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Nonlinear waves and the Inverse Scattering Transform

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ABSTRACT

Solitons are a class of nonlinear stable, localized waves. They arise widely in physical problems; applications include water waves, plasma physics, Bose–Einstein condensation and nonlinear optics. Such localized water waves can be traced back to research in the 1800s. In fiber optics 'bright and dark' solitons were discovered in 1973 by Hasegawa and Tappert. In the 1970s a general theory emerged which allows one to linearize and explicitly find soliton solutions to a class of nonlinear wave equations including physically significant equations such as the Korteweg–deVries, nonlinear Schrödinger (NLS) and sine-Gordon equations. Solitons are connected to eigenvalues/bound states of underlying linear scattering equations. The theory, termed the Inverse Scattering Transform (IST) by Ablowitz, Kaup, Newell, Segur, leads to linearization/solutions to broad classes of nonlinear wave equations. In 2013 the theory was extended to novel classes of nonlocal nonlinear wave equations including PT symmetric and reverse-space–time nonlinear equations. In 2022 the theory was shown to encompass fractional integrable nonlinear systems; the fractional KdV and NLS equations are paradigms.

1. Introduction

In the late 1800s D. Korteweg and G. deVries [1] derived an equation for long waves (shallow water) with weak nonlinearity from water/gravity waves. They also found a periodic wave solution in terms of Jacobian elliptic functions; they called this solution the 'cnoidal' wave. This elliptic function solution has an associated modulus m, $0 < m < 1$. In the limit $m \to 1$ the solution tends to a localized wave which can be written in terms of a hyperbolic secant function. Such a localized wave was predicted many years earlier by J.S. Russell [2] who called it the 'great wave of translation'; this wave came to be known as the solitary wave. For many years this equation, called the Korteweg–deVries (KdV) equation, was used primarily by water wave researchers. In normalized form the KdV equation is written as

$$q_t + 6qq_x + q_{xxx} = 0. \tag{1.1}$$

Although the KdV equation was originally derived in water waves, it arises universally; i.e. it is derived in weakly dispersive, weakly quadratic nonlinear systems cf. [3]. But seventy years after the KdV paper, while studying the so-called Fermi–Pasta–Ulam problem [4], N. Zabusky and M. Kruskal [5] using numerical simulation (which was primitive at that time) to solve the KdV equation, found a remarkable result. They found that when two solitary waves collide their amplitudes/speeds attain the same values they had before the interaction. The interaction was essentially elastic. They termed these waves *solitons*. This discovery has had profound implications in physics and mathematics.

Two years after the (numerical) discovery of solitons C. Gardner, J. Greene, M. Kruskal and R. Miura (GGKM) [6] found a method to linearize the KdV equation with decaying data on the infinite line; they obtained soliton solutions as a special case. The solution technique was surprising; it related the KdV equation to a linear scattering problem and employed direct and inverse scattering.

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Remarkably the associated linear scattering problem was the linear time independent Schrödinger scattering problem. Soltions were shown to be related to eigenvalues of bound states of this linear Schrödinger scattering problem.

At the time Kruskal was communicating with some close colleagues including Peter Lax. Motivated by these results Lax [7] generalized the approach of GGKM. He found if one has compatible linear equations

$$Lv = \lambda v \tag{1.2}$$

$$v_t = Mv \tag{1.3}$$

then taking the time derivative of (1.2) and assuming that the eigenvalue λ does not change with time (i.e. $\lambda_t = 0$)

$$L_t v + L v_t = \lambda_t v + \lambda v_t => \tag{1.4}$$

$$L_t v + LMv = \lambda M v = M\lambda v => \tag{1.5}$$

$$(L_t + LM - ML)v = 0 \quad \text{or} \tag{1.6}$$

$$L_t + [L, M] = 0 \tag{1.7}$$

For the KdV equation the operators can be written as

$$L = \partial_x^2 + q(x,t), \quad M = \gamma - 3q_x - 6q\partial_x - 4\partial_x^3$$

with the L operator associated with the time independent Schrödinger scattering problem and γ is an arbitrary constant.

In 1972, using Lax's approach and direct and inverse scattering, V. Zakharov and A. Shabat [8] linearized and found soliton solutions of the nonlinear Schrödinger (NLS) equation written below. The associated linear problem was a 2×2 Dirac scattering system.

The KdV and NLS equations are 'universal' equations. As mentioned above, the KdV equation arises in cases where weak dispersion and nonlinearity balance. The NLS equation in obtained when the slowly variations in the envelope of a periodic wave is balanced by weak nonlinearity. The generality of these equations indicates that they occur widely in physics. In 1973 A. Hasegawa and F. Tappert [9,10] showed that the NLS equation and soliton solutions have important applications in nonlinear fiber optics. There were key approximations, e.g. averaging over the transverse radial coordinate, in reducing the equations in fiber waveguides to the NLS equation. Direct numerical simulation of the NLS equation was via the newly developed split-step Fourier algorithm [11]. Since that time there has been a vast literature on the NLS equation and associated soliton solutions and their applications in fiber optics cf [12–14].

The NLS equation, written in normalized form is given by

$$iq_t = q_{xx} + 2\sigma q^2 q^* = 0, \ \sigma = \pm 1 \tag{1.8}$$

where $*$ represents the complex conjugate. As found in [8–10,15] the 'focusing' NLS equation with $\sigma = 1$ has 'bright' decaying soliton solutions and with $\sigma = -1$ is called the defocusing NLS equation and has 'dark' soliton solutions which tend to a nonzero constant at infinity. Both the focusing and defocusing NLS equations and their soliton solutions have been used extensively in physics, engineering and math.

In the summer of 1972 at a conference on nonlinear wave motion at Clarkson University in Potsdam NY organized by A. Newell with support from M. Ablowitz and H. Segur, the early research on the KdV equation was presented by one of the key invited speakers: M. Kruskal. One afternoon R. Miura and F. Tappert came to a discussion room and said they had just seen the paper by Zakharov and A. Shabat [8] finding results similar to that found for the KdV equation: linearization and soliton solutions.

So this meant that two key equations of math physics, the KdV and NLS equations, had been linearized/solved. 'Solving' two such universal equations is remarkable. No question this was a major advance, but it was not well understood at the time. Ablowitz, Newell, Segur began a regular seminar in the fall of 1972 and were joined by D. Kaup. This led to our work in 1973–74 where we developed methods to linearize and obtain soliton solutions to a class of nonlinear equations [16–18]; this work is often referred to by the acronym AKNS (for: Ablowitz, Kaup, Newell, Segur). Notable examples in this class of equations include the KdV, NLS, sine-Gordon and modified KdV equations. Shortly thereafter M. Ablowitz and J. Ladik [19–22] showed how these ideas could be applied to nonlinear differential-difference and partial difference equations. One of the notable integrable differential-difference equations is an integrable discretization of Eq. (1.8) given by

$$iQ_{n,t} = Q_{n+1} + Q_{n-1} - 2Q_n + \sigma Q_n Q_n^*(Q_n + Q_{n-1}) = 0, \ \sigma = \pm 1 \tag{1.9}$$

Soon after these ideas were extended to other interesting nonlinear equations (e.g. the three wave/six wave equations) and to multidimensions cf. [23–28]. It took a few years before the solution methods to multidimensional equations were developed, cf. [29]. Mathematically speaking the methods rely heavily on the theory of complex variables including Riemann-Hilbert and DBAR techniques.

Well-known integrable multi-dimensional equations include a multi-dimensional extension of the KdV equation

$$(q_t + 6qq_x + q_{xxx})_x + 3\gamma^2 q_{yy} = 0, \ \gamma^2 = \pm 1 \tag{1.10}$$

which is sometimes referred to as the KPI equation (when $\gamma^2 = -1$) or the KPII equation (when $\gamma^2 = 1$). Like the KdV equation, the KP equation is a universal equation arising in weakly dispersive, weakly nonlinear systems with slow transverse variation [3]. For

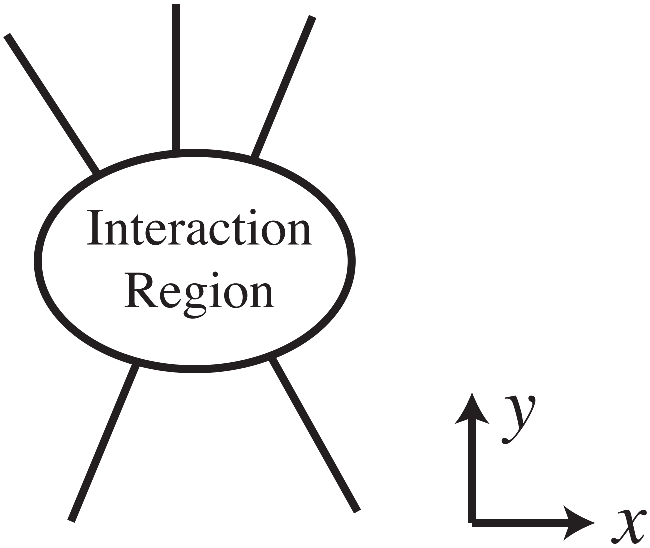

Fig. 1. Line soliton interactions with $M = 3$, $N = 2$.

example it arises in water/gravity waves with surface tension. For small surface tension one finds the KPII equation and for large enough surface tension the KPI equation applies [25]. There is also an integrable multi-dimensional extension of the NLS equation, termed the Davey–Stewartson (DS) equation cf. [29]

$$iq_t + \frac{1}{2}\left[\gamma^2 q_{xx} + q_{yy}\right] + \sigma q^2 q^* = \phi q \qquad (1.11)$$
$$\phi_{xx} - \gamma^2 \phi_{yy} = 2\sigma \left(qq^*\right)_{xx}, \quad \gamma^2 = \pm 1, \sigma = \pm 1.$$

Both the KP (1.10) and DS equations (1.11) have important applications in water/gravity waves with surface tension [25].

In this regard, we mention an interesting application of the KP equation with small surface tension such as occurs in the ocean. There has been important research involving a special class of soliton solutions to the KPII equation. Such solutions, sometimes referred to as web solutions, are generalizations of the so-called 'X-interaction' and 'Y-interaction' solutions of the KP equation (cf. [30–32]). These solutions are described by a series of rays that intersect in an interaction, or web, region. Far enough away from the interaction region, the rays are well described by soliton solutions of the KdV equation with say, M solitons as $y \to \infty$ and N line solitons as $y \to -\infty$; cf. the diagram in Fig. 1, where $M = 3$, $N = 2$.

Observations by MJA on shallow, nearly-flat beaches in Nuevo Vallarta (NV), Mexico indicated that 'X-interaction' ($M = 2, N = 2$) and 'Y-interaction' ($M = 1, N = 2$) structures appear frequently in ocean waves. The PI and former student, D. Baldwin, took numerous photos and videos of these interactions on beaches in Nuevo Vallarta, Mexico and Venice Beach, California, which are some 2000 km apart, and found the same phenomena [33]. One photo of an X-interaction taken in Mexico is given in Fig. 2.

The photos/videos indicate that the actual waves bear strong resemblance to KP-II solutions.

Over the years many nonlinear equations have been shown to be integrable and have soliton solutions. But even the classical AKNS results have been extended to encompass novel nonlocal systems (see Section 4) and fractional integrable nonlinear systems (see Section 5).

Localized waves/solitary waves have become a key concept in physics. They have been theoretically predicted and widely observed in laboratory experiments. Examples include water waves, temporal and spatial optics, Bose–Einstein condensation, magnetics, plasma physics to name a few — see [34–38] and references therein. Physics researchers usually refer to solitary waves as solitons. Here we will use the term soliton in a more mathematical sense; e.g. solitons are related to eigenvalues of an associated linear eigenvalue problem.

The outline of this paper is as follows. In Section 2 the AKNS methods and generalizations to find integrable/soliton equations are discussed. The solution technique of direct and inverse scattering is briefly outlined in Section 3. Section 4 describes research studies related to novel nonlocal soliton equations that began in 2013 and Section 5 discusses the recent research that originated in 2022 on fractional integrable nonlinear wave equations with fractional KdV being the prototype. The Section 6 summarizes this paper.

2. AKNS 2 × 2 linear problems

The method described below is sometimes refereed to as the AKNS method [17,18]. Consider the 2×2 scattering problem given by

$$v_{1,x} = -ikv_1 + q(x,t)v_2, \qquad (2.1)$$
$$v_{2,x} = ikv_2 + r(x,t)v_1,$$

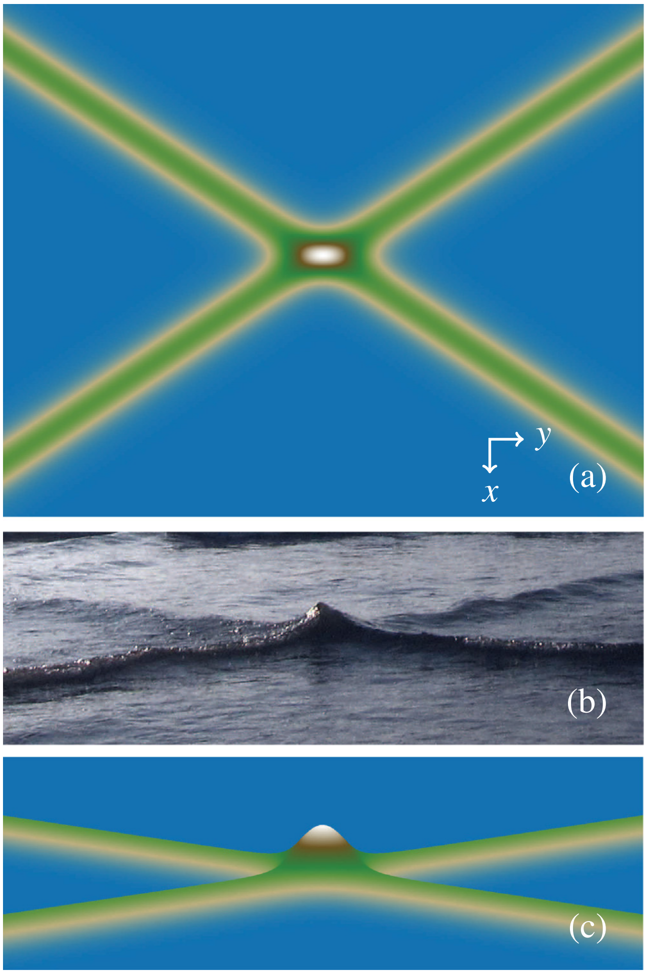

Fig. 2. A plot and photographs of an 'X' interaction with small stem. (a) Contour plot from analytic solutions. (b) Taken in Mexico. (c) Typical figure of analytic solution.

and an associated linear time dependence is given by

$$\begin{aligned} v_{1,t} &= Av_1 + Bv_2, \\ v_{2,t} &= Cv_1 + Dv_2, \end{aligned} \tag{2.2}$$

where A, B, C and D are scalar functions of $q(x,t)$, $r(x,t)$, their derivatives and the parameter k. The parameter k plays the role of an eigenvalue when one carries out the scattering analysis of Eq. (2.1). Note that if there were any x-derivatives on the right-hand side of (2.2) then they can be eliminated by use of Eq. (2.1) . Furthermore, when $r = -1$, then Eq. (2.1) reduces to the linear Schrödingerscattering problem

$$v_{2,xx} + (k^2 + q(x,t))v_2 = 0 \tag{2.3}$$

which was found by [6] to be related to the KdV equation. Further, when $r = \pm q^*$ then Eq. (2.1) this is the Dirac scattering problem used by Zakharov and Shabat [8] in their analysis of the NLS equation.

The AKNS methods allows us the find nonlinear evolution equations. The compatibility of Eqs. (2.1)–(2.2), that is requiring that $v_{j,xt} = v_{j,tx}$, for $j = 1, 2$, and assuming that the eigenvalue k is time-independent, that is $k_t = 0$, imposes a set of conditions that A, B, C and D must satisfy. Sometimes the nonlinear evolution equations obtained this way are referred to as *isospectral* flows. Namely

$$\begin{aligned} v_{1,xt} &= -ikv_{1,t} + q_t v_2 + qv_{2,t}, \\ &= -ik(Av_1 + Bv_2) + q_t v_2 + q(Cv_1 + Dv_2), \\ v_{1,tx} &= A_x v_1 + Av_{1,x} + B_x v_2 + Bv_{2,x}, \\ &= A_x v_1 + A(-ikv_1 + qv_2) + B_x v_2 + B(ikv_2 + rv_1). \end{aligned}$$

Hence by equating the coefficients of v_1 and v_2, we obtain

$$\begin{aligned} A_x &= qC - rB, \\ B_x + 2ikB &= q_t - (A - D)q, \end{aligned} \tag{2.4}$$

respectively. Similarly

$$\begin{aligned} v_{2,xt} &= ikv_{2,t} + r_t v_1 + r v_{1,t}, \\ &= ik(Cv_1 + Dv_2) + r_t v_1 + a(Av_1 + Bv_2), \\ v_{2,tx} &= C_x v_1 + C v_{1,x} + D_x v_2 + D v_{2,x}, \\ &= C_x v_1 + C(-ikv_1 + qv_2) + D_x v_2 + D(ikv_2 + rv_1), \end{aligned}$$

and equating the coefficients of v_1 and v_2 we obtain

$$\begin{aligned} C_x - 2ikC &= r_t + (A - D)r, \\ (-D)_x &= qC - rB. \end{aligned} \tag{2.5}$$

Therefore, from Eq. (2.4), without loss of generality we may assume $D = -A$, and hence it is seen that A, B and C necessarily satisfy the compatibility conditions

$$\begin{aligned} A_x &= qC - rB, \\ B_x + 2ikB &= q_t - 2Aq, \\ C_x - 2ikC &= r_t + 2Ar. \end{aligned} \tag{2.6}$$

We want to solve Eq. (2.6) for A, B and C thus ensuring that (2.1) and (2.2) are compatible. In general, this can only be done if another condition is satisfied, this being the nonlinear evolution equation. AKNS theory provides two approaches to obtain compatible nonlinear equations.

2.1. Power series expansions

The first approach makes use of the fact that k is a free parameter. We may find solvable evolution equations by seeking finite power series expansions for A, B and C:

$$A = \sum_{j=0}^{n} A_j k^j, \qquad B = \sum_{j=0}^{n} B_j k^j, \qquad C = \sum_{j=0}^{n} C_j k^j. \tag{2.7}$$

Substituting Eq. (2.7) into Eq. (2.6) and equating coefficients of powers of k, we obtain $3n+5$ equations. There are $3n+3$ unknowns, A_j, B_j, C_j, $j = 0, 1, \ldots, n$, and we also obtain two nonlinear evolution equations for r and q. Now let us consider some examples.

Suppose that A, B and C are quadratic polynomials in k, that is

$$\begin{aligned} A &= A_2 k^2 + A_1 k + A_0, \\ B &= B_2 k^2 + B_1 k + B_0, \\ C &= C_2 k^2 + C_1 k + C_0. \end{aligned} \tag{2.8}$$

Substitute Eq. (2.8) into Eq. (2.6) and equate powers of k. The coefficients of k^3 immediately give $B_2 = C_2 = 0$. At order k^2, we obtain $A_2 = a$, constant; we could have allowed a to be a function of time, but for simplicity we do not do so. We also find $B_1 = iaq$, $C_1 = iar$. At order k, we obtain $A_1 = b$, constant, for simplicity we set $b = 0$ (if $b \neq 0$ then a more general evolution equation is obtained), then $B_0 = -\frac{1}{2}aq_x$ and $C_0 = \frac{1}{2}ar_x$. Finally, at order k^0, we obtain $A_0 = \frac{1}{2}arq + c$, with c a constant (again for simplicity we set $c = 0$). Therefore we obtain the following evolution equations

$$\begin{aligned} -\tfrac{1}{2}aq_{xx} &= q_t - aq^2 r, \\ \tfrac{1}{2}ar_{xx} &= r_t + aqr^2. \end{aligned} \tag{2.9}$$

If in Eq. (2.9) we set $r = \mp q^*$ and $a = 2i$, then we obtain the nonlinear Schrödinger equation

$$iq_t = q_{xx} \pm 2q^2 q^*. \tag{2.10}$$

for both focusing (+) and defocusing (−) cases. In summary for the NLS equation, setting $r = \mp q^*$, then we find that

$$\begin{aligned} A &= 2ik^2 \mp iqq^*, \\ B &= 2qk + iq_x, \\ C &= \pm 2q^* k \mp iq_x^*, \end{aligned} \tag{2.11}$$

satisfy Eq. (2.6) provided that $q(x,t)$ satisfies the nonlinear Schrödinger Eq. (2.10).

If we substitute a third order polynomials in k after some algebra we find

$$A = a_3k^3 + a_2k^2 + \tfrac{1}{2}(a_3qr + a_1)k + \tfrac{1}{2}a_2qr - \tfrac{1}{4}ia_3(qr_x - rq_x) + a_0,$$
$$B = ia_3qk^2 + \left(ia_2q - \tfrac{1}{2}a_3q_x\right)k + \left[ia_1q - \tfrac{1}{2}a_2q_x + \tfrac{1}{4}ia_3(2q^2r - q_{xx})\right],$$
$$C = ia_3rk^2 + \left(ia_2r + \tfrac{1}{2}a_3r_x\right)k + \left[ia_1r + \tfrac{1}{2}a_2r_x + \tfrac{1}{4}ia_3(2r^2q - r_{xx})\right],$$

in Eq. (2.6), with a_3, a_2, a_1 and a_0 constants, then we find that $q(x,t)$ and $r(x,t)$ satisfy the evolution equations

$$q_t + \tfrac{1}{4}ia_3(q_{xxx} - 6qrq_x) + \tfrac{1}{2}a_2(q_{xx} - 2q^2r) - ia_1q_x - 2a_0q = 0,$$
$$r_t + \tfrac{1}{4}ia_3(r_{xxx} - 6qrr_x) - \tfrac{1}{2}a_2(r_{xx} - 2qr^2) - ia_1r_x + 2a_0r = 0. \tag{2.12}$$

For special choices of the constants a_3, a_2, a_1 and a_0 in Eq. (2.12) we find additional physically interesting evolution equations. If $a_0 = a_1 = a_2 = 0$, $a_3 = -4i$ and $r = -1$, then we obtain the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0.$$

If $a_0 = a_1 = a_2 = 0$, $a_3 = -4i$ and $r = q$, then we obtain the mKdV equation

$$q_t - 6q^2q_x + q_{xxx} = 0.$$

(Note that if $a_0 = a_1 = a_3 = 0$, $a_2 = -2i$ and $r = -q^*$, then we obtain the nonlinear Schrödinger equation (2.10).)

We can also consider expansions of A, B and C in inverse powers of k.

Suppose that

$$A = \frac{a(x,t)}{k}, \qquad B = \frac{b(x,t)}{k}, \qquad C = \frac{c(x,t)}{k},$$

then the compatibility conditions Eq. (2.6) are satisfied if

$$a_x = \tfrac{i}{2}(qr)_t, \qquad q_{xt} = -4iaq, \qquad r_{xt} = -4iar.$$

Special cases of these are (i),

$$a = \tfrac{1}{4}i\cos u, \qquad b = -c = \tfrac{1}{4}i\sin u, \qquad q = -r = -\tfrac{1}{2}u_x,$$

then u satisfies the Sine–Gordon equation

$$u_{xt} = \sin u,$$

and (ii),

$$a = \tfrac{1}{4}i\cosh u, \qquad b = -c = -\tfrac{1}{4}i\sinh u, \qquad q = r = \tfrac{1}{2}u_x,$$

where u satisfies the Sinh–Gordon equation

$$u_{xt} = \sinh u.$$

These examples only show a few of the denumerably infinite nonlinear evolution equations that may be obtained by this procedure. We saw above that when $r = -1$, the scattering problem Eq. (2.1) reduced to the Schrödingerequation (2.3). In this case we can take he associated time dependence

$$v_t = Av + Bv_x \tag{2.13}$$

By requiring that Eq. (2.13) and the time independent Schrödinger equation (in Eq. (2.3) take $\lambda = k^2$)

$$v_{xx} + (\lambda + q(x,t))v = 0, \tag{2.14}$$

are compatible and assuming that $d\lambda/dt = 0$ yields equations for A and B analogous to Eqs. (2.6). Then by expanding in powers of λ we can obtain a general class of equations associated with the linear Schrödinger scattering problem.

2.2. General evolution operators

A natural question is: *can we find a general class on nonlinear evolution equations?* Ablowitz, Kaup, Newell and Segur [18] answered this question by considering the system of Eqs. (2.6) for the functions A, B, C. By requiring that as $|x| \to \infty$ q and r vanish rapidly and $A \to A_0(k), B \to 0, C \to 0$ puts restrictions on $q(x,t), r(x,t)$ which in turn lead to a general class of nonlinear evolution equations.

The general evolution equation associated with (2.1) is given by

$$\begin{pmatrix} r \\ -q \end{pmatrix}_t + 2A_0(L)\begin{pmatrix} r \\ q \end{pmatrix} = 0 \tag{2.15a}$$

where $A_0(k) = \lim_{|x|\to\infty} A(x,t,k)$. $A_0(k)$ can be a ratio of two entire functions and L is the integro-differential operator given by

$$L = \frac{1}{2i}\begin{pmatrix} \partial_x - 2r(I_-q) & 2r(I_-r) \\ -2q(I_-q) & -\partial_x + 2q(I_-r) \end{pmatrix}, \tag{2.15b}$$

where $\partial_x \equiv \partial/\partial x$ and

$$(I_-f)(x) \equiv \int_{-\infty}^{x} f(y)dy. \tag{2.15c}$$

Note that L operates on (r,q), and I_- operates both on the functions immediately to its right and also on the functions to which L is applied. Eq. (2.15a) may be written in matrix form

$$\sigma_3\mathbf{u}_t + 2A_0(L)\mathbf{u} = 0,$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} r \\ q \end{pmatrix}.$$

It is significant that $A_0(k)$ is closely related to the dispersion relation of the associated *linearized* problem. If $f(x)$ and $g(x)$ vanish sufficiently rapidly as $|x|\to\infty$, then in the limit $|x|\to-\infty$,

$$f(x)(I_-g)(x) \equiv f(x)\int_{-\infty}^{x} g(y)dy \to 0,$$

and so for infinitesimal q and r, L is the diagonal differential operator

$$L = \frac{1}{2i}\frac{\partial}{\partial x}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence in this limit Eq. (2.15a) yields

$$r_t + 2A_0(-\frac{1}{2}i\partial_x)r = 0,$$
$$-q_t + 2A_0(\frac{1}{2}i\partial_x)q = 0.$$

The above equations are linear (decoupled) partial differential equations solvable by Fourier transform methods. Substituting the wave solutions

$$r(x,t) = e^{i(kx-\omega_r(2k)t)}, \quad q(x,t) = e^{i(kx-\omega_q(-2k)t)}$$

into the above equations, we obtain the relationship

$$A_0(k) = \frac{1}{2}i\omega_r(2k) = -\frac{1}{2}i\omega_q(-2k). \tag{2.16}$$

Therefore a general evolution equation is given by

$$\begin{pmatrix} r \\ -q \end{pmatrix} = -i\omega(2L)\begin{pmatrix} r \\ q \end{pmatrix} \tag{2.17}$$

with $\omega_q(k) = -\omega(k)$; in matrix form this equation takes the form

$$\sigma_3\mathbf{u}_t + i\omega(2L)\mathbf{u} = 0,$$

Hence, the form of the nonlinear evolution equation is characterized by the dispersion relation of its associated linearized equations and an integro-differential operator.

For the nonlinear Schrödinger equation

$$iq_t = q_{xx} \pm 2q^2q^*, \tag{2.18}$$

the associated linear equation is

$$iq_t = q_{xx}.$$

Therefore the dispersion relation is $\omega_q(k) = -k^2$, and so from Eq. (2.16), $A_0(k)$ is given by

$$A_0(k) = 2ik^2. \tag{2.19}$$

The evolution is obtained from either Eq. (2.15a) or Eq. (2.17); therefore we have

$$\begin{pmatrix} r \\ -q \end{pmatrix}_t = -4iL^2\begin{pmatrix} r \\ q \end{pmatrix} = -2L\begin{pmatrix} r_x \\ q_x \end{pmatrix} = i\begin{pmatrix} r_{xx} - 2r^2q \\ q_{xx} - 2q^2r \end{pmatrix}.$$

When $r = \mp q^*$, both of these equations are equivalent to the nonlinear Schrödinger equation (2.18). Note that Eq. (2.19) is in agreement with Eq. (2.9) with $a = 2i$ which implies that $\lim_{|x|\to\infty} A = 2ik^2$. This explains *á posteriori* why the expansion of A, B and

C in powers of k are related so closely to the dispersion relation. Indeed, now in retrospect, the fact that the nonlinear Schrödinger equation is related to Eq. (2.19) implies that an expansion starting at k^2 is a judicious choice. Similarly, the modified KdV equation when $r = -q$ (focusing), $r = -q$ (defocusing) and Sine–Gordon/Sinh-Gordon equations when $r = -q$ (sine-G), $r = -q$ (sinh-G) can be obtained from the operator Eq. (2.17) using the dispersion relations $\omega(k) = -k^3$ and $\omega(k) = k^{-1}$ respectively. These dispersion relations suggest expansions commencing in powers of k^3 and k^{-1} respectively that indeed we saw to be the case in the earlier section.

The derivation of Eq. (2.17) required that q and r tend to 0 as $|x| \to 0$ and therefore we cannot simply set $r = -1$ in order to obtain the equivalent result for the Schrödinger scattering problem (2.3). However the essential ideas are similar and Ablowitz, Kaup, Newell, and Segur [18] also showed that a general evolution equation in this case is

$$q_t + \gamma(L)q_x = 0, \tag{2.20}$$

where

$$L = -\frac{1}{4}\partial_x^2 - q + \frac{1}{2}q_x I_+, \qquad I_+ f(x) = \int_x^\infty f(y)dy. \tag{2.21}$$

where $\partial_x = \partial/\partial x$ $\gamma(k^2) = \frac{\omega(2k)}{2k}$ and $\omega(k)$ is the dispersion relation of the associated linear equation with $q = e^{i(kx-\omega(k)t)}$.

For the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0,$$

the associated linear equation is

$$q_t + q_{xxx} = 0.$$

Therefore $\omega(k) = -k^3$ and so $\gamma(k^2) = -4k^2$, thus $\gamma(L) = -4L$. Hence Eq. (2.20) yields

$$q_t - 4Lq_x = 0,$$

thus

$$q_t - 4(-\frac{1}{4}\partial_x^2 - q + \frac{1}{2}q_x I_+)q_x = 0,$$

which is the KdV equation!

Associated with the operator L there is a hierarchy of equations (sometimes referred to as the Lenard hierarchy) given by

$$u_t + L^k u_x = 0, \qquad k = 1, 2, \ldots$$

The first two subsequent higher order equations in the KdV hierarchy are given by

$$u_t - \frac{1}{4}(u_{5x} + 10uu_{3x} + 20u_x u_{xx} + 30u^2 u_x) = 0,$$
$$u_t + \frac{1}{16}(u_{7x} + 14uu_{5x} + 42u_x u_{4x} + 70u_{xx}u_{3x} + 70u^2 u_{3x}$$
$$+ 280uu_x u_{xx} + 70u_x^3 + 140u^3 n_x) = 0$$

where $u_{nx} = \partial^n u/\partial x^n$.

In the above we studied two different scattering problems, namely the classical time-independent Schrödinger equation

$$v_{xx} + (u + k^2)v = 0, \tag{2.22}$$

which can be used to linearize the KdV equation, and the 2×2 scattering problem

$$\mathbf{v}_x = k\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\mathbf{v} + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}\mathbf{v}, \tag{2.23}$$

which can linearize the nonlinear Schrödinger, mKdV and Sine–Gordon equations. While Eq. (2.22) can be interpreted as a special case of Eq. (2.23) (when $r = -1$), however, from the point of view of possible generalizations, we regard them as different scattering problems.

2.3. Extensions of AKNS: 1. Higher order linear scattering problems

There have been numerous applications and generalizations of this method over the years, far too numerous for us to go through in this paper. We mention only one result. In [23] an $N \times N$ matrix generalization of the scattering problem Eq. (2.23) was studied:

$$\frac{\partial v}{\partial x} = ik\mathbf{J}\mathbf{v} + \mathbf{Q}\mathbf{v}, \tag{2.24}$$

where the matrix of potentials $\mathbf{Q}(x) \in M_N(\mathbb{C})$ [the space of $N \times N$ matrices over $\mathbb{C}$] with $Q^{ii} = 0$, $\mathbf{J} = \mathrm{diag}(J^1, J^2, \ldots, J^N)$, with $J^i \neq J^j$ for $i \neq j$, $i, j = 1, 2, \ldots, N$, and $\mathbf{v}(x,t)$ is an N-dimensional vector eigenfunction. In this case to obtain evolution equations, the associated time dependence is chosen to be

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{T}\mathbf{v}, \tag{2.25}$$

where $\mathbf{T}$ is also an $N \times N$ matrix. The compatibility of Eqs. (2.25) yield

$$\mathbf{T}_x = \mathbf{Q}_t + ik[\mathbf{J}, \mathbf{T}] + [\mathbf{Q}, \mathbf{T}].$$

In the same way for the 2 × 2 case discussed earlier in the chapter, associated nonlinear evolution equations may be found by assuming an expansion for $\mathbf{T}$ in powers or inverse powers of the eigenvalue k

$$\mathbf{T} = \sum_{j=0}^{n} k^j \mathbf{T}_j. \tag{2.26}$$

Ablowitz and Haberman showed that this scattering problem can be used to solve several physically interesting equations such as the three-wave interaction equations in one special dimension (with $N = 3$ and $n = 1$) and the Boussinesq equation (with $N = 3$ and $n = 2$). The associated general (recursion) operators have been obtained by various authors, cf. [39–41].

2.4. Extensions of AKNS: 2. Discrete scattering systems

in [19,20] the following linear scattering problem was introduced

$$\begin{aligned} v_{1,n+1} &= z v_{1,n} + Q_n v_{2,n} \\ v_{2,n+1} &= \frac{1}{z} v_{2,n} + R_n v_{1,n}, \end{aligned} \tag{2.27}$$

and the linear time dependence is given by

$$\begin{aligned} \partial_t v_{1,n} &= A_n v_1 + B_n v_{2,n}, \\ \partial_t v_{2,n} &= C_n v_1 + D_n v_{2,n}. \end{aligned} \tag{2.28}$$

Employing power series in z and $1/z$ Ablowitz and Ladik found the integrable discrete NLS equation (1.9) written earlier, when $R_n = -\sigma Q_n, \sigma = \pm 1$, integrable discretizations of the modified KdV, sine-Gordon and many other equations were obtained. Since that time many integrable discrete equations have been studied, and a large field of integrable discrete systems has emerged cf. [42].

2.5. Extensions of AKNS: 3. Multidimensional scattering systems

In 1975 Ablowitz and Haberman [24] considered the following linear two space-one time system

$$\frac{\partial v}{\partial x} = \mathbf{J}\frac{\partial v}{\partial y} + \mathbf{Q}\mathbf{v}, \tag{2.29}$$

where the matrix of potentials $\mathbf{Q}(x, y) \in M_N(\mathbb{C})$ [the space of $N \times N$ matrices over $\mathbb{C}$] with $Q^{ii} = 0$, $\mathbf{J} = (J^1, J^2, \ldots, J^N)$, with $J^i \neq J^j$ for $i \neq j$, $i, j = 1, 2, \ldots, N$, and $\mathbf{v}(x, y, t)$ is an N -dimensional vector eigenfunction. In this case to obtain evolution equations, the associated time dependence is chosen to be

$$\frac{\partial \mathbf{v}}{\partial t} = \sum_{j=1}^{M} \frac{\partial^j \mathbf{T}_j}{\partial y^j} \mathbf{v}, \tag{2.30}$$

where $\mathbf{T_j}$ are N x N matrices. Ablowitz and Haberman [24] showed that amongst others one can find the KP equation (1.10), the DS equation (1.11) and the multi-dimensional three wave equations (TWI) given by

$$\begin{aligned} Q_{1,t}(\mathbf{x}, t) + \mathbf{C}_1 \cdot \nabla Q_1(\mathbf{x}, t) &= \sigma_3 Q_2^*(\mathbf{x}, t) Q_3^*(\mathbf{x}, t), \quad \sigma_3 = \pm 1 \\ Q_{2,t}(\mathbf{x}, t) + \mathbf{C}_2 \cdot \nabla Q_2(\mathbf{x}, t) &= -\sigma_2 Q_1^*(\mathbf{x}, t) Q_3^*(\mathbf{x}, t), \quad \sigma_2 = \pm 1 \\ Q_{3,t}(\mathbf{x}, t) + \mathbf{C}_3 \cdot \nabla Q_3(\mathbf{x}, t) &= \sigma_1 Q_1^*(\mathbf{x}, t) Q_2^*(\mathbf{x}, t), \quad \sigma_1 = \pm 1 \end{aligned} \tag{2.31}$$

Multi-dimensional three wave interaction

with distinct $\mathbf{C}_j, j = 1, 2, 3,\ \sigma_1 \sigma_3 / \sigma_2 = 1.$

The KP, DS and TWI equations all come from water/gravity wave system with surface tension [25,43]. Multi-dimensional direct/inverse scattering theory is discussed in [29] and references therein.

3. Remarks on linearization: direct and inverse scattering

The Korteweg–deVries (KdV) equation is the result of compatibility between

$$v_{xx} + (\lambda + q(x, t))v = 0 \tag{3.1}$$

and

$$v_t = (\gamma + u_x)v + (4\lambda - 2q)v_x, \tag{3.2}$$

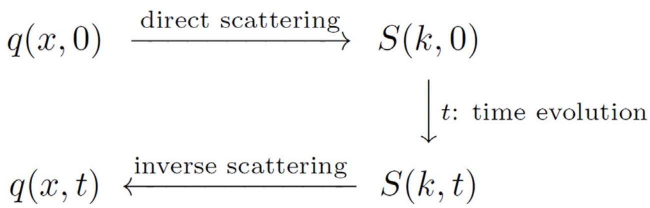

Fig. 3. Solution scheme.

where γ=constant. More precisely, the equality of the mixed derivatives $v_{xxt} = v_{txx}$ with $\lambda_t = 0$ ("isospectrality") leads to the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0 \tag{3.3}$$

It is outside the scope of this paper to go through the details of the solution process corresponding to rapidly decaying data, i.e. $u(x,t) \to 0$ rapidly as $|x| \to \infty$, which can be found in various books e.g. [3].

For Eq. (3.1), the eigenvalues and the behavior of the eigenfunctions as $|x| \to \infty$ determine what we call the **scattering data** at any time t : $S(\lambda, t)$, which depends upon the potential $u(x,t)$. The **direct scattering problem** maps the potential into the scattering data. The **inverse scattering problem** reconstructs the potential $u(x,t)$ from the scattering data $S(\lambda,t)$. The initial value problem for the KdV equation is analyzed as follows. At $t = 0$ we give initial data $u(x,0)$ that we assumed to decay sufficiently rapidly at infinity. The initial data is mapped to $S(\lambda, t = 0)$ via Eq. (3.1). The evolution of the scattering data: $S(\lambda,t)$, is determined from Eq. (3.2). Then $u(x,t)$ is recovered from inverse scattering.

This method is conceptually analogous to the Fourier transform method for solving linear equations, except however the step of solving the inverse scattering problem requires one to solve a linear integral equation (i.e., a Riemann–Hilbert boundary value problem) that adds complications. Schematically the solution is shown in the diagram in Fig. 3, where $S(k,0)$ is the scattering data at $t = 0$ and $S(k,t)$ is the scattering data at a later time t. In the analogy of IST to the Fourier transform method for linear evolution equations; the direct scattering problem and the scattering data play the role of the Fourier transform and the inverse scattering problem the inverse Fourier transform.

The scattering data $S(k,0)$ consists of the reflection coefficient, bound state eigenvalues and normalization coefficients associated with the bound state eigenfunctions. To find $S(k,0)$ we need the time dependence of the reflection coefficient and normalization coefficients. These are related to the linear dispersion relation; the eigenvalues are constants of the motion. The inverse scattering leads to a linear equation either via a Gel'fand-Levitan-Marchenko integral equation or a matrix Riemann-Hilbert problem. Pure soliton solutions can be obtained explicitly. In this case the reflection coefficient vanishes and the inverse scattering problem is reduced to a linear algebraic system. A single soliton for the KdV equation (3.3) is given by the formula

$$q(x,t) = 2\kappa_1^2 \,\mathrm{sech}^2(x - 4\kappa_1^2 t - x_1) \tag{3.4}$$

where κ_1 is a bound state eigenvalue, x_1 is related to the normalization coefficient and the reflection coefficient is zero; i.e. this is a so-called reflectionless potential.

4. Novel nonlocal wave equations

Earlier we have seen that the NLS equation, which we write in the form

$$iq_t(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q^*(x,t), \quad \sigma = \mp 1\ , \tag{4.1}$$

is integrable and has soliton solutions. In 2013, a new nonlocal reduction of the AKNS scattering problem was found [44] which gave rise to an integrable nonlocal NLS equation

$$iq_t(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q^*(-x,t), \quad \sigma = \mp 1 \tag{4.2}$$

under the symmetry condition $r(x,t) = \mp q^*(-x,t)$ in the q, r system (2.9). This is a new symmetry reduction in the AKNS formulation.

Remarkably, Eq. (4.2) has a self-induced nonlinear " PT symmetric potential", thus, it is a PT symmetric equation [45]. In other words, one can view (4.2) as a linear Schrödinger equation

$$iq_t(x,t) = q_{xx}(x,t) + V[q,x,t]q(x,t)\ , \tag{4.3}$$

with a self induced potential $V[q,x,t] \equiv -2\sigma q(x,t)q^*(-x,t)$ satisfying the PT symmetry condition $V[q,x,t] = V^*[q,-x,t]$. We point out that PT symmetric systems, which allow for lossless-like propagation due to their balance of gain and loss [46,47], have attracted considerable attention in recent years; see [48] and references therein for an extensive review on linear and nonlinear waves in PT symmetric systems. Eq. (4.2), referred to as the PT-symmetric NLS equation, was derived in [44] from physical intuition. The IST was carried out in 2016 [49] and extensions to a large class of nonlocal systems was reported in 2017 [50].

Some of the recently obtained nonlocal systems are listed below.

$$iq_t(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q(-x,-t), \tag{4.4}$$

Reverse-space–time nonlocal NLS [44,49],

$$i\mathbf{q}_t(x,t) = \mathbf{q}_{xx}(x,t) - 2\sigma[\mathbf{q}(x,t)\cdot\mathbf{q}(-x,-t)]\mathbf{q}(x,t), \tag{4.5}$$

Reverse space–time vector nonlocal NLS [50],

$$iq_t(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q(x,-t), \tag{4.6}$$

Reverse time nonlocal NLS, [50],

$$q_t(x,t) = iq_{xx}(x,t) + \sigma(q^2(x,t)q(-x,-t))_x, \tag{4.7}$$

Reverse space–time derivative nonlocal NLS, [50,51],

$$q_t(x,t) + q_{xxx}(x,t) - 6\sigma q(x,t)q^*(-x,-t)q_x(x,t) = 0, \tag{4.8}$$

Complex reverse space–time nonlocal mKdV, [50],

$$q_t(x,t) + q_{xxx}(x,t) - 6\sigma q(x,t)q(-x,-t)q_x(x,t) = 0, \tag{4.9}$$

Real reverse space–time nonlocal mKdV, [50],

$$q_{xt}(x,t) + 2s(x,t)q(x,t) = 0, \quad q \in \mathbb{R} \tag{4.10}$$

$$s_x(x,t) = (q(x,t)q(-x,-t))_t, \tag{4.11}$$

Real reverse space–time nonlocal sine-Gordon, [50],

$$iq_t(\mathbf{x},t) + \frac{1}{2}\left[\gamma^2 q_{xx}(\mathbf{x},t) + q_{yy}(\mathbf{x},t)\right] + \sigma q^2(\mathbf{x},t)q^*(-\mathbf{x},t) = \phi(\mathbf{x},t)q(\mathbf{x},t), \tag{4.12}$$

$$\phi_{xx}(\mathbf{x},t) - \gamma^2\phi_{yy}(\mathbf{x},t) = 2\sigma\left[q(\mathbf{x},t)q^*(-\mathbf{x},t)\right]_{xx}, \tag{4.13}$$

nonlocal PT Davey–Stewartson, [50,52]

$$iq_t(\mathbf{x},t) + \frac{1}{2}\left[\gamma^2 q_{xx}(\mathbf{x},t) + q_{yy}(\mathbf{x},t)\right] + \sigma q^2(\mathbf{x},t)q(-\mathbf{x},-t) = \phi(\mathbf{x},t)q(\mathbf{x},t), \tag{4.14}$$

$$\phi_{xx}(\mathbf{x},t) - \gamma^2\phi_{yy}(\mathbf{x},t) = 2\sigma\left[q(\mathbf{x},t)q(-\mathbf{x},-t)\right]_{xx}, \tag{4.15}$$

Reverse space–time nonlocal Davey–Stewartson, [50],

$$Q_{1,t}(x,t) + c_1Q_{1,x}(x,t) = \sigma_3 Q_2(-x,-t)Q_3(-x,-t), \quad \sigma_3 = \pm 1 \tag{4.16}$$

$$Q_{2,t}(x,t) + c_2Q_{2,x}(x,t) = -\sigma_2 Q_1(-x,-t)Q_3(-x,-t), \quad \sigma_2 = \pm 1$$

$$Q_{3,t}(x,t) + c_3Q_{3,x}(x,t) = \sigma_1 Q_1(-x,-t)Q_2(-x,-t) \quad \sigma_1 = \pm 1, \tag{4.17}$$

Reverse space–time nonlocal three wave interaction
with $c_3 > c_2 > c_1$, $\sigma_1\sigma_3/\sigma_2 = 1$, [50,53,54],

$$Q_{1,t}(\mathbf{x},t) + \mathbf{C}_1\cdot\nabla Q_1(\mathbf{x},t) = \sigma_3 Q_2^*(-\mathbf{x},-t)Q_3^*(-\mathbf{x},-t), \quad \sigma_3 = \pm 1 \tag{4.18}$$

$$Q_{2,t}(\mathbf{x},t) + \mathbf{C}_2\cdot\nabla Q_2(\mathbf{x},t) = -\sigma_2 Q_1^*(-\mathbf{x},-t)Q_3^*(-\mathbf{x},-t), \quad \sigma_2 = \pm 1$$

$$Q_{3,t}(\mathbf{x},t) + \mathbf{C}_3\cdot\nabla Q_3(\mathbf{x},t) = \sigma_1 Q_1^*(-\mathbf{x},-t)Q_2^*(-\mathbf{x},-t), \quad \sigma_1 = \pm 1, \tag{4.19}$$

Multi-dimensional reverse space–time nonlocal three wave interaction
with distinct $\mathbf{C}_j, j = 1,2,3$, $\sigma_1\sigma_3/\sigma_2 = 1$, [50,54],

$$i\frac{dQ_n(t)}{dt} = Q_{n+1}(t) - 2Q_n(t) + Q_{n-1}(t) - \sigma Q_n(t)Q^*_{-n}(t)\left[Q_{n+1}(t) + Q_{n-1}(t)\right], \tag{4.20}$$

Nonlocal PT discrete NLS, [55,56],

$$i\frac{dQ_n(t)}{dt} = Q_{n+1}(t) - 2Q_n(t) + Q_{n-1}(t) - \sigma Q_n(t)Q_{-n}(-t)\left[Q_{n+1}(t) + Q_{n-1}(t)\right], \tag{4.21}$$

Reverse discrete-time nonlocal discrete NLS, [56],

$$i\frac{dQ_n(t)}{dt} = Q_{n+1}(t) - 2Q_n(t) + Q_{n-1}(t) - \sigma Q_n(t)Q_n(-t)\left[Q_{n+1}(t) + Q_{n-1}(t)\right], \tag{4.22}$$

Reverse time nonlocal discrete NLS, [56]. The nonlocal nonlinear Schrödinger equation (4.2) was derived in physical applications [57], the nonlocal derivative NLS equation (4.7) was solved and derived in [51] and the nonlocal three wave Eqs. (4.16)–(4.18) were derived from water waves/gravity waves with surface tension in [43].

In [55,56] an integrable discrete PT symmetric " discretization" of Eq. (4.2) was obtained from a new nonlocal PT symmetric reduction of the Ablowitz–Ladik scattering problem [19,20]. In [49] the detailed IST associated with the nonlocal NLS system (4.2) was carried out and integrable nonlocal versions of the modified KdV and sine-Gordon equations were introduced. An extension to a $(2+1)$ dimensional integrable nonlocal NLS type equations was discussed in [52]. These findings have led to renewed interest in integrable systems.

5. Fractional integrable equations

Fractional equations have been used effectively to describe physical systems with power law behavior such as in anomalous diffusion cf. [58–61]. This form of transport has been observed extensively in amorphous materials [62–64], porous media [65–67], climate science [68], fractional Schrödinger equation in quantum mechanics and optics [69–72], amongst others. Equations in multiscale media can express fractional derivatives in any governing term [73], including dispersion, such as found in the 1D nonlinear Schrödinger equation (NLS) in lasers in fiberoptics [3,25] and the Korteweg–de Vries equation (KdV) in water waves [1]. In the case of integer derivatives, NLS and KdV are famously integrable equations, leading to solitonic solutions and an infinite set of conservation laws [29]. Integrable equations are important in nonlinear dynamics yielding exactly solvable cases and are also a critical element of Kolmogorov–Arnold–Moser (KAM) theory underlying our understanding of chaos. While in the space of possible nonlinear evolution equations, integrable cases are extremely rare, they arise frequently in application. The fundamental solution of 1D dispersive integrable equations is the soliton, a robust nondispersing localized wave.

Recently we found a new class of integrable *fractional* nonlinear evolution equations which predict sub- and super-dispersive transport in fractional media [74–76]. Fractional media is "rough" or multi-scale media that is neither regular nor random. The fractional operators in the fNLS and fKdV equations are nonlinear generalizations of the Riesz fractional derivative. In fact, the linear limit of the fNLS equation is the fractional Schrödinger equation derived in [69] using a Feynman path integral over Lévy flights. Fractional equations defined using the Riesz fractional derivative (alternately termed the Riesz transform [77] or fractional Laplacian [78]) are effective tools when describing behavior in complex systems because the Riesz fractional derivative is closely related to non-Gaussian statistics [79].

The fractional KdV (fKdV) and fractional NLS (fNLS) are prototypes; these are the first reported fractional integrable nonlinear wave systems. The analysis employs completeness, anomalous dispersion relations and IST.

It is well known that one dimensional linear evolution equations of the form

$$q_t + \gamma(\partial_x) q_x = 0, \tag{5.1}$$

can be solved by Fourier transforms when $\gamma(\partial_x)$ is a rational function cf. [3]. We can do this because the completeness of plane waves gives an integral representation of $\gamma(\partial_x)$ where $\gamma(k)$ is related to the dispersion relation of the wave equation (5.1) The solution to (5.1) is given by

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\hat{q}(k,0)e^{ikx-ik\gamma(ik)t}. \tag{5.2}$$

However, as Riesz showed [77], the solution (5.2) makes sense for much more general γ. Specifically, Fourier Transforms can be used to solve linear fractional evolution equations, e.g., $\gamma(\partial_x) = |-\partial_x^2|^\alpha$, $0 < \alpha < 1$.

A similar analysis applies to nonlinear evolution equations using IST. We do this by associating a class of integrable nonlinear equations with a linear scattering problem via IST, characterizing the fractional equation with an anomalous dispersion relation and defining the fractional operator associated with this dispersion relation using the completeness of squared eigenfunctions of the scattering equation.

We will apply these ideas in the context fKdV; discussion of fNLS, fractional modified KdV, sine-Gordon and discrete NLS can be found in [74–76]. As mentioned earlier associated with the non-dimensionalized time-independent Schrödinger equation for $v(x)$ with potential $q(x,t)$

$$v_{xx} + \left(k^2 + q(x,t)\right) v = 0, \quad |x| < \infty, \tag{5.3}$$

is the following class of integrable nonlinear equations for $q(x,t)$ [18]

$$q_t + \gamma(L) q_x = 0, \quad L = -\frac{1}{4}\partial_x^2 - q + \frac{1}{2}q_x\int_x^\infty dy \tag{5.4}$$

where $\gamma(L)$ is related to the dispersion relation of the linearization of (5.4). Specifically, if we put $q = e^{i(kx-\omega(k)t)}$ into the linearizations of (5.4) we have

$$\gamma(k^2) = \frac{\omega(2k)}{2k} \tag{5.5}$$

where $\omega(k)$ is the dispersion relation for the linear fKdV equation. Here the linear fKdV equation we will consider is

$$q_t + \left|-\partial_x^2\right|^\epsilon q_{xxx} = 0, \quad |\epsilon| < 1 \tag{5.6}$$

where $|-\partial_x^2|^\epsilon$ is the Riesz fractional derivative and the corresponding dispersion relations is $\omega(k) = -k^3|k|^{2\epsilon}$ which leads to $\gamma(L)$.

To define the fKdV equations we need to understand what operating on a function with $\gamma(L)$ means. We do this by using completeness of the associated linear scattering system.

In [17] it was shown that the eigenfunctions of L are any of the three functions: $\{\partial_x\varphi^2, \partial_x\psi^2, \partial_x(\varphi\psi)\}$ each with eigenvalue $\lambda = k^2$. Here ψ and φ solve the time-independent Schrodinger equation (5.3) subject to the following asymptotic boundary conditions at $x = \pm\infty$

$$\phi \to e^{ikx} \text{ as } x \to -\infty, \quad \psi \to e^{ikx} \text{ as } x \to \infty. \tag{5.7}$$

It was shown that the squared eigenfunctions are complete. Assuming the functions $h(x)$ and $q(x,t)$ are sufficiently decaying and smooth in x, h may be expanded in terms of the squared eigenfunctions by

$$h(x) = \int_{\Gamma_\infty} dk\, \frac{\tau^2(k)}{4\pi ik} \int_{-\infty}^{\infty} dy\, G(x,y,k)h(y), \tag{5.8}$$

where $\Gamma_\infty = \lim_{R\to\infty} \Gamma_R$ with Γ_R the semicircular contour in the upper half plane evaluated from $k = -R$ to $k = R$, $\tau(k), \rho(k)$ are the transmission, reflection coefficients relating the solutions $\varphi(x,k)\tau(k) = \psi(x,-k) + \rho(k)\psi(x,k)$ and

$$G(x,y,k) = \partial_x(\psi^2(x,k)\varphi^2(y,k) - \varphi^2(x,k)\psi^2(y,k)). \tag{5.9}$$

We note that $\tau(k,t)$ has a finite number of simple poles along the imaginary axis denoted $k_j = i\kappa_j$ for $j = 1, 2, \dots, J$, so the above representation can also be evaluated by contour integration thereby evaluating the pole contributions and bringing the contour to the real axis. Eq. (5.8) provides an explicit representation of fKdV, i.e. Eq. (5.4) with $\gamma(L^A) = -4L\,|4L|^{\epsilon}$, which may be written as

$$q_t + \int_{\Gamma_\infty} dk|4k^2|^{1+\epsilon} \frac{\tau^2(k)}{4\pi ik} \int_{-\infty}^{\infty} dy G(x,y,k)q_y = 0, \tag{5.10}$$

or alternatively

$$q_t + \int_{\Gamma_\infty} dk|4k^2|^{\epsilon} \frac{\tau^2(k)}{4\pi ik} \int_{-\infty}^{\infty} dy G(x,y,k)\left(6qq_y + q_{yyy}\right) = 0. \tag{5.11}$$

In the linear limit $q \to 0$, $\gamma(L) \to \gamma\left(-\partial_x^2/4\right)$ so, for fKdV, $\gamma(L) \to -\partial_x^2\left|-\partial_x^2\right|^{\epsilon}$ which is the Riesz fractional derivative. If we then set $\epsilon = 0$, we recover the KdV equation:

$$q_t + 6qq_x + q_{xxx} = 0. \tag{5.12}$$

Given an initial state $q(x,0)$ with sufficient smoothness and decay we can solve fKdV to obtain $q(x,t)$ using IST. To do this, we first map the initial state into scattering space, evolve the resulting scattering data in time, and reconstruct the solution in physical space from these data. It turns out that solving fKdV is remarkably similar to solving the KdV equation with only the time dependence in the inverse problem changing.

For example, an exact fractional soliton solution of fKdV which corresponds to a bound state with one complex eigenvalue $k = i\kappa$ of the Schrö dinger scattering problem is given by

$$q(x,t) = 2\kappa^2 \operatorname{sech}^2\{\kappa((x - x_1) - (4\kappa^2)^{1+\epsilon}t)\}, \tag{5.13}$$

This soliton illustrates superdispersive transport because the velocity increases as ϵ increases. It can also be shown that this fractional soliton solves the fKdV equation using contour integration methods; higher order solitons can also be calculated and their interactions are elastic.

We note that, given the explicit representation of fKdV in Eq. (5.11) alternatively to IST these equations can also be solved numerically in discrete time by finding the kernel $G(x,y,k)$ and evaluating the integrals with respect to y and k at each time step.

6. Conclusion

Solitons and the associated theory is widely considered to be a major development in mathematics and physics. The original papers in the 1960s [5–7] and early 1970s [8,15] set the foundation underlying a method of solution to a class of nonlinear equations. The method termed the Inverse Scattering Transform (IST) by Ablowitz, Kaup, Newell, Segur [17,18] finds classes of ‘integrable’ equations where soliton solutions are connected to eigenvalues/bound states of associated linear scattering problems. This article traces some of the background and developments including recent research on PT symmetric, reverse-space–time nonlocal wave equations and fractional integrable nonlinear systems such as the fractional KdV and NLS equations. Soliton theory has been applied broadly in physics from water waves to fiber optics. Key solutions to KdV equation go back to the 1895 water wave study by Korteweg and deVries [1] and early 1970s in fiber optics by Hasegawa and Tappert [9,10].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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