



Fractional integrable and related discrete nonlinear Schrödinger equations



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ABSTRACT

Integrable fractional equations such as the fractional Korteweg-deVries and nonlinear Schrödinger equations are key to the intersection of nonlinear dynamics and fractional calculus. In this manuscript, the first discrete/differential difference equation of this type is found, the fractional integrable discrete nonlinear Schrödinger equation. This equation is linearized; special soliton solutions are found whose peak velocities exhibit more complicated behavior than other previously obtained fractional integrable equations. This equation is compared with the closely related fractional *averaged* discrete nonlinear Schrödinger equation which has simpler structure than the integrable case. For positive fractional parameter and small amplitude waves, the soliton solutions of the integrable and averaged equations have similar behavior.

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1. Introduction

Integrable systems play a central role in nonlinear dynamics because they provide exactly solvable models for important physical systems. Notable examples of integrable equations are the Korteweg-deVries (KdV), applicable to shallow water waves, plasma physics, and lattice dynamics among others [1–3], and the nonlinear Schrödinger (NLS) equation, which finds applications in nonlinear optics, Bose-Einstein condensates, spin waves in ferromagnetic films, plasma physics, water waves, etc. [2–5]. These integrable nonlinear evolution equations have an infinite number of conservation laws and soliton solutions [6]. Solitons, the fundamental solutions of such equations, are stable, localized nonlinear waves which propagate without dispersing and interact elastically with other solitons. Nonlinear integrable evolution equations have these surprising properties because of their deep mathematical structure described by the inverse scattering transform (IST).

IST is a method of solving nonlinear equations which generalizes Fourier transforms. It solves these equations in three steps: mapping the initial condition into scattering space, evolving the initial data in scattering space in time, and mapping the evolved

scattering data back to physical space; i.e., inverse scattering. This process gives the solution to nonlinear equations solvable by IST in terms of linear integral equations; such nonlinear equations are called integrable. Recently, we used the mathematical structure of IST associated with the Korteweg-deVries (KdV) and nonlinear Schrödinger (NLS) equations to develop a method of finding and solving the fractional KdV (fKdV) and fractional NLS (fNLS) equations [7]. We also showed that this method could be applied to find fractional extensions of the modified KdV, sine-Gordon, and sinh-Gordon equations [8]. These equations represent the first known fractional integrable nonlinear evolution equations with smooth (physical) solutions and deeply connect the fields of nonlinear dynamics and fractional calculus.

Fractional calculus is a mathematical structure originally designed to define non-integer derivatives and integrals. It has since become an effective way of modeling many physical processes that exist in multi-scale media [9,10] or exhibit non-Gaussian statistics or power law behavior [11–13]. A particularly important example is anomalous diffusion, where the mean squared displacement is proportional to t^α , $\alpha > 0$ [11,14–16]. Transport that follows this rule has been observed in biology [17–20], amorphous materials [21–23], porous media [24–27], climate science [28], and attenuation in materials [29] amongst others. As we have shown, the merger of fractional and nonlinear characteristics in integrable equations such as fKdV and fNLS predict *anomalous dispersion*,

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where the velocity and amplitude of solitonic solutions are related by a power law [7].

In this article, we demonstrate how the method introduced in Ref. [7] can be applied to discrete (or differential-difference) systems to define integrable discrete fractional nonlinear evolution equations by presenting a fractional generalization of the integrable discrete nonlinear Schrödinger (IDNLS) equation. We do this by demonstrating the three key mathematical ingredients of our method – IST, power law dispersion relations, and completeness relations – for the Ablowitz-Ladik (AL) discrete scattering problem. In the linear limit, the fractional IDNLS (fIDNLS) equation is a discretization of the fractional Schrödinger equation which was derived with Feynman path integrals over Lévy flights [30,31].

The KdV equation was the first equation shown to be solvable by IST in Ref. [32]; it was soon followed by the NLS equation in Ref. [33]. These two equations were then found to be contained in a general class of equations solvable by IST when associated to the Ablowitz-Kaup-Newell-Segur (AKNS) system [3,34]. Shortly thereafter IST was used to solve families of discrete (or differential difference) problems like the self-dual network [35]. In particular, it was discovered that the AKNS system could be discretized while maintaining integrability, leading to the AL scattering problem which was used to solve a family of discrete nonlinear evolution equations [36]. This family contained important discrete evolution equations – continuous in time but discretized in space – such as integrable discretizations of the nonlinear Schrödinger, KdV, modified KdV, and sine Gordon equations. Further, this family of equations was shown to have soliton solutions and an infinite number of conservation laws [36].

We derive the fIDNLS equation from the AL scattering problem using three key components: linear dispersion relations, completeness relations, and IST. IST is used to linearize the equation and obtain special soliton solutions.

We also show how the characteristics of the fractional IDNLS (fIDNLS) equation reach beyond integrability by comparing the one-soliton solution of the fIDNLS equation to the solitary wave solution of the fractional averaged discrete nonlinear Schrödinger (fADNLS) equation. This equation is a different fractional generalization of the IDNLS equation in which the linear second order difference is replaced by the discrete fractional Laplacian [37–40]. The fADNLS equation can be understood as a discretization of a fractional NLS equation involving the Riesz derivative which has been extensively studied in, e.g., [41–45]; it is also closely related to the (likely) non-integrable fractional DNLS equation, recently studied in [37,46]. Though the fADNLS equation is likely not integrable to our knowledge (apart from the limiting case when fADNLS reduces to IDNLS), the similarity between the two equations suggests that some of the physical predictions of fractional integrable equations are shared by equations which are simpler to realize computationally.

2. The discrete fractional linear Schrödinger equation

Consider the family of discrete linear evolution equations

$$\partial_t q_n + \gamma(-\Delta_n)q_n = 0 \quad (1)$$

for the function $q_n(t)$ which depends on the discrete variable $n \in \mathbb{Z}$ and the continuous variable $t \in \mathbb{R}$. Here, γ is a sufficiently regular function of the discrete laplacian, $-\Delta_n$, defined by

$$(-\Delta_n)q_n(t) = \frac{1}{h^2}(-q_{n+1}(t) + 2q_n(t) - q_{n-1}(t)) \quad (2)$$

where h is the distance between lattice sites. Using the Z-transform, which is equivalent to the discrete Fourier transform, the solution to Eq. (1) can be explicitly written as

$$q_n(t) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} dk \hat{q}(k, 0) e^{iknh - \gamma(4 \sin^2(kh/2)/h^2)t} \quad (3)$$

where $\hat{q}(k, 0) = h \sum_{n=-\infty}^{\infty} q_n(0) e^{-iknh}$ is the Z-transform of $q_n(t)$ at $t = 0$ and $4 \sin^2(kh/2)/h^2$ is the Fourier symbol of $-\Delta_n$. Note that the Z-transform is often written in terms of z with the substitution $z = e^{-ikh}$ where integration in k becomes integration with respect to z on the unit circle. If we choose γ to be power law, then Eq. (1) becomes a fractional discrete equation in terms of the discrete fractional laplacian. For example, if we put $\gamma(-\Delta_n) = -i(-\Delta_n)^{1+\epsilon}$, $|\epsilon| < 1$, then we obtain the linear fractional discrete Schrödinger equation

$$i\partial_t q_n + (-\Delta_n)^{1+\epsilon} q_n = 0. \quad (4)$$

Here, $(-\Delta_n)^{1+\epsilon}$ is the discrete fractional laplacian of order $1 + \epsilon$ which is defined in terms of its Fourier symbol $[4 \sin^2(kh/2)/h^2]^{1+\epsilon}$ and the Z-transform/discrete Fourier transform as

$$(-\Delta_n)^{1+\epsilon} q_n = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} dk \hat{q}(k) e^{iknh} [4 \sin^2(kh/2)/h^2]^{1+\epsilon}. \quad (5)$$

Notice that the k integral above can be evaluated to express the discrete fractional laplacian as a summation over m of q_m multiplied by a weight vector. The solution to Eq. (4) can still be written in the form Eq. (3) with

$$\gamma(4 \sin^2(kh/2)/h^2) = -i[4 \sin^2(kh/2)/h^2]^{1+\epsilon}$$

and, because $4 \sin^2(kh/2)/h^2$ is real and positive, the solution to equation (1) with this choice of γ is well posed. In defining and solving the linear fractional discrete Schrödinger equation, we used a power law dispersion relation, ingredient 1 of our method, and we defined the fractional operator using completeness of the discrete Fourier transform/Z-transform, ingredient 2. Then we solve the equation by the inverse discrete Fourier transform, the analog of ingredient 3.

3. The fractional integrable discrete Schrödinger equation

To develop the fIDNLS equation, the integrable nonlinear analog of Eq. (4), and solve it, we apply the three key ingredients of our method, starting with writing the equation in terms of a linear dispersion relation. Note that $h = 1$ is taken in this section without loss of generality; to recover the scaling factor for $h \neq 1$, replace q_n by hq_n and r_n by hr_n .

As in the linear case, Eq. (1), we have a family of nonlinear evolution equations for the solutions $q_n(t)$ and $r_n(t)$ [47], see also [48],

$$\sigma_3 \frac{d\mathbf{u}_n}{dt} + \gamma(\Lambda_+) \mathbf{u}_n = 0, \quad \mathbf{u}_n = (q_n, -r_n)^T \quad (6)$$

where T represents transpose, $\sigma_3 = \text{diag}(1, -1)$, and Λ_+ is

$$\Lambda_+ \mathbf{x}_n = h_n \begin{pmatrix} E_n^+ & 0 \\ 0 & E_n^- \end{pmatrix} \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \end{pmatrix} \quad (7)$$

$$+ \begin{pmatrix} q_n \sum_{n-1}^+ r_{k-1} & q_n \sum_{n-2}^+ q_{k+1} \\ -r_n \sum_{n-1}^+ r_{k-1} & -r_n \sum_{n-2}^+ q_{k+1} \end{pmatrix} \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \end{pmatrix} \quad (8)$$

$$+ h_n \begin{pmatrix} q_{n+1} \sum_{n+1}^+ \frac{r_k}{h_k} & q_{n+1} \sum_{n+1}^+ \frac{q_k}{h_k} \\ -r_{n-1} \sum_{n-1}^+ \frac{r_k}{h_k} & -r_{n-1} \sum_{n-1}^+ \frac{q_k}{h_k} \end{pmatrix} \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \end{pmatrix} \quad (9)$$

where $h_n = 1 - r_n q_n$, $\sum_n^+ = \sum_{k=n}^\infty$, and $E_n^\pm x_k^{(q)} = x_{n\pm 1}^{(q)}$ with $q = 1, 2$. The inverse of this operator is

$$\Lambda_+^{-1} \mathbf{x}_n = h_n \begin{pmatrix} E_n^- & 0 \\ 0 & E_n^+ \end{pmatrix} \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \end{pmatrix} \quad (10)$$

$$+ \begin{pmatrix} -q_n \sum_n^+ r_{k+1} & -q_n \sum_{n+1}^+ q_{k-1} \\ r_n \sum_{n-1}^+ r_{k+1} & r_n \sum_n^+ q_{k-1} \end{pmatrix} \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \end{pmatrix} \quad (11)$$

$$+ h_n \begin{pmatrix} -q_{n-1} \sum_n^+ \frac{r_k}{h_k} & -q_{n-1} \sum_n^+ \frac{q_k}{h_k} \\ r_{n+1} \sum_{n+1}^+ \frac{r_k}{h_k} & -r_{n+1} \sum_{n+1}^+ \frac{q_k}{h_k} \end{pmatrix} \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \end{pmatrix}. \quad (12)$$

Here, γ is a sufficiently regular function of the operator Λ_+ and is connected with the linearized dispersion relation. Specifying this dispersion relation, or γ directly, picks out particular equations from this family. For example, if we take

$$\gamma(\Lambda_+) = -i(2 - \Lambda_+ - \Lambda_+^{-1})$$

and let $r_n = \mp q_n^*$, then we obtain the IDNLS equation

$$i\partial_t q_n + \Delta_n q_n \pm |q_n|^2 (q_{n+1} + q_{n-1}) = 0. \quad (13)$$

We can relate γ to the dispersion relation of the linearization of (6) by considering the linear limit $q_n \rightarrow 0$. In this limit, we have

$$\Lambda_+ \rightarrow \begin{pmatrix} E_n^+ & 0 \\ 0 & E_n^- \end{pmatrix} \equiv \mathbf{D}_n, \quad (14)$$

so the linearization of the nonlinear evolution equation is

$$\sigma_3 \frac{d\mathbf{u}_n}{dt} + \gamma(\mathbf{D}_n) \mathbf{u}_n = 0. \quad (15)$$

Because $\mathbf{D}_n$ is a diagonal matrix, we have

$$\gamma(\mathbf{D}_n) = \begin{pmatrix} \gamma(E_n^+) & 0 \\ 0 & \gamma(E_n^-) \end{pmatrix}. \quad (16)$$

Taking the first component of (15) with

$$q_n = z^{2n} e^{-i\omega(z)t}$$

gives

$$\gamma(z^2) = i\omega(z). \quad (17)$$

Therefore, by specifying the linear limit of the nonlinear evolution equation, we obtain the nonlinear equation itself. To define the fDNLS equation, we choose the linear limit to be the discrete linear fractional Schrödinger equation in (4), which gives the dispersion relation $\omega(z) = -(2 - z^2 - z^{-2})^{1+\epsilon}$ and, hence, $\gamma(z^2) = -i(2 - z^2 - z^{-2})^{1+\epsilon}$. So, the fDNLS equation is

$$i\partial_t \mathbf{u}_n + (2 - \Lambda_+ - \Lambda_+^{-1})^{1+\epsilon} \mathbf{u}_n(t) = 0. \quad (18)$$

In fact, by choosing $\gamma(z^2) = -i(2 - z^2 - z^{-2})^{m+\epsilon}$, for integer m , we generate a hierarchy of fractional equations

$$i\partial_t \mathbf{u}_n + (2 - \Lambda_+ - \Lambda_+^{-1})^{m+\epsilon} \mathbf{u}_n(t) = 0. \quad (19)$$

It can be shown that the limit of (18) as $\epsilon \rightarrow 0$ is the IDNLS equation (13). Notice that to define the fDNLS equation, we used a power law dispersion relation, ingredient 1 of the method. However, this dispersion relation leads to the operator $(2 - \Lambda_+ - \Lambda_+^{-1})^{1+\epsilon}$ the meaning of which is currently unclear. To define this operator, we will need to use the 2nd ingredient: appropriate completeness relations. The third ingredient will be making use of IST to find solutions of the fDNLS equation.

4. Completeness of squared eigenfunctions and fractional operators

In this section we define the fDNLS equation in (18) and, in fact, any equation of the form (1) that is well-posed in physical space. We do this using the observation that $\gamma(\Lambda_+)$ is a multiplication operator when acting on the eigenfunctions of Λ_+ and the fact that the eigenfunctions of Λ_+ are complete. This result is known as completeness of squared eigenfunctions, and is the second ingredient in our method. The resulting representation of $\gamma(\Lambda_+)$ will be similar to that of the discrete fractional laplacian in (5). The eigenfunctions of Λ_+ are $\Psi_n(z)$ and $\bar{\Psi}_n(z)$ each with eigenvalue z^2 (note that time t is suppressed throughout this section). Therefore, the operation of $\gamma(\Lambda_+)$ on these eigenfunctions is given by

$$\gamma(\Lambda_+) \Psi_n = \gamma(z^2) \Psi_n, \quad \gamma(\Lambda_+) \bar{\Psi}_n = \gamma(z^2) \bar{\Psi}_n. \quad (20)$$

Because Λ_+ is not a self-adjoint operator, completeness of squared eigenfunctions involves both Ψ_n , $\bar{\Psi}_n$ and the adjoint functions Ψ_n^A , $\bar{\Psi}_n^A$ where

$$\gamma(\Lambda_+) \Psi_n^A = \gamma(z^2) \Psi_n^A, \quad \gamma(\Lambda_+) \bar{\Psi}_n^A = \gamma(z^2) \bar{\Psi}_n^A \quad (21)$$

and Λ_+^A is the adjoint, with respect to $\ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$, of Λ_+ . The eigenfunctions and adjoint eigenfunctions can be written in terms of solutions to the Ablowitz-Ladik scattering problem which is a 2×2 eigenvalue problem for the discrete vector-valued function $\mathbf{v}_n = (v_n^{(1)}, v_n^{(2)})^T$

$$\mathbf{v}_{n+1} = \begin{pmatrix} z & q_n \\ r_n & z^{-1} \end{pmatrix} \quad (22)$$

where q_n and r_n act as potentials and z is an eigenvalue. Through this association, one can solve the family of nonlinear evolution equations in (22) (see Appendix for more details).

In [47], it was shown that the arbitrary discrete function $\mathbf{H}_n = (H_n^{(1)}, H_n^{(2)})^T \in l_1$ can be written as

$$\mathbf{H}_n = \sum_{p=1}^2 \oint_{S^{(p)}} \frac{dz}{z} f^{(p)}(z) \sum_{m=-\infty}^{\infty} \mathbf{G}_{n,m}^{(p)}(z) \mathbf{H}_m \quad (23)$$

where $S^{(1)} = S_R$ ($S^{(2)} = S_\delta$) is a circular contour evaluated counterclockwise centered at the origin of radius $R > 1$ ($\delta < 1$) such that all of the singularities of the integrand are inside (outside) of the contour and

$$\mathbf{G}_{n,m}^{(1)}(z) = \Psi_n(z) \Psi_m^A(z)^T / h_n, \quad f^{(1)}(z) = \frac{i}{2\pi a^2(z)} \quad (24)$$

$$\mathbf{G}_{n,m}^{(2)}(z) = \bar{\Psi}_n(z) \bar{\Psi}_m^A(z)^T / h_n, \quad f^{(2)}(z) = \frac{-i}{2\pi \bar{a}^2(z)} \quad (25)$$

with $h_n = 1 - r_n q_n$. The eigenfunctions $\Psi_n(z)$, $\Psi_n^A(z)$, $\bar{\Psi}_n(z)$, $\bar{\Psi}_n^A(z)$ (see appendix) and scattering data $a(z)$, $\bar{a}(z)$ are defined in terms of solutions to the Ablowitz-Ladik scattering problem (see Appendix). With this completeness relation, and the operation of $\gamma(\Lambda_+)$ on Ψ_n and $\bar{\Psi}_n$ in Eq. (20), we have

$$\gamma(\Lambda_+) \mathbf{H}_n = \sum_{p=1}^2 \oint_{S^{(p)}} \frac{dz}{z} f^{(p)}(z) \gamma(z^2) \sum_{m=-\infty}^{\infty} \mathbf{G}_{n,m}^{(p)}(z) \mathbf{H}_m. \quad (26)$$

Therefore, the nonlinear evolution equation in (6) can be explicitly characterized in physical space as

$$\sigma_3 \frac{d\mathbf{u}_n}{dt} = - \sum_{p=1}^2 \oint_{S(p)} \frac{dz}{z} f^{(p)}(z) \gamma(z^2) \sum_{m=-\infty}^{\infty} \mathbf{C}_{n,m}^{(p)}(z) \mathbf{u}_m. \quad (27)$$

In particular, if we put $\gamma(z^2) = -i(2 - z^2 - z^{-2})^{1+\epsilon}$ and $r_n = \mp q_n^*$, the fDNLS equation is the first component of (27). Using the symmetries of the eigenfunctions (see appendix), this is

$$i\partial_t q_n = \sum_{p=1}^2 \oint_{S(p)} \frac{dz}{z} f^{(p)}(z) \gamma(z^2) \sum_{m=-\infty}^{\infty} g_{n,m}^{(p)}(z) \quad (28)$$

with

$$g_{n,m}^{(1)}(z) = -i \frac{v_n v_m}{h_n} \psi_n^{(1)}(z) \psi_{n+1}^{(1)}(z) \quad (29)$$

$$\cdot \left(\phi_m^{(2)}(z) \phi_{m+1}^{(2)}(z) q_m \mp \phi_m^{(1)}(z) \phi_{m+1}^{(1)}(z) q_m^* \right) \\ g_{n,m}^{(2)}(z) = -i \frac{v_n v_m}{h_n} \left(\psi_n^{(2)}(1/z^*) \psi_{n+1}^{(2)}(1/z^*) \right)^* \quad (30)$$

$$\cdot \left(\phi_m^{(1)}(1/z^*) \phi_{m+1}^{(1)}(1/z^*) q_m^* \mp \phi_m^{(2)}(1/z^*) \phi_{m+1}^{(2)}(1/z^*) q_m \right)^*$$

where t has been suppressed.

In the appendix we show how this equation can be linearized via Gel'fand-Levitan-Marchenko type summation equations. After long time the kernel of the summation equation contains only discrete spectra, i.e., the soliton solutions. Multisoliton solutions can be found by standard methods.

5. Solitons and solitary wave solutions of the fDNLS and fADNLS equations

The fDNLS equation in (18) is not the only fractional generalization of the IDNLS equation in (13). A simpler generalization is to replace the discrete laplacian $-\Delta_n$ in (13) with the discrete fractional laplacian $(-\Delta_n)^{1+\epsilon}$ defined in (5) to give the fractional averaged DNLS (fADNLS) equation

$$i\partial_t q_n + (-\Delta_n)^{1+\epsilon} q_n \pm |q_n|^2 (q_{n+1} + q_{n-1}) = 0. \quad (31)$$

Notice that in the figure captions we refer to the fDNLS equation as the fractional integrable equation and the fADNLS equation as the fractional averaged equation.

The fADNLS equation is not known to be integrable, but in the limit $\epsilon \rightarrow 0$, it becomes the IDNLS equation, Eq. (13), which is integrable; therefore, we expect Eq. (31) to have some similarity the fDNLS equation. To characterize this similarity, we will compare the solitons and solitary waves predicted by these equations. The fDNLS equation has an exact one-soliton solution, derivable by the IST. To find the solitary wave solutions to the fADNLS equation we use the same initial condition as that of the fDNLS equation.

Even though this solitary wave initially deforms from the exact secant profile, emitting radiation in the process, its solutions have nearly constant velocity, propagate with nearly constant amplitude, and have comparable velocities to the fDNLS equation in certain regimes. These integrable-like properties of this equation are stronger for positive ϵ than negative ϵ and stronger for smaller wave amplitudes than larger wave amplitudes. Soliton solutions to the fDNLS equation can be derived using the IST (see appendix and [49]); they are of the form

$$q_n(t) = \frac{\sinh(2\eta h)}{h} e^{2i(v_i(z_1^2)t - \xi h n) - i(\psi - \pi/2)} \\ \times \operatorname{sech}\left(2\eta h(n - n_0) - 2v_r(z_1^2)t\right) \quad (32)$$

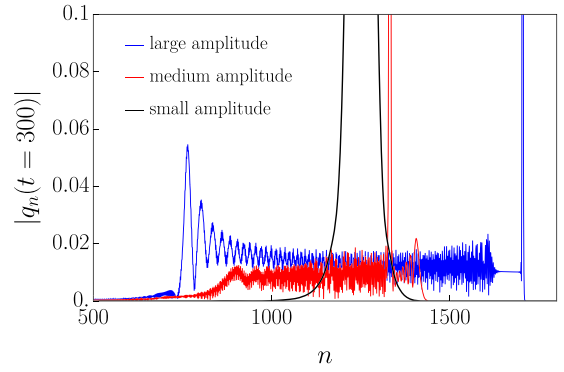


Fig. 1. Radiation emission for small, medium, and large initial data. Solitary wave solutions to the fractional averaged equation emit more radiation for larger amplitude initial conditions and larger fractional order ϵ ($\epsilon = 0.1$ is shown here). The initial amplitudes corresponding to the small, medium, and large solutions are $A = 0.100$, 1.175 , and 3.627 , respectively; however, each solitary wave solution is normalized to peak height 1. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

where $v_i(z_1^2) = \frac{1}{2}\operatorname{Im}\gamma(z_1^2)$, $v_r(z_1^2) = \frac{1}{2}\operatorname{Re}\gamma(z_1^2)$, and $z_1 = e^{h(\eta - i\xi)}$. Here we choose $\gamma(z_1^2) = -i(2 - z_1^2 - z_1^{-2})^{1+\epsilon}$ in accordance with Eq. (18) though (32) holds for all sufficiently regular γ . The free parameters in (32) are ϵ , h , η , ξ , ψ , and n_0 .

To find the localized wave solutions to the fADNLS equation, we numerically evolved the equation at discrete time steps $\{t_m\}_{m=0}^M$ with $t_0 = 0$ using a Fourier split-step scheme. The initial condition $q_n(t_0) = q_n(0)$ is given by (32) with $t = 0$. The Fourier split-step scheme propagates the approximation from t_m to t_{m+1} by separately evaluating the linear and nonlinear parts of the equation; cf. Refs. [50–52]. Explicitly, we compute

$$q_n(t_{m+1}) = e^{-i\Delta t_m \mathcal{L}/2} e^{i \int_{t_m}^{t_{m+1}} d\xi \mathcal{N}} e^{-i\Delta t_m \mathcal{L}/2} q_n(t_m) \quad (33)$$

where $\mathcal{L}q_n = (-\Delta_n)^{1+\epsilon} q_n$ and $\mathcal{N}q_n = \pm |q_n|^2 (q_{n+1} + q_{n-1})$. The particular operator splitting in equation (33) makes the solution method $\mathcal{O}(\Delta t^2)$ accurate [53,54]. The linear step, $e^{-i\Delta t_m \mathcal{L}/2}$, is evaluated using discrete Fourier transforms, while the nonlinear step, $e^{i\Delta t_m \mathcal{N}}$, is evaluated by solving the associated differential equation, equation (31) with $(-\Delta_n)^{1+\epsilon} q_n \rightarrow 0$, using a fourth-order Runge-Kutta scheme. Throughout this manuscript, solutions to the fADNLS equation were computed with the parameters $h = 1$, $\xi = 0.5$, and $\psi = \pi/2$ and with $N = 2,000$ grid points and time discretization $\Delta t = 0.01$.

The fADNLS equation initialized with the soliton solution to fDNLS, i.e., putting $t = 0$ into Eq. (32), leads to radiation emission for non-zero ϵ . Fig. 1 shows this radiation for small ($\eta = 0.05$), medium ($\eta = 0.5$), and large ($\eta = 1$) amplitude initial conditions at simulation time $T = 300$ with $\epsilon = 0.1$. Recall that amplitude is related to the parameters η and h (h is taken to be 1) by $A = \sinh(2\eta h)/h$. The heights of the three solutions are normalized to 1 to compare the relative amount of radiation; the radiation increases with increasing amplitude, with the large amplitude solution having radiation about 2% of the height of the solution, the medium amplitude having 1%, and the small amplitude having negligible radiation.

The positions of the peaks of the fADNLS equation (solid lines) are given along with linear fits (dashed lines) in Fig. 2 for medium amplitude initial conditions and $\epsilon = -0.25, 0.0, +0.25$. The linear fit shows that the positive ϵ solution propagates with nearly constant velocity, while the negative ϵ one has quadratic character which causes it to slow down over time. The amplitudes of these localized wave solutions have breathing patterns. Fig. 3 shows that when we average over these oscillations, the amplitude settles down to a constant for $\epsilon = 0.25$ after deformation from the

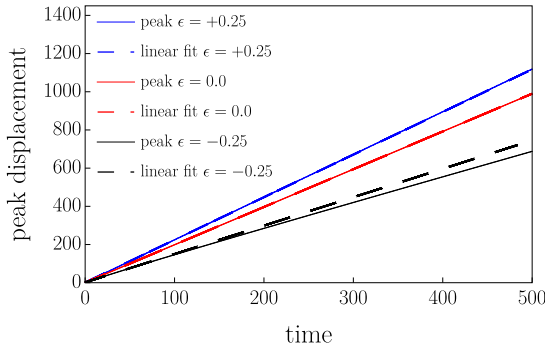


Fig. 2. Linearity of solitary wave peak displacement. Medium amplitude solitary wave solutions of the fractional averaged equation have a nearly linear relationship between displacement and time for positive and zero ϵ . For negative ϵ , the solitary wave slows down over time.

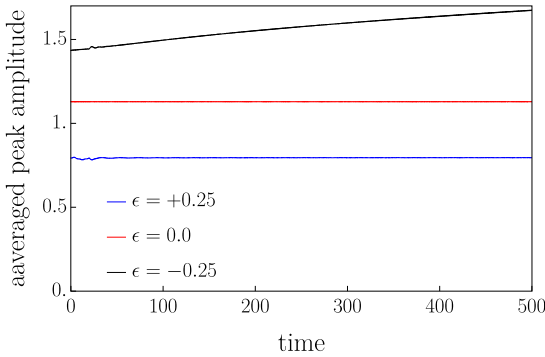


Fig. 3. Time-averaged solitary wave peak amplitude. The time-averaged amplitudes of medium solitary wave solutions of the fractional averaged equation are nearly constant for positive and zero ϵ and grow slightly for negative ϵ . The results in this plot and Fig. 2 suggest that the solitary waves for positive ϵ are closer to solitons than for negative ϵ .

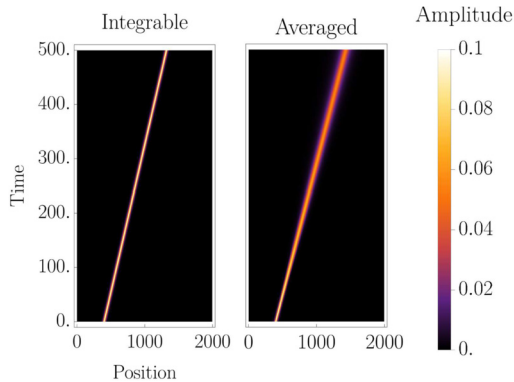


Fig. 4. Integrable and averaged dynamics for small initial conditions. The soliton solution to the fractional integrable equation propagates at a constant velocity without dissipating for $\epsilon = 0.1$. Even though the profile of the solitary wave solution to the fractional averaged equation deforms from the initial solitonic profile, its peak propagates at a nearly identical velocity to the soliton; the soliton has velocity 1.83864 and the solitary wave $1.83871 \pm 1 \times 10^{-6}$.

secant profile, but grows a little bit over time for $\epsilon = -0.25$. The averaged amplitude was obtained by taking the mean of the amplitude for ± 10 time units around each point. These results suggest that for ϵ positive and sufficiently small the localized wave solutions to the fADNLS equation have structure similar to integrable solitons, while those for ϵ negative are less similar.

A comparison of a small amplitude soliton solution to the fIDNLS equation and solitary wave solution to the fADNLS equation is given in Fig. 4 for $\epsilon = 0.1$. The solitary wave spreads out, de-

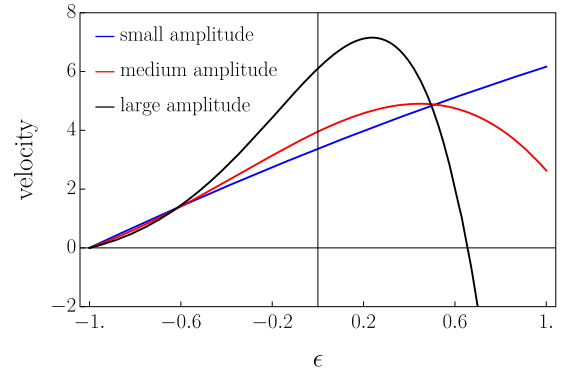


Fig. 5. Fractional soliton velocity. Velocity of the one-soliton solution to the fractional integrable equation exhibits super-dispersive transport for small amplitudes ($A = 0.100$). However, for medium ($A = 1.175$) and large ($A = 3.627$) amplitudes, the velocity has a turning point where increasing ϵ decreases the velocity. This is a fundamentally discrete phenomenon not shared by known continuous fractional integrable equations; cf. [7,8].

forming from the hyperbolic secant profile of the soliton. However, the peak velocities of the two waves are nearly identical, 1.83864 for the soliton and $1.838713 \pm 1 \times 10^{-6}$ for the solitary wave. The soliton moves with exactly constant velocity, but the solitary wave does have an acceleration of $(-1.513 \pm 0.002) \times 10^{-6}$. However, this acceleration is small enough that we can still compare the velocities of these two waves. The velocity and acceleration were estimated by fitting a quadratic curve to the solitary wave peak position and error bounds were obtained by doubling the time discretization, i.e., computing the difference between the results for $\Delta t = 0.01$ and $\Delta t = 0.02$. For larger values of ϵ and for larger amplitude waves the agreement between these two equations diverges.

The peak velocity for the one soliton solution to the fIDNLS equation is given by

$$c_p(\eta, \xi, h) = \frac{v_r}{\eta h}, \quad v_r = -2\text{Im}(\sinh^{1+\epsilon}(h[\eta - i\xi]/2)) \quad (34)$$

which is determined analytically from the form of the soliton in equation (32). The peak velocity of the fIDNLS soliton is related to its amplitude in a much more complicated manner than for the fKdV and fNLS equations which have power law relationships between their amplitude and velocity, i.e., anomalous dispersion. Fig. 5 shows this velocity as a function of ϵ for $h = 1$; $\xi = 0.5$; and small, medium, and large amplitudes.

6. Conclusion

In this paper, the fractional integrable discrete nonlinear Schrödinger equation was obtained and its properties were investigated. We did this by applying three principal mathematical constituents which were introduced in our earlier work [7], [8]: the inverse scattering transform, power law dispersion relations, and completeness relations, to the Ablowitz-Ladik scattering problem. We linearized the equation via Gel'fand-Levitan-Marchenko type summation equations. After long time the kernel of the summation equation contains only discrete spectra; we then obtained an explicit one-soliton solution to this equation, showing that its velocity depends on the fractional parameter ϵ in a more complicated way than its continuous counterpart in the fractional nonlinear Schrödinger equation. Multi-soliton solutions can be obtained by standard methods; but they are outside the scope of this paper. Using a Fourier split step method, we compared the predictions of the integrable discretization to the fractional averaged nonlinear Schrödinger equation, a related non-integrable equation. We demonstrated that for small amplitude initial data, the two

equations predicted nearly identical velocities and similar structure, while for large amplitudes they exhibited qualitatively similar characteristics. This work shows that fractional integrability can be substantially extended beyond the continuous nonlinear systems first studied in [7]. It suggests new areas of research such as fractional integrability for fully discrete systems. It also opens new opportunities for detailed comparison between fractional nonlinear equations which are integrable to those that are (likely) non-integrable.

CRediT authorship contribution statement

Mark J. Ablowitz: Writing – review & editing, Methodology, Formal analysis, Conceptualization. **Joel B. Been:** Writing – review & editing, Writing – original draft, Software, Methodology, Formal analysis, Conceptualization. **Lincoln D. Carr:** Writing – review & editing, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A

A.1. Scattering theory for the Ablowitz-Ladik system

Here, we define eigenfunctions, scattering data, etc. that are used to define the fractional integrable discrete nonlinear Schrödinger (fDNLS) equation and solve it by the IST. The Ablowitz-Ladik scattering problem

$$\mathbf{v}_{n+1} = \begin{pmatrix} z & q_n \\ r_n & z^{-1} \end{pmatrix} \mathbf{v}_n \quad (\text{A.35})$$

is associated to the following family of nonlinear evolution equations

$$\sigma_3 \frac{d\mathbf{u}_n}{dt} + \gamma(\Lambda_+) \mathbf{u}_n = 0, \quad \mathbf{u}_n = (q_n, -r_n)^T \quad (\text{A.36})$$

where T represents transpose, $\sigma_3 = \text{diag}(1, -1)$, $h_n = 1 - r_n q_n$, $\sum_n^+ = \sum_{k=n}^\infty$, and the operator Λ_+ is defined in the main manuscript. Eigenfunctions of the Ablowitz-Ladik scattering system are solutions to equation (A.35) subject to the boundary conditions

$$\phi_n(z, t) \sim \begin{pmatrix} z^n \\ 0 \end{pmatrix}, \quad \bar{\phi}_n(z, t) \sim \begin{pmatrix} 0 \\ z^{-n} \end{pmatrix}, \quad n \rightarrow -\infty, \quad (\text{A.37})$$

$$\psi_n(z, t) \sim \begin{pmatrix} 0 \\ z^{-n} \end{pmatrix}, \quad \bar{\psi}_n(z, t) \sim \begin{pmatrix} z^n \\ 0 \end{pmatrix}, \quad n \rightarrow +\infty. \quad (\text{A.38})$$

Because the “right” eigenfunctions ψ_n and $\bar{\psi}_n$ are linearly independent, we can write the “left” eigenfunctions as

$$\phi_n(z, t) = a(z, t) \bar{\psi}_n(z, t) + b(z, t) \psi_n(z, t), \quad (\text{A.39})$$

$$\bar{\phi}_n(z, t) = \bar{a}(z, t) \psi_n(z, t) + \bar{b}(z, t) \bar{\psi}_n(z, t). \quad (\text{A.40})$$

These relations define the scattering data a , b , $\bar{a}$, and $\bar{b}$. We can write the scattering data explicitly in terms of the eigenfunctions as

$$a(z, t) = v_n W(\phi_n, \psi_n), \quad \bar{a}(z, t) = v_n W(\bar{\psi}_n, \bar{\phi}_n), \quad (\text{A.41})$$

$$b(z, t) = v_n W(\bar{\psi}_n, \phi_n), \quad \bar{b}(z, t) = v_n W(\bar{\phi}_n, \psi_n), \quad (\text{A.42})$$

with the Wronskian $W(\mathbf{u}_n, \mathbf{v}_n) \equiv u_n^{(1)} v_n^{(2)} - u_n^{(2)} v_n^{(1)}$ and $v_n \equiv \prod_{k=n}^\infty h_k$, $h_k = 1 - r_k q_k$. The transmission and reflection coefficients, $\tau(z, t)$, $\bar{\tau}(z, t)$ and $\rho(z, t)$, $\bar{\rho}(z, t)$, respectively, are defined by

$$\tau(z, t) = \frac{1}{a(z, t)}, \quad \rho(z, t) = \frac{b(z, t)}{a(z, t)}, \quad (\text{A.43})$$

$$\bar{\tau}(z, t) = \frac{1}{\bar{a}(z, t)}, \quad \bar{\rho}(z, t) = \frac{\bar{b}(z, t)}{\bar{a}(z, t)}. \quad (\text{A.44})$$

Often, the functions τ , $\bar{\tau}$, ρ , and $\bar{\rho}$ are equivalently referred to as the scattering data. The eigenfunctions

$$\phi_n(z, t) z^{-n}, \quad \psi_n(z, t) z^n \quad (\text{A.45})$$

are analytic and bounded for $|z| > 1$ and continuous for $|z| \geq 1$ and

$$\bar{\phi}_n(z, t) z^n, \quad \bar{\psi}_n(z, t) z^{-n} \quad (\text{A.46})$$

are analytic and bounded for $|z| < 1$ and continuous for $|z| \leq 1$. Hence a and $\bar{a}$ are analytic inside and outside the unit circle, respectively.

The Ablowitz-Ladik scattering system can have discrete eigenvalues, corresponding to bound states. These occur at the zeros of a and $\bar{a}$ – which we notate by z_j for $j = 1, 2, \dots, J$ and $\bar{z}_j$ for $j = 1, 2, \dots, \bar{J}$, respectively – such that $|z_j| > 1$ and $|\bar{z}_j| < 1$. We assume that these eigenvalues are proper, i.e., the zeros of a and $\bar{a}$ are simple (not on the unit circle and finite in number). At these discrete eigenvalues, the eigenfunctions are related by

$$\phi_n(z_j, t) = b(z_j, t) \psi_n(z_j, t), \quad \bar{\phi}_n(\bar{z}_j, t) = \bar{b}(\bar{z}_j, t) \bar{\psi}_n(\bar{z}_j, t). \quad (\text{A.47})$$

We also define the norming constants by

$$c_j(t) = \frac{b(z_j, t)}{a'(z_j, t)}, \quad \bar{c}_j(t) = \frac{\bar{b}(\bar{z}_j, t)}{\bar{a}'(\bar{z}_j, t)} \quad (\text{A.48})$$

where $a'(z_j, t) = \partial_z a(z, t)|_{z=z_j}$, etc. When $r_n = \mp q_n^*$ in (A.35), we have the symmetry reductions

$$\bar{\phi}_n(z, t) = \mathbf{P}_\mp \phi_n^*(1/z^*, t), \quad \bar{\psi}_n(z, t) = \mp \mathbf{P}_\mp \psi_n^*(1/z^*, t) \quad (\text{A.49})$$

for the eigenfunctions and $\bar{a}(z, t) = a^*(1/z^*, t)$ and $\bar{b}(z, t) = \mp b^*(1/z^*, t)$ where

$$\mathbf{P}_\mp = \begin{pmatrix} 0 & \mp 1 \\ 1 & 0 \end{pmatrix}. \quad (\text{A.50})$$

The relation $\bar{a}(z, t) = a^*(1/z^*, t)$ implies that if z_j is a zero (eigenvalue) of $a(z, t)$, then $\bar{z}_j = 1/z_j^*$, $j = 1, 2, \dots$ is a zero of $\bar{a}(z, t)$ and hence $J = \bar{J}$. From the eigenfunctions, solutions of (A.35), we can construct the eigenfunctions of the nonlinear operator Λ_+ , $\Psi_n(z, t)$ and $\bar{\Psi}_n(z, t)$, and its adjoint Λ_+^A , $\Psi_n^A(z, t)$ and $\bar{\Psi}_n^A(z, t)$ by

$$\Psi_n = v_n \psi_n \circ \psi_{n+1}, \quad \Psi_n^A = -v_n \mathbf{P}_- (\phi_n \circ \phi_{n+1}) \quad (\text{A.51})$$

$$\bar{\Psi}_n = v_n \bar{\psi}_n \circ \bar{\psi}_{n+1}, \quad \bar{\Psi}_n^A = -v_n \mathbf{P}_- (\bar{\phi}_n \circ \bar{\phi}_{n+1}) \quad (\text{A.52})$$

where $\mathbf{u}_n \circ \mathbf{v}_m = (u_n^{(1)} v_m^{(1)}, u_n^{(2)} v_m^{(2)})^T$.

A.2. Solving the nonlinear evolution equation using the IST

Solving nonlinear discrete evolution equations with the IST is analogous to solving linear discrete evolution equations with the Z-transform. The IST has three distinct steps: direct scattering, time evolution, and inverse scattering which are analogous to taking the Z-transform, evolving the solution in frequency space, and taking the inverse Z-transform, respectively. In direct scattering, the initial condition is mapped into scattering space by solving the scattering problem (A.35). The time evolution of the scattering data, which represents the solution in scattering space, is evolved in time by solving a simple set of differential equations. Finally, in inverse scattering, the solution in physical space is reconstructed from the scattering data by solving a system of algebraic and summation equations. In the following, we briefly outline direct scattering, time evolution, and inverse scattering for the Ablowitz-Ladik scattering system.

A.2.1. Direct scattering

To perform direct scattering, we use the scattering problem in (A.35) to solve for the eigenfunctions ϕ , $\bar{\phi}$, ψ , and $\bar{\psi}$ at $t = 0$. Existence and uniqueness of these solutions can be proven by converting equation (A.35) and the appropriate boundary conditions into linear summation equations which have uniformly convergent Neumann series [49]. These series also provide an alternative method of constructing these eigenfunctions. Then, the scattering data, a , b , $\bar{a}$, and $\bar{b}$, at $t = 0$ are obtained from the Wronskian relations in equations (A.41) and (A.42).

A.2.2. Time evolution

The scattering data evolves in time according to [47]

$$\frac{d\rho}{dt} - \gamma(z^2)\rho(z, t) = 0, \quad \frac{d\bar{\rho}}{dt} + \gamma(z^2)\bar{\rho}(z, t) = 0, \quad (\text{A.53})$$

$$\frac{dc_j}{dt} - \gamma(z_j^2)c_j(t) = 0, \quad \frac{d\bar{c}_j}{dt} + \gamma(\bar{z}_j^2)\bar{c}_j(t) = 0 \quad (\text{A.54})$$

for $j = 1, 2, \dots, J$ and $j = 1, 2, \dots, \bar{J}$, respectively. We recall that γ is the function of an operator in equation (A.36) and is related to a linear dispersion relation. Also note that z_j and $\bar{z}_j$ are independent of time. To fully characterized the spectral representation of the operator $\gamma(\Lambda_+)$, and find the solution $q_n(t)$, we need the eigenfunctions at time t in addition to the scattering data. These functions are found using inverse scattering.

A.2.3. Inverse scattering

To reconstruct the solutions to the nonlinear evolution equation (A.36) and eigenfunctions at time t , we solve the following Gel'fand-Levitan-Marchenko (GLM) type summation equations for $\kappa(n, m, t)$ [49]

$$\kappa(n, m, t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{F}(m+n, t) \quad (\text{A.55})$$

$$+ \sum_{j=n+1}^{\infty} \bar{\kappa}(n, j, t) \bar{F}(m+j, t) = 0,$$

$$\bar{\kappa}(n, m, t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(m+n, t) \quad (\text{A.56})$$

$$+ \sum_{j=n+1}^{\infty} \kappa(n, j, t) F(m+j, t) = 0$$

where

$$F(n, t) = \sum_{j=1}^J z_j^{-n-1} c_j(t) + \frac{1}{2\pi i} \oint_{S_1} z^{-n-1} \rho(z, t) dz, \quad (\text{A.57})$$

$$\bar{F}(n, t) = \sum_{j=1}^{\bar{J}} \bar{z}_j^{-n-1} \bar{c}_j(t) + \frac{1}{2\pi i} \oint_{S_1} z^{n-1} \bar{\rho}(z, t) dz. \quad (\text{A.58})$$

Then, the potentials can be obtained from

$$q_n(t) = -\kappa^{(1)}(n, n+1, t), \quad r_n(t) = -\bar{\kappa}^{(2)}(n, n+1, t), \quad (\text{A.59})$$

and the right eigenfunctions from

$$\psi_n(z, t) = \sum_{j=n}^{\infty} z^{-j} \mathbf{K}(n, j, t), \quad (\text{A.60})$$

$$\bar{\psi}_n(z, t) = \sum_{j=n}^{\infty} z^j \bar{\mathbf{K}}(n, j, t) \quad (\text{A.61})$$

where

$$\mathbf{K}(n, m, t) = v_n \kappa(n, m, t), \quad (\text{A.62})$$

$$\bar{\mathbf{K}}(n, m, t) = v_n \bar{\kappa}(n, m, t). \quad (\text{A.63})$$

The left eigenfunctions $\phi_n(z, t)$ and $\bar{\phi}_n(z, t)$ can be constructed using the relations in equations (A.39) and (A.40). If $r_n(t) = \mp q_n^*(t)$, then equations (A.55) and (A.56) both reduce to

$$\kappa^{(1)}(n, m, t) - \bar{F}(n+m, t) \pm \sum_{n''=n+1}^{\infty} \sum_{n'=n+1}^{\infty} \kappa^{(1)}(n, n'', t) \cdot \bar{F}^*(n''+n', t) \bar{F}(n'+m, t) = 0 \quad (\text{A.64})$$

We note that under $r_n(t) = \mp q_n^*(t)$ there are induced symmetries: $\bar{\rho}(z) = \mp q_n^*(1/z^*)$ and for $r_n(t) = -q_n^*(t)$ there can be discrete states with $\bar{z}_j = 1/z_j^*$ (hence $\bar{J} = J$), $\bar{c}_j = (z_j^*)^{-2} c_j^*$, $j = 1, 2, \dots, J$. The above GLM summation equations provide a linearization of the fIDNLS equation. Moreover, as $t \rightarrow \infty$, the integral terms in the kernels F , $\bar{F}$ given by equations (A.57) and (A.58) vanish. Hence, we are left with only discrete spectra which yields the multisoliton solutions.

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