975). On the decomposition of tensor products of principal series representations for real-rank one semisimple groups. *Trans. Amer. Math. Soc.* 201:177–211.
[21] Montgomery, S. (1993). *Hopf Algebras and their Actions on Rings*, volume 82 of CBMS Regional Conference Series in Mathematics. Washington, DC: Published for the Conference Board of the Mathematical Sciences.
[22] Müger, M. (2004). On the center of a compact group. *Int. Math. Res. Not.* 51:2751–2756.

[23] Mumford, D., Fogarty, J., Kirwan, F. (1994). *Geometric Invariant Theory*, volume 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], 3rd ed. Berlin: Springer-Verlag.
[24] Nagata, M. (1961/62). Complete reducibility of rational representations of a matric group. *J. Math. Kyoto Univ.* 1:87–99.
[25] Parshall, B., Wang, J. P. (1991). Quantum linear groups. *Mem. Amer. Math. Soc.* 89(439):vi+157.
[26] Radford, D. E. (2012). *Hopf Algebras*, volume 49 of Series on Knots and Everything. Hackensack, NJ: World Scientific Publishing.
[27] Repka, J. (1976). Tensor products of unitary representations of $SL_2(R)$.. *Bull. Amer. Math. Soc.* 82(6):930–932.
[28] Robert, A. (1983). *Introduction to the Representation Theory of Compact and Locally Compact Groups*, volume 80 of London Mathematical Society Lecture Note Series. Cambridge-New York: Cambridge University Press.
[29] Schauenburg, P. (2000). Faithful flatness over Hopf subalgebras: counterexamples. In: *Interactions between Ring Theory and Representations of Algebras (Murcia)*, Volume 210 of Lecture Notes in Pure and Applied Mathematics. New York: Dekker, pp. 331–344.
[30] Schneider, H.-J. (1990). Representation theory of Hopf Galois extensions. *Isr. J. Math.* 72:196–231. Hopf algebras.
[31] Sweedler, M. E. (1969). *Hopf Algebras*. Mathematics Lecture Note Series. New York: W. A. Benjamin, Inc.
[32] Takeuchi, M. (1979). Relative Hopf modules—equivalences and freeness criteria. *J. Algebra* 60(2):452–471.
[33] Van Oystaeyen, F., Zhang, Y. (1993). Induction functors and stable Clifford theory for Hopf modules. *J. Pure Appl. Algebra* 107:337–351. 1996. Contact Franco-Belge en Algèbre (Diepenbeek, 1993).
[34] Wang, S. (2014). Equivalent notions of normal quantum subgroups, compact quantum groups with properties *F* and *FD*, and other applications. *J. Algebra* 397:515–534.
[35] Williams, F. L. (1973). *Tensor Products of Principal Series Representations. Reduction of Tensor Products of Principal Series. Representations of Complex Semisimple Lie Groups.* Lecture Notes in Mathematics, Vol. 358. Berlin-New York: Springer-Verlag.
[36] Witherspoon, S. J. (1999). Clifford correspondence for finite-dimensional Hopf algebras. *J. Algebra* 218(2):608–620.
[37] Witherspoon, S. J. (2002). Clifford correspondence for algebras. *J. Algebra* 256(2):518–530.
[38] Woronowicz, S. L. (1998). Compact quantum groups. In: *Symétries quantiques (Les Houches, 1995)*. Amsterdam: North-Holland, pp. 845–884.