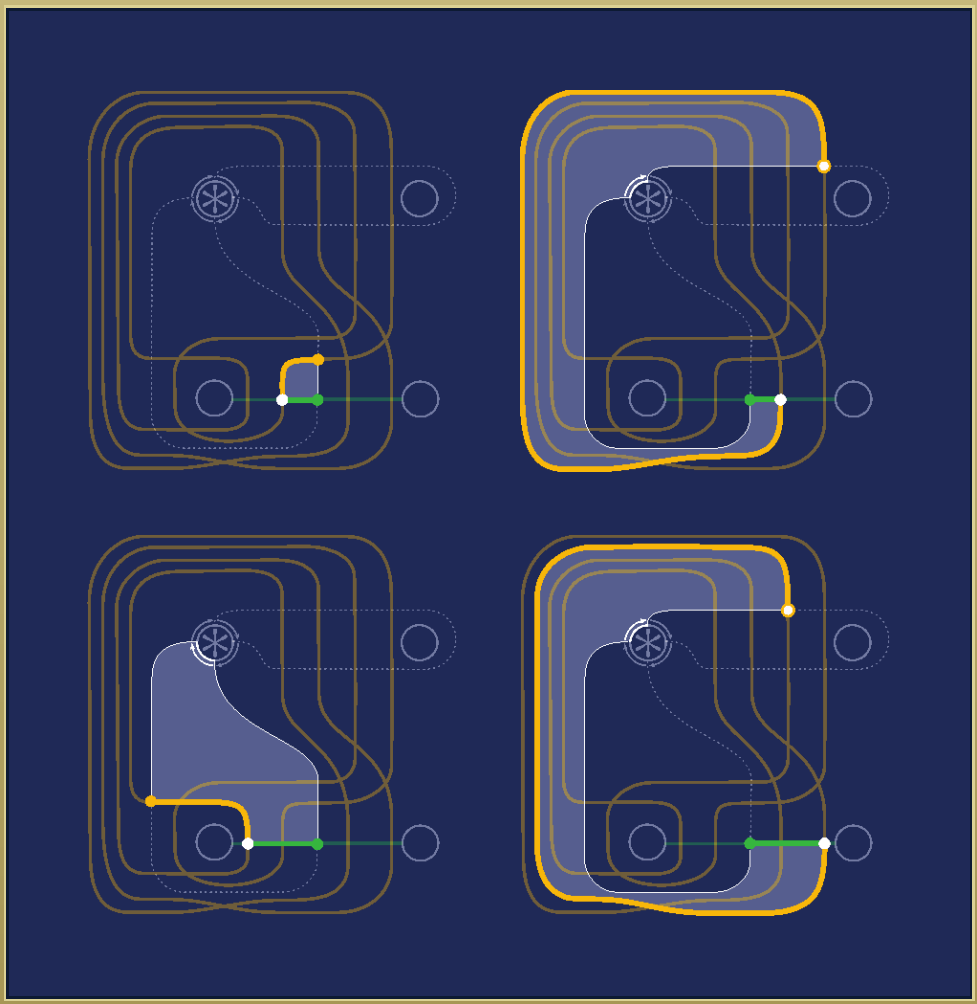
THE OPEN BOOK SERIES 5

Gauge Theory and Low-Dimensional Topology: Progress and Interaction

On naturality of the Ozsváth–Szabó contact invariant

Matthew Hedden and Lev Tovstopyat-Nelip



msp

THE OPEN BOOK SERIES 5 (2022)
Gauge theory and low-dimensional topology: progress and interaction
https://doi.org/10.2140/obs.2022.5.123

On naturality of the Ozsváth–Szabó contact invariant

Matthew Hedden and Lev Tovstopyat-Nelip

We discuss functoriality properties of the Ozsváth–Szabó contact invariant, and expose a number of results which seemed destined for folklore. We clarify the (in)dependence of the invariant on the basepoint, prove that it is functorial with respect to contactomorphisms, and show that it is strongly functorial under Stein cobordisms.

1. Introduction

Heegaard Floer homology provides a seemingly ever-growing number of invariants for low-dimensional topology. Its influence has perhaps most firmly been felt within the realm of 3-dimensional contact geometry, upon which the Ozsváth– Szabó contact invariant [30] and its refinements have had a profound impact. In its most basic form, the contact invariant of a closed contact 3-manifold (Y,ξ) is an element residing in the Heegaard Floer homology group $HF(-Y)$ of the underlying manifold, equipped with the opposite orientation to the one it receives from the contact structure (a group which, perhaps more naturally, can be identified with the Floer cohomology of Y). A decade after its initial development by Ozsváth and Szabó [27; 28], Juhász, Thurston, and Zemke discovered a subtle dependence of Heegaard Floer homology on a choice of basepoint underlying its definition [19]. Indeed, they showed that Floer homology cannot associate a well-defined group to a 3-manifold alone, but only to a 3-manifold equipped with a basepoint. This raises the questions of whether the contact element is well defined, how it depends on the basepoint, and how it behaves under diffeomorphisms, questions raised but not pursued in [19] and [22, pg. 1360].

The purpose of this article is to examine these questions, and further explore functoriality properties of the contact invariant. As a first step, we show that the

Hedden gratefully acknowledges support from NSF grants DMS-1709016 and DMS-2104664.
MSC2020: primary 57K18, 57K31, 57K33, 57K43; secondary 53D05, 53D10.
Keywords: Heegaard Floer homology, contact structure, contact invariant, functoriality, contactomorphism, naturality, Weinstein, Stein.

contact element is a well-defined element (as opposed to an orbit under the group of graded automorphisms) in the Floer homology of a pointed contact 3-manifold; see Theorem 2.3. To prove this, we first establish an appropriate definition and notion of equivalence for pointed contact 3-manifolds and incorporate these ideas into the Giroux correspondence. We then revisit Ozsváth and Szabó's proof of invariance within the naturality framework of [19], ensuring that Heegaard surfaces, links, open books, basepoints, etc. can be arranged to be explicitly embedded in a fixed 3-manifold.

Having checked the aforementioned details, we turn to a refined understanding of invariance of the contact class, showing that it is functorial with respect to pointed contactomorphisms.

Theorem 1.1. Suppose f is a pointed contactomorphism between pointed contact 3-manifolds (Y, ξ, w) and (Y', ξ', w'). Then the induced map on Floer homology

$$f_* : \widehat{HF}(-Y, w) \to \widehat{HF}(-Y', w')$$

carries $c(\xi, w)$ to $c(\xi', w')$.

The functoriality above is an immediate consequence of the functoriality of Floer homology under pointed diffeomorphisms from [19], provided one parses the Giroux correspondence in a categorical framework, which we clarify with Proposition 2.6.

We then show that, while the group in which the contact element lives depends on the basepoint, the contact element itself does not. This can be explained as follows: the dependence of the Floer homology of Y on the basepoint is determined by a functor

$$HF(Y, -) : \pi_1(Y) \to \mathrm{iGrp}$$

from the fundamental groupoid of Y to the isomorphism subcategory of groups. Concretely, this just means that there is a well-defined isomorphism between Floer groups $\widehat{HF}(Y, w)$ and $\widehat{HF}(Y, w')$ associated to a homotopy class of a path between w and w', which is compatible with concatenation (see the next section for more details). If one restricts to the subgroups of $\widehat{HF}(Y, w)$ spanned by contact classes, which we denote $cHF(Y, w)$, this functor yields a transitive system indexed by points in Y; that is, the isomorphisms $cHF(Y, w) \to cHF(Y, w')$ are independent of paths. We can therefore consider the direct limit of the transitive system, which we call the contact subgroup of Floer homology, and denote $cHF(Y)$.

Theorem 1.2. The contact subgroup $cHF(Y)$ is a well-defined invariant of an (unpointed) 3-manifold, functorial with respect to diffeomorphisms. There is an element $c(\xi) \in cHF(Y)$, associated to a contact structure ξ on Y, and the map $f_* : cHF(Y) \to cHF(Y')$ induced by a contactomorphism $f : (Y, \xi) \to (Y', \xi')$ sends $c(\xi)$ to $c(\xi')$.

One can view this result in two ways. On the one hand, it follows from the functoriality established in Theorem 1.1, together with the fact that we can realize the change-of-basepoint diffeomorphism associated to a homotopy class of path by a contactomorphism (see Proposition 2.9). On the other, it can be viewed as a consequence of Zemke's calculation of the representation of the fundamental group on Floer homology in terms of the $H_1(Y)/\mathrm{Tor}$ action and the basepoint action δ_w, together with the fact that contact classes are in the kernel of the $H_1(Y)/\mathrm{Tor}$ action. Adopting the latter perspective, we see that the contact subgroup is a subgroup of a larger basepoint independent subgroup, arising as the kernel of the $H_1(Y)/\mathrm{Tor}$ action. Note that these considerations dash any naive hope that Heegaard Floer homology is generated by contact classes, much less by elements associated to taut foliations, and indicate that such a conjecture might more reasonably be made in the context of a twisted coefficient system in which the $H_1(Y)/\mathrm{Tor}$ action vanishes, e.g., totally twisted coefficients, or for the subgroup arising as the intersection of the reduced Floer homology $HF^-_{red}(Y, w)$ and the kernel of the $H_1(Y)/\mathrm{Tor}$ action. For a rational homology sphere the latter action vanishes, and the question lands back within a similar realm to the L-space conjecture.

Having clarified the definition and invariance properties of the contact element, we then show that it is functorial under Stein cobordisms in a precise way.

Theorem 1.3. Suppose (W, J, ϕ) is a Stein cobordism from a contact 3-manifold (Y_1, ξ_1) to a contact 3-manifold (Y_2, ξ_2). Then

$$F_{W^\dagger,k}(c(Y_2, \xi_2)) = c(Y_1, \xi_1),$$

where $W^\dagger$ indicates the 4-manifold W, viewed as a cobordism from $-Y_2$ to $-Y_1$, and k is the canonical Spin^c structure associated to J. Moreover,

$$F_{W^\dagger,s}(c(Y_2, \xi_2)) = 0$$

for $s = k$.

In a weaker form, such a result follows fairly easily from the existing literature, and was widely known to experts. See Section 3 for a discussion. In the present level of specificity, the proof is slightly more involved than one might initially expect, owing largely to the nature of the composition law for cobordism maps in Heegaard Floer theory. We remark that the incoming and outgoing boundaries of W are not assumed to be connected, and that Theorem 1.3 immediately yields a generalization of Plamenevskaya's independence result for contact invariants from [33]; see Corollary 3.9.

It would be interesting to know how much naturality of the contact element persists as one weakens assumptions on the cobordism. It is known, for instance, that the contact element in monopole Floer homology is natural under strong symplectic cobordisms [2, Theorem 1]; see also [24]. One would thus expect an affirmative answer to the following:

Question 1.4. Is the contact element natural under strong symplectic cobordisms?

Finally, we note that we work with Heegaard Floer homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients throughout. The naturality results of [19] have been extended to projective $\mathbb{Z}$ coefficients (i.e., $\mathbb{Z}/\pm 1$) [7], but at the moment these extensions have not been established for the graph cobordism maps. Assuming they will be, the results at hand should immediately extend to the refined setting.

2. Functoriality of the contact class under diffeomorphisms

In this section we clarify the dependence of the contact class on the basepoint used in the definition of Heegaard Floer homology, and highlight how the results of Juhász, Thurston and Zemke [19] and Zemke [34] couple with Ozsváth and Szabó's argument from [30] to imply that the contact class is a well-defined invariant of a pointed contact 3-manifold, up to pointed isotopy (Theorem 2.3). This invariant is shown to be functorial under pointed contactomorphisms (Theorem 2.7). We then show that while, according to [19], the Heegaard Floer homology group in which the contact element lives depends in an essential way upon the basepoint, the contact invariant is essentially independent from the basepoint (Theorem 1.2). Unless otherwise specified, all 3-manifolds are assumed to be closed and oriented, and contact structures assumed to be cooriented.

Recall from [19] that the Heegaard Floer homology group of a pointed 3-manifold (Y, w) is defined as the direct limit of a transitive system of groups and isomorphisms defined by pointed Heegaard diagrams $(\Sigma, \alpha, \beta, w)$ embedded in (Y, w) and pointed Heegaard moves passing between them (together with auxiliary choices of almost complex structures). See [19, Theorem 1.5] and the surrounding discussion. The contact invariant should therefore be interpreted as an element in the aforementioned direct limit. As such, it would appear to depend on the basepoint, and we therefore make the following definition

Definition 2.1. A pointed contact 3-manifold is a 3-manifold Y equipped with a contact structure ξ and a distinguished basepoint w.

By Gray's theorem, an isotopy between contact structures is induced by an isotopy of the underlying (compact) 3-manifold, and two contact structures on (Y, w) will be considered equivalent if we can find such an isotopy fixing the basepoint w.

Remark 2.2. One could consider a more restrictive definition of equivalence where the isotopy fixes the contact plane at w. This differs from the present notion only by the choice of oriented plane at w, a choice parametrized by a 2-sphere, and would have no effect on our results. See [19, Lemma 2.45] for more details.

In [30], Ozsváth and Szabó defined an invariant of contact structures utilizing the Giroux correspondence between isotopy classes of contact structures on Y and

equivalence classes of fibered links in Y under Hopf plumbing. Given a fibered knot representing ξ, its knot Floer homology has a distinguished filtered subcomplex in bottommost Alexander grading, whose homology is rank one [30, Theorem 1.1]. Inclusion of this subcomplex in $\widehat{CF}(-Y)$ defines an element $c(\xi) \in \widehat{HF}(-Y)$ [30, Definition 1.2], which they showed does not depend on the particular choice of fibered knot representing ξ [30, Theorem 1.3]. Absent from the literature at that time, however, was an understanding of the dependence of the Floer homology group in which $c(\xi)$ resides on the choice of basepoint. We restate their theorem so that this dependence is explicit, and outline the elements of their proof of invariance which should be refined accordingly.

Theorem 2.3 [30, Theorem 1.3]. Suppose two contact structures ξ, η on the pointed 3-manifold (Y, w) are equivalent. Then $c(\xi, w) = c(\eta, w) \in \widehat{HF}(-Y, w)$.

Proof. We begin by observing that the Giroux correspondence [6; 9] has the pointed analogue

$$\frac{\{\text{pointed open books } (L, \pi_L) \text{ in } (Y^3, w)\}}{\{\text{(pointed) isotopy and positive Hopf plumbing}\}} = \frac{\{\text{contact structures on } (Y^3, w)\}}{\{\text{isotopy fixing } w\}}$$

where we emphasize that on the left-hand side we are considering concrete open book decompositions of Y, by which we mean an embedded link $L \subset Y$ together with a fibration on its exterior $\pi_L : Y \setminus L \to S^1$ for which the boundary of the closure of each fiber is L. In these terms a pointed open book for the pointed contact manifold is an open book supporting ξ for which the basepoint is contained in L. Two open books are considered equivalent if they differ by a sequence consisting of ambient isotopies of links and ambient (de)plumbings with positive Hopf bands, where isotopies and (de)plumbings are required to fix the basepoint. The pointed statement follows easily from the unpointed statement. We remark that the inclusion of ambient isotopies of the open book is essential, though typically omitted or implicit in the literature.

Ozsváth and Szabó's proof relies on two lemmas. If we denote the element associated to a fibered knot $K \subset Y$ by $c(K) \in \widehat{HF}(-Y)$, then [30, Lemma 4.1] states that this element is unchanged under connected summing with the right-handed trefoil, T; that is, $c(K \# T) = c(K)$ for any fibered knot $K \subset Y$. Then [30, Lemma 4.4] shows that the element associated to a fibered knot obtained by plumbing $2h$ right-handed Hopf bands to $K \subset Y$ is independent of the choice of plumbings. Using plumbings which realize iterated connected sums with T, the result follows.

The proof of the latter lemma goes by realizing the element associated to any genus h stabilization as the image of a fixed class in $\widehat{HF}(-Y)$ under a map induced by a cobordism W which is diffeomorphic to $Y \times [0, 1]$. To do this, one observes that a genus h stabilization can be obtained by attaching canceling 4-dimensional 1- and

2-handles to $Y \times I$, where the former add handles to the page of the open book and the latter enact Dehn twists to the monodromy. One then uses [30, Theorem 4.2], which states that the element is carried naturally under the 2-handle cobordisms that add left-handed Dehn twists to the monodromy.

According to [19], one should refine these arguments so that they use embedded Heegaard diagrams in Y, with the basepoint lying on the embedded Heegaard surface. When appealing to the functoriality with respect to cobordisms used in the proof of Ozsváth and Szabó's second lemma, one must also be careful about the embedded path in $W \subset Y \times [0,1]$ from the basepoint to itself.

To address the first issue, consider a concrete pointed open book (K, π_K) in Y supporting (a contact structure isotopic to) ξ, with connected binding. From this, one can construct Ozsváth and Szabó's Heegaard diagram adapted to K from [30, Section 3]. This construction can be done so that the diagram is embedded in Y, built from the union of the closure of two fibers by a stabilization, and so that it contains the basepoint. The proof of naturality for knot Floer homology adapts to produce a functorial invariant from the category of pointed knots to a category whose objects are transitive systems of $\mathbb{Z}$-filtered complexes (where the maps in such systems are certain canonical filtered homotopy classes of filtered homotopy equivalences). See [14, Proposition 2.3] for the adaptation of the proof of naturality in [19] to the context of transitive systems of complexes, and [13, Proposition 2.8] for a discussion on how to apply this to knots. Naturality implies that the generator of the homology of the bottommost filtered subcomplex defined by the embedded Heegaard diagram for the knot (K, π_K) produces a well-defined invariant $c(K, w) \in \mathit{HFL}(-Y, w)$, by consideration of the inclusion-induced map.

To argue that the element $c(K, w)$ is invariant under connected summing with a trefoil, we observe that one can form the connected sum of the open book with the trefoil knot ambiently, by embedding the trefoil and its fiber surface in a small ball near the basepoint, but in the complement of the Heegaard surface for K. One can then form an embedded Heegaard diagram adapted to $K \# T$, which is a connected sum of embedded diagrams. The Künneth theorem for the knot Floer homology of a connected sum [26, Theorem 7.1], together with the fact that the new diagram is obtained from the initial diagram by a sequence of pointed embedded Heegaard moves, shows that $c(K \# T, w) = c(K, w)$.

The second lemma from Ozsváth and Szabó's proof also goes through in the context of pointed 3-manifolds and concrete open books. Indeed, if we are given a concrete pointed open book (K', π') for (Y, w) which is obtained from (K, π) by ambiently plumbing $2h$ positive Hopf bands, we cancel the additional right-handed Dehn twists in the monodromy by attaching $2h$ 4-dimensional 2-handles along curves in the page. This results in a cobordism whose outgoing boundary is diffeomorphic to $Y \#^{2h} S^1 \times S^2$. Further attaching $2h$ 4-dimensional 3-handles

to cancel the 2-handles results in a composite cobordism which is diffeomorphic, rel boundary, to $Y \times [0,1]$. Moreover, since the attaching regions for the 2- and 3-handles lie in the complement of the basepoint, the path traced by the basepoint in the cobordism is sent, under a diffeomorphism to $Y \times [0,1]$, to the trivial path $w \times [0,1]$. We can reverse the orientation of Y before performing the aforementioned handle attachments, and the resulting map on the Floer homology of $\mathit{HF}(-Y, w)$ is the identity. The result follows as in [30] by appealing to the naturality of the contact invariant under addition of Dehn twists. □

Remark 2.4. One could alternatively approach the pointed invariance of the contact element using its interpretation by Honda, Kazez and Matić [17]. Such an approach seems necessary to establish naturality of the contact invariants in sutured Floer homology defined using partial open books for 3-manifolds with convex boundary [16].

To understand the functoriality of the contact class, we observe that pointed contact 3-manifolds form the objects of a category whose morphisms are pointed isotopy classes of contactomorphisms. With respect to this structure, the (pointed) Giroux correspondence is functorial. To understand this, we make the following definition:

Definition 2.5. A (pointed) diffeomorphism between concrete (pointed) open books is an orientation-preserving diffeomorphism of pairs $f : (Y, L) \to (Y', L')$ which intertwines the fibrations on the link complements, i.e., $\pi_L = \pi_{L'} \circ f_{Y \setminus L}$ (and which maps the basepoint on L to the basepoint on L').

If the open book (L, π_L) supports a contact structure ξ_L on Y, then a diffeomorphic open book (L', π_L') supports a contact structure $\xi_{L'}$ on Y' satisfying $f_*(\xi_L) = \xi_{L'}$. Since the contact structure induced by an open book is only well defined up to isotopy, we will regard diffeomorphisms of open books up to isotopy without loss of information. In this way, a diffeomorphism of open books defines an isotopy class of contactomorphisms.

Conversely, given an isotopy class of contactomorphisms $f : (Y, \xi) \to (Y', \xi')$, we can push-forward an open book (L, π_L) supporting ξ under f, yielding an open book $(f(L), \pi_L \circ f^{-1})$ supporting (Y', ξ'). The evident diffeomorphism of open books induces the given contactomorphism, up to isotopy. In this way, the Giroux correspondence can be lifted to an isomorphism of categories:

Proposition 2.6 (functorial Giroux correspondence). There is an isomorphism of categories between the category of (pointed) contact 3-manifolds and (pointed) isotopy classes of contactomorphisms and the concrete open book category, whose objects are 3-manifolds equipped with concrete (pointed) open books up to ambient (pointed) Hopf plumbing and whose morphisms are (pointed) isotopy classes of (pointed) diffeomorphisms between open books.

Proof. The (pointed) Giroux correspondence yields a bijection between objects which, in one direction, sends an (equivalence class of (pointed)) open book to the ((pointed) isotopy class of a) contact structure supporting it. The discussion above shows that there are corresponding bijections between morphism sets. □

Using this, we can show that the contact class is functorial with respect to pointed contactomorphisms; see Theorem 1.1.

Theorem 2.7 (functoriality under contactomorphisms). Suppose f is a pointed contactomorphism between pointed contact manifolds (Y, ξ, w) and (Y', ξ', w'). Then the map on Floer homology $f_\sharp : \widehat{HF}(-Y, w) \to \widehat{HF}(-Y', w')$ carries $c(\xi, w)$ to $c(\xi', w')$.

Proof. This follows easily from the definition of the map on Floer homology associated to a pointed diffeomorphism, as described in [19, Section 2.5, Definition 2.42], together with the functorial Giroux correspondence. More precisely, according to Section 2.5 of [19], the map between Floer homology groups associated to a pointed isotopy class of diffeomorphism is defined by the map on transitive systems induced by pushing forward embedded pointed Heegaard diagrams in (Y, w) (and moves between them) to (Y', w'). A pointed Heegaard diagram adapted to a concrete open book supporting (Y, ξ) is mapped, via f, to a pointed Heegaard diagram adapted to a diffeomorphic concrete open book supporting (Y', ξ'). Taking homology of these complexes gives rise to representatives for the direct limit of the transitive systems that define $\widehat{HF}(-Y, w)$ and $\widehat{HF}(-Y', w')$, respectively. Under the induced map, the cycle representing the contact element for $c(\xi, w)$ is taken to that representing $c(\xi', w')$. The result follows. □

Since the hat Floer homology groups depend on the basepoint, the above refinements are necessary in order to understand the invariance of the contact class. Having addressed this, however, we will now show that the contact class is essentially independent of the basepoint, relying on it only insomuch as it is required to define the group in which the class resides.

To explain this, recall that the map $\mathrm{Diff}(Y) \xrightarrow{ev_w} Y$ which evaluates a diffeomorphism at a basepoint is a Serre fibration, and the fiber over w is the pointed diffeomorphism group $\mathrm{Diff}(Y, w)$. The associated long exact sequence on homotopy terminates in

$$\pi_1(Y, w) \to \pi_0(\mathrm{Diff}(Y, w)) \to \pi_0(\mathrm{Diff}(Y)) \to 1.$$

Concretely, this implies that if a pointed diffeomorphism is (unpointed) isotopic to the identity, then it is isotopic to a "point-pushing map" about a loop representing an element in $\pi_1(Y, w)$. If one considers instead the fiber of ev_w over a different basepoint, w', we see that any diffeomorphism of Y sending w to w' which is (unpointed) isotopic to the identity is isotopic, through diffeomorphisms sending

w to w', to a point-pushing map defined by a choice of arc γ from w to w'. Moreover, any two such diffeomorphisms differ, up to pointed isotopy, by a point-pushing map along an element in $\pi_1(Y, w)$. In light of this, the dependence of Floer homology on the basepoint is captured by a representation $\pi_1(Y, w) \to \mathrm{Aut}(\widehat{HF}(Y, w))$ defined by isomorphisms associated to isotopy classes of point-pushing diffeomorphisms. While this representation can be nontrivial, the following proposition implies that it acts trivially on the subspace spanned by contact elements.

Proposition 2.8. Suppose that contact structures ξ and η on the pointed 3-manifold (Y, w) are isotopic, induced by an isotopy of Y which does not necessarily fix the basepoint. Then $c(\xi, w) = c(\eta, w) \in \widehat{HF}(-Y, w)$.

We note that the functoriality of the contact invariant only implies $f_*(c(\xi, w)) = c(\eta, w)$, where f is the endpoint of the isotopy.

Proof. Let $\phi_t : Y \times [0, 1] \to Y$ denote the isotopy carrying ξ to η, where $\phi_0 = \mathrm{Id}_Y$ and $\phi_1 = f$ is a diffeomorphism fixing w, but where ϕ_t may not fix the basepoint for $0 < t < 1$. The discussion preceding the proposition indicates that f is isotopic to a point-pushing map along a curve γ representing an element $[\gamma] \in \pi_1(Y, w)$. More precisely, a loop γ based at w can be regarded as an isotopy of embeddings of a point into Y which, by the isotopy extension theorem, can be extended to an isotopy of Y which is the identity outside a neighborhood of the image of γ. The endpoint of this latter isotopy is a pointed diffeomorphism $f_\gamma : (Y, w) \to (Y, w)$ whose pointed isotopy class depends only on the homotopy class $[\gamma] \in \pi_1(Y, w)$, by another application of the isotopy extension theorem (or, rather its interpretation in terms of the homotopy lifting property of the map $\mathrm{Diff}(M) \to \mathrm{Diff}(N, M)$ which evaluates a diffeomorphism at a submanifold; see [20; 32]). According to the main theorem of [19], there is an induced automorphism $(f_\gamma)_*$ of the Floer homology group $\widehat{HF}(Y, w)$, and the functoriality of the contact class under pointed contactomorphisms implies

$$(f_\gamma)_*(c(\xi, w)) = c(f_{\gamma *}(\xi), w) = c(\eta, w).$$

The automorphism $(f_\gamma)_*$ will, in general, be nontrivial; indeed, Zemke shows that it can be computed via the formula [34, Theorem D]

$$(f_\gamma)_* = \mathrm{Id} + (\delta_w)_* \circ (A_\gamma)_*,$$

where A_γ is the chain level map defining the $H_1(Y)/\mathrm{Tor}$ action on $\widehat{HF}(Y, w)$, and δ_w is the basepoint action which, in the case of $\widehat{CF}(Y, w)$, counts J-holomorphic disks which pass through the hypersurface specified by the basepoint exactly once. The proposition will follow if we can show that contact classes are in the kernel of the $H_1(Y)/\mathrm{Tor}$ action. But this is an easy consequence of their definition. Letting $c \in H_*(\widehat{CF}(-Y, K, w, \mathrm{bot})) \cong \widehat{HF}$ denote the generator of the homology of the

bottommost nontrivial filtered subcomplex in the filtration of $\widehat{CF}(-Y, w)$ induced by the binding of a pointed open book supporting ξ, the contact class is defined as

$$c(\xi, w) := \iota_*(c),$$

where $\iota : F(-Y, K, w, bot) \to CF(-Y, w)$ is the inclusion map. The chain map A_γ on $\widehat{CF}(-Y, w)$ respects the filtration induced by K, defined as it is by counting J-holomorphic disks which avoid w (see [13, Proof of Proposition 5.8]). It follows that A_γ maps $F(-Y, K, w, bot)$ to $F(-Y, K, w, bot)$, but since it shifts the relative $\mathbb{Z}/2\mathbb{Z}$ homological grading, and the homology of the latter subcomplex is one-dimensional, the map on homology must be trivial. Therefore the automorphism on Floer homology induced by a point-pushing map acts as the identity on any contact elements, and we have $c(\xi, w) = (f_\gamma)_*(c(\xi, w)) = c(\eta, w)$ as claimed. □

The above proposition can be used to show that change-of-basepoint maps on Floer homology induced by pushing points along arcs act on contact elements in a canonical way, i.e., $(f_\gamma)_*(c(\xi, w)) \in \widehat{HF}(Y, w')$ is independent of the choice of arc used to construct a diffeomorphism $f_\gamma : (Y, w) \to (Y, w')$. This indicates an independence of the contact class from the choice of basepoint. We can make this independence more precise. To do this, we show that the point-pushing maps along arcs can be refined to pointed contactomorphisms.

Proposition 2.9 (cf. [11]). Given $w, w' \in Y$, there exists a contactomorphism $\phi : (Y, \xi) \to (Y, \xi)$, which is isotopic to the identity and maps w to w'.

Proof. Let $\gamma : [0, 1] \to Y$ denote a smooth embedded path from w to w'. After a C^∞-small isotopy we may assume the path γ is transverse to ξ. Let $\nu(\gamma([0, 1]))$ denote a neighborhood of the transverse arc. A standard neighborhood theorem gives a contact embedding

$$\phi : (\nu(\gamma([0, 1])), \xi) \to (\mathbb{R}^3, \ker(\alpha)), \quad \text{where } \alpha = dz + r^2 d\theta,$$

which takes the image of the arc to the segment $\{(0, 0)\} \times [0, 1]$ along the z-axis; in particular $\phi(w) = (0, 0, 0)$ and $\phi(w') = (0, 0, 1)$.

Let β denote a contact 1-form for ξ which is an extension of $\phi^*\alpha$. The time one flow of the Reeb vector field R_β is then the desired contactomorphism taking w to w'. □

Corollary 2.10. The contact class is independent of the basepoint in the following sense: given two basepoints $w, w' \in Y$, a path γ between them induces an isomorphism $\gamma_* : \widehat{HF}(-Y, w) \to \widehat{HF}(-Y, w')$. For any choice of γ, the contact class satisfies $\gamma_*(c(\xi, w)) = c(\xi, w')$.

Proof. Suppose γ is a path connecting w to w'. As in the proof of Proposition 2.8, the based homotopy class of γ gives rise to a well-defined pointed isotopy class

of pointed diffeomorphism $f_\gamma : (Y, w) \to (Y, w')$, whose associated isomorphism between Floer homology groups we denote γ_*. The contactomorphism constructed using γ in the proof of the preceding proposition is a representative of this pointed isotopy class. Functoriality of the invariant under pointed contactomorphisms, Theorem 2.7, then implies that $\gamma_*(c(\xi, w)) = c(\xi, w')$. □

Proof of Theorem 1.2. In light of the corollary, if we let the subgroup of $\widehat{HF}(-Y, w)$ spanned by contact elements be denoted $\widehat{cHF}(-Y, w)$, we obtain a transitive system (in the sense of [3, Definition 6.1]) of groups indexed by points in Y, for which the isomorphism $f_{w,w'} : \widehat{cHF}(-Y, w) \to \widehat{cHF}(-Y, w')$ is the map on Floer homology associated to the point-pushing map along any arc from w to w'. We call the direct limit of this transitive system the contact subgroup associated to Y, and denote it $\widehat{cHF}(-Y)$:

$$\widehat{cHF}(-Y) := \varinjlim \widehat{cHF}(-Y, w).$$

The corollary shows that it is well defined, independent of any choice of basepoint, and that a contact structure ξ on Y receives an associated element $c(\xi) \in \widehat{cHF}(-Y)$ defined as the image of $c(\xi, w) \in \widehat{cHF}(-Y, w)$ under the canonical inclusion-induced isomorphism $\widehat{cHF}(-Y, w) \to \widehat{cHF}(-Y)$. The contact subgroup is functorial with respect to (unpointed) diffeomorphisms of Y by the main theorem of [19], and an (unpointed) contactomorphism $f : (Y, \xi) \to (Y', \xi')$ sends $c(\xi)$ to $c(\xi')$ by Theorem 2.7. □

Remark 2.11. There is a contact invariant $c^+(\xi) \in HF^+(-Y)$ defined as the image of $c(\xi)$ under the map on homology induced by the inclusion of complexes $\iota : \widehat{CF} \to CF^+$, [25, Section 4]. Corresponding results for $c^+(\xi)$ follow from Theorems 1.1 and 1.2, together with the naturality of ι_* implied by [19, Theorem 1.5].

3. Functoriality of the contact class under Stein cobordisms

In this section we provide a proof of the well-known folk theorem that the Ozsváth–Szabó contact invariant is natural with respect to Stein cobordisms. Nontriviality of the contact invariant of a Stein fillable contact structure was proved in [30, Theorem 1.5]. The proof relied on a naturality result [30, Theorem 4.2] for the invariants of contact structures represented by open book decompositions which differ by a single Dehn twist. This latter result implicitly showed that the contact invariant is natural with respect to a Stein cobordism associated to a Weinstein 2-handle attachment along a Legendrian knot, a fact made more clear in [21, Theorem 2.3] (though stated there in terms of contact $+1$ surgery). These naturality results for Weinstein 2-handles consider the sum of maps associated to all the Spin^c structures on the cobordism. Together with a calculation for 1-handles, they immediately yield a weak naturality of the contact invariant under Stein cobordisms, where one

sums over all Spin^c structures. This is spelled out in [18, Theorem 11.24], under an additional topological restriction on the 1-handles.

The Spin^c refinement of naturality for the contact invariant of a Stein filling, viewed as a Stein cobordism to the standard structure on the 3-sphere, was established in [33, Theorem 4]. A Spin^c refinement of naturality for the contact invariant under general Weinstein 2-handle attachments along a Legendrian link was stated in [8, Lemma 2.11]. The proof relied crucially on a naturality result for the cobordism map associated to a Lefschetz fibration over an annulus. The latter was attributed to Ozsváth and Szabó, who only proved the result for Lefschetz fibrations over a disk. We spell out the proof of the required naturality in Lemma 3.6 below, and use it to establish naturality of the contact class under Weinstein 2-handle cobordism following the strategy in [8]. Given the body of literature on topological aspects of Stein surfaces and domains, exposed beautifully in [1; 4; 10], the only remaining piece necessary for the Spin^c refinement of naturality under a general Stein cobordism (Theorem 1.3) is a discussion of 1-handles, particularly those with feet in different path components.

Recall, then, that a Stein cobordism from a contact 3-manifold (Y_1, ξ_1) to (Y_2, ξ_2) is a smooth 4-manifold W with $\partial W = -Y_1 \sqcup Y_2$, oriented by a complex structure J for which the oriented complex lines of tangency on ∂W agree with ξ_1 and ξ_2, respectively, and which admits a J-convex Morse function ϕ, defined by the requirement that $-d d^C \phi = \omega_\phi$ is symplectic. Such a manifold comes equipped with a Liouville vector field, X_ϕ, defined as the gradient of ϕ with respect to the metric induced by ω_ϕ. See [1] for an introduction.

Theorem 3.1. Suppose (W, J, ϕ) is a Stein cobordism from a contact 3-manifold (Y_1, ξ_1) to a contact 3-manifold (Y_2, ξ_2). Then

$$F_{W^\dagger, k}(c(Y_2, \xi_2)) = c(Y_1, \xi_1),$$

where $W^\dagger$ denotes the 4-manifold W, viewed as a cobordism from $-Y_2$ to $-Y_1$, and k is the canonical Spin^c structure associated to J. Moreover,

$$F_{W^\dagger, s}(c(Y_2, \xi_2)) = 0$$

for $s = k$.

Remark 3.2. The result is equally valid for Weinstein cobordisms, which [1] shows are equivalent to Stein cobordisms for the present purposes.

Remark 3.3. Strictly speaking, the Stein cobordism should be equipped with a properly embedded graph, in the sense of [34]. In this context, the graph is obtained from the basepoints present on the incoming end of the cobordism by their image under the flow of the Liouville vector field. We pick basepoints on the incoming ends which flow to the outgoing ends, with some extra care taken in the case that

components of the boundary merge via Stein 1-handles so that all components of the boundary have a single basepoint (see Lemma 3.5 below). In light of the naturality results from the previous section, and the resulting independence of the contact class of the choice of basepoint, we can safely omit basepoints from most of the discussion and obtain a naturality result for the contact invariant which is basepoint independent.

Proof. By [4, Theorem 1.3.3], the cobordism can be decomposed as a composition of elementary cobordisms corresponding to Stein 0-, 1-, and 2-handle attachments, with the latter two attached along framed points and Legendrian curves, respectively. In this dimension, the subtleties involved with 2-handle framings were clarified by [10]. Though we could avoid it with a more cumbersome inductive argument, we can and will assume that the attachments are ordered by their indices, arising from a self-indexing plurisubharmonic Morse function. For a smooth manifold, this follows from the standard rearrangement theorem for Morse functions [23, Theorem 4.8]. The proof of that theorem, however, modifies the gradient-like vector field for the Morse function so that the stable manifold of an index λ critical point is disjoint from the unstable manifold of an index $\lambda' \geq \lambda$ critical point (achieving the Morse–Smale condition for the manifolds associated to these critical points). In the Stein setting, the gradient vector field and metric are coupled, and one cannot vary one without changing the other. Rearranging critical levels is therefore more subtle. These subtleties are nicely exposed, and dispatched with, in Chapter 10 of [1]. Of particular relevance are Proposition 10.10 and 10.1. Proposition 10.10 allows one to vary the critical values of the J-convex Morse function specifying the handle decomposition, provided the stable and unstable manifolds of the points of interest are disjoint, the Stein analogue of [23, Theorem 4.1]. Proposition 10.1 allows one to vary an isotropic submanifold of a given contact type hypersurface by an isotropic isotopy compatible with a family of J-convex Morse functions, the Stein analogue of [23, Lemma 4.7]. Applying the latter to the attaching spheres of the Stein handles allows us to assume, as in the classical case, that the stable manifold of an index λ critical point is disjoint from the unstable manifold of an index $\lambda' \geq \lambda$ critical point. Thus we can proceed by induction to order the handles and further ensure that all critical points of a given index have the same critical value. The existence of such an ordering for a 2-dimensional Stein domain (a Stein cobordism with $Y_1 = \emptyset$) is implicit in the statement of [10, Theorem 1.3], a result which itself is attributed as implicit in Eliashberg [4].

We assume then, that the cobordism is decomposed as a sequence of elementary 0- and 1-handle cobordisms, followed by a cobordism associated to a collection of Weinstein 2-handle attachments along Legendrian curves, equipped with a J-convex Morse function with a unique critical value. We will show that the contact invariant is mapped in the specified way under a single 0- or 1-handle attachment,

and similarly for a simultaneous collection of Stein 2-handle attachments. The result will then follow from the composition law for cobordism-induced maps on Heegaard Floer homology:

$$F_{W_1^\dagger,t_1} \circ F_{W_2^\dagger,t_2} = \sum_{\{s \in Spin^c(W) \mid s|_{W_i} = t_i\}} F_{W^\dagger,s}.$$

Examining the law, one observes that naturality of the contact invariant for Stein cobordisms W_1 and W_2 sharing a common intermediate boundary does not imply either of the conclusions in the statement of the theorem for their union $W = W_1 \cup W_2$, if $Spin^c$ structures on the latter are not uniquely determined by their restrictions to W_1 and W_2. The following standard lemma makes this precise:

Lemma 3.4. Suppose $W = W_1 \cup_Y W_2$ is a 4-manifold glued along a 3-manifold Y arising as a connected component of ∂W_i (with boundary orientation of Y different for $i = 1, 2$). Then the set of $Spin^c$ structures on W restricting to $t_i \in Spin^c(W_i)$, provided it is nonempty, is in affine correspondence with $\delta H^1(Y)$, where δ is the connecting homomorphism in the Mayer–Vietoris sequence.

In particular, if either W_i is a cobordism associated to a 0- or 1-handle attachment, then a $Spin^c$ structure on W is uniquely determined by its restrictions to the pieces. This is because 0- and 1-handle cobordisms have the property that the restriction $H^1(W_i) \to H^1(\partial W_i)$ is surjective, which implies $\delta H^1(Y)$ is trivial, by exactness. Therefore, we can treat 0- and 1-handles individually. It is certainly possible, however, that a $Spin^c$ structure on a 4-manifold composed of two or more 2-handle cobordisms will not be determined by its restrictions to the pieces. It is therefore not sufficient to prove the naturality of $c(\xi)$ with respect to a single Stein 2-handle. For this reason, we group the index 2 critical points giving rise to the 2-handles together into a single critical level, and prove naturality for such a 2-handle cobordism.

We turn to our treatment of the handles in each dimension. The fact that the contact invariant is natural under Stein 0-handle attachment follows immediately from the definition of the associated map on Floer homology, which is simply the map induced by the canonical isomorphism between Heegaard Floer chain complexes under taking disjoint union with a Heegaard diagram whose surface is a pointed 2-sphere with no curves [34, Section 11.1]. This definition, together with the fact that the contact class of the Stein fillable contact structure on the 3-sphere is nontrivial, yield, upon taking duals, the stated naturality.

The following lemma establishes naturality under Stein 1-handle cobordisms.

Lemma 3.5. Suppose (W, J) is the cobordism associated to a Stein 1-handle attachment. Then Theorem 3.1 is true for $F_W^\dagger$.

Proof. Unlike the case of a Stein domain, a Stein cobordism can have disconnected boundary. Thus, there are two possibilities (1) the feet of the 1-handle lay in different

components of Y_1, the incoming boundary of W, or (2) the feet of the 1-handle lay in the same component of Y_1. In the former, we may assume without loss of generality that there only two components of Y_1, since Floer homology is manifestly multiplicative under disjoint unions (i.e., groups and homomorphisms associated to disjoint unions of 3-manifolds and cobordisms, respectively, are tensor products), and product cobordisms map contact invariants naturally according to the previous section.[1] Similarly, for the second possibility, we may assume Y_1 is connected.

Naturality for 1-handles that connect two components is a consequence of a calculation for the graph cobordism map of the 1-handle cobordism, endowed with a trivalent (strong ribbon) graph that merges the two basepoints in the incoming components to a single basepoint in their outgoing connected sum. This calculation is the content of [15, Proposition 5.2], which indicates that such a graph cobordism induces a chain homotopy equivalence, and the complex associated to a connected sum is therefore homotopy equivalent to the tensor product of the complexes associated to the factors. This calculation reproved, in a functorial way, Ozsváth and Szabó's earlier connected sum formula [27, Theorem 6.2], under which the contact invariant behaves multiplicatively for contact connected sums [12, product formula]. The claimed naturality for Stein 1-handles is then immediate, provided that Ozsváth and Szabó's chain homotopy equivalence, used by the product formula for the contact invariant, agrees, up to homotopy, with the map Zemke associates to the 1-handle cobordism. But this is precisely the content of [35, Proposition 8.1]. Here, we should point out that the trivalent graph arises naturally from the Stein structure, as the stable manifold of the index one critical point of ϕ with respect to X_ϕ, union a flowline of X_ϕ from the critical point to the outgoing boundary.

The case of 1-handles with feet on the same component of the incoming boundary is simpler and, in this case, follows from Ozsváth and Szabó's definition of the 1-handle map [31, Section 4.3], together again with the fact that the contact invariant is multiplicative under contact connected sums. In this case, the outgoing manifold is contactomorphic to the connected sum $(Y\#(S^1\times S^2), \xi\#\xi_{std})$, so it suffices to show that the image of the dual of $c(\xi)$ under Ozsváth and Szabó's map induced by the 1-handle agrees with the dual $c(\xi\#\xi_{std})$. But the 1-handle map sends $c(\xi)^\vee$ to $c(\xi)^\vee \otimes 2_+$. Thus the problem is reduced to a single calculation, verifying that the dual of the contact class of the standard contact structure on $S^1\times S^2$ satisfies $2_+ = (c(S^1\times S^2, \xi_{std}))^\vee \in HF(S^1\times S^2)$. This calculation can be done in numerous ways; see, e.g., [13, Proof of Proposition 5.19], for an explicit treatment. □

[1] Here, a product cobordism means a 4-manifold diffeomorphic to $Y\times I$, through a diffeomorphism induced by the flow of the Liouville vector field. Since the "holonomy" diffeomorphism from the outgoing boundary to the incoming boundary [1, Definition 9.40] is a contactomorphism, the naturality results of the previous section, together with [34, Theorem B.2], indicate the contact invariants are mapped in the specified way.

Next we turn to 2-handles. While the naturality statement needed here for Stein 2-handle cobordisms was stated by Ghiggini in [8, Lemma 2.11], the proof relied on a naturality result for the cobordism map associated to a Lefschetz fibration over an annulus, attributed to [29, Theorem 5.3]. The latter theorem applies only to Lefschetz fibrations over the disk. The desired result can be derived from Ozsváth and Szabó's by a capping argument, together with the composition law for cobordism induced maps on Floer homology. We spell this out explicitly.

Lemma 3.6 (cf. [29, Theorem 5.3]). Let $\pi : W \to [0,1] \times S^1$ be a relatively minimal Lefschetz fibration over the annulus, viewed as a cobordism from Y_1 to Y_2, whose fiber F has genus $g > 1$. Then there is a unique Spin^c structure s over W for which

$$\langle c_1(s), [F] \rangle = 2 - 2g$$

and the induced map

$$F^+_{W,s} : HF^+(Y_1, s|_{Y_1}) \to HF^+(Y_2, s|_{Y_2})$$

is nontrivial. This is the canonical Spin^c structure k, and its associated map is an isomorphism.

Proof. Suppose that $s \in \text{Spin}^c(W)$ is as in the statement of the lemma, and induces a nontrivial map. We will show that s is the canonical Spin^c structure k on W, and the map is an isomorphism. By [29, Theorem 2.2] the fibration on Y_2 extends to a Lefschetz fibration on a 4-manifold W' over the disk $\pi' : W' \to D^2$, whose fiber is identified with F. Let $V = W \cup_{Y_2} W'$. Then V admits a Lefschetz fibration $\pi \cup \pi'$ over the disk.

The composition law for cobordism maps states that

$$F^+_{W'-B^4,k'} \circ F^+_{W,s} = \sum_{\{t \in \text{Spin}^c(W) \mid t|_W = s,\ t|_{W'-B^4} = k'\}} F^+_{V-B^4,t}, \tag{1}$$

where k' is the canonical Spin^c structure on W'. By [29, Theorem 5.3], the map

$$F^+_{W'-B^4,k'} : HF^+(Y_2, k'|_{Y_2}) \to HF^+(S^3)$$

is an isomorphism. Note that, according to [29, Theorem 5.2], there is a unique Spin^c structure on Y_2 whose Chern class evaluates on the class $[F]$ of the fiber to $2 - 2g$, and for which the Floer homology is nontrivial. It follows that if the map $F^+_{W,s}$ is nontrivial, as we've assumed, then the composite $F^+_{W'-B^4,k'} \circ F^+_{W,s}$ is also nontrivial, since the restrictions of s and k' to Y_2 must agree.

Now, since the Chern classes of the Spin^c structures t' and s evaluate to $2 - 2g$ on the class of the fiber $[F]$, the same is true for the Spin^c structures considered on V in the sum on the right-hand side of (1). Applying [29, Theorem 5.3] to the Lefschetz fibration on V implies that there is a unique nontrivial contribution to the sum, coming from the canonical Spin^c structure k_V on V, and $F^+_{V-B^4,k_V}$ is an

isomorphism. Since k_V restricts to the canonical Spin^c structure k on W, it follows that $s = k$, and the corresponding map is an isomorphism. □

With this in hand, we modify the argument of [8, Lemma 2.11] to establish naturality with respect to a collection of Stein 2-handles. This boils down to another application of the composition law, together with a theorem of Eliashberg:

Lemma 3.7. Suppose (W, J) is the cobordism associated to a collection of Stein 2-handle attachments. Then Theorem 3.1 is true for F_{W^+}.

Proof. As detailed above, we assume that all critical points of the plurisubharmonic Morse function on W have the same critical value, or equivalently that (W, J) is constructed by attaching a Stein 2-handle along each component of a Legendrian link L in (Y_1, ξ_1).

We may choose an open book decomposition adapted to ξ_1 such that the Legendrian link L sits naturally in a page. After positively stabilizing the open book we may assume that the pages have connected boundary and are of genus greater than one.

For $i \in \{1, 2\}$ let V_i denote the trace cobordism from Y_i to Y_i', the 3-manifold obtained by performing zero surgery along the binding of the open book. Surgery along L gives rise to the cobordism W from Y_1 to Y_2, and a cobordism W_0 from Y_1' to Y_2'. Note that both Y_1' and Y_2' are fibered 3-manifolds with fiber F obtained by capping off the boundary component of a page of the open book. W_0 admits a Lefschetz fibration over the annulus with fiber F.

Let $X = W \cup V_2 \cup -V_1 \cup W_0$ denote the cobordism from Y_1 to Y_2'. Using [5, Theorem 1.1] we may extend the symplectic structure induced by the Lefschetz fibration on W_0 over the 2-handle cobordism V_1, giving a symplectic structure ω on X. The restriction of ω to W agrees with the symplectic structure on W induced by the Legendrian surgery along L; in particular, the canonical Spin^c structure $k_X \in \mathrm{Spin}^c(X)$ of ω restricts to the canonical Spin^c-structure k of (W, J).

Since V_1 can be obtained from surgery along a homologically nontrivial curve in Y_1', restriction induces an isomorphism $H^2(X, Z) \to H^2(W_0, Z)$, so every Spin^c-structure on W_0 admits a unique extension over X. In particular, the extension of the canonical Spin^c-structure $k_0 \in \mathrm{Spin}^c(W_0)$ is k_X. For $i \in \{1, 2\}$, let $t_i = k_0|_{Y'}$.

Ozsváth and Szabó [30] characterize the contact invariant $c^+(\xi_i) \in HF^+(-Y_i, s_{\xi_i})$ as the image of a class $c^+(\pi_i) \in HF^+(-Y_i', t_i)$ associated to the fibration under the map $F^+_{V_i, p_i}$ where $p_i \in \mathrm{Spin}^c(V_i)$ is the unique extension of t_i. Let $s \in \mathrm{Spin}^c(W)$, then

$$F^+_{W^+,s}(c^+(\xi_2)) = F^+_{W^+,s} \circ F^+_{V_2^+}(c^+(\pi_2)) = \sum_{\{s_X \in \mathrm{Spin}^c(X) \mid s_X|_W = s,\ s_X|_{V_2} = p_2\}} F^+_{X^+,s_X}(c^+(\pi_2)),$$

where the last equality is given by the composition law for cobordism maps. Because

every Spin^c-structure on W_0 admits a unique extension over X, another application of the composition law shows that the above sum is equal to

$$\sum_{\{s_X \in \mathrm{Spin}^c(X) \mid s_X|_W = s,\ s_X|_{V_2} = \mathfrak{p}_2\}} F^+_{V_1^\dagger, s_X|_{V_1}} \circ F^+_{W_0^\dagger, s_X|_{W_0}}(c^+(\pi_2)).$$

Note that $\langle c_1(s_X|_{W_0}), [F]\rangle = \langle c_1(t_2), [F]\rangle = 2 - 2g$. Lemma 3.6 implies that there is at most one nonzero contribution to this sum, coming from a term where $s_X|_{W_0}$ is the canonical Spin^c-structure k_0, in which case $s_X = k_X$ and $s = k$ by the preceding discussion. We have

$$F^+_{\dagger W^\dagger, s}(c^+(\xi^2)) = \begin{cases} F^+_{V_1^\dagger} \circ F^+_{W_0^\dagger, k_0}(c^+(\pi_2)) & \text{for } s = k, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, Lemma 3.6 also tells us that $F^+_{W_0^\dagger, k_0}$ is an isomorphism mapping $c^+(\pi_2)$ to $c^+(\pi_1)$, thus

$$F^+_{\dagger W^\dagger, s}(c^+(\xi^2)) = \begin{cases} F^+_{V_1^\dagger}(c^+(\pi_1)) = c^+(\xi_1) & \text{for } s = k, \\ 0 & \text{otherwise.} \end{cases}$$

This proves that the contact invariant in HF^+ satisfies the naturality claimed by Theorem 3.1 under the map induced by a Stein 2-handle cobordism. To establish the result for the contact invariant in $\widehat{HF}$, recall that $c^+(\xi)$ is defined as the image of $c(\xi)$ under the inclusion-induced map $\iota_* : \widehat{HF} \to HF^+$, and that both invariants can be characterized as the image of a particular class under the 2-handle cobordism which caps the fiber of the open book. In the case of the plus invariant, this is the distinguished class $c^+(\pi)$ associated to the fibration considered above, whereas for the hat invariant we consider the element $c(\pi)$ mapping to $c^+(\pi)$ under ι_*. The claimed naturality result for 2-handles now follows from naturality of ι_* with respect to the maps on Floer homology associated to cobordisms [31, Theorem 3.1, Remark 3.2]; cf. [34, Theorem A]. □

Having established the claimed naturality result for a Stein 0- or 1-handle and for a collection of Stein 2-handles, the theorem follows from the composition law for cobordism maps. □

Remark 3.8. Echoing Remark 2.4, one could alternatively approach Theorem 3.1 using the Honda–Kazez–Matić interpretation of the contact invariant. Using their Heegaard diagrams, the proof hinges on (a) showing that there exists a unique pseudoholomorphic triangle contributing to $F^+_{W^\dagger}(c(\xi_2))$ whose domain is a union of small triangles having corners at the components of a generator representing $c(\xi_1)$ and (b) identifying the Spin^c structure associated to this pseudoholomorphic triangle with the canonical one.

We conclude with the following immediate corollary of Theorem 3.1, which generalizes the main result of [33]:

Corollary 3.9 (cf. [33, Theorem 2]). Let W be a smooth 4-manifold with boundary, equipped with two Stein structures J_1, J_2 with associated Spin^c structures s_1, s_2, and let ξ_1, ξ_2 be the induced contact structures on Y, the outgoing boundary of W. Suppose that the contact structure induced on the incoming boundary of W by J_1 has nonvanishing contact invariant. If the Spin^c structures s_1 and s_2 are not isomorphic, then the contact invariants $c(\xi_1), c(\xi_2)$ are distinct elements of $\widehat{HF}(-Y)$.

Acknowledgements

This article stemmed from the first author's work with Katherine Raoux, discussed at the BIRS workshop "Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4", and a particular application of that work which required naturality of the contact class under Stein cobordisms. It is our pleasure to thank Katherine, BIRS, and the workshop organizers for inspiring us to write this article, and Jamie Conway, Patrick Massot, Ian Zemke, and the referee for a number of helpful comments.

References

[1] K. Cieliebak and Y. Eliashberg, From Stein to Weinstein and back, American Mathematical Society Colloquium Publications 59, Amer. Math. Soc., Providence, RI, 2012. Symplectic geometry of affine complex manifolds. MR Zbl

[2] M. Echeverria, "Naturality of the contact invariant in monopole Floer homology under strong symplectic cobordisms", Algebr. Geom. Topol. 20:4 (2020), 1795–1875. MR Zbl

[3] S. Eilenberg and N. Steenrod, Foundations of algebraic topology, Princeton University Press, 1952. MR Zbl

[4] Y. Eliashberg, "Topological characterization of Stein manifolds of dimension > 2", Internat. J. Math. 1:1 (1990), 29–46. MR Zbl

[5] Y. Eliashberg, "A few remarks about symplectic filling", Geom. Topol. 8 (2004), 277–293. MR Zbl

[6] J. B. Etnyre, "Lectures on open book decompositions and contact structures", pp. 103–141 in Floer homology, gauge theory, and low-dimensional topology, Clay Math. Proc. 5, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[7] M. Gartner, "Projective Naturality in Heegaard Floer Homology", 2019. arXiv 1908.06237

[8] P. Ghiggini, "Ozsváth–Szabó invariants and fillability of contact structures", Math. Z. 253:1 (2006), 159–175. MR Zbl

[9] E. Giroux, "Géométrie de contact: de la dimension trois vers les dimensions supérieures", pp. 405–414 in Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 2002. MR Zbl

[10] R. E. Gompf, "Handlebody construction of Stein surfaces", Ann. of Math. (2) 148:2 (1998), 619–693. MR Zbl

[11] Y. Hatakeyama, "Some notes on the group of automorphisms of contact and symplectic structures", Tohoku Math. J. (2) 18 (1966), 338–347. MR Zbl

[12] M. Hedden, "An Ozsváth–Szabó Floer homology invariant of knots in a contact manifold", Adv. Math. 219:1 (2008), 89–117. MR Zbl

[13] M. Hedden and K. Raoux, "Knot Floer homology and relative adjunction inequalities", 2020. arXiv 2009.05462

[14] K. Hendricks and C. Manolescu, "Involutive Heegaard Floer homology", Duke Math. J. 166:7 (2017), 1211–1299. MR Zbl

[15] K. Hendricks, C. Manolescu, and I. Zemke, "A connected sum formula for involutive Heegaard Floer homology", Selecta Math. (N.S.) 24:2 (2018), 1183–1245. MR Zbl

[16] K. Honda, W. H. Kazez, and G. Matić, "The contact invariant in sutured Floer homology", Invent. Math. 176:3 (2009), 637–676. MR Zbl

[17] K. Honda, W. H. Kazez, and G. Matić, "On the contact class in Heegaard Floer homology", J. Differential Geom. 83:2 (2009), 289–311. MR Zbl

[18] A. Juhász, "Cobordisms of sutured manifolds and the functoriality of link Floer homology", Adv. Math. 299 (2016), 940–1038. MR

[19] A. Juhász, D. Thurston, and I. Zemke, "Naturality and mapping class groups in Heegard Floer homology", pp. v+174 , 2021. MR Zbl

[20] E. L. Lima, "On the local triviality of the restriction map for embeddings", Comment. Math. Helv. 38 (1964), 163–164. MR Zbl

[21] P. Lisca and A. I. Stipsicz, "Ozsváth–Szabó invariants and tight contact three-manifolds, I", Geom. Topol. 8 (2004), 925–945. MR Zbl

[22] P. Massot, "Infinitely many universally tight torsion free contact structures with vanishing Ozsváth–Szabó contact invariants", Math. Ann. 353:4 (2012), 1351–1376. MR Zbl

[23] J. Milnor, Lectures on the h-cobordism theorem, Princeton University Press, 1965. MR Zbl

[24] T. Mrowka and Y. Rollin, "Legendrian knots and monopoles", Algebr. Geom. Topol. 6 (2006), 1–69. MR Zbl

[25] P. Ozsváth and Z. Szabó, "Holomorphic disks and genus bounds", Geom. Topol. 8 (2004), 311–334. MR

[26] P. Ozsváth and Z. Szabó, "Holomorphic disks and knot invariants", Adv. Math. 186:1 (2004), 58–116. MR

[27] P. Ozsváth and Z. Szabó, "Holomorphic disks and three-manifold invariants: properties and applications", Ann. of Math. (2) 159:3 (2004), 1159–1245. MR

[28] P. Ozsváth and Z. Szabó, "Holomorphic disks and topological invariants for closed three-manifolds", Ann. of Math. (2) 159:3 (2004), 1027–1158. MR

[29] P. Ozsváth and Z. Szabó, "Holomorphic triangle invariants and the topology of symplectic four-manifolds", Duke Math. J. 121:1 (2004), 1–34. MR

[30] P. Ozsváth and Z. Szabó, "Heegaard Floer homology and contact structures", Duke Math. J. 129:1 (2005), 39–61. MR

[31] P. Ozsváth and Z. Szabó, "Holomorphic triangles and invariants for smooth four-manifolds", Adv. Math. 202:2 (2006), 326–400. MR

[32] R. S. Palais, "Local triviality of the restriction map for embeddings", Comment. Math. Helv. 34 (1960), 305–312. MR Zbl

[33] O. Plamenevskaya, "Contact structures with distinct Heegaard Floer invariants", Math. Res. Lett. 11:4 (2004), 547–561. MR Zbl

[34] I. Zemke, "Graph cobordisms and Heegaard Floer homology", 2015. arXiv 1512.01184

[35] I. Zemke, "Duality and mapping tori in Heegaard Floer homology", 2018. Zbl arXiv 1801.09270

Received 1 Mar 2021. Revised 28 Oct 2021.

MATTHEW HEDDEN: mhedden@math.msu.edu
Department of Mathematics, Michigan State University, East Lansing, MI, United States

LEV TOVSTOPYAT-NELIP: tovstopy@msu.edu
Department of Mathematics, Michigan State University, East Lansing, MI, United States

Volume Editors:

John A. Baldwin
Boston College
Boston, MA
United States

Hans U. Boden
McMaster University
Hamilton, ON
Canada

John B. Etnyre
Georgia Institute of Technology
Atlanta, GA
United States

Liam Watson
University of British Columbia
Vancouver, BC
Canada

The cover image is based on an illustration from the article "Khovanov homology and strong inversions", by Artem Kotelskiy, Liam Watson and Claudius Zibrowius (see p. 232).

Electronic copies can be obtained free of charge from http://msp.org/obs/5 and printed copies can be ordered from MSP (contact@msp.org).

The Open Book Series is a trademark of Mathematical Sciences Publishers.

ISSN: 2329-9061 (print), 2329-907X (electronic)

ISBN: 978-1-935107-11-8 (print), 978-1-935107-10-1 (electronic)

First published 2022.

MATHEMATICAL SCIENCES PUBLISHERS
798 Evans Hall #3840, c/o University of California, Berkeley CA 94720-3840
contact@msp.org https://msp.org

THE OPEN BOOK SERIES 5

Gauge Theory and Low-Dimensional Topology: Progress and Interaction

This volume is a proceedings of the 2020 BIRS workshop Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4. This was the 6th iteration of a recurring workshop held in Banff. Regrettably, the workshop was not held onsite but was instead an online (Zoom) gathering as a result of the Covid-19 pandemic. However, one benefit of the online format was that the participant list could be expanded beyond the usual strict limit of 42 individuals. It seemed to be also fitting, given the altered circumstances and larger than usual list of participants, to take the opportunity to put together a conference proceedings.

The result is this volume, which features papers showcasing research from participants at the 6th (or earlier) Interactions workshops. As the title suggests, the emphasis is on research in gauge theory, contact and symplectic topology, and in low-dimensional topology. The volume contains 16 refereed papers, and it is representative of the many excellent talks and fascinating results presented at the Interactions workshops over the years since its inception in 2007.

TABLE OF CONTENTS