THE FLEX DIVISOR OF A K3 SURFACE

VALERY ALEXEEV AND PHILIP ENGEL

Abstract. The *flex divisor* $R_{\rm flex}$ of a primitively polarized K3 surface (X,L) is, generically, the set of all points $x\in X$ for which there exists a pencil $V\subset |L|$ whose base locus is $\{x\}$. We show that if $L^2=2d$ then $R_{\rm flex}\in |n_dL|$ with

$$n_d=\frac{(2d)!(2d+1)!}{d!^2(d+1)!^2}=(2d+1)C(d)^2,$$

where $C(d)$ is the Catalan number. We also show that there is a well-defined notion of flex divisor over the whole moduli space F_{2d} of polarized K3 surfaces.

1. Introduction

Let (X,L) be a primitively polarized K3 surface of degree $2d$. Recent work of the authors on compactification of the moduli space F_{2d} of such surfaces has highlighted the importance of a *canonical choice of polarizing divisor*: An algebraically varying choice of divisor $R\in|nL|$ on the generic polarized K3 surface. If this choice of divisor extends over all of F_{2d} then it gives rise to a modular compactification

$$F_{2d}\hookrightarrow \overline{F}^R_{2d}.$$

The compactification is constructed by taking the closure of the space of pairs $(X,\epsilon R)$ in the moduli space of stable slc pairs, for some small $\epsilon>0$.

By the main theorem of [AE21], the normalization of $\overline{F}^R_{2d}$ is semitoroidal whenever R satisfies a property called *recognizability*. Thus, the search for modular toroidal compactifications of F_{2d} is intimately related to finding canonical choices of polarizing divisor, and verifying that those choices are recognizable.

One infinite series of divisors, ranging over all degrees $2d$, is the *rational curve divisor*. On a generic K3 surface (X,L) it can be defined as

$$R_{\rm rc}:=\sum_{\substack{C\in|L|\\ \text{rational}}} C$$

and was proven to be recognizable in [AE21]. By the famous Yau-Zaslow formula [YZ96, Bea99], $R_{\rm rc}\in|n_dL|$ where $n_d=[q^{d+1}]\prod_{n\geq1}(1-q^n)^{-24}$.

Claire Voisin suggested to the authors a second series of divisors, which we call here the *flex divisor* $R_{\rm flex}$. It was first considered by Welters [Wel81] for a quartic K3 surface, who called it the curve of hyperflexes. On the generic (X,L) it is defined as the set of all points $x\in X$ for which there exists a pencil $V\in|L|$ whose set-thereotic base locus is $\{x\}$. When $|L|$ defines an embedding $X\hookrightarrow\mathbb{P}^g$ with $g=d+1$, which is the case for generic $(X,L)\in F_{2d}$ when $d\geq2$, the flex divisor can be concretely thought of as the set of points $x\in X\subset\mathbb{P}^g$ for which there exists a *flex space*: A codimension 2 linear subspace of $\mathbb{P}^g$ intersecting X at only the point x.

Our first result hints towards a positive answer on the question of whether R_{flex} is recognizable. Concretely, we show:

Theorem 1.1. *There is a canonical choice of divisor R_{flex} varying algebraically over all of F_{2d} and giving the flex divisor on the generic K3 surface (X, L).*

The theorem is not obvious, because it is not clear if the flex points form a subvariety of X of the expected dimension, which is one. Additionally, the flex divisor may have multiple components and one must determine their multiplicities. Perhaps most importantly, sometimes points in R_{flex} as in Theorem 1.1 are not flex under the naive definition! This occurs on a quartic surface containing a line—the points on the line are not flex according to the naive definition because the relevant pencil V contains the whole line as a base curve. But the line appears as a component of the flex divisor, see Example 3.14.

The flex divisor is notably an example of a *constant cycle curve* [Huy14]: One whose points all have the same class in the Chow group of zero-cycles $\mathrm{CH}_0(X)$. The method of proof of Theorem 1.1 suggests strongly:

Conjecture 1.2. *Let R be a canonical choice of polarizing divisor for F_{2d}. If R is a constant cycle curve, then it is recognizable.*

A resolution of this conjecture would unify various results about recognizable divisors, such as [AET19], [ABE20], [AE21], and [AEH21].

Our second result is an analogue of the Yau-Zaslow formula. That is, we determine in what multiple of the polarization the flex divisor lives, generalizing known results in the cases $d = 1, 2$.

Theorem 1.3. *Let (X, L) be a K3 surface of degree $2d$. Then $R_{\mathrm{flex}} \in |n_d L|$ with*

$$n_d = \frac{(2d)!(2d+1)!}{d!^2(d+1)!^2} = (2d+1)C(d)^2,$$

where $C(d)$ is the Catalan number.

d	1	2	3	4	5	6	7	8	9
n_d	3	20	175	1764	19404	226512	2760615	34763300	449141836

Table 1. Flex divisor classes

Table 1 tabulates the first nine values of n_d. The value $n_1 = 3$ is well-known, see Example 3.13, while the value $n_2 = 20$ has been computed by various authors [Huy14, Prop. 8.8], [Wit14, Cor. 2.4.6].

The summary of the paper is as follows: Section 2 shows that the flex divisor is well-defined and extends to a divisor over all of F_{2d} and Section 3 computes the multiple n_d of the primitive polarization in which the flex divisor lives, using intersection theory on the Hilbert scheme $X^{[2d]}$ of the K3 surface.

2. Well-definedness

Definition 2.1. We say (X, L) is a *polarized K3 surface of degree* $2d$ if X is a K3 surface with ADE singularities, and $L \to X$ is a primitive, ample line bundle satisfying $L^2 = 2d$.

Let F_{2d} denote the moduli stack of polarized K3 surfaces over $\mathbb{C}$. It is a smooth, irreducible, Deligne-Mumford stack of dimension 19.

Definition 2.2. A point $x \in X$ is *flex* if there exists a pencil $V \subset |L|$ whose base locus is the singleton $\{x\}$.

Let L_{K3} denote the unique even, unimodular lattice of signature $(3, 19)$ and fix a primitive vector $v \in L_{K3}$ of norm $v^2 = 2d$. Define

$$\mathbb{D} := \mathbb{P}\{x \in v^{\perp} \otimes \mathbb{C} \,\big|\, x \cdot x = 0,\, x \cdot \overline{x} > 0\} \text{ and}$$
$$\Gamma := \{\gamma \in O(L_{K3}) \,\big|\, \gamma(v) = v\}.$$

By the Torelli theorem, the coarse space of F_{2d} is the arithmetic quotient $\mathbb{D}/\Gamma$.

Definition 2.3. A *Heegner divisor* in $\mathbb{D}/\Gamma$ is the image of a hyperplane section $w^{\perp} \cap \mathbb{D}$ for some vector $w \in L_{K3} \setminus \mathbb{Z}v$.

Proposition 2.4. *Let $S \subset F_{2d}$ denote the polarized K3 surfaces (X, L) for which there exists a pencil $V \subset |L|$ containing a base curve. Then S is contained in a finite union of Heegner divisors.*

Proof. The condition that $|L|$ contain a pencil with a base curve is an algebraic condition, which is easily seen to be closed on F_{2d}.

Let C be the base curve of a pencil $V \subset |L|$. Fix $H \in V$ and note $H = C + D$ for some non-empty effective divisor D. Since L is ample, we have $0 < L \cdot C < 2d$. Thus $[C] \notin \mathbb{Z}L$. By the primitivity of L, the rank of $\mathrm{Pic}(X)$ is least 2. Hence any point of S lies in some Heegner divisor. Since S is algebraic, we conclude that S is contained in a finite union of Heegner divisors. $\square$

Lemma 2.5. *Let (X, L) be a polarized K3 surface. The flex points $\{x \in X \,\big|\, x \text{ flex}\}$ form a constructible subset of X of dimension at most* 1*.*

Proof. Constructibility is elementary. To show the second statement, it suffices to make the following observation: Any flex point $x \in X$ lies in the Beauville-Voisin class $[x] = c_X \in \mathrm{CH}_0(X)$, defined in [BV04] as the class of any point on a rational curve in X. This follows because:

(1) $2d[x] = H_1 \cdot H_2$ for hyperplane sections H_1, H_2 spanning the pencil V,
(2) the intersection of two curves is some multiple of c_X [BV04, Thm. 1], and
(3) $\mathrm{CH}_0(X)$ is torsion-free [Roj80].

If a Zariski-open subset of points of X were flex, we would have that $\mathrm{CH}_0(X) = \mathbb{Z}$, contradicting Mumford's theorem [Mum69] on the uncountability of the Chow group. So the constructible set of flex points has dimension at most 1. $\square$

Given a smooth surface X, denote by $X^{[k]}$ the Hilbert scheme of k points on X. Let F_{2d}^{sing} denote the substack of F_{2d} parameterizing singular ADE K3 surfaces, which is also a finite union of Heegner divisors.

Definition 2.6. Define the Zariski open subset $T := F_{2d} \setminus (S \cup F_{2d}^{\mathrm{sing}})$. We assume for the remainder of the text that $(X, L) \in T$, unless otherwise stated.

Let $G := \mathrm{Gr}(g-1, g+1)$ be the Grassmannian of codimension 2 linear spaces in $H^0(X, L)^*$, or equivalently pencils in $|L|$. Consider the map

$$i \colon G \to X^{[2d]}, \qquad V \mapsto \mathbb{P}V \cap X$$

sending a codimension 2 linear space to its scheme-theoretic intersection with X, or equivalently sending a pencil to its scheme-theoretic basic locus.

Proposition 2.7. *The mapping $i\colon G \to X^{[2d]}$ is a closed immersion.*

Proof. First, we show that i is a set-theoretic embedding. Suppose, for the sake of contradiction, that two pencils $\mathbb{P}V_1 \cap X = \mathbb{P}V_2 \cap X$ intersect at the same length $2d$ subscheme $Z \subset X$. Consider the set of codimension 2 linear spaces

$$\mathbb{P} := \{V \in G \,\big|\, V \supset V_1 \cap V_2\}.$$

We necessarily have that $\mathbb{P}V \cap X = Z$ for all $V \in \mathbb{P}$ because $Z \subset \mathbb{P}V_1 \cap \mathbb{P}V_2 \cap X$. Hence $i(\mathbb{P})$ consists of a single point. Since $\mathbb{P}$ contains a curve, we conclude that i contracts a curve. But any morphism from a Grassmannian to a projective variety contracting a curve must be constant. So i is constant, which is absurd.

Next, we show that the differential di is injective. Recall that the tangent space $T_V G = \mathrm{Hom}(V, H^0(X,L)^*/V)$ whereas $T_{[Z]}X^{[2d]} = \mathrm{Hom}(I_Z/I_Z^2, \mathcal{O}_Z)$. We may write $\mathbb{P}V = \{x \in \mathbb{P}^g \,\big|\, s_1(x) = s_2(x) = 0\}$ for two sections $s_1, s_2 \in H^0(X,L)$. A tangent vector $T_V G$ can be represented as the vanishing locus of $(s_1 + \epsilon t_1, s_2 + \epsilon t_2)$, where $t_1, t_2 \in H^0(X,L)/\mathbb{C}s_1 \oplus \mathbb{C}s_2$. Then di maps it to $(s_1 \mapsto t_1|_Z, s_2 \mapsto t_2|_Z)$, which uniquely determines an element of $\mathrm{Hom}(I_Z/I_Z^2, \mathcal{O}_Z)$ because $I_Z = (s_1, s_2)$.

Supposing some nonzero $\phi \in T_V G$ satisfies $di(\phi) = 0$, at least one of $t_1, t_2 \in H^0(X,L)/\mathbb{C}s_1 \oplus \mathbb{C}s_2$ is nonzero and satisfies $t_i\big|_Z = 0$. So Z is contained in the codimension 3 linear space $\{x \in \mathbb{P}^g \,\big|\, s_1(x) = s_2(x) = t_i(x) = 0\}$. But then the argument of the first paragraph applies to show i is constant. Contradiction. $\square$

Consider the Hilbert-Chow morphism $HC\colon X^{[2d]} \to X^{(2d)}$. Let $\Delta \subset X^{(2d)}$ be the small diagonal of effective zero cycles of the form $2d[x]$ for some $x \in X$. Define a subscheme $P_{2d} \subset X^{[2d]}$ as the scheme-theoretic fiber $P_{2d} := HC^{-1}(\Delta)$. Let

$$\mathrm{supp}\colon P_{2d} \to X$$

be the support morphism, sending a scheme to the point at which it is supported. Finally, let $i(G) \subset X^{[2d]}$ denote the image of the Grassannian $G = \mathrm{Gr}(g-1, g+1)$ under the morphism i, endowed with its natural structure of a reduced, smooth subscheme. Finally, we may now describe the flex divisor, at least generically.

Definition 2.8. The *flex divisor* on a K3 surface $(X, L) \in T$ is the algebraic cycle

$$R_{\mathrm{flex}} := \mathrm{supp}_*[P_{2d} \cap i(G)].$$

Here supp_* denotes the proper pushforward of algebraic cycles, and the brackets $[\cdot]$ denote the effective algebraic cycle underlying a subscheme.

Note that the cycle class is being taken in P_{2d} to make supp_* sensical.

Lemma 2.9. *The subschemes P_{2d} and $i(G)$ intersect properly in $X^{[2d]}$, i.e. their intersection has pure dimension* 1*. Furthermore,* $[P_{2d}] \cdot [i(G)] = [P_{2d} \cap i(G)]_{X^{[2d]}}$.

Proof. We have that $i(G) \subset X^{[2d]}$ is a smooth subscheme of dimension $2d$. By a result of Haiman [Hai98, Prop. 2.10], $P_{2d} \subset X^{[2d]}$ is a reduced and irreducible Cohen-Macaulay scheme of dimension $2d+1$. Hence, each component of their intersection has dimension at least 1. We claim additionally that each component has dimension at most 1. Note $\mathrm{supp}(P_{2d} \cap i(G)) \subsetneq X$ by Lemma 2.5.

The restriction of supp to $P_{2d} \cap i(G)$ contracts no curves becuase no flex point $x \in X$ has a curve-worth of flex spaces: If $x \in X$ supported a curve-worth of flex spaces, the morphism $HC \circ i : G \to X^{(2d)}$ would contract a curve and hence, as before, G would collapse to a point in $X^{(2d)}$. This is absurd. So $\mathrm{supp}|_{P_{2d}\cap i(G)}$ is finite onto its image in X, which has dimension at most 1.

Hence, each component of $P_{2d} \cap i(G)$ has dimension exactly 1, that is, P_{2d} and $i(G)$ intersect properly. Since $i(G)$ is smooth and P_{2d} is Cohen-Macaulay, [Ful16, Prop. 7.1] gives the second statement. $\square$

Remark 2.10. The proof of Lemma 2.9 implies that every component of the scheme $P_{2d} \cap i(G)$ contributes nontrivially to R_{flex}. Hence, for $(X, L) \in T$, R_{flex} is, *as a set*, exactly the set of flex points.

Proposition 2.11. *Let $u\colon \mathfrak{X} \to T$ be the restriction of the universal family of polarized K3 surfaces. Then the flex divisors $\mathfrak{R}_{\text{flex}} \subset \mathfrak{X}$ form a flat subfamily of curves, specializing to R_{flex} on any fiber $X = \mathfrak{X}_t$.*

Proof. It suffices to relativize the construction of R_{flex} and check flatness of the resulting family of algebraic cycles.

Let $\mathfrak{G}$ be the relative Grassmannian of codimension 2 linear subspaces of $\mathbb{P}(u_*\mathfrak{L})^*$ where $\mathfrak{L} \to \mathfrak{X}$ is the universal polarization. Let $\mathfrak{X}^{[2d]}$ be the relative Hilbert scheme of $2d$ points, and let $\mathfrak{P}_{2d}$ be the subfamily of the relative Hilbert scheme consisting of schemes supported at a single point of the fiber and $\mathfrak{i}$ the relative inclusion $\mathfrak{G} \hookrightarrow \mathfrak{X}^{[2d]}$. Let $\mathfrak{supp} : \mathfrak{P}_{2d} \to \mathfrak{X}$ be the relative support morphism. Consider the algebraic cycle

$$\mathfrak{R}_{\text{flex}} := \mathfrak{supp}_*[\mathfrak{P}_{2d} \cap \mathfrak{i}(\mathfrak{G})] \subset \mathfrak{X}.$$

This cycle is a divisor in the smooth DM stack $\mathfrak{X}$. Any fiber $X = \mathfrak{X}_t \hookrightarrow \mathfrak{X}$ intersects $\mathfrak{R}_{\text{flex}}$ properly by Lemma 2.9. Hence $\mathfrak{R}_{\text{flex}}$ forms a flat family of divisors in $\mathfrak{X}$.

It remains to show that $\mathfrak{R}_{\text{flex}}$ specializes to R_{flex} as defined above on a fiber $X = \mathfrak{X}_t$. The pushforward $\mathfrak{supp}_*$ of algebraic cycles and the cycle class map $[\cdot]$ commute with taking fibers over t because the fibers $X^{[2d]} \hookrightarrow \mathfrak{X}^{[2d]}$ are smoothly immersed and properly intersecting the cycles $\mathfrak{G}$ and $\mathfrak{P}_{2d}$. Hence, $(\mathfrak{R}_{\text{flex}})_t = R_{\text{flex}}$. $\square$

Question 2.12. *For a sufficiently generic $(X, L) \in F_{2d}$ is R_{flex} an irreducible divisor? What is its geometric genus, generically?*

Remark 2.13. Based off [Wel81], Huybrechts [Huy14, Prop. 8.8] shows that when $L^2 = 4$, $R_{\text{flex}} \in |20L|$ is generically irreducible of geometric genus 201. Strangely, this is the genus of a smooth element of $|10L|$. This is not an error: R_{flex} is generically singular for a quartic surface.

We recall now the notion of a constant cycle curve:

Definition 2.14. Let X be a smooth K3 surface, and let $R \subset X$ be a curve. We say that R is a *constant cycle curve* if every point $p \in R$ represents the same class in $\mathrm{CH}_0(X)$. This definition extends to curves $R \subset X$ in an ADE K3 surface by taking the inverse image of R in the minimal resolution of X.

It is known that if R is constant cycle, then $[p] = c_X \in \mathrm{CH}_0(X)$ for all $p \in R$.

Lemma 2.15. *For $(X, L) \in T$, the divisor R_{flex} is a constant cycle curve.*

Proof. This follows immediately from Remark 2.10 and items (1), (2), (3) in the proof of Lemma 2.5. $\square$

Lemma 2.16. *Let $\mathcal{X} \to (C, 0)$ be a family of polarized K3 surfaces and let $\mathcal{R} \subset \mathcal{X}$ be a flat family of curves over C. Suppose that $\mathcal{R}_t$ is a constant cycle curve for all $t \neq 0$. Then $\mathcal{R}_0 \subset \mathcal{X}_0$ is also a constant cycle curve.*

Proof. Replacing $\mathcal{X}$ with a finite base change, there is a simultaneous resolution of singularities which is the minimal resolution on any fiber. So we may assume $\mathcal{X} \to (C, 0)$ is smooth. Any two points $p, q \in \mathcal{R}_0$ can be realized as specializations of points over a finite extension of $\mathbb{C}(C)$. The lemma follows because the specializations of rationally equivalent cycles are rationally equivalent [Ful16, Cor. 20.3]. $\square$

Theorem 2.17. *Let $u\colon \mathfrak{X} \to F_{2d}$ be the universal K3 surface, $T \subset F_{2d}$ a Zariski open subset, and let $\mathfrak{R}^* \subset \mathfrak{X}^* := \mathfrak{X}\big|_T$ be a flat family of divisors, which is a constant cycle curve $R = \mathfrak{R}_t$ on every fiber $X = \mathfrak{X}_t$. Then $\mathfrak{R}^*$ extends to a flat family of divisors $\mathfrak{R}$ over the universal K3 surface $\mathfrak{X} \to F_{2d}$.*

Proof. Let $\mathfrak{L}$ be an extension of $\mathcal{O}_{\mathfrak{X}^*}(\mathfrak{R}^*)$ to $\mathfrak{X}$ and define the projective bundle $\mathbb{P}(u_*\mathfrak{L}) \to F_{2d}$. By assumption, we have a section of $\mathbb{P}(u_*\mathfrak{L})$ over the open subset T defined by $\mathfrak{R}^*$. Let $0 \in F_{2d} \setminus T$. Given any arc $(C, 0) \subset F_{2d}$ with $C \setminus \{0\} \subset T$, there is a unique flat family of curves $\mathcal{R} \subset \mathfrak{X}\big|_C$ extending $\mathfrak{R}^*\big|_{C\setminus\{0\}}$.

By Lemma 2.16, the central fiber $\mathcal{R}_0$ is constant cycle. As noted in [Huy14, Sec. 2.3], Mumford's theorem [Mum69] implies constant cycle curves are rigid. So the flat limit $\mathcal{R}_0$ doesn't deform as the arc $(C, 0)$ deforms. Since F_{2d} is smooth, in particular normal, we conclude by a well-known argument [AET19, Lem. 3.16] that the section of $\mathbb{P}(u_*\mathfrak{L})$ over T extends, as a morphism, over 0. The result follows. $\square$

Corollary 2.18. *$\mathfrak{R}_{\text{flex}}$ extends to a flat family of divisors in the universal K3 surface over F_{2d}.*

3. Degree of the Flex Divisor

In this section, we compute the degree of the flex divisor. We follow [EG00] as a general reference on the cohomology of Hilbert schemes.

Definition 3.1. Let $n > 0$ be a positive integer and let $\alpha \in H^*(X)$ be a cohomology class of pure degree. Define

$$\mathbb{L} := \bigoplus_{m,\, k \geq 0} H^m(X^{[k]}).$$

The *Nakajima (raising) operator* $q_{-n}(\alpha) : \mathbb{L} \to \mathbb{L}$ is defined by the following correspondence: Let $a \geq 0$ and define $b := a + n$. Let $X^{[a,b]}$ be the incidence correspondence of nested pairs of zero-dimensional subschemes $Z_1 \subset Z_2 \subset X$ for which $\operatorname{len} Z_1 = a$ and $\operatorname{len} Z_2 = b$, and let π_a and π_b be the projections to $X^{[a]}$ and $X^{[b]}$. Let S be the residual support morphism $X^{[a,b]} \to X^{(n)}$ sending

$$S\colon (Z_1, Z_2) \mapsto \operatorname{supp}(Z_2) - \operatorname{supp}(Z_1)$$

and let $W_{a,b} \subset S^{-1}(\Delta)$ be the irreducible component of $S^{-1}(\Delta)$ which is the Zariski closure of the $Z_1 \subset Z_2$ for which $\operatorname{supp}(Z_1)$ and $\operatorname{supp}(Z_2) - \operatorname{supp}(Z_1)$ are disjoint. Let $s : W_{a,b} \to \Delta \cong X$ denote the restriction of S and let $t : W_{a,b} \to X^{[a,b]}$ be the inclusion. Then for any $c \in H^r(X^{[a]})$ we define

$$q_{-n}(\alpha)(c) := (\pi_b)_*(\pi_a^* c \cdot t_* s^* \alpha) \in H^{r+2n-2+\deg \alpha}(X^{[b]}).$$

By definition, we declare $H^*(X^{[0]}) = \mathbb{C}\mathbf{1}$ where $\mathbf{1}$ is called the *vacuum element.*

The bidegree of the operator $q_{-n}(\alpha)$ is $(2n - 2 + \deg \alpha, n)$, where the first degree is cohomological degree, and the second is number of points.

Remark 3.2. Definition 3.1 can be intuitively rephrased as follows: The operator $q_{-n}(\alpha)$ takes a family of subschemes of length a (i.e. a cycle in $X^{[a]}$) and tacks on a subscheme of length n supported at a single point lying on the cycle α.

Theorem 3.3 (Nakajima [Nak97], Grojnowski [Gro96])**.** *Let $\{e_i\}_{i=1}^{24}$ be a basis of $H^*(X)$. Then $q_{-n_1}(e_{i_1})\cdots q_{-n_k}(e_{i_k})\mathbf{1}$ (up to reordering operators) are a basis of $\mathbb{L}$.*

More precisely, these Nakajima operators extend to an action of the Heisenberg algebra of $H^*(X)$ on $\mathbb{L}$, which becomes identified with the bosonic Fock space.

Remark 3.4. It follows directly from the definition of the Nakajima operators that $[P_{2d}] = q_{-2d}(1)\mathbf{1}$. Similarly, the schemes supported on a single point of a hyperplane section $H \subset X$ have class $q_{-2d}(h)\mathbf{1}$, with $[H] = h \in H^2(X)$.

Lemma 3.5. *The degree of the flex divisor is* $\deg(i^* q_{-2d}(h)\mathbf{1})$.

Proof. By push-pull formula,

$$\begin{aligned}\deg(R_{\text{flex}}) = R_{\text{flex}} \cdot_X H :&= \operatorname{supp}_*[P_{2d} \cap i(G)] \cdot_X H = [P_{2d} \cap i(G)] \cdot_{P_{2d}} \operatorname{supp}^* H \\ &= i(G) \cdot_{X^{[2d]}} q_{-2d}(h)\mathbf{1} = \deg(i^* q_{-2d}(h)\mathbf{1}).\end{aligned}$$

□

Let σ_i for $i = 1, 2$ denote the Schubert classes in $H^{2i}(G)$ consisting of linear spaces meeting a line and a point in $\mathbb{P}^g$, respectively.

Proposition 3.6. *The degree of the flex divisor is $\sigma_1 \cdot i^* q_{-2d}(1)\mathbf{1}$.*

Proof. The first step is to verify the intersection product

$$q_{-1}(h)q_{-1}(1)^{2d-1}\mathbf{1} \cdot q_{-2d}(1)\mathbf{1} = 2d q_{-2d}(h)\mathbf{1}$$

on $X^{[2d]}$ and the second step is to verify that $i^*(q_{-1}(h)q_{-1}(1)^{2d-1}\mathbf{1}) = 2d\sigma_1$. Then we can apply Lemma 3.5. The first step is set-theoretically clear; the intersection multiplicity $2d$ follows quickly from the description of the ring structure on $H^*(X^{[2d]})$ due to Lehn and Sorger [LS03, Thm. 1.1 and Prop. 2.13].

To verify the second step, note that $q_{-1}(h)q_{-1}(1)^{2d-1}\mathbf{1}$ is represented by the divisor $D_H \subset X^{[2d]}$ of schemes whose support intersects $H \subset X$. Thus $[i^{-1}(D_H)]$ represents $i^*(q_{-1}(h)q_{-1}(1)^{2d-1}\mathbf{1})$. But $i^{-1}(D_H)$ is simply the locus of codimension 2 linear spaces passing through some point of H. Since $[H]_{\mathbb{P}^g} = 2d\ell$ where ℓ is the line class in $\mathbb{P}^g$, we conclude that $[i^{-1}(D_H)] = 2d\sigma_1$. □

Let $\mathcal{Z} \subset X^{[2d]} \times X$ denote the universal subscheme of length $2d$. Let $\mathcal{Z}_G \subset G \times X$ denote the restriction of this subscheme to G (along the inclusion i). Let $\pi_{X^{[2d]}}$ and π denote the projections from $X^{[2d]} \times X$ and $G \times X$ to the first factor, respectively. The *tautological bundle* $\mathcal{O}^{[2d]} \to X^{[2d]}$ is the pushforward $(\pi_{X^{[2d]}})_* \mathcal{O}_{\mathcal{Z}}$ and is a vector bundle of rank $2d$ on $X^{[2d]}$. Let

$$\mathcal{O}_G^{[2d]} := i^* \mathcal{O}^{[2d]} = \pi_* \mathcal{O}_{\mathcal{Z}_G}$$

denote the restriction of this vector bundle to the Grassmannian G.

Proposition 3.7. *We have $i^* q_{-2d}(1) = -2d c_{2d-1}(\mathcal{O}_G^{[2d]})$.*

Proof. Applying [EG00, Thm. 12.4] to the line bundle $\mathcal{O}$ gives the formula

$$\sum_n c(\mathcal{O}^{[n]}) = \exp\left(\textstyle\sum_{m\geq 1} \frac{(-1)^{m-1}}{m} q_{-m}(c(\mathcal{O}))\right).$$

Note $c(\mathcal{O})=1$ and that $q_{-m}(1)$ has bidegree $(2m-2,m)$. So the only term on the right-hand side landing in $H^{2n-2}(X^{[n]})$ is $(-1)^{n-1}n^{-1}q_{-n}(1)$. We conclude

$$i^*q_{-2d}(1) = -2di^*c_{2d-1}(\mathcal{O}^{[2d]}) = -2dc_{2d-1}(\mathcal{O}_G^{[2d]})$$

which follows via commutativity of taking Chern classes with pullback. □

Let Q denote the rank 2 universal quotient bundle on G. To compute the Chern class $c_{2d-1}(\mathcal{O}_G^{[2d]})$ we make use of the following exact sequence:

Proposition 3.8. *There is a resolution of $\mathcal{O}_{\mathcal{Z}_G}$ by vector bundles on $G\times X$:*

$$0\to \det(Q^*)\boxtimes(-2L)\to Q^*\boxtimes(-L)\to\mathcal{O}\to\mathcal{O}_{\mathcal{Z}_G}\to 0.$$

Proof. This exact sequence is simply the global version of the Koszul resolution of $\mathcal{O}_{X\cap\mathbb{P}V}$ where $\mathbb{P}V=\{x\in\mathbb{P}^g \,\big|\, s_1(x)=s_2(x)=0\}$ is a codimension 2 linear space:

$$0\to(s_1s_2)\to(s_1)\oplus(s_2)\to\mathcal{O}_X\to\mathcal{O}_{X\cap\mathbb{P}V}\to 0.$$

On a given fiber of π the restrictions of $\det(Q^*)\boxtimes(-2L)$ and $Q^*\boxtimes(-L)$ are (s_1s_2) and $(s_1)\oplus(s_2)$ respectively, because $Q^*=\mathbb{C}s_1\oplus\mathbb{C}s_2$. □

Let r_1 and r_2 denote the Chern roots of Q.

Proposition 3.9. $\mathrm{ch}(\mathcal{O}_G^{[2d]}) = 2-(d+2)e^{-r_1}-(d+2)e^{-r_2}+(4d+2)e^{-r_1-r_2}$.

Proof. Consider the (derived) pushforward $R\pi_*$ of the exact sequence of Proposition 3.8. Computing the derived pushforward sheaves of each term gives

$$R^i\pi_*\mathcal{O}_{\mathcal{Z}_G} = \begin{cases}\mathcal{O}_G^{[2d]} & \text{if } i=0\\ 0 & \text{if } i>0\end{cases}\qquad R^i\pi_*\mathcal{O} = \begin{cases}\mathcal{O} & \text{if } i=0,2\\ 0 & \text{if } i=1\end{cases}$$

$$R^i\pi_*(Q^*\boxtimes(-L)) = \begin{cases}0 & \text{if } i=0,1\\ Q^*\otimes H^0(X,L)^* & \text{if } i=2\end{cases}$$

$$R^i\pi_*(\det(Q^*)\boxtimes(-2L)) = \begin{cases}0 & \text{if } i=0,1\\ \det(Q^*)\otimes H^0(X,2L)^* & \text{if } i=2.\end{cases}$$

The first equation follows from the definition of $\mathcal{O}_G^{[2d]}$ and that $\mathcal{Z}_G$ is finite over G, and the last three equations all follows from relative Serre duality applied to π. From these computations, and the fact that $h^0(X,L)=d+2$ and $h^0(X,2L)=4d+2$, we get the following equality in the K-group of G:

$$[\mathcal{O}_G^{[2d]}]-2[\mathcal{O}]+(d+2)[Q^*]-(4d+2)[\det(Q^*)]=0.$$

Since the Chern character ch is a homomorphism from K-theory to cohomology, the proposition follows from the equalities $\mathrm{ch}(\mathcal{O})=1$, $\mathrm{ch}(Q^*)=e^{-r_1}+e^{-r_2}$, $\mathrm{ch}(\det(Q^*))=e^{-r_1-r_2}$. □

Corollary 3.10. *The total Chern character of $\mathcal{O}_G^{[2d]}$ is*

$$c(\mathcal{O}_G^{[2d]}) = \frac{(1-r_1-r_2)^{4d+2}}{(1-r_1)^{d+2}(1-r_2)^{d+2}} = \frac{(1-\sigma_1)^{4d+2}}{(1-\sigma_1+\sigma_2)^{d+2}}.$$

Proof. Since $\mathcal{O}_G^{[2d]}$ is a vector bundle, we can compute the total Chern character using the splitting principle and the set of "virtual Chern roots"

$$\{\underbrace{-r_1-r_2}_{4d+2},0,0\}-\{\underbrace{-r_1}_{d+2},\underbrace{-r_2}_{d+2}\}.$$

The theorem then follows from the equalities $r_1+r_2=\sigma_1$ and $r_1r_2=\sigma_2$. □

Remark 3.11. Let $X \subset \mathbb{P}^g$ be Cohen-Macaulary of degree d and codimension r. If X intersects any r-plane in $\mathbb{P}^g$ properly, there is a map $\mathrm{Gr}(r+1, g+1) \to X^{[d]}$ which sends an r-plane to its intersection with X. There is a rank d tautological vector bundle $\mathcal{O}^{[d]} \to X^{[d]}$ and the Chern classes of its pullback to $\mathrm{Gr}(r+1, g+1)$ can be computed in the same manner as above, via the Koszul resolution.

Theorem 3.12. *Let $X \subset \mathbb{P}^g$ be a smooth K3 surface embedded by a primitive ample line bundle L of square $L^2 = 2d = 2g-2$, for which no pencil in $|L|$ has a base curve. Then, the flex divisor satisfies $R_{\mathrm{flex}} \in |n_d L|$ where*

$$n_d = \frac{(2d)!(2d+1)!}{d!^2(d+1)!^2}.$$

Proof. By Propositions 3.6 and 3.7, we have the formula

$$n_d = -\sigma_1 \cdot c_{2d-1}(\mathcal{O}_G^{[2d]}).$$

From the formula of Corollary 3.10 for $c(\mathcal{O}_G^{[2d]})$, plus the fact that the minus signs cancel in any contribution to top degree, we conclude

$$n_d = \left[\sigma_1 \cdot \frac{(1+\sigma_1)^{4d+2}}{(1+\sigma_1+\sigma_2)^{d+2}}\right]_{\mathrm{top}}.$$

The Pieri and Giambelli formulae imply that

$$\sigma_1^m \cdot \sigma_2^n = \frac{m!}{(m/2)!(m/2+1)!}$$

when $m + 2n = 2d$ add up to the correct dimension to give a top class on G. After performing binomial expansion in σ_1 then σ_2, collecting terms of top degree, and plugging in the above formula, we get the ugly expression

$$n_d = \sum_{j=0}^{d} \sum_{\ell=1}^{d-j} (-1)^{j+1} \binom{4d+2}{j} \binom{3d-j}{2d+\ell} \binom{2d+\ell}{2\ell-1} \binom{2\ell}{\ell} \frac{1}{\ell+1}.$$

Applying automated choose identity verification gives the result. □

Example 3.13. Let (X, L) be any ADE K3 surface of degree $L^2 = 2$. The linear system $|L|$ defines a $2:1$ morphism from X onto either $\mathbb{P}^2$ or $\mathbb{F}_4^0$ and R_{flex} is naturally the ramification divisor of this map. The double cover of $\mathbb{P}^2$ is branched in a sextic B. One has $R_{\mathrm{flex}}^2 = B^2/2 = 18 = (3L)^2$, so $n_1 = 3$.

Example 3.14. For a quartic surface, one can compute the flex divisor directly from the definition. Here are some results:

The Fermat quartic $X = V(x_0^4 + x_1^4 + x_2^4 + x_3^4) \subset \mathbb{P}^3$ contains 48 lines. Each line appears with multiplicity one in R_{flex}. The intersections of X with the coordinate hyperplanes $x_i = 0$ appear with multiplicity 2 in R_{flex}. So R_{flex} is cut out by $(x_0^4 + x_1^4)(x_0^4 + x_2^4)(x_0^4 + x_3^4)x_0^2x_1^2x_2^2x_3^2$.

The maximal number of 64 lines on a smooth quartic surface is realized by the Schur quartic $X = V(x_0^4 - x_0x_1^3 + x_2x_3^3 - x_3^4) \subset \mathbb{P}^3$. These lines come in two types. The first type, of which there are 16, are the lines joining the $4+4$ points lying on the skew lines $V(x_0, x_1)$, $V(x_2, x_3)$. They appear in R_{flex} with multiplicity two, while the remaining 48 lines of the second type appear with multiplicity one. So R_{flex} consists only of lines. Thus X has no "flex points" in the naive sense.

Remark 3.15. Based on the $d = 1$ case, the authors hoped that R_{flex} would be a canonical choice of polarizing divisor living in a reasonably small multiple of the polarization class, at least compared to the rational curve divisor R_{rc}. But in fact, the formula of Theorem 3.12 grows *significantly faster* than the Yau-Zaslow formula, with the switch occurring between $d = 8$ and $d = 9$. Asymptotically, $n_d \sim 2^{4d+1}/\pi d^2$ while Yau-Zaslow$_d \sim e^{4\pi\sqrt{d}}/\sqrt{2}d^{27/4}$.

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Email address: `valery@uga.edu`

Department of Mathematics, University of Georgia, Athens GA 30602, USA

Email address: `philip.engel@uga.edu`

Department of Mathematics, University of Georgia, Athens GA 30602, USA