Separable Hamiltonian PDEs and Turning Point Principle for Stability of Gaseous Stars

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Abstract

We consider stability of nonrotating gaseous stars modeled by the Euler-Poisson system. Under general assumptions on the equation of states, we proved a turning point principle (TPP) that the stability of the stars is entirely determined by the mass–radius curve parametrized by the center density. In particular, the stability can only change at extrema (i.e., local maximum or minimum points) of the total mass. For a very general equations of state, TPP implies that for increasing center density the stars are stable up to the first mass maximum and unstable beyond this point until the next mass extremum (a minimum). Moreover, we get a precise counting of unstable modes and exponential trichotomy estimates for the linearized Euler-Poisson system. To prove these results, we develop a general framework of separable Hamiltonian PDEs. The general approach is flexible and can be used for many other problems, including stability of rotating and magnetic stars, relativistic stars, and galaxies. © 2021 Wiley Periodicals LLC.

Contents

1 Introduction

Consider a self-gravitating gaseous star satisfying the 3D Euler-Poisson system

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1.1}$$

$$\rho(v_t + u \cdot \nabla u) = -\nabla p - \rho \nabla V, \tag{1.2}$$

$$\Delta V = 4\pi\rho, \quad \lim_{|x|\to\infty} V(t,x) = 0, \tag{1.3}$$

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where $\rho \geq 0$ is the density, $u(t,x) \in \mathbb{R}^3$ is the velocity, $p = P(\rho)$ is the pressure, and V is the self-consistent gravitational potential. Assume $P(\rho)$ satisfies

$$P(s) \in C^1(0,\infty), \quad P' > 0, \tag{1.4}$$

and there exists $\gamma_0 \in (\frac{6}{5}, 2)$ such that

$$\lim_{s\to 0+} s^{1-\gamma_0} P'(s) = K > 0. \tag{1.5}$$

The assumptions (1.5) implies that the pressure $P(\rho) \approx K\rho^{\gamma_0}$ for ρ near 0.

We consider the stability of nonrotating stars. Throughout the paper, *nonrotating stars* refer to static equilibria of (1.2)–(1.3) with $u = \vec{0}$. Note that any traveling solution of (1.2)–(1.3) with u to be a constant vector $\vec{c}$ becomes static under the Galilean transformation

$$(\rho(x,t), u(x,t)) \to \big((\rho(x + \vec{c}t, t), u(x + \vec{c}t, t) - \vec{c})\big).$$

The density function of a compactly supported nonrotating star can be shown to be radially symmetric [14].

By Lemma 3.2, there exists $\mu_{\max} \in (0, +\infty]$ such that for any center density $\rho_\mu(0) = \mu \in (0, \mu_{\max})$, there exists a unique nonrotating star with the density $\rho_\mu(|x|)$ supported inside a ball with radius $R_\mu = R(\mu) < \infty$. In particular, $\mu_{\max} = \infty$ when $\gamma_0 \geq \frac{4}{3}$ [19] (see also [34, 36, 38] for the proof when $\gamma_0 > \frac{4}{3}$). Denote

$$M(\mu) = \int_{R^3} \rho_\mu \, dx = \int_{S_\mu} \rho_\mu \, dx$$

to be the total mass of the star, where $S_\mu = \{|x| < R_\mu\}$ is the support of ρ_μ. We consider the linear stability of this family of nonrotating gaseous stars $\rho_\mu(|x|)$ for $\mu \in (0, \mu_{\max})$. Our main result is the following turning point principle.

THEOREM 1.1. *The linear stability of ρ_μ is fully determined by the mass–radius curve parametrized by μ. Let $n^u(\mu)$ be the number of unstable modes, namely the total algebraic multiplicities of unstable eigenvalues. For small μ, we have*

$$n^u(\mu) = \begin{cases} 1 \text{ (linear instability)} & \text{when } \gamma_0 \in \left(\frac{6}{5}, \frac{4}{3}\right), \\ 0 \text{ (linear stability)} & \text{when } \gamma_0 \in \left(\frac{4}{3}, 2\right). \end{cases} \tag{1.6}$$

The number $n^u(\mu)$ can only change at mass extrema. For increasing μ, at a mass extrema point where $M'(\mu)$ changes sign, $n^u(\mu)$ increases by 1 if $M'(\mu)R'(\mu)$ changes from $-$ to $+$ (i.e., the mass–radius curve bends counterclockwise) and $n^u(\mu)$ decreases by 1 if $M'(\mu)R'(\mu)$ changes from $+$ to $-$ (i.e., the mass–radius curve bends clockwise).

Here, the mass–radius curve is oriented in a coordinate plane where the horizontal and vertical axes correspond to the support radius and mass of the star, respectively. Theorem 1.1 shows that the stability of nonrotating stars and the number of unstable modes are entirely determined by the mass–radius curve parametrized by the center density μ. In particular, the stability can only change at a center density with extremal mass (i.e., maxima or minima of $M(\mu)$). The change of stability at mass extrema is called the turning point principle (TPP) in the astrophysical literature for both Newtonian and relativistic stars. It was usually based on heuristic arguments. As an example, we quote the following arguments in [40] for relativistic stars: "Suppose that for a given equilibrium configuration a radial mode changes its stability property; i.e., the frequency ω of this mode passes through zero. This implies that there exist infinitesimally nearby equilibrium configurations into which the given 1 can be transformed, without changing the total mass. Hence if ω passes trough zero we have $M'(\mu) = 0$." The same arguments can also be found in other astrophysical textbooks such as [15, 39, 44]. In Theorem 1.1, we give a rigorous justification of TPP for Newtonian stars. Moreover, we obtain the precise counting of unstable modes from the mass–radius curve. For relativistic stars, similar results can also be obtained [16].

Besides the above stability criteria, we obtain more detailed information about the spectra of the linearized Euler-Poisson operator and exponential trichotomy estimates for the linearized Euler-Poisson system, which will be useful for the future study of nonlinear dynamics near the nonrotating stars. To state these results, first we introduce some notations. Let X_μ, Y_μ be the weighted spaces $L^2_{\Phi''(\rho_\mu)}(S_\mu)$ and $(L^2_{\rho_\mu}(S_\mu))^3$, where the enthalpy $\Phi(\rho) > 0$ is defined by

$$\Phi(0) = \Phi'(0) = 0, \quad \Phi''(\rho) = \frac{P'(\rho)}{\rho}. \tag{1.7}$$

Denote $\mathbf{X} = X_\mu \times Y_\mu$. The linearized Euler-Poisson system at $(\rho_\mu, \vec{0})$ is

$$\sigma_t = -\nabla \cdot (\rho_\mu v), \tag{1.8}$$

$$v_t = -\nabla\big(\Phi''(\rho_\mu)\sigma + V\big), \tag{1.9}$$

with $\Delta V = 4\pi\rho$. Here, $(\sigma, v) \in \mathbf{X}$ are the density and velocity perturbations.

Define the operators

$$L_\mu = \Phi''(\rho_\mu) - 4\pi(-\Delta)^{-1} : X_\mu \to X_\mu^*, \quad A_\mu = \rho_\mu : Y_\mu \to Y_\mu^* \tag{1.10}$$

and

$$B_\mu = -\nabla\cdot = -\operatorname{div} : Y_\mu^* \to X_\mu, \quad B'_\mu = \nabla : X_\mu^* \to Y_\mu. \tag{1.11}$$

Here, for $\sigma \in X_\mu$, we denote

$$(-\Delta)^{-1}\sigma = \int_{S_\mu} \frac{1}{4\pi|x-y|}\sigma(y)dy \mid_{S_\mu}.$$

Then (1.8)-(1.9) can be written in the Hamiltonian form

$$\partial_t \begin{pmatrix} \sigma \\ v \end{pmatrix} = \begin{pmatrix} 0 & B_\mu \\ -B'_\mu & 0 \end{pmatrix} \begin{pmatrix} L_\mu & 0 \\ 0 & A_\mu \end{pmatrix} \begin{pmatrix} \sigma \\ v \end{pmatrix} = \mathcal{J}_\mu \mathcal{L}_\mu \begin{pmatrix} \sigma \\ v \end{pmatrix}, \tag{1.12}$$

where the operators

$$\mathcal{J}_\mu = \begin{pmatrix} 0 & B_\mu \\ -B'_\mu & 0 \end{pmatrix} : \mathbf{X}^* \to \mathbf{X}, \quad \mathcal{L}_\mu = \begin{pmatrix} L_\mu & 0 \\ 0 & A_\mu \end{pmatrix} : \mathbf{X} \to \mathbf{X}^*, \tag{1.13}$$

are off-diagonal anti-self-dual and diagonal self-dual, respectively. We call systems like (1.12) "separable Hamiltonian systems."

In the following theorems and throughout this paper, we follow the tradition in the astrophysics literature that "nonradial" perturbations refer to those modes corresponding to nonconstant spherical harmonics. See the more precise Definition 3.16 of the subspaces $\mathbf{X}_r$ and $\mathbf{X}_{nr}$ of radial and nonradial perturbations in Section 3.4.

THEOREM 1.2.

(i) *The steady state* ρ_μ*, which is parametrized by the* C^1 *parameter* μ*, is spectrally stable to nonradial perturbations in* $\mathbf{X}_{nr}$ *with isolated purely imaginary eigenvalues. The zero eigenvalue is isolated with an infinite-dimensional kernel space*

$$\ker(\mathcal{J}_\mu \mathcal{L}_\mu) = \left\{ \begin{pmatrix} 0 \\ u \end{pmatrix} \;\middle|\; \int \rho_\mu |u|^2 dx < \infty,\ \nabla \cdot \left(\rho_\mu u\right) = 0 \right\}$$
$$\oplus \operatorname{span}\left\{ \begin{pmatrix} \partial_{x_i} \rho_\mu \\ 0 \end{pmatrix},\ i = 1, 2, 3 \right\},$$

and the only generalized eigenvectors of 0 *are given by* $(0, \partial_{x_i} \nabla \tilde{\zeta})^T$ *with*

$$\mathcal{J}_\mu \mathcal{L}_\mu \begin{pmatrix} 0 \\ \partial_{x_i} \nabla \tilde{\zeta} \end{pmatrix} = \begin{pmatrix} \partial_{x_i} \rho_\mu \\ 0 \end{pmatrix}, \quad i = 1, 2, 3,$$

where $\tilde{\zeta}$ *is defined in* (3.47) *and* (3.48).

(ii) *Under radial perturbations in* $\mathbf{X}_r$*, the spectra of the linearized system* (1.8)–(1.9) *are isolated eigenvalues with finite multiplicity,*

$$\ker(\mathcal{J}_\mu \mathcal{L}_\mu) \cap \mathbf{X}_r = \operatorname{span}\left\{ (\partial_\mu \rho_\mu, 0)^T \right\},$$

and the steady state ρ_μ *is spectrally stable to radial perturbations if and only if* $n^-(D_\mu^0) = 1$ *and* $i_\mu = 1$*. Here, the self-adjoint operator* D_μ^0 *is defined in* (3.27) *and*

$$i_\mu = \begin{cases} 1 & \text{if } M'(\mu) \frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right) > 0 \text{ or } M'(\mu) = 0, \\ 0 & \text{if } M'(\mu) \frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right) < 0 \text{ or } \frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right) = 0. \end{cases} \tag{1.14}$$

Moreover, the number of growing modes is

$$n^u(\mu) = n^-\left(D_\mu^0\right) - i_\mu. \tag{1.15}$$

The index i_μ in (1.14) is well-defined, since $M'(\mu)$ and $\frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right)$ cannot be zero at the same point (Lemma 3.13). The stability of nonrotating stars under nonradial perturbations was known in the astrophysics literature as the Antonov-Lebowitz theorem [2, 25]. Theorem 1.2 implies that the spectra of the linearized Euler-Poisson equation at ρ_μ are contained in the imaginary axis except finitely many unstable (stable) eigenvalues with finite algebraic multiplicity.

THEOREM 1.3. *The operator* $\mathcal{J}_\mu \mathcal{L}_\mu$ *generates a* C^0 *group* $e^{t\mathcal{J}_\mu \mathcal{L}_\mu}$ *of bounded linear operators on* $\mathbf{X}$*, and there exists a decomposition*

$$\mathbf{X} = E^u \oplus E^c \oplus E^s,$$

with the following properties:

(i) $E^u(E^s)$ *consists only of eigenvectors corresponding to negative (positive) eigenvalues of* $\mathcal{J}_\mu \mathcal{L}_\mu$ *and*

$$\dim E^u = \dim E^s = n^-\big(D^0_\mu\big) - i_\mu. \tag{1.16}$$

(ii) *The quadratic form* $(\mathcal{L}_\mu \cdot, \cdot)_{\mathbf{X}}$ *vanishes on* $E^{u,s}$*, but is nondegenerate on* $E^u \oplus E^s$*, and*

$$E^c = \left\{ \begin{pmatrix} \sigma \\ v \end{pmatrix} \in \mathbf{X} \;\middle|\; \left\langle \mathcal{L}_\mu \begin{pmatrix} \sigma \\ v \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ v_1 \end{pmatrix} \right\rangle = 0 \;\; \forall \begin{pmatrix} \sigma_1 \\ v_1 \end{pmatrix} \in E^s \oplus E^u \right\}.$$

(iii) E^c, E^u, E^s *are invariant under* $e^{t\mathcal{J}_\mu \mathcal{L}_\mu}$*. Let*

$$\lambda_u = \min\{\lambda \mid \lambda \in \sigma(\mathcal{J}_\mu \mathcal{L}_\mu|_{E^u})\} > 0.$$

Then there exist $C_0 > 0$ *such that*

$$\begin{aligned} \left|e^{t\mathcal{J}_\mu \mathcal{L}_\mu}|_{E^s}\right| &\le C_0 e^{-\lambda_u t}, \; t \ge 0, \\ \left|e^{t\mathcal{J}_\mu \mathcal{L}_\mu}|_{E^u}\right| &\le C_0 e^{\lambda_u t}, \; t \le 0, \end{aligned} \tag{1.17}$$

$$\left|e^{t\mathcal{J}_\mu \mathcal{L}_\mu}|_{E^c}\right| \le C_0(1 + |t|), \; t \in \mathbb{R} \quad \text{if } M'(\mu) \ne 0, \tag{1.18}$$

and

$$\left|e^{t\mathcal{J}_\mu \mathcal{L}_\mu}|_{E^c}\right| \le C_0(1 + |t|)^2, \; t \in \mathbb{R} \quad \text{if } M'(\mu) = 0. \tag{1.19}$$

(iv) *Suppose that* $M'(\mu) \ne 0$*. Then*

$$\left|e^{t\mathcal{J}_\mu \mathcal{L}_\mu}\big|_{E^c \cap \mathbf{X}_r}\right| \le C \tag{1.20}$$

for some constant C*. In particular, when* $n^-\big(D^0_\mu\big) = 1$ *and* $M'(\mu)\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big) > 0$*, Lyapunov stability is true for radial perturbations in the sense that*

$$\left|e^{t\mathcal{J}_\mu \mathcal{L}_\mu}\big|_{\mathbf{X}_r}\right| \le C. \tag{1.21}$$

Above linear estimates will be useful for the future study of nonlinear dynamics, particularly, the construction of invariant (stable, unstable, and center) manifolds for the nonlinear Euler-Poisson system. The $O(|t|)$ growth in (1.18) is due to the nonradial generalized kernel associated to the translation modes given in Theorem 1.2 i). At the mass extrema points, the $O(|t|^2)$ growth in (1.19) is due to the radial generalized kernel associated to the mode of varying center density given in Theorem 1.2(ii). Lyapunov stability on the radial center space $E^c \cap \mathbf{X}_r$ (under the nondegeneracy condition $M'(\mu) \ne 0$) hints that the steady state might be nonlinearly stable on the center manifold once constructed.

Theorems 1.2–1.3 are applied to various examples of equations of state. For polytropic stars with $P(\rho) = K\rho^\gamma$ $(\gamma \in (\frac{6}{5}, 2))$, we recover the classical sharp instability criterion [26, 27] that $\gamma \in (\frac{6}{5}, \frac{4}{3})$. Even for this case, our results give some new information not found in the literature that there is only one unstable mode and Lyapunov stability is true on the center space. Next, we consider more practical white dwarf stars with $P(\rho) = Af(B^{1/3}\rho^{1/3})$, where A, B are two constants and $f(x)$ is defined in (3.70). It is proved in Corollary 3.26 that white dwarf stars $\rho_\mu(|x|)$ are linearly Lyapunov stable for any center density $\mu > 0$. For stars with a general equations of state, we prove in Corollary 3.28 that they are stable up to the first mass maximum and unstable beyond this point until the next mass extrema (a minimum). Examples for which the first mass maximum is obtained at a finite center density including the asymptotically polytropic equations of state satisfying that $P(\rho) \approx \rho^{\gamma_1}$ (for ρ large) with $\gamma_1 \in (0, \frac{6}{5})$ or $(\frac{6}{5}, \frac{4}{3})$. We refer to Corollary 3.29 for more details.

There exist huge astrophysical literature on the stability of gaseous stars (e.g., [7, 10, 23, 26, 39, 43] and references therein). We briefly mention some more recent mathematical works. Linear instability of polytropic stars was studied in [27]. Nonlinear instability for polytropic stars was proved in [20] for $\gamma \in (\frac{6}{5}, \frac{4}{3})$ and in [11] for $\gamma = \frac{4}{3}$. Nonlinear conditional stability was shown in [37] for polytropic stars with $\gamma > \frac{4}{3}$, and for white dwarf stars in [32]. In these works, stable stars were constructed by solving variational problems, for example, by minimizing the energy functional subject to the mass constraint. In a work under preparation [29], we will show that the linear stability criteria in Theorems 1.2 and 1.1 are also true on the nonlinear level.

In the rest of this introduction, we discuss the methods in our proof of Theorems 1.2 and 1.3. Since the nonrotating stars are spherically symmetric, radial and nonradial perturbations are decoupled for the linearized Euler-Poisson equation. The stability for nonradial perturbations was obtained in the astrophysical literature in 1960s [2, 25]. The radial perturbations were usually studied by the Eddington equation (3.67)–(3.68), which is a singular Sturm-Liouville problem.

In this paper, we study stability of nonrotating stars in a Hamiltonian framework. The linearized Euler-Poisson system can be written as a separable Hamiltonian form (1.12). In Section 2, we first study general linear Hamiltonian PDEs of the separable form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & B \\ -B' & 0 \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{JL} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.22}$$

where $u \in X$, $v \in Y$, and X, Y are real Hilbert spaces. The triple (L, A, B) is assumed to satisfy assumptions (G1)–(G4) in Section 2, which roughly speaking require that $B : Y^* \supset D(B) \to X$ is a densely defined closed operator, $L : X \to X^*$ is bounded and self-dual with finitely many negative modes, and $A : Y \to Y^*$ is bounded, self-dual, and nonnegative. Those assumptions qualify (1.22) as a special case of the general linear Hamiltonian systems studied in [30]. However, the special form of such systems ensures certain more specific structure in the linear dynamics, in particular a more explicit formula for unstable dimensions, all nonzero eigenvalues being semisimple, a more detailed block decomposition, and at most cubic bound of the degree of the algebraic growth in E^c.

Adapting the above framework to the linearized Euler-Poisson system (3.51) for radial perturbations, we obtain that the number of unstable modes is equal to $n^-(L_{\mu,r}|_{\overline{R(B_{\mu,r})}})$, where $L_{\mu,r}$ and $B_{\mu,r}$ are the restriction of operators L_μ and B_μ to radial functions. The quadratic form $\langle L_{\mu,r}\cdot,\cdot\rangle$ is exactly the second variation of the energy functional $E_\mu(\rho)$ defined in (3.49), and $\overline{R(B_{\mu,r})}$ is the space of radial perturbations preserving the total mass. The unstable index formula (1.15) follows from these structures. In particular, the index i_μ (defined in (1.14)) measures if the mass constraint can reduce the negative modes of $L_{\mu,r}$ by one or not. The stability condition $L_{\mu,r}|_{\overline{R(B_{\mu,r})}} \geq 0$ amounts to Chandrasekhar's variational principle [6, 8] that the stable states should be energy minimizers under the constraint of constant mass. Moreover, the separable Hamiltonian formulation yields that the Sturm-Liouville operator in (3.67) can be written in a factorized form $B'_{\mu,r}L_{\mu,r}B_{\mu,r}A_{\mu,r}$, where $A_{\mu,r} = \rho_\mu$ is a positive operator on $Y_{\mu,r}$. Compared with the traditional way of treating the singular Sturm-Liouville operator (3.67), this factorized form is more convenient to prove self-adjointness and discreteness of eigenvalues (Lemma 2.9) without relying on ODE techniques. We refer to Remark 3.24 for more details.

To get TPP from Theorem 1.2, it is reduced to find $n^-(L_{\mu,r}) = n^-(D^0_\mu)$, where D^0_μ is a second-order ODE operator from the linearization of the steady state equation. We use a continuity argument to find $n^-(D^0_\mu)$. First, for small μ, $n^-(D^0_\mu)$ is shown to be equal to the corresponding negative index for the Lane-Emden stars with polytropic index γ_0 (defined in (1.5)). For Lane-Emden stars with $\gamma \in \left(\frac{6}{5}, 2\right)$, we show that the negative index is always 1. For general equations of state, it can be shown that $n^-(D^0_\mu) = 1$ for small μ. For increasing μ, we determine $n^-(D^0_\mu)$ by keeping track of its changes. A key observation is that D^0_μ has a one-dimensional kernel only at critical points of the mass–radius ratio $M(\mu)/R_\mu$. Therefore, $n^-(D^0_\mu)$ can only change at critical points of $M(\mu)/R_\mu$. The jump of $n^-(D^0_\mu)$ at such critical points is shown to be exactly the jump of i_μ. This not only gives us a way to find $n^-(D^0_\mu)$ for any $\mu > 0$, but also implies that the number of unstable modes $n^u(\mu)$ does not change when crossing a critical point of $M(\mu)/R_\mu$. At extrema points of total mass $M(\mu)$, $n^-(D^0_\mu)$ remains unchanged but i_μ must change from 0 to 1 (or from 1 to 0) if the bending of the mass–radius curve is counterclockwise (or clockwise). This proves TPP that the number of unstable modes can only change at extrema mass and also gives an explicit way to determine $n^u(\mu)$ from the mass–radius curve. The exponential trichotomy estimates in Theorem 1.3 follow from the general Theorems 2.3 and 2.6.

The general framework of separable Hamiltonian PDEs in Section 2 is flexible and can be used for many other problems. Hamiltonian systems in the separable form of (1.22) appear in many other problems, which include nonlinear Klein-Gordon equations, nonlinear Schrödinger equations, and 3D Vlasov-Maxwell systems for collisionless plasmas. This framework was also used in the recent study of stability of neutron stars modeled by the Euler-Einstein equation [16] and relativistic globular clusters modeled by Vlasov-Einstein equation [17]. In particular, for the Euler-Einstein equation, a similar TPP can be proved [16] for relativistic stars as in Theorem 2.6. More recently the stability of rotating stars of the Euler-Poisson system was studied [28] by the separable Hamiltonian approach.

This paper is organized as follows. Section 2 is about the abstract theory for the separable linear Hamiltonian PDEs. Section 3 is about the stability of nonrotating stars and is divided into several subsections. Section 3.1 is for the existence of nonrotating stars. In Section 3.2 the Hamiltonian structures of linearized Euler-Poisson is studied. Section 3.3 finds the negative index $n^-(D^0_\mu)$ for all $\mu > 0$. In Section 3.4, we derive the equations for nonradial perturbations and prove the Antonov-Lebowitz theorem. In Section 3.5, TPP is proved for radial perturbations. In Section 3.6, more explicit stability criteria are given for several classes of equations of state. In the appendix, we outline the Lagrangian formulation of the Euler-Poisson system (1.1)–(1.3) and its linearization.

2 Separable Linear Hamiltonian PDE

Let X and Y be real Hilbert spaces. We make the following assumptions on (L, A, B) in the Hamiltonian PDE (1.22):

(**G1**) The operator $B : Y^* \supset D(B) \to X$ and its dual operator $B' : X^* \supset D(B') \to Y$ are densely defined and closed (and thus $B'' = B$).

(**G2**) The operator $A : Y \to Y^*$ is bounded and self-dual (i.e., $A' = A$ and thus $\langle Au, v\rangle$ is a bounded symmetric bilinear form on Y). Moreover, there exist $\delta > 0$ and a closed subspace $Y_+ \subset Y$ such that

$$Y = \ker A \oplus Y_+, \quad \langle Au, u\rangle \geq \delta \|u\|_Y^2 \ \forall u \in Y_+.$$

(**G3**) The operator $L : X \to X^*$ is bounded and self-dual (i.e., $L' = L$) and there exists a decomposition of X into the direct sum of three closed subspaces

$$X = X_- \oplus \ker L \oplus X_+, \quad n^-(L) \triangleq \dim X_- < \infty \tag{2.1}$$

satisfying

(**G3.a**) $\langle Lu, u\rangle < 0$ for all $u \in X_- \backslash \{0\}$;

(**G3.b**) there exists $\delta > 0$ such that

$$\langle Lu, u\rangle \geq \delta \|u\|^2 \quad \text{for any } u \in X_+.$$

(**G4**) The above $X_\pm$ and Y_+ satisfy

$$\ker(i_{X_+ \oplus X_-})' \subset D(B'), \quad \ker(i_{Y_+})' \subset D(B).$$

Remark 2.1. We adopt the notations as in [46]. For a densely defined linear operator $A : X \to Y$ between Hilbert spaces X, Y, we use $A' : Y^* \to X^*$ and $A^* : Y \to X$ for the dual and adjoint operators of A, respectively. The operators A' and A^* are related by

$$A^* = I_X A' I_Y^{-1},$$

where $I_X : X^* \to X$ and $I_Y : Y^* \to Y$ are the isomorphisms defined by the Riesz representation theorem. Given a closed subspace X_1 of a Hilbert space X, $i_{X_1} : X_1 \to X$ denotes the embedding and $(i_{X_1})' : X^* \to X_1^*$ the dual operator with

$$\ker(i_{X_1})' = \{ f \in X^* \mid \langle f, x \rangle = 0, \ \forall x \in X_1 \}.$$

Remark 2.2. The assumption (**G4**) for L (or for A) is satisfied automatically if $\dim \ker L < \infty$ (or $\dim \ker A < \infty$). See Remark 2.3 in [30] for details.

In this paper, the above abstract framework will be applied to the linearized Euler-Poisson system to be studied in detail, where A is actually positive definite. The more general semipositive definiteness assumption on A is partially motivated by the focusing nonlinear Schrödinger equation (NLS) with energy subcritical or critical power nonlinearity,

$$\text{(NLS)} \qquad i u_t = \Delta u + |u|^p u, \quad u : \mathbb{R}^{1+d} \to \mathbb{C} = \mathbb{R}^2, \ p \in \left(1, \frac{4}{d-2}\right],$$

with the Hamiltonian

$$H(u) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u|^2 - \frac{1}{p+2}|u|^p \, dx.$$

There exist standing waves and steady waves in the subcritical and critical cases, respectively,

$$U_\omega(t, x) = e^{-i\omega t}\phi_\omega(x), \quad -\Delta\phi_\omega + \omega\phi_\omega - \phi_\omega^{p+1} = 0.$$

For ground states, $\phi_\omega(x)$ is always radially symmetric and positive, where $\omega > 0$ if $p < \frac{4}{d-2}$ and $\omega = 0$ if $p = \frac{4}{d-2}$. The linearization of (NLS) in the rotation frame $u(t, x) = e^{-i\omega t} v(t, x)$ at $v_\omega = \phi_\omega$ with v viewed as a vector in $\mathbb{R}^2$ takes the form of (1.22) where

$$B = I, \quad L = -\Delta + \omega - (p+1)\phi_\omega^p, \quad A = -\Delta + \omega - \phi_\omega^p,$$

on the energy space H^1 in the subcritical case and $\dot{H}^1$ in the critical case. Clearly $\phi_\omega > 0$ spans $\ker A$ and thus $A \geq 0$. Viewing L and A as perturbations to $-\Delta + \omega$, a simple argument based on the compactness shows (**G1-G4**) are satisfied.

Equation (1.22) is of the Hamiltonian form

$$\partial_t w = \mathbf{JL} w, \tag{2.2}$$

where $\mathbf{u} = (u, v) \in \mathbf{X} = X \times Y$. Here, the operators

$$\mathbf{J} = \begin{pmatrix} 0 & B \\ -B' & 0 \end{pmatrix} : \mathbf{X}^* \supset D(\mathbf{J}) \to \mathbf{X}$$

and

$$\mathbf{L} = \begin{pmatrix} L & 0 \\ 0 & A \end{pmatrix} : \mathbf{X} \to \mathbf{X}^*.$$

Under assumptions (**G1-G4**), we can check that:

(i) The operator $\mathbf{J}$ is anti-self-dual in the sense that

$$D(\mathbf{J}) = D(B') \times D(B)$$

is dense in $\mathbf{X}^*$ and $\mathbf{J}' = -\mathbf{J}$.

(ii) The operator $\mathbf{L}$ is bounded and self-dual (i.e., $\mathbf{L}' = \mathbf{L}$) such that $\langle \mathbf{Lu}, \mathbf{v} \rangle$ is a bounded symmetric bilinear form on $\mathbf{X}$. For any $\mathbf{u} = (u, v) \in \mathbf{X}$, note that

$$\langle \mathbf{Lu}, \mathbf{u} \rangle = \langle Lu, u \rangle + \langle Av, v \rangle, \quad \ker \mathbf{L} = \ker L \times \ker A.$$

Let

$$\mathbf{X}_- = X_- \times \{0\}, \quad \mathbf{X}_+ = X_+ \times Y_+, \tag{2.3}$$

where $X_\pm$ and Y_+ are as in (**G2**) and (**G3**). Then we have the decomposition

$$\mathbf{X} = \mathbf{X}_- \oplus \ker \mathbf{L} \oplus \mathbf{X}_+, \quad \dim \mathbf{X}_- = n^-(\mathbf{L}) = n^-(L),$$

satisfying: $\langle \mathbf{Lu}, \mathbf{u} \rangle < 0$ for all $\mathbf{u} \in \mathbf{X}_- \backslash \{0\}$ and there exists $\delta_0 > 0$ such that

$$\langle \mathbf{Lu}, \mathbf{u} \rangle \geq \delta_0 \|\mathbf{u}\|^2 = \delta_0 \left(\|u\|_X^2 + \|v\|_Y^2 \right) \quad \text{for any } \mathbf{u} \in \mathbf{X}_+.$$

(iii) Assumption (**G4**) implies

$$\begin{aligned} \ker(i_{\mathbf{X}_+ \oplus \mathbf{X}_-})' &= \{\mathbf{f} \in \mathbf{X}^* \mid \langle \mathbf{f}, \mathbf{u} \rangle = 0, \ \forall \mathbf{u} \in \mathbf{X}_- \oplus \mathbf{X}_+\} \\ &= \ker(i_{X_+ \oplus X_-})' \times \ker(i_{Y_+})' \subset D(\mathbf{J}). \end{aligned}$$

Therefore, $(\mathbf{X}, \mathbf{J}, \mathbf{L})$ satisfies the assumptions (**H1–H3**) in [30], and we can apply the general theory for linear Hamiltonian PDE [30] to study the solutions of (1.22). In particular, the semigroup $e^{t\mathbf{JL}}$ is well-defined. Corollary 12.1 in [30] also implies

$$\mathbf{LJ} = (\mathbf{JL})',\ BA,\ (BA)' = AB',\ B'L,\ (B'L)' = LB \text{ densely defined, closed.} \tag{2.4}$$

Moreover, by using the separable nature of (1.22), we obtain more precise estimates on the instability index and the growth in the center space. Our main theorem for (1.22) is the following, whose proof would be self-contained except for a few technical lemmas in [30] that are cited. We adopt the same notations as in [30]. In particular, for a closed subspace $X_1 \subset X$, we denote

$$L_{X_1} = i'_{X_1} L i_{X_1} : X_1 \to X_1^* \Longrightarrow \langle L_{X_1} u_1, u_2 \rangle = \langle L u_1, u_2 \rangle \quad \forall u_1, u_2 \in X_1. \tag{2.5}$$

THEOREM 2.3. *Assume (***G1–G4***) for (1.22).* The operator $\mathbf{JL}$ generates a C^0 group $e^{t\mathbf{JL}}$ of bounded linear operators on $\mathbf{X}$, and there exists a decomposition

$$\mathbf{X} = E^u \oplus E^c \oplus E^s$$

of closed subspaces $E^{u,s,c}$ with the following properties:

(i) E^c, E^u, E^s are invariant under $e^{t\mathbf{JL}}$.

(ii) $E^u(E^s)$ only consists of eigenvectors corresponding to negative (positive) eigenvalues of $\mathbf{JL}$ and

$$\dim E^u = \dim E^s = n^-\big(L|_{\overline{R(BA)}}\big), \tag{2.6}$$

where $n^-(L|_{\overline{R(BA)}})$ denotes the number of negative modes of $L|_{\overline{R(BA)}}$ as defined in (2.1). If $n^-(L|_{\overline{R(BA)}}) > 0$, then there exists $M > 0$ such that

$$|e^{t\mathbf{JL}}|_{E^s}| \le Me^{-\lambda_u t},\ t \ge 0; \quad \big|e^{t\mathbf{JL}}|_{E^u}\big| \le Me^{\lambda_u t},\ t \le 0, \tag{2.7}$$

where $\lambda_u = \min\{\lambda \mid \lambda \in \sigma(\mathbf{JL}|_{E^u})\} > 0$.

(iii) The quadratic form $\langle \mathbf{L}\,\cdot, \cdot\rangle$ vanishes on $E^{u,s}$, i.e., $\langle \mathbf{Lu}, \mathbf{u}\rangle = 0$ for all $\mathbf{u} \in E^{u,s}$ but is nondegenerate on $E^u \oplus E^s$, and

$$E^c = \{\mathbf{u} \in \mathbf{X} \mid \langle \mathbf{Lu}, \mathbf{v}\rangle = 0,\ \forall \mathbf{v} \in E^s \oplus E^u\}. \tag{2.8}$$

(iv) There exist closed subspaces $\mathbf{X}_j$, $j = 0, \dots, 5$, such that

$$E^c = \ker L \oplus \ker A \oplus (\oplus_{j=1}^5 \mathbf{X}_j), \quad \dim \mathbf{X}_1 = \dim \mathbf{X}_5 \le n^-(L) - \dim E^u,$$

$$\mathbf{X}_1, \mathbf{X}_4, \mathbf{X}_5 \subset X \times \{0\}, \quad \mathbf{X}_2 \subset \{0\} \times Y.$$

In this decomposition, $\mathbf{JL}|_{E^c}$ and the quadratic form $\mathbf{L}_{E^c}$ take the block form

$$\mathbf{L}_{E^c} \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{L}_{15} \\ 0 & 0 & 0 & \mathbf{L}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{L}_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{L}_4 & 0 \\ 0 & 0 & \mathbf{L}_{51} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{JL}|_{E^c} \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & T_{X2} & T_{X3} & 0 & 0 \\ 0 & 0 & T_{Y1} & 0 & T_{Y3} & T_{Y4} & T_{Y5} \\ 0 & 0 & 0 & T_{12} & T_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & T_{25} \\ 0 & 0 & 0 & 0 & T_3 & 0 & T_{35} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

All the nontrivial blocks of $\mathbf{L}_{E^c}$ are nondegenerate and

$$\mathbf{L}_2 \ge \epsilon, \quad \mathbf{L}_3 \ge \epsilon,$$

for some $\epsilon > 0$. All the blocks of $\mathbf{JL}$ are bounded except T_3, which is anti-self-adjoint with respect to the equivalent inner product $\langle \mathbf{L}_3\,\cdot, \cdot\rangle$ satisfying $\ker T_3 = \{0\}$. Consequently, there exists $M > 0$ such that

$$\big|e^{t\mathbf{JL}}|_{E^c}\big| \le M(1 + |t|)^3,\ t \in \mathbb{R}. \tag{2.9}$$

(v) Denote Z to be the space $D(BA)$ with the graph norm

$$\|y\|_Z = \|y\|_Y + \|BAy\|_X.$$

If the embedding $Z \hookrightarrow Y$ is compact, then the spectra of T_3 are nonzero, isolated with finite multiplicity, and have no accumulating point except for $+\infty$. Moreover, the eigenfunctions of T_3 form an orthonormal basis of $\mathbf{X}_3$ with respect to $\langle \mathbf{L}_3 \cdot, \cdot \rangle$. Consequently, the spectra $\sigma(\mathbf{JL})\backslash\{0\}$ are isolated with finite multiplicity, and have no accumulating point except for $+\infty$.

Remark 2.4. Here the nondegeneracy of a bounded symmetric quadratic form $B(u,v) : Z \otimes Z \to \mathbb{R}$ on a real Banach space Z is defined as the induced bounded linear operator $v \longrightarrow f = B(\cdot, v) \in Z^*$ is an isomorphism from X to X^*.

The above theorem implies that the solutions of (1.22) are spectrally stable (i.e., nonexistence of exponentially growing solution) if and only if $L|_{\overline{R(BA)}} \geq 0$. Moreover, $n^-(L|_{\overline{R(BA)}})$ gives the dimension of the subspaces of exponentially growing solutions. The exponential trichotomy estimates (2.7)–(2.9) are important in the study of nonlinear dynamics near an unstable steady state for which the linearized equation (1.22) is derived. If the spaces $E^{u,s}$ have higher regularity, then the exponential trichotomy can be lifted to more regular spaces. We refer to theorem 2.2 in [30] for more precise statements.

Compared to [30], the separable Hamiltonian form of (1.22) yields a simpler block form. In particular, the anti-self-adjointness of T_3 ensures the semisimplicity of any eigenvalue $\lambda \in i\mathbb{R}\backslash\{0\}$ and the nondegeneracy of $\mathbf{L}$ restricted to its subspace of generalized eigenvectors. This is not true for general linear Hamiltonian systems; see examples in [30]. The separable Hamiltonian form also implies the order $O(|t|^3)$ of the growth in the center direction, which is better than the general cases in [30]. These properties hold essentially due to the second-order equation (2.17) satisfied by v.

Remark 2.5. As the only nontrivial block T_3 in the block decomposition of $\mathbf{JL}$ is anti-self-adjoint with respect to an equivalent norm, it is clear that all the possible algebraic growth of $e^{t\mathbf{JL}}$ must be associated to the possible zero eigenvalue. The second-order equation (2.17) allows at most $O(|t|)$ growth as in the case of wave equations. So it is natural to guess that the solutions of the first-order system (1.22) may also grow no faster than $O(|t|)$. However, the possible degeneracy of B and A indeed creates more growth and the above $O(|t|^3)$ growth is optimal. Consider the following example:

$$X = \mathbb{R}^2, \quad Y = \mathbb{R}^3, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

One may compute

$$\mathbf{JL} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\mathbf{JL})^2 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(\mathbf{JL})^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\mathbf{JL})^4 = 0.$$

Therefore $e^{t\mathbf{JL}}$ exhibits $O(|t|^3)$ growth.

In the following theorem, we given some conditions on (L, A, B) that yield a better growth estimate of $e^{t\mathbf{JL}}$ on the center subspace E^c.

THEOREM 2.6. *Assume* (**G1-G4**) *for* (1.22). *The following hold under additional assumptions*:

(i) *If A is injective on $\overline{R(B'LBA)}$, then $|e^{t\mathbf{JL}}|_{E^c}| \leq M(1+t^2)$ for some $M > 0$.*

(ii) *If $\overline{R(BA)} = X$, then $|e^{t\mathbf{JL}}|_{E^c}| \leq M(1+|t|)$ for some $M > 0$.*

(iii) *Suppose $\langle L\cdot,\cdot\rangle$ and $\langle A\cdot,\cdot\rangle$ are nondegenerate on $\overline{R(B)}$ and $\overline{R(B')}$, respectively. Then $|e^{t\mathbf{JL}}|_{E^c}| \leq M$ for some $M > 0$. Namely, there is Lyapunov stability on the center space E^c.*

Remark 2.7. Motivated by the second-order equation (2.17), when $L|_{\overline{R(BA)}}$ has a negative mode, it is tempting to find the most unstable eigenvalue $\lambda_0 > 0$ of (1.22) satisfying $B'LBAv = -\lambda_0^2 v$ by solving the variational problem

$$-\lambda_0^2 = \min_{\langle Av,v\rangle=1, v\in D(A)} \langle B'LBAv, Av\rangle. \tag{2.10}$$

However, in many applications, particularly to kinetic models such as Vlasov-Maxwell and Vlasov-Einstein systems, it is difficult to solve the variational problem (2.10) directly due to the lack of compactness. In Theorem 2.3, the existence of unstable eigenvalues follows from the self-adjointness of the operator $B'LBA$ and the assumption $n^-(L) < \infty$.

The proof of Theorem 2.3 will be split into several lemmas and propositions. We start with a general functional analysis result that might be of independent interest.

PROPOSITION 2.8. *Let X, Y be Hilbert spaces, $L : X \to X$ a bounded self-adjoint operator, and $A: Y \supset D(A) \to X$ a densely defined and closed operator. In addition, assume that:*

(1) *The adjoint operator $A^* : X \supset D(A^*) \to Y$ is also densely defined.*

(2) $\exists\ \delta > 0$ *and a closed subspace* $X_+ \subset X$ *such that* $(Lx, x) \geq \delta \|x\|^2$ $\forall x \in X_+$ *and* $X_+^{\perp} \subset D(A^*)$.

Then: (i) *the operator* $A^* L A$ *is self-adjoint on* Y *with domain*

$$D(A^* L A) \subset D(A).$$

(ii) *Denote* Z *to be the space* $D(A)$ *equipped with the graph norm*

$$\|y\|_Z = \|y\|_Y + \|Ay\|_X.$$

If the embedding $Z \hookrightarrow Y$ *is compact, then the spectra of* $A^* L A$ *are purely discrete, and have no accumulating point except for* $+\infty$. *Moreover, the eigenfunctions of* $A^* L A$ *form a basis of* Y.

PROOF. Let

$$X_1 = \{x \in X \mid \langle Lx, x' \rangle = 0,\ \forall x' \in X_+\}.$$

The uniform positivity of L on X_+ and lemma 12.2 in [30] imply

$$X = X_+ \oplus X_1, \quad P_1^* L P_+ = P_+^* L P_1 = 0,$$

where $P_{+,1}$ are the associated projections. Therefore,

$$L = P_+^* L P_+ + P_1^* L P_1 \triangleq L_+ - L_1$$

with symmetric bounded $L_{+,1}$ and $L_+ \geq 0$. Since $R(P_1^*) = X_+^{\perp} \subset D(A^*)$, the closed graph theorem implies that $A^* P_1^*$ is bounded. Therefore, $P_1 A$ has a continuous extension $(A^* P_1^*)^* = (P_1 A)^{**}$; i.e., $P_1 A$ is bounded. Thus $P_+ A$ is closed and densely defined. Let $S_+ : X \to X$ be a bounded symmetric linear operator such that

$$S_+^* S_+ = S_+^2 = L_+, \quad S_+ \geq 0.$$

Moreover, for any $x \in X_+$,

$$\|S_+ x\|_X^2 = (L_+ x, x) = (Lx, x) \geq \delta \|x\|_X^2,$$

which implies that

$$\|S_+ x\|_X \geq \sqrt{\delta} \|x\|_X \quad \forall x \in X_+. \tag{2.11}$$

This lower bound of S_+ implies that $T_+ \triangleq S_+ P_+ A$ is also closed with the dense domain $D(T_+) = D(A)$ and thus $T_+^* T_+$ is self-adjoint. We note that

$$\begin{aligned} A^* L A &= A^* P_+^* L_+ P_+ A - A^* P_1^* L_1 P_1 A \\ &= (A^* P_+^* S_+)(S_+ P_+ A) - A^* P_1^* L_1 P_1 A \triangleq T_+^* T_+ - B_1. \end{aligned} \tag{2.12}$$

Here, B_1 is bounded and symmetric. Therefore, by the Kato-Rellich theorem $A^* L A$ is self-adjoint with

$$D\big(A^* L A\big) \subset D(T_+) = D(A).$$

By theorem 4.2.9 in [13], to prove conclusions in (ii), it suffices to show that the embedding $Z_1 \hookrightarrow Y$ is compact. Here, the space Z_1 is $D(T_+) = D(A)$ with the graph norm

$$\|y\|_{Z_1} = \|y\|_Y + \|T_+ y\|_X.$$

We show that $\|\cdot\|_{Z_1}$ and $\|\cdot\|_Z$ are equivalent. Indeed, since A and T_+ are closed with the same domain, $A : Z_1 \to X$ and $T_+ : Z \to X$ are also apparently closed and thus bounded, which immediately implies the equivalence of $\|\cdot\|_{Z_1}$ and $\|\cdot\|_Z$. □

In the above proposition, we can allow $n^-(L) = \infty$, but the condition $X_+^\perp \subset D(A^*)$ need to be verified. The next lemma shows that this condition is implied by our assumptions (**G1–G4**).

LEMMA 2.9. *Suppose the operators L, B, B', A satisfy assumptions* (**G1–G4**). *Then:*

(1) $\widetilde{L} = AB'LBA : Y \supset D(\widetilde{L}) \to Y^*$ *and* $\widetilde{A} = LBAB'L : X \supset D(\widetilde{A}) \to X^*$ *are self-dual, namely* $\widetilde{L}' = \widetilde{L}$ *and* $\widetilde{A}' = \widetilde{A}$.

(2) *In addition to* (**G1–G4**), *assume* $\ker A = \{0\}$; *then* $\widetilde{\mathbb{L}} = B'LBA$ *is self-adjoint on* $(Y, [\cdot,\cdot])$ *with the equivalent inner product* $[\cdot,\cdot] = \langle A\cdot,\cdot\rangle$.

(3) *Denote Z to be the space $D(BA)$ with the graph norm*

$$\|y\|_Z = \|y\|_Y + \|BAy\|_X.$$

If the embedding $Z \hookrightarrow Y$ is compact and $\ker A = \{0\}$, *then the spectra of $\widetilde{\mathbb{L}}$ are purely discrete with finite multiplicity, and have no accumulating point except for $+\infty$. Moreover, the eigenfunctions of $\widetilde{\mathbb{L}}$ form a basis of Y.*

PROOF. Recall that $I_X : X^* \to X$, $I_Y : Y^* \to Y$ are isomorphisms defined by the Riesz representation theorem. Define the operators

$$\mathbb{A} = BA : Y \supset D(\mathbb{A}) \to X, \quad L_1 = I_X L : X \to X.$$

The adjoint operators are

$$L_1^* = L_1, \quad \mathbb{A}^* = I_Y AB' I_X^{-1}.$$

According to (2.4), $\mathbb{A}^*$ is densely defined and closed.

Since $(X_+ \oplus X_-)^\perp \subset X_+^\perp$ is a closed subspace of codimension that is equal to $\dim X_- < \infty$, we have

$$\dim W = \dim X_- < \infty.$$

where

$$W = X_+^\perp \cap (X_+ \oplus X_-), \quad X_+^\perp = W \oplus (X_+ \oplus X_-)^\perp.$$

Assumption (**G4**) implies that $D(\mathbb{A}^*) \cap (X_+ \oplus X_-)$ is dense in $X_+ \oplus X_-$. Approximate W by $\widetilde{W} \subset D(\mathbb{A}^*) \cap (X_+ \oplus X_-)$ such that $\dim W = \dim \widetilde{W}$, which is possible since $\dim W < \infty$. Let

$$\widetilde{X}_+ = \{x \in X_+ \oplus X_- \mid (x, y) = 0, \ \forall y \in \widetilde{W}\}.$$

The quadratic form $\langle L\cdot,\cdot\rangle$ is uniformly positive definite on the approximation $\widetilde{X}_+$ of X_+ and

$$\widetilde{X}_+^{\perp} = (X_+ \oplus X_-)^{\perp} \oplus \widetilde{W} \subset D(\mathbb{A}^*).$$

Therefore, all conditions in Proposition 2.8 are satisfied by $\widetilde{X}_+$, L_1, and $\mathbb{A}$ and thus $\mathbb{A}^* L_1 \mathbb{A} = I_Y AB'LBA$ are self-adjoint. This implies that $\widetilde{L} = AB'LBA$ satisfies $\widetilde{L}' = \widetilde{L}$. It follows from the same argument that $\widetilde{A}' = \widetilde{A}$.

Statement (ii) and (iii) are direct corollaries of (i) and Proposition 2.8. □

We shall start the proof of Theorem 2.3 with several steps of decomposition of X and Y.

LEMMA 2.10. *Assume* (**G1–G4**). Suppose $X_{1,2}$ are closed subspaces of X and $Y_{1,2}$ are closed subspaces of Y such that $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$. Let $P_{1,2} : X \to X_{12}$ and $Q_{1,2} : Y \to Y_{1,2}$ be the associated projections and, for $j,k = 1,2$,

$$L_j = (i_{X_j})' L i_{X_j}, \quad A_j = (i_{Y_j})' A i_{Y_j}, \quad B_{jk} = P_j B Q_k'^{jk} = Q_j B' P_k',$$

$$\mathbf{X}_1 = X_1 \times Y_1, \quad \mathbf{L}_1 = \begin{pmatrix} L_1 & 0 \\ 0 & A_1 \end{pmatrix}, \quad \mathbf{J}_1 = \begin{pmatrix} 0 & B_{11} \\ -B^{11} & 0 \end{pmatrix},$$

$$\mathbf{X}_2 = X_2 \times Y_2, \quad \mathbf{L}_2 = \begin{pmatrix} L_2 & 0 \\ 0 & A_2 \end{pmatrix}, \quad \mathbf{J}_2 = \begin{pmatrix} 0 & B_{22} \\ -B^{22} & 0 \end{pmatrix}$$

In addition, we assume

$$\langle Lx_1, x_2\rangle = 0 \ \ \forall x_1 \in X_1,\ x_2 \in X_2; \quad \langle Ay_1, y_2\rangle = 0 \ \ \forall y_1 \in Y_1,\ y_2 \in Y_2;$$

$$P_1'(X_1^*) \subset D(B'), \quad Q_1'(Y_1^*) \subset D(B).$$

Then we have

(1) In this decomposition, **JL** takes the form

$$\mathbf{JL} \longleftrightarrow \begin{pmatrix} \mathbf{J}_1\mathbf{L}_1 & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{J}_2\mathbf{L}_2 \end{pmatrix},$$

where

$$\mathbf{T}_{12} = \begin{pmatrix} 0 & B_{12}A_2 \\ -B^{12}L_2 & 0 \end{pmatrix}, \quad \mathbf{T}_{21} = \begin{pmatrix} 0 & B_{21}A_1 \\ -B^{21}L_1 & 0 \end{pmatrix}.$$

(2) We have that B_{22} and B^{22} are densely defined closed operators and B_{jk} and B^{jk}, $(j,k) \neq (2,2)$, and thus $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, and $\mathbf{J}_1\mathbf{L}_1$ are all bounded. Here, we abuse the notations slightly in using B_{jk} and B^{jk} for $(j,k) \neq (2,2)$, to also denote their continuous extensions.

(3) $B^{jk} = B_{kj}'$ for all $j,k = 1,2$.

(4) (L_1, A_1, B_{11}) and (L_2, A_2, B_{22}) satisfy (**G1–G4**) and

$$n^-(L) = n^-(L_1) + n^-(L_2),\ \ker L = \ker L_1 \oplus \ker L_2,\ \ker A = \ker A_1 \oplus \ker A_2.$$

PROOF. The assumptions $P_1'(X_1^*) \subset D(B')$ and $Q_1'(Y_1^*) \subset D(B)$ imply that $B'P_1'$ and BQ_1' are closed operators defined on Hilbert spaces X_1^* and Y_1^*. The Closed Graph Theorem yields that $B'P_1'$ and BQ_1' are bounded operators. Therefore, $P_1 B$ and $Q_1 B'$ are also both bounded as they have continuous extensions $(P_1 B)'' = (B'P_1')'$ and $(Q_1 B')'' = (BQ_1')'$. Consequently the second statement, as well as the closedness of $P_2 B$ and $B'P_2'$ with dense domains, follows.

For $(j,k) \neq (2,2)$, it is easy to verify $B^{jk} = B'_{jk}$ as they are compositions of bounded operators. To show $B^{22} = B_{22}$, we notice that the closedness and the density of the domains of $P_2 B$ and $B'P_2' = (P_2 B)'$ imply

$$P_2 B = (P_2 B)'' = (B'P_2')',$$
$$(B^{22})' = (Q_2 B' P_2')' = (B'P_2')' Q_2' = P_2 B Q_2' = B_{22}.$$

The closedness of B_{22} and B^{22} again yields $B^{22} = (B^{22})'' = B'_{22}$. It completes the proof of the third statement.

The L-orthogonality of the splitting $X_1 \oplus X_2$ and the A-orthogonality of $Y = Y_1 \oplus Y_2$ yield block diagonal forms of L and A in these splittings. The block form of **JL** follows from straightforward calculations.

It has been proved in the above that B_{11} and B_{22} satisfy (**G1**), while (**G2**) for A_1 and A_2 and (**G3**) for L_1 and L_2 are proved in lemma 12.3 in [30]. Apparently (**G4**) is satisfied by (L_1, A_1, B_{11}) as B_{11} is a bounded operator. Finally, (**G4**) for (L_2, A_2, B_{22}) also follows directly from the proof of lemma 12.3 in [30]. □

Remark 2.11. Even though the framework in [30] is slightly different, those properties of J and L used in the proof of Lemma 12.3 therein are all satisfied by L_2, A_2, and B_{22} here. Therefore, the same proof works to show that (**G4**) is satisfied by (L_2, A_2, B_{22}).

The following three lemmas focus on decomposing a subspace of the center subspace.

LEMMA 2.12. *Assume* (**G1–G3**) *and that* L *is nondegenerate* (*in the sense of Remark* 2.4). *Let* $\widetilde{X} \subset X$ *be a closed subspace such that* $\ker(i_{\widetilde{X}})' \subset D(B')$; *then there exist closed subspaces* X_j, $j = 1,2,3,4$, *such that*

$$\widetilde{X} = X_1 \oplus X_2, \quad \widetilde{X}^{\perp_L} \triangleq \{x \in X \mid \langle Lx, \tilde{x}\rangle = 0 \ \forall \tilde{x} \in \widetilde{X}\} = X_1 \oplus X_3,$$
$$X = \oplus_{j=1}^4 X_j, \quad n_1 \triangleq \dim X_1 = \dim X_4 \leq n^-(L).$$

Moreover, let P_j, $j = 1,2,3,4$, *be the associated projections and it holds that*

$$P_1'(X_1^*) \oplus P_3'(X_3^*) \oplus P_4'(X_4^*) = \ker(i_{X_2})' \subset D(B').$$

In this decomposition, the quadratic form L *takes the block form*

$$L \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & L_{14} \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ L_{41} & 0 & 0 & 0 \end{pmatrix}, \quad L_{jk} = (i'_{X_j}) L i_{X_k} : X_k \to X_j^*, \quad L_j = L_{jj},$$

with $L_{14} = L_{41}'$, L_2, and L_3 all nondegenerate.

As stated in Remark 2.2, assumption (**G4**) holds for nondegenerate L.

PROOF. Let

$$X_1 = \widetilde{X} \cap \widetilde{X}^{\perp_L} = \big(\widetilde{X} + \widetilde{X}^{\perp_L}\big)^{\perp_L},$$

where the nondegeneracy of L was used in the second equality. Since $\langle Lx, x\rangle = 0$ for all $x \in X_1 \subset X$,

$$n_1 = \dim X_1 = \operatorname{codim}\big(\widetilde{X} + \widetilde{X}^{\perp_L}\big) \le n^-(L)$$

is a direct consequence of the nondegeneracy assumption of L and theorem 5.1 in [30]. According to the density of $D(B')$, there exist $f_j \in D(B')$, $j = 1, \dots, n_1$, such that $(i_{X_1})' f_j \in X_1^*$, $j = 1, \dots, n_1$, form a basis of X_1^*. Let $x_j \in X_1$, $j = 1, \dots, n_1$, be the basis of X_1 dual to $\{(i_{X_1})' f_j\}_{j=1}^{n_1}$, namely, $\langle f_j, x_k\rangle = \delta_{jk}$. Let

$$X_4 = \operatorname{span}\left\{L^{-1} f_j - \frac{1}{2}\sum_{k=1}^{n_1} \langle f_j, L^{-1} f_k\rangle x_k,\ j = 1, \dots, n_1\right\}.$$

It is easy to verify that

$$\dim X_4 = n_1, \quad \langle Lx, \tilde{x}\rangle = 0,\ \forall x, \tilde{x} \in X_4,$$

and $L_{14} = L_{41}'$ is nondegenerate. Let

$$X_2 = \big\{x \in \widetilde{X} \mid \langle f_j, x\rangle = 0,\ j = 1, \dots, n_1\big\},$$

and

$$X_3 = \big\{x \in \widetilde{X}^{\perp_L} \mid \langle f_j, x\rangle = 0,\ j = 1, \dots, n_1\big\}.$$

The direct sum relations and the block form of L stated in the lemma follow straightforwardly. The nondegeneracy of L and the definitions of X_2 and X_3 imply that L_{X_2} and L_{X_3} (as defined in (2.5)) are injective. Therefore, lemma 12.2 in [30] yields the nondegeneracy of $L_2 = L_{X_2}$ and $L_3 = L_{X_3}$. Finally, observing

$$L(X_1) = P_4'(X_4^*) \subset P_3'(X_3^*) \oplus P_4'(X_4^*) = \ker(i_{\widetilde{X}})' \subset D(B') \tag{2.13}$$

and

$$P_1'(X_1^*) \oplus P_4'(X_4^*) = \ker(i_{X_2 \oplus X_3})' = \operatorname{span}\{f_1, \dots, f_{n_1}\} + L(X_1) \subset D(B'),$$

the proof of the lemma is complete. □

LEMMA 2.13. *In addition to* (**G1–G4**)*, assume* $\ker A = \{0\}$ *and* $n^-(L|_{\overline{R(B)}}) = 0$, *the latter of which implies* $L|_{\overline{R(B)}} \ge 0$ *and* $A \ge \delta > 0$. *Let* $Y_1 = \ker \widetilde{\mathbb{L}}$ *and*

$$Y_2 = Y_1^{\perp_A} = \{y \in Y \mid \langle Ay, \tilde{y}\rangle = 0,\ \forall \tilde{y} \in Y_1\},$$

where $\widetilde{\mathbb{L}} = B' L B A$ *is defined as in Lemma* 2.9. *Then it holds that*

$$Y_1 = \ker(L_{\overline{R(B)}} BA), \quad Y_2 = \overline{R(\widetilde{\mathbb{L}})} = \overline{R(B' L_{\overline{R(B)}})}, \quad Y = Y_1 \oplus Y_2.$$

In this decomposition, the quadratic form A takes the block form

$$A \longleftrightarrow \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad A_j = (i_{Y_j})' A i_{Y_j} : Y_j \to Y_j^*.$$

Here $L_{\overline{R(B)}} : \overline{R(B)} \to \big(\overline{R(B)}\big)^*$ is defined as in (2.5). In the following, we also view B as a closed operator from Y^* to $\overline{R(B)}$.

PROOF. Observing that Y_2 is defined as the orthogonal complement of Y_1 in Y_A and $\widetilde{\mathbb{L}}$ is self-adjoint on Y_A, it follows immediately that $Y_2 = \overline{R(\widetilde{\mathbb{L}})}$ and $Y = Y_1 \oplus Y_2$. We shall show the remaining alternative representations of Y_1 and Y_2 in the rest of the proof.

On the one hand, since

$$B = i_{\overline{R(B)}} B \quad \text{and} \quad B' = B'(i_{\overline{R(B)}})' \Longrightarrow \widetilde{\mathbb{L}} = B' L_{\overline{R(B)}} BA, \tag{2.14}$$

clearly $\ker(L_{\overline{R(B)}} BA) \subset Y_1$ according to their definitions. On the other hand, each $y \in Y_1$ satisfies

$$\langle L_{\overline{R(B)}} BAy, BAy \rangle = [\widetilde{\mathbb{L}} y, y] = 0.$$

Due to the assumption $L|_{\overline{R(B)}} \geq 0$, a standard variational argument implies that $L_{\overline{R(B)}} BAy = 0$ and thus $y \in \ker(L_{\overline{R(B)}} BA)$. We obtain $Y_1 = \ker(L_{\overline{R(B)}} BA)$.

For any $x \in D(B' L_{\overline{R(B)}})$ and $y \in Y_1 = \ker(L_{\overline{R(B)}} BA)$, we have

$$[B' L_{\overline{R(B)}} x, y] = \langle Ay, B' L_{\overline{R(B)}} x \rangle = \langle L_{\overline{R(B)}} BAy, x \rangle = 0,$$

which along with the closedness of Y_2, implies $\overline{R(B' L_{\overline{R(B)}})} \subset Y_2$. Obviously, $R(\widetilde{\mathbb{L}}) \subset R(B' L_{\overline{R(B)}})$ and thus $Y_2 \subset \overline{R(B' L_{\overline{R(B)}})}$. Therefore, the equal sign holds and this completes the proof of the lemma. □

Applying the above lemmas (Lemma 2.12 to $\widetilde{X} = \overline{R(B)}$), we obtain the following decomposition.

PROPOSITION 2.14. *In addition to* (**G1–G4**), *assume* (a) *L is nondegenerate,* (b) $n^-(L_{\overline{R(B)}}) = 0$, *and* (c) $A \geq \delta > 0$. *Let*

$$\begin{cases} \mathbf{X}_1 = X_1 \times \{0\}, & \mathbf{X}_2 = \{0\} \times Y_1, \quad \mathbf{X}_3 = X_2 \times Y_2, \\ \mathbf{X}_4 = X_3 \times \{0\}, & \mathbf{X}_5 = X_4 \times \{0\} \end{cases} \tag{2.15}$$

as defined in Lemmas 2.12 *and* 2.13. *Then in the decomposition* $\mathbf{X} = \bigoplus_{j=1}^5 \mathbf{X}_j$, $\mathbf{JL}$ *and the quadratic form* $\mathbf{L}$ *take the form*

$$\mathbf{L} \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{L}_{15} \\ 0 & \mathbf{L}_2 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{L}_3 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{L}_4 & 0 \\ \mathbf{L}_{51} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{JL} \longleftrightarrow \begin{pmatrix} 0 & T_{12} & T_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & T_{25} \\ 0 & 0 & T_3 & 0 & T_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

All the nontrivial blocks of $\mathbf{L}$ *are nondegenerate*:

$$\mathbf{L}_{15} = L_{14}, \quad \mathbf{L}_{51} = L_{41}, \quad \mathbf{L}_2 = A_1 \geq \delta, \quad \mathbf{L}_3 = \begin{pmatrix} L_2 & 0 \\ 0 & A_2 \end{pmatrix} \geq \epsilon, \quad \mathbf{L}_4 = L_3,$$

for some $\epsilon > 0$. *All the blocks of* $\mathbf{JL}$,

$$T_{12} = P_1 B Q_1' A_1, \quad T_{13}\begin{pmatrix} x \\ y \end{pmatrix} = P_1 B Q_2' A_2 y, \quad T_{25} = -Q_1 B' P_1' L_{14},$$

$$T_{35} = \begin{pmatrix} 0 \\ -Q_2 B' P_1' L_{14} \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & P_2 B Q_2' A_2 \\ -Q_2 B' P_2' L_2 & 0 \end{pmatrix}, \quad \ker T_3 = \{0\},$$

are bounded except for T_3, *which is anti-self-adjoint with respect to the equivalent inner product* $\langle \mathbf{L}_3 \cdot, \cdot \rangle$. *Here* $P_{1,2,3,4}$ *and* $Q_{1,2}$ *are the projections associated to the decomposition of* X *and* Y *given in Lemmas* 2.12 *and* 2.13. *Finally, denote* Z *to be the space* $D(BA)$ *with the graph norm*

$$\|y\|_Z = \|y\|_Y + \|BAy\|_X.$$

If the embedding $Z \hookrightarrow Y$ *is compact, then the spectra of* T_3 *are nonzero, isolated with finite multiplicity, and have no accumulating point except for* $+\infty$. *Moreover, the eigenfunctions of* T_3 *form an orthonormal basis of* $\mathbf{X}_3$ *with respect to* $\langle \mathbf{L}_3 \cdot, \cdot \rangle$.

Remark 2.15. One should notice that P_1 in T_{12} and Q_2 in the lower left entry of T_3 are put there only to specify the target spaces, but do not change any values.

PROOF. Since Lemma 2.12 and 2.13 imply

$$P_3'(X_3^*) \oplus P_4'(X_4^*) = \ker B' \text{ and } X_2^* = R(L_{\overline{R(B)}}) \Longrightarrow B' P_2'(X_2^*) \subset Y_2, \tag{2.16}$$

in such decompositions of X and Y, the operator

$$B' : \bigoplus_{j=1}^{4} P_j'(X_j^*) = X^* \supset D(B') \to Y = Y_1 \oplus Y_2$$

takes the form

$$B' \longleftrightarrow \begin{pmatrix} Q_1 B' P_1' & 0 & 0 & 0 \\ Q_2 B' P_1' & Q_2 B' P_2' & 0 & 0 \end{pmatrix}.$$

The block forms of $\mathbf{L}$ and $\mathbf{JL}$ follow from those of L, A, B', and B through a direct calculation. From Lemma 2.12, L_2 is nondegenerate, which along with $\overline{R(B)} = X_1 \oplus X_2$, $X_1 = \ker L_{\overline{R(B)}}$, and the additional assumption $L_{\overline{R(B)}} \geq 0$, we obtain the uniform positivity of L_2, and thus that of $\mathbf{L}_3$.

The proof of the boundedness of T_{jk} and the anti-self-adjointness of T_3 is much like that in the proof of Lemma 2.10. In fact, according to Lemma 2.12, $B' P_j' : X_j^* \to Y$, $j \neq 2$, is a closed operator on the domain X_j^*, and the closed graph theorem implies that it is also bounded. Since $B' P_j' = (P_j B)'$, $j \neq 2$, $P_j B$ also has a continuous extension given by $(B' P_j')'$; therefore $P_j B$, $j \neq 2$, is also bounded. The boundedness of T_{jk}, and the closedness and the density of the domains of P_2B and $B'P_2'$ follow immediately. Moreover, $Q_2B'P_2' : X_2^* \to Y_2$ is also closed since $B'P_2'(X_2^*) \subset Y_2$ and thus $Q_2B'P_2' = B'P_2'$. Consequently,

$$P_2B = (P_2B)'' = (B'P_2')', \quad (Q_2B'P_2')' = (B'P_2')'Q_2' = P_2BQ_2',$$

and

$$Q_2B'P_2' = (Q_2B'P_2')'' = \big((B'P_2')'Q_2'\big)' = (P_2BQ_2')'.$$

Since A_2 and L_2 are isomorphisms satisfying $A_2' = A_2$ and $L_2' = L_2$, we obtain

$$(L_2P_2BQ_2'A_2)' = A_2Q_2B'P_2'L_2 \quad \text{and} \quad (A_2Q_2B'P_2'L_2)' = L_2P_2BQ_2'A_2.$$

Thus, T_3 is anti-self-adjoint with respect to the equivalent inner product $\langle \mathbf{L}_3 \cdot, \cdot \rangle$. Finally, (2.16) imply that $\ker(Q_2B'P_2') = \ker(B'P_2') = \{0\}$ and thus $Q_2B'P_2'L_2$ is injective due to the nondegeneracy of L_2. Moreover, $R(L_{\overline{R(B)}}) = P_2'(X_2^*)$ and $Y_2 = \overline{R(B'L_{\overline{R(B)}})}$ also yield that $R(Q_2B'P_2') = R(B'P_2') \subset Y_2$ is dense. Therefore, the dual operator P_2BQ_2' is injective and the injectivity of T_3 follows.

Finally, let us make the additional assumption of the compact embedding of Z into Y. Let $Z_2 = D(P_2BQ_2'A_2) \subset Y_2$. Since

$$(BA - P_2BQ_2'A_2)|_{Z_2} = P_1BA|_{Z_2} \in L(Y_2, X)$$

is bounded due to the boundedness of P_1B, we have that Z_2 is also compactly embedded in Y_2. As A_2 is uniformly positive definite, Lemma 2.10 and Lemma 2.9 imply that $Q_2B'P_2'L_2P_2BQ_2'A_2$ is self-adjoint on $(Y_2, \langle A_2, \cdot, \cdot \rangle)$ with an orthonormal basis of eigenvectors $\{y_n\}_{n=1}^\infty$ associated to a sequence of eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots$ of finite multiplicity accumulating only at $+\infty$. Here the eigenvalues are positive due to $L_2 \ge \epsilon > 0$ and $\ker T_3 = \{0\}$. Let

$$\begin{aligned}\mathbf{u}_n^{\pm} = {} & \big(\langle L_2P_2BQ_2'A_2y_n, P_2BQ_2'A_2y_n\rangle + \lambda_n\langle Ay_n, y_n\rangle\big)^{-\frac{1}{2}} \\ & \cdot (\pm P_2BQ_2'A_2y_n, \lambda_n y_n).\end{aligned}$$

It is easy to see that $\{\mathbf{u}_n^{\pm}\}$ form an orthonormal basis of $\mathbf{X}_2$ by using $\ker T_3 = \{0\}$ and $T_3\mathbf{u}_n^{\pm} = \mp\lambda_n\mathbf{u}_n^{\mp}$. This completes the proof of the lemma. □

With these preparations, we are ready to prove Theorem 2.3.

PROOF OF THEOREM 2.3. We will prove the theorem largely based on Lemma 2.9 and the observation that solutions to (1.22) satisfy a second-order equation

$$\partial_{tt}v + B'LBAv = 0. \tag{2.17}$$

• *Step 1. Preliminary removal of* $\ker L$ *and* $\ker A$. Let

$$\widetilde{X}_1 = \ker L, \quad \widetilde{X}_2 = X_+ \oplus X_-, \quad \widetilde{Y}_1 = \ker A, \quad \widetilde{Y}_2 = Y_+$$

$$\widetilde{L}_j = (i_{\widetilde{X}_j})'Li_{\widetilde{X}_j}, \quad \widetilde{A}_j = (i_{\widetilde{Y}_j})'Ai_{\widetilde{Y}_j}, \quad \widetilde{B}_{jk} = \widetilde{P}_jB\widetilde{Q}_k', \quad \widetilde{B}^{jk} = \widetilde{Q}_jB'\widetilde{P}_k',$$

where $j, k = 1, 2$ and $\widetilde{P}_{1,2}$ are projections associated to $X = \widetilde{X}_1 \oplus \widetilde{X}_2$ and $\widetilde{Q}_{1,2}$ to $Y = \widetilde{Y}_1 \oplus \widetilde{Y}_2$. Assumptions (**G1–G4**) imply that hypotheses of Lemma 2.10 are satisfied. Therefore, in the splitting

$$\mathbf{X} = (\widetilde{X}_1 \oplus \widetilde{Y}_1) \oplus (\widetilde{X}_2 \oplus \widetilde{Y}_2)$$

the operator **JL** takes the form

$$\mathbf{JL} \leftrightarrow \begin{pmatrix} 0 & \widetilde{\mathbf{T}}_{12} \\ 0 & \widetilde{\mathbf{J}}_2 \widetilde{\mathbf{L}}_2 \end{pmatrix}, \tag{2.18}$$

where

$$\widetilde{\mathbf{J}}_2 \leftrightarrow \begin{pmatrix} 0 & \widetilde{B}_{22} \\ -\widetilde{B}^{22} & 0 \end{pmatrix}, \quad \widetilde{\mathbf{L}}_2 \leftrightarrow \begin{pmatrix} \widetilde{L}_2 & 0 \\ 0 & \widetilde{A}_2 \end{pmatrix}, \quad \widetilde{\mathbf{T}}_{12} \leftrightarrow \begin{pmatrix} 0 & \widetilde{B}_{12} \widetilde{A}_2 \\ -\widetilde{B}^{12} \widetilde{L}_2 & 0 \end{pmatrix}$$

and $(\widetilde{L}_2, \widetilde{J}_2, \widetilde{B}_{22})$ satisfy (**G1–G4**). Moreover, the same lemma also implies that $\widetilde{T}_{12}$ is bounded and both $\widetilde{L}_2$ and $\widetilde{A}_2$ are nondegenerate.

- *Step 2. Hyperbolic subspaces.* As $\widetilde{A}_2 \geq \epsilon$ for some $\epsilon > 0$, according to Lemma 2.9, $\widetilde{\mathbb{L}} = \widetilde{B}_{22}' \widetilde{L}_2 \widetilde{B}_{22} \widetilde{A}_2$ is self-adjoint on $\widetilde{Y}_2$ with respect to the inner product $[\cdot, \cdot] = \langle \widetilde{A}_2 \cdot, \cdot \rangle$. Since for any $v_1, v_2 \in D(\widetilde{\mathbb{L}}) \subset \widetilde{Y}_2$,

$$\big[\widetilde{\mathbb{L}} v_1, v_2\big] = \big\langle \widetilde{A}_2 \widetilde{B}_{22}' \widetilde{L}_2 \widetilde{B}_{22} \widetilde{A}_2 v_1, v_2 \big\rangle = \big\langle \widetilde{L}_2 \widetilde{B}_{22} \widetilde{A}_2 v_1, \widetilde{B}_{22} \widetilde{A}_2 v_2 \big\rangle,$$

and

$$BA = \begin{pmatrix} \widetilde{B}_{11} & \widetilde{B}_{12} \\ \widetilde{B}_{21} & \widetilde{B}_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{A}_2 \end{pmatrix} = \begin{pmatrix} 0 & \widetilde{B}_{12} \widetilde{A}_2 \\ 0 & \widetilde{B}_{22} \widetilde{A}_2 \end{pmatrix} \Longrightarrow R(\widetilde{B}_{22} \widetilde{A}_2) = \widetilde{P}_2(R(BA)),$$

along with the definition of $\widetilde{X}_1$, we obtain the dimension of the eigenspace of negative eigenvalues of the operator $\widetilde{\mathbb{L}}$ given by

$$n_1 \triangleq n^-(\widetilde{\mathbb{L}}) = n^-\big(\widetilde{L}_2|_{\overline{R(\widetilde{B}_{22} \widetilde{A}_2)}}\big) = n^-\big(L|_{\overline{R(BA)}}\big) \leq n^-(L).$$

Let $\widetilde{v}_j$ be the eigenvectors of $\widetilde{\mathbb{L}}$ associate with eigenvalues $-\lambda_j^2 < 0$, $j = 1, \ldots, n_1$, which might be repeated, such that

$$[\widetilde{v}_j, \widetilde{v}_{j'}] = \delta_{jj'}, \quad [\widetilde{\mathbb{L}} \widetilde{v}_j, \widetilde{v}_{j'}] = -\lambda_j^2 \delta_{jj'}.$$

Let

$$\widetilde{u}_j = \lambda_j^{-1} \widetilde{B}_{22} \widetilde{A}_2 \widetilde{v}_j, \;\; \widetilde{\mathbf{u}}_j^{\pm} = (\widetilde{u}_j, \pm \widetilde{v}_j) \Longrightarrow \widetilde{\mathbf{J}}_2 \widetilde{\mathbf{L}}_2 \widetilde{\mathbf{u}}_j^{\pm} = \pm \lambda_j \widetilde{\mathbf{u}}_j^{\pm},$$
$$\langle \widetilde{L}_2 \widetilde{u}_j, \widetilde{u}_k \rangle = -\delta_{jk}.$$

To return to **JL**, let

$$\begin{aligned} \mathbf{u}_j^{\pm} = (u_j, \pm v_j) &\triangleq \widetilde{\mathbf{u}}_j^{\pm} \pm \lambda_j^{-1} \widetilde{\mathbf{T}}_{12} \widetilde{\mathbf{u}}_j^{\pm} \\ &= \big(\widetilde{u}_j + \lambda_j^{-1} \widetilde{B}_{12} \widetilde{A}_2 \widetilde{v}_j, \pm \big(v_j - \lambda_j^{-1} \widetilde{B}^{12} \widetilde{L}_2 \widetilde{u}_j\big)\big), \end{aligned}$$

which are the eigenvectors of **JL** with eigenvalue $\pm \lambda_j$ satisfying

$$\mathbf{JL}\mathbf{u}_j^{\pm} = \pm \lambda_j \mathbf{u}_j^{\pm}, \quad \langle L u_j, u_k \rangle = -\delta_{jk}, \quad \langle A v_j, v_k \rangle = \delta_{jk}.$$

Define the hyperbolic subspaces as

$$E^u = \operatorname{span}\{\mathbf{u}_j^+ \mid j = 1, \dots, n_1\}, \quad E^s = \operatorname{span}\{\mathbf{u}_j^- \mid j = 1, \dots, n_1\},$$

and statement (ii) follows.

• *Step 3. Reduction to the center subspace.* Let

$$X_h = \operatorname{span}\{u_j \mid j = 1, \dots, n_1\} \subset R(B), \quad X_c = \{u \in X \mid \langle Lu, \widetilde{u}\rangle, \ \widetilde{u} \in X_h\},$$
$$Y_h = \operatorname{span}\{v_j \mid j = 1, \dots, n_1\} \subset R(B'), \quad Y_c = \{v \in Y \mid \langle Av, \widetilde{v}\rangle, \ \widetilde{v} \in Y_h\},$$

and

$$E^c = X_c \times Y_c \Longrightarrow \mathbf{X} = (X_h \times Y_h) \oplus E^c = E^s \oplus E^u \oplus E^c.$$

Due to the invariance of $E^{s,u}$ under $e^{t\mathbf{JL}}$ and that of E^c, the rest of statements (i) and (iii) follow from standard arguments (see, e.g., [30], for more details). Apparently $\ker L \subset X_c$ and $\ker A \subset Y_c$.

The above calculations show that L_{X_h} and A_{Y_h} are nondegenerate and thus lemma 12.2 in [30] yields

$$X = X_h \oplus X_c, \quad Y = Y_h \oplus Y_c,$$

with associated projections $P_{h,c}$ and $Q_{h,c}$. By their definitions we have

$$P_h'(X_h^*) = \ker(i_{X_c})' = L(X_h) \subset D(B'),$$
$$Q_h'(Y_h^*) = \ker(i_{Y_c})' = A(Y_h) \subset D(B).$$

Therefore, these decompositions satisfy the assumptions of Lemma 2.10 and thus (1.22) restricted on the invariant E^c also has the separable Hamiltonian form with

$$(L_{X_c}, A_{Y_c}, B_c = P_c B Q_c')$$

satisfying (**G1–G4**). The invariance of E^c and $X_h \times Y_h$ and the block form in Lemma 2.10 imply

$$R(B_c A_{Y_c}) = BA(Y_c) \subset X_c.$$

Due to the L-orthogonality between X_c and X_h, we also have the L-orthogonality between X_h and $\overline{R(B_c A_c)}$ both of which are contained in $\overline{R(BA)}$. As L is negative definite on X_h, we obtain

$$n^-(L|_{\overline{R(BA)}}) \geq n^-(L_{\overline{R(B_c A_c)}}) + \dim X_h = n^-(L_{\overline{R(B_c A_c)}}) + n^-(L|_{\overline{R(BA)}}),$$

which implies

$$n^-(L_{X_c}|_{\overline{R(B_c A_{Y_c})}}) = 0. \tag{2.19}$$

Remark 2.16. Due to the invariance of $E^{u,s,c}$ under $e^{t\mathbf{JL}}$ and the nondegeneracy of $\mathbf{JL}$ and A on the finite-dimensional $E^{u,s}$ and Y_h, respectively, it follows that

(a) A_{Y^c} is injective on $\overline{R(B_c' L_{X_c} B_c A_{Y_c})} = \overline{B' LBA(X_c)}$ if A is injective on $\overline{R(B' LBA)}$;

(b) $\overline{R(B_c A_{Y_c})} = X_c$ if $\overline{R(BA)} = X$.

• *Step 4. Reduction (again) of* $\ker L_{X_c}$ *and* A_{Y_c} *in* E^c. We shall basically redo Step 1 in $E^c = X_c \times Y_c$. It would be a much cleaner exposition if we could find a way to combine these two steps together. However, we were not able to manage that as the positivity of A is required in Lemma 2.9 to identify the hyperbolic directions, and meanwhile there is not a clear simple way to separate the kernels in a decomposition invariant under $e^{t\mathbf{JL}}$.

Let

$$\begin{aligned} \mathbf{X}_{0L} &= \ker L_{X_c} \times \{0\} = \ker L \times \{0\}, \\ \mathbf{X}_{0A} &= \{0\} \times \ker A_{Y_c} = \{0\} \times \ker A. \end{aligned} \tag{2.20}$$

According to Lemma 2.10, X_c and Y_c satisfies (**G1–G4**), so there exist closed subspaces of $\widetilde{X} \subset X_c$ and $\widetilde{Y} \subset Y_c$ such that

$$X_c = \widetilde{X} \oplus \ker L, \quad \ker(i_{\widetilde{X}})' \subset D(B_c'), \quad Y_c = \ker A \oplus \widetilde{Y}, \quad \ker(i_{\widetilde{Y}})' \subset D(B_c).$$

Let

$$\widetilde{\mathbf{X}} = \widetilde{X} \times \widetilde{Y}.$$

Applying Lemma 2.10 again to the decomposition $E^c = (\mathbf{X}_{0L} \oplus \mathbf{X}_{0A}) \oplus \widetilde{\mathbf{X}}$, we obtain the block forms of (1.22) restricted on the invariant E^c and its energy $\mathbf{L}_{E^c}$,

$$\mathbf{L}_{E^c} \longleftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\mathbf{L}} \end{pmatrix}, \quad \mathbf{JL}|_{E^c} \longleftrightarrow \begin{pmatrix} 0 & T_{0\sim} \\ 0 & \widetilde{\mathbf{J}}\widetilde{\mathbf{L}} \end{pmatrix},$$

where $T_{0\sim}$ is bounded and $\widetilde{\mathbf{J}}\widetilde{\mathbf{L}}$ has the separable Hamiltonian form with

$$\big(L_{\widetilde{X}}, A_{\widetilde{Y}}, \widetilde{B} = \widetilde{P} B_c \widetilde{Q}'\big), \quad L_{\widetilde{X}} \text{ and } A_{\widetilde{Y}} \text{ nondegenerate.}$$

Here $\widetilde{P} : X_c \to \widetilde{X}$ and $\widetilde{Q} : Y_c \to \widetilde{Y}$ are the associated projections. Finally, Lemma 2.10 implies $\widetilde{B} A_{\widetilde{Y}} = \widetilde{P} B_c A_{Y_c}|_{\widetilde{Y}}$, which along with the definition of $\mathbf{X}_{0L,0A}$, the fact that $A_{\widetilde{Y}} : \widetilde{Y} \to \widetilde{Y}^*$ is isomorphic and (2.19) yield

$$\begin{aligned} n^-(L_{\widetilde{X}}|_{\overline{R(\widetilde{B})}}) = n^-(L_{\widetilde{X}}|_{\overline{R(\widetilde{B} A_{\widetilde{Y}})}}) &= n^-(L_{\widetilde{X}}|_{\overline{\widetilde{P} R(B_c A_{Y_c})}}) \\ &= n^-(L_{\widetilde{X}}|_{\overline{R(B_c A_{Y_c})}}) = 0. \end{aligned}$$

Therefore, $(L_{\widetilde{X}}, A_{\widetilde{Y}}, \widetilde{B})$ satisfy all the assumptions in Proposition 2.14.

Remark 2.17. Due to the upper triangular block form of $\mathbf{JL}|_{\mathbf{X}_c}$ and the remark at the end of the last step, we have:

(a) $\mathbf{X}_{0A} = \{0\}$ if A is injective on $\overline{R(B'LBA)}$;

(b) $\overline{R(\widetilde{B} A_{\widetilde{Y}})} = \overline{R(\widetilde{P} B_c A_{Y_c})} = \widetilde{X}$ if $\overline{R(BA)} = X$.

• *Step 5. Proof of statement* (iv). The block form decomposition of $\mathbf{L}$ and $\mathbf{JL}$ on E^c follows from the above splitting and Proposition 2.14. As in Lemma 2.12, here $\mathbf{X}_1 = X_1 \times \{0\}$ and $X_1 = L_{\overline{R(\widetilde{B})}}$. Those zero blocks in the bounded operator

$$T_{0\sim} : \widetilde{\mathbf{X}} \to \ker L \times \ker A$$

are due to the facts that $\mathbf{JL}$ maps $X \times \{0\}$ to $\{0\} \times Y$ and vice versa. The well-posedness of $e^{t\mathbf{JL}}$ and its $O(1 + |t|^3)$ growth estimate follow from direct computation based on the block form of $\mathbf{JL}$ where the only unbounded operator T_3 generates a unitary group e^{tT_3}.0

• *Statement* (v) follows directly from Proposition 2.14. □

In order to obtain the better estimates of $e^{t\mathbf{JL}}$, we only need to refine or modify the decomposition under various assumptions.

PROOF OF THEOREM 2.6. According to the remark at the end of the above Step 4, $\mathbf{X}_{0A} = \{0\}$ under the assumption of (i) and thus the second row and column in the block form of $\mathbf{JL}_{E^c}$ disappear, which immediately implies the $O(1 + t^2)$ growth of $e^{t\mathbf{JL}}|_{E^c}$. The same remark and Lemma 2.12 imply that, under the assumption of (ii), $\mathbf{X}_1 = \mathbf{X}_5 = \{0\}$, the $O(1 + |t|)$ growth of $e^{t\mathbf{JL}}|_{E^c}$ follows from the reduced block form of $\mathbf{JL}_{E^c}$ readily.

• *Proof of statement* (iii). Under the nondegeneracy assumptions of $L_{\overline{R(B)}}$ and $A_{\overline{R(B')}}$, the decomposition of X can be carried out in a different, but much simpler, way. In fact, lemma 12.2 in [30] implies

$$X = X_0 \oplus \widetilde{X}, \quad \widetilde{X} = \overline{R(B)}, \quad X_0 = \ker(B'L) = \{u \in X \mid \langle Lu, \widetilde{u}\rangle = 0, \, \widetilde{u} \in \widetilde{X}\},$$
$$Y = Y_0 \oplus \widetilde{Y}, \quad \widetilde{Y} = \overline{R(B')}, \quad Y_0 = \ker(BA) = \{u \in Y \mid \langle Av, \widetilde{v}\rangle = 0, \, \widetilde{v} \in \widetilde{Y}\},$$

associated with the projection $\widetilde{P}$ on X and $\widetilde{Q}$ on Y, respectively. In the decomposition

$$\mathbf{X} = \mathbf{X}_0 \oplus \widetilde{\mathbf{X}}, \quad \mathbf{X}_0 = X_0 \times Y_0, \quad \widetilde{\mathbf{X}} = \widetilde{X} \times \widetilde{Y},$$

which is invariant under $\mathbf{JL}$, we have

$$\mathbf{JL} \Longleftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\mathbf{J}}\widetilde{\mathbf{L}} \end{pmatrix}, \quad \widetilde{\mathbf{L}} = \begin{pmatrix} L_{\widetilde{X}} & 0 \\ 0 & A_{\widetilde{Y}} \end{pmatrix}, \quad \widetilde{\mathbf{J}} = \begin{pmatrix} 0 & \widetilde{P} B \widetilde{Q}' \\ -\widetilde{Q} B' \widetilde{P}' & 0 \end{pmatrix},$$

where $\widetilde{\mathbf{J}}\widetilde{\mathbf{L}}$ is also in the separable Hamiltonian form $(L_{\widetilde{X}}, A_{\widetilde{Y}}, \widetilde{B} = \widetilde{P} B \widetilde{Q}')$. In particular, $L_{\widetilde{X}}$ and $A_{\widetilde{Y}}$ are nondegenerate on $\widetilde{X}$ and $\widetilde{Y}$ and $\widetilde{\mathbf{J}}\widetilde{\mathbf{L}}$ is injective on $\widetilde{\mathbf{X}}$, the last of which implies

$$\overline{R(\widetilde{B} A_{\widetilde{Y}})} = \widetilde{X}.$$

From the above theorem, the system $\widetilde{\mathbf{J}}\widetilde{\mathbf{L}}$ has the trichotomy decomposition

$$\widetilde{\mathbf{X}} = \widetilde{E}^u \oplus \widetilde{E}^s \oplus \widetilde{E}^c, \quad \dim \widetilde{E}^{u,s} = n^-(L_{\overline{R(\widetilde{B} A_{\widetilde{Y}})}}) = n^-(L_{\widetilde{X}}).$$

Lemma 12.2 in [30] implies $\widetilde{\mathbf{L}}$ is uniformly positive definite on $\widetilde{E}^c$, and thus we obtain the Lyapunov stability on $\widetilde{E}^c$. Clearly

$$E^{u,s} = \widetilde{E}^{u,s}, \quad E^c = \mathbf{X}_0 \oplus \widetilde{E}^c,$$

give the trichotomy decomposition of $\mathbf{JL}$ and thus its Lyapunov stability on E^c follows. □

To end the section, we prove the following result on perturbations to L.

PROPOSITION 2.18. *Suppose X is a Hilbert space and $L : X \to X^*$ satisfies* (**G3**) *and $n_0 = \dim\ker L < \infty$. It holds that there exists $C, \delta > 0$ such that any bounded $\widetilde{L} : X \to X$ with $\widetilde{L}^* = \widetilde{L}$ and $\|\widetilde{L} - L\| < \delta$ also satisfies* (**G3**). *Moreover, there exists $\widetilde{L}_0 : \ker L \to (\ker L)^*$ such that*

$$\dim\ker\widetilde{L} = \dim\ker\widetilde{L}_0, \quad n^-(\widetilde{L}) - n^-(L) = n^-(\widetilde{L}_0),$$
$$\left\|\widetilde{L}_0 - (\widetilde{L} - L)_{\ker L}\right\| < C\|\widetilde{L} - L\|^2,$$

where the notation $(\widetilde{L} - L)_{\ker L}$ is in the fashion of (2.5).

COROLLARY 2.19. *If, in addition, L is nondegenerate, then $\widetilde{L}$ is also nondegenerate and $n^-(\widetilde{L}) = n^-(L)$.*

PROOF OF PROPOSITION 2.18. Let $X_\pm \subset X$ be closed subspaces ensured by (**G3**) for L. Denote

$$X_0 = \ker L, \quad X_1 = X_+ \oplus X_-, \quad \widetilde{X}_0 = X_1^{\perp_{\widetilde{L}}} = \{x \in X \mid \langle \widetilde{L}x, x_1\rangle = 0, \ \forall x_1 \in X_1\}.$$

Clearly $L_{X_1} = i_{X_1}^* L i_{X_1} : X_1 \to X_1^*$ is an isomorphism. The closeness between $\widetilde{L}$ and L implies that $\widetilde{L}_{X_1} : X_1 \to X_1^*$ is also an isomorphism and $n^-(\widetilde{L}_{X_1}) = n^-(L_{X_1})$. Therefore, we have

$$\dim\ker\widetilde{L} = \dim\ker\widetilde{L}_{\widetilde{X}_0}, \quad n^-(\widetilde{L}) - n^-(L) = n^-(\widetilde{L}_{\widetilde{X}_0}).$$

To analyze $\widetilde{L}_{\widetilde{X}_0}$, a standard argument yields a unique bounded linear operator $S : X_0 \to X_1$ such that

$$\|S\| \le C\|\widetilde{L} - L\|, \quad \widetilde{X}_0 = \operatorname{graph}(S) = \{x_0 + Sx_0 \mid x_0 \in X_0\}.$$

Using the isomorphism $I + S : X_0 \to \widetilde{X}_0$ as a conjugacy map, let

$$\widetilde{L}_0 = (I + S^*)\widetilde{L}(I + S) : X_0 \to X_0^*.$$

We have, for $x_0, x_0' \in X_0$,

$$\begin{aligned}
\left\langle \widetilde{L}_0 x_0, x_0'\right\rangle &= \left\langle \widetilde{L}(x_0 + Sx_0), (x_0' + Sx_0')\right\rangle \\
&= \left\langle \widetilde{L}_{X_0}x_0, x_0'\right\rangle + \left\langle \widetilde{L}Sx_0, x_0'\right\rangle + \left\langle \widetilde{L}x_0, Sx_0'\right\rangle + \left\langle \widetilde{L}Sx_0, Sx_0'\right\rangle \\
&= \left\langle (\widetilde{L} - L)_{X_0}x_0, x_0'\right\rangle + \left\langle (\widetilde{L} - L)_{X_0}x_0', Sx_0\right\rangle + \left\langle (\widetilde{L} - L)_{X_0}x_0, Sx_0'\right\rangle \\
&\quad + \left\langle S^*\widetilde{L}Sx_0, x_0'\right\rangle
\end{aligned}$$

where we used $L_{X_0} = 0$. Therefore, the estimate on $\widetilde{L}_0$ follows from that on S. □

3 Stability of Nonrotating Stars

In this section, we study stability of nonrotating stars. We divide it into several steps.

3.1 Existence of nonrotating stars

Nonrotating stars are steady solutions $(\rho, u) = (\rho_0(|x|), 0)$ of (1.1)–(1.3), where $\rho_0(r)$ satisfies

$$-\nabla P(\rho_0) - \rho_0 \nabla V_0 = 0 \tag{3.1}$$

with $\Delta V_0 = 4\pi\rho_0$. For the consideration of the existence of nonrotating stars, we assume $P(\rho)$ satisfies assumption (1.4) and

$$\lim_{s\to 0+} s^{1-\gamma_0} P'(s) = K > 0 \quad \text{for some } \gamma_0 > \frac{6}{5}. \tag{3.2}$$

Note that the enthalpy function Φ defined by (1.7) is convex since $P'(\rho) > 0$ for $\rho > 0$ by assumption (1.4). Let $F(s) = (\Phi')^{-1}(s)$ for $s \in (0, s_{\max})$, where

$$s_{\max} = \int_0^\infty \frac{P'(\rho)}{\rho} d\rho \in (0, \infty].$$

We extend $F(s)$ to $s \in (-\infty, 0)$ by zero extension and denote the extended function by $F_+(s) : \mathbb{R} \to [0, \infty)$. We consider physically realistic nonrotating stars ρ_0 with compact support

$$\{\rho_0 > 0\} = \{|x| < R\} \triangleq B_R,$$

where $R > 0$ is the radius of the support. Then by (3.1), we have

$$V_0 + \Phi'(\rho_0) = V_0(R) \tag{3.3}$$

and $\rho_0 = F(V_0(R) - V_0)$ inside B_R. Since $V_0'(r) > 0$ by the Poisson equation, when $r > R$, we have

$$\rho_0(r) = 0 = F_+(V_0(R) - V_0(r)).$$

Therefore, the steady potential $V_0(|x|)$ satisfying the nonlinear elliptic equation in radial coordinates is

$$\Delta V_0 = V_0'' + \frac{2}{r} V_0' = 4\pi F_+(V_0(R) - V_0). \tag{3.4}$$

Define $y(r) = V_0(R) - V_0(r) = \Phi'(\rho_0)$. Then y satisfies the ODE

$$y'' + \frac{2}{r} y' = -4\pi F_+(y). \tag{3.5}$$

Let $\mu = \rho_0(0)$ to be the center density. We solve (3.5) with the initial condition

$$y(0) = \Phi'(\rho_0(0)) = \Phi'(\mu) > 0, \quad y'(0) = 0, \tag{3.6}$$

or equivalently the first-order equation

$$y'(r) = -\frac{4\pi}{r^2} \int_0^r s^2 F_+(y(s)) ds, \quad y(0) = \Phi'(\mu). \tag{3.7}$$

It is easy to see that the unique solution $y_\mu(r)$ of the above ODE exists for $r \in (0,+\infty)$ and $y'_\mu(r) < 0$. If there exists a finite number $R_\mu > 0$ such that $y_\mu\big(R_\mu\big) = 0$, define

$$\rho_\mu(|x|) = \begin{cases} F\big(y_\mu(|x|)\big) & \text{if } |x| < R_\mu, \\ 0 & \text{if } |x| \geq R_\mu, \end{cases} \tag{3.8}$$

and $V_\mu = 4\pi\Delta^{-1}\rho_\mu$. Then $\big(\rho_\mu, 0\big)$ is a nonrotating steady solution of (1.1)–(1.3) with compact support and R_μ is the support radius.

Remark 3.1. Since F_+ is actually a C^1 function for $\gamma \in (\frac{6}{5}, 2)$, the solution (y, y') to (3.5) and (3.6) is C^1 in both r and in μ with $y' < 0$. Therefore, the implicit function theorem implies that R_μ is C^1 in μ and thus so is ρ_μ.

Below we give some conditions to ensure that the ODE (3.5) has solutions with compact support. Assume $P(\rho)$ satisfies (1.4)–(1.5). For polytropic stars with $P(\rho) = K\rho^\gamma$ $\big(\gamma > \frac{6}{5}\big)$, it is well-known [7] that for any center density $\mu > 0$, there exists compact supported solutions. Let $\gamma = 1 + \frac{1}{n}$; (3.5) becomes the classical Lane-Emden equation

$$y'' + \frac{2}{r}y' = -4\pi\left(\frac{\gamma-1}{K\gamma}\right)^n y_+^n = -C_\gamma y_+^n, \tag{3.9}$$

where $0 < n < 5$, $y_+ = \max\{y, 0\}$, and

$$C_\gamma = 4\pi\left(\frac{\gamma-1}{K\gamma}\right)^{\frac{1}{\gamma-1}}.$$

Let $y_\mu(r) = \Phi'(\rho_\mu(r))$ be the solution of (3.9) with

$$y_\mu(0) = \Phi'(\mu) = \frac{K\gamma}{\gamma-1}\mu^{\gamma-1} =: \alpha.$$

Denote the transformation

$$y_\mu(r) = \alpha\theta\big(\alpha^{\frac{n-1}{2}}r\big), \quad s = \alpha^{\frac{n-1}{2}}r; \tag{3.10}$$

then $\theta(s)$ satisfies the same equation

$$\theta'' + \frac{2}{s}\theta' = -C_\gamma\theta_+^n, \quad \theta(0) = 1,\ \theta'(0) = 0. \tag{3.11}$$

The function $\theta(s)$ is called the Lane-Emden function.

The next lemma shows that under assumption (1.4)–(3.2), nonrotating stars with compact support exist for small center density.

LEMMA 3.2. *Assume* (1.4) *and* (3.2). *There exists* $\mu_0 > 0$ *such that for any* $\mu \in (0, \mu_0)$, $y_\mu(R_\mu) = 0$ *for some* $R_\mu > 0$. *Here,* $y_\mu(r)$ *is the solution of* (3.5) *with the initial condition* (3.6). *Then* $\rho_\mu(|x|)$ *defined by* (3.8) *is a nonrotating star with support radius* R_μ.

PROOF. It is equivalent to prove the statement for $\alpha = y_\mu(0) = \Phi'(\mu)$ sufficiently small. Motivated by (3.2) and (3.10), we define

$$y_\mu(r) = \alpha\theta_\alpha\big(\alpha^{\frac{n_0-1}{2}} r\big), \quad s = \alpha^{\frac{n_0-1}{2}} r, \tag{3.12}$$

where $n_0 = \frac{1}{\gamma_0 - 1}$. Then $\theta_\alpha(s)$ satisfies the equation

$$\theta_\alpha'' + \frac{2}{s}\theta_\alpha' = -4\pi\frac{1}{\alpha^{n_0}}F_+(\alpha\theta_\alpha) = -g_\alpha(\theta_\alpha), \tag{3.13}$$

with the initial condition $\theta_\alpha(0) = 1$, $\theta_\alpha'(0) = 0$. Denote

$$g_\alpha(\theta) = 4\pi\frac{1}{\alpha^{n_0}}F_+(\alpha\theta), \quad \theta \in [0,1], \tag{3.14}$$

and

$$g_0(\theta) = C_{\gamma_0}\theta_+^{n_0}, \quad C_{\gamma_0} = 4\pi\left(\frac{\gamma_0 - 1}{K\gamma_0}\right)^{\frac{1}{\gamma_0-1}}. \tag{3.15}$$

Then by assumption (3.2) and the definition of F_+, it is easy to show that when $\alpha \to 0+$, $g_\alpha \to g_0$ in $C^1([0,1])$ and in $C^0((-\infty,1])$. Let $\theta_0(s)$ be the Lane-Emden function satisfying

$$\theta_0'' + \frac{2}{s}\theta_0' = -C_{\gamma_0}(\theta_0)_+^{n_0} = g_0(\theta_0), \quad \theta_0(0) = 1, \ \theta_0'(0) = 0. \tag{3.16}$$

Then for any $R > 0$, we have $\theta_\alpha \to \theta_0$ in $C^1(0,R)$. Define $G(\alpha,s) = \theta_\alpha(s)$ for $\alpha > 0$, $s > 0$, and $G(0,s) = \theta_0(s)$. Let R_0 be the support radius of θ_0; then $G(0,R_0) = \theta_0(R_0) = 0$ and $\frac{\partial}{\partial s}G(0,R_0) = \theta_0'(R_0) < 0$. By the implicit function theorem, there exists $\alpha_0 > 0$ such that when $\alpha \in (0,\alpha_0)$, $G(\alpha,s)$ has a unique zero S_α near R_0. Then S_α is the support radius of θ_α. Therefore, for any $0 < \mu < \mu_0 = F(\alpha_0)$, there exists a unique nonrotating solution $y_\mu(r)$ defined by (3.12) with the support radius $R_\mu = \alpha^{-(n_0-1)/2}S_\alpha$. □

Let

$$\mu_{\max} = \sup\{\mu \mid \exists \text{ solution } \rho_{\mu'} \text{ is compactly supported}, \ \forall \mu' \in (0,\mu]\} \in (0,+\infty].$$

For any center density $\rho_\mu(0) = \mu \in (0,\mu_{\max})$, let $R_\mu = R(\mu) < \infty$ be the support radius of the density $\rho_\mu(|x|)$ of the unique nonrotating stars and

$$M(\mu) = \int_{R^3}\rho_\mu\,dx = \int_{|x|<R_\mu}\rho_\mu\,dx$$

to be the total mass of the star.

Remark 3.3. For polytropic stars with $P(\rho) = K\rho^\gamma$ $\left(\gamma > \frac{6}{5}\right)$, we have $\mu_{\max} = +\infty$. The scaling relation (3.10) implies the classical formulae ([7])

$$M(\mu) = C_1\mu^{\frac{1}{2}(3\gamma-4)}, \quad R_\mu = C_2\mu^{\frac{1}{2}(\gamma-2)}. \tag{3.17}$$

for positive constants C_1, C_2 depending only on γ.

For the general equations of state satisfying (1.4) and (3.2) with $\gamma_0 \geq \frac{4}{3}$, it was shown in [19] that $\mu_{\max} = +\infty$. See also [34, 36, 38] for the case $\gamma_0 > \frac{4}{3}$. On the other hand, for $\gamma_0 \in (\frac{6}{5}, \frac{4}{3})$, counterexamples of $P(\rho)$ with $\mu_{\max} < \infty$ were constructed in [38]. For physically realistic equations of state such as white dwarf stars, $\gamma_0 = \frac{5}{3}$ (see [7, 39]).

3.2 Linearized Euler-Poisson equation

We assume $P(\rho)$ satisfies (1.4)–(1.5). Near a nonrotating star $(\rho_\mu, 0)$ with center density μ, the linearized Euler-Poisson system is

$$\sigma_t = -\nabla \cdot (\rho_\mu v), \tag{3.18}$$

$$v_t = -\nabla(\Phi''(\rho_\mu)\sigma + V), \tag{3.19}$$

with $\Delta V = 4\pi\rho$. Here, σ, v are the density and velocity perturbations, respectively. In the linear approximation, we take the density perturbation σ and the velocity perturbation v with the same support as ρ_μ, that is,

$$\operatorname{supp}(\sigma), \operatorname{supp}(v) \subset \overline{S_\mu} = \{|x| \leq R_\mu\}.$$

This is reasonable in view of the underlying Lagrangian formulation of the problem. See the Appendix for more details. Formally, the above linearized system has an invariant energy functional

$$H_\mu(\sigma, v) = \frac{1}{2}\int_{S_\mu} \left(\rho_\mu |v|^2 + \Phi''(\rho_\mu)\sigma^2\right) dx - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla V|^2 \, dx. \tag{3.20}$$

To ensure $H_\mu(\sigma, v) < \infty$, we consider the natural energy space $X_\mu = L^2_{\Phi''(\rho_\mu)}$ for σ and $Y_\mu = (L^2_{\rho_\mu})^3$ for v. Here, $L^2_{\Phi''(\rho_\mu)}, L^2_{\rho_\mu}$ are the $\Phi''(\rho_\mu), \rho_\mu$-weighted L^2 spaces in S_μ and thus (3.18)–(3.19) form a linear evolution system on $X_\mu \times Y_\mu$. For $\sigma \in L^2_{\Phi''(\rho_\mu)}$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla V|^2 \, dx &= -4\pi \int_{S_\mu} \rho V \, dx \leq 4\pi \|\sigma\|_{L^2_{\Phi''(\rho_\mu)}} \left(\int_{S_\mu} \frac{V^2}{\Phi''(\rho_\mu)} \, dx\right)^{\frac{1}{2}} \\ &\lesssim \|\sigma\|_{L^2_{\Phi''(\rho_\mu)}} \|V\|_{L^6(\mathbb{R}^3)} \lesssim \|\sigma\|_{L^2_{\Phi''(\rho_\mu)}} \|\nabla V\|_{L^2(\mathbb{R}^3)} \end{aligned} \tag{3.21}$$

and thus $\|\nabla V\|_{L^2(\mathbb{R}^3)} \lesssim \|\sigma\|_{L^2_{\Phi''(\rho_\mu)}}$. In above estimates, we use the fact that $\frac{1}{\Phi''(\rho_\mu)}$ is bounded in $\overline{S_\mu}$ since $\frac{1}{\Phi''(\rho_\mu)} \approx \rho_\mu^{2-\gamma_0}$ $(\gamma_0 < 2)$ for $\rho_\mu \ll 1$. The notation $P \lesssim Q$ means $P \leq C_\mu Q$ for some constant C_μ depending only on μ.

Remark 3.4. Since $\gamma_0 \in (\frac{6}{5}, 2)$ and

$$\rho_\mu = O\left((R_\mu - r)^{\frac{1}{\gamma_0 - 1}}\right), \quad \Phi''\left(\rho_\mu(r)\right) = O\left((R_\mu - r)^{\frac{\gamma_0 - 2}{\gamma_0 - 1}}\right),$$

in such weighted spaces, as $r \to R_\mu -$, $v \in Y_\mu$ allows v to approach infinity, while $\sigma \in X_\mu$ may approach infinity for $\gamma_0 \in (\frac{3}{2}, 2)$ or must satisfy $\liminf_{r \to R_\mu -} \sigma(r) = 0$ for $\gamma_0 \in (\frac{6}{5}, \frac{3}{2}]$. Recalling that $\operatorname{supp}(\rho)$ is the domain occupied by the fluid, the vanishing of $\sigma(R_\mu-)$ in the latter case does not mean that the domain does not evolve, but is only not reflected in the linear order of the density perturbation due to its degeneracy near the boundary for $\gamma_0 \in (\frac{6}{5}, \frac{3}{2}]$. In fact, the variation of the domain is clearly indicated in that v does not have to vanish near $r = R_\mu$.

Define the operators

$$L_\mu = \Phi''(\rho_\mu) - 4\pi(-\Delta)^{-1} : X_\mu \to X_\mu^*, \quad A_\mu = \rho_\mu : Y_\mu \to Y_\mu^*,$$

and

$$B_\mu = -\nabla\cdot = -\operatorname{div} : Y_\mu^* \to X_\mu, \quad B'_\mu = \nabla : X_\mu^* \to Y_\mu.$$

Here, for $\sigma \in X_\mu$, we denote

$$(-\Delta)^{-1}\sigma = \int_{S_\mu} \frac{1}{4\pi|x-y|}\sigma(y)dy\bigg|_{S_\mu}.$$

Then the linearized system (3.18)–(3.19) can be written in a separable Hamiltonian form

$$\partial_t \begin{pmatrix} \sigma \\ v \end{pmatrix} = \begin{pmatrix} 0 & B_\mu \\ -B'_\mu & 0 \end{pmatrix} \begin{pmatrix} L_\mu & 0 \\ 0 & A_\mu \end{pmatrix} \begin{pmatrix} \sigma \\ v \end{pmatrix} = \mathcal{J}_\mu \mathcal{L}_\mu \begin{pmatrix} \sigma \\ v \end{pmatrix},$$

which will be checked to satisfy assumptions (**G1–G4**) in the general framework of Section 2. First, (**G2**) is obvious for the operator A_μ defined in (1.10) with $\ker A_\mu = \{0\}$. We note that

$$S_1 = \sqrt{\rho_\mu} : \left(L^2(S_\mu)\right)^3 \to Y_\mu^* = \left(L^2_{1/\rho_\mu}\right)^3, \quad S_2 = \sqrt{\Phi''(\rho_\mu)} : X_\mu \to L^2(S_\mu),$$

are isomorphisms. Therefore, to show $B_\mu : Y_\mu^* \to X_\mu$ is densely defined and closed, it is equivalent to check

$$\widetilde{B}_\mu = S_2 B S_1 = -\sqrt{\Phi''(\rho_\mu)}\operatorname{div}\left(\sqrt{\rho_\mu}\cdot\right) : \left(L^2(S_\mu)\right)^3 \to L^2(S_\mu)$$

is densely defined and closed. The domain of $\widetilde{B}_\mu$ is

$$D(\widetilde{B}_\mu) = \left\{u \in \left(L^2(S_\mu)\right)^3 \,\middle|\, \sqrt{\Phi''(\rho_\mu)}\nabla\cdot(\sqrt{\rho_\mu}u) \in L^2 \text{ in the distribution sense}\right\}.$$

It is clear that any C^1 function with compact support inside S_μ is in $D(\widetilde{B}_\mu)$; thus $D(\widetilde{B}_\mu)$ is dense in $(L^2(S_\mu))^3$. Define

$$\widetilde{C}_\mu = \sqrt{\rho_\mu}\,\nabla\left(\sqrt{\Phi''(\rho_\mu)}\cdot\right) : L^2(S_\mu) \to \left(L^2(S_\mu)\right)^3,$$

with

$$D(\widetilde{C}_\mu) = \left\{\sigma \in L^2(S_\mu) \mid \sqrt{\rho_\mu}\,\nabla\left(\sqrt{\Phi''(\rho_\mu)}\sigma\right) \in \left(L^2(S_\mu)\right)^3\right\}.$$

Then $\widetilde{C}_\mu$ is also densely defined.

LEMMA 3.5. *The above-defined operators satisfy $\widetilde{C}_\mu = \widetilde{B}_\mu^*$ and $\widetilde{B}_\mu = (\widetilde{C}_\mu)^* = (\widetilde{B}_\mu)^{**}$. Thus $\widetilde{B}_\mu$ and $\widetilde{B}_\mu^*$ are both closed.*

PROOF. We start the proof of the lemma with a basic property of functions in $D(\widetilde{C}_\mu)$. Namely, for any $f \in D(\widetilde{C}_\mu)$, there exists $M > 0$ such that for any $r \in (\frac{1}{2}R_\mu, R_\mu)$, it holds that

$$\left\| \sqrt{\rho_\mu \Phi''(\rho_\mu)} f \right\|_{L^2(\partial S(r))} \leq M(R_\mu - r)^{\frac{1}{2}}, \tag{3.22}$$

where $\partial S(r)$ is the sphere with radius r. In fact, by the definition of $D(\widetilde{C}_\mu)$, the trace of f on any sphere $\partial S(r)$ with radius $r < R_\mu$ belongs to $L^2(\partial S(r))$ and

$$g \triangleq \sqrt{\rho_\mu}\partial_r\left(\sqrt{\Phi''(\rho_\mu)} f\right) \in L^2(S_\mu).$$

Since for any $\theta \in S^2$,

$$\left(\sqrt{\Phi''(\rho_\mu)} f\right)(r\theta) = \left(\sqrt{\Phi''(\rho_\mu)} f\right)\left(\frac{1}{2}R_\mu\theta\right) + \int_{\frac{1}{2}R_\mu}^{r} (\rho_\mu^{-1/2} g)(r'\theta)dr',$$

it follows that

$$\begin{aligned}\left\| \sqrt{\Phi''(\rho_\mu)} f \right\|_{L^2(\partial S(r))} &\leq M\left(1 + \|g\|_{L^2(S_\mu)}\left(\int_{\frac{1}{2}R_\mu}^{r} \left(R_\mu - r'\right)^{\frac{1}{1-\gamma_0}} dr'\right)^{\frac{1}{2}}\right) \\ &\leq M\left(1 + (R_\mu - r)^{\frac{2-\gamma_0}{2(1-\gamma_0)}}\right)\end{aligned}$$

and thus (3.22) follows.

By the definition of adjoint operators, $f \in D(\widetilde{B}_\mu^*) \subset L^2(S_\mu)$ and $w = \widetilde{B}_\mu^* f$ if and only if, for any $v \in D(\widetilde{B}_\mu)$,

$$\int_{S_\mu} w \cdot v\, dx = \langle f, \widetilde{B}_\mu v\rangle = -\int_{S_\mu} \sqrt{\Phi''(\rho_\mu)} f \nabla \cdot \left(\sqrt{\rho_\mu} v\right) dx. \tag{3.23}$$

By taking compacted supported v and integrating by parts, we obtain that $f \in D(\widetilde{C}_\mu^*)$ and $w = \widetilde{C}_\mu f$ is necessary. To show this is also sufficient, for any $v \in D(\widetilde{B}_\mu)$, we integrate on smaller balls and take the limit,

$$\begin{aligned}&-\int_{S_\mu} \sqrt{\Phi''(\rho_\mu)} f \nabla \cdot \left(\sqrt{\rho_\mu} v\right) dx \\ &= -\lim_{n\to\infty} \int_{S(R_\mu - \epsilon_n)} \sqrt{\Phi''\big(\rho_\mu\big)} f \nabla \cdot (\sqrt{\rho_\mu} v) dx \\ &= \langle f, \widetilde{C}_\mu v\rangle - \lim_{n\to\infty} \int_{\partial S(R_\mu - \epsilon_n)} \sqrt{\Phi''(\rho_\mu)\rho_\mu}\, f v \cdot \frac{x}{R_\mu - \epsilon_n}\, dS,\end{aligned}$$

where $\epsilon_n \to 0+$. According to (3.22),

$$\left| \int_{\partial S(R_\mu - \epsilon_n)} \sqrt{\Phi''\big(\rho_\mu\big)\rho_\mu}\, f v \cdot \frac{x}{R_\mu - \epsilon_n}\, dS \right| \leq M\epsilon_n^{1/2} \|v\|_{L^2(\partial S(R_\mu - \epsilon_n))}.$$

Since $v \in L^2(S_\mu)$, there exists a sequence $\epsilon_n \to 0+$ such that

$$\epsilon_n^{1/2} \|v\|_{L^2(\partial S(R_\mu - \epsilon_n))} \to 0,$$

and thus (3.23) holds, which implies $\widetilde{B}_\mu^* = \widetilde{C}_\mu$.

By a similar argument to that above, $\widetilde{C}_\mu^* = \widetilde{B}_\mu$, and this completes the proof of the lemma. □

We now check that L_μ defined by (1.10) satisfies (**G3**). Let

$$I_{X_\mu} = \frac{1}{\Phi''(\rho_\mu)} : X_\mu^* \to X_\mu$$

be the isomorphism from Riesz representation theorem, and define the operator

$$\mathbb{L}_\mu = I_{X_\mu} L_\mu = \mathrm{Id} - \frac{1}{4\pi\Phi''(\rho_\mu)}(-\Delta)^{-1} : X_\mu \to X_\mu. \tag{3.24}$$

LEMMA 3.6. $\mathbb{L}$ *is bounded and self-adjoint on* X_μ *and* $\mathbb{L}_\mu - \mathrm{Id}$ *is compact.*

PROOF. Let

$$\mathbb{K} = \mathbb{L}_\mu - \mathrm{Id} = -\frac{1}{4\pi\Phi''(\rho_\mu)}(-\Delta)^{-1} : X_\mu \to X_\mu.$$

We first show that $\mathbb{K}$ is compact. Indeed, for any $\sigma \in X_\mu$, we have

$$\|\mathbb{K}\sigma\|_{X_\mu} = \left(\int_{S_\mu} \frac{V^2}{\Phi''(\rho_\mu)}\, dx\right)^{\frac{1}{2}} \lesssim \left(\int_{S_\mu} V^2\, dx\right)^{\frac{1}{2}},$$

where $\Delta V = 4\pi\rho$. By the previously established estimate $\|V\|_{\dot{H}^1} \lesssim \|\sigma\|_{X_\mu}$ and the compactness of $\dot{H}^1(\mathbb{R}^3)$ to $L^2(S_\mu)$, the compactness of $\mathbb{K}$ follows. Since $\mathbb{K}$ is symmetric on X_μ, the self-adjointness of $\mathbb{L}_\mu$ follows from the Kato-Rellich theorem. □

Assumption (**G3**) follows from the above lemma. To compute $n^-(L_\mu|_{X_\mu})$, we define the elliptic operator

$$D_\mu = -\Delta - 4\pi F'_+\big(V_\mu(R_\mu) - V_\mu\big) : \dot{H}^1(\mathbb{R}^3) \to \dot{H}^{-1}(\mathbb{R}^3).$$

Then for $\phi \in \dot{H}^1(\mathbb{R}^3)$,

$$\langle D_\mu\phi, \phi\rangle = \int_{\mathbb{R}^3} |\nabla\phi|^2 dx - 4\pi \int_{S_\mu} F'\big(V_\mu(R_\mu) - V_\mu\big)|\phi|^2\, dx$$

defines a bounded bilinear symmetric form on $\dot{H}^1(\mathbb{R}^3)$. The next lemma shows that the study of the quadratic form

$$\langle L_\mu\sigma, \sigma\rangle = \int_{S_\mu} \Phi''(\rho_\mu)\sigma^2 - \frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla V|^2 dx, \quad \sigma \in X_\mu,$$

can be reduced to study D_μ on $\dot{H}^1(\mathbb{R}^3)$.

LEMMA 3.7. *It holds that* $n^-(L_\mu|_{X_\mu}) = n^-(\mathbb{L}_\mu) = n^-(D_\mu)$ *and* $\dim\ker L_\mu = \dim\ker \mathbb{L}_\mu = \dim\ker D_\mu$.

PROOF. The proof of the lemma is largely based on the observation $D_\mu = F' L_\mu(-\Delta)$ in S_μ.

First, for any $\rho \in X_\mu$, we can show that

$$\langle L_\mu \rho, \rho\rangle \geq \frac{1}{4\pi}(D_\mu V, V), \quad \Delta V = 4\pi\rho.$$

Indeed, inside S_μ we have $F'(V_\mu(R_\mu) - V_\mu) = \frac{1}{\Phi''(\rho_\mu)}$. Then

$$\begin{aligned}\langle L_\mu \rho, \rho\rangle &= \int_{S_\mu} \frac{1}{F'}\rho^2\,dx - \frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla V|^2\,dx \\ &= \int_{\mathbb{R}^3} \frac{1}{4\pi}|\nabla V|^2\,dx + \int_{S_\mu}\left(2V\rho + \frac{1}{F'}\rho^2\right)dx \\ &\geq \int\left(\frac{1}{4\pi}|\nabla V|^2 - F'V^2\right)dx = \frac{1}{4\pi}\langle D_\mu V, V\rangle.\end{aligned}$$

Denote by $n^{\leq 0}(L_\mu)$ and $n^{\leq 0}(D_\mu)$ the maximal dimensions of nonpositive subspaces of L_μ and D_μ, respectively. Then the above inequality implies that

$$n^{\leq 0}(L_\mu) \leq n^{\leq 0}(D_\mu).$$

Second, for any $\phi \in \dot{H}^1(\mathbb{R}^3)$, let $\rho_\phi = F'_+\phi \in X_\mu$ and $\Delta V_\phi = 4\pi\rho_\phi$. Then

$$\begin{aligned}\langle D_\mu \phi, \phi\rangle &= \int_{\mathbb{R}^3} |\nabla\phi|^2\,dx - 4\pi\int_{S_\mu} F'|\phi|^2\,dx \\ &= 4\pi\left(\int_{S_\mu}\frac{|\rho_\phi|^2}{F'}\,dx + \frac{1}{4\pi}\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx - 2\int_{S_\mu}\rho_\phi\bar{\phi}\,dx\right) \\ &= 4\pi\left(\int_{S_\mu}\frac{|\rho_\phi|^2}{F'}dx + \frac{1}{4\pi}\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx - \frac{1}{2\pi}\int_{\mathbb{R}^3}\nabla V_\phi\cdot\nabla\bar{\phi}\,dx\right) \\ &\geq 4\pi\left(\int_{S_\mu}\frac{|\rho_\phi|^2}{F'}dx - \frac{1}{4\pi}\int_{\mathbb{R}^3}|\nabla V_\phi|^2\,dx\right) = 4\pi\big\langle L_\mu\rho_\phi, \rho_\phi\big\rangle.\end{aligned}$$

Thus $n^{\leq 0}(L_\mu) \geq n^{\leq 0}(D_\mu)$ and a combination with the previous inequality yields

$$n^{\leq 0}(L_\mu) = n^{\leq 0}(D_\mu). \tag{3.25}$$

We note that $L_\mu\rho = 0$ for $\rho \in X_\mu$ is equivalent to $D_\mu V = 0$ where $\Delta V = 4\pi\rho$, and $D_\mu\phi = 0$ for $\phi \in \dot{H}^1$ is equivalent to $L_\mu\rho_\phi = 0$ ($\rho_\phi = F'_+\phi$). Thus we have $\dim\ker L_\mu = \dim\ker D_\mu$ and consequently $n^-(L_\mu) = n^-(D_\mu)$ follows from (3.25). □

In the rest of this subsection, we study some basic properties of the operator D_μ. Since the potential term in D_μ is radially symmetric, we can use spherical harmonic functions to decompose D_μ into operators on radially symmetric spaces. Let $Y_{lm}(\theta)$ be the standard spherical harmonics on $\mathbb{S}^2$ where $l = 0, 1, \ldots;$

$m = -l, \dots, l$. Then $\Delta_{\mathbb{S}^2} Y_{lm} = -l(l+1)Y_{lm}$. For any function $u(x) \in \dot{H}^1$, we decompose

$$u(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} u_{lm}(r) Y_{lm}(\theta), \quad u_{lm}(r) = \int_{\mathbb{S}^2} u(r\theta) Y_{lm}(\theta) dS_\theta.$$

Then we have

$$D_\mu u = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} D_\mu^l u_{lm}(r)\, Y_{lm}(\theta),$$

where

$$D_\mu^l = -\Delta_r + \frac{l(l+1)}{r^2} - 4\pi F_+'\big(V_\mu(R_\mu) - V_\mu(r)\big), \tag{3.26}$$

and $\Delta_r = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}$. In particular, the operator

$$D_\mu^0 = -\Delta_r - 4\pi F_+'\big(V_\mu(R_\mu) - V_\mu(r)\big) \tag{3.27}$$

is D_μ restricted to radial functions.

The study of D_μ is reduced to the study of operators D_μ^l $(l \geq 0)$ for radial functions.

LEMMA 3.8.

(i) $\ker D_\mu^1 = \{V_\mu'(r)\}$ *and* $D_\mu^1 \geq 0$.

(ii) *For* $l \geq 2$, $D_\mu^l > 0$.

(iii) $n^-(D_\mu) = n^-\big(D_\mu^0\big) \geq 1$.

PROOF. The arguments are rather standard. Taking ∂_{x_i} of the steady equation

$$\Delta V_\mu = V_\mu'' + \frac{2}{r} V_\mu' = 4\pi F_+\big(V_\mu(R_\mu) - V_\mu(r)\big), \tag{3.28}$$

we get $D_\mu \partial_{x_i} V_\mu = 0$, $i = 1, 2, 3$. Thus $D_\mu^1 V_\mu'(r) = 0$. Since $V_\mu'(r) > 0$ for $r > 0$, (i) follows from the Sturm-Liouville theory for the ODE operator D_μ^1. Then for $l \geq 2$,

$$D_\mu^l = D_\mu^1 + \frac{l(l+1) - 2}{r^2} > 0.$$

By (i) and (ii), we have $n^-(D_\mu) = n^-(D_\mu^0)$. Since $D_\mu \partial_{x_i} V_\mu = 0$ and $\partial_{x_i} V_\mu$ changes sign, 0 cannot be the first eigenvalue of D_μ. Thus $n^-(D_\mu) \geq 1$. This proves (iii). □

3.3 The negative index of D_μ

We find the negative index $n^-(D_\mu) = n^-(D_\mu^0)$ in this subsection. Although D_μ is defined as an operator $\dot{H}^1 \to \dot{H}^{-1}$, the eigenfunctions with negative eigenvalues of D_μ decay exponentially fast at infinity and are in H^2. Thus, when computing $n^-(D_\mu)$ below, we can treat D_μ as an operator $H^2 \to L^2$ and $D_\mu^0 : H_r^2 \to L_r^2$.

The following formula for the surface potential $V_\mu\big(R_\mu\big)$ will be used later.

LEMMA 3.9. *It holds that*

$$V_\mu(R_\mu) = -\frac{M(\mu)}{R_\mu}. \tag{3.29}$$

PROOF. Since

$$V_\mu'' + \frac{2}{r}V_\mu' = \frac{1}{r^2}\frac{d}{dr}\big(r^2 V_\mu'(r)\big) = 4\pi\rho_\mu,$$

we have

$$V_\mu'(r) = \frac{4\pi}{r^2}\int_0^r \rho_\mu(r) r^2\, dr = \frac{M(\mu)}{r^2} \quad \text{for } r \ge R_\mu. \tag{3.30}$$

Thus

$$V_\mu(r) = -\frac{M(\mu)}{r} \quad \text{for } r \ge R_\mu,$$

and formula (3.29) follows. □

To find $n^-(D_\mu^0)$, our key observation is that D_μ^0 has a kernel only at critical points of the surface potential $V_\mu(R_\mu)$ or, equivalently, at points where

$$\frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right) = 0$$

by the above lemma.

LEMMA 3.10. *When* $\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big) \ne 0$, $\ker D_\mu^0 = \{0\}$; *when* $\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big) = 0$, $\ker D_\mu^0 = \left\{\frac{\partial}{\partial\mu}V_\mu\right\}$.

PROOF. Let $y_\mu(r) = V_\mu(R_\mu) - V_\mu(r)$; then

$$\Delta_r y_\mu = y_\mu'' + \frac{2}{r}y_\mu' = -4\pi F_+\big(y_\mu(r)\big).$$

Observing that F_+ is actually a C^1 function for $\gamma \in (\frac{6}{5}, 2)$, denote $u_\mu(r) = \frac{\partial}{\partial\mu}y_\mu(r)$, and by taking $\frac{\partial}{\partial\mu}$ of above equation for y_μ, we get

$$u_\mu'' + \frac{2}{r}u_\mu' = -4\pi F_+'(y_\mu(r))u_\mu. \tag{3.31}$$

Suppose $D_\mu^0 v(r) = 0$ with $v(|x|) \in \dot{H}^1(\mathbb{R}^3)$. Then

$$v'' + \frac{2}{r}v' = \frac{1}{r^2}\frac{d}{dr}\big(r^2 v'(r)\big) = -4\pi F_+'\big(y_\mu(r)\big)v(r) \tag{3.32}$$

and

$$v'(r) = -\frac{4\pi}{r^2}\int_0^r s^2 F_+'\big(y_\mu(s)\big)v(s)ds,$$

which implies that $v \in C^1(0, +\infty)$. Since both $u_\mu(r)$ and $v(r)$ satisfy the same second-order ODE (3.31) and (3.32) with zero derivative at $r = 0$, we have $v(r) =$

$Cu_\mu(r)$ for some constant $C \neq 0$. This implies $u_\mu \in \dot{H}^1(\mathbb{R}^3)$ harmonic outside S_μ. Along with $\lim_{r\to\infty} V(r) = 0$ we obtain

$$0 = \lim_{r\to+\infty} u_\mu(r) = \frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right).$$

Therefore, D_μ^0 has a kernel only when $\frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right) = 0$, and in this case it follows from the above analysis that $\ker D_\mu^0 = \left\{\frac{\partial}{\partial\mu} V_\mu\right\}$. □

To find $n^-(D_\mu^0)$, we use a continuity approach to follow its changes when μ is increased from 0 to $\mu_{\max}$. First, we find $n^-(D_\mu^0)$ for small μ. By the above lemma, for increasing μ, the negative index $n^-(D_\mu^0)$ can only change at critical points of $\frac{M(\mu)}{R(\mu)}$. Then we find the jump formula of $n^-(D_\mu^0)$ at those critical points. Combining these steps, we get $n^-(D_\mu^0)$ for any $\mu > 0$.

By the proof of Lemma 3.2, for small μ the steady state ρ_μ is close (up to a scaling) to the Lane-Emden stars. So we first find $n^-(D_\mu^0)$ for Lane-Emden stars. We treat the cases $\gamma \in (\frac{6}{5}, \frac{4}{3})$ and $\gamma \in [\frac{4}{3}, 2)$ separately.

LEMMA 3.11. *Let $P(\rho) = K\rho^\gamma$, $\gamma \in (\frac{6}{5}, 2)$; then $n^-(D_\mu^0) = 1$ for any $\mu > 0$.*

PROOF. Let $y_\mu(r)$ be the solution of (3.9) with $y_\mu(0) = \alpha = \Phi'(\mu)$. Recall that $y_\mu(r) = \alpha\theta(\alpha^{(n-1)/2} r)$, where $\theta(s)$ is the Lane-Emden function satisfying (3.11). Then

$$D_\mu^0 = -\Delta_r - C_\gamma n (y_\mu)_+^{n-1}, \quad n = \frac{1}{\gamma - 1}.$$

Let $\psi(r)$ be an eigenfunction satisfying $D_\mu^0 \psi = \lambda\psi$ with $\lambda < 0$. Define $\psi(r) = \phi(\alpha^{(n-1)/2} r)$ and $s = \alpha^{(n-1)/2} r$. Then $\phi(s)$ satisfies the equation

$$\left(-\Delta_s - C_\gamma n \theta_+^{n-1}\right)\phi = \alpha^{-(n-1)}\lambda\phi.$$

Thus $n^-(D_\mu^0) = n^-(B_n)$, where

$$B_n = -\Delta_s - C_\gamma n \theta_+^{n-1}. \tag{3.33}$$

It suffices to show that $n^-(B_n) = 1$.

We first consider the case $\gamma \in (\frac{4}{3}, 2)$ where $n \in (1, 3]$. Define $\theta_a(s) = a\theta(a^{(n-1)/2} s)$, $a > 0$, and

$$w(s) = \frac{d}{da}(\theta_a(s))\bigg|_{a=1} = \theta(s) + \frac{n-1}{2} s\theta'(s).$$

Note that $\theta_a(s)$ satisfies the Lane-Emden equation

$$\theta_a'' + \frac{2}{s}\theta_a' = -C_\gamma \theta_{a,+}^n, \quad \theta_a(0) = a,\ \theta_a'(0) = 0. \tag{3.34}$$

Let R_n be the support radius of $\theta(s)$, then $\theta(R_n) = 0$ and $\theta(s) > 0, \theta'(s) < 0$ for $s \in (0, R_n)$. By taking $\frac{d}{d\alpha}$ of (3.34), we have

$$w'' + \frac{2}{s}w' = -C_\gamma n\theta_+^{n-1}w, \quad s \in (0, R_n), \tag{3.35}$$

with $w(0) = 1, w'(0) = 0$. We show that $w(s)$ has a unique zero in $(0, R_n)$. Indeed, since $w(0) = 1$ and $w(R_n) = \frac{n-1}{2}R_n\theta'(R_n) < 0$, by continuity of $w(s)$ there exists $s_0 \in (0, R_n)$ such that $w(s_0) = 0$. Moreover, for $s \in (0, R_n)$ we have

$$\begin{aligned} w'(s) &= \frac{n+1}{2}\theta'(s) + \frac{n-1}{2}s\theta''(s) \\ &= \frac{n+1}{2}\theta'(s) + \frac{n-1}{2}\left(-2\theta'(s) - C_\gamma s\theta(s)^n\right) \\ &= \frac{3-n}{2}\theta'(s) - \frac{n-1}{2}C_\gamma s\theta(s)^n < 0. \end{aligned}$$

Thus $w(s)$ is monotone decreasing with exactly one zero s_0 in $(0, R_n)$. We extend $w(s)$ to be a $C^1(0, \infty)$ function by solving the ODE (3.35) in (R_n, ∞). Noting that the right-hand side of (3.35) is zero in (R_n, ∞), we get

$$w(s) = \frac{C_1}{s} + C_2, \quad s \in (R_n, +\infty),$$

where

$$C_1 = -R_n^2 w'(R_n) > 0, \quad C_2 = w(R_n) - \frac{C_1}{R_n} < 0.$$

Thus $w(s) < 0$ in (R_n, ∞) and $w(s) \searrow C_2$ as $s \to +\infty$. Therefore, $w(s)$ only has one zero in $(0, +\infty)$. We show $n^-(B_n) = 1$ by comparison arguments. Suppose $n^-(B_n) \geq 2$. Let $\lambda_1 < 0$ be the second negative eigenvalue of B_n and $\xi(s) \in H_r^1$ be the corresponding eigenfunction, that is,

$$\left(\xi'' + \frac{2}{s}\xi'\right) = -C_\gamma n\theta_+^{n-1}\xi - \lambda_1\xi. \tag{3.36}$$

Then $\xi(s) = cs^{-1}e^{-\sqrt{-\lambda_1}s}$ for $s > R_n$. By Sturm-Liouville theory, $\xi(s)$ has exactly one zero $s_1 \in (0, +\infty)$. We claim that this would lead to $w(s)$ having two zeros, one in $(0, s_1)$ and the other in (s_1, ∞). We can assume $\xi(s) > 0$ in $(0, s_1)$; then $\xi'(s_1) < 0$. Suppose $w(s)$ has no zero in $(0, s_1)$; then $w(s) > 0$ in $(0, s_1)$ and $w'(s) < 0$ in $[0, s_1]$. The integration of

$$\int_0^{s_1} [(3.35)\xi(s) - (3.36)w(s)]s^2\, ds$$

and an integration by parts yield

$$-s_1^2\xi'(s_1)w(s_1) = \lambda_1 \int_0^{s_1} \xi(s)w(s)ds.$$

This is a contradiction since the left-hand side is positive and the right-hand side is negative. Thus $w(s)$ must have one zero in $(0, s_1)$. By the same argument, $w(s)$ has another zero in (s_1, ∞). This is in contradiction to the fact that $w(s)$ has exactly one zero in $(0, \infty)$. Thus $n^-(B_n) < 2$, which together with Lemma 3.8(iii) shows that $n^-(B_n) = 1$.

We complete the proof of the lemma by a continuation argument. According to Corollary 2.19, $n^-(D_\mu^0)$ is locally constant in μ and γ on the set $\{(\mu, \gamma) \mid \ker D_\mu^0 = \{0\}\}$. For polytropic stars with $P(\rho) = K\rho^\gamma$ $\left(\frac{6}{5} < \gamma < 2\right)$, by (3.17) we have

$$\frac{M(\mu)}{R_\mu} = \frac{C_1}{C_2}\mu^{\gamma-1} \quad \text{and thus} \quad \frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right) > 0$$

for any $\mu > 0$ and $\gamma \in (\frac{6}{5}, 2)$. Therefore, by Lemma 3.10, $\ker D_\mu^0 = \{0\}$ for any $\gamma \in \left(\frac{6}{5}, 2\right)$ and thus $n^-(D_\mu^0) = 1$ for all $\mu > 0$. □

For a general equations of state, by Corollary 2.19, Lemma 3.10, and Lemma 3.11, we have the following:

LEMMA 3.12. *Assume* (1.4)–(1.5) *for* $P(\rho)$. *There exists* $\mu_0 > 0$ *such that for any* $\mu \in (0, \mu_0)$, $n^-(D_\mu^0) = 1$. *Moreover, as a function of* $\mu \in (0, \mu_{\max})$, $n^-(D_\mu^0)$ *is locally constant.*

PROOF. We use the notations in Lemma 3.2, where the nonrotating stars with small center density μ are constructed. Define the operator

$$B_\alpha = -\Delta_s - g_\alpha'(\theta_\alpha) : \dot{H}_r^1 \to \dot{H}_r^{-1},$$

where θ_α, g_α are defined in (3.12) and (3.14). As in the proof of Lemma 3.11, we have $n^-(D_\mu^0) = n^-(B_\alpha)$ where $\alpha = \Phi'(\mu)$. We also define

$$B_0 = -\Delta_s - g_0'(\theta_0) = -\Delta_s - C_{\gamma_0} n_0 (\theta_0)_+^{n_0-1},$$

where θ_0 is the Lane-Emden function satisfying (3.16) and g_0 is defined in (3.15). By the proof of Lemma 3.2, when $\alpha \to 0+$, $g_\alpha \to g_0$ in $C^1(0,1)$ and $\theta_\alpha \to \theta_0$ in $C^1(0, R)$ for any $R > 0$. By Lemma 3.11, we have $n^-(B_0) = 1$. Corollary 2.19 implies that there exists $\alpha_0 > 0$ such that when $\alpha < \alpha_0$ we have $n^-(B_\alpha) = 1$. This proves the lemma by letting $\mu_0 = (\Phi')^{-1}(\alpha_0)$. Moreover, $n^-(D_\mu^0)$ changes only at critical points of $\frac{M(\mu)}{R_\mu}$ due to Corollary 2.19. □

We first prove the following lemma of the nondegeneracy of the mass–radius curve of the nonrotating stars, which will be crucial in the analysis of the change of the Morse index $n^-(D_\mu^0)$.

LEMMA 3.13. *There exists no point* $\mu \in (0, \mu_{\max})$ *such that*

$$M'(\mu) = \frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right) = 0.$$

PROOF. Suppose otherwise; then $M'(\mu) = \frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big) = 0$ at some $\mu \in (0, \mu_{\max})$. Then by Lemma 3.10, $D_\mu^0 \frac{\partial V_\mu}{\partial \mu} = 0$, i.e.,

$$\left(\frac{\partial V_\mu}{\partial \mu}\right)'' + \frac{2}{r}\left(\frac{\partial V_\mu}{\partial \mu}\right)' = -4\pi F_+'\big(y_\mu(r)\big)\frac{\partial V_\mu}{\partial \mu}, \quad r > 0, \tag{3.37}$$

and $\frac{\partial V_\mu}{\partial \mu} = -\frac{\partial y_\mu}{\partial \mu}$ in S_μ. By (3.31) and $\rho_\mu = F_+(y_\mu)$, we have

$$\left(\frac{\partial y_\mu}{\partial \mu}\right)'(R_\mu) = -\frac{1}{R_\mu^2}\int_0^{R_\mu} s^2 4\pi F_+'\big(y_\mu(s)\big)\frac{\partial y_\mu}{\partial \mu} ds = -\frac{1}{R_\mu^2} M'(\mu) = 0.$$

Then $\big(\frac{\partial V_\mu}{\partial \mu}\big)'(R_\mu) = 0$ and by (3.37) it follows that $\big(\frac{\partial V_\mu}{\partial \mu}\big)'(r) = 0$ for any $r > R_\mu$. Therefore, $\frac{\partial V_\mu}{\partial \mu}(r) = 0$ for any $r \geq R_\mu$. By (3.37), this implies that $\frac{\partial V_\mu}{\partial \mu}(r) = 0$ for any $r > 0$. But this is impossible since

$$\frac{\partial V_\mu}{\partial \mu}(0) = -\frac{\partial y_\mu}{\partial \mu}(0) = -\Phi''(\mu) \neq 0. \qquad \square$$

Finally, we give the following proposition on the change of $n^-(D_\mu^0)$ at critical points of $\frac{M(\mu)}{R_\mu}$.

PROPOSITION 3.14. *Let μ^* be a critical point of $\frac{M(\mu)}{R_\mu}$, then for μ near μ_* it holds that*

$$n^-\big(D_\mu^0\big) = n^-\big(D_{\mu_*}^0\big) + i_\mu \tag{3.38}$$

where the index i_μ is defined in (1.14). *Therefore, the jump of $n^-(D_\mu^0)$ at μ^* equals that of i_μ.*

PROOF. To prove (3.38), we need to study the perturbation of zero eigenvalue of $D_{\mu^*}^0$ for μ near μ^*. The idea is similar to the proof of Proposition 2.18, but with a more concrete decomposition. For μ near μ_*, let

$$Z(\mu) = \big\{u \in \dot H^1(\mathbb{R}^3) \mid \big\langle F_+'\big(V_\mu(R_\mu) - V_\mu(r)\big), u\big\rangle = 0\big\}.$$

Using $F_+(0) = 0$, one may compute

$$\begin{aligned}
&\big\langle F_+'\big(V_\mu(R_\mu) - V_\mu(r)\big), \partial_\mu V_\mu\big\rangle \\
&= \int_{S_\mu} F_+'\big(V_\mu(R_\mu) - V_\mu(r)\big)\partial_\mu V_\mu(r) dx \\
&= -\partial_\mu \int_{S_\mu} F_+\big(V_\mu(R_\mu) - V_\mu(r)\big) dx \\
&\quad + \partial_\mu\big(V_\mu(R_\mu)\big)\int_{S_\mu} F_+'\big(V_\mu(R_\mu) - V_\mu(r)\big) dx
\end{aligned}$$

$$= -M'(\mu) - \partial_\mu\left(\frac{M(\mu)}{R_\mu}\right)\int_{S_\mu} F'_+\big(V_\mu(R_\mu) - V_\mu(r)\big)\,dx.$$

Lemma 3.13 yields that $M'(\mu) \neq 0$ for μ near μ_* and thus

$$\dot{H}^1(\mathbb{R}^3) = Z(\mu) \oplus \mathbb{R}\{\partial_\mu V_\mu\}. \tag{3.39}$$

Moreover, differentiating (3.28) and using Lemma 3.9, we obtain

$$D^0_\mu \partial_\mu V_\mu = 4\pi\,\partial_\mu\left(\frac{M(\mu)}{R_\mu}\right) F'_+\big(V_\mu(R_\mu) - V_\mu(r)\big).$$

Therefore, (3.39) is a D^0_μ-orthogonal decomposition. From Lemma 3.10, D^0_μ is nondegenerate on $Z(\mu)$ for μ close to μ_* and thus

$$n^-\big(D^0_\mu\big) - n^-\big(D^0_{\mu_*}\big) = n^-\big(D^0_\mu|_{\mathbb{R}\{\partial_\mu V_\mu\}}\big).$$

Using the above calculations, we have

$$\begin{aligned}
&\big\langle D^0_\mu \partial_\mu V_\mu, \partial_\mu V_\mu\big\rangle \\
&\quad = 4\pi\,\partial_\mu\left(\frac{M(\mu)}{R_\mu}\right)\big\langle F'_+\big(V_\mu(R_\mu) - V_\mu(r)\big), \partial_\mu V_\mu\big\rangle \\
&\quad = -4\pi M'(\mu)\partial_\mu\left(\frac{M(\mu)}{R_\mu}\right) - 4\pi\left(\partial_\mu\left(\frac{M(\mu)}{R_\mu}\right)\right)^2 \int_{S_\mu} F'_+\big(V_\mu(R_\mu) - V_\mu(r)\big)\,dx.
\end{aligned}$$

Therefore, (3.38) follows for μ near μ_*. □

3.4 Stability for nonradial perturbations

We study the linearized system (3.18)–(3.19) for nonradial and radial perturbations separately. Here we follow the tradition in the astrophysics literature that "nonradial" perturbations refer to those modes corresponding to nonconstant spherical harmonics. See Definition 3.16 for the precise definition.

First, we give a Helmholtz-type decomposition of vector fields in Y_μ.

LEMMA 3.15. *There is a direct sum decomposition $Y_\mu = Y_{\mu,1} \oplus Y_{\mu,2}$, where $Y_{\mu,1}$ is the closure of*

$$\Big\{u \in \big(C^1(S_\mu)\big)^3 \cap Y_\mu \mid \nabla\cdot\big(\rho_\mu u\big) = 0\Big\}$$

in Y_μ, and $Y_{\mu,2}$ is the closure of

$$\big\{u \in Y_\mu \mid u = \nabla p \text{ for some } p \in C^1(S_\mu)\big\}$$

in Y_μ.

PROOF. Define the space Z to be the closure of

$$\left\{p \in C^1(S_\mu)\ \middle|\ \int_{S_\mu} \rho_\mu |\nabla p|^2\,dx < \infty\right\}$$

under the norm $\|p\|_Z = \left(\int_{S_\mu} \rho_\mu |\nabla p|^2\, dx\right)^{1/2}$, modulo the constant functions. The inner product on Z is defined as

$$(p_1, p_2)_Z = \int_{S_\mu} \rho_\mu \nabla p_1 \cdot \nabla p_2\, dx.$$

For any fixed $u \in Y_\mu$, we seek $p_u \in Z$ as a weak solution of the equation

$$\nabla \cdot (\rho_\mu \nabla p) = \nabla \cdot (\rho_\mu u).$$

This is equivalent to

$$\int_{S_\mu} \rho_\mu \nabla p_u \cdot \nabla p\, dx = \int_{S_\mu} \rho_\mu u \cdot \nabla p\, dx \quad \forall p \in Z. \tag{3.40}$$

The right-hand side above defines a bounded linear functional on Z. Thus by the Riesz representation theorem, there exists a unique $p_u \in Z$ satisfying (3.40). Let $u_2 = \nabla p_u \in Y_{\mu,2}$. Then $u_1 = u - u_2 \in Y_{\mu,1}$. Moreover, it is clear that $Y_{\mu,1} \perp Y_{\mu,2}$ in the inner product of Y_μ. This finishes the proof of the lemma. □

The decomposition

$$X_\mu \times Y_\mu = (\{0\} \times Y_{\mu,1}) \oplus (X_\mu \times Y_{\mu,2}),$$

is clearly invariant for the linearized system (3.18)–(3.19). We shall call perturbations in $\{0\} \times Y_{\mu,1}$ and $X_\mu \times Y_{\mu,2}$ to be pseudo-divergence free and irrotational, respectively. In particular, $\{0\} \times Y_{\mu,1}$ is a subspace of steady states for (3.18)–(3.19), where 0 is the only eigenvalue. Thus, we restrict to initial data $(\sigma(0), u(0)) \in X_\mu \times Y_{\mu,2}$. Any solution $(\sigma(t), u(t)) \in X_\mu \times Y_{\mu,2}$ can be written as

$$\sigma(x,t) = \sigma_1(r,t) + \sigma_2(x,t), \tag{3.41}$$

and

$$u(x,t) = \nabla \xi = v_1(r,t)\frac{x}{r} + \nabla \xi_2(x,t), \tag{3.42}$$

where (σ_1, v_1) is the radial component defined by

$$\sigma_1(r,t) = \int_{\mathbb{S}^2} \sigma(r\theta) dS_\theta, \quad \xi_1(r,t) = \int_{\mathbb{S}^2} \xi(r\theta) dS_\theta, \quad v_1(x,t) = \frac{\partial}{\partial r}\xi_1(r,t),$$

and $(\sigma_2, \xi_2) = (\sigma - \sigma_1, \xi - \xi_1)$ are the nonradial components.

The radial component (σ_1, v_1) will be studied in the next subsection. The nonradial component $(\sigma_2(x,t), \xi_2(x,t))$ satisfies the system

$$\begin{aligned} &\partial_t \sigma_2 = -\nabla \cdot \left(\rho_\mu \nabla \xi_2\right) \\ &\xi_{2,t} = -\left(\Phi''(\rho_\mu)\sigma_2 + V_2\right) = -L_\mu \sigma_2, \ \ \Delta V_2 = 4\pi\sigma_2. \end{aligned}$$

This is of the Hamiltonian form

$$\partial_t \begin{pmatrix} \sigma_2 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} L_\mu & 0 \\ 0 & \widetilde{A}_\mu \end{pmatrix} \begin{pmatrix} \sigma_2 \\ \xi_2 \end{pmatrix}, \tag{3.43}$$

where $\widetilde{A}_\mu = -\nabla \cdot (\rho_\mu \nabla)$, $(\sigma_2, \ \xi_2) \in X_{\mu,n} \times Y_{\mu,n}$ with

$$X_{\mu,n} = \left\{ \rho \in X_\mu \;\middle|\; \int_{\mathbb{S}^2} \rho(r\theta) dS_\theta = 0 \right\}$$

and

$$Y_{\mu,n} = \left\{ \xi \in Y_{\mu,2} \;\middle|\; \int_{S_\mu} \rho_\mu |\nabla \xi|^2 \, dx < \infty, \ \int_{\mathbb{S}^2} \xi(r\theta) dS_\theta = 0 \right\},$$

$$\|\xi\|_{Y_{\mu,n}} = \|\nabla \xi\|_{L^2_{\rho_\mu}}.$$

We take this opportunity to define the following terminology.

DEFINITION 3.16. Define the subspaces of radial and nonradial perturbations for the linearized Euler-Poisson system (3.18)–(3.19) as

$$\mathbf{X}_r = \left\{ \left(\rho(|x|), v(|x|) \frac{x}{|x|} \right) \in X_\mu \times Y_\mu \right\},$$

$$\mathbf{X}_{nr} = \big(\{0\} \times Y_{\mu 1}\big) \oplus \{(\rho, u = \nabla \xi) \in X_\mu \times Y_\mu \mid \rho \in X_{\mu,n}, \ \xi \in Y_{\mu,n}\}.$$

Clearly we have that the decomposition $X_\mu \times Y_\mu = \mathbf{X}_r \oplus \mathbf{X}_{nr}$ is invariant under $e^{t\mathcal{J}_\mu \mathcal{L}_\mu}$.

By using spherical harmonics, for any $\rho \in X_{\mu,n}$, we write

$$\rho(x) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \rho_{lm}(r) Y_{lm}(\theta);$$

then

$$L_\mu \rho = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} L_{\mu,l} \rho_{lm} \, Y_{lm}(\theta),$$

where

$$L_{\mu,l} = \left(\Phi''(\rho_\mu) - 4\pi \left(-\Delta_r + \frac{l(l+1)}{r^2} \right)^{-1} \right) : X_{\mu,r} \to X^*_{\mu,r}. \tag{3.44}$$

By Lemma 3.8 and the proof of Lemma 3.7, we have

$$n^-\big(L_{\mu,l}|_{X_{\mu,r}}\big) = n^-\big(D^l_\mu\big) = 0 \quad \forall l \geq 1.$$

Therefore,

$$n^-(L_\mu|_{X_{\mu,n}}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} n^-(L_{\mu,l}|_{X_{\mu,r}}) = 0.$$

Since $\widetilde{A}_\mu > 0$ on $Y_{\mu,n}$, by Theorem 2.3, there is no unstable eigenvalue for the system (3.43). Moreover, we shall show that all the eigenvalues of (3.43) are isolated with finite multiplicity. Define the space

$$Z_{\mu,n} = \big\{ \xi \in Y_{\mu,n} \mid \widetilde{A}_\mu \xi \in X_{\mu,n} \big\}$$

with the norm

$$\|\xi\|_{Z_{\mu,n}} = \|\nabla\xi\|_{L^2_{\rho_\mu}} + \left\|\widetilde{A}_\mu\xi\right\|_{L^2_{\Phi''(\rho_\mu)}}.$$

Then by Theorem 2.3, it suffices to show that the embedding $Z_{\mu,n} \hookrightarrow Y_{\mu,n}$ is compact. This follows from proposition 12 in [21].

By using spherical harmonics, we can further decompose (3.43). For $(\sigma_2, \xi_2) \in X_{\mu,n} \times Y_{\mu,n}$, let

$$\sigma_2(x) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \sigma_{lm}(r,t)Y_{lm}(\theta), \quad \xi_2(x,t) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \xi_{lm}(r,t)Y_{lm}(\theta).$$

For each $l \geq 1$, $-l \leq m \leq l$, the component $(\sigma_{lm}(r,t), \xi_{lm}(r,t))$ satisfies the separable Hamiltonian system

$$\partial_t \begin{pmatrix}\sigma_{lm}\\ \xi_{lm}\end{pmatrix} = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}L_{\mu,l} & 0\\ 0 & A_{\mu,l}\end{pmatrix}\begin{pmatrix}\sigma_{lm}\\ \xi_{lm}\end{pmatrix}, \tag{3.45}$$

on the space $X_{\mu,r} \times \widetilde{Y}_{\mu,r}$, where

$$\widetilde{Y}_{\mu,r} = \left\{p(r) \,\middle|\, \int_0^{R_\mu} \rho_\mu\big(r^2(\partial_r p)^2 + p^2\big)dr < \infty\right\}, \tag{3.46}$$

the operator $L_{\mu,l}$ is defined in (3.44), and

$$A_{\mu,l} = -\frac{1}{r^2}\partial_r\big(\rho_\mu r^2\partial_r\big) + \frac{\rho_\mu l(l+1)}{r^2} : \widetilde{Y}_{\mu,r} \to \widetilde{Y}^*_{\mu,r}.$$

By the properties of the operators $L_{\mu,l}$ (equivalently the operators D^l_μ) given in Lemma 3.8, it is easy to see that, when $l > 1$, all the eigenvalues of (3.45) are nonzero and purely imaginary. When $l = 1$, (3.45) has a kernel space spanned by $(\rho'_\mu(r), 0)^T$ corresponding to translation modes $(\partial_{x_i}\rho_\mu, 0)^T$ for the linearized Euler-Poisson system (3.18)–(3.19). According to Theorem 2.3, all eigenvalues of $\mathcal{J}_\mu\mathcal{L}_\mu$ restricted to the invariant subspace $X_\mu \times Y_{\mu,2}$, and thus of (3.45), are semisimple except for possibly the zero eigenvalue. Since 0 is an isolated eigenvalue, Theorem 2.3 applied to $\mathcal{J}_\mu\mathcal{L}_\mu|_{X_{\mu,n}\times Y_{\mu,n}}$ implies that the eigenspace of 0 only consists of generalized eigenvectors with finite multiplicity.

Indeed, (3.45) does have a nontrivial generalized eigenvector and thus nontrivial Jordan blocks associated to 0. To see this, for any $\zeta \in \widetilde{Y}_{\mu,r}$, we have

$$\left|\int_{S_\mu} \rho'_\mu\zeta\, dx\right| \leq \|\zeta\|_{\widetilde{Y}_{\mu,r}}\left(\int_{S_\mu}\big(\rho'_\mu\big)^2\rho_\mu^{-1}\, dx\right)^{\frac{1}{2}} \lesssim \|\zeta\|_{\widetilde{Y}_{\mu,r}}$$

where we used $\gamma_0 \in (\frac{6}{5}, 2)$ and

$$\rho_\mu = O\big(|R_\mu - r|^{\frac{1}{\gamma_0-1}}\big), \quad \rho'_\mu = O(|R_\mu - r|^{\frac{1}{\gamma_0-1}-1}), \quad \text{for } |R_\mu - r| \ll 1.$$

Therefore, $\rho'_\mu \in \widetilde{Y}^*_{\mu,r}$ and thus the Lax-Milgram theorem imply that there exists a unique $\zeta(r) \in \widetilde{Y}_{\mu,r}$ such that

$$\rho'_\mu = A_{\mu,1}\zeta = -\frac{1}{r^2}\partial_r\left(\rho_\mu r^2 \partial_r \zeta\right) + \frac{2\rho_\mu}{r^2}\zeta. \tag{3.47}$$

Therefore, $(0, \zeta Y_{1m}(\theta))^T$, $m = 0, \pm 1$, belong to the generalized kernel of (3.45), which corresponds to $(0, \partial_{x_j}(\zeta \frac{x}{r}))^T$, $j = 1, 2, 3$, in the generalized kernel of $\mathcal{J}_\mu \mathcal{L}_\mu$ with

$$\mathcal{J}_\mu \mathcal{L}_\mu\left(0, \partial_{x_j} \nabla \widetilde{\zeta}(|x|)\right)^T = \left(\partial_{x_j}\rho_\mu, 0\right)^T, \quad \widetilde{\zeta}' = \zeta. \tag{3.48}$$

Moreover, these functions in the generalized kernel of $\mathcal{J}_\mu \mathcal{L}_\mu$ do *not* belong to the range $R(\mathcal{J}_\mu \mathcal{L}_\mu)$. In fact, suppose

$$(\mathcal{J}_\mu \mathcal{L}_\mu)(\rho, u)^T = \left(0, \partial_{x_j} \nabla \widetilde{\zeta}\right)^T, \quad (\rho, u)^T \in X_\mu \times Y_\mu.$$

Then one may compute

$$\begin{aligned}\left\langle A_\mu \partial_{x_j} \nabla \widetilde{\zeta}, \partial_{x_j} \nabla \widetilde{\zeta}\right\rangle &= \left\langle L_\mu \begin{pmatrix} 0 \\ \partial_{x_j} \nabla \widetilde{\zeta} \end{pmatrix}, \begin{pmatrix} 0 \\ \partial_{x_j} \nabla \widetilde{\zeta} \end{pmatrix} \right\rangle \\ &= -\left\langle L_\mu J_\mu L_\mu \begin{pmatrix} 0 \\ \partial_{x_j} \nabla \widetilde{\zeta} \end{pmatrix}, \begin{pmatrix} \rho \\ u \end{pmatrix} \right\rangle = -\left\langle L_\mu \begin{pmatrix} \partial_{x_j}\rho_\mu \\ 0 \end{pmatrix}, \begin{pmatrix} \rho \\ u \end{pmatrix} \right\rangle = 0,\end{aligned}$$

which is a contradiction. Therefore, we may conclude that the zero eigenvalue of $\mathcal{J}_\mu \mathcal{L}_\mu|_{X_{\mu,n} \times Y_{\mu,n}}$ has a six-dimensional eigenspace with geometric multiplicity 3 and algebraic multiplicity 6.

The above discussions are summarized below.

PROPOSITION 3.17. *Any nonrotating star ρ_μ is spectrally stable under nonradial perturbations in $\mathbf{X}_{nr}$. All nonzero eigenvalues of* (3.43) *are isolated and of finite multiplicity. The zero eigenvalue of the linearized Euler-Poisson operator $\mathcal{J}_\mu \mathcal{L}_\mu|_{\mathbf{X}_{nr}}$ is isolated with an infinite-dimensional space*

$$(\{0\} \times Y_{\mu 1}) \oplus \operatorname{span}\left\{\left(\partial_{x_j}\rho_\mu, 0\right)^T, \ \left(0, \partial_{x_j}\left(\zeta \frac{x}{r}\right)\right)^T \ \middle| \ j = 1, 2, 3\right\}$$

where $\mathcal{J}_\mu \mathcal{L}_\mu$ has three 2×2 *Jordan blocks associated to* (3.48) *generated by the translation symmetry.*

Remark 3.18. For irrotational perturbations, the eigenvalues of (3.43) were shown to be purely discrete in [21] by a different approach. In [3–5], the spectrum for nonradial perturbations were shown to be countable, and it was conjectured in [3] that zero is the only accumulation point. This is indeed true for barotropic equations of states $P(\rho)$ by the above proposition or results in [21].

Remark 3.19. In the astrophysics literature [1, 2, 25], the stability of nonrotating stars under nonradial perturbations (the Antonov-Lebowitz theorem) was shown by using the physical principle that the stable states should be energy minimizers under the constraint of constant mass. We discuss such an energy principle below.

The steady density ρ_μ has the following variational structure. Define the functional

$$E_\mu(\rho) = \int \Phi(\rho)dx - \frac{1}{8\pi}\int |\nabla V|^2\,dx - V_\mu(R_\mu)\int \rho\,dx, \tag{3.49}$$

with $\Delta V = 4\pi\rho$. Then ρ_μ is a critical point of $E_\mu(\sigma)$, that is, $E'_\mu(\rho_\mu) = 0$, which is exactly the equation (3.57). The second-order variation of E_μ at ρ_μ is

$$\left\langle E''_\mu(\rho_\mu)\rho, \rho\right\rangle = \int \left(\Phi''(\rho_\mu)\rho^2 - \frac{1}{4\pi}|\nabla V|^2\right)dx = \langle L_\mu \rho, \rho\rangle. \tag{3.50}$$

We note that the energy functional

$$E(\rho, u) = \frac{1}{2}\int \rho|u|^2\,dx + \int \Phi(\rho)dx - \frac{1}{8\pi}\int |\nabla V|^2\,dx$$

is conserved for the nonlinear Euler-Poisson equation (1.1)–(1.3). Let $M(\rho) = \int \rho\,dx$ to be the total mass and define

$$I_\mu(\rho, u) = E(\rho, u) - V_\mu(R_\mu)M(\rho) = \frac{1}{2}\int \rho|v|^2\,dx + E_\mu(\rho).$$

Then $(\rho_\mu, 0)$ is a critical point of $I_\mu(\rho, u)$. The second-order variation of $I_\mu(\rho, u)$ at $(\rho_\mu, 0)$ is given by the functional

$$H_\mu(\sigma, v) = \frac{1}{2}\int_{S_\mu} \rho_\mu|v|^2\,dx + \frac{1}{2}\langle L_\mu\sigma, \sigma\rangle$$

as defined in (3.20), which is a conserved quantity of the linearized Euler-Poisson system (3.18)–(3.19).

By the above variational structures, the physical principle that stable stars should be energy minimizers under the constraint of constant mass is equivalent to the statement that ρ_μ is stable only when $\langle E''_\mu(\rho_\mu)\sigma, \sigma\rangle \geq 0$ for all perturbations σ supported in S_μ satisfying the mass constraint $\int \sigma\,dx = 0$. This was also called Chandrasekhar's variational principle [8] in the astrophysical literature [6].

3.5 Turning point principle for radial perturbations

Denote $X_{\mu,r}$ and $Y_{\mu,r}$ to be the radially symmetric subspace of $L_{\Phi''(\rho_\mu)}(S_\mu)$ and $L^2_{\rho_\mu}(S_\mu)$, respectively. By (1.12), the radial component (σ_1, v_1) of (σ, v) as defined in (3.41)–(3.42) satisfies

$$\begin{aligned}\partial_t(\sigma_1 v_1) &= \begin{pmatrix} 0 & -\frac{1}{r^2}\partial_r\left(r^2\cdot\right) \\ -\partial_r & 0\end{pmatrix}\begin{pmatrix}\Phi''(\rho_\mu) - 4\pi(-\Delta_r)^{-1} & 0 \\ 0 & \rho_\mu\end{pmatrix}\begin{pmatrix}\sigma_1 \\ v_1\end{pmatrix} \\ &= \begin{pmatrix} 0 & B_{\mu,r} \\ -B'_{\mu,r} & 0\end{pmatrix}\begin{pmatrix} L_{\mu,r} & 0 \\ 0 & A_{\mu,r}\end{pmatrix}\begin{pmatrix}\sigma_1 \\ v_1\end{pmatrix} = J^\mu L^\mu \begin{pmatrix}\sigma_1 \\ v_1\end{pmatrix}.\end{aligned}$$

Here, $\sigma_1 \in X_{\mu,r}$, $v_1 \in Y_{\mu,r}$, and the operators

$$L_{\mu,r} = \Phi''(\rho_\mu) - 4\pi(-\Delta_r)^{-1} : X_{\mu,r} \to X^*_{\mu,r}, \tag{3.51}$$

$$A_{\mu,r} = \rho_\mu : Y_{\mu,r} \to Y^*_{\mu,r}, \tag{3.52}$$

$$B_{\mu,r} = -\frac{1}{r^2}\partial_r(r^2\cdot) : Y^*_{\mu,r} \to X_{\mu,r}, \quad B'_{\mu,r} = \partial_r : X^*_{\mu,r} \to Y_{\mu,r}, \tag{3.53}$$

and

$$J^\mu = \begin{pmatrix} 0 & B_{\mu,r} \\ -B'_{\mu,r} & 0 \end{pmatrix} : X^*_{\mu,r} \times Y^*_{\mu,r} \to X_{\mu,r} \times Y_{\mu,r}, \tag{3.54}$$

$$L^\mu = \begin{pmatrix} L_{\mu,r} & 0 \\ 0 & A_{\mu,r} \end{pmatrix} : X_{\mu,r} \times Y_{\mu,r} \to X^*_{\mu,r} \times Y^*_{\mu,r}. \tag{3.55}$$

As the triple $\big(L_\mu, A_\mu, B_\mu\big)$ in (1.12) satisfies assumptions (**G1–G4**) in Section 2, the above reduction procedure and Lemma 2.10 imply that the triple $(L_{\mu,r}, A_{\mu,r}, B_{\mu,r})$ satisfies (**G1–G4**) as well. Thus, (3.51) is a separable Hamiltonian system, for which Theorem 2.3 is applicable.

PROOF OF THEOREM 1.2 II). By Theorem 2.3, the linear stability/instability of (3.51) is reduced to finding $n^u(\mu) = n^-(L_{\mu,r}|_{\overline{R(B_{\mu,r})}})$. By the proof of Lemma 3.7 restricted to radial spaces, we have $n^-(L_{\mu,r}) = n^-(D^0_\mu)$ where D^0_μ is defined by (3.27). Moreover, it holds that

$$\overline{R\big(B_{\mu,r}\big)} = \big(\ker B'_{\mu,r}\big)^\perp = (\ker \partial_r)^\perp = \left\{\rho \in X_{\mu,r} \;\middle|\; \int_{S_\mu} \rho\, dx = 0\right\}. \tag{3.56}$$

Therefore, to find $n^-(L_{\mu,r}|_{\overline{R(B_{\mu,r})}})$ it is equivalent to determine the negative dimensions of the quadratic form $\big\langle L_{\mu,r}\cdot,\cdot\big\rangle$ under the mass constraint $\int_{S_\mu} \rho\, dx = 0$. We divide the argument into three cases.

Case 1. $\frac{d}{d\mu}\Big(\frac{M(\mu)}{R(\mu)}\Big) \neq 0$. By (3.3) and Lemma 3.9, the steady density ρ_μ satisfies the equation

$$\Phi'(\rho_\mu) - 4\pi(-\Delta)^{-1}\rho_\mu = V_\mu(R_\mu) = -\frac{M(\mu)}{R_\mu}, \tag{3.57}$$

inside the support S_μ. Applying ∂_μ to the above equation, we get

$$L_\mu \frac{\partial \rho_\mu}{\partial \mu} = \Phi''(\rho_\mu)\frac{\partial \rho_\mu}{\partial \mu} - 4\pi(-\Delta)^{-1}\frac{\partial \rho_\mu}{\partial \mu} = -\frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right) \quad \text{in } S_\mu, \tag{3.58}$$

which implies that

$$\overline{R\big(B_{\mu,r}\big)} = \left\{\rho \;\middle|\; \left\langle L_{\mu,r}\frac{\partial \rho_\mu}{\partial \mu}, \rho\right\rangle = 0\right\}$$

and

$$\left\langle L_{\mu,r}\frac{\partial \rho_\mu}{\partial \mu}, \frac{\partial \rho_\mu}{\partial \mu}\right\rangle = -\frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right)\int_{S_\mu}\frac{\partial \rho_\mu}{\partial \mu}dx = -\frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right)M'(\mu). \tag{3.59}$$

Case 1a. $M'(\mu) \neq 0$. The above properties immediately yield

$$n^-\big(L_{\mu,r}|_{\overline{R(B_{\mu,r})}}\big) = \begin{cases} n^-(L_{\mu,r})r - 1 & \text{if } M'(\mu)\frac{d}{d\mu}\Big(\frac{M(\mu)}{R_\mu}\Big) > 0, \\ n^-(L_{\mu,r}) & \text{if } M'(\mu)\frac{d}{d\mu}\Big(\frac{M(\mu)}{R_\mu}\Big) < 0, \end{cases}$$
$$= n^-\big(D_\mu^0\big) - i_\mu.$$

Case 1b. $M'(\mu) = 0$. In this case, we have

$$\left\langle L_{\mu,r}\frac{\partial \rho_\mu}{\partial \mu}, \frac{\partial \rho_\mu}{\partial \mu}\right\rangle = 0, \quad \frac{\partial \rho_\mu}{\partial \mu} \in \overline{R(B_{\mu,r})}, \quad \ker L_{\mu,r} = \{0\},$$

where Lemma 3.10 was used. There exists $\psi \notin \overline{R(B_{\mu,r})}$. Let

$$Z_0 = \operatorname{span}\left\{\psi, \frac{\partial \rho_\mu}{\partial \mu}\right\}, \quad Z_1 = \{\rho \in \overline{R(B_{\mu,r})} \mid \langle L_{\mu,r}\psi, \rho\rangle\} = 0,$$

and we have

$$X_{\mu,r} = Z_0 \oplus Z_1, \quad \overline{R(B_{\mu,r})} = Z_1 \oplus \mathbb{R}\frac{\partial \rho_\mu}{\partial \mu}.$$

We obtain from lemma 12.3 in [30] and (3.59) that

$$n^-(L_{\mu,r}|_{\overline{R(B_{\mu,r})}}) = n^-(L_{\mu,r}|_{Z_1}), \quad n^-(L_{\mu,r}) = n^-(L_{\mu,r}|_{Z_1}) + n^-(L_{\mu,r}|_{Z_0}).$$

It is straightforward to compute $n^-(L_{\mu,r}|_{Z_0}) = 1$ and thus

$$n^-(L_{\mu,r}|_{\overline{R(B_{\mu,r})}}) = n^-(L_{\mu,r}) - 1 = n^-\big(D_\mu^0\big) - i_\mu.$$

Case 2. $\frac{d}{d\mu}\Big(\frac{M(\mu)}{R(\mu)}\Big) = 0$. By Lemma 3.13, we have

$$\int_{S_\mu} \frac{\partial \rho_\mu}{\partial \mu}\, dx = M'(\mu) \neq 0 \Longrightarrow \frac{\partial \rho_\mu}{\partial \mu} \notin \overline{R(B_{\mu,r})}.$$

Therefore,

$$X_{\mu,r} = \overline{R(B_{\mu,r})} \oplus \mathbb{R}\frac{\partial \rho_\mu}{\partial \mu},$$

which implies $n^-(L_{\mu,r}|_{\overline{R(B_{\mu,r})}}) = n^-(L_{\mu,r})$. □

Remark 3.20. If μ belongs to a stable interval, we must have $\frac{d}{d\mu}\Big(\frac{M(\mu)}{R(\mu)}\Big) \neq 0$. Indeed, when $\frac{d}{d\mu}\Big(\frac{M(\mu)}{R(\mu)}\Big) = 0$, by (1.16) and Lemma 3.8, we have $n^u(\mu) = n^-\big(D_\mu^0\big) \geq 1$.

To prove Theorem 1.2(iii), by Proposition 3.17 it remains to show that the eigenvalues of the operator $\mathcal{J}^\mu\mathcal{L}^\mu$ defined in (3.51) are purely isolated and

$$\ker J^\mu L^\mu = \operatorname{span}\left\{\begin{pmatrix} \partial_\mu \rho_\mu \\ 0 \end{pmatrix}\right\}. \tag{3.60}$$

We first prove (3.60) and leave the proof of the discreteness of eigenvalues of $J^{\mu}L^{\mu}$ to the end of this section. By (3.58) we have

$$L^{\mu}\begin{pmatrix}\partial_{\mu}\rho_{\mu}\\ 0\end{pmatrix} = -\frac{d}{d\mu}\left(\frac{M(\mu)}{R(\mu)}\right)\begin{pmatrix}1\\ 0\end{pmatrix};$$

thus $\operatorname{span}\left\{\begin{pmatrix}\partial_{\mu}\rho_{\mu}\\ 0\end{pmatrix}\right\} \subset \ker J^{\mu}L^{\mu}$. To prove $\ker J^{\mu}L^{\mu} \subset \operatorname{span}\left\{\begin{pmatrix}\partial_{\mu}\rho_{\mu}\\ 0\end{pmatrix}\right\}$, we consider two cases. Suppose $J^{\mu}L^{\mu}\begin{pmatrix}\sigma\\ v\end{pmatrix} = 0$ for some nonzero $\begin{pmatrix}\sigma\\ v\end{pmatrix} \in X_{\mu,r} \times Y_{\mu,r}$. It is easy to check that $v = 0$ and $L_{\mu,r}\sigma = c$ for some constant c.

Case 1. $\frac{d}{d\mu}\left(\frac{M(\mu)}{R(\mu)}\right) \neq 0$. Then

$$L_{\mu,r}\left(\sigma + \frac{c}{\frac{d}{d\mu}\left(\frac{M(\mu)}{R_{\mu}}\right)}\partial_{\mu}\rho_{\mu}\right) = 0.$$

This implies that

$$\sigma = -\frac{c}{\frac{d}{d\mu}\left(\frac{M(\mu)}{R_{\mu}}\right)}\partial_{\mu}\rho_{\mu},$$

since by Lemma 3.10, $\dim L_{\mu,r} = \dim\ker D_{\mu}^{0} = 0$.

Case 2. $\frac{d}{d\mu}\left(\frac{M(\mu)}{R(\mu)}\right) = 0$. Then $\ker L_{\mu,r} = \operatorname{span}\{\partial_{\mu}\rho_{\mu}\}$ and $M'(\mu) \neq 0$ by Lemma 3.13. From $L_{\mu,r}\sigma = c$ we have

$$0 = \langle L_{\mu,r}\partial_{\mu}\rho_{\mu}, \sigma\rangle = \langle L_{\mu,r}\sigma, \partial_{\mu}\rho_{\mu}\rangle = cM'(\mu).$$

Thus $c = 0$ and $L_{\mu,r}\sigma = 0$, which again imply that $\sigma \in \operatorname{span}\{\partial_{\mu}\rho_{\mu}\}$. This proves (3.60).

Next, we prove the turning point principle by using Theorem 1.2.

PROOF OF THEOREM 1.1. By Lemma 3.12, when μ is small enough, $n^{-}(D_{\mu}^{0}) = 1$. By the proof of Lemma 3.2, when μ is small, we have

$$\rho_{\mu} = F_{+}\left(\alpha\theta_{\alpha}\left(\alpha^{\frac{n_0-1}{2}}r\right)\right), \quad \alpha = \Phi'(\mu),\ n_0 = \frac{1}{\gamma_0 - 1}.$$

Here, $\theta_{\alpha} \to \theta_0$ in $C^1(0, R)$ for any $R > 0$, and θ_0 is the Lane-Emden function satisfying (3.16). The support radius of ρ_{μ} is

$$R_{\mu} = \alpha^{-\frac{n_0-1}{2}}S_{\alpha} = \alpha^{-\frac{2-\gamma_0}{2(\gamma_0-1)}}S_{\alpha},$$

where S_{α} is C^1 in α, and when $\alpha \to 0$, $S_{\alpha} \to R_0$, the support radius of θ_0. The total mass is

$$\begin{aligned} M(\mu) &= 4\pi\int_0^{R_{\mu}} F_{+}\left(\alpha\theta_{\alpha}\left(\alpha^{\frac{n_0-1}{2}}r\right)\right)r^2\,dr \\ &= \alpha^{\frac{1}{2}\frac{(3\gamma_0-4)}{\gamma_0-1}}\int_0^{S_{\alpha}} g_{\alpha}(\theta_{\alpha}(s))s^2\,ds, \end{aligned}$$

where $g_\alpha \to g_0$ in $C^1(0,1)$ with g_α, g_0 defined in (3.14) and (3.15). So

$$\int_0^{S_\alpha} g_\alpha(\theta_\alpha(s))s^2\,ds \to \int_0^{R_0} g_0(\theta_0(s))s^2\,ds > 0 \quad \text{when } \alpha \to 0.$$

Thus for μ small, we have: (i) $\frac{M(\mu)}{R(\mu)} \approx \alpha = \Phi'(\mu)$ and $\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big) > 0$; (ii) $M'(\mu) > 0$ when $\gamma_0 \in \big(\frac{4}{3}, 2\big)$ and $M'(\mu) < 0$ when $\gamma_0 \in \big(\frac{6}{5}, \frac{4}{3}\big)$. Thus when μ is small, the formula (1.6) for $n^u(\mu)$ follows from Theorem 1.2.

Next, we keep track of the changes of $n^u(\mu)$ along the mass–radius curve by increasing μ. We consider four cases.

Case 1. No critical points of $\frac{M(\mu)}{R(\mu)}$ or $M(\mu)$ are met. Then $\frac{d}{d\mu}\Big(\frac{M(\mu)}{R_\mu}\Big)$ and $M'(\mu)$ do not change sign. By Lemma 3.12 and (1.15), $n^u(\mu)$ is unchanged.

Case 2. At a critical point μ^* of $\frac{M(\mu)}{R_\mu}$. The jump formula (3.38) implies that

$$n^u(\mu^*+) = n^-\big(D^0_{\mu^*+}\big) - i_{\mu^*+} = n^-\big(D^0_{\mu^*-}\big) - i_{\mu^*-} = n^u\big(\mu^*-\big).$$

That is, the number of unstable modes remains unchanged when crossing μ^*.

Case 3. At an extremum (i.e., maximum or minimum) point $\bar{\mu}$ of $M(\mu)$ where $M'(\mu)$ changes sign, then $\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big)|_{\mu=\bar{\mu}} \neq 0$ and $n^-(D^0_\mu)$ is the same in a neighborhood of $\bar{\mu}$. But $M'(\mu)$ changes sign when crossing $\bar{\mu}$; thus we have

$$n^u(\bar{\mu}+) - n^u(\bar{\mu}-) = -(i_{\mu+} - i_{\mu-}) = \pm 1,$$

when $M'(\mu)\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big)$ changes from $\pm$ to $\mp$ at $\bar{\mu}$. Since

$$M'(\bar{\mu}) = 0 \quad \text{and} \quad \frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right)\Big|_{\mu=\bar{\mu}} \neq 0,$$

when μ is near $\bar{\mu}$, we have $R'\big(\bar{\mu}\big) \neq 0$ and the sign of $M'(\mu)\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big)$ is the same as $-M'(\mu)R'(\mu)$. Thus $n^u(\bar{\mu}+) - n^u(\bar{\mu}-) = \pm 1$ when $M'(\mu)R'(\mu)$ changes from $\mp$ to $\pm$ at $\bar{\mu}$ or, equivalently the mass–radius curve bends counterclockwise (clockwise) at $\bar{\mu}$.

Case 4. At a critical but nonextremum point of $M(\mu)$, it holds that

$$\frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right)\Big|_{\mu=\tilde{\mu}} \neq 0,$$

and $n^-(D^0_\mu)$ is the same for μ near $\tilde{\mu}$. Since $\tilde{\mu}$ is not an extremum point of $M(\mu)$, the sign of $M'(\mu)$ does not change when crossing $\tilde{\mu}$. Then by (1.15) $n^u(\mu)$ does not change when crossing $\tilde{\mu}$. However, we should note that if

$$M'(\mu)\frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right) > 0$$

or equivalently $M'(\mu)R'(\mu) < 0$ in a neighborhood of $\tilde{\mu}$ excluding $\tilde{\mu}$, then $n^u(\mu)$ has a removable jump discontinuity at $\tilde{\mu}$ where $n^u(\mu)$ is reduced by one.

Summing up the above discussions, we finish the proof of Theorem 1.1. □

Below, we prove Theorem 1.3 about exponential trichotomy estimates of (3.51).

PROOF OF THEOREM 1.3. Conclusion (i) is by Theorems 1.2 and 2.3. Conclusion (ii) and (1.17) follow directly from Theorem 2.3. To prove (1.18) and (1.19), we consider radial and nonradial perturbations separately. For nonradial perturbations, Proposition 3.17 implies that all eigenvalues are discrete and on the imaginary axis. Hence according to the block decomposition and the anti-self-adjointness of $\mathbf{T}_3$ in Theorem 2.3, the algebraic growth can only arise from the generalized kernel. By Theorem 1.2(i), we have

$$\left|e^{t\mathcal{J}_\mu\mathcal{L}_\mu}|_{E^c\cap\mathbf{X}_{nr}}\right| = \left|e^{t\mathcal{J}_\mu\mathcal{L}_\mu}|_{\mathbf{X}_{nr}}\right| \le C_0(1+|t|). \tag{3.61}$$

For radial perturbations, when $M'(\mu)=0$, by Theorem 2.6(i), we have

$$\left|e^{t\mathcal{J}_\mu\mathcal{L}_\mu}|_{E^c\cap\mathbf{X}_r}\right| \le C_0(1+|t|)^2,$$

and (1.19) follows by combining it with (3.61). When $M'(\mu)\neq 0$, we check that $L_{\mu,r}|_{\overline{R(B_{\mu,r})}}$ is nondegenerate. Let $W_1 = \operatorname{span}\{\frac{\partial\rho_\mu}{\partial\mu}\}$. Since $\int \frac{\partial\rho_\mu}{\partial\mu}\,dx = M'(\mu)\neq 0$, there is an invariant decomposition $\mathbf{X}_r = R(B_{\mu,r})\oplus W_1$. When $M'(\mu)\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big)\neq 0$, by the proof of Theorem 1.2(ii), $\overline{R(B_{\mu,r})}$ is the $L_{\mu,r}$-orthogonal complement space of $W_1 = \operatorname{span}\{\frac{\partial\rho_\mu}{\partial\mu}\}$. The nondegeneracy of

$$L_{\mu,r}|_{\overline{R(B_{\mu,r})}}$$

follows since $\ker L_{\mu,r} = \{0\}$, and $L_{\mu,r}|_{W_1}$ is nondegenerate by (3.59). When $\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big) = 0$, we have $\ker L_{\mu,r} = W_1$ and the nondegeneracy of $L_{\mu,r}|_{\overline{R(B_{\mu,r})}}$ also follows. Thus by Theorem 2.6(iii), we have $|e^{t\mathcal{J}_\mu\mathcal{L}_\mu}|_{E^c\cap\mathbf{X}_r}| \le C_0$, which implies Conclusion (iv) and (1.18). □

It remains to prove that the eigenvalues of the linearized problem (3.51) for radial perturbations are all discrete by Theorem 2.3. We need the following Hardy's inequality [20, 24].

LEMMA 3.21 (Hardy's inequality). *Let k be a real number and g be a function satisfying*

$$\int_0^1 s^k\big(g^2+|g'|^2\big)\,ds<\infty.$$

(i) *If $k>1$, then we have*

$$\int_0^1 s^{k-2}g^2\,ds \lesssim \int_0^1 s^k\big(g^2+|g'|^2\big)\,ds.$$

(ii) *If $k<1$, then g has a trace at $x=0$ and*

$$\int_0^1 s^{k-2}(g-g(0))^2\,ds \lesssim C\int_0^1 s^k|g'|^2\,ds. \tag{3.62}$$

Define the function space $Z_{\mu,r}$ to be the closure of $D(B_{\mu,r}A_{\mu,r}) \subset Y_{\mu,r}$ under the graph norm

$$\begin{aligned}\|v\|_{Z_{\mu,r}} &= \|v\|_{Y_{\mu,r}} + \|B_{\mu,r}A_{\mu,r}v\|_{X_{\mu,r}} \\ &= \left(\int_0^{R_\mu} \rho_\mu |v|^2 r^2\,dr\right)^{\frac{1}{2}} + \left(\int_0^{R_\mu} \Phi''(\rho_\mu)\left|\frac{1}{r^2}\partial_r\big(r^2\rho_\mu v\big)\right|^2 r^2\,dr\right)^{\frac{1}{2}}.\end{aligned}$$

By Theorem 2.3, to show the discreteness of eigenvalues for radial perturbations, it suffices to show the following compactness lemma.

LEMMA 3.22. *The embedding $Z_{\mu,r} \hookrightarrow Y_{\mu,r}$ is compact.*

PROOF. First, near the support radius R_μ we have $\rho_\mu(r) \approx (R_\mu - r)^{1/(\gamma-1)}$. This is well-known for Lane-Emden stars. To be self-contained, we give a proof for general equations of state. By (3.7), we have

$$y_\mu'(R_\mu) = -\frac{4\pi}{R_\mu^2}\int_0^{R_\mu} s^2 F_+\big(y_\mu(s)\big)\,ds = -\frac{1}{R_\mu^2}M(\mu) < 0.$$

Thus for r near R_μ, $y_\mu(r) \approx R_\mu - r$. Since $\rho_\mu(r) = F_+(y_\mu(r))$ and $F_+(y) \approx y^{1/(\gamma-1)}$ for $0 < y \ll 1$, we deduce that for r near R_μ,

$$\rho_\mu(r) \approx \big(y_\mu(r)\big)^{\frac{1}{\gamma-1}} \approx (R_\mu - r)^{\frac{1}{\gamma-1}}. \tag{3.63}$$

Then for r near R_μ,

$$\Phi''\big(\rho_\mu(r)\big) \approx \rho_\mu(r)^{\gamma-2} \approx (R_\mu - r)^{\frac{\gamma-2}{\gamma-1}}. \tag{3.64}$$

Let $r_2 < R_\mu$ and $R_\mu - r_2$ be small enough so that (3.63) and (3.64) are valid in (r_2, R_μ). Then for any $v \in Z_{\mu,r}$, we have

$$\begin{aligned}&\int_{r_2}^{R_\mu} \Phi''(\rho_\mu)\left|\frac{1}{r^2}\partial_r\big(r^2\rho_\mu v\big)\right|^2 r^2\,dr \\ &\gtrsim \int_{r_2}^{R_\mu} (R_\mu - r)^{\frac{\gamma-2}{\gamma-1}}\left|\partial_r\big(r^2\rho_\mu v\big)\right|^2 dr \\ &\gtrsim \int_{r_2}^{R_\mu} (R_\mu - r)^{\frac{\gamma-2}{\gamma-1}-2}\left|r^2\rho_\mu v\right|^2 dr \text{ (by Hardy's inequality (3.62))} \\ &\gtrsim \int_{r_2}^{R_\mu} (R_\mu - r)^{-1}\rho_\mu v^2\,dr \text{ (by (3.63))} \\ &\gtrsim (R_\mu - r_2)^{-1}\int_{r_2}^{R_\mu} \rho_\mu v^2\,dr.\end{aligned}$$

Thus,

$$\int_{r_2}^{R_\mu} \rho_\mu |v|^2 r^2\,dr \lesssim (R_\mu - r_2)\|B_{\mu,r}A_{\mu,r}v\|_{X_{\mu,r}}^2. \tag{3.65}$$

Let $r_1 \in (0, R_\mu)$ be small enough so that

$$\frac{1}{2}\mu \le \rho_\mu(r) \le \mu \quad \forall r \in (0, r_1);$$

then

$$0 < \delta_1(\mu) \le \Phi''(\rho_\mu) \le \delta_2(\mu) \quad \forall r \in (0, r_1),$$

where $\delta_1(\mu) = \min_{\rho\in(\frac{1}{2}\mu,\mu)} \Phi''(\rho)$ and $\delta_2(\mu) = \max_{\rho\in(\frac{1}{2}\mu,\mu)} \Phi''(\rho)$. We have

$$\begin{aligned}&\int_0^{r_1} \Phi''(\rho_\mu)\left|\frac{1}{r^2}\partial_r\left(r^2\rho_\mu v\right)\right|^2 r^2\,dr\\ &\quad\gtrsim \int_0^{r_1}\frac{1}{r^2}\left|\partial_r\left(r^2\rho_\mu v\right)\right|^2 dr \gtrsim \int_0^{r_1}(\rho_\mu v)^2\,dr \text{ (by (3.62))}\\ &\quad\gtrsim r_1^{-2}\int_0^{r_1} v^2r^2\,dr.\end{aligned}$$

Thus,

$$\int_0^{r_1}\rho_\mu|v|^2r^2\,dr \lesssim r_1^2\|B_{\mu,r}A_{\mu,r}v\|^2_{X_{\mu,r}}. \tag{3.66}$$

Denote $B_Z = \{v \in Z_{\mu,r} \mid \|v\|_{Z_{\mu,r}} \le 1\}$ to be the unit ball in $Z_{\mu,r}$. Then for any $\varepsilon > 0$, by estimates (3.65) and (3.66), we can choose $0 < r_1 < r_2 < R_\mu$ such that

$$\int_0^{r_1}\rho_\mu|v|^2r^2\,dr + \int_{r_2}^{R_\mu}\rho_\mu|v|^2r^2\,dr \le \varepsilon \quad \forall v \in B_Z.$$

The compactness of $Z_{\mu,r} \hookrightarrow Y_{\mu,r}$ follows from the above estimate and the compactness of the embedding $Z_{\mu,r} \hookrightarrow L^2(r_1, r_2)$. □

Remark 3.23. The stability criterion $L_{\mu,r}|_{\overline{R(B_{\mu,r})}} \ge 0$ has the following physical meaning. By (3.50), the quadratic form $\langle L_{\mu,r}\rho, \rho\rangle$ is the second-order variation of the energy functional $E_\mu(\rho)$ defined in (3.49). By (3.56), the space $\overline{R(B_{\mu,r})}$ consists of perturbations satisfying the mass constraint. Thus, our stability criterion verifies Chandrasekhar's variational principle that stable states should be energy minimizers under the mass constraint (see also Remark 3.19).

Remark 3.24. In the astrophysical literature, the linear radial oscillations were usually studied through the singular Sturm-Liouville equation

$$\frac{d}{dr}\left(\Gamma_1P_\mu\frac{1}{r^2}\frac{d}{dr}(r^2\xi)\right) - \frac{4}{r}\frac{dP_\mu}{dr}\xi + \omega^2\rho_\mu\xi = 0, \tag{3.67}$$

with the boundary conditions

$$\xi(0) = 0 \quad\text{and}\quad \xi(R_\mu) \text{ is finite.} \tag{3.68}$$

Here, ξ is the linearized Lagrangian displacement in the radial direction, $P_\mu = P(\rho_\mu)$, $\Gamma_1 = \frac{\rho_\mu P'(\rho_\mu)}{P(\rho_\mu)}$ is the local polytropic index, and $i\omega$ is the eigenvalue. The equation (3.67) was first derived by Eddington in 1918 [12] and had been widely used in later works (e.g., [10, 20, 26, 27, 35]). For polytropic stars $P(\rho) = K\rho^{\gamma}$, $\Gamma_1 = \gamma$, (3.67) is greatly simplified and can be used to show $\gamma = \frac{4}{3}$ is the critical index for stability [26, 27]. However, for a general equation of states, it is difficult to get explicit stability criteria such as TPP in Theorem 1.1 by (3.67). Moreover, since the Sturm-Liouville problem (3.67) is singular near $r = 0$ and R_μ, it is highly nontrivial [3–5, 21, 27, 35]) to prove self-adjointness and discreteness of eigenvalues, which were taken for granted in the astrophysical literature.

By the separable Hamiltonian formulation (3.51), the eigenvalue equation can be written as (see (2.17))

$$B'_{\mu,r} L_{\mu,r} B_{\mu,r} A_{\mu,r} v = \omega^2 v, \tag{3.69}$$

which is equivalent to (3.67) by explicit calculations. There are several advantages of the factorized form (3.69) over (3.67). First, each factor in (3.69) has a clear physical meaning related to the variational structures of steady states or the physical constraint. Second, the form in (3.69) makes it convenient to prove properties of the operator $B'_{\mu,r} L_{\mu,r} B_{\mu,r} A_{\mu,r}$ such as the self-adjointness and discreteness of eigenvalues. This approach is rather flexible and has been used in recent works on the stability of rotating stars [28] and relativistic stars [16, 17].

3.6 Examples

We apply the stability criteria for several examples of gaseous stars.

1. Polytropic stars

For polytropic stars, $P(\rho) = K\rho^{\gamma}$ with $\gamma \in (\frac{6}{5}, 2)$. Then by Lemma 3.11, we have $n^-(D^0_\mu) = 1$ for any $\mu > 0$. The functions $M(\mu)$ and R_μ are given by (3.17). For any $\gamma > 1$, we have

$$\frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right) > 0 \quad \text{for all } \mu > 0.$$

When $\gamma \in (\frac{6}{5}, \frac{4}{3})$ we have $M'(\mu) < 0$ and thus $i_\mu = 0$. Then it follows from Theorems 1.2 and 1.3 that for any $\mu > 0$, ρ_μ is unstable with $n^u(\mu) = 1$ and there is Lyapunov stability on the codim 2 center space. When $\gamma \in (\frac{4}{3}, 2)$, we have $M'(\mu) > 0$ and thus $i_\mu = 1$. By Theorems 1.3(iv), linear Lyapunov stability holds for any $\mu > 0$. The case $\gamma = \frac{4}{3}$ is the critical index for stability. In this case, we have $M'(\mu) = 0$. Thus, $i_\mu = 1$ and we have spectral stability. In [11], nonlinear instability was shown for $\gamma = \frac{4}{3}$ in the sense that for any small perturbation with positive total energy of stationary solutions, either the support of the density will go to infinity or singularity forms in the solution in finite time.

2. White dwarf stars

Next, we consider white dwarf stars [7] with $P_{\mathrm{w}}(\rho) = Af(x)$ and $\rho = Bx^3$, where A, B are two constants and

$$f(x) = x(x^2+1)^{\frac{1}{2}}(2x^2-3) + 3\ln\big(x + \sqrt{1+x^2}\big) = 8\int_0^x \frac{u^4\,du}{\sqrt{1+u^2}}. \tag{3.70}$$

Then $P_{\mathrm{w}}(\rho)$ satisfies (1.5) with $\gamma_0 = \frac{5}{3}$. Therefore, for any center density $\mu \in (0,\infty)$, there exists a unique nonrotating star $\rho_\mu(|x|)$ (see Remark 3.3). It was shown in [42] (see also [19]) that $M'(\mu) > 0$ for any $\mu > 0$.

LEMMA 3.25. *Assume $P(\rho)$ satisfies (1.5) with $\gamma_0 \in (\frac{4}{3}, 2)$. Let $\mu_0 \in (0, +\infty]$ be such that $M'(\mu) \geq 0$ on $[0, \mu_0)$. Then*

$$\frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right) > 0 \quad \textit{for any } \mu \in (0, \mu_0).$$

PROOF. By the proof of Theorem 1.1 we have

$$\frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right) > 0 \quad \text{and} \quad M'(\mu) > 0$$

when μ is small enough. Suppose the conclusion of the lemma is not true. Let $\mu_1 \in (0, \mu_0)$ be the first zero of $\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big)$. Then $\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big) > 0$ for all $\mu \in (0, \mu_1)$. Consequently, by Lemma 3.12, $n^-(D^0_\mu) = 1$ for all $\mu \in (0, \mu_1)$. At $\mu = \mu_1$, we have $n^-(D^0_{\mu_1}) \geq 1$ (by Lemma 3.8(iii)) and 0 is an eigenvalue of $D^0_{\mu_1}$. Since $M'(\mu_1) > 0$ due to our assumption and Lemma 3.10, when $\mu < \mu_1$ and $|\mu - \mu_1|$ is small enough, we have $\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big)M'(\mu) > 0$. Therefore $i_\mu = 1$ according to (1.14) and thus Proposition 3.14 implies $n^-(D^0_\mu) = n^-(D^0_{\mu_1}) + 1 \geq 2$. This is in contradiction to the fact that $n^-(D^0_\mu) = 1$ for $\mu \in (0, \mu_1)$. ☐

COROLLARY 3.26. *White dwarf stars $\rho_\mu(|x|)$ are linearly stable for any center density $\mu > 0$.*

PROOF. Lemmas 3.25 and 3.12 imply that $n^-(D^0_\mu) = 1$ for all $\mu > 0$. Since $M'(\mu) > 0$ and $\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big) > 0$ for $\mu \in (0,\infty)$, linear Lyapunov stability of ρ_μ follows from Theorem 1.3(iv). ☐

Remark 3.27. The mass of white dwarf stars has a finite upper bound $M_\infty = \lim_{\mu\to\infty} M(\mu)$, which is known as Chandrasekhar's limit [7, 9]. We note that for white dwarf stars, $P_{\mathrm{w}}(\rho) \approx 2AB^{-4/3}\rho^{4/3}$ when ρ is large. The Chandrasekhar limit M_∞ is exactly the mass of the polytropic star with $P(\rho) = 2AB^{-4/3}\rho^{4/3}$, which is independent of μ by (3.17).

3. More general equations of state

Last, we consider general equations of state $P(\rho)$ satisfying (1.4)–(1.5). Assume $\gamma_0 \in (\frac{4}{3}, 2)$ in (1.5). Indeed, $\gamma_0 = \frac{5}{3}$ for most physical equations of state including white dwarf stars. Then for μ small, we have

$$n^-\big(D_\mu^0\big) = 1, \quad M'(\mu) > 0, \quad \frac{d}{d\mu}\left(\frac{M(\mu)}{R_\mu}\right) > 0.$$

Let

$$\mu_0 = \inf\{\mu > 0 \mid M'(\mu) < 0\} \in (0, +\infty].$$

If $\mu_0 < +\infty$ and $M'(\mu) < 0$ for $0 < \mu - \mu_0 \ll 1$, we denote

$$\mu_1 = \sup\{\mu > \mu_0 \mid M'(\mu') < 0 \quad \forall \mu' \in (\mu_0, \mu)\} \in (\mu_0, +\infty].$$

COROLLARY 3.28. *Assume $P(\rho)$ satisfies* (1.5) *with $\gamma_0 \in (\frac{4}{3}, 2)$. Then the non-rotating star $\rho_\mu(|x|)$ is linearly stable for $\mu \in (0, \mu_0)$. If $\mu_0 < +\infty$, then ρ_μ is linearly unstable for $\mu \in (\mu_0, \mu_1)$ and $n^u(\mu) = 1$.*

PROOF. Linear stability of $\rho_\mu(|x|)$ for $\mu \in (0, \mu_0)$ follows as in Corollary 3.26. When $\mu_0 < \infty$, linear instability of ρ_μ for $\mu \in (\mu_0, \mu_1)$ and $n^u(\mu) = 1$ follows from Theorem 1.1. □

If $M(\mu)$ has isolated extremum points, then μ_0, μ_1 are the first maximum and minimum points, respectively. Below we give examples of $P(\rho)$ for which the maximum of $M(\mu)$ is obtained at a finite center density, which gives the first transition point of stability. As in [19], we consider asymptotically polytropic equations of state satisfying that, for some positive constants $a_0, a_1, n_0, n_1, c_-, c_+$,

(i)

$$P(\rho) = c_- \rho^{\frac{n_0+1}{n_0}}\big(1 + O\big(\rho^{\frac{a_0}{n_0}}\big)\big) \quad \text{when } \rho \to 0; \tag{3.71}$$

(ii)

$$P(\rho) = c_+ \rho^{\frac{n_1+1}{n_1}}\big(1 + O\big(\rho^{-\frac{a_1}{n_1}}\big)\big) \quad \text{when } \rho \to +\infty. \tag{3.72}$$

Denote $\gamma_0 = \frac{n_0+1}{n_0}$ and $\gamma_\infty = \frac{n_1+1}{n_1}$. By theorem 5.5 in [19], when $n_1 \in (0, 5)$, to first order, the mass–radius relation for high central pressures is approximated by the mass–radius relation for an exact polytrope with polytropic index n_1. That is, when μ is large enough,

$$M(\mu) \propto \mu^{\frac{3-n_1}{2n_1}} = \mu^{\frac{1}{2}(3\gamma_1 - 4)}, \quad R_\mu \propto \mu^{\frac{1-n_1}{2n_1}} = \mu^{\frac{1}{2}(\gamma_1 - 2)}.$$

Therefore, when $n_1 > 3$ (i.e., $\gamma_\infty < \frac{4}{3}$), $\frac{d}{d\mu}\big(\frac{M(\mu)}{R_\mu}\big) > 0$ and $M'(\mu) < 0$ for sufficiently large μ. Thus for large μ, we have $i_\mu = 0$ and ρ_μ is linearly unstable by Theorem 1.2. When $n_0 < 3$ (i.e., $\gamma_0 > \frac{4}{3}$), we have $M'(\mu) > 0$ for μ small enough. Thus, the transition of stability must occur at some $\mu > 0$. the maximum of $M(\mu)$ is obtained at $\mu_0 < \infty$ which is the first transition of stability by Corollary 3.28.

By theorem 5.4 in [19], when $\gamma_0 > \frac{4}{3}$ and $\gamma_\infty < \frac{6}{5}$ (i.e., $n_1 > 5$), the mass–radius relation for high central pressures possesses a spiral structure, with the spiral given by

$$\begin{pmatrix} R_\mu \\ M(\mu) \end{pmatrix} = \begin{pmatrix} R_0 \\ M_0 \end{pmatrix} + \left(\frac{1}{\alpha}\right)^{\gamma_1} \mathcal{B}\mathcal{J}\left(\gamma_2 \ln \frac{1}{\alpha}\right) b + o\left(\left(\frac{1}{\alpha}\right)^{\gamma_1}\right), \quad \mu \gg 1, \tag{3.73}$$

where $\alpha = \Phi'(\mu)$, R_0 and M_0 are constants, $\mathcal{B}$ is a nonsingular matrix, and b a nonzero vector. The matrix $\mathcal{J}(\varphi) \in \mathrm{SO}(2)$ describes a rotation by an angle φ, and the constants γ_1 and γ_2 are given by

$$\gamma_1 = \frac{1}{4}(n_1 - 5), \quad \gamma_2 = \frac{1}{4}\sqrt{7n_1^2 - 22n_1 - 1}\,.$$

Thus, when $\mu \to \infty$, the mass $M(\mu)$ has infinitely many extremum points. We claim that at each of these extremum points, the number of unstable modes $n^u(\mu)$ must increase by 1, and in particular $n^u(\mu) \to \infty$ when $\mu \to \infty$. Indeed, for large μ the mass–radius curve must spiral counterclockwise, and then by Theorem 1.1 $n^u(\mu)$ increases by 1 when crossing any mass extremum of $M(\mu)$ on the spiral. Suppose not, and the mass–radius curve spirals clockwise when $\mu \to \infty$. Then by Theorem 1.1 $n^u(\mu)$ decreases by 1 when crossing each mass extremum of $M(\mu)$ on the spiral. Therefore, after crossing finitely many mass extrema in the spiral, $n^u(\mu)$ must become zero. Let μ^* be the first mass extremum in the spiral such that $n^u(\mu) = 0$ for μ slightly less than μ^*. Then for μ slightly less than μ^*, we have $n^-(D_\mu^0) = 1$ and $i_\mu = 1$, which implies that $M'(\mu)R'(\mu) < 0$. Thus when crossing μ^*, the sign of $M'(\mu)R'(\mu)$ must change from $-$ to $+$, which contradicts the assumption that the spiral is clockwise. This proves that the mass–radius spiral can only be counterclockwise.

We summarize the above discussions in the following.

COROLLARY 3.29. *Consider asymptotically polytropic $P(\rho)$ satisfying* (3.71)–(3.72). *Assume $\gamma_0 \in (\frac{4}{3}, 2)$ (i.e., $n_0 \in (1,3)$ in* (3.71)). *Then when $n_1 \in (3,5)$ or $n_1 > 5$ with n_1 defined in* (3.72)*, there must be a transition point of stability in the sense of Corollary* 3.28. *Moreover, ρ_μ is unstable when μ is large enough. When $n_1 > 5$, $n^u(\mu) \to \infty$ when $\mu \to \infty$.*

Remark 3.30. White dwarf stars are supported by the pressure due to cold degenerate electrons, as given by the equation of state (3.70). When the density is high enough, the pressure due to cold degenerate neutrons should be taken into account. For such modified equations of state, the maximal mass (Chandrasekhar's limit) is indeed achieved at a finite center density $\mu_0 < \infty$. Then by Corollary 3.28 μ_0 is the first transition point of stability and nonrotating stars with center density slightly larger than μ_0 become unstable. We refer to figure 11.2 and section 11.4 in [44] for such a mass–radius curve and physical explanations. If the stars are much more compact than the one with a Chandrasekhar limit, then relativistic effects cannot be ignored and the Euler-Einstein model should be used. A similar turning point principle can be derived for stability of relativistic compact stars modeled by the Euler-Einstein equation [16, 17].

Appendix: Lagrangian Formulation and Hamiltonian Structure

In this appendix we formally outline the Lagrangian formulation of the Euler-Poisson system (1.1)–(1.3) and its linearization. Let $(\rho_\mu(|x|), u(x) \equiv 0)$ be the nonrotating star supported on the ball $S_\mu \subset \mathbb{R}^3$ with radius $R(\mu)$, where $\mu = \rho_\mu(0)$. We simply take S_μ as the reference domain in the Lagrangian framework and define the (abstract) configuration space of Lagrangian maps as

$$\Lambda = \{\text{diffeomorphism } \mathcal{X} : S_\mu \to \mathcal{X}(S_\mu) \subset \mathbb{R}^3\}.$$

For any reference density $\rho_* : S_\mu \to \mathbb{R}^+ \cup \{0\}$ and a path of Lagrangian maps $\mathcal{X}(t) \in \Lambda$, the action functional $\mathcal{A}$ is given by

$$\mathcal{A} = \int \left(\int_{S_\mu} \frac{1}{2} |\mathcal{X}_t|^2 \rho_* \, dy - \int_{\mathcal{X}(t,S_\mu)} \Phi(\rho) dx + \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla V|^2 \, dx \right) dt,$$

where the enthalpy $\Phi(\rho)$ is defined in (1.7), the gravitational potential $V(t,x)$ by (1.3) (or equivalently $V = |x|^{-1} * \rho$), and the physical density ρ in the Eulerian coordinates is given by

$$\rho(t,\cdot) = \left(\frac{\rho_*}{\det D\mathcal{X}(t,\cdot)} \right) \circ \mathcal{X}(t,\cdot)^{-1} : \mathcal{X}(t, S_\mu) \to \mathbb{R}^+ \cup \{0\}$$

and extended as 0 outside $\mathcal{X}(t, S_\mu) \subset \mathbb{R}^3$. Through a standard calculus of variation procedure (with respect to $\mathcal{X}$), it is straightforward to verify that $\mathcal{X}(t)$ is a critical path of $\mathcal{A}$ if and only if $(\rho, u = \mathcal{X}_t \circ \mathcal{X}^{-1})$, which is supported on $\mathcal{X}(t, S_\mu)$, solves the Euler-Poisson system (1.1)–(1.3). The reference density ρ_* plays the role of a parameter not evolved in t. The conserved energy of this Lagrangian system is

$$\begin{aligned} E &= \int_{S_\mu} \frac{1}{2} |\mathcal{X}_t|^2 \rho_* \, dy + \int_{\mathcal{X}(t,S_\mu)} \Phi(\rho) dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla V|^2 \, dx \\ &= \int_{\mathbb{R}^3} \frac{1}{2} \rho |u|^2 + \Phi(\rho) - \frac{1}{8\pi} |\nabla V|^2 \, dx. \end{aligned}$$

One observes that the potential energy consisting of the enthalpy and gravity depends on $\mathcal{X}$ only through the density ρ. Therefore the action functional is invariant under the transformation $\mathcal{X}(t) \to \mathcal{X}(t) \circ \mathcal{T}$, where $\mathcal{T}$ belongs to the group $\mathcal{G}$ of diffeomorphisms on S_μ preserving ρ_*, namely,

$$\mathcal{G} = \{\text{diffeomorphism } \mathcal{T} : S_\mu \to S_\mu \mid (\rho_* \circ \mathcal{T}) \det D\mathcal{T} = \rho_*\}.$$

The Euler-Poisson system (1.1)–(1.3) in the Eulerian formulation is essentially a reduction of the Lagrangian system due to this relabeling symmetry where $\rho(t,\cdot)$ and $u(t,\cdot)$ are supported on $\mathcal{X}(t, S_\mu)$.

The nonrotating star $(\rho_\mu, u \equiv 0)$ corresponds to the stationary solution $\mathcal{X} \equiv id$ along with $\rho_* = \rho_\mu(|x|)$, which is a critical point of the potential energy. Let

$\mathcal{X}(t,x,\epsilon)$ be a family of solutions (parametrized by ϵ) in the Lagrangian formulation with the reference density $\rho_*(x,\epsilon)$ such that $\mathcal{X}(t,x,0) = x$, $\rho_*(x,0) = \rho_\mu(x)$, for all $x \in S_\mu$. The linearized system at $\mathcal{X} = id$ and ρ_μ governs the dynamics of the leading-order variation $\widetilde{\mathcal{X}} = \partial_\epsilon \mathcal{X}|_{\epsilon=0}$, which also involves $\sigma = \partial_\epsilon \rho|_{\epsilon=0}$. The corresponding quantities in the Eulerian formulation are

$$\sigma = \partial_\epsilon \rho|_{\epsilon=0} = \big(\partial_\epsilon \rho_* - \nabla\cdot(\rho_\mu \widetilde{\mathcal{X}})\big), \quad v = \partial_\epsilon u|_{\epsilon=0} = \partial_\epsilon(\mathcal{X}_t \circ \mathcal{X}^{-1})|_{\epsilon=0} = \widetilde{\mathcal{X}}_t,$$

which are supported on S_μ. The associated action of the linearized Lagrangian system, which is simply the quadratic part of $\mathcal{A}$, can be expressed more conveniently using σ as

$$\mathcal{A}_2 = \frac{1}{2}\int_{S_\mu} \rho_\mu |\widetilde{\mathcal{X}}_t|^2 - \Phi''(\rho_\mu)\sigma^2\,dy + \frac{1}{8\pi}\int_{\mathbf{R}^3} |\nabla(|x|^{-1} * \sigma)|^2\,dx.$$

Using the above formula of σ, which also implies (1.8), one obtains the linearized equation through the variation of $\mathcal{A}_2$ with respect to $\widetilde{\mathcal{X}}$,

$$-\widetilde{\mathcal{X}}_{tt} - \nabla\cdot\big(\Phi''(\rho_\mu)\sigma + |x|^{-1} * \sigma\big) = 0,$$

which is equivalent to (1.9). The quadratic part

$$E_2 = \frac{1}{2}\int_{S_\mu} \rho_\mu |v|^2 + \Phi''(\rho_\mu)\sigma^2\,dx - \frac{1}{8\pi}\int_{\mathbf{R}^3} |\nabla(|x|^{-1} * \sigma)|^2\,dx = H_\mu(\sigma, v)$$

of the nonlinear energy E, which is equal to the Hamiltonian $H_\mu(\sigma, v)$ of the linearized Euler-Poisson system defined in (3.20), is conserved by these linearized solutions.

Through the Legendre transformation $U = \rho_* \mathcal{X}_t$, the Lagrangian structure with the action $\mathcal{A}$ induces a natural Hamiltonian structure of the Euler-Poisson system with the Hamiltonian $\mathcal{H}$ and the standard symplectic structure J:

$$\mathcal{H}(\mathcal{X}, U) = \int_{S_\mu} \frac{1}{2\rho_*}|U|^2\,dy + \int_{\mathcal{X}(t,S_\mu)} \Phi(\rho)dx - \frac{1}{8\pi}\int_{\mathbf{R}^3} |\nabla V|^2\,dx,$$
$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It might be possible to apply the general results in Section 2 to analyze the linearized Euler-Poisson system at $(\rho_\mu, 0)$ as a linear Hamiltonian system of the linearized Lagrangian map $\partial_\epsilon \mathcal{X}$ and momentum $\partial_\epsilon U$. As in the nonlinear case, one could expect such a system to be reduced to (1.8)–(1.9) through a reduction due to the relabeling symmetry. We carried out the analysis directly on (1.8)–(1.9) with the different symplectic structure $\mathcal{J}_\mu$, where the large symmetry group (corresponding to additional infinite kernel dimensions) has been reduced and stability/instability is directly on the linearized density and velocity.

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