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Sharp Exponential Decay Rates for Anisotropically Damped Waves

Blake Keeler and Perry Kleinhenz

Abstract. In this article, we study energy decay of the damped wave equation on compact Riemannian manifolds where the damping coefficient is anisotropic and modeled by a pseudodifferential operator of order zero. We prove that the energy of solutions decays at an exponential rate if and only if the damping coefficient satisfies an anisotropic analogue of the classical geometric control condition, along with a unique continuation hypothesis. Furthermore, we compute an explicit formula for the optimal decay rate in terms of the spectral abscissa and the long-time averages of the principal symbol of the damping over geodesics, in analogy to the work of Lebeau for the isotropic case. We also construct genuinely anisotropic dampings which satisfy our hypotheses on the flat torus.

1. Introduction

Let (M, g) be a smooth, compact Riemannian manifold without boundary and let Δ_g be the associated Laplace–Beltrami operator (taken with the convention that $\Delta_g \leqslant 0$). Suppose $W : L^2(M) \to L^2(M)$ is bounded and nonnegative. We consider the generalized damped wave equation given by

$$\begin{cases} \partial_t^2 u - \Delta_g u + 2W\partial_t u = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \tag{1.1}$$

for $(u_0, u_1)^T \in \mathscr{H} := H^1(M) \oplus L^2(M)$, where $\mathscr{H}$ is taken with the natural norm

$$\|(u_0, u_1)^T\|_{\mathscr{H}}^2 = \|(1 - \Delta_g)^{\frac{1}{2}} u_0\|_{L^2(M)}^2 + \|u_1\|_{L^2(M)}^2.$$

We study the asymptotic properties of the energy of solutions to (1.1) as $t \to \infty$. Here, the energy is defined by

$$E(u, t) = \frac{1}{2} \int_M |\nabla_g u(t, x)|^2 + |\partial_t u(t, x)|^2 \, \mathrm{d}v_g(x), \tag{1.2}$$

where dv_g is the Riemannian volume form on M. It is straightforward to compute that

$$\frac{d}{dt}E(u,t) = -2\mathrm{Re}\,\langle W\partial_t u, \partial_t u\rangle \leqslant 0, \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $L^2(M,g)$. Thus, the assumption that W is a nonnegative operator guarantees that the energy of solutions to (1.1) experiences dissipation, but (1.3) does not indicate how quickly the energy decays as $t\to\infty$. The most straightforward type of decay is uniform stabilization, i.e., when there exists a constant $C>0$ and a real-valued function $t\mapsto r(t)$ with $r(t)\to 0$ as $t\to\infty$ such that

$$E(u,t) \leqslant Cr(t)E(u,0). \tag{1.4}$$

It is worth noting that if (1.4) is satisfied, standard semigroup theory implies that the decay rate $r(t)$ must be exponential.

In the case where W acts via multiplication by a bounded, nonnegative function b, a great deal is known about energy decay rates. Perhaps the most well-known result states that solutions to (1.1) experience uniform stabilization if and only if W satisfies the geometric control condition (GCC) [26,28]. The GCC is satisfied if there exists some $T>0$ such that every geodesic with length at least T intersects the set where b is bounded below by some positive constant. In the setting where the GCC is not satisfied, many other works have proved weaker decay rates with respect to higher regularity initial data (c.f. [2–7,21,23]). With more restrictive assumptions on W and M, one can show that some of these weaker decay rates are in fact sharp (c.f. [1,10,11,14,18,19,22,30,32]).

A distinct shortcoming of the multiplicative case is that the damping force is sensitive only to positional information and not to the direction in which the solution propagates. For this reason, one can classify multiplicative damping as an isotropic force, but many physical systems which experience *anisotropic* damping forces are studied in materials science, physics, and engineering [8,15,16]. However, a general analysis of the damped wave equation in the anisotropic case has not yet been done. This article aims to address this gap in the literature by studying the case where the anisotropic damping force is modeled by a pseudodifferential operator.

It is common in analysis of the generalized damped wave Eq. (1.1) to assume that W takes the form of a square, i.e., $W=B^*B$ for some bounded operator B (c.f. [1]). This guarantees that W is nonnegative and enables the use of certain techniques from spectral theory. We allow for a slightly more general assumption here, namely that W takes the form

$$W = \sum_{j=1}^{N} B_j^* B_j$$

for some finite collection $\{B_j\}_{j=1}^N \subset \Psi^0_{c\ell}(M)$, where $\Psi^0_{cl}(M)$ denotes the space of classical pseudodifferential operators on M of order zero with polyhomogeneous symbol expansions (c.f. [33, Ch. 7]). The corresponding space of symbols is denoted $S^0_{c\ell}(T^*M)$. We note that allowing W to take the form of a sum of squares is indeed a generalization, as it is not generically possible to write $\sum_{j=1}^N B_j^* B_j$ as B^*B for some $B \in \Psi^0_{c\ell}(M)$, since the pseudodifferential calculus only allows for the computation of square roots modulo a smoothing remainder. Indeed, even at the level of symbols it is in general not possible to take a smooth square root of a sum of squares of smooth functions. We denote by $w \in S^0_{c\ell}(T^*M)$ the principal symbol of W, taken to be positively fiber-homogeneous of degree 0 outside a small neighborhood of the zero section in T^*M. That is, $w(x, s\xi) = w(x, \xi)$ for all $s \geqslant 1$ and all $|\xi| \geqslant c$ for some $c > 0$ which can be chosen to be arbitrarily small. This homogeneity allows us to treat w as a function on the co-sphere bundle

$$S^*M := \{(x, \xi) \in T^*M : |\xi|_g = 1\}.$$

We now state the required assumptions for the main theorem. The first is an anisotropic analogue of the classical geometric control condition, given in terms of the long-time averages of w over lifted geodesics.

Assumption 1 (**Anisotropic Geometric Control Condition**). Let φ_t denote the lift of the geodesic flow to T^*M. Assume that there exists a compact neighborhood K of the zero section in T^*M and constants $T_0, c > 0$ such that for every $(x_0, \xi_0) \in T^*M \backslash K$,

$$\frac{1}{T}\int_0^T w(\varphi_t(x_0, \xi_0))\,\mathrm{d}t \geqslant c, \quad \text{for } T \geqslant T_0.$$

That is, the long-time averages of w over geodesics are uniformly bounded below. In this case, we say W satisfies the *anisotropic* geometric control condition (AGCC).

Remark 1.1. One can equivalently state the AGCC as requiring that every lifted geodesic intersects the elliptic set of W in time $T < T_0$, but the above characterization is more useful for our purposes. Also, note that in the case of multiplicative damping, Assumption 1 is equivalent to classical geometric control condition stated in [28].

The second key assumption requires that the kernel of W contain no nontrivial eigenfunctions of Δ_g.

Assumption 2. If $v \in L^2(M)$ satisfies $-\Delta_g v = \lambda^2 v$ with $\lambda \neq 0$, then $Wv \neq 0$.

In the case where $W = b(x)$, Assumption 2 is satisfied when b is supported on any open set, since eigenfunctions of Δ_g cannot vanish on open sets by the unique continuation principle (c.f. [28]). It is for this reason that we sometimes refer to Assumption 2 as a “unique continuation hypothesis.”

With these assumptions stated, we then have the following equivalence.

Theorem 1. *All solutions u to (1.1) with $W \in \Psi^0_{c\ell}(M)$ satisfy*

$$E(u,t) \leqslant Ce^{-\beta t}E(u,0) \tag{1.5}$$

for some C, $\beta > 0$ and for all $t \geqslant 0$ if and only if W satisfies Assumptions 1 and 2.

In other words, solutions experience uniform stabilization if and only if W satisfies Assumptions 1 and 2.

The existing literature on anisotropic damping coefficients is quite limited. In the context of pseudodifferential W, Sjöstrand [29] studied the asymptotic distribution of eigenvalues of the stationary damped wave equation. Christianson, Schenck, Vasy, and Wunsch [9] showed that a polynomial resolvent estimate for a related complex absorbing potential problem gives another polynomial resolvent estimate of the same order for the stationary damped wave equation. However, these results do not consider anisotropic damping in a time-dependent setting and so do not provide energy decay results. Theorem 1 addresses this gap in the literature by providing conditions which guarantee exponential uniform stabilization, in analogy to the classical result of Rauch and Taylor [28].

Since Theorem 1 only claims the existence of some exponential decay rate β, a natural question is to determine the optimal rate of decay for a given damping coefficient. Given a fixed $W \in \Psi^0_{c\ell}(M)$, we define the *best exponential decay rate* as in [21] via

$$\alpha := \sup\{\beta \in \mathbb{R} :\, \exists C > 0 \text{ such that } E(u,t) \leqslant Ce^{-\beta t}E(u,0)\ \ \forall u \text{ which solve (1.1)}\}. \tag{1.6}$$

Our next result shows that α can be expressed in terms of two fundamental quantities: the spectral abscissa and the long-time averages of w over lifted geodesics. The spectral abscissa is defined with respect to the operator

$$A_W := \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta_g & -2W \end{pmatrix},$$

which is the infinitesimal generator of the solution semigroup for (1.1). For each $R > 0$, we set

$$D(R) = \sup\{\mathrm{Re}\,(\lambda) :\, |\lambda| > R,\, \lambda \in \mathrm{Spec}(A_W)\}.$$

We then define the spectral abscissa as

$$D_0 = \lim_{R \to 0^+} D(R). \tag{1.7}$$

We also define for $t \in \mathbb{R}$ the time-average of the damping along geodesics

$$L(t) = \inf_{(x,\xi) \in S^*M} \frac{1}{t} \int_0^t w(\varphi_s(x,\xi))\, \mathrm{d}s,$$

and the long-time limit

$$L_\infty = \lim_{t \to \infty} L(t). \tag{1.8}$$

We can then characterize α as follows.

Theorem 2. *The best exponential decay rate for solutions to* (1.1) *with* $W \in \Psi^0_{c\ell}(M)$ *is*

$$\alpha = 2\min\{-D_0, L_\infty\},$$

where D_0 *and* L_∞ *are defined by* (1.7) *and* (1.8), *respectively.*

Remark 1.2. It is noteworthy that the optimal decay rate here is an exact analogy of the multiplicative case studied by Lebeau (c.f. [21, Theorem 2]). While the broad structure of our proof is similar, there are portions of the analysis which diverge greatly, particularly in Sect. 2 where we investigate the action of pseudodifferential operators on Gaussian beams.

Remark 1.3. Theorem 2 is significantly stronger than Theorem 1, although this is not immediately obvious. The main portion of this article is dedicated to the proof of Theorem 2. We then show that Theorem 2 implies Theorem 1 in Sect. 5.

Theorems 1 and 2 fit into a broad range of existing results which attempt to reproduce the equivalence of the GCC and exponential decay under modified hypotheses on the damping. It is not uncommon for such statements to be somewhat inconclusive. For example, when the damping is allowed to be time dependent, [24] showed that for time periodic damping, the GCC indeed guarantees exponential decay. In recent work, [20] showed that a generalization of the GCC implies exponential energy decay for non-periodic, time-dependent damping, but the converse is not known in either the periodic or non-periodic case. In the setting where the damping is allowed to take negative values (commonly called "indefinite damping"), the state of the art is similarly mixed. If M is an open domain in $\mathbb{R}^n$ with C^2 boundary, [25] proves an exponential decay rate provided that the damping is positive in a neighborhood of ∂M (which implies the GCC) and $\inf_{x\in M} W(x)$ is not too negative. However, it is currently not known if an appropriate generalization of the GCC is equivalent to exponential stability in the indefinite case. The limitations of these results illustrate that seemingly simple changes to hypotheses on the damping coefficient can create substantial barriers to reproducing the classical equivalence theorem. So, the fact that Theorem 1 provides a direct analogy of the GCC for pseudodifferential damping which is equivalent to exponential decay is somewhat exceptional. Note also that these other generalizations do not possess an analogy of Theorem 2. To the authors knowledge, there are no results providing explicit exponential decay rates for general time-dependent damping on manifolds, and although [25] provides a rate for the exponential decay in the indefinite case, it is not shown to be sharp.

Our final result concerns Assumption 2, which is necessary in order to obtain Theorem 1. To see this, suppose that v satisfies $-\Delta_g v = \lambda^2 v$ with $\lambda \neq 0$ and $Wv = 0$. Then, the function

$$u(t, x) = e^{it\lambda} v(x),$$

solves (1.1) with initial data $(v, i\lambda v)^T$, but has energy $E(u,t) = \lambda^2 \|v\|^2_{L^2(M)}$ for all t. As previously mentioned, when W is a multiplication operator supported on any open set, unique continuation results guarantee that W does not annihilate any eigenfunctions of Δ_g, making Assumption 2 unnecessary. However, in the pseudodifferential setting, verifying this assumption is more difficult.

A special case in which Assumption 2 is easy to check is when W is constructed from functions of Δ_g. Suppose $W = B^*B$ with $B = f(-\Delta_g)$, where $f : \mathbb{R} \to \mathbb{R}$ satisfies a "symbol-type" estimate of the form

$$|\partial_s^k f| \leqslant C(1 + |s|)^{-k}$$

for any k. The functional calculus of Strichartz [31] shows that W is pseudodifferential of order 0 when constructed in this way. The calculus also immediately implies that Assumption 2 holds as long as f does not vanish on the spectrum of $-\Delta_g$, since for any eigenfunction v with eigenvalue λ, we have $Wv = f(\lambda)^2 v$. However, damping coefficients constructed in this fashion are somewhat uninteresting in the sense that the principal symbol is a function of $|\xi|^2_g$, and therefore independent of direction. Thus, examples of this type are not truly anisotropic. In general, it is not obvious that one can always construct nontrivial anisotropic examples satisfying Assumption 2, although we expect that a rich class of examples do indeed exist. The following theorem demonstrates that one can always produce such examples when (M, g) is real analytic.

Theorem 3. *If (M, g) is compact and real analytic, then there exists $W \in \Psi^0_{cl}(M)$ of the form $W = \sum_{j=1}^N B_j^* B_j$, such that for each $x \in M$, the principal symbol of W vanishes on an open cone in $T^*_x M$, and for any eigenfunction v of Δ_g, we have $Wv \neq 0$.*

The fact that the principal symbol vanishes in an open cone of directions at each point implies that the W in this theorem is not built from functions of Δ_g, which rules out the trivial case discussed above. Using the machinery developed in the proof of Theorem 3, we are able to produce explicit examples on the flat 2-torus of operators $W \in \Psi^0_{cl}$ which satisfy both Assumptions 1 and 2. This construction is presented in Sect. 6.

Remark 1.4. As mentioned previously, Assumption 2 follows directly from the geometric control condition in the multiplicative case. One might hope that this could be generalized to the scenario where W is pseudodifferential, but this problem is exceedingly difficult in general. In fact, there is no result known to the authors which addresses this question fully. One scenario in which there is some progress is manifolds with Anosov geodesic flow. In this setting, Dyatlov, Jin and Nonnenmacher were able to show a semiclassical lower bound on $\|Av_h\|$ for $A \in \Psi^0_h(M)$ and $v \in \ker(h^2\Delta_g - 1)$ when the principal symbol of A does not vanish identically on S^*M [10]. However, even this result only proves unique continuation in the high-frequency limit. One still cannot exclude the possibility that a low-frequency eigenfunction is annihilated by the damping term. Furthermore, even in this specialized setting, the proof involves highly sophisticated techniques. An analysis of the general case is an open problem, and we suspect that it would be a significant undertaking.

1.1. Outline of the Article

The majority of this article is devoted to the proof of Theorem 2, which spans Sects. 2, 3, and 4. We begin in Sect. 2 with a detailed study of the action of certain pseudodifferential operators on coherent states, which is a critical component of constructing quasimodes for (1.1). This analysis is a key point where the pseudodifferential case becomes significantly more difficult than the multiplicative setting. In Sect. 3, we then use the result of Sect. 2 to produce quasimodes for the damped wave equation whose energy is strongly localized near a fixed geodesic. The analysis of these quasimodes allows us to prove that $\alpha \leqslant 2\min\{-D_0, L_\infty\}$. The proof of Theorem 2 is completed in Sect. 4, where we prove the lower bound $\alpha \geqslant 2\min\{-D_0, L_\infty\}$. This section primarily utilizes spectral theory arguments and follows in close analogy to [21], so we omit some of the more technical details.

In Sect. 5, we show that Theorem 2 implies Theorem 1. This follows directly from spectral theory analysis.

Finally, in Sect. 6, we restrict to the case of real analytic manifolds to produce some examples. We provide a fairly generic condition on pseudodifferential operators which guarantees that they satisfy Assumption 2. We also show that one can always produce examples which fall into this category, thus proving Theorem 3. We then conclude by constructing some explicit examples on the flat torus which satisfy both Assumptions 1 and 2.

2. Pseudodifferential Operators Acting on Coherent States

A key component of the proof of Theorem 2 is to build quasimodes for (1.1) using Gaussian beams, which are strongly localized along a given geodesic. In this section, we obtain precise estimates for pseudodifferential operators acting on slightly simpler objects, namely coherent states. A coherent state on $\mathbb{R}^n$ is a sequence of smooth functions $\{h_k\}$ taking the form

$$h_k(x) = k^{\frac{n}{4}} e^{ik\langle x-x_0,\xi_0\rangle} e^{\frac{ik}{2}\langle A(x-x_0),(x-x_0)\rangle} b(x) \tag{2.1}$$

for some fixed $(x_0, \xi_0) \in S^*\mathbb{R}^n$, where $b \in C_c^\infty(\mathbb{R}^n)$ and $A \in \mathbb{C}^{n\times n}$ has positive definite imaginary part. Heuristically, one thinks of h_k as being strongly microlocalized near (x_0, ξ_0). The objective of this section is to show that if a symbol $a \in S^m_{c\ell}(\mathbb{R}^{2n})$ vanishes to some finite order at (x_0, ξ_0), then $\|\mathrm{Op}(a)h_k\|_{L^2(\mathbb{R}^n)}$ satisfies a bound which depends on the symbol order m and on the order of vanishing. Here, $\mathrm{Op}(a)$ denotes the standard quantization given by

$$\mathrm{Op}(a)f(x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,\xi) f(y)\,\mathrm{d}y\,\mathrm{d}\xi. \tag{2.2}$$

Proposition 2.1. *Fix $(x_0, \xi_0) \in S^*\mathbb{R}^n$, $b \in C_c^\infty(\mathbb{R}^n)$, and a matrix $A \in \mathbb{C}^{n\times n}$, with positive definite imaginary part. Then, for any $k \geqslant 1$, let h_k be given by*

(2.1). *Let* $a \in S^m_{c\ell}(\mathbb{R}^{2n})$ *have compact support in* x *and a polyhomogeneous expansion given by*

$$a \sim \sum_{j \geqslant 0} a_{m-j},$$

where each $a_{m-j} \in S^{m-j}_{c\ell}(\mathbb{R}^{2n})$. *Suppose there exists an* $\ell \in \mathbb{N}$ *such that* a_{m-j} *vanishes to order* $\ell - 2j$ *at* (x_0, ξ_0) *for all* $j \leqslant \frac{\ell}{2}$. *Then, for each* $\varepsilon > 0$, *there exists a* $C > 0$, *depending on* ε, m, *and* ℓ, *so that*

$$\|\mathrm{Op}(a)h_k\|_{L^2(\mathbb{R}^n)} \leqslant Ck^{m-\frac{\ell}{2}+\varepsilon}. \tag{2.3}$$

Remark 2.2. In the subsequent sections, we only strictly need Proposition 2.1 in the case where $m = 0$ and $\ell = 1$. However, the proof in the general case is not much more difficult, and so we believe it to be worthwhile to state the full strength of the estimate.

Proof. By the polyhomogeneity of a, for any $N_0 \geqslant 0$ there exists $r_{N_0} \in S^{m-N_0}_{c\ell}$ such that

$$a = \sum_{j=0}^{N_0-1} a_{m-j} + r_{N_0}. \tag{2.4}$$

We begin with the following lemma, which handles the remainder term in this expansion.

Lemma 2.3. *Let* $r \in S^{-s}_{c\ell}(\mathbb{R}^{2n})$ *with* $s \geqslant 0$. *Then, there exists a* $C > 0$ *such that*

$$\|\mathrm{Op}(r)h_k\|_{L^2(\mathbb{R}^n)} \leqslant Ck^{-\frac{s}{2}}. \tag{2.5}$$

Proof. By the quantization formula (2.2), we have

$$\|\mathrm{Op}(r)h_k\|^2_{L^2(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^n} \left| \int\limits_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} r(x,\xi)h_k(y)\,\mathrm{d}y\,\mathrm{d}\xi \right|^2 \mathrm{d}x.$$

Assume without loss of generality that $x_0 = 0$. We then change variables via $x \mapsto k^{-\frac{1}{2}}x$, $y \mapsto k^{-\frac{1}{2}}y$, and $\xi \mapsto k^{\frac{1}{2}}\xi$. Recalling the definition of h_k, we obtain

$$\begin{aligned}&\|\mathrm{Op}(r)h_k\|^2_{L^2(\mathbb{R}^n)} \\ &= \int\limits_{\mathbb{R}^n} \left| \int\limits_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} r(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi) e^{ik^{\frac{1}{2}}\langle y,\xi_0\rangle} e^{\frac{i}{2}\langle Ay,y\rangle} b(k^{-\frac{1}{2}}y)\,\mathrm{d}y\,\mathrm{d}\xi \right|^2 \mathrm{d}x.\end{aligned} \tag{2.6}$$

For notational convenience, we define $g_A(y) = e^{\frac{i}{2}\langle Ay,y\rangle}$ and let $\tau_s : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ denote dilation by $s > 0$. That is, $\tau_s f(y) = f(sy)$. Then, using $\hat{}$ to denote the standard Fourier transform, we define

$$F_k(\xi) := \int\limits_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} g_A(y) b(k^{-1/2}y)\mathrm{d}y = k^{n/2}\left[\widehat{g}_A * \tau_{k^{\frac{1}{2}}}\widehat{b}\right](\xi). \tag{2.7}$$

Thus, we can rewrite (2.6) as

$$\|\mathrm{Op}(r)h_k\|^2_{L^2(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^n} \left| \int\limits_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} r(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi) F_k(\xi - k^{\frac{1}{2}}\xi_0) \mathrm{d}\xi \right|^2 \mathrm{d}x. \quad (2.8)$$

We claim that for any $N \in \mathbb{N}$ and any multi-index β, there exists a constant $C_{N,\beta} > 0$ which is independent of k, such that

$$\left|\partial_\xi^\beta F_k(\xi)\right| \leqslant C_{N,\beta}(1+|\xi|)^{-N} \quad \text{for all } k \in \mathbb{N}. \quad (2.9)$$

To see this, consider the case where $|\beta| = 0$ and note that

$$\begin{aligned} &\left|(1+|\xi|)^N k^{n/2}[\widehat{g}_A * \tau_{k^{\frac{1}{2}}}\widehat{b}](\xi)\right| \\ &\quad \leqslant C_N k^{n/2} \int\limits_{\mathbb{R}^n} (1+|\xi-\eta|)^N (1+|\eta|)^N |\widehat{g}_A(\xi-\eta)|\,|\widehat{b}(k^{\frac{1}{2}}\eta)|\,\mathrm{d}\eta \\ &\quad \leqslant C_N k^{n/2} \int\limits_{\mathbb{R}^n} (1+|\eta|)^N |\widehat{b}(k^{\frac{1}{2}}\eta)|\,\mathrm{d}\eta, \end{aligned}$$

where the last inequality follows from the fact that $\widehat{g}_A$ is Schwartz class, since A has positive definite imaginary part. Now, observe that

$$\begin{aligned} k^{n/2}\int\limits_{\mathbb{R}^n}(1+|\eta|)^N|\widehat{b}(k^{\frac{1}{2}}\eta)\,\mathrm{d}\eta| &\leqslant k^{n/2}\int\limits_{\mathbb{R}^n} \frac{(1+k^{\frac{1}{2}}|\eta|)^{N+n+1}}{(1+k^{\frac{1}{2}}|\eta|)^{n+1}}|\widehat{b}(k^{\frac{1}{2}}\eta)|\,\mathrm{d}\eta \\ &\leqslant C_N k^{n/2}\int\limits_{\mathbb{R}^n}(1+k^{\frac{1}{2}}|\eta|)^{-n-1}\,\mathrm{d}\eta, \end{aligned}$$

for some new $C_N > 0$, since $\widehat{b}$ is Schwartz class. Changing variables via $\eta \mapsto k^{-\frac{1}{2}}\eta$, we obtain that

$$\left|(1+|\xi|)^N k^{n/2}[\widehat{g}_A * \tau_{k^{\frac{1}{2}}}\widehat{b}](\xi)\right| \leqslant C_N$$

after potentially increasing C_N. Dividing through by $(1+|\xi|)^N$ completes the proof of (2.9) for $|\beta| = 0$. To obtain the estimate when $|\beta| \neq 0$, simply repeat the above proof with $\widehat{g}_A$ replaced by $\partial_\xi^\beta \widehat{g}_A$.

Now, in order to estimate (2.8) we introduce a smooth cutoff function χ which is identically one in a neighborhood of $x = 0$. We then write

$$\|\mathrm{Op}(r)h_k\|^2_{L^2(\mathbb{R}^n)} = I + I\!I,$$

where I is defined by

$$I = \int\limits_{\mathbb{R}^n} \left| \int\limits_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \chi(x) r(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi) F_k(\xi - k^{\frac{1}{2}}\xi_0) d\xi \right|^2 \mathrm{d}x,$$

and $I\!I$ is defined analogously with $\chi(x)$ replaced by $1-\chi(x)$. To estimate I, we note that when $|\xi| \leqslant 1$,

$$|r(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi)F_k(\xi - k^{\frac{1}{2}}\xi_0)| \leqslant C_N(1+|\xi - k^{\frac{1}{2}}\xi_0|)^{-N} \leqslant C'_N k^{-N/2},$$

for some C_N, $C'_N > 0$ and any N, by (2.9) and the fact that r has nonpositive order and is therefore uniformly bounded. Thus,

$$I = \int\limits_{\mathbb{R}^n} \left| \int\limits_{|\xi|\geqslant 1} e^{i\langle x,\xi\rangle}\chi(x) r(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi) F_k(\xi - k^{\frac{1}{2}}\xi_0) \mathrm{d}\xi \right|^2 \mathrm{d}x + \mathcal{O}(k^{-\infty}). \tag{2.10}$$

Now, when $|\xi| \geqslant 1$,

$$|r(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi)| \leqslant C(1 + k^{\frac{1}{2}}|\xi|)^{-s} \leqslant Ck^{-\frac{s}{2}}.$$

Combining this with (2.10), we have

$$I \leqslant Ck^{-s}\|F_k\|^2_{L^1(\mathbb{R}^n)} \leqslant C'k^{-s}, \tag{2.11}$$

where the final inequality follows from (2.9).

Now, consider $I\!I$. Since $1-\chi$ vanishes in a neighborhood of $x = 0$, we may integrate by parts arbitrarily many times in ξ using the operator $\frac{\langle x,\nabla_\xi\rangle}{i|x|^2}$, which preserves $e^{i\langle x,\xi\rangle}$. That is, for any $\nu \geqslant 0$, we have

$$I\!I = \int\limits_{\mathbb{R}^n} \left| \int\limits_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} (1-\chi(x)) \left(\frac{i\langle x,\nabla_\xi\rangle}{|x|^2}\right)^\nu \left(r(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi) F_k(\xi - k^{\frac{1}{2}}\xi_0)\right) d\xi \right|^2 \mathrm{d}x.$$

By (2.9) and the fact that $r \in S^{-s}_{c\ell}$, we have for any multi-index β and any N,

$$\begin{aligned} &\left|\partial_\xi^\beta \left(r(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi) F_k(\xi - k^{\frac{1}{2}}\xi_0)\right)\right| \\ &\quad \leqslant \sum_{|\gamma|\leqslant|\beta|} C_{\gamma,N} k^{\frac{|\gamma|}{2}} (1 + k^{\frac{1}{2}}|\xi|)^{-s-|\gamma|} (1 + |\xi - k^{\frac{1}{2}}\xi_0|)^{-N}. \end{aligned}$$

In the region where $|\xi| \geqslant 1$, the above is bounded by $C_N k^{-\frac{s}{2}}(1+|\xi - k^{\frac{1}{2}}\xi_0|)^{-N}$ for some $C_N > 0$. Alternatively, when $|\xi| \leqslant 1$, we have a bound of the form $C_N k^{-N/2}$, since $1 + |\xi - k^{\frac{1}{2}}\xi_0| \geqslant Ck^{\frac{1}{2}}$. Combining these facts, we have

$$\begin{aligned} &\int\limits_{\mathbb{R}^n} \left| \int\limits_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} (1-\chi(x)) \left(\frac{i\langle x,\nabla_\xi\rangle}{|x|^2}\right)^\nu \left(r(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi) F_k(\xi - k^{\frac{1}{2}}\xi_0)\right) \mathrm{d}\xi \right|^2 \mathrm{d}x \\ &\qquad \leqslant C_N k^{-s}, \end{aligned}$$

for some $C_N > 0$, provided $\nu > \frac{n+1}{2}$ so that the integral in x is convergent. Therefore,

$$I\!I \leqslant C_N k^{-s}.$$

Combining this with (2.11) and taking square roots of both sides completes the proof. $\square$

We now return to the proof of Proposition 2.1. We aim to estimate each of the terms in the sum in (2.4) separately. For each $j \leqslant N_0 - 1$, we have

$$\|\mathrm{Op}(a_{m-j})h_k\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a_{m-j}(x,\xi) h_k(y) \mathrm{d}y \mathrm{d}\xi \right|^2 \mathrm{d}x.$$

As before, we change variables via $x \mapsto k^{-\frac{1}{2}}x, y \mapsto k^{-\frac{1}{2}}y$, and $\xi \mapsto k^{\frac{1}{2}}\xi$. This gives

$$\|\mathrm{Op}(a_{m-j})h_k\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a_{m-j}(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi) F_k(\xi - k^{\frac{1}{2}}\xi_0) \mathrm{d}\xi \right|^2 \mathrm{d}x,$$

where F_k is given by (2.7). Now, let $\chi(x,\xi) = \chi(\xi)$ be a smooth function which is identically one for $|\xi| \leqslant \frac{1}{2}$ and supported where $|\xi| \leqslant 1$. Then, for any $0 < \delta < 1/2$ and any $k > 0$, define

$$\chi_{k,\delta}(\xi) = \chi(k^{-\delta}(\xi - k^{\frac{1}{2}}\xi_0)),$$

so that $\chi_{k,\delta}$ is identically one on the ball of radius $\frac{1}{2}k^{\delta}$ centered at $k^{\frac{1}{2}}\xi_0$ and zero outside the corresponding ball of radius k^{δ}. Since $|\xi| > 2$ on the support of $\chi_{k,\delta}$ for sufficiently large k, the homogeneity of a_{m-j} implies that

$$\chi_{k,\delta}(\xi) a_{m-j}(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi) = k^{\frac{m-j}{2}} \chi_{k,\delta}(\xi) a_{m-j}(k^{-\frac{1}{2}}x, \xi). \tag{2.12}$$

Recall that a_{m-j} vanishes to order $\ell_j := \ell - 2j$ at $(x_0 = 0, \xi_0)$ for all $j \leqslant \frac{\ell}{2}$, so we can Taylor expand to write

$$a_{m-j}(x,\xi) = \sum_{|\gamma|=\ell_j} x^{\gamma} f_{m-j,\gamma}(x,\xi) + \left(\frac{\xi}{|\xi|} - \xi_0\right)^{\gamma} g_{m-j,\gamma}(x,\xi)$$

where $f_{m-j,\gamma}$, $g_{m-j,\gamma} \in S^{m-j}_{c\ell}(\mathbb{R}^{2n})$ have compact support in x for each multi-index γ. Combining this with (2.12), we have

$$\begin{aligned} &\chi_{k,\delta}(\xi) a_{m-j}(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi) \\ &\quad = k^{\frac{m-j}{2}} \chi_{k,\delta}(\xi) \sum_{|\gamma|=\ell_j} \left(k^{-\frac{\ell_j}{2}} x^{\gamma} f_{m-j,\gamma}(k^{-\frac{1}{2}}x,\xi) + \left(\frac{\xi}{|\xi|} - \xi_0\right)^{\gamma} g_{m-j,\gamma}(k^{-\frac{1}{2}}x,\xi) \right). \end{aligned}$$

Then, we define

$$\mathcal{A}_{j,1}(x) = k^{\frac{m-j-\ell_j}{2}} \sum_{|\gamma|=\ell_j} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \chi_{k,\delta}(\xi) x^{\ell_j} f_{m-j,\gamma}(k^{-\frac{1}{2}}x,\xi) F_k(\xi - k^{\frac{1}{2}}\xi_0) \mathrm{d}\xi, \tag{2.13}$$

$$\mathcal{A}_{j,2}(x) = k^{\frac{m-j}{2}} \sum_{|\gamma|=\ell_j} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \chi_{k,\delta}(\xi) \left(\frac{\xi}{|\xi|} - \xi_0\right)^{\gamma} g_{m-j,\gamma}(k^{-\frac{1}{2}}x,\xi) F_k(\xi - k^{\frac{1}{2}}\xi_0) \mathrm{d}\xi. \tag{2.14}$$

and

$$\mathcal{R}_j(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \left(1 - \chi_{k,\delta}(\xi)\right) a_{m-j}(k^{-\frac{1}{2}}, k^{\frac{1}{2}}\xi) F_k(\xi - k^{\frac{1}{2}}\xi) \, \mathrm{d}\xi, \tag{2.15}$$

so that

$$\mathrm{Op}(a_{m-j})h_k = \mathcal{A}_{j,1} + \mathcal{A}_{j,2} + \mathcal{R}_j.$$

We claim that $\mathcal{R}_j$ is negligible for large k. Since $F_k(\xi - k^{\frac{1}{2}}\xi_0)$ is a Gaussian centered at $k^{\frac{1}{2}}\xi_0$ and $1-\chi_{k,\delta}$ is supported at least k^δ away from that center, we are able to show that $\mathcal{R}_j$ is controlled by an arbitrarily large negative power of k.

Lemma 2.4. *For any $N' \in \mathbb{N}$ there exists $C_{N'} > 0$ such that*

$$\|\mathcal{R}_j\|_{L^2(\mathbb{R}^n)} \leqslant C_{N'} k^{-N'}. \tag{2.16}$$

Proof. To begin, note that for any multi-index β

$$\partial_\xi^\beta \chi_{k,\delta}(\xi) = k^{-\delta|\beta|}(\partial_\xi^\beta \chi)(k^{-\delta}(\xi - k^{\frac{1}{2}}\xi_0)). \tag{2.17}$$

Combining (2.17) with (2.9) shows that for any $N \in \mathbb{N}$ and any multi-index β, there exists $C_{N,\beta} > 0$ such that

$$\partial_\xi^\beta \left[(1-\chi_{k,\delta})F_k(\xi)\right] \leqslant C_{N,\beta} \mathbb{1}_{\mathrm{supp}\,(1-\chi_{k,\delta})}(\xi)(1+|\xi - k^{\frac{1}{2}}\xi_0|)^{-N},$$

where for any set $E \subset \mathbb{R}^n$, $\mathbb{1}_E$ denotes the indicator function of E.

Now, when $|x| \geqslant 1$, for $\nu \geqslant 0$ we may integrate by parts in (2.15) as in the proof of Lemma 2.3 to obtain

$$\mathcal{R}_j(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \left(\frac{i\langle x, \nabla_\xi\rangle}{|x|^2}\right)^\nu \left[(1-\chi_{k,\delta}(\xi)) a_{m-j}(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi) F_k(\xi)\right] \mathrm{d}\xi.$$

Since $a_{m-j} \in S^{m-j}_{c\ell}(\mathbb{R}^{2n})$, we have that for any multi-index β,

$$|\partial_\xi^\beta a_{m-j}(k^{-\frac{1}{2}}x, k^{\frac{1}{2}}\xi)| \leqslant C k^{\frac{|\beta|}{2}}(1+k^{\frac{1}{2}}|\xi|)^{m-j-|\beta|} \leqslant C k^{\frac{|\beta|}{2}}(1+k^{\frac{1}{2}}|\xi|)^m$$

Thus, for any $N \in \mathbb{N}$, there exists a constant C_N such that whenever $|x| \geqslant 1$,

$$|\mathcal{R}_j(x)| \leqslant C_N \sup_{|\beta| \leqslant \nu} \frac{1}{|x|^\nu} \int_{\mathbb{R}^n} \mathbb{1}_{\mathrm{supp}\,(1-\chi_{k,\delta})}(\xi)(1+|\xi - k^{\frac{1}{2}}\xi_0|)^{-N} k^{\frac{|\beta|}{2}}(1+k^{\frac{1}{2}}|\xi|)^m \,\mathrm{d}\xi.$$

Recall that $|\xi_0| = 1$, and so by the triangle inequality

$$1 + k^{\frac{1}{2}}|\xi| \leqslant 1 + k + k^{\frac{1}{2}}|\xi - k^{\frac{1}{2}}\xi_0| \leqslant Ck(1+|\xi - k^{\frac{1}{2}}\xi_0|).$$

Thus, when $|x| \geqslant 1$,

$$|\mathcal{R}_j(x)| \leqslant C_N \frac{1}{|x|^\nu} \int_{\mathbb{R}^n} \mathbb{1}_{\mathrm{supp}\,(1-\chi_{k,\delta})}(\xi)(1+|\xi - k^{\frac{1}{2}}\xi_0|)^{-N+m} k^{m+\frac{\nu}{2}} \,\mathrm{d}\xi. \tag{2.18}$$

Using polar coordinates $\xi - k^{\frac{1}{2}}\xi_0 = r\omega$ with $r \in \mathbb{R}^+$, $\omega \in \mathbb{S}^{n-1}$, we compute

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbb{1}_{\mathrm{supp}\,(1-\chi_{k,\delta})}(\xi)(1+|\xi - k^{\frac{1}{2}}\xi_0|)^{-N+m} \,\mathrm{d}\xi &\leqslant \int_{\mathbb{S}^{n-1}} \int_{k^\delta/2}^{\infty} (1+r)^{-N+m} r^{n-1} \,\mathrm{d}r \,\mathrm{d}\omega \\ &\leqslant Ck^{\delta(m+n-N)}.\end{aligned}$$

Combining this with (2.18), we have

$$|\mathcal{R}_j(x)| \leqslant C_N |x|^{-\nu} k^{\delta(m+n-N)+m+\frac{\nu}{2}}, \quad \text{if } |x| \geqslant 1.$$

Since ν and N were both arbitrary, given any $N' \geqslant 0$ we can choose $\nu \geqslant \frac{n+1}{2}$ and N sufficiently large so that

$$|\mathcal{R}_j(x)| \leqslant C_{N'} k^{-N'} |x|^{-\frac{n+1}{2}}, \quad \text{if } |x| \geqslant 1.$$

By an analogous argument when $|x| \leqslant 1$, except without integration by parts, we have

$$|\mathcal{R}_j(x)| \leqslant C_{N'} k^{-N'} \quad \text{if } |x| \leqslant 1.$$

Combining these inequalities and taking the L^2 norm completes the proof of (2.16). $\square$

It remains to estimate $\mathcal{A}_{j,1}$ and $\mathcal{A}_{j,2}$. It is here that we take advantage of the compatibility of the vanishing of a_{m-j} with the particular form of the coherent state h_k. We first consider $\mathcal{A}_{j,1}$.

Lemma 2.5. *For any $j \geqslant 0$, there exists $C_j > 0$ such that*

$$\|\mathcal{A}_{j,1}\|_{L^2(\mathbb{R}^n)} \leqslant C_j k^{m-j-\frac{\ell_j}{2}}. \tag{2.19}$$

Proof. Let $\chi_{k,\delta}$ be as in the proof of Lemma 2.4, and observe that on the support of $\chi_{k,\delta}$

$$k^{\frac{1}{2}} - k^{\delta} \leqslant |\xi| \leqslant k^{\frac{1}{2}} + k^{\delta}.$$

Also, recall that $f_{m-j,\gamma} \in S^{m-j}_{c\ell}(\mathbb{R}^{2n})$ has compact support in x, so

$$|\partial_\xi^\beta f_{m-j,\gamma}(x,\xi)| \leqslant C_\beta |\xi|^{m-j-|\beta|} \quad \text{for all } x \in \mathbb{R}^n.$$

Therefore,

$$\sup_{x\in\mathbb{R}^n} |\partial_\xi^\beta f_{m-j,\gamma}(x,\xi)| \leqslant C_\beta k^{\frac{m-j-|\beta|}{2}}, \quad \text{for all } \xi \in \operatorname{supp}\chi_{k,\delta}. \tag{2.20}$$

Now, when $|x| \geqslant 1$, we may integrate by parts as before to obtain that for any $\nu \geqslant 0$ and any N sufficiently large,

$$\begin{aligned}
|\mathcal{A}_{j,1}(x)| &\leqslant k^{\frac{m-j-\ell_j}{2}} \sum_{|\gamma|=\ell_j} \left| \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \chi_{k,\delta}(\xi) x^\gamma f_{m-j,\gamma}(k^{-\frac{1}{2}}x,\xi) F_k(\xi - k^{\frac{1}{2}}\xi_0) \mathrm{d}\xi \right| \\
&\leqslant k^{\frac{m-j-\ell_j}{2}} |x|^{\ell_j} \int_{\mathbb{R}^n} \left| \left(\frac{i\langle x, \nabla_\xi\rangle}{|x|^2} \right)^\nu \left[\chi_{k,\delta}(\xi) f_{m-j,\gamma}(k^{-\frac{1}{2}}x,\xi) F_k(\xi - k^{\frac{1}{2}}\xi_0) \right] \right| \mathrm{d}\xi \\
&\leqslant C_{\nu,N} k^{m-j-\frac{\ell_j}{2}} |x|^{\ell_j-\nu} \int_{\mathbb{R}^n} (1 + |\xi - k^{\frac{1}{2}}\xi_0|)^{-N} \mathrm{d}\xi \\
&\leqslant C'_{\nu,N} k^{m-j-\frac{\ell_j}{2}} |x|^{\ell_j-\nu},
\end{aligned}$$

where the second-to-last inequality follows from (2.9), (2.17) and (2.20). When $|x| \leqslant 1$, by the same argument with $\nu = 0$, we obtain

$$|\mathcal{A}_{j,1}(x)| \leqslant C k^{m-j-\frac{\ell_j}{2}}$$

for some $C > 0$. Thus, for each $\nu \geqslant 0$, there exists a constant $C_\nu > 0$ so that

$$|\mathcal{A}_{j,1}(x)| \leqslant C_\nu k^{m-j-\frac{\ell}{2}}(1+|x|)^{\ell_j-\nu} \quad \text{for all } x \in \mathbb{R}^n.$$

Choosing ν so that $\ell_j - \nu \leqslant -\frac{n+1}{2}$ and taking the L^2 norm gives the desired inequality. □

Finally, we turn our attention to $\mathcal{A}_{j,2}$. The estimation of this term is the most subtle of the three, and requires some very technical analysis of the relationship between the vanishing factor $\left(\frac{\xi}{|\xi|} - \xi_0\right)$ and the structure of the support of $\chi_{k,\delta}$.

Lemma 2.6. *For any $j \geqslant 0$, there exists $C_j > 0$ such that*

$$\|\mathcal{A}_{j,2}\|_{L^2(\mathbb{R}^n)} \leqslant C_j k^{m-j+(\delta-\frac{1}{2})\ell_j}. \tag{2.21}$$

Proof. Once again, we let $\chi_{k,\delta}$ be as in the proof of Lemma 2.4. We first consider the case where $|x| \geqslant 1$, and the case $|x| \leqslant 1$ follows from an analogous argument. When $|x| \geqslant 1$, we may again use integration by parts to see that for any $\nu \geqslant 0$ and any N sufficiently large,

$$\begin{aligned}|\mathcal{A}_{j,2}(x)| \leqslant k^{\frac{m-j}{2}} \sum_{|\gamma|=\ell_j} \Bigg| \int_{\mathbb{R}^n} & e^{i\langle x,\xi\rangle} \left(\frac{i\langle x, \nabla_\xi\rangle}{|x|^2}\right)^\nu \\ & \times \left[\chi_{k,\delta}(\xi)\left(\frac{\xi}{|\xi|} - \xi_0\right)^\gamma g_{m-j,\gamma}(k^{-\frac{1}{2}}x,\xi) F_k(\xi - k^{\frac{1}{2}}\xi_0)\right] \mathrm{d}\xi \Bigg|.\end{aligned}$$

Recall that on $\operatorname{supp}\chi_{k,\delta}$, $k^{\frac{1}{2}} - k^\delta \leqslant |\xi| \leqslant k^{\frac{1}{2}} + k^\delta$, and so $|\partial_\xi^\beta g_{m-j,\gamma}| \leqslant Ck^{\frac{m-j-|\beta|}{2}} \leqslant Ck^{\frac{m-j}{2}}$. Combining this with (2.9) gives

$$|\mathcal{A}_{j,2}(x)| \leqslant Ck^{m-j}|x|^{-\nu} \sum_{\beta \leqslant \nu} \sum_{|\gamma|=\ell_j} \int_{\mathbb{R}^n} \left|\partial_\xi^\beta \left(\chi_{k,\delta}(\xi)\left(\frac{\xi}{|\xi|} - \xi_0\right)^\gamma\right)\right| (1+|\xi - k^{\frac{1}{2}}\xi_0|)^{-N}\,\mathrm{d}\xi, \tag{2.22}$$

when $|x| \geqslant 1$. Therefore, it is sufficient to show that for any multi-indices β, γ with $|\beta| \leqslant \nu$ and $|\gamma| = \ell_j$, there exists $C > 0$ such that

$$\left|\partial_\xi^\beta \left(\chi_{k,\delta}(\xi)\left(\frac{\xi}{|\xi|} - \xi_0\right)^\gamma\right)\right| \leqslant Ck^{(\delta-\frac{1}{2})\ell_j}. \tag{2.23}$$

To show this, it is convenient to choose coordinates on $\mathbb{R}^n$ so that $\xi_0 = (1, 0, \ldots, 0)$. Writing $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, we have that on the support of $\chi_{k,\delta}$,

$$|\xi - k^{1/2}\xi_0| = \sqrt{(\xi_1 - k^{1/2})^2 + \xi_2^2 + \cdots + \xi_n^2} \leqslant k^\delta,$$

by the definition of $\chi_{k,\delta}$. Thus,

$$|\xi_r| \leqslant k^\delta \text{ for } r \neq 1 \quad \text{and} \quad |\xi| \geqslant k^{1/2} - k^\delta \quad \text{for all } \xi \in \operatorname{supp}\chi_{k,\delta}. \tag{2.24}$$

Also, note that $\xi_1 > 0$ on $\operatorname{supp}\chi_{k,\delta}$ for k large enough, and so we can write

$$|\xi| - \xi_1 = (|\xi| + \xi_1)^{-1}(\xi_2^2 + \cdots + \xi_n^2).$$

Combining these facts, we have that for large k,

$$|\xi|-\xi_1=\frac{\xi_2^2+\cdots+\xi_n^2}{|\xi|+\xi_1}\leqslant\frac{(n-1)k^{2\delta}}{k^{1/2}-k^{\delta}}\leqslant Ck^{2\delta-\frac{1}{2}}.$$

We can now show (2.23) for $\beta=0$. Recalling that $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)\in\mathbb{N}^n$ is a multi-index with $|\gamma|=\gamma_1+\cdots+\gamma_n=\ell_j$, we make use of the above inequality and (2.24) to obtain

$$\begin{aligned}\left|\left(\frac{\xi}{|\xi|}-\xi_0\right)^{\gamma}\right|&=\frac{|\xi_1-|\xi||^{\gamma_1}|\xi_2|^{\gamma_2}\cdots|\xi_n|^{\gamma_n}}{|\xi|^{\ell_j}}\\&\leqslant Ck^{\left(2\delta-\frac{1}{2}\right)\gamma_1}\cdot\frac{k^{\delta\gamma_2}\cdots k^{\delta\gamma_n}}{k^{\frac{\ell_j}{2}}}\\&=Ck^{(\delta-\frac{1}{2})\ell_j+(\delta-\frac{1}{2})\gamma_1}\\&\leqslant Ck^{(\delta-\frac{1}{2})\ell_j},\end{aligned}$$

since $\delta<\frac{1}{2}$, which proves (2.23) in the case where $|\beta|=0$. To handle the case where $\beta\neq0$ we let $\beta^{(1)},\beta^{(2)},\ldots,\beta^{(n)}$ be multi-indices which partition β, and expand via the product rule to obtain

$$\begin{aligned}&\left|\partial_\xi^\beta\left(\frac{\xi}{|\xi|}-\xi_0\right)^{\gamma}\right|\\&\leqslant\sum_{|\beta^{(1)}+\beta^{(2)}+\cdots+\beta^{(n)}|=\beta}C_{\beta_j}\left|\partial_\xi^{\beta^{(1)}}\left(\frac{\xi_1-|\xi|}{|\xi|}\right)^{\gamma_1}\partial_\xi^{\beta^{(2)}}\left(\frac{\xi_2}{|\xi|}\right)^{\gamma_2}\cdots\partial_\xi^{\beta^{(n)}}\left(\frac{\xi_n}{|\xi|}\right)^{\gamma_n}\right|.\end{aligned}\tag{2.25}$$

Since $\frac{\xi_1-|\xi|}{|\xi|}$ and $\frac{\xi_j}{|\xi|}$ are homogeneous of degree zero, we have that for any multi-index θ,

$$\left|\partial_\xi^\theta\left(\frac{\xi_1-|\xi|}{|\xi|}\right)\right|\leqslant\frac{C}{|\xi|^{|\theta|}},\quad\text{and}\quad\left|\partial_\xi^\theta\left(\frac{\xi_r}{|\xi|}\right)\right|\leqslant\frac{C}{|\xi|^{|\theta|}}\ \text{ if } r\neq1.$$

Furthermore, we recall that $|\xi|\geqslant k^{\frac{1}{2}}-k^{\delta}\geqslant Ck^{\frac{1}{2}}$ on $\operatorname{supp}\chi_{k,\delta}$, and so

$$\left|\chi_{k,\delta}(\xi)\partial_\xi^\theta\left(\frac{\xi_1-|\xi|}{|\xi|}\right)\right|\leqslant Ck^{-\frac{|\theta|}{2}},\quad\text{and}\quad\left|\chi_{k,\delta}(\xi)\partial_\xi^\theta\left(\frac{\xi_r}{|\xi|}\right)\right|\leqslant Ck^{-\frac{|\theta|}{2}},\ r\neq1.\tag{2.26}$$

Now consider $\partial_\xi^{\beta^{(r)}}\left(\frac{\xi_r}{|\xi|}\right)^{\gamma_r}$. Expanding via the product rule, we can rewrite this as a linear combination of terms of the form

$$\left(\frac{\xi_r}{|\xi|}\right)^{t_r}\partial_\xi^{\sigma^{(1)}}\left(\frac{\xi_r}{|\xi|}\right)\cdots\partial_\xi^{\sigma^{(q)}}\left(\frac{\xi_r}{|\xi|}\right),$$

where each $\sigma^{(i)}$ is a multi-index with $|\sigma^{(i)}|\geqslant1$, $\sigma^{(1)}+\cdots+\sigma^{(q)}=\beta^{(r)}$, and $t_r+q=\gamma_r$. That is, there are t_r factors of $\frac{\xi_r}{|\xi|}$ which do not have any derivatives, and the remaining γ_r-t_r factors each have at least 1 derivative applied to them. Note then that $t_r\geqslant\max(0,\gamma_r-|\beta^{(r)}|)$. By (2.24) each factor with no derivatives is bounded by $k^{\delta-\frac{1}{2}}$. The homogeneous estimate (2.26) controls the factors with derivatives, giving

$$\begin{aligned}\left|\chi_{k,\delta}(\xi)\left(\frac{\xi_r}{|\xi|}\right)^{t_r}\partial_\xi^{\sigma^{(1)}}\left(\frac{\xi_r}{|\xi|}\right)\cdots\partial_\xi^{\sigma^{(q)}}\left(\frac{\xi_r}{|\xi|}\right)\right| &\leqslant k^{(\delta-\frac{1}{2})t_r}k^{-\frac{|\sigma^{(1)}|}{2}}\cdots k^{-\frac{|\sigma^{(q)}|}{2}}\\ &\leqslant Ck^{(\delta-\frac{1}{2})t_r-\frac{|\beta^{(r)}|}{2}}.\end{aligned}$$

Thus, by the triangle inequality, we have

$$\left|\chi_{k,\delta}(\xi)\partial_\xi^{\beta^{(r)}}\left(\frac{\xi_r}{|\xi|}\right)^{\gamma_r}\right| \leqslant Ck^{(\delta-\frac{1}{2})t_r-\frac{|\beta^{(r)}|}{2}}.$$

When $\gamma_r-|\beta^{(r)}|>0$, we have

$$k^{(\delta-\frac{1}{2})t_r}k^{-\frac{|\beta^{(r)}|}{2}} \leqslant k^{(\delta-\frac{1}{2})(\gamma_r-|\beta^{(r)}|)}k^{-\frac{|\beta^{(r)}|}{2}} \leqslant Ck^{(\delta-\frac{1}{2})\gamma_r},$$

since $t_r \geqslant \gamma_r-|\beta^{(r)}|>0$ and $\delta-\frac{1}{2}<0$. On the other hand, when $\gamma_r \leqslant |\beta^{(r)}|$, we still have $t_r \geqslant 0$ and so

$$k^{(\delta-\frac{1}{2})t_r}k^{-\frac{|\beta^{(r)}|}{2}} \leqslant Ck^{-\frac{|\beta^{(r)}|}{2}} \leqslant Ck^{-\frac{\gamma_r}{2}} \leqslant Ck^{(\delta-\frac{1}{2})\gamma_r},$$

since $\delta>0$. Thus, there exists a $C_\beta>0$ so that

$$\left|\chi_{k,\delta}(\xi)\partial_\xi^{\beta^{(r)}}\left(\frac{\xi_r}{|\xi|}\right)^{\gamma_r}\right| \leqslant C_\beta k^{(\delta-\frac{1}{2})\gamma_r}. \tag{2.27}$$

An analogous argument shows

$$\left|\chi_{k,\delta}(\xi)\partial_\xi^{\beta^{(1)}}\left(\frac{\xi_1-|\xi|}{|\xi|}\right)^{\gamma_1}\right| \leqslant C_\beta k^{(\delta-\frac{1}{2})\gamma_1}, \tag{2.28}$$

for some potentially different $C_\beta>0$. Combining (2.27) and (2.28) with (2.17) and (2.25) yields

$$\begin{aligned}\left|\partial_\xi^\beta\left(\chi_{k,\delta}(\xi)\left(\frac{\xi}{|\xi|}-\xi_0\right)^\gamma\right)\right| &\leqslant C_\beta k^{(\delta-\frac{1}{2})(\gamma_1+\gamma_2+\cdots+\gamma_n)}\\ &= \quad C_\beta k^{(\delta-\frac{1}{2})|\gamma|}\\ &= \quad C_\beta k^{(\delta-\frac{1}{2})\ell_j}.\end{aligned}$$

We have therefore proved (2.23).

Combining (2.22) and (2.23), we have that for any $\nu \geqslant 0$ and any N large enough, there exists $C_\nu, C'_\nu>0$ so that

$$\begin{aligned}|\mathcal{A}_{j,2}(x)| &\leqslant C_\nu|x|^{-\nu}k^{\frac{m-j}{2}+(\delta-\frac{1}{2})\ell_j}\sum_{|\gamma|=\ell_j}k^{\frac{m-j}{2}}\int_{\mathbb{R}^n}(1+|\xi-k^{\frac{1}{2}}\xi_0|)^{-N}\mathrm{d}\xi\\ &\leqslant C'_\nu|x|^{-\nu}k^{m-j+(\delta-\frac{1}{2})\ell_j}.\end{aligned}$$

We then have

$$|\mathcal{A}_{j,2}(x)| \leqslant C_\nu|x|^{-\nu}k^{m-j+(\delta-\frac{1}{2})\ell_j}, \quad \text{for all } |x| \geqslant 1, \tag{2.29}$$

for some $C_\nu>0$.

To estimate $\mathcal{A}_{j,2}(x)$ when $|x| \leqslant 1$, we repeat the above argument without integrating by parts. From this, we obtain

$$|\mathcal{A}_{j,2}(x)| \leqslant Ck^{m-j+(\delta-\frac{1}{2})\ell_j}, \quad \text{for all } |x| \leqslant 1 \tag{2.30}$$

for some $C > 0$. Choosing $\nu > \frac{n-1}{2}$, we can combine (2.29) with (2.30), then take L^2 norms to obtain (2.21) as desired. $\square$

Recalling the definitions of $\mathcal{R}_j, \mathcal{A}_{j,1}$, and $\mathcal{A}_{j,2}$, we combine Lemmas 2.4, 2.5, and 2.6 to obtain that for each $0 \leqslant j \leqslant N_0 - 1$,

$$\begin{aligned} \|\mathrm{Op}(a_{m-j})h_k\|_{L^2(\mathbb{R}^n)} &\leqslant \|\mathcal{A}_{j,1}\|_{L^2(\mathbb{R}^n)} + \|\mathcal{A}_{j,2}\|_{L^2(\mathbb{R}^n)} + \|\mathcal{R}_j\|_{L^2(\mathbb{R}^n)} \\ &\leqslant C_j k^{m-j+(\delta-\frac{1}{2})\ell_j}, \end{aligned} \tag{2.31}$$

for some $C_j > 0$. Since $\ell_j = \ell - 2j$, (2.5) and (2.31) imply that for any fixed $N_0 \geqslant m$, there exist $C_1, C_2 > 0$ so that

$$\begin{aligned} ||\mathrm{Op}(a)h_k||_{L^2} &\leqslant \sum_{j=0}^{N_0-1} ||\mathrm{Op}(a_{m-j})h_k||_{L^2} + ||\mathrm{Op}(r_{N_0})h_k||_{L^2} \\ &\leqslant \sum_{j=0}^{N_0-1} C_1 k^{m-j+(\delta-\frac{1}{2})\ell_j} + C_2 k^{\frac{m-N_0}{2}} \\ &= \sum_{j=0}^{N_0-1} C_1 k^{m-\frac{\ell}{2}+(\ell-2j)\delta} + C_2 k^{\frac{m-N_0}{2}} \\ &\leqslant C_1 k^{m-\frac{\ell}{2}+\ell\delta} + C_2 k^{\frac{m-N_0}{2}}. \end{aligned}$$

Choosing $N_0 \geqslant \ell - m - 2\varepsilon$ and $\delta \leqslant \frac{\varepsilon}{\ell}$ completes the proof of Proposition 2.1. $\square$

3. The Upper Bound for α

In this section, we show that $\alpha \leqslant 2\min\{-D_0, L_\infty\}$, where D_0 and L_∞ are defined as in Sect. 1. That $-2D_0$ is an upper bound is straightforward to show. To see this, let $\lambda_j \in \mathrm{Spec}(A_W)\backslash\{0\}$. Then, there exists $u = (u_0, u_1)^T \neq 0$ such that $A_W u = \lambda_j u$, where we recall

$$A_W = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta_g & -2W \end{pmatrix}.$$

It is then immediate that $u(x) = e^{t\lambda_j} u_0(x)$ solves the damped wave equation with initial data $(u_0, u_1)^T$, and

$$E(u,t) = e^{2t\mathrm{Re}\,(\lambda_j)} E(u,0).$$

Since $E(u,0) \neq 0$, we have that $\alpha \leqslant -2\mathrm{Re}\,(\lambda_j)$ for all j. Furthermore, by the definition of D_0, there must either exist some λ_{j_0} with $\mathrm{Re}\,(\lambda_{j_0}) = D_0$, or a sequence of λ_j with $\mathrm{Re}\,(\lambda_j) \to D_0$. In either case, we must have $\alpha \leqslant -2D_0$.

Showing that $2L_\infty$ is also an upper bound is more complicated. Our technique for this is inspired by the method of Gaussian beams introduced by

Ralston [26,27]. Using Gaussian beams, one can produce quasimodes for the wave equation with energy strongly localized near a single geodesic. Intuitively, solutions to (1.1) should decay only when they interact with the damping coefficient. Motivated by this, we modify the Gaussian beam construction to obtain solutions whose energy decays at a rate proportional to strength of the damping along that geodesic, in analogy to [17,21].

To begin, we recall Ralston's original Gaussian beam construction on $\mathbb{R}^n$ with a Riemannian metric g. Let $A(t)$ be an $n \times n$ symmetric matrix-valued function with positive definite imaginary part. Let $t \mapsto (x_t, \xi_t)$ denote a geodesic trajectory and set

$$\psi(x,t) = \langle \xi_t, x - x_t \rangle + \frac{1}{2} \langle A(t)(x - x_t), x - x_t \rangle.$$

Let $b \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$. Then, we define

$$u_k(x,t) = k^{-1+n/4} b(t,x) e^{ik\psi(x,t)}. \tag{3.1}$$

The work of [27] guarantees that there exist appropriate choices of b and $A(t)$ so that u_k is a quasimode of the undamped wave equation with positive energy, which is concentrated along the geodesic (x_t, ξ_t). We summarize some notable facts from [27] in the following Lemma.

Lemma 3.1 ([27]). *Fix $T > 0$ and $(x_0, \xi_0) \in S^*M$. For $\varphi_t(x_0, \xi_0) = (x_t, \xi_t)$, there exists a $b \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ and an $n \times n$ symmetric matrix-valued function $t \mapsto A(t)$ so that if u_k is given by (3.1), we have*

$$\sup_{t \in [0,T]} \|\partial_t^2 u_k(\cdot, t) - \Delta_g u_k(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leqslant Ck^{-\frac{1}{2}} \quad \text{for } k \geqslant 1. \tag{3.2}$$

Furthermore, for all $t \in [0, T]$,

$$\lim_{k \to \infty} E(u_k, t) > 0, \tag{3.3}$$

and the limit is always finite and independent of t.

Remark 3.2. By (3.3), we may assume without loss of generality that $\lim\limits_{k \to \infty} E(u_k, t) = 1$ for all $t \in [0, T]$.

Remark 3.3. Using coordinate charts and a partition of unity, we can extend this construction to the case of manifolds, which results in a sequence $\{u_k\} \subset C^\infty(\mathbb{R}^+ \times M)$ such that $\lim_{k \to \infty} E(u_k, t) = 1$ and the appropriate analogue of (3.2) holds.

Next, we modify $\{u_k\}$ to obtain a sequence of quasimodes for the damped wave equation as follows. For each $(x, \xi) \in S^*M$, we define $G_t(x, \xi)$ to be the solution of the initial value problem

$$\begin{cases} G_0^+(x,\xi) = 1 \\ \partial_t G_t(x,\xi) = -w(\varphi_t(x,\xi)) G_t(x,\xi), \end{cases}$$

which has solution

$$G_t(x,\xi) = \exp\left(-\int_0^t w(\varphi_s(x,\xi))\,\mathrm{d}s\right). \tag{3.4}$$

In [17,21], an analogue of G_t in the multiplicative case is constructed as the propagator for defect measures associated with the damped wave equation. We note that it is possible to derive G_t in the anisotropic case following this argument with little modification. However, our argument does not rely on the nature of the construction, and so we simply define G_t to have the appropriate form.

It is clear from the definition that $G_t(x,\xi)$ decays exponentially along any geodesic which intersects the damping region. Motivated by this, we fix $(x_0,\xi_0) \in S^*M$ and set

$$v_k(t,x) = G_t(x_0,\xi_0)u_k(t,x).$$

We now show that for any $\varepsilon > 0$, v_k is an $\mathcal{O}(k^{-\frac{1}{2}+\varepsilon})$ quasimode of (1.1).

Proposition 3.4. *Given $(x_0,\xi_0) \in S^*M$, let $u_k(t,x)$ be as specified in Remark 3.3 and set $v_k(t,x) = G_t(x_0,\xi_0)u_k(t,x)$. For any $T > 0$ and $\varepsilon > 0$, there exists a constant $C_{\varepsilon,T} > 0$ so that*

$$\sup_{t\in[0,T]} \|(\partial_t^2 - \Delta_g + 2W\partial_t)v_k(t,\cdot)\|_{L^2(M)} \leqslant C_{\varepsilon,T}k^{-\frac{1}{2}+\varepsilon}. \tag{3.5}$$

Proof. By direct computation, we have

$$\begin{aligned}
(\partial_t^2 - \Delta_g + 2W\partial_t)v_k &= G_t(\partial_t^2 - \Delta_g)u_k + 2\partial_t G_t\partial_t u_k + (\partial_t^2 G_t)u_k + 2W\partial_t(G_t u_k)\\
&= G_t(\partial_t^2 - \Delta_g)u_k - 2w(x_t,\xi_t)G_t\partial_t u_k - \partial_t(w(x_t,\xi_t)G_t)u_k\\
&\quad + 2WG_t\partial_t u_k - 2w(x_t,\xi_t)WG_t u_k\\
&= G_t(\partial_t^2 - \Delta_g)u_k + 2(W - w(x_t,\xi_t))G_t\partial_t u_k\\
&\quad + \Big(w(x_t,\xi_t)^2 - 2w(x_t,\xi_t)W - \partial_t w(x_t,\xi_t)\Big)\,G_t u_k.
\end{aligned}$$

By the construction of u_k and the boundedness of G_t, we have

$$\sup_{t\in[0,T]} \|G_t(\partial_t^2 - \Delta_g)u_k(t,\cdot)\|_{L^2(M)} \leqslant O(k^{-\frac{1}{2}}).$$

Since W is order zero, and therefore bounded on $L^2(M)$, we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]} \big|\big|(w(x_t,\xi_t)^2 - 2w(x_t,\xi_t)W - \partial_t w(x_t,\xi_t))G_t u_k\big|\big|_{L^2(M)}\\
&\qquad \leqslant C\sup_{t\in[0,T]} ||u_k(t,\cdot)||_{L^2} = \mathcal{O}(k^{-1}),
\end{aligned}$$

where the final equality follows from the fact that $\int_{\mathbb{R}^n} k^{\frac{n}{2}} e^{-k|y|^2}\,\mathrm{d}y$ is uniformly bounded in k.

To estimate $(W - w(x_t,\xi_t))\,G_t\partial_t u_k(t,\cdot)$ we will apply Proposition 2.1 with $m = 0$ and $\ell = 1$. Note that $W - w(x_t,\xi_t)$ is an order zero pseudodifferential operator whose symbol vanishes to first order at (x_t,ξ_t), and so it satisfies the hypotheses of Proposition 2.1. Furthermore

$$\partial_t u_k(x,t) = k^{-1+\frac{n}{4}} \partial_t b(t,x) e^{ik\psi(x,t)} + ik^{\frac{n}{4}} b(t,x) \partial_t \psi(x,t) e^{ik\psi(x,t)},$$

and for fixed t, both of these terms take the form of a coherent state h_k as defined by (2.1) (the fact that the first has an extra factor of k^{-1} is irrelevant, as it only improves the estimate). Since all quantities depend on t in a C^∞ fashion, we have that for any $\varepsilon > 0$,

$$\sup_{t\in[0,T]} ||2(W - w(x_t,\xi_t))G_t \partial_t u_k(t,\cdot)||_{L^2(M)} \leqslant C(k^{-1/2+\varepsilon}). \tag{3.6}$$

By the triangle inequality, we obtain (3.5), which completes the proof. □

The next step in the proof of the upper bound for α is to produce from a given point $(x_0,\xi_0) \in S^*M$ a sequence of *exact* solutions to (1.1) whose energy approaches $|G_t(x_0,\xi_0)|^2$.

Proposition 3.5. *Given any $T > 0$, any $\varepsilon > 0$, and any $(x_0,\xi_0) \in S^*M$, there exists an exact solution u of the generalized damped wave equation (1.1) with*

$$|E(u,0) - 1| < \varepsilon$$

and

$$\big|E(u,T) - |G_T(x_0,\xi_0)|^2\big| < \varepsilon. \tag{3.7}$$

Proof. Let u_k and v_k be as defined previously. Then, define ω_k as the unique solution of the damped wave equation with initial conditions $\omega_k(x,0) = v_k(x,0)$ and $\partial_t\omega_k(x,0) = \partial_t v_k(x,0)$. It is immediate that

$$E(\omega_k,0) = E(v_k,0) = E(u_k,0) \to 1, \quad \text{as } k\to\infty.$$

To see (3.7), first note by the triangle inequality

$$|E(\omega_k,t)^{\frac{1}{2}} - E(v_k,t)^{\frac{1}{2}}| \leqslant E(\omega_k - v_k,t)^{\frac{1}{2}}. \tag{3.8}$$

Thus, it suffices to prove that $\lim_{k\to\infty} E(v_k,t) = |G_t(x_0,\xi_0)|^2$ and that $\lim\limits_{k\to\infty} E(\omega_k - v_k,t) = 0$. To see that $\lim_{k\to\infty} E(v_k,t) = |G_t(x_0,\xi_0)|^2$, note that by the definition of v_k and properties of G_t

$$\begin{aligned} E(v_k,t) = \frac{1}{2}\int\limits_M & |G_t(x_0,\xi_0)\partial_t u_k(x,t) - w(x_t,\xi_t)G_t(x_0,\xi_0)u_k(x,t)|^2 \\ & + |G_t(x_0,\xi_0)\nabla_g u_k(x,t)|^2 \, \mathrm{d}v_g(x). \end{aligned}$$

Now since $w(x_t,\xi_t)$ and G_t are bounded

$$\int\limits_M |w(x_t,\xi_t)G_t(x_0,\xi_0)u_k(x,t)|^2 \, \mathrm{d}v_g(x) \leqslant C\,||u_k(t,\cdot)||_{L^2}^2 \leqslant C'k^{-2},$$

for some C, $C' > 0$. Thus,

$$\begin{aligned} \lim_{k\to\infty} E(v_k,t) &= \lim_{k\to\infty} \frac{1}{2}\int\limits_M |G_t(x_0,\xi_0)\partial_t u_k(x,t)|^2 + |G_t(x_0,\xi_0)\nabla_g u_k(x,t)|^2 \, \mathrm{d}v_g(x) \\ &= |G_t(x_0,\xi_0)|^2 \lim_{k\to\infty} E(u_k,t) \end{aligned}$$

$$= |G_t(x_0,\xi_0)|^2, \tag{3.9}$$

where in the final equality we used that $\lim_{k\to\infty} E(u_k,t)=1$.

To control $E(\omega_k - v_k, t)$, let $f_k = (\partial_t^2 - \Delta + 2W\partial_t)v_k$. Then

$$(\partial_t^2 - \Delta + 2W\partial_t)(v_k - \omega_k) = f_k.$$

By Proposition 3.4, for any $\varepsilon, T > 0$ there exists a $C_{\varepsilon,T} > 0$ such that

$$\sup_{t\in[0,T]} \|f_k(t,\cdot)\|_{L^2(M)} \leqslant C_{\varepsilon,T} k^{-\frac{1}{2}+\varepsilon}. \tag{3.10}$$

By direct computation, we have

$$\begin{aligned}
\partial_t E(\omega_k - v_k, t) &= \int_M (\partial_t^2 - \Delta_g)(\omega_k - v_k)\partial_t \overline{(\omega_k - v_k)} \\
&\quad + (\partial_t^2 - \Delta_g)\overline{(\omega_k - v_k)}\partial_t(\omega_k - v_k)\, \mathrm{d}v_g(x) \\
&= 2\mathrm{Re} \int_M [f_k - 2W\partial_t(\omega_k - v_k)]\partial_t\overline{(\omega_k - v_k)}\, \mathrm{d}v_g(x) \\
&= 2\mathrm{Re} \int_M f_k \cdot \partial_t\overline{(\omega_k - v_k)}\, \mathrm{d}v_g(x) - 4\mathrm{Re}\, \langle W\partial_t(\omega_k - v_k), \\
&\quad \partial_t(\omega_k - v_k)\rangle_{L^2(M)}.
\end{aligned}$$

Note that the second term on the right-hand side above is nonpositive, since W is a nonnegative operator. Also, note that

$$\|\partial_t(\omega_k - v_k)\|_{L^2(M)} \leqslant E(\omega_k,t) + E(v_k,t) \leqslant E(\omega_k,0) + E(v_k,0) = 2E(u_k,0),$$

which is uniformly bounded by a constant since $E(u_k,0)\to 1$. Combining this with (3.10), we know that there exists $C'_{\varepsilon,T} > 0$ such that

$$\begin{aligned}
\sup_{t\in[0,T]} \left| 2\mathrm{Re} \int_M f_k \partial_t \overline{(\omega_k - v_k)}\, \mathrm{d}v_g(x)\right| &\leqslant 2\|f_k(t,\cdot)\|_{L^2} \|\partial_t(\omega_k - v_k)(t,\cdot)\|_{L^2} \\
&\leqslant C'_{\varepsilon,T} k^{-\frac{1}{2}+\varepsilon}.
\end{aligned}$$

Thus, for any $\varepsilon > 0$

$$\sup_{t\in[0,T]} |\partial_t E(\omega_k - v_k, t)| \leqslant C'_{\varepsilon,T} k^{-\frac{1}{2}+\varepsilon}.$$

Since $E(v_k - \omega_k, 0) = 0$, integrating in t gives

$$\sup_{t\in[0,T]} E(v_k - \omega_k, t) \leqslant C'_{\varepsilon,T} T k^{-\frac{1}{2}+\varepsilon}.$$

Combining the above with (3.8) and (3.9) yields (3.7). □

For the penultimate step in the proof of the upper bound for α, we show that $t \mapsto tL(t)$ is superadditive. That is, for $r,t \geqslant 0$, we claim that

$$(t+r)L(t+r) \geqslant tL(t) + rL(r).$$

Recall that the time averaging function $t \mapsto L(t)$ is defined by

$$L(t) = \frac{1}{t} \inf_{(x_0,\xi_0)\in S^*M} \int_0^t w(x_s,\xi_s)\,\mathrm{d}s.$$

To see that $tL(t)$ is superadditive, observe that

$$\begin{aligned}
(t+r)L(t+r) &= \inf_{(x_0,\xi_0)\in S^*M} \int_0^{r+t} w(x_s,\xi_s)\mathrm{d}s \\
&= \inf_{(x_0,\xi_0)\in S^*M} \left(\int_0^t w(x_s,\xi_s)\mathrm{d}s + \int_t^{t+r} w(x_s,\xi_s)\mathrm{d}s \right) \\
&\geqslant \inf_{(x_0,\xi_0)\in S^*M} \int_0^t w(x_s,\xi_s)\mathrm{d}s + \inf_{(x_0,\xi_0)\in S^*M} \int_t^{t+r} w(x_s,\xi_s)\mathrm{d}s \\
&= \inf_{(x_0,\xi_0)\in S^*M} \int_0^t w(x_s,\xi_s)\mathrm{d}s + \inf_{(x_0,\xi_0)\in S^*M} \int_0^{r} w(x_s,\xi_s)\mathrm{d}s \\
&= tL(t) + rL(r).
\end{aligned}$$

Then, by Fekete's lemma, $L_\infty := \lim_{t\to\infty} L(t) = \sup_{t\in[0,\infty)} L(t)$, and thus $L(t) \leqslant L_\infty$ for all t. That the supremum is not infinite follows from the fact that $w(x,\xi)$ is uniformly bounded on T^*M.

We are now ready to show that $\alpha \leqslant 2L_\infty$. Assume for the sake of contradiction that $\alpha = 2L_\infty + 3\eta$ for some $\eta > 0$. Then since $2(L_\infty + \eta) < \alpha$, there exists a $C > 0$ such that for all $t \geqslant 0$ and all solutions u of (1.1),

$$E(u,t) \leqslant CE(u,0)e^{-2t(L_\infty+\eta)}. \tag{3.11}$$

For the next step, it is convenient to remove the factor of C. To accomplish this, choose $T > 0$ large enough so that $\max(C,1) < e^{T\eta}$. Then

$$Ce^{-2T(L_\infty+\eta)} < e^{-T(2L_\infty+\eta)}.$$

Since $L(t) \leqslant L_\infty$ for all t, we obtain

$$Ce^{-2T(L_\infty+\eta)} < e^{-2TL_\infty - T\eta} \leqslant e^{-2TL(T)-T\eta}. \tag{3.12}$$

Note that $L(T)$ can be rewritten in terms of G_T as

$$L(T) = -\frac{1}{T} \sup_{(x,\xi)\in S^*M} \ln\left(G_T(x,\xi)\right).$$

Thus, there exists a point $(x_0,\xi_0) \in S^*M$ such that $\ln G_T(x_0,\xi_0) > -TL(T) - \frac{1}{2}T\eta$. Therefore,

$$e^{-2TL(T)-T\eta} < |G_T(x_0,\xi_0)|^2.$$

So by (3.12) there exists a $\delta > 0$ such that

$$Ce^{-2T(L_\infty+\eta)} < |G_T(x_0,\xi_0)|^2 - \delta.$$

Now, by Proposition 3.5, there exists an exact solution u of (1.1) such that

$$1 > E(u,0) - \frac{\delta}{2} \quad \text{and} \quad E(u,T) > |G_T(x_0,\xi_0)|^2 - \frac{\delta}{2}.$$

Thus,

$$\begin{aligned}
E(u,T) &> E(u,T)\left(E(u,0) - \frac{\delta}{2}\right) \\
&> E(u,0)\left(|G_T(x_0,\xi_0)|^2 - \frac{\delta}{2}\right) - \frac{\delta}{2}E(u,T) \\
&> E(u,0)\left(|G_T(x_0,\xi_0)|^2 - \frac{\delta}{2}\right) - \frac{\delta}{2}E(u,0) \\
&= E(u,0)\left(|G_T(x_0,\xi_0)|^2 - \delta\right).
\end{aligned}$$

Therefore,

$$E(u,T) > E(u,0)(|G_T(x_0,\xi_0)|^2 - \delta) > CE(u,0)e^{-2T(L_\infty+\eta)},$$

but this contradicts (3.11). Thus, we must have $\alpha \leqslant 2L_\infty$. Combining this with the discussion at the beginning of this section, we have proved the upper bound

$$\alpha \leqslant 2\min\{-D_0, L_\infty\}.$$

We complete the proof of Theorem 2 in the next section by proving the corresponding lower bound for α.

4. The Lower Bound for α

In this section, we prove that the best exponential decay rate satisfies

$$\alpha \geqslant 2\min\{-D_0, L_\infty\}, \tag{4.1}$$

which is the final component of the proof of Theorem 2. In contrast to the proof of the upper bound, this section proceeds in direct analogy to the work of Lebeau, and so we omit many of the details which can be found in [17,21]. While the proofs presented here are not new, we include them to introduce notation that is used later in Sect. 5, where we use Theorem 2 to prove Theorem 1.

We begin with the following energy inequality, which for the multiplicative case is presented as Lemma 3.1 in [21].

Lemma 4.1. *For every $T > 0$ and every $\varepsilon > 0$, there exists a constant $c(\varepsilon, T) > 0$ so that for every solution u of (1.1),*

$$E(u,T) \leqslant (1+\varepsilon)e^{-2TL(T)}E(u,0) + c(\varepsilon,T)\|(u_0,u_1)\|^2_{L^2\bigoplus H^{-1}}, \tag{4.2}$$

This inequality is proved using straightforward properties of the propagation of the defect measure, so the proof from [21] goes through with no modification. To obtain the desired lower bound on α we must further control the $\|(u_0,u_1)\|^2_{L^2\bigoplus H^{-1}}$ on the right-hand side.

Given Lemma 4.1, we proceed by introducing the adjoint $A_W^* = \begin{pmatrix} 0 & -\mathrm{Id} \\ -\Delta_g & -2W \end{pmatrix}$ of the semigroup generator A_W. Note that the spectrum of A_W^* is the conjugate of the spectrum of A_W. We denote by E_{λ_j} and $E^*_{\lambda_j}$ the generalized eigenspaces of A_W and A_W^* with associated eigenvalues λ_j and $\overline{\lambda_j}$, respectively. We note that by an exact analogy of the proof of [1, Lemma 4.2], one can show that the spectrum of A_W, and hence A_W^*, contains only isolated eigenvalues λ_j with $\mathrm{Re}\,(\lambda_j) \leqslant 0$. Thus, each E_{λ_j} and $E^*_{\lambda_j}$ is finite dimensional. Recall that $\mathscr{H} = H^1(M) \oplus L^2(M)$, equipped with the natural norm. It is also useful to introduce the $\dot{\mathscr{H}}$ seminorm defined for elements of $\mathscr{H}$ by

$$\|(u_0,u_1)^T\|^2_{\dot{\mathscr{H}}} = \|\nabla u_0\|^2_{L^2} + \|u_1\|^2_{L^2}.$$

For each $N \geqslant 1$, define the subspace

$$H_N = \left\{ \varphi \in \mathscr{H} : \langle \varphi, \psi \rangle_{\mathscr{H}} = 0,\, \forall \psi \in \bigoplus_{|\lambda_j| \leqslant N} E^*_{\lambda_j} \right\}.$$

Our first observation is that H_N is invariant under the action of the semigroup e^{tA_W}. To demonstrate this, let $\{\psi_k\}$ be a basis of the finite-dimensional space $\bigoplus_{|\lambda_j| \leqslant N} E^*_{\lambda_j} \subset D(A_W^*)$. Now, since $E^*_{\lambda_j}$ is invariant under A_W^*, we can express each $A_W^*\psi_\ell$ as a finite linear combination of the $\{\psi_k\}$. Thus, for each ℓ and any $\varphi \in H_N$, we have

$$\partial_t \langle e^{tA_W}\varphi, \psi_\ell\rangle_{\mathscr{H}}\big|_{t=0} = \langle e^{tA_W}\varphi, A_W^*\psi_\ell\rangle_{\mathscr{H}}\big|_{t=0} = \sum c_{\ell,k}\langle \varphi, \psi_k\rangle_{\mathscr{H}} = 0,$$

by the definition of H_N. Repeating this argument, we see that $\partial_t^j \langle e^{tA_W}\varphi, \psi_\ell\rangle_{\mathscr{H}} \big|_{t=0} = 0$ for all j. Observing that $\langle e^{tA_W}\varphi, \psi_\ell\rangle_{\mathscr{H}}$ is an analytic function of t, we have $\langle e^{tA_W}\varphi, \psi_\ell\rangle_{\mathscr{H}} = 0$ for all $t \in \mathbb{R}$. Therefore $e^{tA_W}\varphi \in H_N$.

Now, define $\mathscr{H}' = L^2 \oplus H^{-1}$ and let θ_N denote the norm of the embedding of H_N in $\mathscr{H}'$, which is well defined since M is compact. Since W is bounded on L^2, it is compact as an operator from $L^2 \to H^{-1}$. Therefore $A_W^* : \mathscr{H} \to \mathscr{H}'$ is a compact perturbation of the skew-adjoint operator $\begin{pmatrix} 0 & -\mathrm{Id} \\ -\Delta_g & 0 \end{pmatrix}$. Thus, the family $\{E^*_{\lambda_j}\}_{j=0}^{\infty}$ is total in $\mathscr{H}$, and so $\lim_{N\to\infty}\theta_N = 0$ (c.f. [12, Ch. 5, Theorem 10.1]).

We can now proceed with the proof of (4.1). Assume that $2\min\{-D_0, L_\infty\} > 0$, otherwise the statement is trivial. Choose $\eta > 0$ small enough so that $\beta = 2\min\{-D_0, L_\infty\} - \eta > 0$ and take T large enough so that $4|L_\infty - L(T)| < \eta$ and $e^{\frac{\eta T}{2}} > 3$. Then, by Lemma 4.1 with $\varepsilon = 1$, there exists a constant $c(1,T)$ such that for every solution u of (1.1)

$$E(u,t) \leqslant 2e^{-2TL(T)}E(u,0) + c(1,T)\|(u_0,u_1)\|^2_{\mathscr{H}'}. \tag{4.3}$$

Next, choose N large enough so that $c(1,T)\theta_N^2 \leqslant e^{-2TL(T)}$. Then, for solutions u of (1.1) with initial data $(u_0,u_1)^T \in H_N$

$$E(u,T) \leqslant 3e^{-2TL(T)}E(u,0).$$

Since H_N is invariant under evolution by e^{tA_W}

$$E(u,kT) \leqslant 3^k e^{-2kTL(T)} E(u,0), \quad \forall k \in \mathbb{N}.$$

Then, we can use the fact that $4|L_\infty - L(T)| < \eta$ and $\frac{\eta T}{2} > \ln 3$ to obtain that

$$\begin{aligned} E(u,kT) &\leqslant 3^k e^{-2kT(L_\infty - \eta/4)} E(u,0) \\ &\leqslant \left(e^{\ln 3 - \frac{\eta T}{2}}\right)^k e^{-2kTL_\infty} E(u,0) \\ &\leqslant e^{-kT\beta} E(u,0), \end{aligned}$$

where the final inequality follows from the fact that $\beta \leqslant 2L_\infty - \eta < 2L_\infty$ by definition. Since the energy is nondecreasing, it follows that

$$E(u,t) \leqslant Ce^{-\beta t} E(u,0) \quad \forall t \geqslant 0, \tag{4.4}$$

for some constant $C > 0$.

To extend (4.4) to all solutions of (1.1), let Π denote the orthogonal projection from $\mathscr{H}$ onto $\bigoplus_{|\lambda_j| \leqslant N} E_{\lambda_j}$. Then for any $v = (u_0, u_1)^T \in \mathscr{H}$, there is an orthogonal decomposition of the form $v = \Pi v + (\mathrm{Id} - \Pi)v$. Since E_{λ_j} and $E^*_{\lambda_k}$ are orthogonal for $\lambda_j \neq \lambda_k$, we have that $(\mathrm{Id} - \Pi)v \in H_N$, and hence $H_N^\perp = \bigoplus_{|\lambda_j| \leqslant N} E_{\lambda_j}$. Since E_{λ_j} is invariant under e^{tA_W} and $H_N^\perp$ is finite dimensional, we have that there exists a $C > 0$ so that for all solutions u of (1.1) with initial data in $H_N^\perp$,

$$E(u,t) \leqslant Ce^{2D_0} E(u,0) \leqslant Ce^{-\beta t} E(u,0), \quad \forall t \geqslant 0. \tag{4.5}$$

Finally, since Π and $\mathrm{Id} - \Pi$ are continuous with respect to the $\dot{\mathscr{H}}$ seminorm, for some $C > 0$

$$E(\Pi u, 0) + E((\mathrm{Id} - \Pi)u, 0) \leqslant CE(u,0).$$

Therefore, using the decomposition $\Pi + (\mathrm{Id} - \Pi)$ on the initial data of any solution u we can apply (4.4) and (4.5) to obtain

$$E(u,t) \leqslant Ce^{-\beta t} E(u,0), \quad \forall t \geqslant 0, \tag{4.6}$$

for some possibly larger $C > 0$. By definition of the best possible decay rate, $\alpha \geqslant \beta = 2\min\{-D_0, L_\infty\} - \eta$. Since η can be taken arbitrarily small, this proves (4.1). Combining this with the upper bound obtained in Sect. 3 completes the proof of Theorem 2.

5. Proof of Theorem 1

In this section, we show that Theorem 2 implies Theorem 1. First, we will assume both Assumptions 1 and 2 are satisfied. We will show this implies $\alpha > 0$, which is equivalent to exponential energy decay. Note that Assumption 1 immediately implies that $L_\infty \geqslant c > 0$. Thus, we only need to show that $D_0 < 0$. For this, we introduce the quantity

$$D_\infty := \lim_{R \to \infty} \sup\{\mathrm{Re}\,(\lambda) : |\lambda| > R, \lambda \in \mathrm{Spec}\, A_W\}.$$

We claim that $D_\infty \leqslant -L_\infty$. To show this, first recall the definitions of E_{λ_j} and H_N from Sect. 4. Let u be a solution to (1.1) with initial data $(u_0, u_1)^T \in E_{\lambda_j}$ with $|\lambda_j| > N$. Then $u = e^{tA_W}(u_0, u_1)^T = e^{t\lambda_j}(u_0, u_1)^T$. Note that $E_{\lambda_j} \subset H_N$ whenever $|\lambda_j| > N$. Combining this with the proof of (4.4), we obtain

$$e^{2\mathrm{Re}\,(\lambda_j)t}E(u,0) = E(u,t) \leqslant Ce^{-\beta t}E(u,0),$$

for every $0 < \beta < 2L_\infty$. Hence, $2\mathrm{Re}\,(\lambda_j) \leqslant -\beta$ whenever $|\lambda_j| \geqslant N$, and so $\mathrm{Re}\,(\lambda_j) \leqslant -L_\infty$ for such λ_j. It immediately follows that $D_\infty \leqslant -L_\infty < 0$.

Recall also that the spectrum of A_W consists only of isolated eigenvalues, and $\mathrm{Re}\,(\lambda) \leqslant 0$ for all $\lambda \in \mathrm{Spec}(A_W)$. Thus, in order to have $D_0 = 0$, either $D_\infty = 0$ or there exists a nonzero eigenvalue of A_W on the imaginary axis. Since we have already shown $D_\infty < 0$, we need only rule out nonzero imaginary eigenvalues. Suppose $i\lambda \in \mathrm{Spec}(A_W)$ with $\lambda \in \mathbb{R}$ and corresponding eigenvector $(v_0, v_1)^T$. Then $v_1 = \lambda v_0$, and

$$\Delta_g v_0 + \lambda^2 v_0 - 2i\lambda W v_0 = 0. \tag{5.1}$$

Taking the L^2 inner product of both sides with v_0 and then taking the imaginary part gives

$$-2\lambda\langle Wv_0, v_0\rangle = 0.$$

If $\lambda = 0$, the equation is trivially satisfied. However, if $\lambda \neq 0$, then $\langle Wv_0, v_0\rangle = 0$. Recalling that $W = \sum B_j^* B_j$ for some collection of operators B_j, we must have $Wv_0 = 0$. Then by (5.1) v_0 is an eigenfunction of Δ_g with eigenvalue $-\lambda^2$ and $v_0 \in \ker W$. But by Assumption 2, this is impossible. Thus, the only possible eigenvalue of A_W on the imaginary axis is zero and we cannot have $D_0 = 0$. Combining this with the fact that $L_\infty > 0$, we have shown that Assumptions 1 and 2 imply $\alpha > 0$, which in turn demonstrates that solutions to (1.1) experience exponential energy decay.

We now prove the reverse implication in Theorem 1. For this, we assume that (1.5) holds with some $\beta > 0$ for all solutions u and we want to see that Assumptions 1 and 2 hold. By definition, $\alpha \geqslant \beta > 0$, and hence both $-D_0$ and L_∞ are strictly positive. Because $L_\infty \geqslant \alpha/2 > 0$ Assumption 1 holds. Similarly, since $D_0 < 0$, there cannot be any eigenvalues of A_W on the imaginary axis except possibly at zero. Now suppose that $v \in L^2$ satisfies $-\Delta_g v = \lambda^2 v$ with $\lambda \neq 0$ and $Wv = 0$. Then $(v, i\lambda v)^T$ is an eigenvector of A_W with eigenvalue $i\lambda \neq 0$, which is a contradiction. Thus, Assumption 2 must also hold, which completes the proof of Theorem 1.

6. A Class of Examples on Analytic Manifolds

One of the key hypotheses of Theorem 1 was that the damping coefficient W must not annihilate any eigenfunctions of Δ_g associated with nonzero eigenvalues. In the case where W is a multiplication operator which satisfies the classical geometric control condition, this is always satisfied by the unique continuation properties of elliptic operators [28]. However, when the damping is pseudodifferential it is much more difficult to check this hypothesis.

In this section, we produce a collection of operators on real analytic manifolds which satisfy Assumption 2 and are neither multiplication operators nor functions of Δ_g. We also construct two examples of explicit pseudodifferential damping coefficients on $\mathbb{T}^2$ which satisfy Assumptions 1 and 2. The primary tool in this discussion is the notion of the analytic wavefront set, and so we begin by providing some background definitions for the reader's convenience. More details can be found in [13, §8.4-8.6].

Given a set $X \subseteq \mathbb{R}^n$ and a distribution $u \in \mathcal{D}'(X)$, if u is real analytic on an open neighborhood of x_0, we write that $u \in C^a$ near $x_0 \in X$. In analogy with the relationship between the standard wavefront set and C^∞ singularities, one can resolve C^a singularities by defining the *analytic* wavefront set, written $\mathrm{WF}_A(u)$ and defined as follows.

Definition 6.1. We say that a point $(x_0, \xi_0) \in T^*X\backslash 0$ is not in $\mathrm{WF}_A(u)$, if there exists an open neighborhood U of x_0, a conic neighborhood Γ of ξ_0 and a bounded sequence $u_N \in \mathcal{E}'(X)$, which are equal to u on U and which satisfy

$$|\widehat{u}_N(\xi)| \leqslant C\left(\frac{N+1}{|\xi|}\right)^N, \tag{6.1}$$

for all $\xi \in \Gamma$.

By [13, Prop. 8.4.2], we have that $u \in C^a$ near x_0 if and only if $\mathrm{WF}_A(u)$ contains no points of the form (x_0, ξ) with $\xi \neq 0$.

We also introduce a set which we can be thought of as the analytically invertible directions of u, denoted by $\Gamma_A(u)$. Its complement is commonly called the (analytic) characteristic set of u [13].

Definition 6.2. We say that $\xi_0 \in \mathbb{R}^n\backslash 0$ is in $\Gamma_A(u)$ if there exists a complex conic neighborhood V of ξ_0 and a function Φ, which is holomorphic in $\{\xi \in V : |\xi| > c\}$ for some $c > 0$, satisfying $\Phi\widehat{u} = 1$ in $V \cap \mathbb{R}^n$ and there exists $C, N > 0$ such that

$$|\Phi(\zeta)| \leqslant C|\zeta|^N,$$

for $\zeta \in V$.

The final preliminary we require is the notion of the normal set of a closed region F contained within a manifold M. For the purposes of this definition, we only require that M be C^2.

Definition 6.3. Let F be a closed region in a C^2 manifold M. The exterior normal set, $N_e(F)$, is defined as the set of all $(x_0, \xi_0) \in T^*M\backslash 0$ such that $x_0 \in F$ and such that there exists a real-valued function $f \in C^2(M)$ with $df(x_0) = \xi_0 \neq 0$ and

$$f(x) \leqslant f(x_0), \quad x \in F.$$

The interior normal set of F is then defined by $N_i(F) = \{(x, \xi) : (x, -\xi) \in N_e(F)\}$ and the full normal set is defined as $N(F) = N_e(F) \bigcup N_i(F)$. We write $\overline{N}(F)$ to denote the closure of the normal set of F.

Note that the projection of $N_e(F)$ onto M is dense in ∂F but might not be equal to ∂F [13, Prop. 8.5.8].

With these definitions in hand, we are able to describe a class of pseudodifferential operators which do not annihilate any eigenfunctions of Δ_g.

To produce the desired class of examples, let (M, g) be a compact, real analytic manifold of dimension n. Suppose $\chi, \widetilde{\chi} \in C_c^\infty(M)$ are cutoff functions supported entirely within a single coordinate patch, with χ not identically zero and $\widetilde{\chi} \equiv 1$ on an open neighborhood of the support of χ. Let $b \in C^\infty(\mathbb{R}^n)$ be homogeneous of degree 0 outside a compact neighborhood of the origin, and define $B \in \Psi^0_{cl}(M)$ in local coordinates by $Bu = \widetilde{\chi}\mathrm{Op}(b(\xi))\chi u$. Let $\check{b}$ denote the inverse Fourier transform of b and $\pi_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ denote the natural projection onto the fiber variables ξ. We show that any such B cannot annihilate eigenfunctions of Δ_g under the following condition on χ and b.

Lemma 6.4. *If $\pi_2(\overline{N}(\operatorname{supp}\chi)) \cap \Gamma_A(\check{b}) \neq \emptyset$, then for any eigenfunction u of Δ_g, we have $Bu \neq 0$.*

Proof. We proceed by contradiction, so assume $Bu = 0$ for some eigenfunction u of Δ_g. Thus $\mathrm{WF}_A(Bu) = \emptyset$ and we aim to show there exists some $(x_0, \xi_0) \in \mathrm{WF}_A(Bu)$. First, by [13, Thm 8.5.6'], we have

$$\overline{N}(\operatorname{supp}\chi u) \subseteq \mathrm{WF}_A(\chi u). \tag{6.2}$$

Since u is an eigenfunction, it cannot vanish identically on any open set. We claim that this implies

$$\partial(\operatorname{supp}\chi) \subseteq \partial(\operatorname{supp}\chi u). \tag{6.3}$$

To see this, suppose $x \in \partial(\operatorname{supp}\chi)$ and let V be any open neighborhood of x. Since $\chi(x) = 0$, we have that $\chi(x)u(x) = 0$, so it is enough to show that χu is not identically zero on all of V. Without loss of generality, we may assume that V lies entirely within the same coordinate patch containing $\operatorname{supp}\chi$. Since x is a boundary point of the support, χ does not vanish identically on V. By the continuity of χ, this implies the existence of a smaller open neighborhood $\widetilde{V} \subset V$ (not containing x) where χ is never zero. Since u is an eigenfunction, it cannot vanish identically on $\widetilde{V}$, and hence χu is not identically zero on $\widetilde{V} \subseteq V$, which proves (6.3).

Next, we want to show $\overline{N}(\operatorname{supp}\chi) \subset \overline{N}(\operatorname{supp}\chi u)$. Take $(x_0, \xi_0) \in N(\operatorname{supp}\chi)$, so x_0 maximizes a function f on $\operatorname{supp}\chi$ with $df(x_0) \neq 0$. Thus, $x_0 \in \partial(\operatorname{supp}\chi) \subseteq \partial(\operatorname{supp}\chi u)$. That is, x_0 is not an interior point. Furthermore, since $\operatorname{supp}\chi \supseteq \operatorname{supp}\chi u$ and f is maximized at x_0 in $\operatorname{supp}\chi$ it must also be maximized at x_0 when restricted to the smaller set $\operatorname{supp}\chi u$. Therefore, $N(\operatorname{supp}\chi) \subseteq N(\operatorname{supp}\chi u)$, and hence $\overline{N}(\operatorname{supp}\chi) \subseteq \overline{N}(\operatorname{supp}\chi u)$. Thus, by (6.2),

$$\overline{N}(\operatorname{supp}\chi) \subseteq \mathrm{WF}_A(\chi u). \tag{6.4}$$

Since the cutoff function χ is supported in a single coordinate patch, we can treat χu and $\mathrm{Op}(b)\chi u$ as functions on $\mathbb{R}^n$. Now, observe that $\check{b} * \chi u =$

$\mathrm{Op}(b)\chi u$, where $*$ denotes standard convolution. This, along with [13, Thm 8.6.15] gives

$$\mathrm{WF}_A(\chi u) \subseteq \mathrm{WF}_A(\mathrm{Op}(b)\chi u) \cup (\mathbb{R}^n \times \Gamma_A(\check{b})^c). \tag{6.5}$$

Applying (6.4), we obtain

$$\overline{N}(\operatorname{supp}\chi) \subseteq \mathrm{WF}_A(\mathrm{Op}(b)\chi u) \cup (\mathbb{R}^n \times \Gamma_A(\check{b})^c),$$

and therefore,

$$\overline{N}(\operatorname{supp}\chi) \cap (\mathbb{R}^n \times \Gamma_A(\check{b})) \subseteq \mathrm{WF}_A(\mathrm{Op}(b)\chi u).$$

By hypothesis, there exists a point

$$(x_0,\xi_0) \in \overline{N}(\operatorname{supp}\chi) \cap (\mathbb{R}^n \times \Gamma_A(\check{b})) \subseteq \mathrm{WF}_A(\mathrm{Op}(b)\chi u).$$

In particular, $x_0 \in \operatorname{supp}\chi$, and since $\widetilde{\chi} \equiv 1$ on a neighborhood of $\operatorname{supp}\chi$, we see that (x_0,ξ_0) must also lie inside $\mathrm{WF}_A(\widetilde{\chi}\mathrm{Op}(b)\chi u) = \mathrm{WF}_A(Bu)$. This contradicts the assumption that $Bu = 0$, and thus the proposition is proved. □

Remark 6.5. It is worth noting that the argument of this lemma works when Δ_g is replaced by P, an elliptic second-order pseudodifferential operator, since the eigenfunctions of such P have the unique continuation property.

Given Proposition 6.4, the proof of Theorem 3 is straightforward.

Proof of Theorem 3. Given a real analytic manifold (M,g), take $\chi,\widetilde{\chi}$ as in the statement of Proposition 6.4. Let $(x_0,\xi_0) \in N_e(\operatorname{supp}\chi)$ be an arbitrary exterior normal. Then, take any $b \in C^\infty(\mathbb{R}^n)$ which is identically one in a conic neighborhood of ξ_0, zero on the complement of a slightly larger conic neighborhood, and homogeneous of degree 0 outside a compact neighborhood of the origin. Then $\Gamma_A(\check{b})$ contains ξ_0 because $b \equiv 1$ on a conic neighborhood of ξ_0, and so one may take $\Phi \equiv 1$ in the definition of Γ_A. Proposition 6.4 then guarantees that $B = \widetilde{\chi}\mathrm{Op}(b)\chi$ does not annihilate any eigenfunctions of Δ_g, and thus neither does $W = B^*B$. One can repeat this process in any finite number of coordinate patches to show that there exists $W = \sum_{j=1}^N B_j^* B_j$ with the same property. □

We now construct a pseudodifferential damping coefficient on $\mathbb{T}^2$ which satisfies Assumptions 1 and 2.

Example 6.6. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ denote the two-dimensional torus equipped with the flat metric, and let Δ be the associated Laplace–Beltrami operator. Let $\delta > 0$ and let $\chi_1 \in C_c^\infty(\mathbb{T}^2)$ be supported in the vertical strip $\{(x^{(1)},x^{(2)}) \in \mathbb{T}^2 : \frac{1}{2}-\delta \leqslant x^{(1)} \leqslant \frac{1}{2}+\delta\}$ and equal to one on a smaller vertical strip. Define $\widetilde{\chi}_1$ in a similar way, but with $\widetilde{\chi}_1 \equiv 1$ on the support of χ_1. Analogously, let $\chi_2 \in C_c^\infty(\mathbb{T}^2)$ be supported in the horizontal strip $\{(x^{(1)},x^{(2)}) \in \mathbb{T}^2 : \frac{1}{2}-\delta \leqslant x^{(2)} \leqslant \frac{1}{2}+\delta\}$ and equal to one on a smaller horizontal strip, and define $\widetilde{\chi}_2$ similarly with $\widetilde{\chi}_2 \equiv 1$ on the support of χ_2.

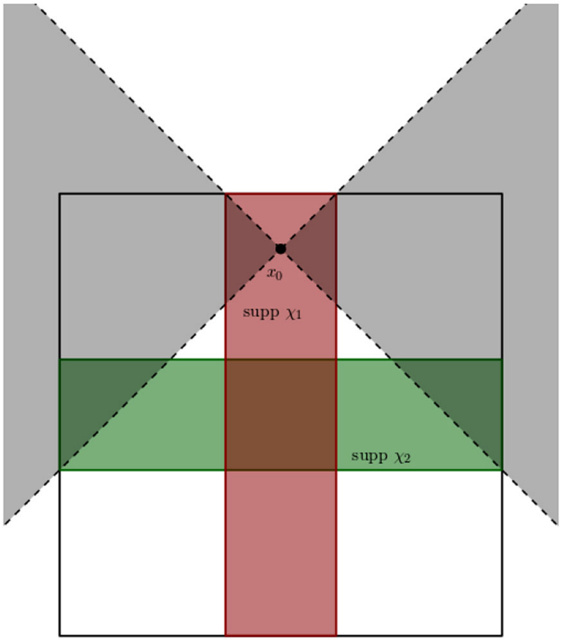

Figure 1. The cone of directions containing the support of $b_1(x_0,\cdot)$ in $T^*_{x_0}\mathbb{T}^2$

Now, let $\varepsilon > 0$ and let $b_1 \in C^\infty(\mathbb{S}^1)$ be supported in the set

$$\Theta_{2\varepsilon} = \left(-\frac{\pi}{4} - 2\varepsilon, \frac{\pi}{4} + 2\varepsilon\right) \cup \left(\frac{3\pi}{4} - 2\varepsilon, \frac{5\pi}{4} + 2\varepsilon\right)$$

and equal to one on the smaller set Θ_ε. Similarly, let $b_2 \in C^\infty(\mathbb{S}^1)$ be nonzero on $\Theta_{2\varepsilon} + \frac{\pi}{2}$ and equal to one on $\Theta_\varepsilon + \frac{\pi}{2}$. Choose $\beta \in C^\infty_c(\mathbb{R})$ to be supported in $[\frac{1}{4}, \infty)$ and equal to one on $[\frac{1}{2}, \infty)$. Then define symbols $b_j \in S^0_{cl}(T^*\mathbb{T}^2)$ by

$$b_j(\xi) = b_j(\theta)\beta(r), \quad j = 1, 2,$$

where $\xi = (r, \theta)$ in standard polar coordinates on $T^*_x\mathbb{T}^2$. Figure 1 illustrates the cone of directions in $T^*_{x_0}\mathbb{T}^2$ in which b_1 is supported at some arbitrary $x_0 \in \operatorname{supp}\chi_1$. Now define $B_j = \widetilde{\chi}_j \mathrm{Op}(b_j)\chi_j$, and set the damping coefficient W to be

$$W = B_1^* B_1 + B_2^* B_2.$$

To see $\ker W$ contains no nontrivial eigenfunctions of Δ we apply Proposition 6.4. Note $\overline{N}(\operatorname{supp}\chi_1)$ contains all points of the form (x, ξ) with $x \in \partial(\operatorname{supp}\chi_1)$ and $\xi = (r, \theta)$, where $\theta = 0$ or $\theta = \pi$. Since b_1 is constant in a conic neighborhood of both of these cotangent directions, the hypotheses of Proposition 6.4 are satisfied. Thus, $\ker B_1$ contains no eigenfunctions of the Laplacian. An analogous argument holds for B_2, and since $B_1^* B_1$ and $B_2^* B_2$ are nonnegative operators, W cannot annihilate any eigenfunctions of the Laplacian.

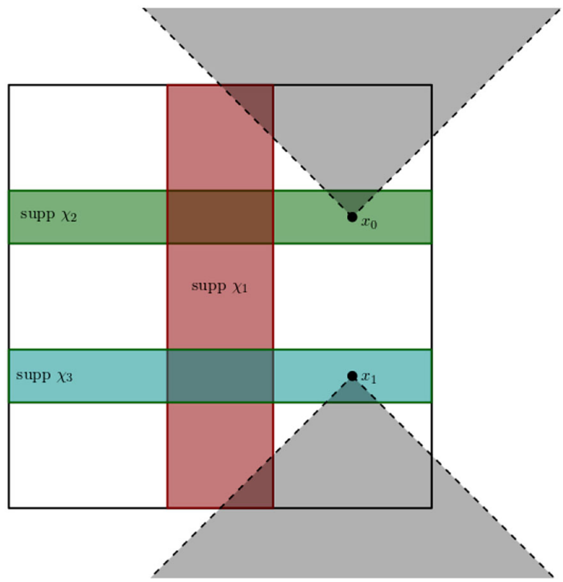

FIGURE 2. The cones containing the supports of $b_2(x_0, \cdot)$ and $b_3(x_1, \cdot)$

To show exponential energy decay with W as the damping coefficient, we must also demonstrate that W satisfies the AGCC. For this, it is convenient to observe that the AGCC is equivalent to the existence of some $T_0 > 0$ and $c > 0$ such that every trajectory $t \mapsto \varphi_t(x_0, \xi_0)$ encounters the set

$$\mathscr{W}_c = \{(x, \xi) \in T^*\mathbb{T}^2 : w(x, \xi) \geqslant c > 0\}$$

in time $T \leqslant T_0$. Recall that the geodesics on $\mathbb{T}^2$ are the projections of straight lines in $\mathbb{R}^2$ under the quotient map. Thus, the geodesic flow on $S^*\mathbb{T}^2$ is given by

$$(x, \xi) \mapsto ((x + t\xi) \bmod \mathbb{Z}^2, \xi).$$

Given an arbitrary point $(x_0, \xi_0) \in S^*\mathbb{T}^2$, we will show that $(\gamma(t), \gamma'(t)) = ((x_0 + t\xi_0) \bmod \mathbb{Z}^2, \xi_0)$ must intersect $\mathscr{W}_c$ in some fixed time $T_0 > 0$. Let us write $\xi_0 \in \mathbb{S}^1$ as $(\cos\theta_0, \sin\theta_0)$, and consider the case where θ_0 lies in Θ_ε. Suppose first that

$$\theta_0 \in \left(-\frac{\pi}{4} - \varepsilon, \frac{\pi}{4} + \varepsilon\right),$$

which implies $b_1(\xi_0) \neq 0$. Then, if $x_0 = (x_0^{(1)}, x_0^{(2)})$, the horizontal coordinate of $\gamma(t)$ is given by

$$(x_0^{(1)} + t\cos\theta_0) \bmod \mathbb{Z},$$

which must reach $\frac{1}{2}$ in some time less than $\frac{1}{\cos\theta_0} \leqslant \frac{1}{\cos(\pi/4+\varepsilon)}$. Therefore, $(\gamma(t), \gamma'(t))$ intersects the region where b_1 is strictly positive in time less than

$\frac{1}{\cos(\pi/4+\varepsilon)}$. The same argument holds if instead $\theta_0 \in (\frac{3\pi}{4} - \varepsilon, \frac{5\pi}{4} + \varepsilon)$, and so whenever $\theta_0 \in \Theta_\varepsilon$, we have that there exists a $c > 0$ such that $(\gamma(t), \gamma'(t))$ intersects $\{b_1(x,\xi) \geqslant \sqrt{c}\}$ in finite time. Analogously, if $\theta_0 \in \Theta_\varepsilon + \frac{\pi}{2}$, then the vertical component of $\gamma(t)$, given by $(x_0^{(2)} + t\sin\theta_0) \bmod \mathbb{Z}$, must equal $\frac{1}{2}$ in some time less than $\frac{1}{\sin(\pi/4-\varepsilon)}$. Therefore, $(\gamma(t), \gamma'(t))$ intersects $\{b_2(x,\xi) \geqslant \sqrt{c}\}$ in finite time. Since

$$\mathbb{T}^2 \times \left(\Theta_\varepsilon \cup \left(\Theta_\varepsilon + \frac{\pi}{2}\right)\right) = S^*\mathbb{T}^2,$$

and since $w(x,\xi) = b_1^2(x,\xi) + b_2^2(x,\xi)$, we have that for every $(x_0, \xi_0) \in S^*\mathbb{T}^2$, the curve $\varphi_t(x_0,\xi_0)$ intersects $\mathscr{W}_c$ in some fixed time $T_0 > 0$. We have therefore shown that W as defined here satisfies both Assumptions 1 and 2. Thus by Theorem 1, all solutions to the damped wave equation on $\mathbb{T}^2$ with damping coefficient W experience exponential energy decay.

Remark 6.7. In the previous example, one may notice that on the intersection of the vertical and horizontal strips, the principal symbol of the damping coefficient is supported in all directions $\xi \in T^*\mathbb{T}^2\backslash 0$. So in this region, W behaves very much like a multiplication operator for frequencies away from zero. A natural question is whether or not there must always be a point of "full microsupport" if the hypotheses of Theorem 1 are to be satisfied. In fact, there need not be such a point. To see this, we can modify our example above as follows.

Define χ_1, $\widetilde{\chi}_1$, χ_2, $\widetilde{\chi}_2$ and b_1 in a similar fashion to the previous example, but now define b_2 to be supported only in the directions with angle $\theta \in (\frac{\pi}{4} - 2\varepsilon, \frac{3\pi}{4} + 2\varepsilon)$ and identically one on $(\frac{\pi}{4} - \varepsilon, \frac{3\pi}{4} + \varepsilon)$. Next, we introduce another horizontal strip, disjoint from the first, with a corresponding pair of cutoff functions $\chi_3, \widetilde{\chi}_3$. Then, define $b_3 \in C^\infty(\mathbb{S}^1)$ to be supported in $(\frac{5\pi}{4} - 2\varepsilon, \frac{7\pi}{4} + 2\varepsilon)$ and equal to one on $(\frac{5\pi}{4} - \varepsilon, \frac{7\pi}{4} + \varepsilon)$, and let $b_3(\xi) = b_3(\theta)\beta(r)$, where $\xi = (r,\theta)$ as before. This is illustrated in Fig. 2. Then, if we define $B_3 = \widetilde{\chi}_3 \mathrm{Op}(b_3)\chi_3$ and set $W = \sum_{j=1}^3 B_j^* B_j$, we can apply arguments similar to those above to see that Assumptions 1 and 2 are still satisfied, but there does not exist any point $x \in \mathbb{T}^2$ where $w(x,\xi)$ is supported in all directions.

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Blake Keeler
Department of Mathematics and Statistics
Dalhousie University
6316 Coburg Road
PO Box 15000, Halifax, NSB3H 4R2
Canada
e-mail: bkeeler2015@gmail.com

Perry Kleinhenz
Department of Mathematics
Michigan State University
619 Cedar River Rd
East Lansing, MI48823
USA
e-mail: pkleinhe@gmail.com

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