Modular Commutators in Conformal Field Theory

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The modular commutator is a recently discovered entanglement quantity that quantifies the chirality of the underlying many-body quantum state. In this Letter, we derive a universal expression for the modular commutator in conformal field theories in $1+1$ dimensions and discuss its salient features. We show that the modular commutator depends only on the chiral central charge and the conformal cross ratio. We test this formula for a gapped $(2+1)$-dimensional system with a chiral edge, i.e., the quantum Hall state, and observe excellent agreement with numerical simulations. Furthermore, we propose a geometric dual for the modular commutator in certain preferred states of the AdS/CFT correspondence. For these states, we argue that the modular commutator can be obtained from a set of crossing angles between intersecting Ryu-Takayanagi surfaces.

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One of the overarching themes of research in theoretical physics over the past few decades has been the study of entanglement in interacting quantum many-body systems. Calculation of the canonical measure of entanglement—entanglement entropy—has played a crucial role in elucidating the physics of topological order [1,2], conformal field theory [3], and holographic duality [4].

Recently, a new entanglement quantity known as the *modular commutator* [5,6] was introduced [7]. The modular commutator is defined as $J(A,B,C)_\rho \coloneqq i\mathrm{Tr}(\rho_{ABC}[\ln\rho_{AB}, \ln\rho_{BC}])$ for a generic tripartite quantum state ρ_{ABC} [8], and, unlike other known entanglement measures, it is odd under time reversal. In the context of topologically ordered systems in $2+1$D, the modular commutator was used to extract the chiral central charge of the edge theory [5,6].

In this Letter, we derive a universal expression for the modular commutator in conformal field theories in $1+1$D and discuss its physical implications. Let A, B, and C be three contiguous spacetime intervals; see Fig. 1(a). In this setup, we derive a general expression for $J(A,B,C)$ in the vacuum. If the subsystems lie in a single time slice, the expression simplifies to

$$J(A,B,C)_{|\Omega\rangle} = \frac{\pi c_-}{6}(2\eta - 1), \tag{1}$$

where $\eta = [(x_2-x_1)(x_4-x_3)/(x_3-x_1)(x_4-x_2)]$ is the cross ratio, $c_- = c_L - c_R$ is the chiral central charge of the CFT, and $|\Omega\rangle$ is the vacuum state. Using a standard conformal mapping from the complex plane to the cylinder, expressions for the modular commutator for finite systems in the vacuum and infinite systems at finite temperature are also derived.

We primarily discuss two applications. First, we argue that Eq. (1) can be a useful tool to study the entanglement structure of $2+1$D chiral gapped systems at their edges. Specifically, consider three contiguous intervals A, B, and C at the edge of a disk; see Fig. 3(a). We propose the following formula—based on an argument utilizing Eq. (1)—for the modular commutator:

$$J(A,B,C)_{|\psi_{2\mathrm{D}}\rangle} = \frac{\pi \mathfrak{c}_-}{3}\eta, \tag{2}$$

where $\mathfrak{c}_-$ is the chiral central charge of the $2+1$D system (defined as a coefficient appearing in the edge energy current [9–11]) and $|\psi_{2\mathrm{D}}\rangle$ is the ground state. We test Eq. (2) numerically for the Chern insulator and $p+ip$ topological superconductor, demonstrating excellent agreement.

When A, B, and C cover the entire edge [see Fig. 3(b)], i.e., $\eta = 1$, we provide an independent information-theoretic argument for a stronger result:

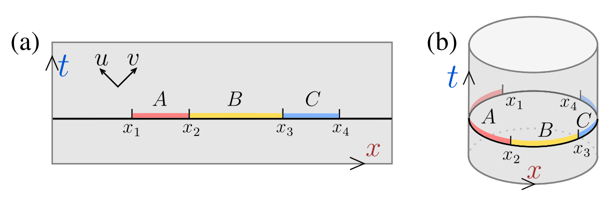

FIG. 1. (a) Three contiguous intervals A, B, and C, on a single time slice. (b) Contiguous intervals on a circle S^1 with circumference L.

$$J(A,B,C)_{|\tilde{\psi}_{2D}\rangle} = \frac{\pi}{3}\mathfrak{c}_-, \tag{3}$$

where $|\tilde{\psi}_{2D}\rangle$ is any state which is indistinguishable from $|\psi_{2D}\rangle$ in the bulk region. We emphasize the generality of Eq. (3) in two directions. First, this equation holds even if there is an excitation localized at the edge. Second, the argument continues to hold even if the shape of the edge is deformed continuously. The underlying argument—based on the properties of modular commutator [5,6] and techniques from the entanglement bootstrap [12]—reveals that the robustness of this result originates from the entanglement area law of the bulk.

Second, we propose a holographic interpretation of Eq. (1). Our interpretation rests on an observation that Eq. (1) can be recast as

$$J(A,B,C)_{|\Omega\rangle} = \frac{\pi c_-}{6}\cos\theta, \tag{4}$$

where θ is the crossing angle of the two geodesics (i.e., two Ryu-Takayanagi surfaces [4]) in AdS_3, each anchored at the boundaries of AB and BC, respectively. We verify this correspondence at both zero and finite temperature and propose a generalization to any state whose bulk geometry has a "moment of time symmetry" [13,14].

Our approach to derive Eq. (1) will be geometric in nature. The main advantage of this derivation is that it makes the generalization of Eq. (1) to arbitrary spacetime intervals straightforward. Alternative derivations shall be discussed in Supplemental Material [16] as well.

Geometric derivation.—Our derivation of Eq. (1) is based on the following two observations. First, the modular commutator $J(A,B,C)$ can be viewed as the linear response of the BC entanglement entropy under the AB modular flow [6,44,45]. Second, for a $1+1$D CFT, the modular flow for a finite interval generates a special conformal transformation that keeps the two ends of the interval fixed [46–48]. Thus, we will compute the modular commutator $J(A,B,C)$ by the change of the entropy S_{BC} from the infinitesimal conformal transformation generated by the modular flow corresponding to AB.

The *modular flow* of an operator O with respect to a state ρ and a subsystem A is defined as $O(s) \coloneqq \rho_A^{is} O \rho_A^{-is}$ for $s \in \mathbb{R}$, where $\rho_A^{is} \coloneqq e^{is\log\rho_A}$ is the unitary operator generated by the modular Hamiltonian. We consider the action of the modular flow associated with the interval AB in the vacuum. Define the following one-parameter family of density matrices: $\rho_{ABC}(s) \coloneqq \rho_{AB}^{is}\rho_{ABC}\rho_{AB}^{-is}$. The response of the von Neumann entropy of $\rho_{BC}(s) = \mathrm{Tr}_A[\rho_{ABC}(s)]$ under this flow is related to the modular commutator by [6]

$$\left.\frac{dS[\rho_{BC}(s)]}{ds}\right|_{s=0} = -J(A,B,C)_\rho, \tag{5}$$

with $S(\rho) \coloneqq -\mathrm{Tr}[\rho\ln\rho]$.

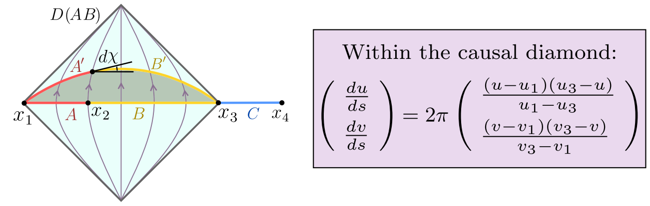

FIG. 2. Modular flow in the interior of the causal diamond $D(AB)$ and the associated vector field. Under an infinitesimal flow by a parameter ds, interval AB becomes $A'B'$ and a boost angle $d\chi$ develops at the left end of B'.

In quantum field theory, the observables restricted to the interval AB completely determine the observables in the full *causal diamond* $D(AB)$, i.e., the domain of dependence of AB. In $1+1$D CFT, the modular flow associated to a spacelike interval in the vacuum is a local transformation of observables lying within its causal diamond [46]. The relevant vector fields are illustrated in Fig. 2.

Now we can use the following regulated form of the single-interval entanglement entropy for chiral CFTs in $1+1$D [49,50]:

$$S_{BC} = \frac{c_L}{12}\ln\frac{(v_4-v_2)^2}{\epsilon_{v2}\epsilon_{v4}} + \frac{c_R}{12}\ln\frac{(u_4-u_2)^2}{\epsilon_{u2}\epsilon_{u4}}, \tag{6}$$

where $u = t - x$ and $v = t + x$ are light-cone coordinates, $u_2 = t_2 - x_2$, $v_2 = t_2 + x_2$, $u_4 = t_4 - x_4$, $v_4 = t_4 + x_4$, and ϵ_{u2}, ϵ_{u4}, ϵ_{v2}, and ϵ_{v4} denote the UV cutoffs in the u and v directions at the end points x_2 and x_4. Details of the cutoff prescription are discussed in Supplemental Material [16].

Note that the point x_4 is unaffected by the modular flow with respect to AB, because it is outside $D(AB)$. Thus, u_4, v_4 and $\epsilon_{u4}, \epsilon_{v4}$ remain unchanged; the change occurs only at x_2. Importantly, the cutoffs $\epsilon_{u(v)2(4)}$ transform nontrivially under local diffeomorphisms. They are rescaled by the local boost angle (see Fig. 2):

$$d\ln\epsilon_{v2} = -d\ln\epsilon_{u2} = d\chi, \tag{7}$$

where $d\chi = [2\pi(x_{23} - x_{12})/x_{13}]ds$ is the boost angle at x_2. Here, we use the convention $x_{ij} = x_j - x_i$. Differentiating Eq. (6) and using Eq. (7), we obtain

$$J(A,B,C)_{|\Omega\rangle} = \frac{\pi c_-}{6}(2\eta - 1), \tag{8}$$

where the chiral central charge is $c_- = c_L - c_R$ and the cross ratio is $\eta = (x_{12}x_{34}/x_{13}x_{24})$. Generalization of Eq. (8) to general Cauchy surfaces is straightforward and can be used to determine c_L and c_R individually in terms of the modular commutator; see Supplemental Material [16] for details.

Equation (8) for $J(A,B,C)_{|\Omega\rangle}$ possesses a set of important properties, summarized below. First, J is odd under time reversal, which exchanges c_L and c_R. This is in contrast with other entanglement measures such as the entanglement entropy, which are even under time reversal. Second, J is odd under the map $\eta \to 1-\eta$. In particular, $J=0$ at $\eta = 1/2$, where the modular commutator changes sign. Third, as the length of one interval gets small, J does not vanish but takes on universal values. As $x_1 \to x_2$ or $x_3 \to x_4$, $\eta \to 0$ and $J \to -\pi c_-/6$, and similarly, as $x_2 \to x_3$, $\eta \to 1$ and $J \to \pi c_-/6$. In fact, we shall later see that the universal difference $J(\eta=1) - J(\eta=0) = \pi c_-/3$ is exactly the modular commutator for 2D chiral topological order. Last, if $c_- \neq 0$, we have $J = \pi c_-/6 \neq 0$ when ABC is the entire circle. This distinguishes $|\Omega\rangle$ from any pure state on a Hilbert space factorized into a tensor product on spatial regions, as the latter necessarily has $J=0$. Thus, $c_- \neq 0$ is incompatible with any lattice regularization (see also [51] for an alternative argument) [52].

More generally, one can consider a thermal state at inverse temperature β on a circle of circumference L, denoted as $\rho^{(\beta;L)}$. Through standard conformal mappings from planes to cylinders [53], one can show that the modular commutator $J(A,B,C)$ remains to be in the form in Eq. (1) in two limits $\beta/L \to 0, \infty$, with the cross ratio η replaced by $\eta_{\mathrm{eff}}^{(\beta;L)}$:

$$\eta_{\mathrm{eff}}^{(\beta;L)} = \begin{cases} \frac{\sin(\pi x_{12}/L)\sin(\pi x_{34}/L)}{\sin(\pi x_{13}/L)\sin(\pi x_{24}/L)}, & \beta/L \to \infty, \\ \frac{\sinh(\pi x_{12}/\beta)\sinh(\pi x_{34}/\beta)}{\sinh(\pi x_{13}/\beta)\sinh(\pi x_{24}/\beta)}, & L/\beta \to \infty. \end{cases} \tag{9}$$

Chiral thermal state.—The modular commutator can be nonzero even for nonchiral CFTs, provided that the temperatures for the left- and the right-moving modes are unequal. We refer to such states as *chiral thermal states* [54–56]:

$$\rho^{(\beta_L,\beta_R;L)} = \frac{1}{Z} e^{-\beta_L H_L - \beta_R H_R}. \tag{10}$$

Here, H_L and H_R are the Hamiltonians of the left- and right-moving sectors, respectively. Similarly, (β_L, β_R) represent inverse temperatures for the respective modes.

There are a few reasons to study chiral thermal states. First, a chiral thermal state can be obtained by applying the Lorentzian boost to a thermal state. Second, there are concrete lattice models whose underlying state at low temperature can be well described by a chiral thermal state. For instance, it was noted that the reduced density matrix near the edge of a chiral topological order in 2 + 1D can be represented by a chiral thermal state with $(\beta_L, \beta_R) = (\infty, \mathrm{finite})$ [55]. Third, as we show in Supplemental Material [16], one can sometimes explicitly construct chiral thermal states in lattice models, making the numerical verification tractable.

From Eq. (9), for a general chiral thermal state $\rho^{(\beta_L,\beta_R;L)}$ we have

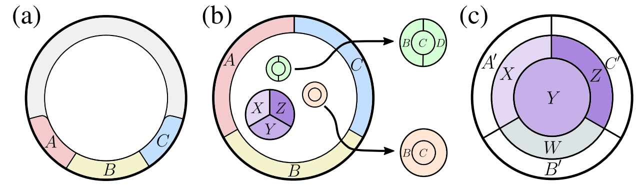

FIG. 3. A 2 + 1D gapped chiral system on a disk and various choices of subsystems. The sizes (width) for subsystems within the bulk (adjacent to the edge) are large compared to the bulk correlation length.

$$J(A,B,C)_{\rho^{(\beta_L,\beta_R;L)}} = \frac{\pi}{3} c\left(\eta_{\mathrm{eff}}^{(\beta_L;L)} - \eta_{\mathrm{eff}}^{(\beta_R;L)}\right), \tag{11}$$

where $c = c_R = c_L$. We construct chiral thermal states for the free fermion CFT on the lattice and compute the modular commutators for various choices of parameters. The numerical result agrees excellently with Eq. (11) (see Supplemental Material [16] for details).

Edge of 2 + 1D *chiral topological order.*—The chiral thermal state can provide insights into the edges of 2 + 1D gapped systems with nonzero chiral central charge, denoted as $\mathfrak{c}_-$ [9–11,55]. (We choose a different font to distinguish two concepts: the chiral central charge $\mathfrak{c}_-$ of a 2 + 1D gapped phase versus c_- for a 1 + 1D chiral CFT.)

Consider a ground state $|\psi_{2\mathrm{D}}\rangle$ on a disk for concreteness; see Fig. 3. For an annulus which covers the entire edge, e.g., the annulus in Fig. 3(a), the reduced density matrix of $|\psi_{2\mathrm{D}}\rangle$ can be viewed as a 1 + 1D system. If the edge is completely chiral (that is when, e.g., it has only left-moving modes but not right-moving modes), it is expected to be described by a chiral thermal state whose c equals $\mathfrak{c}_-$ [55].

Then, by applying Eq. (11) to the interval choice in Fig. 3(a) and taking $\beta_L = \infty$, $\beta_R \ll L_A, L_B, L_C$ (the lengths of the regions), we arrive at a prediction:

$$J(A,B,C)_{|\psi_{2\mathrm{D}}\rangle} = \frac{\pi}{3}\mathfrak{c}_-\eta. \tag{12}$$

We have tested this formula numerically for a Chern insulator and observed excellent agreement; see Fig. 4. We propose this formula to hold for general translation invariant topologically ordered systems in 2 + 1D.

Topological argument.—When the union of intervals A, B, and C is the entire annulus, as shown in Fig. 3(b), Eq. (12) becomes $J = (\pi/3)\mathfrak{c}_-$. Here, we present an entirely different argument for this formula, based on the entanglement area law of the 2 + 1D bulk [1,2]. Our argument reveals an extra degree of robustness of this expression:

$$J(A,B,C)_{|\tilde{\psi}_{2\mathrm{D}}\rangle} = \frac{\pi}{3}\mathfrak{c}_- \quad \text{for Fig.3(b)}. \tag{13}$$

We show that Eq. (13) holds for any state $|\tilde{\psi}_{2\mathrm{D}}\rangle$ locally indistinguishable from the ground state within the bulk.

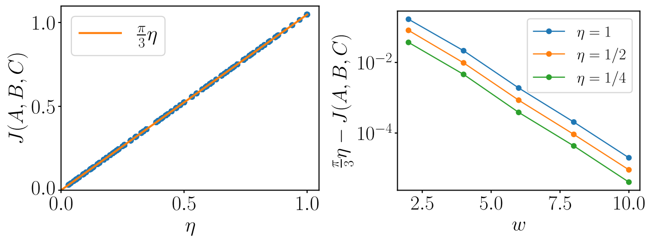

FIG. 4. $J(A, B, C)$ versus η for the Chern insulator, which is realized by filling the lowest band of the Hofstadter model with flux $\pi/2$. We use a square lattice on a cylinder with circumference $L = 144$ and height $W = 32$. A, B, and C are rectangular strips on the boundary with length L_A, L_B, and L_C, respectively, and width w. Left: We fix $w = 10$ and vary the lengths L_A, L_B, and L_C. Blue dots represent numerical data, and the orange line represents the analytical prediction Eq. (12). Right: We choose several (L_A, L_B, L_C) and vary w. The three choices $(L_A, L_B, L_C) = (48, 48, 48), (36, 36, 36), (24, 48, 24)$ correspond to $\eta = 1, 1/2, 1/4$, respectively.

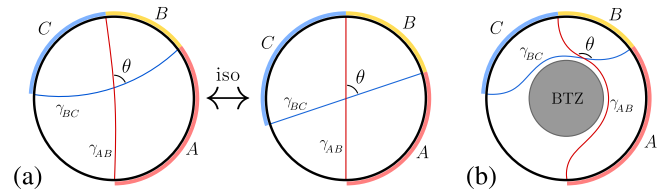

FIG. 5. Verified cases of the holographic conjecture: (a) At zero temperature. Each disk is a Poincaré disk, and the two are related by an isometry. (b) At a finite (high) temperature such that $\beta \ll L$.

Note that we need not assume $|\tilde{\psi}_{2D}\rangle$ to be the ground state; our argument applies even if there are edge excitations, as long as the global state is pure.

The key observation that leads to Eq. (13) is an equivalence we will establish between the edge and the bulk modular commutator for the set of subsystems shown in Fig. 3(b):

$$J(A, B, C)_{|\tilde{\psi}_{2D}\rangle} = -J(X, Y, Z)_{|\tilde{\psi}_{2D}\rangle}. \quad (14)$$

Note that the regions A, B, and C lie at the edge, while the regions X, Y, and Z lie entirely in the bulk. Once this relation is established, one can use the formula for the bulk modular commutator [5], i.e., $J(X, Y, Z)_{|\tilde{\psi}_{2D}\rangle} = -(\pi/3)\mathfrak{c}_-$, to complete the derivation.

The equivalence of the two modular commutators directly follows from Sec. VI in Ref. [6], as we explain below. (See Supplemental Material [16] for a more detailed explanation.) First of all, the state $|\tilde{\psi}_{2D}\rangle$, being indistinguishable from the ground state in the bulk, satisfies the axioms of entanglement bootstrap [12]. Of particular importance to us is Axiom A1 in Ref. [12], which holds for local disklike regions away from the edge; it says $(S_{BC} + S_{CD} - S_B - S_D)_{|\psi_{2D}\rangle} = 0$ for the green disk BCD shown in Fig. 3(b), where $|\psi_{2D}\rangle$ is the ground state. This axiom, applied to the bulk disk $XYZW$ in Fig. 3(c), gives $I(A'\colon Y|X) = I(C'\colon Y|Z) = 0$, where $I(X\colon Z|Y) \equiv S_{XY} + S_{YZ} - S_{XYZ} - S_Y$ is the conditional mutual information. It then follows that, for state $|\tilde{\psi}_{2D}\rangle$,

$$J(X, Y, Z) = J(A'X, Y, C'Z) = -J(A'X, WB', C'Z).$$

Letting $A = A'X$, $B = B'W$, and $C = C'Z$, we conclude Eq. (14).

Let us emphasize the generality of the argument above. Note that nowhere in the derivation did we use any symmetry (e.g., translation or rotation symmetry), nor did we use any condition of the state in the vicinity of the edge. For instance, even in the presence of strong disorder, even though the conformal symmetry does not hold—not even approximately—formula (13) still holds; this is numerically verified for integer quantum Hall states—see Supplemental Material [16]. Moreover, the argument holds as long as $|\tilde{\psi}_{2D}\rangle = U_{\text{edge}}|\psi_{2D}\rangle$, where U_{edge} is *any* unitary operator along the edge which is thin compared to the width of the subsystems; specifically, U_{edge} should be supported within the annulus $A'B'C'$ for the choice of ABC in Fig. 3(c). (Under a plausible assumption, the unitarity assumption can be dropped. See Supplemental Material [16] for the details.)

Holographic interpretation.—In the AdS/CFT correspondence [57], entanglement quantities of the boundary CFT are mapped to geometric quantities in the bulk of an asymptotic AdS space. For example, the Ryu-Takayanagi (RT) formula [4] implies that, in ordinary nonchiral AdS/CFT, the entanglement entropy of a boundary region A in a time-symmetric state is given by the minimal length of the bulk geodesic γ_A (also known as the RT surface) homologous to the region. Some examples are shown in Fig. 5.

Here, we propose to extend the holographic dictionary to the modular commutator for chiral realizations of AdS/CFT, e.g., Ref. [15]. In states whose bulk geometries are locally AdS_3 [58] with a moment of time symmetry, we propose

$$J(A, B, C) = \frac{\pi c_-}{6} \sum_i \cos\theta_i, \quad (15)$$

where $\{\theta_i\}$ is the set of crossing angles of the RT surfaces, i.e., geodesics γ_{AB} and γ_{BC}. Each θ_i is chosen such that γ_{AB}, seen inwardly, lies at the right side of the angle; see Fig. 5 for examples. In general, AB and BC may have multiple connected components; see Supplemental Material [16] for the relevant discussion.

We can verify the conjecture for a few simple cases shown in Fig. 5. The vacuum state of chiral AdS_3/CFT_2 is described by the ordinary vacuum AdS_3 spacetime [15]. On the $t = 0$ slice of this spacetime, we can apply a bulk isometry to place the intersection point of any two geodesics at the center of the Poincaré disk. Then, the two geodesics become straight lines with a crossing angle θ. Since the cross ratio η—given by $\eta = (x_{12}x_{34})/(x_{13}x_{24})$—is preserved under this isometry, the identity $2\eta - 1 = \cos\theta$ follows from simple trigonometry. Thus, we arrive at

$$J(A,B,C)_{|\Omega\rangle} = \frac{\pi c_-}{6}\cos\theta. \tag{16}$$

At high temperatures $\beta \ll L$, thermal states in CFT are dual to Banados-Teitelboim-Zanelli (BTZ) black holes [60] in the bulk; see Fig. 5(b). An analogous derivation applies, because the BTZ black hole can be viewed as a quotient of global AdS_3. The result confirms our conjecture. (See Supplemental Material [16] for details.)

In the semiclassical limit of AdS/CFT, a boundary modular Hamiltonian K is dual to a bulk geometric operator which, in nonchiral AdS/CFT, is proportional to the area of the RT surface [61,62]. In chiral AdS/CFT, the operator has additional terms [59]; we will call the full operator F. The modular commutator of contiguous intervals can be written in terms of commutators of F operators. This commutator is zero in the vacuum for a single time slice if the chiral central charge is zero [63], which matches Eq. (1). However, for chiral theories, Eq. (15) implies the uncertainty relation

$$\Delta F(AB) \cdot \Delta F(BC) \geq \frac{\pi c_-}{12}|\cos\theta|. \tag{17}$$

Thus, the uncertainty in the geometric operator F grows parametrically with the chiral central charge.

Discussion.—In this Letter, we computed the modular commutator [5,6] in $1+1$D CFTs, arriving at a simple formula Eq. (1) and discussing its applications in condensed matter systems and holography. For future work, it will be interesting to verify our conjecture in AdS/CFT to more general setups, e.g., disconnected intervals, states whose bulk geometries have no moment of time symmetry, and states with bulk quantum matter. Another interesting open problem is how our conjecture generalizes to higher dimensions. On the condensed matter side, it would be interesting to understand how Eqs. (12) and (13) generalize when the sector of the chiral edge is modified by an anyon in the bulk.

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Note added.—Recently, we noticed a related work [64], which has some overlap with this Letter.

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