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RESEARCH ARTICLE

Syntomic complexes and p-adic étale Tate twists

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Abstract

The primary goal of this paper is to identify syntomic complexes with the p-adic étale Tate twists of Geisser–Sato–Schneider on regular p-torsion-free schemes. Our methods apply naturally to a broader class of schemes that we call 'F-smooth'. The F-smoothness of regular schemes leads to new results on the absolute prismatic cohomology of regular schemes.

Contents

1. Introduction

Let X be a scheme. In [BL22a, Sec. 8], the first author and Lurie, following the earlier work [BMS19], define and study certain *syntomic complexes* $\mathbb{Z}_p(i)(X) = R\Gamma_{\mathrm{syn}}(X, \mathbb{Z}_p(i))$ for $i \in \mathbb{Z}$, extending earlier constructions in the literature [FM87, Kat87]. These syntomic complexes yield a generalization of the p-adic étale cohomology (with Tate twisted coefficients) for $\mathbb{Z}[1/p]$-schemes to arbitrary schemes, and exhibit quite different behaviour in positive and mixed characteristic, where they are obtained from prismatic cohomology. We refer to [CN17, Sec. 1.1] for a survey of applications of syntomic cohomology.

The purpose of this paper is to identify the syntomic complexes as étale sheaves on X in a class of examples. In doing so, we generalize a number of existing results in the literature, including those of [Kur87, Kat87, Tsu99, CN17], and recover the p-adic étale Tate twists of [Sch94, Gei04, Sat07].

1.1. What is syntomic cohomology?

To formulate our results, it is convenient to name the restriction of syntomic cohomology to the small étale site.

Notation 1.1 (The complexes $\mathbb{Z}/p^n(i)_X$). For any scheme X and integer $i \in \mathbb{Z}$, write $\mathbb{Z}/p^n(i)_X \in \mathcal{D}(X_{\mathrm{et}}, \mathbb{Z}/p^n)$ for the object of the derived ∞-category of étale sheaves of $\mathbb{Z}/p^n$-modules on X obtained

by restricting the syntomic complexes[1] $\mathbb{Z}/p^n(i)(-)$ of [BL22a, §8] to the small étale site X_{et} of X. Thus, we have a defining identification $R\Gamma(X, \mathbb{Z}/p^n(i)_X) \simeq \mathbb{Z}/p^n(i)(X)$.

Let us describe this object in the key examples.

Example 1.2 (Syntomic cohomology in characteristic $\neq p$). For any X, the restriction of $\mathbb{Z}/p^n(i)_X$ to the locus $X[1/p] \subset X$ is given by $\mu_{p^n}^{\otimes i} \simeq (\mathbb{Z}/p^n(i)_X)|_{X[1/p]}$. In particular, if p is invertible on X, then $\mathbb{Z}/p^n(i)_X$ is simply the usual étale Tate twist $\mu_{p^n}^{\otimes i}$.

Example 1.3 (Syntomic complexes via logarithmic de Rham–Witt sheaves in characteristic p). When X is a regular $\mathbb{F}_p$-scheme, we have isomorphisms $\mathbb{Z}/p^n(i)_X \simeq W_n\Omega^i_{\log,X}[-i]$ for $W_n\Omega^i_{\log,X}$ the logarithmic Hodge–Witt sheaves considered in [Mil76, Ill79, Gro85], cf. [BMS19, Sec. 8].

Example 1.4 (Syntomic cohomology of p-adic formal schemes). For any scheme X, the pullback of $\mathbb{Z}/p^n(i)_X$ to the étale site of the p-adic completion $\widehat{X}$ (or equivalently that of X/p) is constructed as a filtered Frobenius eigenspace of prismatic cohomology first studied in [BMS19]. That is, if $X = \mathrm{Spec}(R)$ for R a p-Henselian animated ring, then one has an expression

$$\mathbb{Z}_p(i)(X) = \mathrm{eq}\left(\mathrm{can}, \phi_i : \mathcal{N}^{\geq i}\mathbb{\Delta}_X\{i\} \rightrightarrows \mathbb{\Delta}_X\{i\}\right). \tag{1}$$

Here, $\mathbb{\Delta}_X\{i\}$ denotes the Breuil–Kisin twisted (absolute) prismatic cohomology of X, $\mathcal{N}^{\geq *}$ denotes the Nygaard filtration, ϕ_i denotes the ith divided Frobenius and can denotes the inclusion map. We refer to [BL22a, Sec. 7] for a detailed treatment of these objects.

Earlier versions of this construction (which agree with the above for $i \leq p-2$ or up to isogeny; cf. [AMMN22, Sec. 6] for comparisons) were introduced in [FM87, Kat87] using crystalline cohomology and the Hodge filtration instead of prismatic cohomology and the Nygaard filtration.

Examples 1.2 and 1.4 essentially suffice to describe syntomic cohomology in general via a gluing procedure: If R is a commutative ring with p-henselization $R^h_{(p)}$, one has a fiber square

$$\begin{array}{ccc} \mathbb{Z}/p^n(i)(\mathrm{Spec}(R)) & \longrightarrow & R\Gamma_{\text{et}}(\mathrm{Spec}(R[1/p]), \mu_{p^n}^{\otimes i}) \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^n(i)(\mathrm{Spec}(R^h_{(p)})) & \longrightarrow & R\Gamma_{\text{et}}(\mathrm{Spec}(R^h_{(p)}[1/p]), \mu_{p^n}^{\otimes i}), \end{array}$$

where the terms on the right are usual étale cohomology (cf. Example 1.2), the term on the bottom left is computed via prismatic cohomology as in Example 1.4, and the bottom horizontal map is obtained from the prismatic logarithm and the étale comparison theorem for prismatic cohomology in [BL22a, §8.3]. In fact, this approach was used as the definition of the top-left vertex in [BL22a, §8.4].

For any scheme X, the complex $\mathbb{Z}/p^n(0)_X$ identifies with the constant sheaf $\mathbb{Z}/p^n$ on X_{et}. One can also make the complex explicit in weight 1:

Example 1.5 (Cf. [BL22a, Prop. 8.4.14]). For any scheme X, one has that $\mathbb{Z}/p^n(1)_X$ is the derived pushforward of μ_{p^n} from the fppf site to the étale site (or equivalently the fiber of $p^n : \mathbb{G}_m \to \mathbb{G}_m$ in the derived category of étale sheaves).

Finally, for completeness, we recall that syntomic cohomology also has a close connection to p-adic K-theory, yielding a simple construction of the former which appears in [Niz12]. For this, we recall ([BMS19, Sec. 4] or [BL22a, App. C]) that a ring R is *p-quasisyntomic* if it has bounded p-power torsion and $L_{R/\mathbb{Z}} \otimes^{\mathbb{L}}_R R/pR \in \mathcal{D}(R/pR)$ has Tor-amplitude in $[-1, 0]$; for instance, any locally complete intersection (lci) noetherian ring has this property.

[1] In [BL22a, §8], the object $\mathbb{Z}_p(i)(X)$ is defined in the p-complete derived ∞-category; by $\mathbb{Z}/p^n(i)(X)$, we mean the reduction mod p^n. Note also that the construction $X \mapsto \mathbb{Z}/p^n(i)(X)$ on affine schemes carries filtered colimits of rings to filtered colimits [BL22a, Cor. 8.4.11].

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Example 1.6 (The $\mathbb{Z}/p^n(i)_X$ via algebraic K-theory). Let X be a p-quasisyntomic scheme. In this case, one can give a direct construction of the $\mathbb{Z}/p^n(i)_X$ using algebraic K-theory for $i \geq 0$. Namely, $\mathbb{Z}/p^n(i)_X \in \mathcal{D}(X_{\text{et}}, \mathbb{Z}/p^n)$ is the derived pushforward of the sheafification of the presheaf $K_{2i}(-; \mathbb{Z}/p^n)$ from the syntomic site of X to the étale site of X. This is essentially a consequence of the work [BMS19] and rigidity [Gab92, Sus83, CMM21] and will be discussed in more detail separately.

The connection to algebraic K-theory does not play a direct role in this article; nonetheless, the connection to topological Hochschild homology provided by the K-theoretic approach inspired many of the arguments in this paper.

1.2. Results

Syntomic cohomology is essentially p-adic étale motivic cohomology where the latter is defined, cf. [Gei04, Niz06, EN19]. For example, syntomic cohomology admits a robust theory of Chern classes. However, the syntomic complexes are defined for arbitrary schemes through the theory of prismatic cohomology, without any explicit use of algebraic cycles. We will identify syntomic cohomology for a class of p-torsion-free 'F-smooth' schemes and obtain a formula related to the Beilinson–Lichtenbaum conjecture in motivic cohomology. To begin, let us formulate the definition of F-smoothness.

Definition 1.7 (F-smoothness, Definition 4.1 below). We say that a p-quasisyntomic ring R is *F-smooth* if for each i, the prismatic divided Frobenius $\phi_i : \mathcal{N}^i \mathbb{\Delta}_R \to \overline{\mathbb{\Delta}}_R\{i\}$ has fiber in $\mathcal{D}(R)$ with p-complete Tor-amplitude in degrees $\geq i+2$, and if the Nygaard filtration on the (twisted) prismatic cohomology $\mathbb{\Delta}_R\{i\}$ is complete. This definition globalizes to schemes in a natural way.

The terminology 'F-smooth' is meant to evoke both the Frobenius (used in the definition) as well as the hypothetical 'field with one element': For p-complete rings, we view F-smoothness roughly as an absolute version of the smoothness condition in algebraic geometry. Correspondingly, the class of F-smooth rings contains smooth algebras over perfectoid rings (Proposition 4.12) and for p-complete noetherian rings F-smoothness is equivalent to regularity (Theorem 4.15). The verification that regular rings are F-smooth (and in particular the Nygaard-completeness of the prismatic cohomology) has a further application: Under excellence assumptions, we verify a cohomological bound on the Hodge–Tate stack of a regular local ring suggested in [BL22b, Conj. 10.1]. In equal characteristic p, F-smoothness is equivalent to the condition of Cartier smoothness identified in [KM21, KST21]. Over a perfectoid base, this condition has been studied independently in the work of V. Bouis [Bou22]; most of the following identification (Theorem 1.8) of the $\mathbb{Z}/p^n(i)_X$ in this case has also been proved by Bouis, cf. [Bou22, Th. 4.14].

Let us now formulate the main comparison. By adjunction and Example 1.2, for any scheme X, we have a natural map $\mathbb{Z}/p^n(i)_X \to Rj_*(\mu_{p^n}^{\otimes i})$, for $j : X[1/p] \subset X$ the open inclusion. For $i \geq 0$, results of [AMMN22] give that $\mathbb{Z}/p^n(i)_X \in \mathcal{D}^{[0,i]}(X_{\text{et}}, \mathbb{Z}/p^n)$, whence we obtain a canonical comparison $\mathbb{Z}/p^n(i)_X \to \tau^{\leq i} Rj_*(\mu_{p^n}^{\otimes i})$. In general, the Kummer map (obtained from Example 1.5 and the cup product) induces a map $(\mathcal{O}_X^\times)^{\otimes i} \to \mathcal{H}^i(\mathbb{Z}/p^n(i)_X)$ which one can show to be surjective; see also [LM21] for more on the target. This determines the image of $\mathcal{H}^i(\mathbb{Z}/p^n(i)_X) \to R^i j_*(\mu_{p^n}^{\otimes i})$ as the subsheaf generated by $\mathcal{O}_X^\times$-symbols.[2]

Theorem 1.8. *Let X be a p-torsion-free F-smooth scheme (e.g., a regular scheme flat over $\mathbb{Z}$). For $i \geq 0$, the comparison map $\mathbb{Z}/p^n(i)_X \to \tau^{\leq i} Rj_*(\mu_{p^n}^{\otimes i})$ is an isomorphism on cohomology in degrees $< i$. On $\mathcal{H}^i$, the comparison map is injective with an image generated by the symbols, using the map of étale sheaves $(\mathcal{O}_X^\times)^{\otimes i} \to \mathcal{H}^i(Rj_*\mu_{p^n}^{\otimes i})$.*

[2]Note that, by [BK86, Hyo88, SS20], the sheaf $R^i j_*(\mu_{p^n}^{\otimes i})$ is generated by symbols from $\mathcal{O}^\times_{X[1/p]}$ in a wide variety of settings.

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In particular, $\mathbb{Z}/p^n(i)_X$ is obtained by modifying the truncated p-adic nearby cycles $\tau^{\leq i}Rj_*(\mu_{p^n}^{\otimes i})$ in the top cohomological degree by taking the image of $(\mathcal{O}_X^\times)^{\otimes i}$: One has a fiber square

$$\begin{array}{ccc} \mathbb{Z}/p^n(i)_X & \longrightarrow & \tau^{\leq i}Rj_*(\mu_{p^n}^{\otimes i}) \\ \downarrow & & \downarrow \\ \text{image}\left((\mathcal{O}_X^\times)^{\otimes i} \to R^i j_*(\mu_{p^n}^{\otimes i})\right)[-i] & \longrightarrow & R^i j_*(\mu_{p^n}^{\otimes i})[-i] \end{array}$$

in $\mathcal{D}(X_{\text{et}})$. On schemes which are smooth or regular with semistable reduction over a discrete valuation ring (DVR), Theorem 1.8 identifies $\mathbb{Z}/p^n(i)_X$ with the 'p-adic étale Tate twists' considered in [Sat07], and earlier in the smooth case in [Gei04, Sch94]; cf. [Sat05] for a survey.

Many special cases of Theorem 1.8 have previously appeared in the literature. As above, the $\mathbb{Z}/p^n(i)_X$ always restrict to the usual Tate twists on $X[1/p]$, so the main task is to identify $i^*\mathbb{Z}/p^n(i)_X$ for $i : X/p \subset X$, or equivalently the complexes defined in [BMS19]. In low weights or up to isogeny (i.e., using the approach of [FM87, Kat87]), comparisons between syntomic cohomology and p-adic vanishing cycles have been proved in a variety of settings, including smooth and semistable schemes over a DVR or its absolute integral closure, in [Kur87, Kat87, Tsu99, CN17]. Theorem 1.8 integrally in all weights for smooth $\mathcal{O}_C$-algebras, for C an algebraically closed complete non-Archimedean field of mixed characteristic $(0, p)$, is proved in [BMS19, Sec. 10] (see also [CDN21] for a semistable analog).

Theorem 1.8 is also closely related (via [BMS19]) to the calculations of topological cyclic homology for smooth algebras over the ring of integers in a p-adic field, cf. [HM03, HM04, GH06], and the recent revisiting in [LW22]. We do not calculate the topological cyclic homology but rather its associated graded terms, and the methods are at least superficially different; it would be interesting to make a direct connection.[3]

Our proof of Theorem 1.8 is based on some calculations in prismatic cohomology. In particular, it is based on the étale comparison theorem (cf. [BS22, Th. 9.1], [BL22a, Th. 8.5.1] and Theorem 5.1 below), which states that for any scheme X, one can always recover the Tate twists $\mu_p^{\otimes i}$ on the generic fiber by inverting a suitable class $v_1 \in H^0(\mathbb{F}_p(p-1)(\mathbb{Z}))$ in the syntomic cohomology of X. One can identify the image of the class v_1 in the prismatic cohomology of $\mathbb{Z}_p$, after which the result follows from a linear algebraic argument.

Conventions

Throughout, we use the theory of (absolute) prismatic cohomology as developed in [BL22a, BMS19, Dri20, BS22].

We will simply write $\hat{R}$ for the p-adic completion if there is no potential for confusion. If R is p-complete, we write $R\langle t\rangle$ for the p-completed polynomial ring and $R\left\langle t^{1/p^\infty}\right\rangle$ for the p-completion of $R[t^{1/p^\infty}]$.

For an animated ring R, we let $\mathcal{D}(R)$ denote the ∞-category of R-modules (i.e., if R is an ordinary ring, $\mathcal{D}(R)$ is the derived ∞-category of R).

Given an object $M \in \mathcal{D}(R)$ and an element $x \in R$, we will write M/x or $\frac{M}{x}$ for the mapping cone of $x : M \to M$. In particular, even when M is a discrete R-module, the object M/x need not live in degree 0.

2. Some calculations in prismatic cohomology

In this section, we recall some basic calculations in absolute prismatic cohomology. Our goal is to name some elements $v_1, \widetilde{\theta}, \theta$ in the prismatic cohomology of $\mathbb{Z}_p$, which will play a basic role in the sequel.

[3] The Segal conjecture at the level of topological Hochschild homology, which is closely related to the condition of F-smoothness, is often used in these calculations.

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2.1. Prismatic sheaves

Let us first recall the construction of the prismatic sheaves, after [BL22a, BS22]; their Nygaard completion was first constructed in [BMS19].

Following [BMS19, Sec. 4], we use the *quasisyntomic site* $\mathrm{qSyn}_{\mathbb{Z}_p}$. An object of $\mathrm{qSyn}_{\mathbb{Z}_p}$ is a p-complete, p-torsion-free ring A such that $L_{A/\mathbb{Z}_p} \otimes^{\mathbb{L}}_A (A/p) \in \mathcal{D}(A/p)$ has Tor-amplitude in $[-1, 0]$. There is a basis $\mathrm{qrsPerfd}_{\mathbb{Z}_p} \subset \mathrm{qSyn}_{\mathbb{Z}_p}$ of p-torsion-free *quasiregular semiperfectoid rings*, that is, those objects in $\mathrm{qSyn}_{\mathbb{Z}_p}$ which admit a surjection from a perfectoid ring.

Construction 2.1 (Prismatic sheaves). Let $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$ be a p-torsion-free quasiregular semiperfectoid ring. Then we have naturally associated to R the following:

1. A prism $(\mathbb{\Delta}_R, \phi, I)$ together with a map $R \to \mathbb{\Delta}_R/I$ (which is in fact the initial prism with this structure). We write $\overline{\mathbb{\Delta}}_R = \mathbb{\Delta}_R/I$ and call it the Hodge–Tate cohomology.
2. An invertible $\mathbb{\Delta}_R$-module $\mathbb{\Delta}_R\{1\}$ with a natural ϕ-linear map $\phi_1 : \mathbb{\Delta}_R\{1\} \to I^{-1}\mathbb{\Delta}_R\{1\}$ whose ϕ-linearization is an isomorphism; the reduction $\overline{\mathbb{\Delta}}_R\{1\}$ is identified with I/I^2. We let $\mathbb{\Delta}_R\{n\} = \mathbb{\Delta}_R\{1\}^{\otimes n}$ and obtain $\phi_n : \mathbb{\Delta}_R\{n\} \to I^{-n}\mathbb{\Delta}_R\{n\}$.
3. A descending, multiplicative Nygaard filtration $\{\mathcal{N}^{\geq i}\mathbb{\Delta}_R\}$ on the ring $\mathbb{\Delta}_R$ given by $\mathcal{N}^{\geq i}\mathbb{\Delta}_R = \phi^{-1}(I^i\mathbb{\Delta}_R)$; we write $\mathcal{N}^i\mathbb{\Delta}_R = \mathrm{gr}^i(\mathcal{N}^{\geq *}\mathbb{\Delta}_R)$.
4. A map of graded rings $\bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_R \to \bigoplus_{i\in\mathbb{Z}}(I/I^2)^{\otimes i} = \bigoplus_{i\in\mathbb{Z}}\overline{\mathbb{\Delta}}_R\{i\}$, obtained by passing to associated graded terms of the map of filtered rings $\phi : \{\mathcal{N}^{\geq *}\mathbb{\Delta}_R\} \to \{I^*\mathbb{\Delta}_R\}$.
5. The prismatic logarithm $\log_{\mathbb{\Delta}} : T_p(R^\times) \to \mathbb{\Delta}_R\{1\}$, whose image consists precisely of those elements $y \in \mathbb{\Delta}_R\{1\}$ such that $\phi_1(y) = y$.

All of the above define sheaves of p-torsion-free, p-complete abelian groups with trivial higher cohomology on $\mathrm{qrsPerfd}_{\mathbb{Z}_p}$; by descent, one obtains $\widehat{\mathcal{D}}(\mathbb{Z}_p)$-valued sheaves on $\mathrm{qSyn}_{\mathbb{Z}_p}$ with the same notation. Moreover, we will also need to consider the prismatic complexes for arbitrary animated rings; these can be defined starting from the above using animation (compare [BL22a, Sec. 4.5]).

Construction 2.2 (Syntomic sheaves). One has also, for each $i \geq 0$, the $\mathcal{D}(\mathbb{Z}_p)^{\geq 0}$-valued sheaf of abelian groups $\mathbb{Z}_p(i)(-)$ on $\mathrm{qrsPerfd}_{\mathbb{Z}_p}$ which carries R to the fiber of $\mathrm{can} - \phi_i : \mathcal{N}^{\geq i}\mathbb{\Delta}_R\{i\} \to \mathbb{\Delta}_R\{i\}$ for can the inclusion map, as originally introduced in [BMS19]. By [BS22, Th. 14.1], there is a basis for $\mathrm{qrsPerfd}_{\mathbb{Z}_p}$ on which the $\mathbb{Z}_p(i)(-)$ are discrete.

By animation, one extends the $\mathbb{Z}_p(i)(-)$ to all p-complete animated rings. In [BL22a, Sec. 8], the syntomic sheaves $\mathbb{Z}_p(i)(-)$ are extended to all animated rings, and by Zariski descent to all schemes, by gluing the above construction on the p-completion and the usual Tate twists on the generic fiber. On p-quasisyntomic rings, the $\mathbb{Z}_p(i)(-)$ are concentrated in nonnegative degrees.

Example 2.3 (The case of $\mathbb{Z}_p^{\mathrm{cycl}}$). In the particular case where $R = \mathbb{Z}_p^{\mathrm{cycl}} \overset{\mathrm{def}}{=} \widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$, then we have an identification $\mathbb{\Delta}_R = \widehat{\mathbb{Z}_p[q^{1/p^\infty}]}_{(p,q-1)}$, $I = [p]_q := \frac{q^p-1}{q-1}$. In this case, the choice of p-power roots $(1, \zeta_p, \zeta_{p^2}, \dots)$ determines an element $\epsilon \in T_p(R^\times)$ such that $\log_{\mathbb{\Delta}}(\epsilon) \in \mathbb{\Delta}_R\{1\}$ is divisible by $(q-1)$ and such that $\frac{\log_{\mathbb{\Delta}}(\epsilon)}{q-1}$ is a generator for the module $\mathbb{\Delta}_R\{1\}$, cf. [BL22a, Sec. 2.6].

Construction 2.4 (The Hodge–Tate cohomology of $\mathbb{Z}_p$). Let us recall the calculation of the Hodge–Tate cohomology of $\mathbb{Z}_p$. In fact, we have an isomorphism of bigraded $\mathbb{F}_p$-algebras,

$$H^*\left(\frac{\overline{\mathbb{\Delta}}_{\mathbb{Z}_p}}{p}\{*\}\right) \simeq E(\alpha) \otimes P(\theta^{\pm 1}),$$

where $|\alpha| = (1, p)$ and $\theta = (0, p)$ (we write the cohomological grading first and the internal grading next). In fact, this follows from the treatment in [BL22a, Sec. 3]. The Hodge–Tate cohomology of $\mathbb{Z}_p$ is given by the coherent cohomology of the sheaves $\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{i\}$ on the stack $\mathrm{WCart}^{\mathrm{HT}} \simeq B\mathbb{G}_m^\sharp$. As in *loc. cit.*, p-torsion sheaves on $B\mathbb{G}_m^\sharp$ are simply $\mathbb{F}_p$-vector spaces V equipped with an endomorphism

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$\Theta : V \to V$ such that the generalized eigenvalues of Θ live in $\mathbb{F}_p \subset \overline{\mathbb{F}_p}$, and $\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{i\}$ corresponds to the endomorphism $i : \mathbb{F}_p \to \mathbb{F}_p$. With this identification in mind, the calculation follows.

Using [BL22a, Prop. 5.7.9], we also find

$$H^*\left(\bigoplus_{i\geq 0} \frac{\mathcal{N}^i \mathbb{\Delta}_{\mathbb{Z}_p}}{p}\right) \simeq E(\alpha) \otimes P(\theta)$$

such that the natural map $\bigoplus_{i\geq 0} \frac{\mathcal{N}^i \mathbb{\Delta}_{\mathbb{Z}_p}}{p} \to \bigoplus_{i\in\mathbb{Z}} \frac{\overline{\mathbb{\Delta}}_{\mathbb{Z}_p}}{p}\{i\}$ on cohomology carries $\alpha \mapsto \alpha, \theta \mapsto \theta$.[4]

Example 2.5. Let R be a p-torsionfree perfectoid ring. We have $R \xrightarrow{\sim} \overline{\mathbb{\Delta}}_R$, so one forms the Breuil–Kisin twists $R\{i\}$. The map $\bigoplus_{i\geq 0} \mathcal{N}^i \mathbb{\Delta}_R \to \bigoplus_{i\in\mathbb{Z}} \overline{\mathbb{\Delta}}_R\{i\}$ is identified with the inclusion map $\bigoplus_{i\geq 0} R\{i\} \to \bigoplus_{i\in\mathbb{Z}} R\{i\}$. Under these identifications, θ maps to a generator of $\mathcal{N}^p \frac{\mathbb{\Delta}_R}{p}$; in fact, this is evident because θ is a unit in the Hodge–Tate cohomology.

Proposition 2.6. *Let A be any animated ring. Then the map of graded E_∞-rings over $\mathbb{F}_p$,*

$$\bigoplus_{i\geq 0} \mathcal{N}^i \frac{\mathbb{\Delta}_A}{p} \to \bigoplus_{i\in\mathbb{Z}} \frac{\overline{\mathbb{\Delta}}_A\{i\}}{p}$$

exhibits the target as the localization of the source at the element θ.

Proof. By quasisyntomic descent and left Kan extension, it suffices to treat the case where A is a smooth algebra over a p-torsion-free perfectoid ring so that one is in the setting of relative prismatic cohomology [BS22]. In this case, one can trivialize the Breuil–Kisin twists, and one knows that the map $\phi_i : \mathcal{N}^i \mathbb{\Delta}_A \to \overline{\mathbb{\Delta}}_A\{i\}$ is the ith stage of the conjugate filtration on the Hodge–Tate cohomology $\overline{\mathbb{\Delta}}_A \simeq \overline{\mathbb{\Delta}}_A\{i\}$, cf. [BS22, Th. 12.2]. Since the conjugate filtration is exhaustive and since θ maps to a unit in the target, the result easily follows from the Hodge–Tate comparison [BS22, Th. 4.11]. □

2.2. *The elements v_1, $\widetilde{\theta}$*

In this subsection, we construct two further elements in the prismatic cohomology of $\mathbb{Z}$.

Construction 2.7 (The class v_1). We define a class $v_1 \in H^0(\mathbb{F}_p(p-1)(\mathbb{Z}))$ as follows.

Let R be the ring $\mathbb{Z}[\zeta_{p^\infty}]$. Then by flat descent [BL22a, Prop. 8.4.6], $H^0(\mathbb{F}_p(p-1)(\mathbb{Z}))$ is the equalizer of the two maps

$$H^0(\mathbb{F}_p(p-1)(R)) \rightrightarrows H^0(\mathbb{F}_p(p-1)(R \otimes R)). \tag{2}$$

The element $(1, \zeta_p, \zeta_{p^2}, \dots) \in T_p(R^\times)$ determines a class $\epsilon \in H^0(\mathbb{Z}_p(1)(R))$ via the identification of [BL22a, Prop. 8.4.14]. We claim that the image of $\epsilon^{p-1} \in H^0(\mathbb{F}_p(p-1)(R))$ belongs to the equalizer of the two maps (2).

To see this, it suffices to map $R \otimes R$ to both its p-adic completion and to $R \otimes R[1/p]$. The images of ϵ^{p-1} in the latter are identical, as one sees using the trivialization of the sheaf $\mu_p^{\otimes p-1}$ on $\mathbb{Z}[1/p]$-algebras. Thus, it suffices to calculate in $\mathbb{F}_p(p-1)(\widehat{R \otimes R})$. Equivalently, we may do this calculation in $\mathbb{\Delta}_{R\otimes R}/p\{p-1\}$. By construction, the two images of ϵ yields classes $\epsilon_1, \epsilon_2 \in T_p\left((\widehat{R \otimes R})^\times\right)$. The images under the prismatic logarithm mod p yield elements

$$\log_{\mathbb{\Delta}}(\epsilon_1), \log_{\mathbb{\Delta}}(\epsilon_2) \in \mathbb{\Delta}_{R\otimes R}\{1\}/p.$$

[4] Under the motivic filtrations of [BMS19], this calculation is also closely related to Bökstedt's calculation of $\mathrm{THH}_*(\mathbb{Z})$.

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As in Example 2.3, $\mathbb{\Delta}_R$ is canonically identified with $\widehat{\mathbb{Z}_p[q^{1/p^\infty}]}_{(p,q-1)}$. Let $q_1, q_2 \in \mathbb{\Delta}_{R\otimes R}$ denote the images of q under the two maps $\mathbb{\Delta}_R \rightrightarrows \mathbb{\Delta}_{R\otimes R}$.

Since the maps are (p, I)-completely flat, the elements $(q_1 - 1), (q_2 - 1) \in \mathbb{\Delta}_{R\otimes R}/p$ are nonzero divisors, by the conjugate filtration and the Hodge–Tate comparison [BS22, Th. 4.11]. To see that $\log_{\mathbb{\Delta}}(\epsilon_1)^{p-1} = \log_{\mathbb{\Delta}}(\epsilon_2)^{p-1} \in \mathbb{\Delta}_{R\otimes R}\{p-1\}/p$, we may thus invert $(q_1 - 1)(q_2 - 1)$, after which both $\log_{\mathbb{\Delta}}(\epsilon_1)$ and $\log_{\mathbb{\Delta}}(\epsilon_2)$ become generators of the invertible $\mathbb{\Delta}_{R\otimes R}/p[\frac{1}{(q_1-1)(q_2-1)}]$-module $\mathbb{\Delta}_{R\otimes R}\{1\}/p[\frac{1}{(q_1-1)(q_2-1)}]$. But then there exists a unit $x \in \mathbb{\Delta}_{R\otimes R}/p[\frac{1}{(q_1-1)(q_2-1)}]$ with $x \log_{\mathbb{\Delta}}(\epsilon_1) = \log_{\mathbb{\Delta}}(\epsilon_2)$. Since $\log_{\mathbb{\Delta}}(\epsilon_i), i = 1, 2$ are fixed points of the divided Frobenius ϕ_1, we find that $\phi(x) = x$, or $x^p = x$. Since x is a unit, this gives $x^{p-1} = 1$, so $\log_{\mathbb{\Delta}}(\epsilon_1)^{p-1} = \log_{\mathbb{\Delta}}(\epsilon_2)^{p-1}$ in $\mathbb{\Delta}_{R\otimes R}/p\{p-1\}[\frac{1}{(q_1-1)(q_2-1)}]$, as desired.

The class $v_1 \in H^0(\mathbb{F}_p(p-1)(\mathbb{Z}_p))$ also appears (in a different language) in [Dri20, Prop. 8.11.2].

Although it will not play a role in the sequel, let us remark on the connection to the element v_1 in stable homotopy theory. Suppose $p > 2$ for simplicity. The topological class $v_1^{\mathrm{top}} \in \pi_{2p-2}(\mathbb{S}/p)$ in the stable stems gives a nonzero class in $\pi_{2p-2}\mathrm{TC}(\mathbb{Z}_p; \mathbb{F}_p)$; under the motivic spectral sequence of [BMS19], this is detected (up to nonzero scalar) by the class denoted v_1 above. In fact, we can check this after passage from $\mathbb{Z}_p$ to $\mathcal{O}_{\mathbb{C}_p}$; then, the description $ku/p = \mathrm{TC}(\mathcal{O}_{\mathbb{C}_p}; \mathbb{F}_p)$ (cf. [HN20] for an account) easily implies the claim.

Construction 2.8 (The element $\widetilde{\theta}$). The element $v_1 \in H^0(\mathbb{F}_p(p-1)(\mathbb{Z}))$ maps to $H^0\left(\mathcal{N}^{\geq p-1}\frac{\mathbb{\Delta}_{\mathbb{Z}_p}\{p-1\}}{p}\right)$. In fact, since $\mathcal{N}^{p-1}\mathbb{\Delta}_{\mathbb{Z}_p} = 0$ (Construction 2.4), we obtain a unique lift to an element $\widetilde{\theta} \in H^0\left(\mathcal{N}^{\geq p}\frac{\mathbb{\Delta}_{\mathbb{Z}_p}\{p-1\}}{p}\right)$.

Proposition 2.9. *The image of $\widetilde{\theta}$ in $H^0(\mathcal{N}^p\mathbb{\Delta}_{\mathbb{Z}_p}/p)$ is a generator (which, up to normalization, we can take to be θ).*

Proof. It suffices to show that the image of $\widetilde{\theta}$ is nonzero in $H^0(\mathcal{N}^p\mathbb{\Delta}_{\mathbb{Z}_p}/p)$. We may do this calculation in $\mathbb{Z}_p^{\mathrm{cycl}}$. Let $\epsilon \in T_p((\mathbb{Z}_p^{\mathrm{cycl}})^\times)$ be the canonical element $(1, \zeta_p, \zeta_{p^2}, \dots)$. We have $v_1 = \log_{\mathbb{\Delta}}(\epsilon)^{p-1}$, which is $(q-1)^{p-1} \equiv (q^{1/p} - 1)^{p(p-1)} \pmod p$ times a generator of $\mathbb{\Delta}_{\mathbb{Z}_p^{\mathrm{cycl}}}\{p-1\}/p$. Noting that the Nygaard filtration is the filtration by powers of $[p]_{q^{1/p}} \equiv (q^{1/p} - 1)^{p-1} \pmod p$, we find that v_1 maps to a nonzero element of $\mathcal{N}^p\frac{\mathbb{\Delta}_{\mathbb{Z}_p^{\mathrm{cycl}}}\{p-1\}}{p}$, as desired. □

Remark 2.10 (A direct prismatic construction). Let us now describe another construction of the image of v_1 in $H^0(\mathbb{F}_p(p-1)(\mathbb{Z}_p))$ that does not rely on the explicit use of the ring $\mathbb{Z}[\zeta_{p^\infty}]$ or the prismatic logarithm. Given any p-torsion-free prism (A, I, ϕ) such that A/I is also p-torsion-free, we have as in [BL22a, Sec. 2.2] a natural invertible module $A\{1\}$ together with a ϕ-linear map $\phi_1 : A\{1\} \to I^{-1}A\{1\}$ which becomes an isomorphism upon ϕ-linearization. We also have the tensor powers $A\{i\}$ and the maps $\phi_i : A\{i\} \to I^{-i}A\{i\}$. Specifying an element of $H^0(\mathbb{F}_p(p-1)(\mathbb{Z}_p))$ is equivalent to specifying, for each such prism (A, I), an element of $A/p\{p-1\}$ which is fixed under ϕ_{p-1}.

Let us construct an element in $IA/p\{p-1\}$ which is a fixed point for $\phi_{p-1} : A/p\{p-1\} \to I^{-(p-1)}A/p$, as follows. Choose a generator $y \in A/p\{1\}$. By the above, $\phi_1(y)/y$ is a generator for the invertible A/p-module I^{-1}/p, so '$y/\phi_1(y)$' is a generator for the ideal $I/p \subset A/p$. Now, consider the element $\frac{y}{\phi_1(y)}y^{p-1} \in IA/p\{p-1\}$. Unwinding the definitions, it follows that ϕ_{p-1} carries this element to $\frac{y^{\otimes p}}{\phi_1(y)^{\otimes p}}\phi_{p-1}(y^{\otimes p-1}) = \frac{y}{\phi_1(y)} \otimes y^{\otimes p-1}$, that is, we have a fixed point for ϕ_{p-1}. It is easy to check that this does not depend on the choice of generator y and that it produces a fixed point for ϕ_{p-1} (modulo p) as desired. One can check that this construction reproduces the image of v_1 in $H^0(\mathbb{F}_p(p-1)(\mathbb{Z}_p))$ at least up to scalars by calculating explicitly for the prism corresponding to the perfectoid ring $\mathbb{Z}_p^{\mathrm{cycl}}$.

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3. The Nygaard filtration on Hodge–Tate cohomology

In this section, we define the Nygaard filtration on Hodge–Tate cohomology and study some of its basic properties.

3.1. Definitions

Construction 3.1. Let $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$. Consider the prism $(\mathbb{\Delta}_R, I)$ and the Nygaard filtration $\mathcal{N}^{\geq *}\mathbb{\Delta}_R$. The image of the Nygaard filtration yields a filtered ring $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_R$. The ideal $I \subset \mathbb{\Delta}_R$ maps via the canonical augmentation $\mathbb{\Delta}_R \to R$ to the ideal p (e.g., by calculating explicitly for $R = \mathbb{Z}_p^{\mathrm{cycl}}$). Therefore, we have a canonical isomorphism of graded rings

$$\mathrm{gr}^* \overline{\mathbb{\Delta}}_R \simeq \bigoplus_{i \geq 0} \mathcal{N}^i \frac{\mathbb{\Delta}_R}{p}. \tag{3}$$

Note here the composite of $R \to \overline{\mathbb{\Delta}}_R \to \mathrm{gr}^0 \overline{\mathbb{\Delta}}_R \simeq R/p$ is the Frobenius. In particular, if we consider the filtration (3) as one of R-modules, then $\mathrm{gr}^i \overline{\mathbb{\Delta}}_R \simeq \mathcal{N}^i \mathbb{\Delta}_R / p^{(-1)}$, with the superscript denoting restriction along Frobenius. We highlight the special case of an isomorphism of R-algebras,

$$\mathrm{gr}^0 \overline{\mathbb{\Delta}}_R \simeq R/p^{(-1)}, \tag{4}$$

for $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$ and then by descent and left Kan extension for all animated rings R. We can also do the same with the Breuil–Kisin twists $\overline{\mathbb{\Delta}}_R\{i\}$, which yield invertible $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_R$-modules $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_R\{i\}$, with associated gradeds the same as above.

By descent and Kan extension, we construct for any animated ring A the commutative algebra object $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_A$ of the filtered derived ∞-category.

In the remainder of the subsection, we detect the element p in the Nygaard filtration on Hodge–Tate cohomology and obtain a twisted form of the filtration for Hodge–Tate cohomology modulo p which will sometimes be easier to work with.

Example 3.2 (Detection of the element p). We show that the element $p \in H^0(\overline{\mathbb{\Delta}}_{\mathbb{Z}_p})$ is detected in filtration p of the Nygaard filtration on $\overline{\mathbb{\Delta}}_{\mathbb{Z}_p}$, by the class $\theta \in H^0(\mathcal{N}^p \frac{\mathbb{\Delta}_{\mathbb{Z}_p}}{p})$ (up to units).

To see this, we may replace $\mathbb{Z}_p$ by the perfectoid ring $R = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$, and it suffices to show that $p \in \mathcal{N}^{\geq p}\overline{\mathbb{\Delta}}_R \setminus \mathcal{N}^{\geq p+1}\overline{\mathbb{\Delta}}_R$. Since R is perfectoid, $\mathbb{\Delta}_R = W(R^\flat)$. Let $p^\flat \in R^\flat$ be given by the system of elements $(p, p^{1/p}, p^{1/p^2}, \dots)$ in R. The prismatic ideal $I \subset \mathbb{\Delta}_R = W(R^\flat)$ is $I = (p - [p^\flat])$, and the map $R \to \mathbb{\Delta}_R/I$ is an isomorphism whose inverse given by the Fontaine map $W(R^\flat) \to R$ (whose kernel is I). Now, $\mathcal{N}^{\geq i}\mathbb{\Delta}_R = \phi^{-1}(I)^i = (p - [p^{\flat,1/p}])^i$. The image of this ideal in $\overline{\mathbb{\Delta}}_R$ is $p^{i/p}$, since $[p^{\flat,1/p}]$ maps to $p^{1/p}$. The claim now follows.

Construction 3.3 (The twisted Nygaard filtration on $\frac{\overline{\mathbb{\Delta}}_R}{p}$). Let R be any animated ring. Then there is a natural decreasing, multiplicative $\mathbb{Z}_{\geq 0}^{op}$-indexed filtration $\widetilde{\mathcal{N}}^{\geq *}\frac{\overline{\mathbb{\Delta}}_R}{p}$ on $\frac{\overline{\mathbb{\Delta}}_R}{p}$ with associated graded given as

$$\mathrm{gr}^* \frac{\overline{\mathbb{\Delta}}_R}{p} \simeq \left(\bigoplus_{i \geq 0} \mathcal{N}^i \frac{\mathbb{\Delta}_R}{p} \right) / \theta, \tag{5}$$

where θ lives in grading p. Furthermore, for any $i \in \mathbb{Z}$, we can construct a similar filtration $\widetilde{\mathcal{N}}^{\geq *}\frac{\mathbb{\Delta}_R\{i\}}{p}$, which is a module over the filtration on $\frac{\mathbb{\Delta}_R}{p}$; the associated graded terms are given individually as

$$\mathrm{gr}^j \frac{\overline{\mathbb{\Delta}}_R\{i\}}{p} \simeq \mathrm{cofib}\left(\theta : \mathcal{N}^{j-p} \frac{\mathbb{\Delta}_R}{p} \to \mathcal{N}^j \frac{\mathbb{\Delta}_R}{p} \right), \tag{6}$$

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where $\mathcal{N}^j \frac{\overline{\mathbb{\Delta}}_R}{p} = 0$ for $j < 0$. In fact, by descent from $\mathrm{qrsPerfd}_{\mathbb{Z}_p}$ and left Kan extension, these claims follow from Construction 3.1 combined with the identification of Example 3.2.

Remark 3.4. The twisted Nygaard filtration $\widetilde{\mathcal{N}}^{\geq *} \frac{\overline{\mathbb{\Delta}}_R\{i\}}{p}$ is complete if and only if the Nygaard filtration $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_R\{i\}$ is complete, as follows by p-completeness.

3.2. Relative perfectness

In the sequel, we will study how the above filtration varies as R does. To begin, for future reference we include here a special case of this result based on the notion of *relative perfectness*.

Definition 3.5 (Relatively perfect maps). Let A be an animated ring, and let B be an animated A-algebra. We say that B is *relatively perfect* over A if the diagram

$$\begin{array}{ccc} A/p & \longrightarrow & B/p \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ A/p & \longrightarrow & B/p \end{array}$$

is a pushout square of animated rings. This implies that the cotangent complex $L_{B/A}$ vanishes p-adically, cf. [Bha12, Cor. 3.8], so $L_{A/\mathbb{Z}} \otimes_A B \to L_{B/\mathbb{Z}}$ is a p-adic equivalence.

Remark 3.6. Suppose A, B are discrete rings and $A \to B$ is p-completely flat. Then $A \to B$ is relatively perfect in the above sense if and only if the analogous diagram involving the *ordinary* quotients of A, B by (p) is co-Cartesian. In fact, we claim that if $R \to S$ is any flat map of animated $\mathbb{F}_p$-algebras, then $R \to S$ is relatively perfect in the animated sense if and only if $\pi_0(R) \to \pi_0(S)$ is relatively perfect in the classical sense. The 'only if' direction is clear as applying $\pi_0(-)$ preserves pushout squares. For the reverse implication, observe that $R \to S$ is relatively perfect in the animated sense exactly when the relative Frobenius $(S/R)^{(1)} := S \otimes_{R,\varphi} R \to S$ is an isomorphism of animated R-algebras. Now base change along $R \to \pi_0(R)$ is conservative on connective R-modules, so it suffices to check that $(S/R)^{(1)} \otimes_R \pi_0(R) \to S \otimes_R \pi_0(R)$ is an isomorphism in $\mathcal{D}(\pi_0(R))$. Noting that the formation of the relative Frobenius commutes with arbitrary base change along maps of animated rings, it remains to observe that $\pi_0(R) \to \pi_0(S)$ identifies with the base change $\pi_0(R) \to S \otimes^L_R \pi_0(R)$ of $R \to S$ by the flatness assumption and that the Frobenius twist of a flat $\pi_0(R)$-algebra is automatically discrete.

Proposition 3.7. *Let $A \to B$ be a relatively perfect map of animated rings. Then the natural map induces an equivalence (after p-completion) of filtered objects $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_A\{i\} \otimes_A B \simeq \mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_B\{i\}$, and similarly for the twisted Nygaard filtrations on $\frac{\overline{\mathbb{\Delta}}_{(-)}\{i\}}{p}$. Moreover, for each i, we have a p-adic equivalence $\mathcal{N}^i\mathbb{\Delta}_A \otimes_A B \simeq \mathcal{N}^i\mathbb{\Delta}_B$.*

Proof. We have that $\overline{\mathbb{\Delta}}_A\{i\} \otimes_A B \to \overline{\mathbb{\Delta}}_B\{i\}$ is an equivalence by the p-complete vanishing of the cotangent complex, for example, by comparing the absolute conjugate filtrations, [BL22a, Sec. 4.5]. This also yields the claim about the Nygaard pieces $\mathcal{N}^i\mathbb{\Delta}$, using the Nygaard fiber sequence [BL22a, Rem. 5.5.8]. Finally, the claim about $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}$ now follows from the claims about $\overline{\mathbb{\Delta}}$ and $\mathcal{N}^i\mathbb{\Delta}$; note that we need relative perfectness and not only p-adic vanishing of the relative cotangent complex because of the restrictions along Frobenius involved in equation (3). □

3.3. Polynomial rings

The purpose of this subsection is to identify explicitly the Hodge–Tate cohomology of a polynomial ring, together with its Nygaard filtration (Proposition 3.12). We also treat the easier case of the Nygaard graded pieces of prismatic cohomology (Proposition 3.11).

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In the sequel, we use the following. Let $\{A^{\geq *}\}$ be a filtered ring. Then the ∞-category $\mathcal{D}(A^{\geq *})$ of $A^{\geq *}$-modules in the filtered derived ∞-category admits a t-structure, where (co)connectivity is checked levelwise, and such that the heart consists of modules over $A^{\geq *}$ in the category $\mathrm{Fun}(\mathbb{Z}^{op}, \mathrm{Ab})$; we will sometimes simply refer to these as $A^{\geq *}$-modules.

In addition, for future reference, it will be helpful to keep track of the naturally arising internal gradings, which we first review.

Remark 3.8 (Automatic internal gradings). Let $\mathcal{F}$ be a functor from $\mathrm{qrsPerfd}_{\mathbb{Z}_p}$ to p-complete abelian groups with (for simplicity) bounded p-power torsion. Suppose that, for any $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$, we are given an R-module structure on $\mathcal{F}(R)$ which is natural in R in the evident sense. Suppose further that for any such R, the natural map $\mathcal{F}(R) \otimes_R R[t^{1/p^\infty}] \to \mathcal{F}(R\langle t^{1/p^\infty}\rangle)$ is a p-adic equivalence.

Then for any $R' \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$ with a $\mathbb{Z}[1/p]_{\geq 0}$-grading (in the p-complete sense), the R'-module $\mathcal{F}(R')$ also inherits a canonical $\mathbb{Z}[1/p]_{\geq 0}$-grading for essentially diagrammatic reasons. We have a map $\mathrm{coact} : R' \to R'\langle t^{1/p^\infty}\rangle$ carrying a homogeneous element $x \in R'_i$ to $x \otimes t^i$. An element $y \in \mathcal{F}(R')$ is homogeneous of degree $i \in \mathbb{Z}[1/p]_{\geq 0}$ if and only if it maps under coact to $y \otimes t^i \in \mathcal{F}(R'\langle t^{1/p^\infty}\rangle) \simeq \mathcal{F}(R')\langle t^{1/p^\infty}\rangle$.

Construction 3.9 (Internal gradings on Hodge–Tate cohomology). Let R be a $\mathbb{Z}[1/p]_{\geq 0}$-graded animated ring. In this case, the (twisted) Hodge–Tate cohomology together with its Nygaard filtration $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_R\{i\}$ naturally inherits the structure of a $\mathbb{Z}[1/p]_{\geq 0}$-graded object of $\widehat{\mathcal{D}(\mathbb{Z}_p)}$. Explicitly, one uses quasisyntomic descent, animation, Remark 3.8 and that the natural map

$$\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_R\{i\} \otimes_{\mathbb{Z}} \mathbb{Z}[t^{1/p^\infty}] \to \mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_{R\otimes_{\mathbb{Z}}\mathbb{Z}[t^{1/p^\infty}]}\{i\}$$

is an isomorphism p-adically by relative perfectness (Proposition 3.7).[5] Similarly, in the above setting, Remark 3.8 yields an additional grading on $\mathcal{N}^i\mathbb{\Delta}_R, i \geq 0$. Since there will be multiple gradings at the same time, we will refer to these internal gradings as weight gradings.

Remark 3.10. Let R be a $\mathbb{Z}[1/p]_{\geq 0}$-graded ring. If R is concentrated in degrees $\mathbb{Z}_{\geq 0}$, then $\mathcal{N}^i\mathbb{\Delta}_R$ and $\overline{\mathbb{\Delta}}_R\{i\}$ are concentrated in degrees $\mathbb{Z}_{\geq 0}$, as one sees using the conjugate filtration over a perfectoid base. However, the associated graded terms of the Nygaard filtration are in degrees $\frac{1}{p}\mathbb{Z}_{\geq 0}$: This follows from equation (3) noting that there is a restriction along Frobenius involved, which divides degrees by p.

Proposition 3.11. *Let R be a p-torsion-free quasiregular semiperfectoid ring. Then there are natural isomorphisms of graded $A^* = \bigoplus_{i\geq 0}\widehat{\mathcal{N}^i\mathbb{\Delta}_R \otimes_R R[x]}$-modules*

$$H^j\Big(\bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_{R[x]}\Big) \simeq \begin{cases} A^*, & j = 0 \\ A^{*-1}, & j = 1. \end{cases} \tag{7}$$

With respect to the internal weight grading with $|x| = 1$ and R in weight zero, then the generator in H^0 has weight zero and the generator in H^1 has weight 1.

Proof. The generator in H^0 is simply the unit. The generator in $H^1(\mathcal{N}^1\mathbb{\Delta}_{R[x]})$ comes from the class dx, via the isomorphism $\mathcal{N}^1\mathbb{\Delta}_S \simeq \widehat{L_{S/\mathbb{Z}}}[-1]$ for any animated ring S, cf. [BL22a, Prop. 5.5.12]. Having named the classes, it suffices by base-change (since for any perfectoid ring R_0, the functor $R \mapsto \bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_R$ is a symmetric monoidal functor from animated R_0-algebras to p-complete graded objects) to verify the isomorphism when R is perfectoid, where the result follows from the isomorphisms with the conjugate filtration: For any R-algebra S (in particular, $R[x]$), $\mathcal{N}^i\mathbb{\Delta}_S \simeq \mathrm{Fil}^{\mathrm{conj}}_{\leq i}\overline{\mathbb{\Delta}}_S$ by [BS22, Th. 12.2], and using the Hodge–Tate comparison for the latter [BS22, Th. 4.11]. □

[5] In the language of [BL22b], the Hodge–Tate stack associated to the group scheme $\mathbb{G}_m^{\mathrm{perf}} = \varprojlim_p \mathbb{G}_m$ is $\mathbb{G}_m^{\mathrm{perf}} \times \mathrm{WCart}^{\mathrm{HT}}$ by relative perfectness, so if a scheme X is equipped with a $\mathbb{G}_m^{\mathrm{perf}}$-action, then so is its Hodge–Tate stack $\mathrm{WCart}^{\mathrm{HT}}_X$.

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Proposition 3.12. *Let $R \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$ be a p-torsion-free quasiregular semiperfectoid ring. Let $A^{\geq *}$ be the p-completion of the filtered ring $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_R \otimes_R R[x]$. Then there are isomorphisms of $A^{\geq *}$-modules*

$$H^0(\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_{R[x]}) \simeq A^{\geq *}, \quad H^1(\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_{R[x]}) \simeq A^{\geq *-1}\{-1\} \oplus \bigoplus_{i=1}^{p-1} A^{\geq *-1}/A^{\geq *}. \tag{8}$$

With respect to the internal weight grading with $|x| = 1$ and R in degree zero, the generator of H^0 is in weight zero, the generator of $A^{\geq *-1}\{-1\}$ is in weight one and the ith copy of $A^{\geq *-1}/A^{\geq *}$ has generator in weight $\frac{i}{p}$.

Proof. Let us first name the generators. The generator of H^0 is simply 1. The first generator in H^1 is the class $dx \in H^1(\overline{\mathbb{\Delta}}_{R[x]}\{1\})$ constructed via the boundary map $\overline{\mathbb{\Delta}}_{R[x]}\{1\} \to \mathbb{\Delta}_{R[x]}/I^2 \to \overline{\mathbb{\Delta}}_{R[x]}$ as the image of x (note that this boundary map is how one produces the Hodge–Tate comparison, [BS22, Cons. 4.9]); it lifts uniquely to $H^1(\mathcal{N}^{\geq 1}\overline{\mathbb{\Delta}}_{R[x]}\{1\})$ and thus produces a map of $A^{\geq *}$-modules $A^{\geq *-1}\{-1\} \to H^1(\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_{R[x]})$. Next, we have the fiber sequence of $R[x]$-modules

$$\mathcal{N}^{\geq 1}\overline{\mathbb{\Delta}}_{R[x]} \to \overline{\mathbb{\Delta}}_{R[x]} \to R/p^{(-1)}[x^{1/p}],$$

from the description (4) (and quasisyntomic descent) to identify $\mathrm{gr}^0\overline{\mathbb{\Delta}}_{R[x]} = R/p^{(-1)}[x^{1/p}]$. For each $0 < i < p$, the boundary map applied to $x^{i/p}$ gives a class in $H^1(\mathcal{N}^{\geq 1}\overline{\mathbb{\Delta}}_{R[x]})$ of weight i/p; by construction, this class is annihilated by $A^{\geq 1}$ since $R/p^{(-1)}[x^{1/p}]$ is by definition, whence we obtain maps in from $A^{\geq *-1}/A^{\geq *}$.

Since we have named the generating classes, to prove the isomorphism, we may assume (by base-change) that R is a p-torsion-free *perfectoid* ring.[6] Moreover, by descent in R, we may assume that R contains a pth root of p, for example, using André's lemma in the form of [BS22, Th. 7.14]. We make this assumption for the rest of the argument. This implies that the Nygaard filtration on $R \simeq \overline{\mathbb{\Delta}}_R$ is the filtration by powers of $p^{1/p}$, cf. Example 3.2 and equation (4).

In this case, we have isomorphisms (via the Hodge–Tate comparison [BS22, Th. 4.11])

$$H^i(\overline{\mathbb{\Delta}}_{R[x]}) \simeq \begin{cases} R\langle x\rangle, & i = 0 \\ R\{-1\}\langle x\rangle\, dx, & i = 1, \end{cases}$$

where the class dx arises from the image of the class x under the connecting map in the cofiber sequence $\overline{\mathbb{\Delta}}_{R[x]}\{1\} \to \mathbb{\Delta}_{R[x]}/I^2 \to \overline{\mathbb{\Delta}}_{R[x]}$.

Using the expression (3) for the Nygaard filtration (which is complete in this case since the algebra is smooth over a perfectoid, so we can check on associated graded terms), we find that multiplication by $p^{1/p}$ induces isomorphisms $p^{1/p} : \mathcal{N}^{\geq i}\overline{\mathbb{\Delta}}_{R[x]} \simeq \mathcal{N}^{\geq i+1}\overline{\mathbb{\Delta}}_{R[x]}$ for $i > 0$, also using the comparison between the associated graded pieces of the Nygaard filtration and the Hodge–Tate filtration [BS22, Th. 12.2]. As above, we can identify the map $R\langle x\rangle \to \overline{\mathbb{\Delta}}_{R[x]} \to \mathrm{gr}^0\overline{\mathbb{\Delta}}_{R[x]}$ with the R-linear map $R\langle x\rangle \to R/p^{1/p}\langle x^{1/p}\rangle = (R/p\langle x\rangle)^{(-1)}, x \mapsto x$ (unwinding the restriction along Frobenius as in equation (4)). This yields

$$H^*(\mathcal{N}^{\geq 1}\overline{\mathbb{\Delta}}_{R[x]}) \simeq \begin{cases} p^{1/p}R\langle x\rangle, & * = 0 \\ R\{-1\}\langle x\rangle\, dx \oplus \bigoplus_{i\geq 0, p\nmid i} R/p^{1/p}\cdot x^{i/p}, & * = 1. \end{cases} \tag{9}$$

[6] Let R_0 be a p-torsion-free perfectoid ring. Then the construction $R \mapsto \mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_R$, from animated R_0-algebras to p-complete filtered $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_{R_0}$-algebras, preserves colimits and in particular preserves coproducts. In fact, this holds for $R \mapsto \overline{\mathbb{\Delta}}_R$ itself by the Hodge–Tate comparison, and on associated graded terms by equation (3) and [BS22, Th. 12.2].

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It follows that, as filtered $A^{\geq *} \overset{\text{def}}{=} \widehat{\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_R \otimes_R R[x]}$-modules in $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathrm{Ab})$, the classes specified yield a natural isomorphism

$$H^1(\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_{R[x]}) \simeq A^{\geq *}\{-1\} \oplus \bigoplus_{i=1}^{p-1} A^{\geq *-1}/A^{\geq *}. \tag{10}$$

□

3.4. The Hodge–Tate cohomology of a quotient

In this subsection, we use the results of the previous subsection on polynomial rings to get an expression (via a fiber sequence) of the Hodge–Tate cohomology of a quotient (Corollary 3.16) and some control of the Nygaard filtration too (Corollary 3.15). To begin, we start with the (easier) case of the Nygaard pieces themselves.

Proposition 3.13. *Let R be any animated $\mathbb{Z}[x]$-algebra. Then there exists a natural fiber sequence of graded $\bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_R$-modules*

$$\left(\bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_R\right)/x \to \bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_{R/x} \to \bigoplus_{i\geq 0}\mathcal{N}^{i-1}\mathbb{\Delta}_{R/x}. \tag{11}$$

Proof. First, let $B \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$. We construct a cofiber sequence, naturally in B, of $\bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_{B[x]}/x$-modules

$$\bigoplus_{i\geq 0}\mathcal{N}^{i-1}\mathbb{\Delta}_B[-1] \to \bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_{B[x]}/x \to \bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_B. \tag{12}$$

To construct this, we use Proposition 3.11, which shows that the (bi)graded E_∞-ring $\bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_{B[x]}/x$ is concentrated in weights 0 and 1, using the weight grading on $B[x]$ with $|x| = 1$ and B in weight zero. Now, any weight-graded module over $\bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_{B[x]}/x$ admits a filtration by the weight grading, which gives the cofiber sequence (12), using again Proposition 3.11 to identify the weight zero and weight one components with $\bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_B$ and $\bigoplus_{i\geq 0}\mathcal{N}^{i-1}\mathbb{\Delta}_B$.

By base-change and descent, one now deduces the proposition. In fact, we may assume that R is an $B[x]$-algebra for some $B \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$, provided everything is done independently of the choice of B. Then the desired equation (11) follows from equation (12), using that

$$\bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_R \otimes_{\bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_{B[x]}} \bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_B \to \bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_{R/x}$$

is a p-adic equivalence. □

Proposition 3.14. *Let $B \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$. Then, naturally in B, there is a finite filtration on $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_{B[x]}/x$ in $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_{B[x]}/x$-modules whose associated graded terms are $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_B$, $(p-1)$ copies of $\left(\mathcal{N}^{\geq *-1}\overline{\mathbb{\Delta}}_B/\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_B\right)[-1]$, and $\mathcal{N}^{\geq *-1}\overline{\mathbb{\Delta}}_B\{-1\}[-1]$.*

Proof. In fact, this follows from the natural expression (3.12), noting the weight grading (with $|x| = 1$). In particular, $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_{B[x]}/x$ has weights in $0, \frac{1}{p}, \frac{2}{p}, \ldots, 1$ with the weight zero component being $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_B$, the weight $\frac{i}{p}$ component for $0 < i < p$ being $\left(\mathcal{N}^{\geq *-1}\overline{\mathbb{\Delta}}_B/\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_B\right)[-1]$ and the weight 1 component being $\mathcal{N}^{\geq *-1}\overline{\mathbb{\Delta}}_B\{-1\}[-1]$. □

Corollary 3.15. *Let A be any animated $\mathbb{Z}[x]$-algebra. Then the filtered object $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_A/x\{i\}$ admits a natural finite filtration, whose associated graded terms are $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_{A/x}\{i\}$, then $(p-1)$ copies of $\left(\mathcal{N}^{\geq *-1}\overline{\mathbb{\Delta}}_{A/x}/\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_{A/x}\right)\{i\}[-1]$, and $\mathcal{N}^{\geq *-1}\overline{\mathbb{\Delta}}_{A/x}\{i-1\}[-1]$.*

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Proof. It suffices to replace $\mathbb{Z}[x]$ by $B[x]$ for $B \in \mathrm{qrsPerfd}_{\mathbb{Z}_p}$ and construct the filtration naturally in B by quasisyntomic descent. But then the claim follows from Proposition 3.14. □

We separately record the resulting fiber sequence for Hodge–Tate cohomology itself (forgetting the Nygaard filtration in Corollary 3.15). Such a fiber sequence can also be produced using the description of the Hodge–Tate stack of the affine line, cf. [BL22b, Ex. 9.1].

Corollary 3.16. *Let R be any animated $\mathbb{Z}[x]$-algebra. Then there is a natural fiber sequence*

$$\overline{\mathbb{\Delta}}_R/x\,\{i\} \to \overline{\mathbb{\Delta}}_{R/x}\,\{i\} \to \overline{\mathbb{\Delta}}_{R/x}\,\{i-1\}\,. \tag{13}$$

4. *F*-smoothness

The goal of this section is to formulate the notion of F-smoothness (Definition 4.1). This is a variant of (p-adic) smoothness designed to capture smoothness in an absolute sense. For instance, smooth algebras over any perfectoid ring are F-smooth (Proposition 4.12), as are regular rings (Theorem 4.15); in fact, the latter is the main result of this section. Our idea is to essentially define F-smoothness by demanding a strong form of the $L\eta$-isomorphism in relative prismatic cohomology ([BS22, Theorem 15.3], [BMS18]), adapted to the absolute prismatic context using the Beilinson t-structure interpretation of the $L\eta$ functor as in [BMS19, §5.1] (see Remark 4.11). To work effectively with this notion, we need access to the certain naturally defined elements of the prismatic cohomology (or variants) of $\mathbb{Z}_p$ introduced in §2.

4.1. *F-smoothness: definition*

Let A be a p-quasisyntomic ring. Recall [BMS19, Def. 4.1] that an object $M \in \mathcal{D}(A)$ has *p-complete Tor-amplitude* in degrees $\geq r$ if for every discrete A/p-module N, we have $M \otimes^L_A N \in \mathcal{D}^{\geq r}(A)$.

Definition 4.1 (F-smoothness). We say that A *is F-smooth* if for each $i \in \mathbb{Z}_{\geq 0}$, the map in $\mathcal{D}(A)$,

$$\phi_i : \mathcal{N}^i \mathbb{\Delta}_A \to \overline{\mathbb{\Delta}}_A\,\{i\}$$

induced by the Frobenius on $\mathbb{\Delta}_A$ has fiber $\mathrm{fib}(\phi_i)$ with p-complete Tor-amplitude in degrees $\geq i+2$ and if the Nygaard filtration on $\overline{\mathbb{\Delta}}_A\,\{i\}$ (or equivalently $\mathbb{\Delta}_A\,\{i\}$) is complete. Note that this condition only depends on the p-completion of A.

We say that a p-quasisyntomic scheme is F-smooth if it is covered by the spectra of rings which are F-smooth (note that F-smoothness is preserved by Zariski localization by Proposition 4.6 below).

The condition of Nygaard-completeness in the definition of F-smoothness is slightly delicate. In order to work with it, we will also use the following auxiliary condition.

Definition 4.2 (Weak F-smoothness). We say that a p-quasisyntomic ring A is *weakly F-smooth* if for each i, the object

$$\mathrm{fib}\left(\theta : \mathcal{N}^i \frac{\mathbb{\Delta}_A}{p} \to \mathcal{N}^{i+p} \frac{\mathbb{\Delta}_A}{p}\right) \in \mathcal{D}(A), \tag{14}$$

has p-complete Tor-amplitude in degrees $\geq i+1$. If A is p-torsion-free and weakly F-smooth, then the above fiber is concentrated in degrees $\geq i+2$, as it is p-torsion.

Proposition 4.3 (F-smoothness vs weak F-smoothness). *If a p-quasisyntomic ring A is F-smooth, then A is weakly F-smooth. Conversely, the p-quasisyntomic ring A is F-smooth if and only it is weakly F-smooth and the natural map of graded E_∞-rings*

$$\bigoplus_i \phi_i : \bigoplus_{i\geq 0} \mathcal{N}^i \frac{\mathbb{\Delta}_A}{p} \to \bigoplus_{i\in\mathbb{Z}} \frac{\widehat{\overline{\mathbb{\Delta}}}_A\,\{i\}}{p} \tag{15}$$

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(where the target denotes the direct sum of the Nygaard-completed Hodge–Tate cohomologies mod p) exhibits the target as the localization of the source at θ.

Proof. The first claim follows from the commutative diagram

$$\begin{array}{ccc} \mathcal{N}^i \frac{\mathbb{\Delta}_A}{p} & \xrightarrow{\theta} & \mathcal{N}^{i+p} \frac{\mathbb{\Delta}_A}{p} \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i+p}} \\ \frac{\overline{\mathbb{\Delta}}_A\{i\}}{p} & \xrightarrow{\theta,\simeq} & \frac{\overline{\mathbb{\Delta}}_A\{i+p\}}{p} \end{array}$$

obtained from the map of graded E_∞-rings $\bigoplus_{i\geq 0} \mathcal{N}^i \frac{\mathbb{\Delta}_A}{p} \to \bigoplus_{i\in\mathbb{Z}} \frac{\overline{\mathbb{\Delta}}_A\{i\}}{p}$. The second claim follows from the above and Proposition 2.6: The localization of the source in equation (15) is precisely the mod p Hodge–Tate cohomology. □

Remark 4.4 (Stability of weak F-smoothness under filtered colimits and étale localization). As the construction $A \mapsto \mathcal{N}^i \mathbb{\Delta}_A \in \mathcal{D}(A)$ commutes with p-completed filtered colimits and étale localization, it follows that the collection of weakly F-smooth rings is closed under filtered colimits and étale localizations inside all p-quasisyntomic rings. Moreover, weak F-smoothness can be detected locally for the étale topology.

Remark 4.5 (Essential constancy of the twisted Nygaard filtration under weak F-smoothness). If A is weakly F-smooth, then for any fixed integer n, we have

$$H^n\left(\mathrm{fib}\left(\theta : \mathcal{N}^j \frac{\mathbb{\Delta}_A}{p} \to \mathcal{N}^{j+p} \frac{\mathbb{\Delta}_A}{p}\right)\right) = 0 \quad \text{for} \quad j \gg 0.$$

It follows that the twisted Nygaard filtration on $\frac{\overline{\mathbb{\Delta}}_A\{i\}}{p}$ (Construction 3.3) is essentially constant in each cohomological degree; moreover, the implicit constants are independent of A.

Proposition 4.6 (Stability of F-smoothness under filtered colimits and étale localization). *The property of being F-smooth is stable under filtered colimits.*

Proof. Given a filtered diagram $\{A_i\}$ of F-smooth rings with colimit A, each A_i is weakly F-smooth by Proposition 4.3; it then follows from Remark 4.5 that the p-completion of $\varinjlim_i \widehat{\overline{\mathbb{\Delta}}}_{A_i}\{j\}$ gives $\widehat{\overline{\mathbb{\Delta}}}_A\{j\}$, which easily shows that A is F-smooth. □

For the next result, cf. also [BLM21, Prop. 9.5.11] for the analog in characteristic p. Unlike in *loc. cit.*, we make a (p-complete) flatness hypothesis; we expect that this should be unnecessary but were unable to remove it.

Proposition 4.7. *Let A be a p-quasisyntomic ring. Let B be a p-completely flat A-algebra which is relatively perfect. If A is F-smooth, so is B. Moreover, the converse holds true if B is p-completely faithfully flat over A. In particular, F-smoothness is étale local and passes to étale algebras.*

Proof. We have p-adic equivalences $\mathcal{N}^i \mathbb{\Delta}_A \otimes_A B \xrightarrow{\sim} \mathcal{N}^i \mathbb{\Delta}_B$ by Proposition 3.7. From this, it follows that if A is weakly F-smooth, then so is B; the converse holds if B is p-completely faithfully flat over A.

Next, we have that $\widetilde{\mathcal{N}}^{\geq *} \frac{\overline{\mathbb{\Delta}}_A\{i\}}{p} \otimes_A B \to \widetilde{\mathcal{N}}^{\geq *} \frac{\overline{\mathbb{\Delta}}_B\{i\}}{p}$ is an equivalence, again by Proposition 3.7. If A is F-smooth, then $\widetilde{\mathcal{N}}^{\geq *} \frac{\overline{\mathbb{\Delta}}_A\{i\}}{p}$ is prozero in any range of degrees, whence the same holds true for $\widetilde{\mathcal{N}}^{\geq *} \frac{\overline{\mathbb{\Delta}}_B\{i\}}{p}$ (by p-complete flatness), whence completeness of the Nygaard filtration (Remark 3.4); we conclude B is then F-smooth. The converse if B is p-completely faithfully flat follows similarly. □

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Proposition 4.8. *If a p-quasisyntomic ring A is F-smooth, then the polynomial ring $A[x]$ is also F-smooth.*

Proof. Suppose A is F-smooth. The weak F-smoothness of $A[x]$ follows using the cofiber sequence of $\bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_A$-modules obtained by unfolding Proposition 3.11,

$$\left(\bigoplus_{i\geq 0}\mathcal{N}^{i-1}\mathbb{\Delta}_A\widehat{\otimes_A}A[x]\right)[-1]\to\bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_{A[x]}\to\bigoplus_{i\geq 0}\mathcal{N}^i\mathbb{\Delta}_A\widehat{\otimes_A}A[x].$$

By Proposition 3.12, and quasisyntomic descent, we find that there is a finite filtration on $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_{A[x]}$ (considered as an object of the filtered derived ∞-category) where the associated graded terms are given by the p-completions of $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_A\otimes_A A[x]$, $(p-1)$ copies of $\mathcal{N}^{\geq *-1}\overline{\mathbb{\Delta}}_A\otimes_A A[x]/\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_A\otimes_A A[x]$ and $\mathcal{N}^{\geq *-1}\overline{\mathbb{\Delta}}_A\otimes_A A[x]\{-1\}[-1]$. Thus, it suffices to show that under the F-smoothness hypotheses, $(\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_A\otimes_A A[x])\{i\}$ is complete mod p for each $i\in\mathbb{Z}$. For this, it suffices to prove the analogous completeness with $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_A\{i\}$ replaced in the above tensor product by the twisted Nygaard filtration on the mod p reduction (Construction 3.3); however, this follows from the essential constancy of the twisted Nygaard filtration, Remark 4.5. □

Proposition 4.9. *Let A be a p-quasisyntomic ring, and let B be a p-completely flat A-algebra such that $A/p\to B/p$ is smooth. If A is F-smooth, so is B.*

Proof. Combine Proposition 4.8 and Proposition 4.6. □

Proposition 4.10. *Let A be a p-quasisyntomic ring. Then A is F-smooth if and only if all the localizations $A_{\mathfrak{p}}$ for $\mathfrak{p}\in\operatorname{Spec}(A)$, are F-smooth.*

Proof. If A is F-smooth, then all of its localizations are F-smooth by Proposition 4.7. The converse direction follows similarly as in the proof of Proposition 4.7, noting that p-complete Tor-amplitude can be checked on localizations. □

Remark 4.11 (F-smoothness and the Beilinson t-structure). Assume A is an F-smooth p-quasisyntomic ring. Write $\mathbb{\Delta}_A^{[\bullet]}$ for the complete filtered object defined by the prismatic complex $\mathbb{\Delta}_A$ equipped with the filtration defined by powers of the Hodge–Tate ideal sheaf, so we have a natural identification

$$\operatorname{gr}^*\mathbb{\Delta}_A^{[\bullet]}\simeq\overline{\mathbb{\Delta}}_A\{*\}.$$

By definition of the Nygaard filtration, the Frobenius on $\mathbb{\Delta}_A$ refines to a map

$$\varphi_A:\mathcal{N}^{\geq *}\mathbb{\Delta}_A\to\mathbb{\Delta}_A^{[\bullet]}$$

in the filtered derived category. Using the connectivity bound $\mathcal{N}^i\mathbb{\Delta}_A\in\mathcal{D}^{\leq i}(\mathbb{Z}_p)$ (cf. [BL22a, Rem. 5.5.9]), the F-smoothness hypothesis implies in particular that φ_A induces an equivalence

$$\mathcal{N}^i\mathbb{\Delta}_A\simeq\tau^{\leq i}\operatorname{gr}^i\mathbb{\Delta}_A^{[\bullet]}$$

As both filtrations are complete by assumption, it follows that the map φ_A identifies its source with the connective cover of its target for the Beilinson t-structure on the filtered derived category (see [BMS19, Sec. 5.4] for an account).

4.2. F-smoothness over a base

In this subsection, we study the F-smoothness condition over a perfectoid base. We offer the following characterization; work of V. Bouis [Bou22] has studied F-smoothness over mixed characteristic perfectoid base rings in more detail and yielded important examples.

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Proposition 4.12 (Cf. [Bou22, Th. 2.16, 2.18]). *Let R_0 be a perfectoid ring, and let A be an R_0-algebra. Suppose A is quasisyntomic. Then A is F-smooth if and only if:*

1. *$\widehat{L_{A/R_0}}$ is a p-completely flat A-module.*
2. *The p-completed derived de Rham cohomology $L\Omega_{A/R_0}$ (cf. [Bha12]) is Hodge-complete.*

Proof. The divided Frobenius $\phi_i : \mathcal{N}^i\overline{\mathbb{\Delta}}_A \to \overline{\mathbb{\Delta}}_A$ (where we trivialize the Breuil–Kisin twists since we are over R_0) matches the source with the ith stage of the conjugate filtration (cf. [BS22, Th. 4.11]) on the Hodge–Tate cohomology, [BS22, Th. 12.2].

Now, the condition (2) that the p-completed derived de Rham cohomology is Hodge-complete is equivalent to the condition that the derived prismatic cohomology $\overline{\mathbb{\Delta}}_A$ (over the perfect prism corresponding to R_0) is Nygaard-complete, thanks to [BMS19, Th. 7.2(5)].

Therefore, once one knows the derived prismatic cohomology is Nygaard-complete, the F-smoothness condition amounts to the statement that the conjugate filtration map $\mathrm{Fil}_i\overline{\mathbb{\Delta}}_A \to \overline{\mathbb{\Delta}}_A$ has homotopy fiber (in $\mathcal{D}(A)$) with p-complete Tor-amplitude in degrees $\geq i+2$, for each i. Using the associated gradeds of the conjugate filtration (given by $\mathrm{gr}^j = \widehat{\wedge^j L_{A/R_0}}[-j]$), one easily sees by considering $i = 0, 1$ that this is equivalent to the condition that $\widehat{L_{A/R_0}}$ should be p-completely flat over A. □

In the special case of quasisyntomic $\mathbb{F}_p$-algebras, the condition of F-smoothness had been previously studied under the name Cartier smoothness [KM21, KST21] which we review next.[7]

Definition 4.13 (Cf. [KM21, KST21]). Let A be a quasisyntomic $\mathbb{F}_p$-algebra. We say that A is *Cartier smooth* if:

1. The cotangent complex $L_{A/\mathbb{F}_p}$ is a flat discrete A-module.
2. The inverse Cartier map $C^{-1} : \Omega^i_{A/\mathbb{F}_p} \to H^i(\Omega^*_{A/\mathbb{F}_p})$ is an isomorphism for $i \geq 0$. Here, $\Omega^*_{A/\mathbb{F}_p}$ denotes the classical de Rham complex of A over $\mathbb{F}_p$.

Proposition 4.14. *Let A be a quasisyntomic $\mathbb{F}_p$-algebra. Then A is F-smooth if and only if A is Cartier smooth.*

Proof. Suppose $L_{A/\mathbb{F}_p}$ is a flat A-module. Then the derived de Rham cohomology $L\Omega_{A/\mathbb{F}_p}$ maps to its Hodge completion, which is just the usual algebraic de Rham complex $\Omega^\bullet_{A/\mathbb{F}_p}$. Using the conjugate filtration on the former [Bha12, Prop. 3.5], we see that the condition that this map should be an equivalence is precisely the Cartier isomorphism condition. Therefore, the result follows from Proposition 4.12. □

4.3. F-smoothness of regular rings

In this subsection, we prove the following theorem.

Theorem 4.15. *Let A be a regular (noetherian) ring. Then A is F-smooth. Conversely, if A is a p-complete noetherian ring which is F-smooth, then A is regular.*

We first prove the forward direction. When A is an $\mathbb{F}_p$-algebra, F-smoothness is equivalently to Cartier smoothness (Proposition 4.14) and thus follows at once from regularity via Néron–Popescu desingularization, which implies that A is a filtered colimit of smooth $\mathbb{F}_p$-algebras. One can also prove the result directly [BLM21, Sec. 9.5]. In the case of an *unramified* regular ring, most of the result also appears in [BL22a, Prop. 5.7.9, 5.8.2].

Proposition 4.16. *Let A be a p-quasisyntomic ring, and let $x \in A$ be a nonzero divisor. Suppose A/x and $A[1/x]$ are F-smooth. Then A is F-smooth.*

[7] The second author had previously asked in [Mat22, Question 4.21] whether there could be a notion of Cartier smoothness in mixed characteristic; we are also grateful to Matthew Morrow for discussions on this point.

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Proof. First, we show that A is weakly F-smooth. Write $\mathcal{F}^i_A = \mathrm{fib}(\theta : \mathcal{N}^i \frac{\mathbb{\Delta}_A}{p} \to \mathcal{N}^{i+p} \frac{\mathbb{\Delta}_A}{p})$. Using the cofiber sequence of Proposition 3.13, we find that there is a cofiber sequence $(\mathcal{F}^i_A)/x \to \mathcal{F}^i_{A/x} \to \mathcal{F}^{i-1}_{A/x}$. Moreover, $\mathcal{F}^i_{A[1/x]} = (\mathcal{F}^i_A)[1/x]$. Note that an object $N \in \mathcal{D}(A)$ has p-complete Tor-amplitude in degrees $\geq j$ if and only if $N[1/x] \in \mathcal{D}(A), N/x \in \mathcal{D}(A/x)$ have p-complete Tor-amplitude in degrees $\geq j$. From these observations, it follows easily that A is weakly F-smooth.

Now, we show that A is F-smooth. For this, it suffices to show that the map

$$\overline{\mathbb{\Delta}}_A\{i\} \to \widehat{\overline{\mathbb{\Delta}}}_A\{i\} \tag{16}$$

is an equivalence for each i; here, the latter denotes the Nygaard-completed Hodge–Tate cohomology. By weak F-smoothness of A, the natural map $\widehat{\overline{\mathbb{\Delta}}}_A\{i\}[1/x] \to \widehat{\overline{\mathbb{\Delta}}}_{A[1/x]}\{i\}$ is an equivalence, thanks to Remark 4.5. Therefore, by our assumptions, the comparison map (16) becomes an isomorphism after p-completely inverting x, so its fiber mod p is x-power torsion. It thus suffices to show that equation (16) induces an isomorphism after base-change along $A \to A/x$. But by Corollary 3.15 and our assumption of F-smoothness of A/x, the filtered object $\mathcal{N}^{\geq *}\overline{\mathbb{\Delta}}_A/x$ is complete. □

Corollary 4.17. *Let A be a p-quasisyntomic ring such that A is p-torsion-free and such that the $\mathbb{F}_p$-algebra A/p is Cartier smooth. Then A is F-smooth.*

Proof. Apply Proposition 4.16 with $x = p$. □

Proof that regular rings are F-smooth. Suppose A is regular. Since A is lci, A is p-quasisyntomic. By Proposition 4.10, the ring A is F-smooth if and only if all of its localizations are F-smooth. Consequently, we may assume that A is local with maximal ideal $\mathfrak{m} \subset A$ and in particular of finite Krull dimension. By induction on the Krull dimension, we may assume that any regular ring of smaller Krull dimension (e.g., any further localization of A) is F-smooth. If A is zero-dimensional and hence a field, then we already know that A is F-smooth: More generally, any regular ring in characteristic p is Cartier smooth and hence F-smooth. So suppose $\dim(A) > 0$. Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$; then $A[1/x]$ and A/x are F-smooth by induction on the dimension. By Proposition 4.16, it follows that A is F-smooth. □

For the proof that F-smoothness implies regularity, we will actually need much less than F-smoothness itself. We expect that the result is related to recent works relating regularity to p-derivations [HJ21, Sai22].

Lemma 4.18. *Let $(A, \mathfrak{m}, k)$ be a complete intersection local ring. Then A is regular if and only if the map of k-vector spaces $H^{-1}(L_{A/\mathbb{Z}} \otimes_A k) \to H^{-1}(L_{k/\mathbb{Z}})$ is injective.*

Proof. We have a transitivity triangle (for $\mathbb{Z} \to A \to k$), $L_{A/\mathbb{Z}} \otimes_A k \to L_{k/\mathbb{Z}} \to L_{k/A}$ and $L_{k/\mathbb{Z}}$ is concentrated in degrees $[-1, 0]$. Thus, the injectivity condition of the lemma is equivalent to the statement that $H^{-2}(L_{k/A}) = 0$, whence the result by [Iye07, Prop. 8.12]. □

Proposition 4.19. *Let A be a complete intersection local noetherian ring with residue field k of characteristic p. Then the following are equivalent:*

1. *A is regular.*
2. *$\mathrm{cofib}(\theta : A/p \to \mathcal{N}^p \mathbb{\Delta}_A/p) \otimes^{\mathbb{L}}_{A/p} k \in \mathcal{D}^{\geq 1}(k)$ (e.g., this holds if A is F-smooth by Proposition 4.3 and its proof).*

We remind the reader that reduction mod p is interpreted in the derived sense in this article, including in the statement above and the proof below.

Proof. We have already shown above that regular rings are F-smooth, whence (1) implies (2), so we show the converse. For any animated ring B, the Nygaard fiber sequence of [BL22a, Rem. 5.5.8] and

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the conjugate filtration on diffracted Hodge cohomology [BL22a, Cons. 4.7.1] yields a fiber sequence in $\mathcal{D}(B/p)$,

$$\mathrm{cofib}(\theta : B/p \to \mathcal{N}^p \mathbb{\Delta}_B/p) \to \bigwedge^p L_{B/p/\mathbb{F}_p}[-p] \to B/p. \tag{17}$$

In more detail, if B is a polynomial $\mathbb{Z}$-algebra, then the Nygaard fiber sequence of *loc. cit.* gives a fiber sequence

$$\mathcal{N}^p \mathbb{\Delta}_B \to \mathrm{Fil}_p^{\mathrm{conj}} \hat{\Omega}_B^{\not p} \xrightarrow{\Theta + p} \mathrm{Fil}_{p-1} \hat{\Omega}_B^{\not p}.$$

Using the eigenvalues of the action of the Sen operator Θ on the associated graded terms of $\hat{\Omega}_B^{\not p}$ (note that the conjugate filtration is just the Postnikov filtration in this case) as in [BL22a, Notation 4.7.2], we find that

$$H^*(\mathcal{N}^p \mathbb{\Delta}_B) \simeq \begin{cases} B/p & * = 1 \\ \widehat{\Omega}^p_{B/\mathbb{Z}} & * = p \\ 0 & \text{otherwise} \end{cases}.$$

Moreover, the map $\theta : B/p \to \mathcal{N}^p \mathbb{\Delta}_B/p$ is an isomorphism in H^0 by comparison with the case $B = \mathbb{Z}$ (Construction 2.4); this (together with left Kan extension) easily gives the fiber sequence (17).

Taking $B = A$ and base-changing to $A/p \to k$, we obtain a fiber sequence

$$\mathrm{cofib}(\theta : A/p \to \mathcal{N}^p \mathbb{\Delta}_A/p) \otimes_{A/p} k \to \bigwedge^p L_{A/\mathbb{Z}}[-p] \otimes_A k \to k \tag{18}$$

By the lci hypotheses, $L_{(A/p)/\mathbb{F}_p} \in \mathcal{D}(A/p)$ has Tor-amplitude in $[-1, 0]$. The condition (2) is equivalent to the injectivity of the map (obtained by applying H^0 to the second map in equation (18), using décalage [Ill71, §4.3.2])

$$\Gamma^p H^{-1}(L_{A/\mathbb{Z}} \otimes_A k) = H^{-p}(\bigwedge^p L_{A/\mathbb{Z}} \otimes_A k) \to k. \tag{19}$$

We have constructed the map (19) naturally in the lci ring $(A, \mathfrak{m})$ with residue field k. Moreover, it is injective if $A = k$ since we have seen that regular rings are F-smooth and hence satisfy (2). Conversely, suppose A satisfies (2). It follows by naturality of equation (19) that $H^{-1}(L_{A/\mathbb{Z}} \otimes_A k) \to H^{-1}(L_{k/\mathbb{Z}})$ is injective, whence regularity of A by Lemma 4.18. □

Proof that F-smoothness implies regularity under p-completeness. Let A be a p-complete noetherian ring which is F-smooth. We argue that A is regular. It suffices to check that the localization of A at any maximal ideal is regular since a noetherian ring is regular if and only if its localizations at maximal ideals are regular. Since p belongs to any maximal ideal, we reduce to the case where A is a p-complete local ring which is F-smooth. Our p-quasisyntomicity assumption implies that $L_{A/\mathbb{Z}} \otimes_A k \in \mathcal{D}^{[-1,0]}(k)$; by [Avr99, Prop. 1.8], this implies that A is a complete intersection. Then we can appeal to Proposition 4.19 to conclude that A is regular, as desired. □

4.4. Dimension bounds

As an application, we can obtain some dimension bounds on the Hodge–Tate cohomology of regular rings and verify [BL22b, Conj. 10.1] with an additional assumption of F-finiteness. Let us recall the setup. For a quasisyntomic ring R, we consider the Hodge–Tate stack $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)}$ defined in [BL22b, Cons. 3.7]; recall that this stack comes with a map $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)} \to \mathrm{Spf}(R)$ and line bundles $\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)}}\{i\}$ whose cohomology yields $\overline{\mathbb{\Delta}}_R\{i\}$.

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Before formulating the result, let us also recall some facts about F-finiteness. A noetherian $\mathbb{F}_p$-algebra S is said to be *F-finite* if it is finitely generated over its pth powers. If S is a noetherian local $\mathbb{F}_p$-algebra, F-finiteness is equivalent to the assumption that the residue field of S is F-finite and S is excellent, cf. [Kun76, Cor. 2.6]. Moreover, S is F-finite if and only if the cotangent complex $L_{S/\mathbb{F}_p} \in \mathcal{D}(S)$ is almost perfect, cf. [DM17, Th. 3.6] and [Lur18, Th. 3.5.1].

Corollary 4.20. *Let R be a p-complete regular local ring with residue field k. Suppose that R/pR is F-finite. Let $d = \dim R + \log_p[k : k^p]$. Then the Hodge–Tate stack $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)}$ has cohomological dimension $\leq d$. In particular, $\overline{\mathbb{\Delta}}_R\{i\} \in \mathcal{D}^{\leq d}(\mathbb{Z}_p)$ for each i.*

Proof. Let us first reduce to the case where R is complete. The map $\hat{R}_{\mathfrak{m}} \otimes_{\mathbb{Z}} L_{R/\mathbb{Z}} \to L_{\hat{R}_{\mathfrak{m}}/\mathbb{Z}}$ is an isomorphism after p-completion: In fact, both sides are almost perfect mod p by F-finiteness (as recalled above) and the map is an isomorphism after base-change to the residue field, whence the claim by Nakayama. It follows by [BL22b, Rem. 3.9] (and its proof) that the diagram

$$\begin{array}{ccc} \mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(\hat{R}_{\mathfrak{m}})} & \longrightarrow & \mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)} \\ \downarrow & & \downarrow \\ \mathrm{Spf}(\hat{R}_{\mathfrak{m}}) & \longrightarrow & \mathrm{Spf}(R) \end{array}$$

is Cartesian. Therefore, since $R \to \widehat{R}_{\mathfrak{m}}$ is faithfully flat, it suffices to replace everywhere R by $\hat{R}_{\mathfrak{m}}$, so we may assume that R itself is complete.

Let us now verify the dimension bound on the Hodge–Tate complexes $\overline{\mathbb{\Delta}}_R\{i\}$,that is, that $\overline{\mathbb{\Delta}}_R\{i\} \in \mathcal{D}^{\leq d}(\mathbb{Z}_p)$ for each i. The associated graded terms of the Nygaard filtration on $\overline{\mathbb{\Delta}}_R\{i\}$ (i.e., $\mathcal{N}^j\mathbb{\Delta}_R/p^{(-1)}$) are almost perfect R-modules, whence $\mathfrak{m}$-adically complete, in light of the Nygaard fiber sequences [BL22a, Rem. 5.5.8] and the almost perfectness mod p of $L_{R/\mathbb{Z}}$ recalled above. Using the completeness of the Nygaard filtration (Theorem 4.15), we find that $\overline{\mathbb{\Delta}}_R\{i\}$ is $\mathfrak{m}$-adically complete. If R is zero-dimensional and hence $R = k$, the result follows from the comparison [BL22a, Th. 5.4.2] between Hodge–Tate and de Rham cohomology of $\mathbb{F}_p$-algebras since $\dim \Omega^1_{k/\mathbb{F}_p} = \log_p[k : k^p]$, cf. [Sta19, Tag 07P2]. Otherwise, choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. The ring R/x is also regular local with the same residue field and of dimension one less. To see that $\overline{\mathbb{\Delta}}_R\{i\} \in \mathcal{D}^{\leq d}(R)$, it suffices (by x-adic completeness proved above) to show that $\overline{\mathbb{\Delta}}_R\{i\}/x \in \mathcal{D}^{\leq d}(R)$. However, we have a fiber sequence from Corollary 3.16 which, together with induction on the dimension, implies the claim.

Now, we prove the cohomological dimension bound on $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)}$. First, we prove that the cohomological dimension is at most $d+1$. Let W be a Cohen ring for k. By the Cohen structure theorem, we have a surjection

$$A = W[[t_1, \ldots, t_r]] \to R$$

for $r = \dim(R)$, whose kernel is generated by a nonzero divisor. By choosing a p-basis for k, we see that the ring A is formally étale over a polynomial ring in d variables over $\mathbb{Z}_p$, and consequently that $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(A)} = \mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(\mathbb{Z}_p\langle x_1,\ldots,x_d\rangle)} \times_{\mathrm{Spf}(\mathbb{Z}_p\langle x_1,\ldots,x_d\rangle)} \mathrm{Spf}(A)$. Using the expression for the Hodge–Tate stack of the polynomial $\mathbb{Z}_p$-algebra in [BL22b, Ex. 9.1] as the classifying stack of $(\mathbb{G}_a^{\sharp d} \rtimes \mathbb{G}_m^{\sharp})$, and the explicit description of representations of $\mathbb{G}_a^{\sharp}, \mathbb{G}_m^{\sharp}$ in [BL22a, Sec. 3.5] and [BL22b, Lem. 6.7], one finds that $\mathrm{cd}(\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(A)}) \leq d+1$. By affineness of $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)} \to \mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(A)}$ (Lemma 4.21 below), we obtain $\mathrm{cd}(\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)}) \leq d+1$. It remains to show that H^{d+1} of any quasicoherent sheaf on $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)}$ (which we may assume to be p-torsion) vanishes.

Consider the category of p-torsion sheaves on $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)}$ (we recall that $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)}$ is defined as a functor on p-nilpotent rings, so this case will suffice). We claim that for any p-torsion sheaf $\mathcal{F}$ on $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)}$, we have $H^0(\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)}, \mathcal{F}\{-n\}) \neq 0$ for some n. In fact, using the affine map

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$\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)} \to \mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(A)}$, this claim reduces to the analogous claim for $\mathrm{Spf}(A)$; but this in turn follows from the explicit description of the Hodge–Tate stack for $\mathrm{Spf}(A)$. It follows that the category of p-torsion sheaves on $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)}$ is generated under colimits and extensions by the quotients of the $\mathcal{O}\{n\}/p$. Since H^{d+1} is a right exact functor for quasicoherent sheaves on $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)}$ (as proved above), and since we have the cohomological dimension bound on the $\mathcal{O}\{n\}$, we now conclude as desired. □

Lemma 4.21. *Let R be a quasisyntomic ring and let $t \in R$ be a nonzero divisor. The map $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R/t)} \to \mathrm{Spf}(R/t) \times_{\mathrm{Spf}(R)} \mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(R)}$ is affine.*

Proof. We reduce to the case where R is the p-completion of $\mathbb{Z}_p[t]$. In this case, by [BL22b, Ex. 9.1], the above map is identified with $B\mathbb{G}_m^\sharp \to B(\mathbb{G}_a^\sharp \rtimes \mathbb{G}_m^\sharp)$; in particular, it is affine. □

5. Comparison with p-adic étale Tate twists

In this section, we prove Theorem 1.8 from the introduction. That is, on a F-smooth p-torsion-free scheme, we show that the complex $\mathbb{Z}/p^n(i)_X$ can be obtained via a generalization of the construction of p-adic étale Tate twists [Gei04, Sch94, Sat07], that is, by modifying the truncated p-adic vanishing cycles $\tau^{\leq i} Rj_*(\mu_{p^n}^{\otimes i})$ by taking the subsheaf in degree i generated by symbols from X.

Our strategy is as follows. Since we already know the $\mathbb{F}_p(i)$ for $\mathbb{Z}[1/p]$-schemes are the usual Tate twists, it suffices to treat the p-Henselian case. One needs to show that the map $\mathbb{F}_p(i)(X) \to \mathbb{F}_p(i)(X[1/p])$ is highly coconnected. The étale comparison theorem (Theorem 5.1 below) implies that one may obtain the $\mathbb{F}_p(i)(X[1/p])$ by inverting the operator v_1 on the $\mathbb{F}_p(i)(X)$. Thus, we reduce to showing that the map $v_1 : \mathbb{F}_p(i)(X) \to \mathbb{F}_p(i+p-1)(X)$ is highly coconnected. Here, we use an explicit argument (which was inspired by [HW22]) with the expression of [BMS19] to check the claim. To determine the top-degree cohomology, we use also the classical results of Bloch–Kato [BK86] on p-adic vanishing cycles.

In [KM21, KST21], it is shown that the description of the $\mathbb{Z}/p^n(i)_X$ for regular $\mathbb{F}_p$-schemes via logarithmic Hodge–Witt forms (cf. [BMS19, Sec. 8] and [GH99]) also holds for the Cartier smooth case. Our Theorem 1.8 may be seen as a mixed characteristic analog of this result.

5.1. *The étale comparison*

Let X be a qcqs derived scheme. As in [BL22a, Sec. 8], one associates the graded E_∞-algebra $\bigoplus_{i\in\mathbb{Z}} \mathbb{F}_p(i)(X)$, the mod p syntomic cohomology of X. When X is the spectrum of a p-complete animated ring, this can be obtained (via descent and left Kan extension) from the Frobenius fixed points of prismatic cohomology as in [BL22a, Sec. 7] and [BMS19]. However, when X is a $\mathbb{Z}[1/p]$-scheme, it is the usual Tate twisted étale cohomology $\bigoplus_{i\in\mathbb{Z}} R\Gamma_{\mathrm{et}}(X, \mu_p^{\otimes i})$.

For any X, the class $v_1 \in H^0(\mathbb{F}_p(p-1)(\mathbb{Z}))$ constructed in Construction 2.7 yields a class in $H^0\left(\bigoplus_i \mathbb{F}_p(i)(X)\right)$ which maps to a unit after passage to $X[1/p]$.

Theorem 5.1 (The étale comparison, [BL22a, Th. 8.5.1]). *Let X be any qcqs derived scheme. The natural map of graded E_∞-algebras over $\mathbb{F}_p$,*

$$\bigoplus_{i\in\mathbb{Z}} \mathbb{F}_p(i)(X) \to \bigoplus_{i\in\mathbb{Z}} \mathbb{F}_p(i)(X[1/p]), \tag{20}$$

exhibits the target as the localization of the source at v_1. In particular, for any i, the filtered colimit

$$\mathbb{F}_p(i)(X) \xrightarrow{v_1} \mathbb{F}_p(i+p-1)(X) \xrightarrow{v_1} \mathbb{F}_p(i+2(p-1))(X) \to \dots$$

is canonically identified with $\mathbb{F}_p(i)(X[1/p]) = R\Gamma_{\mathrm{et}}(X[1/p]; \mathbb{F}_p(i))$.

Proof. When X is a scheme over $\mathbb{Z}[\zeta_{p^\infty}]$, the result is proved in [BL22a, Th. 8.5.1]: In that case, one obtains a similar statement for the p-complete E_∞-algebras $\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}_p(i)(X)$, $\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}_p(i)(X[1/p])$,

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when one inverts the class $\epsilon \in H^0(\mathbb{Z}_p(1)(\mathbb{Z}[\zeta_{p^\infty}]))$ arising from the given system of p-power roots of unity. Let us explain how one can deduce the current form of the result.

First, if X is p-quasisyntomic (which is the only case that will be used below), then we observe that both sides of equation (20) are coconnective. Using the sheaf property, one may reduce to the case where X lives over $\mathbb{Z}[\zeta_{p^\infty}]$, which is proved in *loc. cit.*

To prove the result more generally, it suffices to show that the construction which carries an animated ring R to $R\Gamma_{\text{et}}(\text{Spec}(R[1/p]), \mu_p^{\otimes i})$ (i.e., the right-hand side of equation (20)) is left Kan extended from smooth $\mathbb{Z}$-algebras. In fact, the left-hand side is left Kan extended from smooth $\mathbb{Z}$-algebras [BL22a, Prop. 8.4.10], as is its localization after inverting v_1, and for smooth (in particular p-quasisyntomic) algebras we have already seen the result.

Now, we claim that the construction which carries an animated ring R to $\bigoplus_{i\in\mathbb{Z}} R\Gamma_{\text{et}}(\text{Spec}(R \otimes \mathbb{Z}[\zeta_{p^\infty}][1/p]), \mu_p^{\otimes i})$ is left Kan extended from smooth $\mathbb{Z}$-algebras. In fact, by [BL22a, Th. 8.5.1], this construction is the localization of $R \mapsto \bigoplus_{i\in\mathbb{Z}} \mathbb{F}_p(i)(\text{Spec}(R \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_{p^\infty}]))$ at v_1. This construction in turn fits into a fiber sequence [BL22a, Rem. 8.4.8] involving terms that are either rigid for Henselian pairs or which commute with sifted colimits, cf. the proof of [BL22a, Prop. 8.4.10]. As in *loc. cit.*, this implies that $\bigoplus_{i\in\mathbb{Z}} R\Gamma_{\text{et}}(\text{Spec}(R \otimes \mathbb{Z}[\zeta_{p^\infty}][1/p]), \mu_p^{\otimes i})$ is left Kan extended from smooth $\mathbb{Z}$-algebras. Taking $\mathbb{Z}_p^\times$-Galois invariants, we conclude that $\bigoplus_{i\in\mathbb{Z}} R\Gamma_{\text{et}}(\text{Spec}(R[1/p]), \mu_p^{\otimes i})$ has the desired left Kan extension property. □

5.2. Comparison with the generic fiber

In this subsection, we prove the following basic comparison result; over a perfectoid, this has also been proved by Bouis, cf. [Bou22, Th. 4.14].

Proposition 5.2. *Let A be a p-torsion-free p-quasisyntomic ring which is F-smooth. Then for each i, the canonical map $\mathbb{F}_p(i)(A) \to \mathbb{F}_p(i)(A[1/p]) = R\Gamma_{\text{et}}(\text{Spec}(A[1/p]); \mathbb{F}_p(i))$ has fiber in $\mathcal{D}^{\geq i+1}(\mathbb{F}_p)$.*

Without loss of generality, we may assume A is p-Henselian. To prove this result, we use Theorem 5.1. Using this, we are reduced to understanding the effect of multiplying with the class v_1 on the syntomic cohomology of A. Recall that the latter is defined as an equalizer:

$$\mathbb{F}_p(i)(A) = \text{eq}\left(\mathcal{N}^{\geq p-1}\frac{\mathbb{\Delta}_A\{p-1\}}{p} \rightrightarrows \frac{\mathbb{\Delta}_A\{p-1\}}{p}\right)$$

of the Frobenius and canonical maps. To analyze the behaviour of cupping with v_1 with respect to the fiber of the canonical map above, we shall use the relation of v_1 with $\tilde{\theta}$ and the following result.

Lemma 5.3. *Let A be a p-torsion-free p-quasisyntomic ring which is F-smooth. Then for each i, j, the fiber of the multiplication map*

$$\tilde{\theta} : \mathcal{N}^{\geq j}\frac{\mathbb{\Delta}_A\{i\}}{p} \to \mathcal{N}^{\geq j+p}\frac{\mathbb{\Delta}_A\{i+p-1\}}{p} \tag{21}$$

belongs to $\mathcal{D}^{\geq j+2}(\mathbb{F}_p)$.

Proof. The F-smoothness assumption shows that, for each j', the fiber of $\theta : \mathcal{N}^{j'}\frac{\mathbb{\Delta}_A}{p} \to \mathcal{N}^{j'+p}\frac{\mathbb{\Delta}_A}{p}$ belongs to $\mathcal{D}^{\geq j'+2}(\mathbb{F}_p)$. By filtering both sides (by the Nygaard filtration, which is complete by F-smoothness) of equation (21), the conclusion of the lemma follows, in light of Proposition 2.9. □

Proposition 5.4. *Suppose A is a p-torsion-free p-quasisyntomic ring which is F-smooth. For each $i \in \mathbb{Z}$, the Frobenius map*

$$\phi_i : \mathcal{N}^{\geq i}\frac{\mathbb{\Delta}_A\{i\}}{p} \to \frac{\mathbb{\Delta}_A\{i\}}{p} \tag{22}$$

has fiber in $\mathcal{D}^{\geq i+2}(\mathbb{F}_p)$.

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Proof. In fact, this follows because the map (22) admits a complete descending filtration, indexed over $j \geq i$, with gr^j given by $\phi_j : \mathcal{N}^j \frac{\mathbb{\Delta}_A}{p} \to \frac{\overline{\mathbb{\Delta}}_A\{j\}}{p}$; this is clear from the definition of the Nygaard filtration via descent from quasiregular semiperfectoid rings; now, F-smoothness gives the cohomological bound on the fiber of the map on associated graded terms. □

Proof of Proposition 5.2. We will show that the map

$$v_1 : \mathbb{F}_p(i)(A) \to \mathbb{F}_p(i+p-1)(A) \tag{23}$$

has fiber in $\mathcal{D}^{\geq i+1}(\mathbb{F}_p)$; this will suffice thanks to the étale comparison (Theorem 5.1). Without loss of generality, we can assume A is p-Henselian. By construction, the fiber of equation (23) is the equalizer of the two maps (arising from the canonical map and divided Frobenius map)

$$\mathrm{fib}\left(\mathcal{N}^{\geq i}\frac{\mathbb{\Delta}_A\{i\}}{p} \xrightarrow{v_1} \mathcal{N}^{\geq i+p-1}\frac{\mathbb{\Delta}_A\{i+p-1\}}{p}\right) \rightrightarrows \mathrm{fib}\left(\frac{\mathbb{\Delta}_A\{i\}}{p} \xrightarrow{v_1} \frac{\mathbb{\Delta}_A\{i+p-1\}}{p}\right). \tag{24}$$

By assumption, since A is F-smooth, the Frobenius maps

$$\phi_i : \mathcal{N}^{\geq i}\frac{\mathbb{\Delta}_A\{i\}}{p} \to \frac{\mathbb{\Delta}_A\{i\}}{p}, \quad \phi_{i+p-1} : \mathcal{N}^{\geq i+p-1}\frac{\mathbb{\Delta}_A\{i+p-1\}}{p} \to \frac{\mathbb{\Delta}_A\{i+p-1\}}{p}$$

have fibers in $\mathcal{D}^{\geq i+2}(\mathbb{F}_p)$. Therefore, by taking fibers of multiplication by v_1, we find that the fiber of the Frobenius maps in equation (24) belong to $\mathcal{D}^{\geq i+2}(\mathbb{F}_p)$.

Now, consider the canonical map in equation (24); we claim that it induces the zero map in cohomological degrees $\leq i$. To see this, we observe that the canonical map factors through the map

$$\mathrm{fib}\left(\mathcal{N}^{\geq i}\frac{\mathbb{\Delta}_A\{i\}}{p} \xrightarrow{v_1} \mathcal{N}^{\geq i+p-1}\frac{\mathbb{\Delta}_A\{i+p-1\}}{p}\right) \to \mathrm{fib}\left(\mathcal{N}^{\geq i-1}\frac{\mathbb{\Delta}_A\{i\}}{p} \xrightarrow{\tilde{\theta}} \mathcal{N}^{\geq i+p-1}\frac{\mathbb{\Delta}_A\{i+p-1\}}{p}\right) \tag{25}$$

as $v_1 \in \mathcal{N}^{\geq p-1}\frac{\mathbb{\Delta}_{\mathbf{Z}_p}\{p-1\}}{p}$ lifts to $\tilde{\theta} \in \mathcal{N}^{\geq p}\frac{\mathbb{\Delta}_{\mathbf{Z}_p}\{p-1\}}{p}$. However, we have seen that the right-hand side of the above belongs to $\mathcal{D}^{\geq i+1}(\mathbb{F}_p)$ thanks to Lemma 5.3. This implies that the canonical map vanishes in degrees $\leq i$.

Thus, we find that the desired fiber of the map (23) is the equalizer of two maps (24), one of which has fiber in $\mathcal{D}^{\geq i+2}(\mathbb{F}_p)$, and one of which is zero in degrees $\leq i$. This implies the result. □

5.3. *Generation by symbols*

In this section, we complete the proof of Theorem 1.8 from the introduction. First, we prove the following basic symbolic generation result. For more refined results about the connection of the $\{H^i(\mathbb{Z}/p^n(i)(R))\}$ to p-adic Milnor K-theory, cf. [LM21]. In the following, we use that, for any ring R, we have a natural Kummer map $R^\times \to H^1(\mathbb{Z}_p(1)(R))$, cf. Example 1.5. Iterating, we obtain a 'symbol' map $(R^\times)^{\otimes i} \to H^i(\mathbb{Z}_p(i)(R))$.

Proposition 5.5. *For any strictly Henselian local ring R, the symbol map $(R^\times)^{\otimes i} \to H^i(\mathbb{Z}/p^n(i)(R))$ is surjective.*

To prove Proposition 5.5, it clearly suffices to assume that R is p-Henselian and that $n = 1$, using the connectivity bound $\mathbb{Z}/p^n(i)(R) \in \mathcal{D}^{\leq i}(\mathbb{Z}/p^n)$, cf. [AMMN22, Cor. 5.43]. By the left Kan extension property of the $\mathbb{F}_p(i)(-)$ for p-Henselian rings ([AMMN22, Th. 5.1] or [BL22a, Prop. 7.4.8]), we may assume that R is the strict henselization at a characteristic p point of a smooth $\mathbb{Z}$-scheme. In this case, we know by Proposition 5.2 and Theorem 4.15 that the natural map induces an injection

$$H^i(\mathbb{F}_p(i)(R)) \subset H^i(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes i}) = H^i(\mathbb{F}_p(i)(R[1/p])), \tag{26}$$

and we will identify the left-hand side as the subgroup of the right-hand side generated by symbols.

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We now recall some of the work of Bloch–Kato [BK86], which describes the right-hand side of equation (26); it will be convenient to formulate the assertion sheaf-theoretically.

Let X be a smooth $\mathbb{Z}$-scheme; let $j : X[1/p] \subset X, i : Y \overset{\text{def}}{=} X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{F}_p) \subset X$ be the respective open and closed immersions corresponding to the ideal (p). The work [BK86] describes the étale sheaves of $\mathbb{F}_p$-modules on Y,

$$M^i \overset{\text{def}}{=} i^* R^i j_*(\mu_p^{\otimes i}). \tag{27}$$

In particular, using the map $i^* j_* \mathcal{O}_X^\times \to M^1$ arising from the Kummer sequence and the graded ring structure on the $\{M^i\}$, one has a symbol map

$$i^* j_* (\mathcal{O}_{X[1/p]}^\times)^{\otimes i} \to M^i. \tag{28}$$

By [BK86, Th. 14.1], the symbol map is surjective. Moreover, by [BK86, 6.6], one has a surjective residue map of $\mathbb{F}_p$-sheaves

$$\mathrm{res} : M^i \twoheadrightarrow \Omega^{i-1}_{Y,\log}. \tag{29}$$

Proposition 5.6. *If R is a strictly Henselian local ring which is ind-smooth over $\mathbb{Z}$, then the kernel of the surjective residue map (29) $H^i(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes i}) \to \Omega^{i-1}_{R/p,\log}$ is the subgroup of $H^i(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes i})$ generated by symbols from R, that is, by the image of $R^\times \otimes \cdots \otimes R^\times$ under the symbol map $R[1/p]^\times \otimes \ldots R[1/p]^\times \twoheadrightarrow H^i(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes i})$, as in equation (28).*

Proof. Let $B \subset H^i(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes i})$ be the subgroup generated by the symbols from R. The Bloch–Kato filtration [BK86, Cor. 1.4.1] gives a short exact sequence

$$0 \to \Omega^{i-1}_{R/p} \to H^i(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes i}) \to \Omega^i_{R/p,\log} \oplus \Omega^{i-1}_{R/p,\log} \to 0,$$

where the second map $H^i(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes i}) \to \Omega^{i-1}_{R/p,\log}$ is the residue (29). By construction of the filtration and the first map [BK86, 4.3], one sees that B contains the subgroup $\Omega^{i-1}_{R/p}$. As in [BK86, 6.6], the map $H^i(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes i}) \to \Omega^i_{R/p,\log} \oplus \Omega^{i-1}_{R/p,\log}$ carries the symbol $r_1 \otimes \cdots \otimes r_i$ for $r_1, \ldots, r_i \in R^\times$ to $(\frac{dr_1}{r_1} \wedge \cdots \wedge \frac{dr_i}{r_i}, 0)$ and the symbol $r_1 \otimes \cdots \otimes r_{i-1} \otimes p$ to $(0, \frac{dr_1}{r_1} \wedge \cdots \wedge \frac{dr_{i-1}}{r_{i-1}})$. From this, one sees that $H^i(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes i})/B \xrightarrow{\sim} \Omega^{i-1}_{R/p,\log}$ via the residue, as claimed. □

Now, we return to the proof of Proposition 5.5, and identify the image of equation (26). The $\mathcal{D}(\mathbb{F}_p)$-valued sheaf $\mathbb{F}_p(i)(-)$ restricts to an object (with the same notation) on the category of ind-smooth, p-Henselian $\mathbb{Z}$-algebras R. For any such R, we have natural maps from equations (26) and (29),

$$\mathbb{F}_p(i)(R) \to \mathbb{F}_p(i)(R[1/p]) \xrightarrow{\mathrm{res}} R\Gamma_{\mathrm{et}}(\mathrm{Spec}(R/p), \Omega^{i-1}_{\cdot,\log})[-i] = \mathbb{F}_p(i-1)(R/p)[-1],$$

where the last identification is [BMS19, Sec. 8] (and reviewed in Example 1.3). We claim that the composite vanishes. In fact, this is true for any such map.

Proposition 5.7. *Any natural map $\mathbb{F}_p(i)(R) \to \mathbb{F}_p(i-1)(R/p)[-1]$, defined on p-Henselian ind-smooth $\mathbb{Z}$-algebras R, vanishes.*

Proof. By left Kan extension, we can define a natural map on all quasisyntomic $\mathbb{Z}_p$-algebras R, $\mathbb{F}_p(i)(R) \to \mathbb{F}_p(i-1)(R/p)[-1]$. Both sides define $\mathcal{D}(\mathbb{F}_p)$-valued sheaves for the quasisyntomic topology. The source is discrete as a sheaf (by the odd vanishing theorem, [BS22, Th. 4.1]) and the target is concentrated in cohomological degree 1, whence the map must vanish. □

Proof of Proposition 5.5. As before, we may assume that R is ind-smooth over $\mathbb{Z}$ and that $n = 1$. We have seen that the map $H^i(\mathbb{F}_p(i)(R)) \to H^i(\mathbb{F}_p(i)(R[1/p]))$ is injective, and its image must contain

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the image of $(R^\times)^{\otimes i}$. The image of $H^i(\mathbb{F}_p(i)(R))$ is contained in the kernel of the residue map thanks to Proposition 5.7. But by Proposition 5.6, the kernel of the residue maps on $H^i(\mathbb{F}_p(i)(R[1/p]))$ is precisely the image of $(R^\times)^{\otimes i}$. The result follows. □

Proof of Theorem 1.8. Let X be a p-torsion-free scheme which is F-smooth. Thanks to Proposition 5.2, the map $\mathbb{Z}/p^n(i)_X \to Rj_*(\mu_{p^n}^{\otimes i})$ has homotopy fiber in degrees $\geq i+1$. Since $\mathbb{Z}/p^n(i)_X$ is concentrated in degrees $[0,i]$ by [AMMN22, Cor. 5.43], it suffices to identify the image of the (injective) map $\mathcal{H}^i(\mathbb{Z}/p^n(i)_X) \to R^i j_*(\mu_{p^n}^{\otimes i})$. The claim is that it is exactly the subsheaf generated by symbols on X. This follows thanks to the symbolic generation of the source (Proposition 5.5). □

5.4. Comparison with Geisser–Sato–Schneider

In this section, we use the above results to compare the $\mathbb{Z}/p^n(i)_X$ with the complexes defined by Sato [Sat07] for semistable schemes, cf. also the earlier work of Schneider [Sch94] and Geisser [Gei04] for the smooth case; such a comparison was predicted in [BMS19, Rem. 1.16].

Let X be a regular scheme of finite type over a Dedekind domain A such that every characteristic p residue field of A is perfect. Suppose that X is semistable over characteristic p points of $\mathrm{Spec}(A)$. For $n, i \geq 0$, Sato [Sat07] constructs objects $\mathfrak{I}_n(i)_X \in \mathcal{D}^{[0,i]}(X_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z})$ and conjectures [Sat07, Conjecture 1.4.1] that they can be identified with the étale sheafification of the motivic (cycle) complexes mod p^n; in the smooth case, this follows from [Gei04]. Here, we compare the $\mathfrak{I}_n(i)_X$ to the $\mathbb{Z}/p^n(i)_X$.

Theorem 5.8. *There is a canonical, multiplicative equivalence $\mathfrak{I}_n(i)_X \simeq \mathbb{Z}/p^n(i)_X$ of objects in $\mathcal{D}^b(X_{\mathrm{et}}, \mathbb{Z}/p^n)$.*

Proof. As in [Sat07, §4.2], the complex $\mathfrak{I}_n(i)_X$ is built as the mapping fiber of a map from $\tau^{\leq i} Rj_*(\mu_{p^n}^{\otimes i})$ to the $(-i)$-suspension of a discrete sheaf. Therefore, in order to verify the comparison, it suffices (by combining Proposition 5.2, Theorem 4.15 and Proposition 5.5) to show that the étale sheaf $\mathcal{H}^i(\mathfrak{I}_n(i)_X)$ is generated by symbols. We may assume $n = 1$ for this and work stalkwise.

Let R denote the strict henselization of a characteristic p point $x \in X$. We can replace A by its strict henselization, which is a mixed characteristic DVR; let $\pi \in A$ denote the uniformizer. Consider the $\mathbb{F}_p$-vector space $H^i(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes i})$. We have a symbol map $(R[1/p]^\times)^{\otimes i} \to H^i(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes i})$. Let $F \subset H^i(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes i})$ be the subgroup generated by the images of $(R^\times)^{\otimes i}$ and $(1+\pi R)^\times \otimes (R[1/p]^\times)^{\otimes i-1}$ under the symbol map, cf. [Sat07, §3.4]. As in [Sat07, Def. 4.2.4], the image of the injective map $\mathcal{H}^i(\mathfrak{I}_1(i)_X)_x \to H^i(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes i})$ is exactly the subgroup F.

Our observation is that the image of $(1+\pi R)^\times \otimes (R[1/p]^\times)^{\otimes i-1}$ under the symbol map is actually contained in the image of $(R^\times)^{\otimes i}$. Since R is a UFD (as a regular local ring), we have $R[1/p]^\times = \pi^{\mathbb{Z}} \oplus R^\times$. Consider a symbol $(1+\pi a) \otimes b_1 \otimes \cdots \otimes b_{i-1}$ for $b_1, \ldots, b_{i-1} \in R[1/p]^\times$. Using the unique factorization, as well as the fact that $\pi \otimes (-\pi)$ maps to zero in $H^2(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes 2})$, we reduce to the case $i = 2$.

Therefore, it suffices to show that, for $a \in R$, the image of $(1+\pi a) \otimes \pi$ in $H^2(\mathrm{Spec}(R[1/p]), \mu_p^{\otimes 2})$ belongs to the image of $R^\times \otimes R^\times$. By bilinearity, we may assume that $a \in R$ is a unit (e.g., if a is not a unit, we write $(1+\pi a) = \frac{1+\pi a}{1+\pi(a+1)}(1+\pi(a+1))$). In this case, $(1+\pi a) \otimes (-\pi a)$ maps to zero (cf. [Tat76, Th. 3.1]). Using bilinearity again, it follows that $(1+\pi a) \otimes \pi$ maps to an element of $H^2(\mathrm{Spec}(R[1/p], \mu_p^{\otimes 2})$ in the image of $R^\times \otimes R^\times$.

Consequently, it follows that the ring $\bigoplus_{i \geq 0} \mathcal{H}^i(\mathfrak{I}_1(i)_X)_x$ is generated by symbols, whence we conclude. □

Example 5.9. Let K be a discretely valued field of mixed characteristic, and let $\mathcal{O}_K \subset K$ be the ring of integers; let k be the residue field. Let X be a smooth scheme over $\mathcal{O}_K$ with special fiber k. Then the above results (together with the description of p-adic nearby cycles in [BK86], cf. Proposition 5.6) show that we have a natural cofiber sequence in $\mathcal{D}(X_{\mathrm{et}}, \mathbb{Z}/p^n)$,

$$\mathbb{Z}/p^n(i)_X \to \tau^{\leq i} Rj_*(\mu_{p^n}^{\otimes i}) \to W_n\Omega^{i-1}_{X_k,\log}[-i], \tag{30}$$

where the second map is the residue map from [BK86].

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Such results have appeared in the literature before, but usually only in low weights or with some denominators, using the approach to syntomic cohomology of [FM87, Kat87], cf. [AMMN22, Sec. 6] for a comparison. In particular, [Kur87] constructs the above cofiber sequence in low weights. The comparison for semistable schemes and more generally with a log structure after allowing denominators (in all weights) is [CN17]. Integral comparisons for algebras over $\mathcal{O}_C$ appear in the smooth case in [BMS19, Th. 10.1] and in the semistable case (allowing log structures) in [CDN21]; up to isogeny or in low weights, this was previously treated in [Kat87, Tsu99].

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