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# Centers of Categorified Endomorphism Rings

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## Abstract

We prove that for a large class of well-behaved cocomplete categories $\mathcal{C}$ the weak and strong Drinfeld centers of the monoidal category $\mathcal{E}$ of cocontinuous endofunctors of $\mathcal{C}$ coincide. This generalizes similar results in the literature, where $\mathcal{C}$ is the category of modules over a ring $A$ and hence $\mathcal{E}$ is the category of $A$-bimodules.


**Keywords** Drinfeld center · Weak center · Locally presentable category · 2-abelian group · 2-ring · Dualizable · Linearly reductive

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## 1 Introduction

The present note is motivated by the following result from [2] (see Theorem 2.10 therein):

**Theorem 1.1** *For a ring $A$, the weak and strong centers of the monoidal category ${}_A\mathcal{M}_A$ of $A$-bimodules coincide.*

We give a refresher on the terminology in Section 2.2 below, pausing here only for a broad-strokes perspective on the result.

As seen from Definitions 2.8 and 2.10 below, Theorem 1.1 says, essentially, that a certain morphism

$$A \otimes V \to V \otimes A$$

of $A \otimes A$-bimodules (for a bimodule $V \in {}_A\mathcal{M}_A$ underlying a weak-center object) is automatically an isomorphism. The proofs of [2, Propositions 2.5 and 2.6] make it clear that this is the type of rigidity phenomenon familiar from the theory of *descent* in ring theory and / or algebraic geometry [13]. In the latter setup one typically starts with commutative rings $R \to S$ and an $S$-module $M$ and seeks to recover an $R$-module $M_R$ such that

$$M \cong S \otimes_R M_R;$$

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in other words, the goal is to *descend* the $S$-module to an $R$-module. The sort of structure necessary to achieve this in good cases (e.g. $S$ is faithfully flat over $R$ [13, Théorème 3.2]) is a *descent datum* (see [13, discussion preceding Proposition 3.1]): an $S \otimes S$-module morphism

$$g : S \otimes M \to M \otimes S$$

(where '$\otimes$' means '$\otimes_R$') such that

(1) the diagram

$$\begin{array}{ccccc} & \xrightarrow{g_{23}} & S \otimes M \otimes S & \xrightarrow{g_{12}} & \\ S \otimes S \otimes M & & \xrightarrow[g_{13}]{\hspace{6em}} & & M \otimes S \otimes S \end{array}$$

commutes (with the indices indicating the tensorands on which $g$ operates), and

(2) the morphism

$$M \to S \otimes M \xrightarrow{g} M \otimes S \to M$$

is the identity, where the leftmost arrow is the natural inclusion obtained by tensoring the unit $R \to S$ of $S$ with $\mathrm{id}_M$ and the rightmost arrow is multiplication by scalars in $S$.

Under these circumstances it turns out [13, Proposition 3.1] that in fact $g$ is automatically an isomorphism. This is essentially the same phenomenon as that captured in Theorem 1.1 in the broader context of non-commutative rings.

In attempting to isolate precisely what it is about categories of bimodules that occasions such rigidity results one is led to consider the celebrated Eilenberg-Watts theorem ([21, Theorem 1] or [8]):

- ${}_A\mathcal{M}_A$ is equivalent to the category of cocontinuous (i.e. colimit-preserving [15, §V.4]) endofunctors of the category ${}_A\mathcal{M}$ of left $A$-modules (or its right-handed version $\mathcal{M}_A$),
- such that the monoidal structure given by '$\otimes_A$' is identified with endofunctor composition.

This is the starting point for the generalization of Theorem 1.1 appearing as Theorem 3.1 below. The pattern we extrapolate can be summarized as follows (with a forward reference to Section 2.1 below for category-theoretic terminology).

- One can substitute other "well-behaved" cocomplete categories $\mathcal{C}$ for ${}_A\mathcal{M}$;
- and their *duals* $\mathcal{C}^* \cong$ consisting of cocontinuous functors $\mathcal{C} \to$ (some "base" category) for $\mathcal{M}_A$;
- and their *endomorphism 2-rings*

$$\mathcal{E} := \mathcal{C} \boxtimes \mathcal{C}^* \cong \text{cocontinuous endofunctors of } \mathcal{C} \tag{1.1}$$

for ${}_A\mathcal{M}_A$.

For Eq. 1.1 to be both meaningful and valid $\mathcal{C}$ needs to be what in Definition 2.6 (and [4, Definition 1.1]) we refer to as *dualizable* (this is what 'well-behaved' means in the above discussion). With all of this behind us, Theorem 3.1 reads more or less as follows.

**Theorem 1.2** *If $\mathcal{C}$ is a dualizable locally presentable category then the weak center of its category of cocontinuous endofunctors coincides with its strong center.*

In addition to recovering Theorem 1.1, this applies to categories $\mathcal{C}$ going beyond modules, as we recall in Section 3: $\mathcal{C}$ can be, for instance,

- the category $\mathcal{M}^C$ of right-comodules over a *right-semiperfect* [14, p.369] coalgebra over a field;
- the category QCOH$([X/G])$ of quasicoherent sheaves over the quotient stack $[X/G]$ where $X$ is affine and $G$ is a *virtually linearly reductive* [7, §1] linear algebraic group acting on $X$.

## 2 Preliminaries

Some standard background on monoidal categories is needed, as covered for instance in [11, Chapter XI], [5, §5.1], [15, Chapter XI], and any number of other sources.

### 2.1 Some 2-algebra

We reprise some terminology from [6, §2].

**Definition 2.1** (a) A *2-abelian group* is a locally presentable category in the sense of [1, Definition 1.17]. 2-abelian groups form a 2-category 2AB with left adjoints as 1-morphisms and natural transformations as 2-morphisms.
(b) A *2-ring* is a 2-abelian group $\mathcal{C}$ which is in addition a monoidal category with tensor product '$\otimes$', so that all functors of the form $x \otimes -$ and $- \otimes x$ are left adjoints. 2-rings similarly form a 2-category 2RNG with *monoidal* left adjoints as 1-morphisms.
(c) A *commutative 2-ring* is a 2-ring additionally equipped with a symmetry (i.e. it is a symmetric monoidal category). As before, these form the 2-category 2COMRNG with symmetric monoidal left adjoints as 1-morphisms.

It turns out (e.g. [6, Corollary 2.2.5]) that 2AB is symmetric monoidal, being equipped with a tensor product denoted by '$\boxtimes$'. For 2-abelian groups $\mathcal{A}$ and $\mathcal{B}$ their tensor product $\mathcal{A} \boxtimes \mathcal{B}$ is the universal recipient of a bifunctor

$$\mathcal{A} \times \mathcal{B} \to \mathcal{A} \boxtimes \mathcal{B}$$

that is separately cocontinuous (i.e. a "bilinear map" of 2-abelian groups). The symmetric monoidal structure lifts to 2RNG and 2COMRNG in the sense that if $\mathcal{A}$ and $\mathcal{B}$ are 2-rings so is $\mathcal{A} \boxtimes \mathcal{B}$ in a natural fashion, etc.

This machinery allows us to employ the usual language of rings and modules in the context of 2-abelian groups:

**Definition 2.2** Let $\mathcal{R}$ be a 2-ring. A *left (2-)$\mathcal{R}$-module* is a 2-abelian group $\mathcal{X}$ equipped with a morphism $\mathcal{R} \boxtimes \mathcal{X} \to \mathcal{X}$ in 2AB, satisfying the obvious unitality and associativity conditions. *Right* (2-)$\mathcal{R}$-modules are defined analogously, as are bimodules, etc.

The respective 2-categories of left or right or bimodules are denoted by ${}_{\mathcal{R}}\mathcal{M}$, $\mathcal{M}_{\mathcal{R}}$ and ${}_{\mathcal{R}}\mathcal{M}_{\mathcal{S}}$. respectively.

As usual, we have tensor product 2-bifunctors

$${}_{\mathcal{R}}\mathcal{M}_{\mathcal{S}} \times {}_{\mathcal{S}}\mathcal{M}_{\mathcal{T}} \xrightarrow{\boxtimes_{\mathcal{S}}} {}_{\mathcal{R}}\mathcal{M}_{\mathcal{T}}:$$

For 2-$\mathcal{S}$-modules $\mathcal{X}$ and $\mathcal{Y}$ with module-structure functors

$$\mathcal{X} \boxtimes \mathcal{S} \xrightarrow{\triangleleft} \mathcal{X} \quad \text{and} \quad \mathcal{S} \boxtimes \mathcal{Y} \xrightarrow{\triangleright} \mathcal{Y}$$

we can define $\mathcal{X} \boxtimes_{\mathcal{S}} \mathcal{Y}$ as the universal functor $\pi$ admitting a natural isomorphism $\theta$ as depicted below:

$$\begin{array}{ccccc} & \xrightarrow{\triangleleft\boxtimes \mathrm{id}} & \mathcal{X} \boxtimes \mathcal{Y} & \xrightarrow{\pi} & \\ \mathcal{X} \boxtimes \mathcal{S} \boxtimes \mathcal{Y} & & \cong \Downarrow \theta & & \mathcal{X} \boxtimes_{\mathcal{S}} \mathcal{Y} \\ & \xrightarrow[\mathrm{id} \boxtimes \triangleright]{} & \mathcal{X} \boxtimes \mathcal{Y} & \xrightarrow[\pi]{} & \end{array}$$

This is a 2-colimit, and can be obtained as a combination of a *co-inserter* and a *co-equifier*: constructions dual to the inserters and equifiers of [1, §2.71 and Lemma 2.76], for instance. [6, Proposition 2.1.11] outlines a more explicit construction for such 2-colimits in 2AB, which do exist.

In particular, for a *commutative* 2-ring $\mathcal{R}$, the 2-category ${}_{\mathcal{R}}\mathcal{M} \cong \mathcal{M}_{\mathcal{R}}$ is symmetric monoidal under $\boxtimes_{\mathcal{R}}$.

**Definition 2.3** For a commutative 2-ring $\mathcal{R}$ an $\mathcal{R}$-algebra (or 2-$\mathcal{R}$-algebra for extra precision) is an algebra in the symmetric monoidal 2-category ${}_{\mathcal{R}}\mathcal{M}$.

It turns out that 2AB is not only symmetric monoidal but also *monoidal-closed*, i.e. has *internal homs*. More precisely, we have the familiar hom-tensor adjunction in the present higher-categorical setting (see e.g. [12, §6.5], [1, Exercise 1.l]). The following result is a "relative" version of [4, Lemma 2.7] (which cites the preceding two sources), in the sense that it deals with modules over 2-rings rather than plain 2-abelian groups. The techniques involved in the proofs are no different.

**Lemma 2.4** *Let $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ be three 2-rings.*

(a) *For any two bimodules*

- $\mathcal{X} \in {}_{\mathcal{R}}\mathcal{M}_{\mathcal{T}}$
- $\mathcal{Y} \in {}_{\mathcal{S}}\mathcal{M}_{\mathcal{T}}$

*the category*

$$\mathrm{HOM}_{\mathcal{T}}(\mathcal{Y}, \mathcal{X}) := \{\textit{left adjoints } \mathcal{X} \to \mathcal{Y} \textit{ compatible with the 2-module structures}\}$$

*has a natural structure of a $\mathcal{R}$-$\mathcal{S}$-bimodule.*

(b) *This gives us, for each bimodule $\mathcal{Y} \in {}_{\mathcal{S}}\mathcal{M}_{\mathcal{T}}$, a 2-adjunction*

$$ {}_{\mathcal{R}}\mathcal{M}_{\mathcal{S}} \underset{\mathrm{HOM}_{\mathcal{T}}(\mathcal{Y},-)}{\overset{-\boxtimes_{\mathcal{S}}\mathcal{Y}}{\rightleftarrows}} {}_{\mathcal{R}}\mathcal{M}_{\mathcal{T}} $$

*with the top arrow as the left (2-)adjoint.*

*Remark 2.5* We leave it to the reader to formulate analogous versions for tensoring on the left rather than right, etc.

**Definition 2.6** Let $\mathcal{R}$ be a 2-ring and $\mathcal{X}$ a left $\mathcal{R}$-module.

(a) The *dual* $\mathcal{X}^*$ of $\mathcal{X}$ over $\mathcal{R}$ is ${}_{\mathcal{R}}\mathrm{HOM}(\mathcal{X}, \mathcal{R})$; it is a *right* $\mathcal{R}$-module. Similarly, duals of right modules are naturally left modules.
(b) If $\mathcal{R}$ is commutative the 2-$\mathcal{R}$-module $\mathcal{X}$ is *(1-)dualizable over* $\mathcal{R}$ if the canonical morphism

$$\mathcal{X} \boxtimes_{\mathcal{R}} \mathcal{X}^* \xrightarrow{\mathrm{CAN}} \mathrm{END}_{\mathcal{R}}(\mathcal{X}) \tag{2.1}$$

is an isomorphism of 2-$\mathcal{R}$-modules.

*Remark 2.7* For any 2-ring $\mathcal{R}$ and 2-$\mathcal{R}$-module $\mathcal{X}$ $\mathrm{END}_{\mathcal{R}}(\mathcal{X})$ is naturally a 2-ring (and a 2-$\mathcal{R}$-algebra when $\mathcal{R}$ is commutative), with composition as the tensor product and $\mathrm{id}_{\mathcal{X}}$ as the unit.

Dualizable objects (typically over $\mathcal{R} = \mathrm{VECT}_{\mathbb{K}}$ for some field $\mathbb{K}$) were the focus of [4], where we give alternative characterizations of dualizability in [4, Lemma 3.1]. In particular, it is enough to require that the identity

$$\mathrm{id}_{\mathcal{X}} \in \mathrm{END}_{\mathcal{R}}(\mathcal{X})$$

belong to the image of Eq. 2.1.

## 2.2 Centers

Recall (e.g. [11, Definition XIII.4.1] or [10, Definition 3]):

**Definition 2.8** Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a monoidal category. The *(Drinfeld) center* $Z(\mathcal{C})$ of $\mathcal{C}$ is the category of pairs $(x, \theta)$ where $x \in \mathcal{C}$ is an object and

$$\theta : - \otimes x \xrightarrow{\cong} x \otimes - \tag{2.2}$$

is a natural isomorphism satisfying the following conditions (suppressing the associativity constraints in the monoidal category):

(1) For $y, z \in \mathcal{C}$ the diagram

$$\begin{array}{ccccc} & & y \otimes x \otimes z & & \\ & \nearrow^{\mathrm{id}_y \otimes \theta_z} & & \searrow^{\theta_y \otimes \mathrm{id}_z} & \\ y \otimes z \otimes x & & \xrightarrow[\theta_{y \otimes z}]{} & & x \otimes y \otimes z \end{array}$$

commutes, and
(2) the isomorphism

$$\theta_{\mathbf{1}} : \mathbf{1} \otimes x \to x \otimes \mathbf{1}$$

is the canonical one attached to the monoidal structure $(\mathcal{C}, \otimes, \mathbf{1})$.

*Remark 2.9* In fact, in Definition 2.8 condition (2) follows from (1), but this uses the fact that $\theta$ is an *isomorphism*; we have displayed both conditions with an eye towards Definition 2.10 below.

Following [18, Definition 4.3] (where the notion seems to have been introduced) and [2, §1.1], we give

**Definition 2.10** For $(\mathcal{C}, \otimes, \mathbf{1})$ as in Definition 2.8 the *weak right center* $WZ_r(\mathcal{C})$ is the category of pairs $(x, \theta)$ as above, satisfying conditions (1) and (2), but requiring only that Eq. 2.2 be a natural transformation.

One defines the weak *left* center $WZ_\ell(\mathcal{C})$ analogously, requiring a natural transformation

$$x \otimes - \to - \otimes x$$

instead.

Unless specified otherwise *weak center* means weak *right* center, and we simply write $WZ$ for $WZ_r$.

## 3 Main Results

**Theorem 3.1** *Let $\mathcal{R}$ be a commutative 2-ring, $\mathcal{X} \in {}_{\mathcal{R}}\mathcal{M}$ a dualizable $\mathcal{R}$-module, and*

$$\mathcal{E} := \mathcal{X} \boxtimes_{\mathcal{R}} \mathcal{X}^* \cong \mathrm{E{\small ND}}_{\mathcal{R}}(\mathcal{X}) \tag{3.1}$$

*its endomorphism ring. Then, the canonical fully faithful inclusion*

$$Z(\mathcal{E}) \to WZ(\mathcal{E}) \tag{3.2}$$

*is an equivalence.*

This requires some preliminary discussion and tooling, starting with the observation that this weak-equals-strong principle cannot be expected to hold in general: Definition 2.10 is indeed a weakening of Definition 2.8, in the sense that there are examples of monoidal categories (even 2-algebras) $\mathcal{C}$ where the canonical functor Eq. 3.2 is not an equivalence.

*Example 3.2* Let $(\mathcal{C}, \otimes, \mathbf{1})$ be any symmetric, additive monoidal category. Recall (e.g. [9, Definition 2.1]) that its *Bernstein center* is the (commutative) ring of natural endomorphisms of the identity functor. Any element $\theta$ of the Bernstein center gives an element $(\mathbf{1}, \psi)$ of the weak center $WZ(\mathcal{C})$, whereby

$$y \cong y \otimes \mathbf{1} \xrightarrow{\psi_y} \mathbf{1} \otimes y \cong y$$

is simply $\theta_y$, provided $\theta_{\mathbf{1}}$ is the identity. Furthermore, if $\theta_y$ fails to be an isomorphism for any $y$ we obtain an element outside the plain center $Z(\mathcal{C})$.

All of this is easily arranged. Let $\mathcal{C}$, for instance, be the (symmetric, monoidal) category $\mathrm{Rep}_f(G)$ of finite-dimensional complex representations over a finite group $G$. Every $y \in \mathrm{Rep}_f(G)$ has a canonical **1**-*isotypic component*: the space $y^G$ of all $G$-invariant vectors in $y$. There is an element $\theta$ of the Bernstein center that surjects every object onto this isotypic component:

$$y \ni v \xmapsto{\theta_y} \frac{1}{|G|} \sum_{g \in G} gv \in y$$

If $G$ is non-trivial then $\theta_y$ will fail to be an isomorphism on those $y$ that are not sums of copies of **1**, and we have an example as required above.

Since $\mathcal{R}$ is our "base ring" throughout the discussion we henceforth abbreviate '$\boxtimes_{\mathcal{R}}$' to simply '$\boxtimes$', and similarly for $\mathrm{H{\small OM}} := \mathrm{H{\small OM}}_{\mathcal{R}}$. Recall also our notation Eq. 3.1 for the endomorphism 2-ring $\mathcal{E}$ of $\mathcal{X}$.

Because $\mathcal{X}$ is assumed dualizable over $\mathcal{R}$, the canonical morphism

$$\mathcal{X} \to \mathcal{X}^{**}$$

is an isomorphism (of abelian 2-groups, i.e. an equivalence of categories). It follows from this that $\mathcal{E}$ is also dualizable and in fact self-dual, and we can identify

$$\text{END}_{\mathcal{R}}(\mathcal{E}) \;\cong\; \mathcal{E} \boxtimes \mathcal{E}^* \;\cong\; \mathcal{E} \boxtimes \mathcal{E} \;\cong\; \mathcal{X} \boxtimes \mathcal{X}^* \boxtimes \mathcal{X} \boxtimes \mathcal{X}^*. \tag{3.3}$$

Given that 2AB is a *symmetric* monoidal 2-category, there is some choice in how we identify the right and left-hand sides of Eq. 3.3. In the sequel, it will be convenient to make this identification by pairing the two *middle* tensorands on the right-hand side of Eq. 3.3 against $\mathcal{E} \cong \mathcal{X} \boxtimes \mathcal{X}^*$ in the obvious fashion (by pairing each $\mathcal{X}$ to a $\mathcal{X}^*$). Concretely, simple-tensor object

$$x \boxtimes f \boxtimes y \boxtimes g \in \mathcal{X} \boxtimes \mathcal{X}^* \boxtimes \mathcal{X} \boxtimes \mathcal{X}^*$$

corresponds to the element

$$\mathcal{E} \ni \psi \longmapsto f(\psi(y))x \boxtimes g \in \mathcal{X} \boxtimes \mathcal{X}^* \cong \mathcal{E}$$

of $\text{END}_{\mathcal{R}}(\mathcal{E})$.

Now fix an object $(e, \theta) \in WZ(\mathcal{E})$ of the weak right center and consider the two endomorphisms

$$- \otimes e \text{ and } e \otimes - \quad \in \quad \text{END}(\mathcal{E}) \cong \mathcal{E} \boxtimes \mathcal{E}.$$

With the above convention in mind, they are identifiable, respectively, with

$$\mathbf{1} \boxtimes e \quad \text{and} \quad e \boxtimes \mathbf{1} \tag{3.4}$$

where

$$\mathbf{1} := \text{id}_{\mathcal{X}} = \mathbf{1}_{\mathcal{E}} \in \mathcal{E} = \text{END}(\mathcal{X}) \cong \mathcal{X} \boxtimes \mathcal{X}^*$$

is the identity functor on $\mathcal{X}$ (i.e. the monoidal unit of $\mathcal{E}$). We caution the reader that the tensor product in Eq. 3.4 is *external*, i.e. it is not to be confused with the internal tensor product '$\otimes$' of $\mathcal{E}$. Indeed, under the latter we of course have

$$\mathbf{1} \otimes e \cong e \cong e \otimes \mathbf{1}$$

(as in any monoidal category).

The natural transformation

$$- \otimes e \to e \otimes -$$

that constitutes the structure of a weak-center element (Definition 2.10) translates to a morphism

$$\mathbf{1} \boxtimes e \xrightarrow{\theta} e \boxtimes \mathbf{1} \tag{3.5}$$

in $\mathcal{E} \boxtimes \mathcal{E}$ (denoted slightly abusively by the same symbol '$\theta$' we used for the natural transformation in Definition 2.10). The conditions (1) and (2) can then be recast in terms of Eq. 3.5 as we explain presently.

To express condition (1) we need to work in the *triple* tensor product $\mathcal{E}^{\boxtimes 3}$. To that end, we consider morphisms between tensor products of $e$ and two copies of $\mathbf{1}$, with two indices among 1, 2 and 3 indicating where $\theta$ operates. Thus:

$$\theta_{12} := \theta \boxtimes \text{id}_{\mathbf{1}} \;:\; \mathbf{1} \boxtimes e \boxtimes \mathbf{1} \to e \boxtimes \mathbf{1} \boxtimes \mathbf{1},$$
$$\theta_{23} := \text{id}_{\mathbf{1}} \boxtimes \theta \;:\; \mathbf{1} \boxtimes \mathbf{1} \boxtimes e \to \mathbf{1} \boxtimes e \boxtimes \mathbf{1},$$

and similarly,

$$\theta_{13} : \mathbf{1} \boxtimes \mathbf{1} \boxtimes e \to e \boxtimes \mathbf{1} \boxtimes \mathbf{1}$$

is the morphism acting identically on the middle tensorand and as $\theta$ on the two outer ones. (1) in Definition 2.8 can now be recovered simply as

$$\theta_{13} = \theta_{12} \circ \theta_{23} : \mathbf{1} \boxtimes \mathbf{1} \boxtimes e \to e \boxtimes \mathbf{1} \boxtimes \mathbf{1}. \tag{3.6}$$

Next, denote by

$$m : \mathcal{E} \boxtimes \mathcal{E} \to \mathcal{E}$$

the "multiplication" morphism, imparting on $\mathcal{E} = \mathrm{END}(\mathcal{X})$ its monoidal category structure. In terms of the decomposition

$$\mathcal{E} \boxtimes \mathcal{E} \cong \mathcal{X} \boxtimes \mathcal{X}^* \boxtimes \mathcal{X} \boxtimes \mathcal{X}^*$$

$m$ is simply the evaluation of the two middle tensorands $\mathcal{X}$ and $\mathcal{X}^*$ against each other. With this in place, condition (2) in Definition 2.8 simply asks that

$$m(\theta) : \mathbf{1} \otimes e \to e \otimes \mathbf{1}$$

be the canonical isomorphism, i.e. the identity once we have made the usual identifications

$$\mathbf{1} \otimes e \cong e \cong e \otimes \mathbf{1}.$$

In short, for future reference:

$$m(\theta) = \mathrm{id}_e : e \cong \mathbf{1} \otimes e \to e \otimes \mathbf{1} \cong e. \tag{3.7}$$

*Proof of Theorem 3.1* Since we already know that Eq. 3.2 is fully faithful (as is immediate from Definitions 2.8 and 2.10), it remains to show that it is essentially surjective: for an arbitrary object $(e, \theta) \in WZ(\mathcal{E})$ the morphism Eq. 3.5 is an isomorphism in $\mathcal{E} \boxtimes \mathcal{E}$. What we will in fact do is identify the inverse of $\theta$: it is precisely

$$\theta' := \tau \circ \theta \circ \tau : e \boxtimes \mathbf{1} \to \mathbf{1} \boxtimes e,$$

where $\tau$ is the tensorand-reversal functor on $\mathcal{E} \boxtimes \mathcal{E}$.

Denote by

$$m_{13} : \mathcal{E} \boxtimes \mathcal{E} \boxtimes \mathcal{E} \to \mathcal{E} \boxtimes \mathcal{E}$$

the functor that multiplies the outer (first and third) tensorands of the domain onto the second tensorand of the codomain. We then have

$$\begin{aligned} \theta &= m_{13}(\theta_{23}) \quad \text{and} \\ \theta' &= m_{13}(\theta_{12}), \end{aligned}$$

meaning that

$$m_{13}(\theta_{13}) = m_{13}(\theta_{12} \circ \theta_{23}) = \theta' \circ \theta \tag{3.8}$$

(where the first equality uses Eq. 3.6 above). On the other hand though, Eq. 3.7 implies that the left-hand side $m_{13}(\theta_{13})$ of Eq. 3.8 is nothing but the identity, and thus

$$\theta' \circ \theta = \mathrm{id}_{\mathbf{1} \boxtimes e}.$$

The other composition $\theta \circ \theta'$ is treated similarly, so we do not repeat the argument. □

*Remark 3.3* The proof of Theorem 3.1 given above is a paraphrase, in the present categorified context, of an argument familiar from descent theory. See e.g. [13, Proposition 3.1]. Where $\mathcal{X}$ would have been the category of modules over (in those authors' notation) a ring $S$.

In the context of $k$-linear 2-abelian groups (for some field $k$) the examples of dualizable $\mathcal{C}$ in [4] are all, abstractly, of the form

$$k\text{-linear functors } \Gamma^{op} \to {}_k\mathrm{VECT} \tag{3.9}$$

for small $k$-linear categories $\Gamma$. These are also

- the $k$-linear abelian categories admitting a generating set of *small* projective objects [17, §3.6, Corollary 6.4];
- the $k$-linear locally presentable admitting a strongly generating set of small projective objects [12, Theorem 5.26];

recall that the small projective objects in an abelian category $\mathcal{C}$ are simply those $x \in \mathcal{C}$ for which the representable functor

$$\hom(x, -) : \mathcal{C} \to \mathrm{SET}$$

is cocontinuous. This is taken as the definition of the hyphenated term '*strong-projective*' in [12, §5.5], and we reuse that term in Corollary 3.4 below for consistency.

Conversely, [4, Lemma 3.5] shows that *all* categories of the form Eq. 3.9 are dualizable 2-modules over ${}_k\mathrm{VECT}$. The same argument goes through for arbitrary commutative 2-rings $\mathcal{R}$ (in place of ${}_k\mathrm{VECT}$), so we have

**Corollary 3.4** *Let $\mathcal{R}$ be a commutative 2-ring and $\mathcal{X}$ a 2-$\mathcal{R}$-module with a strong generating set of small-projective objects. Then, the weak center of the monoidal category*

$$\mathcal{X} \boxtimes_{\mathcal{R}} \mathcal{X}^* \cong \mathrm{END}_{\mathcal{R}}(\mathcal{X})$$

*coincides with its strong center.*

We end with some examples of categories falling under the scope of Corollary 3.4 (and hence Theorem 3.1).

*Example 3.5* Throughout the present discussion we assume $C$ is a coalgebra over a field.

By [4, Theorem 1.3], Theorem 3.1 applies to categories of right comodules $\mathcal{M}^C$ over *right-semiperfect* coalgebras $C$ in the sense of [14, p.369]: every right $C$-comodule has a projective cover.

This of course includes cosemisimple coalgebras (i.e. those with only projective modules or equivalently, direct sums of simple coalgebras; [20, Definition, p.290] or [16, Definition 2.4.1]).

*Example 3.6* Overlapping Example 3.5 to a degree, consider a linear algebraic group [3, §1.6] $G$ acting on an affine scheme $X$ and the category

$$\mathrm{QCOH}(X)^G \cong \mathrm{QCOH}([X/G]) \tag{3.10}$$

of $G$-equivariant quasicoherent sheaves on $X$, or equivalently, as Eq. 3.10 recalls [19, Tag 06WV], that of quasicoherent sheaves on the quotient stack [19, Tag 044O] $[X/G]$.

According to [4, Theorem 1.5] Eq. 3.10 is dualizable provided $G$ is *virtually linearly reductive* in the sense of [7, §1]: $G$ has a normal linearly reductive closed algebraic subgroup $H \trianglelefteq G$ such that $G/H$ is a finite group scheme. This means that

- the Hopf algebra $\mathcal{O}(H)$ is cosemisimple while $\mathcal{O}(G/H)$ is finite-dimensional;
- equivalently by [7, Theorem, p.76], the Hopf algebra $\mathcal{O}(G)$ of regular functions on $G$ is (left and right) semiperfect in the sense of Example 3.5.

*Example 3.7* One can generalize Example 3.6 as follows. Note that the category Eq. 3.10 can be recovered as that of $\mathcal{O}(X)$-modules (where $\mathcal{O}(X)$ is the algebra of regular functions on $X$) internal to the category of $\mathcal{O}(G)$-comodules. In short:

$$\text{QCOH}([X/G]) \cong \mathcal{M}_{\mathcal{O}(X)}^{\mathcal{O}(G)}.$$

Mimicking this construction, we can take $\mathcal{X}$ in Theorem 3.1 to be the category $\mathcal{M}_A$ of (right, say) modules over an algebra $A$ internal to the commutative 2-algebra $\mathcal{R}$ (so that in Example 3.6 we would have $\mathcal{R} = \mathcal{M}^{\mathcal{O}(G)}$ and $A = \mathcal{O}(X)$).

This means that Theorem 3.1 applies, for instance, to categories of graded modules over graded algebras (for arbitrary grading monoids), etc.

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## Declarations

**Conflict of Interests** There are no conflicts of interest to report.

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