Michigan Math. J. 73 (2023), 571–591
https://doi.org/10.1307/mmj/20205974

Canonical Modules and Class Groups of Rees-Like Algebras

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Abstract. Rees-like algebras have played a major role in settling the Eisenbud–Goto conjecture. This paper concerns the structure of the canonical module of the Rees-like algebra and its class groups. Via an explicit computation based on linkage, we provide an explicit and surprisingly well-structured resolution of the canonical module in terms of a type of double-Koszul complex. Additionally, we give descriptions of both the divisor class group and the Picard group of a Rees-like algebra.

1. Introduction

Rees-like algebras were introduced by Peeva and the second author [17]. Given a homogeneous ideal I in a polynomial ring $S = K[x_1, \dots, x_n]$ over a field K, the *Rees-like algebra* is $\mathcal{RL}(I) := S[It, t^2] \subseteq S[t]$. Rees-like algebras provide a machine taking as input an arbitrary homogeneous ideal I in a standard graded polynomial ring S and producing a homogeneous prime ideal in a nonstandard graded polynomial ring. A particularly nice advantage of the construction is that its defining equations are explicit, unlike for Rees algebras. Among their applications, there are the construction of graded prime ideals with larger than expected regularity, which may then be homogenized to produce a negative answer to the Eisenbud–Goto conjecture [5]. As useful as these algebras are, there remain many questions as to the geometry of the varieties they define. Toward this end, the authors completed a study of the singularities of the Rees-like algebras, where again explicit methods were used to describe the Jacobian and establish various normality properties [15].

A fundamental tool to study the properties of finitely generated algebras over a field is the canonical module. In this paper, we give a complete description of the canonical module of the Rees-like algebra of an ideal of height at least 2 when the characteristic of the base field is not 2. In particular, we give an explicit presentation of $\omega_{\mathcal{RL}(I)}$ via linkage theory by fully describing the minimal free resolution of $\omega_{\mathcal{RL}(I)}$, including explicit differential maps. We show that the resolution has a surprising self-dual structure built from two Koszul complexes. Moreover, we show that even though the Rees-like algebra is not Cohen–Macaulay when I is not

Received August 26, 2020. Revision received June 22, 2021.
The second author was supported by a grant from the Simons Foundation (576107, JGM) and NSF grant DMS-1900792.

principal, its canonical module, defined as an appropriate Ext module, is Cohen–Macaulay; see Section 3.

THEOREM A (Theorem 3.13). *Suppose k is a field with $\mathrm{char}(k) \neq 2$ and S is the polynomial ring $k[x_1, \dots, x_n]$. Let $I = (f_1, \dots, f_m)$ be an ideal of S with $\mathrm{ht}(I) \geq 2$. The canonical module $\omega_{\mathcal{RL}(I)}$ of the Rees-like algebra $\mathcal{RL}(I)$ is Cohen–Macaulay.*

In particular, setting M to be the matrix

$$\mathsf{M} = \left[\begin{array}{cccc|cccc} f_1 t & f_2 t & \cdots & f_m t & f_1 & f_2 & \cdots & f_m \\ \hline f_1 t^2 & f_2 t^2 & \cdots & f_m t^2 & f_1 t & f_2 t & \cdots & f_m t \end{array}\right],$$

the canonical module of the Rees-like algebra $\mathcal{RL}(I)$ is

$$\omega_{\mathcal{RL}(I)} \cong \mathrm{coker}(\mathsf{M}),$$

and thus $\mathrm{type}(\mathcal{RL}(I)) = 2$.

The fact that $\omega_{\mathcal{RL}(I)}$ is Cohen–Macaulay is not overly surprising given that the integral closure $S[t]$ of $\mathcal{RL}(I)$ is Cohen–Macaulay; $\mathcal{RL}(I)$ is canonically Cohen–Macaulay in the language of Schenzel [21]. Nonetheless, we find rather interesting the "double-Koszul complex" structure of its resolution over the presenting polynomial ring.

We next turn our attention to divisor class groups. This is a somewhat delicate topic as the literature on class groups primarily limits itself to normal rings, whereas Rees-like algebras are never normal. Nevertheless, Rees-like algebras are Noetherian domains, so the codimension-1 Chow group or divisor class group (see, e.g., [4, Sec. 11.5]) is well-defined.

First, we prove the following general result about class groups for which we could find no reference in the literature.

THEOREM B (Theorem 4.1). *Let A be an excellent domain satisfying Serre's condition (R_1). Let $\overline{A}$ denote the integral closure of A. Then*

$$\mathrm{Cl}(A) \cong \mathrm{Cl}(\overline{A}).$$

Because Rees-like algebras of ideals of height at least two satisfy the (R_1) condition [15, Thm. 6] (and are excellent domains), it follows that the class groups of these Rees-like algebras are trivial; see Corollary 4.6.

Finally, we consider the Picard group of $\mathcal{RL}(I)$. Our fundamental tool in this investigation is the conductor square, which realizes the Rees-like algebra as a pullback. This is also called a Milnor square, and exploiting a fundamental exact sequence relating Picard groups and groups of units defined using this square, we show that the Picard group of a Rees-like algebra vanishes precisely when I is radical.

THEOREM C (Theorem 4.7). *For a field k and $S = k[x_1, \dots, x_n]$, an S-ideal I is radical if and only if* $\mathrm{Pic}(\mathcal{RL}(I)) = 0$.

Recalling [15, Sec. 5, Thm. 8], where it is shown that $\mathcal{RL}(I)$ is seminormal if and only if I is radical, Theorem C supports a theme suggesting that Rees-like algebras are best behaved for radical ideals.

These results are in stark contrast with the theory of Rees algebras, where the situation is much more complicated; see, for instance, [1; 10; 11; 12; 22; 23; 24] and the references therein.

The rest of the paper is structured as follows. In Section 2, we recall some preliminary results and the definition on Rees-like algebras. In Section 3, we compute a presentation and free resolution of the canonical module of a Rees-like algebra. Finally, in Section 4, we study the divisor class group and Picard group of a Rees-like algebra.

Computations with *Macaulay2* [9] inspired many of the results in this paper.

2. Preliminaries

We reserve the following notation. Throughout, unless otherwise stated, k is a field, and $S = k[x_1, \dots, x_n]$ is a standard graded polynomial ring. We also reserve bold letters $\mathbf{F}_\bullet, \mathbf{D}_\bullet, \dots$ for chain complexes of modules with differentials $d_\bullet^{\mathbf{F}}, d_\bullet^{\mathbf{D}}, \dots$.

For a homogeneous S-ideal I with generators $I = (f_1, \dots, f_m)$, recall that the *Rees-like algebra of I* is $S[It, t^2] \subseteq S[t]$, where t is a new variable. We denote this by $\mathcal{RL}(I) := S[It, t^2]$, and we denote by $\mathcal{RLP}(I)$ the prime ideal arising as the kernel of the map $T \to \mathcal{RL}(I)$, where $T = S[y_1, \dots, y_m, z]$ is a nonstandard graded polynomial ring over S, and the map is determined by sending $y_i \mapsto f_i t$ and $z \mapsto t^2$. In particular, $\mathcal{RL}(I) \cong T/\mathcal{RLP}(I)$, where T has grading defined by $\deg(y_i) = \deg(f_i) + 1$ and $\deg(z) = 2$. (Later we distinguish between different presentations of $\mathcal{RL}(I)$, depending on the choice of generators of I.) We quickly recall the relevant structure theorem for Rees-like algebras.

THEOREM 2.1 (McCullough and Peeva [17, Thm. 1.6, Prop. 2.9]). *The ideal $\mathcal{RLP}(I)$ is the sum $\mathcal{RLP}(I)_{\mathrm{syz}} + \mathcal{RLP}(I)_{\mathrm{gen}}$ with generators*

$$\mathcal{RLP}(I)_{\mathrm{syz}} = \left\{ r_j := \sum_{i=1}^{m} c_{ij} y_i \;\middle|\; \sum_{i=1}^{m} c_{ij} f_i = 0 \right\} \quad \textit{and}$$
$$\mathcal{RLP}(I)_{\mathrm{gen}} = \{ y_i y_j - z f_i f_j \mid 1 \le i, j \le m \}.$$

Moreover,

- $e_{\mathrm{Euler}}(T/\mathcal{RLP}(I)) = 2\prod_{i=1}^{m}(\deg(f_i) + 1)$,
- $\mathrm{pd}_T(T/\mathcal{RLP}(I)) = \mathrm{pd}(S/I) + m - 1$,
- $\mathrm{ht}(\mathcal{RLP}(I)) = m$,

and in particular, $T/\mathcal{RLP}(I)$ is Cohen–Macaulay if and only if $m = 1$.

REMARK 2.2. In the previous theorem, $e_{\mathrm{Euler}}(M)$ denotes the Euler multiplicity of the positively graded T-module M defined as follows. Let

$$E_M(u) = \sum_i \sum_j (-1)^i \beta^T_{i,j}(M) u^j \in \mathbb{Z}[u]$$

denote the Euler polynomial of M. After factoring out a maximal possible power of $(1-u)$, we write $E_M = (1-u)^c h_M(u)$. Finally, we define the Euler multiplicity of M to be $e_{\mathrm{Euler}}(M) = h_M(1)$. When T is a standard graded polynomial ring, this is the usual degree or multiplicity of M.

We refer the reader to [2, Thm. 2.5] for more detail and observe that its proof works for any prime standardization (see [15, Def. 2]), because it preserves graded Betti numbers [15, Proposition 3.4].

3. The Canonical Module

We start this section with a brief summary of the proof of the main theorem concerning the structure of the canonical module. Recall that the Rees-like algebra $S[It, t^2]$ is a quotient of a polynomial ring T. Set $Q := \mathcal{RLP}(I)$. As T/Q is not Cohen–Macaulay if $\mathrm{ht}(I) \geq 2$, we take as our definition of the canonical module $\omega_{T/Q} := \mathrm{Ext}^c_T(T/Q, T)$, where $c = \mathrm{codim}\, Q$. To calculate the canonical module, our approach is based on linkage. Two ideals I and J in S are said to be linked if there is a complete intersection $C \subset I \cap J$ such that $J = C : I$ and $I = C : J$. Many nice properties of ideals persist on linkage, namely if I and J are linked, then I defines a Cohen–Macaulay quotient if and only if J does. The application for us is to compute the canonical module via the following well-known result; for a proof, we refer the reader to [26, Thm. 6.25].

THEOREM 3.1. *For a polynomial ring T, a prime ideal Q of height m, and a $C \subset Q$ a complete intersection of height m,*

$$\omega_{T/Q} \cong (C : Q)/Q.$$

The key observation is that among the generators of Q, we find a natural complete intersection C to work with. We determine the primary decomposition of C in an explicit manner and provide a Rees-like algebra interpretation for it; see Lemma 3.3(4). We then compute the minimal generators for $C : Q$, which also form a Gröbner basis. These generators allow us to relate this calculation of the canonical module to an interesting chain complex, obtained by combining two Koszul complexes, which then serves as the claimed explicit minimal free resolution.

NOTATION 3.2. We assume that k is a field with $\mathrm{char}(k) \neq 2$. Let $S = k[x_1, \dots, x_n]$, and let $f_1, \dots, f_m$ be minimal generators of a homogeneous ideal I. We also assume that $\mathrm{ht}(I) \geq 2$. Denote by $\mathcal{RLP}(f_1, \dots, f_m)$ the Rees-like prime defined in Section 2. There is a distinguished complete intersection in $\mathcal{RLP}(f_1, \dots, f_m)$, namely,

$$C = (y_1^2 - zf_1^2, y_2^2 - zf_2^2, \dots, y_m^2 - zf_m^2).$$

Note that a different choice of a minimal generating set $g_1, \dots, g_m$ of I gives a different but isomorphic Rees–like prime in the same polynomial ring $T = S[y_1, \dots, y_m, z]$. For instance, $\mathcal{RLP}(f_1, -f_2, f_3, \dots, f_m) \neq \mathcal{RLP}(f_1, \dots, f_m)$, whereas $\mathcal{RLP}(f_1, -f_2, f_3, \dots, f_m) \cong \mathcal{RLP}(f_1, \dots, f_m)$.

LEMMA 3.3. *With Notation 3.2, we have the following*:

(i) *For any choice of* $\pm$ *signs,* $C \subset \mathcal{RLP}(\pm f_1, \pm f_2, \dots, \pm f_m)$.
(ii) $\mathcal{RLP}(f_1, f_2, \dots, f_m) = \mathcal{RLP}(-f_1, -f_2, \dots, -f_m)$.
(iii) *If* $m \geq 2$, *then for any choice of* $\pm$ *signs,*

$$\mathcal{RLP}(f_1, f_2, \dots, f_m) \neq \mathcal{RLP}(f_1, -f_2, \pm f_3, \pm f_4, \dots, \pm f_m).$$

(iv) *The complete intersection ideal* C *defined above is radical and has the following primary decomposition*:

$$C = \bigcap \mathcal{RLP}(f_1, \pm f_2, \pm f_3, \dots, \pm f_m),$$

where the intersection is taken over all possible choices of $\pm$ *signs.*

Proof. (1) We simply observe that when we replace y_i by $\pm f_i t$ and z by t^2, we see that $y_i^2 - z f_i^2$ becomes $(\pm f_i t)^2 - t^2 f_i^2 = 0$.

(2) Let $\phi : T \to S[t]$ be the map sending $y_i \mapsto f_i t$ and $z \mapsto t^2$. Then, clearly, $\mathcal{RLP}(f_1, f_2, \dots, f_m) = \operatorname{Ker}(\phi) = \operatorname{Ker}(-\phi) = \mathcal{RLP}(-f_1, -f_2, \dots, -f_m)$.

(3) The element $y_1 y_2 - z f_1 f_2$ is in the left-hand ideal but not the right-hand one.

(4) By (3) there are 2^{m-1} distinct primes in the intersection above; let us write them $Q_1, \dots, Q_{2^{m-1}}$. By (1) C is a subset of the ideal $H = \bigcap_{j=1}^{2^{m-1}} Q_j$. Both C and H are unmixed homogeneous ideals with the grading $\deg(x_j) = 1$, $\deg(y_i) = d_i + 1$ and $\deg(z) = 2$. Since $y_i^2 - z f_i^2$ is homogeneous of degree $2(\deg(f_i) + 1)$, we have $e_{\text{Euler}}(T/C) = 2^m D$, where $D = \prod_{i=1}^m (d_i + 1)$. By Theorem 2.1, $e_{\text{Euler}}(T/Q_i) = 2D$ for every $i = 1, \dots, 2^{m-1}$. Then $e_{\text{Euler}}(T/C) = e_{\text{Euler}}(T/H) = 2^m D$.

Let C^{std} and H^{std} denote prime standardizations of C and H, respectively, in a new standard graded polynomial ring T^{std}. By [2, Thm. 2.5] (see also Remark 2.2), $e(T^{\text{std}}/C^{\text{std}}) = e(T^{\text{std}}/H^{\text{std}})$. Moreover, by [15, Prop. 3.4] both C^{std} and H^{std} are still unmixed ideals of height m with $C^{\text{std}} \subseteq H^{\text{std}}$. Therefore $C^{\text{std}} = H^{\text{std}}$, and $C = H$ by the faithful flatness of the standardization map. $\square$

Next, we want to obtain an explicit description of the link $L = C : \mathcal{RLP}(I)$, where $\mathcal{RLP}(I) = \mathcal{RLP}(f_1, f_2, \dots, f_m)$. To do this, we identify interesting candidate generators, which posses remarkable symmetries. For any subset $A \subseteq [m] := \{1, 2, \dots, m\}$, we define the elements g_A^{even} and g_A^{odd} as follows. For a subset $S \subseteq A$, let $\underline{y}^S$ denote $\prod_{i \in S} y_i$ and set $\overline{S} = A \setminus S$. We define two elements of T,

$$g_A^{\text{even}} := \sum_{i=0}^{\lfloor \#A/2 \rfloor} \sum_{\substack{S \subseteq A \\ \#S = 2i}} \underline{y}^{\overline{S}} \underline{f}^S z^i$$

and

$$g_A^{\text{odd}} := \sum_{i=0}^{\lfloor (\#A-1)/2 \rfloor} \sum_{\substack{S \subseteq A \\ \#S=2i+1}} \underline{y}^{\overline{S}} \underline{f}^S z^i,$$

where $\#A$ denotes the cardinality of A.

For example, when $m = 4$, we get

$$g_{[4]}^{\text{even}} = y_1y_2y_3y_4 + y_1y_2f_3f_4z + y_1f_2y_3f_4z + \cdots + f_1f_2y_3y_4z + f_1f_2f_3f_4z^2,$$

$$g_{[4]}^{\text{odd}} = y_1y_2y_3f_4 + y_1y_2f_3y_4 + \cdots + f_1y_2y_3y_4 + y_1f_2f_3f_4z + \cdots + f_1f_2f_3y_4z.$$

The elements $g_{[j]}^{\text{even}}$ and $g_{[j]}^{\text{odd}}$ are invariant under an S_j-action, which permutes the variables y_i, and they satisfy the following useful identities.

LEMMA 3.4. *For $1 \le j \le m$ and $1 \le h \le j$, we have*

$$g_{[j]}^{\text{odd}} = y_h g_{[j]\setminus\{h\}}^{\text{odd}} + f_h g_{[j]\setminus\{h\}}^{\text{even}},$$

$$g_{[j]}^{\text{even}} = y_h g_{[j]\setminus\{h\}}^{\text{even}} + z f_h g_{[j]\setminus\{h\}}^{\text{odd}},$$

$$y_h g_{[j]}^{\text{even}} = z f_h g_{[j]}^{\text{odd}} + (y_h^2 - z f_h^2) g_{[j]\setminus\{h\}}^{\text{even}},$$

$$f_h g_{[j]}^{\text{even}} = y_h g_{[j]}^{\text{odd}} - (y_h^2 - z f_h^2) g_{[j]\setminus\{h\}}^{\text{odd}}.$$

Proof. The proofs of the first two identities are similar to each other as are the proofs of the last two. We provide the reasoning for the first and third identities and leave the other two for the interested reader.

To prove the first identity, we fix h and isolate the terms involving y_h to obtain

$$\begin{aligned} g_{[j]}^{\text{odd}} &= \sum_{i=0}^{\lfloor (j-1)/2 \rfloor} \sum_{\substack{S \subseteq \{1,\dots,j\} \\ \#S=2i+1}} \underline{y}^{\overline{S}} \underline{f}^S z^i \\ &= y_h \sum_{i=0}^{\lfloor (j-1)/2 \rfloor} \sum_{\substack{S \subseteq \{1,\dots,\widehat{h},\dots,j\} \\ \#S=2i+1}} \underline{y}^{\overline{S}} \underline{f}^S z^i + f_h \sum_{i=0}^{\lfloor (j-1)/2 \rfloor} \sum_{\substack{S \subseteq \{1,\dots,\widehat{h},\dots,j\} \\ \#S=2i}} \underline{y}^{\overline{S}} \underline{f}^S z^i \\ &= y_h \sum_{i=0}^{\lfloor (j-1)/2 \rfloor} \sum_{\substack{S \subseteq \{1,\dots,\widehat{h},\dots,j\} \\ \#S=2i+1}} \underline{y}^{\overline{S}} \underline{f}^S z^i + f_h g_{[j]\setminus\{h\}}^{\text{even}} \\ &= y_h g_{[j]\setminus\{h\}}^{\text{odd}} + f_h g_{[j]\setminus\{h\}}^{\text{even}}. \end{aligned}$$

To see the last equality, note the following observations.

- If j is even, then $\lfloor (j-1)/2 \rfloor = \lfloor (j-2)/2 \rfloor$, so $\sum_{i=0}^{\lfloor (j-1)/2 \rfloor} \sum_{\substack{S \subseteq \{1,\dots,\widehat{h},\dots,j\} \\ \#S=2i+1}} \underline{y}^{\overline{S}} \times y_h \underline{f}^S z^i = g_{[j]\setminus\{h\}}^{\text{odd}}$.
- If j is odd, then for $i = \lfloor (j-1)/2 \rfloor$, there is only one subset $S \subseteq [j]$ with $\#S = 2i+1$, namely $S = [j]$. For this value of i and the only possible associated

S, the variable y_h does not divide $\underline{y}^{\overline{S}}\underline{f}^{S}z^i = f_1 f_2 \cdots f_j z^i$. Thus

$$\sum_{i=0}^{\lfloor (j-1)/2\rfloor} \sum_{\substack{S\subseteq\{1,\dots,\widehat{h},\dots,j\}\\ \#S=2i+1}} \underline{y}^{\overline{S}}\underline{f}^{S}z^i = \sum_{i=0}^{\lfloor (j-2)/2\rfloor} \sum_{\substack{S\subseteq\{1,\dots,\widehat{h},\dots,j\}\\ \#S=2i+1}} \underline{y}^{\overline{S}}\underline{f}^{S}z^i = g^{\text{odd}}_{[j]\smallsetminus\{h\}}.$$

Thus the first identity holds.

As for the third identity, we have

$$\begin{aligned} y_h g^{\text{even}}_{[j]} &= y_h^2 g^{\text{even}}_{[j]\smallsetminus\{h\}} + y_h z f_h g^{\text{odd}}_{[j]\smallsetminus\{h\}} \\ &= y_h^2 g^{\text{even}}_{[j]\smallsetminus\{h\}} + z f_h (g^{\text{odd}}_{[j]} - f_h g^{\text{even}}_{[j]\smallsetminus\{h\}}) \\ &= (y_h^2 - z f_h^2) g^{\text{even}}_{[j]\smallsetminus\{h\}} + z f_h g^{\text{odd}}_{[j]}, \end{aligned}$$

where the first equality follows from the second identity, the middle equality follows from the first identity, and the last simply rearranges the terms. □

To simplify the notation, we write g^{odd}_j for $g^{\text{odd}}_{[j]}$ and g^{even}_j for $g^{\text{even}}_{[j]}$.

LEMMA 3.5. *With Notation 3.2, if $Q = \mathcal{RLP}(f_1, -f_2, \pm f_3, \pm f_4, \dots, \pm f_m)$, then $g^{\text{even}}_m, g^{\text{odd}}_m \in Q$.*

Proof. We show that $g^{\text{even}}_j, g^{\text{odd}}_j \in Q$ by induction on $2 \le j \le m$. First, note that $g^{\text{even}}_2 = y_1 y_2 + z f_1 f_2 = y_1 y_2 - z f_1(-f_2) \in Q$ and, similarly, $g^{\text{odd}}_2 = y_1 f_2 + y_2 f_1 = y_2 f_1 - y_1(-f_2) \in Q$,

Now let $j > 2$ and suppose $g^{\text{even}}_{j-1}, g^{\text{odd}}_{j-1} \in Q$. Then by Lemma 3.4 $g^{\text{even}}_j = y_j g^{\text{even}}_{j-1} + z f_j g^{\text{odd}}_{j-1} \in Q$, and, similarly, $g^{\text{odd}}_j = y_j g^{\text{odd}}_{j-1} + f_j g^{\text{even}}_{j-1} \in Q$. □

COROLLARY 3.6. *If $Q = \mathcal{RLP}(\pm f_1, \pm f_2, \dots, \pm f_m)$, then $g^{\text{even}}_m, g^{\text{odd}}_m \in Q$ for any choice of $\pm$ signs except for $Q = \mathcal{RLP}(f_1, \dots, f_m) = \mathcal{RLP}(-f_1, \dots, -f_m)$.*

Proof. By the symmetry of g^{even}_m and g^{odd}_m, we can assume that the signs on f_1 and f_2 are different. Then the statement follows from Lemma 3.5 and Lemma 3.3(2). □

Our next goal is to prove that $C : \mathcal{RLP}(f_1, \dots, f_m) = C + (g^{\text{even}}_m, g^{\text{odd}}_m)$. From now on we adopt the following notation.

NOTATION 3.7. Let $I = (f_1, \dots, f_m) \subseteq S$, and let $Q = \mathcal{RLP}(f_1, \dots, f_m) \subseteq T$ be its Rees-like prime. We set $L := C : Q \subseteq T$ and $J := C + (g^{\text{even}}_m, g^{\text{odd}}_m) \subseteq T$.

Proving $L = J$ will require a sequence of lemmas. First, we construct two useful short exact sequences.

LEMMA 3.8. *With Notation 3.7, we have short exact sequences*

$$0 \to T/Q \xrightarrow{\cdot g^{\text{odd}}_m} T/C \to T/(C + (g^{\text{odd}}_m)) \to 0$$

and

$$0 \to T/(IT + (y_1, \dots, y_m)) \xrightarrow{\cdot g_m^{\text{even}}} T/(C + (g_m^{\text{odd}})) \to T/J \to 0.$$

In particular, $Q = C : (g_m^{\text{odd}})$ *and* $IT + (y_1, \dots, y_m) = (C + (g_m^{\text{odd}})) : (g_m^{\text{even}})$.

Proof. The first short exact sequence is explained by the equality $C : (g_m^{\text{odd}}) = Q$, which follows by Lemma 3.3(4) and Corollary 3.6.

Analogously, for the second sequence, we need to show $(C + (g_m^{\text{odd}})) : (g_m^{\text{even}}) = IT + (y_1, \dots, y_m)$. First, note that by the third and fourth equalities in Lemma 3.4 we see that f_h and y_h lie in $(C + (g_m^{\text{odd}})) : (g_m^{\text{even}})$ for every $1 \le h \le m$, so $IT + (y_1, \dots, y_m) \subseteq (C + (g_m^{\text{odd}})) : (g_m^{\text{even}})$.

Since $IT \subset (C + (g_m^{\text{odd}})) : (g_m^{\text{even}})$, it suffices to consider the reverse inclusion modulo IT. Let $a \in T$ be such that $a \cdot g_m^{\text{even}} \in (C + (g_m^{\text{odd}}))$ modulo IT. Since $g_m^{\text{even}} \equiv y_1 y_2 \cdots y_m$ modulo IT and $(C + (g_m^{\text{odd}})) \equiv (y_1^2, \dots, y_m^2)$ modulo IT, we have $a y_1 \cdots y_m \in (y_1^2, \dots, y_m^2)$ in T/IT. Because $y_1, \dots, y_m$ is a regular sequence on T/IT, we get $a \in (y_1, \dots, y_m) + IT$. Therefore $IT + (y_1, \dots, y_m) = (C + (g_m^{\text{odd}})) : (g_m^{\text{even}})$. □

Next, we compute the initial ideal of J.

LEMMA 3.9. *Fix* $y_1 > y_2 > \cdots > y_m > z > x_1 > \cdots > x_n$ *and let* $<$ *be the lex order* $<$ *on* T. *Then* $y_1^2 - zf_1^2, \dots, y_m^2 - zf_m^2, g_m^{\text{even}}, g_m^{\text{odd}}$ *form a Gröbner basis of* J *with respect to* $<$. *In particular,*

$$\text{in}_<(J) = (y_1^2, \dots, y_m^2, y_1 \cdots y_m, y_1 \cdots y_{m-1} \text{in}_<(f_m)), \tag{1}$$

and $\text{pd}(T/\text{in}_<(J)) \le m + 1$.

Proof. For the first part of the statement, we show that all S-pairs reduce to 0 using the basic identities from Lemma 3.4.

Clearly, $\text{in}_<(y_h^2 - zf_h^2) = y_h^2$ for all h. By [7, Prop. 2.15] the S-pairs $S(y_h^2 - zf_h^2, y_j^2 - zf_j^2)$ reduce to 0 for all $1 \le h < j \le m$. Since $\text{in}_<(g_m^{\text{even}}) = y_1 \cdots y_m$, we see that

$$\begin{aligned} S(y_h^2 - zf_h^2, g_m^{\text{even}}) &= y_1 \cdots \widehat{y_h} \cdots y_m (y_h^2 - zf_h^2) - y_h g_m^{\text{even}} \\ &= y_1 \cdots \widehat{y_h} \cdots y_m (y_h^2 - zf_h^2) - (y_h^2 - zf_h^2) g_{[m]-\{h\}}^{\text{even}} - zf_h g_m^{\text{odd}} \\ &= (y_1 \cdots \widehat{y_h} \cdots y_m - g_{[m]-\{h\}}^{\text{even}})(y_h^2 - zf_h^2) - zf_m g_m^{\text{odd}}. \end{aligned}$$

Since the two initial terms of $(y_1 \cdots \widehat{y_h} \cdots y_m - g_{[m]-\{h\}}^{\text{even}})(y_h^2 - zf_h^2)$ and $zf_h g_m^{\text{odd}}$ are different (the first one is divisible by y_h^2, and the second one is not), this is a standard expression. Therefore the S-pair $S(y_h^2 - zf_h^2, g_m^{\text{even}})$ reduces to 0.

A similar calculation shows that $S(y_h^2 - zf_h^2, g_m^{\mathrm{odd}})$ also reduces to 0 for all h. Finally, we consider

$$\begin{aligned} S(g_m^{\mathrm{even}}, g_m^{\mathrm{odd}}) &= \mathrm{in}_<(f_m)g_m^{\mathrm{even}} - y_m g_m^{\mathrm{odd}} \\ &= \mathrm{in}_<(f_m)g_m^{\mathrm{even}} - (f_m g_m^{\mathrm{even}} + g_{j-1}^{\mathrm{odd}}(y_m^2 - zf_m^2)) \\ &= (\mathrm{in}_<(f_m) - f_m)g_m^{\mathrm{even}} - g_{j-1}^{\mathrm{odd}}(y_m^2 - zf_m^2), \end{aligned}$$

where the second equality follows from Lemma 3.4. It is easy to see that last line is a standard expression for $S(g_m^{\mathrm{even}}, g_m^{\mathrm{odd}})$, and so it also reduces to 0.

For the second part of the statement, we observe that $y_1, \dots, y_m$ and $a := \mathrm{in}_<(f_m)$ form a regular sequence; thus $A = k[y_1, \dots, y_m, a]$ is isomorphic to a polynomial ring in $m+1$ variables. By the first part of the proof the ideal $\mathrm{in}_<(J)$ is extended from an A-ideal, and so $\mathrm{pd}(T/\mathrm{in}_<(J)) \leq m+1$. □

REMARK 3.10. We record two observations regarding J.

(i) It is not hard to show that $\mathrm{pd}(T/\mathrm{in}_<(J)) = m+1$ and $\beta_{m+1}^T(T/\mathrm{in}_<(J)) = 1$. However, for our intended use of Lemma 3.9, the inequality $\mathrm{pd}(T/\mathrm{in}_<(J)) \leq m+1$ is sufficient; see the proof of Proposition 3.12.

(ii) The equality $J = L$ is easily proved, once we have shown that J is unmixed (see the first paragraph of the proof of Proposition 3.12). In turn, this would follow easily if $\mathrm{in}_<(J)$ were unmixed (e.g., by [3, Lemma 3.6]). Unfortunately, this short route is not available, because we can see easily that $\mathrm{in}_<(J)$ is not unmixed. (For any variable x dividing $\mathrm{in}_<(f_m)$, $(y_1, \dots, y_m, x)$ is a height $m+1$ associated prime of $\mathrm{in}_<(J)$.) Thus we need a bit more work to prove that J is unmixed.

As a step toward proving J is unmixed, we next show that $(y_1, y_2, \dots, y_m, z)$ is not an associated prime of T/J.

LEMMA 3.11. *Let* $\mathfrak{p} = (y_1, y_2, \dots, y_m, z)$. *Then* $\mathfrak{p} \notin \mathrm{Ass}(T/J)$.

Proof. First, we show that $Q_\mathfrak{p}$ is a complete intersection. Recall that we have a decomposition $Q = \mathcal{RLP}(I)_{\mathrm{syz}} + \mathcal{RLP}(I)_{\mathrm{gen}}$ as in Theorem 2.1. The ideal $\mathcal{RLP}(I)_{\mathrm{syz}}$ is generated by elements of the form $\sum_i s_i y_i$ such that $\sum_i s_i f_i = 0$ in S. In particular, the following elements corresponding to Koszul syzygies of I are in $(\mathcal{RLP}(I)_{\mathrm{syz}})_\mathfrak{p}$: $y_1 - \frac{f_1}{f_m}y_m, y_2 - \frac{f_2}{f_m}y_m, \dots, y_{m-1} - \frac{f_{m-1}}{f_m}y_m$. For brevity, set $y_i' = y_i - \frac{f_i}{f_m}y_m$. Since $y_m^2 - zf_m^2 \in \mathcal{RLP}(I)_{\mathrm{gen}}$, it follows that $Q_\mathfrak{p}$ is generated by the regular sequence $y_1', y_2', \dots, y_{m-1}', y_m^2 - zf_m^2$. (These elements, along with y_m, form a regular system of parameters of the regular local ring $S_\mathfrak{p}$.)

Now we compute the link $L_\mathfrak{p} = C_\mathfrak{p} : Q_\mathfrak{p}$. Set $\overline{y_i} = y_i + \frac{f_i}{f_m}y_m$, so that

$$y_i^2 - zf_i^2 = \overline{y_i}y_i' + \frac{f_i^2}{f_m^2}(y_m^2 - zf_m^2).$$

Therefore $[y_1^2 - zf_1^2, \ldots, y_m^2 - zf_m^2] = \mathsf{D}[y_1', \ldots, y_{m-1}', y_m^2 - zf_m^2]^{\mathsf{T}}$, where

$$\mathsf{D} = \begin{bmatrix} \overline{y_1} & 0 & \cdots & 0 & f_1^2/f_m^2 \\ 0 & \overline{y_2} & \cdots & 0 & f_2^2/f_m^2 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \overline{y_{m-1}} & f_{m-1}^2/f_m^2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

By [25, Thm. A.140], $L_p = (C + (\det \mathsf{D}))_{\mathfrak{p}}$. Note that

$$\begin{aligned} \det(\mathsf{D}) &= \prod_{i=1}^{m-1} \overline{y_i} \\ &= \prod_{i=1}^{m-1} \left(y_i + \frac{f_i}{f_m} y_m \right) \\ &= \sum_{S \subseteq \{1,\ldots,m-1\}} \underline{y}^{\overline{S}} \frac{\underline{f}^S}{f_m^{|S|}} y_m^{|S|} \\ &= \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} \left(\sum_{\substack{S \subseteq \{1,\ldots,m-1\} \\ \#S = 2i}} \underline{y}^{\overline{S}} \frac{\underline{f}^S}{f_m^{2i}} y_m^{2i} + \sum_{\substack{S \subseteq \{1,\ldots,m-1\} \\ \#S = 2i+1}} \underline{y}^{\overline{S}} \frac{\underline{f}^S}{f_m^{2i+1}} y_m^{2i+1} \right) \\ &\equiv \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} \left(\sum_{\substack{S \subseteq \{1,\ldots,m-1\} \\ \#S = 2i}} \underline{y}^{\overline{S}} \underline{f}^S z^i + \sum_{\substack{S \subseteq \{1,\ldots,m-1\} \\ \#S = 2i+1}} \underline{y}^{\overline{S}} \underline{f}^S z^i \frac{y_m}{f_m} \right) (\bmod C_{\mathfrak{p}}) \\ &= g_{m-1}^{\text{even}} + \frac{y_m}{f_m} g_{m-1}^{\text{odd}}, \end{aligned}$$

where the third line follows from expanding the product, the fourth line separates the even and odd terms, and the fifth line follows since $z - \frac{y_m^2}{f_m^2} \in C_{\mathfrak{p}}$. Finally, note that

$$f_m \det(\mathsf{D}) \equiv f_m g_{m-1}^{\text{even}} + y_m g_{m-1}^{\text{odd}} \equiv g_m^{\text{odd}} (\bmod C_{\mathfrak{p}}).$$

It follows that $L_{\mathfrak{p}} = (C + g_m^{\text{odd}})_{\mathfrak{p}}$. Since

$$f_m g_m^{\text{even}} = y_m g_m^{\text{odd}} - (y_m^2 - zf_m^2) g_{m-1}^{\text{odd}} \in C + (g_m^{\text{odd}}),$$

we have

$$L_{\mathfrak{p}} = (C + (g_m^{\text{odd}}))_{\mathfrak{p}} = J_{\mathfrak{p}}.$$

Since $Q_{\mathfrak{p}}$ is a complete intersection, in particular, $T_{\mathfrak{p}}/Q_{\mathfrak{p}}$ is Cohen–Macaulay. Since $J_{\mathfrak{p}} = C_{\mathfrak{p}} : Q_{\mathfrak{p}}$ is a link of $Q_{\mathfrak{p}}$, then by linkage, for example, [20, Prop. 2.6], also, $T_{\mathfrak{p}}/J_{\mathfrak{p}}$ is Cohen–Macaulay. In particular, $J_{\mathfrak{p}}$ is unmixed of height m; therefore $\mathfrak{p}T_{\mathfrak{p}} \notin \operatorname{Ass}(T_{\mathfrak{p}}/J_{\mathfrak{p}})$, and so $\mathfrak{p} \notin \operatorname{Ass}(T/J)$. □

We can now prove the following:

PROPOSITION 3.12. *In Notation 3.7, we have $L = J$, that is, $C : Q = C + (g_m^{\text{odd}}, g_m^{\text{even}})$.*

Proof. The containment $L \supseteq J$ follows from Lemma 3.3 and Corollary 3.6. Next, we show that $J^{un} = L$. Since $C = Q \cap L \subseteq J \subseteq L$, since all these ideals have height m, and since Q, L are unmixed, we have $C \subseteq J^{un} \subseteq L$. Since $C \subseteq J^{un}$ are unmixed of the same height, $\operatorname{Ass}(T/J^{un}) \subseteq \operatorname{Ass}(T/C)$, so by Lemma 3.3(4) all associated primes of T/J^{un} have the form $\mathcal{RLP}(f_1, \pm f_2, \ldots, \pm f_m)$. By Theorem 2.1 (or the proof of Lemma 3.11) they are all contained in $\mathfrak{p} = (y_1, \ldots, y_m, z)$. Since $J_{\mathfrak{p}} = L_{\mathfrak{p}}$ by Lemma 3.11, we have $J_{Q_i} = L_{Q_i}$ for each $Q_i \in \operatorname{Ass}(T/J^{un})$. This proves $J^{un} = L$.

It then suffices to prove that J is unmixed. We observe that for any associated prime q of T/J, we have $\operatorname{ht}(q) \leq m+1$, because

$$\operatorname{ht}(q) \leq \operatorname{pd}(T/J) \leq \operatorname{pd}(T/\operatorname{in}_<(C + (g_m^{\text{even}}, g_m^{\text{odd}}))) \leq m+1.$$

The first inequality follows from [13, Lemma 2.6], the second inequality follows from [19, Thm. 22.9], and the last inequality is proved in Lemma 3.9. Therefore we only need to prove that J contains no associated primes of height $m+1$. Our next goal is to prove the following:

CLAIM 1. *There exists a linear form ℓ in $k[y_1, \ldots, y_m]$ that is regular on T/J.*

Proof of Claim 1. It suffices to show that no prime ideal $\mathfrak{p} \in \operatorname{Ass}(T/J)$ of height $m+1$ contains $(y_1, \ldots, y_m)$. Indeed, if any such $\mathfrak{p}$ exists, then since $C \subseteq \mathfrak{p}$, we have $z(f_1^2, \ldots, f_m^2) \subseteq \mathfrak{p}$; since $\operatorname{ht}(f_1^2, \ldots, f_m^2) = \operatorname{ht}(I) > 1$, the only possibility is that $z \in \mathfrak{p}$, and therefore $\mathfrak{p} = (y_1, \ldots, y_m, z)$. This possibility is ruled out by Lemma 3.11. □

CLAIM 2. *We may assume that y_m is regular on T/J.*

Proof of Claim 2. By Claim 1 there is a linear form $0 \neq \ell \in k[y_1, \ldots, y_m]$ that is regular on T/J. By possibly multiplying by a unit and permuting the variables, we may assume that $\ell = y_m + \sum_{i=1}^{m-1} \alpha_i y_i$, where $\alpha_i \in k$. We consider the automorphism ψ of T that fixes all variables except it sends $y_m \mapsto \ell$. It is easy check that $\psi^{-1}(J)$ has the same generators as J except that every instance of f_m is replaced by $f_m + \sum_{i=1}^{m-1} \alpha_i f_i$. Then this corresponds to choosing a different minimal set of generators of I before constructing the Rees-like prime. Since ℓ is not in any associated prime of J, y_m is not in any associated prime of $\psi^{-1}(J)$. □

We now conclude the proof of Proposition 3.12. Since y_m is regular on T/J and $y_m^2 - zf_m^2 \in J$, also f_m is regular on T/J. To prove that J is unmixed, it then suffices to show that J_{f_m} is unmixed in the localization T_{f_m}. Since f_m is a unit in T_{f_m} and $f_m g_m^{\text{even}} = y_m g_m^{\text{odd}} - (y_m^2 - zf_m^2) g_{m-1}^{\text{odd}} \in (C + (g_m^{\text{odd}}))_{f_m}$, the ideal $J_{f_m} = (C + (g_m^{\text{odd}}))_{f_m}$ is an almost complete intersection of height m.

Now in the ring T_{f_m}, we have

$$\begin{aligned} J+(y_m) &= (y_1^2 - zf_1^2, \dots, y_{m-1}^2 - zf_{m-1}^2, y_m^2 - zf_m^2, y_m, g_m^{\text{odd}}) \\ &= (y_1^2 - zf_1^2, \dots, y_{m-1}^2 - zf_{m-1}^2, zf_m^2, y_m, g_m^{\text{odd}}) \\ \text{(because } f_m \text{ is a unit)} &= (y_1^2 - zf_1^2, \dots, y_{m-1}^2 - zf_{m-1}^2, z, y_m, g_m^{\text{odd}}) \qquad (2) \\ &= (y_1^2, y_2^2, \dots, y_{m-1}^2, y_m, z, g_m^{\text{odd}}) \\ \text{(by definition of } g_m^{\text{odd}}) &= (y_1^2, y_2^2, \dots, y_{m-1}^2, y_m, z, y_1 \cdots y_{m-1}). \end{aligned}$$

Since $M = (y_1^2, \dots, y_{m-1}^2, y_1 y_2 \cdots y_{m-1})$ is $(y_1, \dots, y_{m-1})$-primary and extended from $k[y_1, \dots, y_{m-1}]$, M is Cohen–Macaulay of height $m-1$. Since y_m, z is a regular sequence on $(T/M)_{f_m}$, the ideal $(y_1^2, y_2^2, \dots, y_{m-1}^2, y_1 \cdots y_{m-1}, y_m, z)_{f_m} = (J + (y_m))_{f_m}$ is Cohen–Macaulay too. Since y_m is regular on T/J and f_m is regular on T/J, y_m is also regular on $(T/J)_{f_m}$, and thus $(T/J)_{f_m}$ is Cohen–Macaulay. In particular, J_{f_m} is unmixed, and then so is J. □

We are now able to construct a finite T-free resolution of the canonical module of any Rees-like algebra $\mathcal{RL}(I) = S[It, t^2] = T/\mathcal{RLP}(I)$, assuming that $\text{char}(k) \neq 2$ and I has height at least 2. It is built from an amalgamation of the Koszul complexes on the generators $f_1, \dots, f_m$ of I and the variables $y_1, \dots, y_m$.

THEOREM 3.13. *Suppose k is a field with* $\text{char}(k) \neq 2$. *Let $S = k[x_1, \dots, x_n]$, and let $I = (f_1, \dots, f_m)$ be an ideal of S with* $\text{ht}(I) \geq 2$. *Then the canonical module $\omega_{\mathcal{RL}(I)}$ of the Rees-like algebra $\mathcal{RL}(I)$ is a maximal Cohen–Macaulay $\mathcal{RL}(I)$-module. In particular, if* M *is the matrix*

$$\mathsf{M} = \left[\begin{array}{cccc|cccc} y_1 & y_2 & \cdots & y_m & f_1 & f_2 & \cdots & f_m \\ \hline zf_1 & zf_2 & \cdots & zf_m & y_1 & y_2 & \cdots & y_m \end{array}\right],$$

then the canonical module of the Rees-like algebra $\mathcal{RL}(I)$ is

$$\omega_{\mathcal{RL}(I)} \cong \text{coker}(\mathsf{M}),$$

as T-modules, and thus $\text{type}(\mathcal{RL}(I)) = 2$.

Proof. As usual, let $T = S[y_1, \dots, y_m, z]$. Let $\mathbf{K}_\bullet(\underline{y})$ denote the Koszul complex on $y_1, \dots, y_m$ over T with differential maps $d_i^{\underline{y}} : \mathbf{K}_i(\underline{y}) \to \mathbf{K}_{i-1}(\underline{y})$, and let $\mathbf{K}_\bullet(\underline{f})$ denote the Koszul complex on $f_1, \dots, f_m$ over T with differential maps $d_i^{\underline{f}} : \mathbf{K}_i(\underline{f}) \to \mathbf{K}_{i-1}(\underline{f})$. Define the new complex of free T-modules $\mathbf{D}_\bullet$ with $\mathbf{D}_i = T^{2\binom{m}{i}}$ for $0 \leq i \leq m$ with differential given as a matrix by

$$d_i^{\mathbf{D}} = \left[\begin{array}{c|c} d_i^{\underline{y}} & d_i^{\underline{f}} \\ \hline z \cdot d_i^{\underline{f}} & d_i^{\underline{y}} \end{array}\right].$$

It is easy to check that $d_{i-1}^{\mathbf{D}} \circ d_i^{\mathbf{D}} = 0$, and thus $\mathbf{D}_\bullet$ is a complex. We also have the following short exact sequences of complexes:

$$0 \to \mathbf{D}_\bullet \xrightarrow{z} \mathbf{D}_\bullet \to \mathbf{D}_\bullet / z\mathbf{D}_\bullet \to 0$$

and

$$0 \to \mathbf{K}_\bullet(\underline{y}) \otimes_T T/zT \to \mathbf{D}_\bullet/z\mathbf{D}_\bullet \to \mathbf{K}_\bullet(\underline{y}) \otimes_T T/zT \to 0.$$

Because $\mathbf{K}_\bullet(\underline{y}) \otimes_T T/zT$ is acyclic, it follows from the long exact sequence of homology associated with the second short exact sequence that $\mathbf{D}_\bullet/z\mathbf{D}_\bullet$ is also acyclic. Now from the long exact sequence associated with the first short exact sequence we see that multiplication by z induces an isomorphism on $H_i(\mathbf{D}_\bullet)$ for $i > 0$; then by Nakayama's lemma we get $H_i(\mathbf{D}_\bullet) = 0$ for $i > 0$. Note that $d_1^{\mathbf{D}} = \mathsf{M}$.

Now define $d_0^{\mathbf{D}} : \mathbf{D}_0 \to \frac{C:Q}{C}$ as follows. By Proposition 3.12, $\frac{C:Q}{C}$ is minimally generated by g_m^{even} and $-g_m^{\text{odd}}$. Since $\mathbf{D}_0 = T^2$, we map the first basis element to g_m^{even} and the second basis element to $-g_m^{\text{odd}}$. By Lemma 3.4 we have

$$y_m g_m^{\text{even}} + zf_m(-g_m^{\text{odd}}) = g_{m-1}^{\text{even}}(y_m^2 - zf_m^2) \in C$$

and

$$f_m g_m^{\text{even}} + y_m(-g_m^{\text{odd}}) = -g_{m-1}^{\text{odd}}(y_m^2 - zf_m^2) \in C.$$

Therefore $\text{Im}(d_1^{\mathbf{D}}) = \text{Im}(\mathsf{M}) \subseteq \text{Ker}(d_0^{\mathbf{D}})$. To show the reverse inclusion, suppose that $a, b \in T$ such that $d_0^{\mathbf{D}}[a,b]^{\mathsf{T}} = 0 \in \frac{C:Q}{C}$; that is,

$$a \cdot g_m^{\text{even}} + b(-g_m^{\text{odd}}) \in C.$$

Then by Lemma 3.8, $a \in (C + (g_m^{\text{odd}})) : (g_m^{\text{even}}) = IT + (y_1, \ldots, y_m)$. Since the entries in the first row of M generate $IT + (y_1, \ldots, y_m)$, we can use the columns of M to rewrite a and b, and we may assume that $a = 0$. Then $b \in C : (g_m^{\text{odd}}) = Q$. By Theorem 2.1 every element of Q is a linear combination of the elements $y_i y_j - zf_i f_j$, where $1 \le i \le j \le m$ and $\sum_j c_j y_j$, where $\sum_j c_j f_j = 0$. Note that

$$\begin{bmatrix} 0 \\ y_i y_j - zf_i f_j \end{bmatrix} = y_j \begin{bmatrix} f_i \\ y_i \end{bmatrix} - f_i \begin{bmatrix} y_j \\ zf_j \end{bmatrix} \in \text{Im}(d_1^{\mathbf{D}})$$

and

$$\begin{bmatrix} 0 \\ \sum_j c_j y_j \end{bmatrix} = \sum_j c_j \begin{bmatrix} f_j \\ y_j \end{bmatrix} \in \text{Im}(d_1^{\mathbf{D}}),$$

where $\sum_j c_j f_j = 0$. Therefore $[0, b]^{\mathsf{T}} \in \text{Im}(d_1^{\mathbf{D}})$ for any $b \in Q$. It follows that $\text{Im}(d_1^{\mathbf{D}}) = \text{Ker}(d_0^{\mathbf{D}})$ and that $\mathbf{D}_\bullet$ is a minimal T-free resolution of $\frac{C:Q}{C}$. Finally, we have $\omega_{\mathcal{RL}(I)} \cong \frac{C:Q}{C}$, for example, by [13, Lemma 3.1]. □

In retrospect, the fact that the canonical module is Cohen–Macaulay should not be surprising since the integral closure of $S[It, t^2]$ is a polynomial ring, and thus a finite Cohen–Macaulay module over the non-Cohen–Macaulay Rees-like algebra $\mathcal{RL}(I)$. Yet, we find interesting the self-dual nature of the T-free resolution of the canonical module in the previous theorem, especially given that one of the constituent Koszul complexes, $\mathbf{K}_\bullet(\underline{f})$, need not be exact.

As a corollary, we get the following surprising self-duality statement.

COROLLARY 3.14. *Using the notation above,*

$$\omega_{\mathcal{RL}(I)} \cong \operatorname{Ext}_T^m(T/Q, T) \cong \operatorname{Ext}_T^m(\omega_{\mathcal{RL}(I)}, T).$$

Proof. Because $\mathbf{K}_\bullet(\underline{y})$ and $\mathbf{K}_\bullet(f)$ are self-dual, it follows from the definition that $\mathbf{D}_\bullet$ is self-dual as well, that is, $\overline{\mathbf{D}_\bullet} \cong \operatorname{Hom}_T(\mathbf{D}_\bullet, T)$. □

EXAMPLE 3.15. Let $S = k[x_1, x_2]$ and set $I = (x_1, x_2)^2$. We construct the resolution of the canonical module of the Rees-like algebra $\mathcal{RL}(I)$. As such, set $T = S[y_1, y_2, y_3, z]$ and let $Q = \mathcal{RLP}(x_1^2, x_1x_2, x_2^2)$. By the previous theorem, $\omega_{\mathcal{RL}(I)} \cong \frac{C:Q}{C}$, where $C = (y_1^2 - zx_1^4, y_2^2 - zx_1^2x_2^2, y_3^2 - zx_2^4)$ and

$$C : Q = C + (g_3^{\text{odd}}, g_3^{\text{even}}),$$

where

$$g_3^{\text{even}} = y_1y_2y_3 + x_1x_2^3y_1z + x_1^2x_2^2y_2z + x_1^3x_2y_3z,$$
$$g_3^{\text{odd}} = x_2^2y_1y_2 + x_1x_2y_1y_3 + x_1^2y_2y_3 + x_1^3x_2^3z.$$

Moreover, as a T-module, $\omega_{\mathcal{RL}(I)}$ has T-free resolution:

$$T^2 \xleftarrow{d_1} T^6 \xleftarrow{d_2} T^6 \xleftarrow{d_3} T^2 \longleftarrow 0,$$

where

$$d_1 = \left[\begin{array}{ccc|ccc} y_1 & y_2 & y_3 & x_1^2 & x_1x_2 & x_2^2 \\ \hline zx_1^2 & zx_1x_2 & zx_2^2 & y_1 & y_2 & y_3 \end{array}\right],$$

$$d_2 = \left[\begin{array}{ccc|ccc} -y_2 & -y_3 & 0 & -x_1x_2 & -x_2^2 & 0 \\ y_1 & 0 & -y_3 & x_1^2 & 0 & -x_2^2 \\ 0 & y_1 & y_2 & 0 & x_1^2 & x_1x_2 \\ \hline -zx_1x_2 & -zx_2^2 & 0 & -y_2 & -y_3 & 0 \\ zx_1^2 & 0 & -zx_2^2 & y_1 & 0 & -y_3 \\ 0 & zx_1^2 & zx_1x_2 & 0 & y_1 & y_2 \end{array}\right],$$

$$d_3 = \left[\begin{array}{c|c} -y_3 & -x_2^2 \\ y_2 & x_1x_2 \\ -y_1 & -x_1^2 \\ \hline -zx_2^2 & -y_3 \\ zx_1x_2 & y_2 \\ -zx_1^2 & -y_1 \end{array}\right].$$

4. Class Groups

We now turn our attention to the investigation of class groups of Rees-like algebras. A main complication comes from the fact that Rees-like algebras are never normal. On the other hand, the integral closure of $S[It, t^2]$ is the UFD $S[t]$, and when the height of I is at least 2, $S[It, t^2]$ satisfies Serre's (R_1) condition by [15, Thm. 5.1]. We leverage these two facts in our computations.

4.1. Divisor Class Group

We first review class groups in the generality we consider; for details, we refer the reader to [4, Sec. 11.5]. Denote for a ring R the set of height 1 primes by $\operatorname{Spec}_1 R$. Let R be a Noetherian domain. A *Weil divisor* is a formal finite $\mathbb{Z}$-linear combination $\sum_{\mathfrak{p}\in \operatorname{Spec}_1(R)} n_{\mathfrak{p}}[\mathfrak{p}]$ of height 1 primes. These naturally form an Abelian group $\operatorname{Div}(R)$.

If R is normal, then $R_{\mathfrak{q}}$ is a DVR for all height 1 primes $\mathfrak{q}$, leading to the usual notion of linear equivalence. Note, however, that for any (possibly, nonnormal) domain R, the ring $R_{\mathfrak{q}}$ is still a one-dimensional domain. Thus for any nonzero $x \in R$, the $R_{\mathfrak{q}}$-module $R_{\mathfrak{q}}/xR_{\mathfrak{q}}$ has finite length, which we denote $\operatorname{ord}_{\mathfrak{q}}(x) := \lambda(R_{\mathfrak{q}}/xR_{\mathfrak{q}})$. When $R_{\mathfrak{q}}$ is a DVR, $\operatorname{ord}_{\mathfrak{q}}(x)$ agrees with the $\mathfrak{q}$-adic valuation of x, and so this recovers the more familiar definition of class group. This extends in the natural fashion to $\operatorname{Frac}(R)$ and yields a well-defined map $\operatorname{div}_R : \operatorname{Frac}(R) \to \operatorname{Div}(R)$ sending $x/y \in \operatorname{Frac}(R)$ with $x, y \in R$ to $\sum_{\mathfrak{q}\in \operatorname{Spec}_1 R}(\operatorname{ord}_{\mathfrak{q}}(x) - \operatorname{ord}_{\mathfrak{q}}(y))[\mathfrak{q}]$. Elements in the image $\operatorname{Prin}(R)$ of this map are called *principal divisors* and the *divisor class group* or *codimension-1 Chow group* is the quotient

$$\operatorname{Cl}(R) := \operatorname{Div}(R)/\operatorname{Prin}(R).$$

There are few computations in the literature of class groups of nonnormal domains; see, for instance, [8; 6; 18] or Kollár's theory of Mumford divisors [14].

To compute the class group of a Rees-like algebra, we prove a much more general theorem providing sufficient conditions under which the class group of an algebra is isomorphic to that of its integral closure (Theorem 4.1). Since the integral closure of a Rees-like algebra is a polynomial ring, it follows that the class group of a Rees-like algebra is trivial under mild hypotheses.

Theorem 4.1 is likely unsurprising for experts, but we could not locate its statement or proof in the literature, so we provide a proof along with examples to illustrate the necessity of the hypothesis. The proof of [28, Ch. V, Sec. 5, Rem., p. 269] makes essentially similar claims, and we could deduce a quicker argument accepting those, but we opted to provide a more detailed argument.

We work first in the following general setup. Let A be a Noetherian integral domain, and let $\overline{A}$ denote its integral closure.

THEOREM 4.1. *Let A be an excellent domain satisfying Serre's condition (R_1). Let $\overline{A}$ denote the integral closure of A. Then*

$$\operatorname{Cl}(A) \cong \operatorname{Cl}(\overline{A}).$$

Proof. The proof follows by showing that contraction of primes along the inclusion $A \to \overline{A}$ induces a bijection between the sets of height one primes $\operatorname{Spec}_1(A)$ and $\operatorname{Spec}_1(\overline{A})$. Let $\varphi\colon \operatorname{Div}(\overline{A}) \to \operatorname{Div}(A)$ be the function obtained by linearly extending $\varphi(P) := P \cap A$. This map is clearly a group homomorphism. In the following, we will demonstrate an equality of rings $\overline{A}_P = A_{\varphi(P)}$. As $\operatorname{Frac}(\overline{A}) = \operatorname{Frac}(A)$, any principal divisor $\operatorname{div}_{\overline{A}}(f) = \sum a_i[P_i]$ in $\operatorname{Div}(\overline{A})$ has the image

$\sum a_i[P_i \cap A] = \operatorname{div}_A(f)$, which then will guarantee that $\operatorname{Cl}(A) \cong \operatorname{Cl}(\overline{A})$. We establish these in the following claims.

CLAIM 1. *If $P \in \operatorname{Spec}_1(\overline{A})$, then $\mathfrak{p} := P \cap A \in \operatorname{Spec}_1(A)$.*

Since $\operatorname{Frac}(A) = \operatorname{Frac}(\overline{A})$, then obviously $\operatorname{trdeg}_{\operatorname{Frac}(A)}(\operatorname{Frac}(\overline{A})) = 0$, and so $\operatorname{trdeg}_{\kappa(\mathfrak{p})}(\kappa(P)) = 0$.

Since A is excellent, then A is universally catenary, and $\overline{A}$ is a f.g. A-algebra, thus the dimension equality [16, Thm. 15.6] holds, so we have

$$\operatorname{ht}(P) = \operatorname{ht}(\mathfrak{p}) + \operatorname{trdeg}_{\operatorname{Frac}(A)}(\operatorname{Frac}(\overline{A})) - \operatorname{trdeg}_{\kappa(\mathfrak{p})}(\kappa(P)).$$

It follows that $\operatorname{ht}(P) = \operatorname{ht}(\mathfrak{p}) = 1$.

CLAIM 2. *For every $\mathfrak{p} \in \operatorname{Spec}_1(A)$, there is $P \in \operatorname{Spec}_1(\overline{A})$ with $\mathfrak{p} = P \cap A$.*

The existence of a prime $P \in \operatorname{Spec}(A)$ contracting to $\mathfrak{p}$ is guaranteed by the lying-over property of integral extensions. By the dimension formula

$$\operatorname{ht}(P) \le \operatorname{ht}(P) + \operatorname{trdeg}_{\kappa(\mathfrak{p})}(\kappa(P)) = \operatorname{ht}(\mathfrak{p}) + \operatorname{trdeg}_A(\overline{A}) = \operatorname{ht}(\mathfrak{p}) = 1.$$

As P is a nonzero ideal of the domain $\overline{A}$, $\operatorname{ht}(P) = 1$.

CLAIM 3. *We have the equality of rings $A_{\mathfrak{p}} = \overline{A}_P$ inside their common fraction field.*

First, observe that $A_{\mathfrak{p}}$ is a DVR, so we can write $\mathfrak{p}A_{\mathfrak{p}} = fA_{\mathfrak{p}}$ for some $f \in A_{\mathfrak{p}}$, and every element $a \in A_{\mathfrak{p}}$ has the form $a = wf^t$ for some unit $w \in A_{\mathfrak{p}}$ and $t \in \mathbb{N}_0$. If the equality does not hold, then there exists $x \in \overline{A}_P$ with $x \notin A_{\mathfrak{p}}$. Since $\operatorname{Frac}(A) = \operatorname{Frac}(\overline{A})$, we have $\overline{A}_P \subseteq \operatorname{Frac}(\overline{A}_P) = \operatorname{Frac}(A_{\mathfrak{p}})$, so we can write $x = a_1/a_2$ with $a_1, a_2 \in A_{\mathfrak{p}}$. By the above there exist $r_1, r_2 \in \mathbb{N}_0$ and units $u_1, u_2 \in A_{\mathfrak{p}}$ such that $a_i = uf^{r_i}$ for $i = 1, 2$, so $x = uf^r$ for some unit $u = u_1/u_2 \in A_{\mathfrak{p}}$ and $r = r_1 - r_2 \in \mathbb{Z}$. Since $x \notin A_{\mathfrak{p}}$, $r < 0$, and since u^{-1} and f lie in $A_{\mathfrak{p}} \subseteq \overline{A}_P$, also $f^{-1} \in \overline{A}_P$. We use this to prove that $\overline{A}_P$ is a field: any nonzero element $y \in \overline{A}_P$ can be written, as above, in the form $y = vf^s$, where v is a unit in $A_{\mathfrak{p}}$, and $s \in \mathbb{Z}$. By the above, both v and f^s are units in $\overline{A}_P$, thus y is a unit, and therefore $\overline{A}_P$ is a field. This is a contradiction, so $A_{\mathfrak{p}} = \overline{A}_P$ as claimed.

By the above, for every prime $\mathfrak{p} \in \operatorname{Spec}_1(A)$, there is a prime $P \in \operatorname{Spec}_1(\overline{A})$ lying over $\mathfrak{p}$. We now show that P is unique. Let $P' \in \operatorname{Spec}(\overline{A})$ be another height one prime with $P' \cap A = \mathfrak{p}$; then by the arguments above $\overline{A}_{P'} = A_{\mathfrak{p}} = \overline{A}_P$. Let $y \in P'$, and write $y = \frac{a}{b}$ with $a, b \in A_P$ and $b \notin P$. Since $y \in P'\overline{A}_{P'} = P\overline{A}_P$, we have $by = a \in P$, and thus $y \in P$. It follows that $P' \subseteq P$, and then, by symmetry, $P' = P$, which concludes the proof. □

In particular, we obtain that any integral extension of a k-algebra A satisfying (R_1) and with same fraction field as A has the same class group as A.

COROLLARY 4.2. *Let k be a field, and let $A \subseteq B$ be an integral extension of finitely generated k-algebra domains such that* $\mathrm{Frac}(A) = \mathrm{Frac}(B)$ *and A satisfies Serre's condition* (R_1)*. Then*

$$\mathrm{Cl}(A) \cong \mathrm{Cl}(B).$$

Proof. Being k-algebras, both A and B are excellent rings, and by assumption they have the same integral closure $\overline{A}$. We then prove that B satisfies condition (R_1). Since the proof of Claim 2 in the previous theorem does not require A to be (R_1), every prime ideal $\mathfrak{p}'$ of B is contracted from a height one prime ideal P of $\overline{A}$. Then we have natural inclusions $A_{P\cap A} \subseteq B_{\mathfrak{p}} \subseteq \overline{A}_P$. Since A is (R_1), the proof of Claim 3 of the previous theorem implies that $A_{P\cap A} = B_{\mathfrak{p}} = \overline{A}_P$, so B is (R_1).

The conclusion now follows from the previous theorem, because $\mathrm{Cl}(A)$ and $\mathrm{Cl}(B)$ are both isomorphic to $\mathrm{Cl}(\overline{A})$. □

The next examples show the necessity of each assumption in the previous result.

EXAMPLE 4.3 (Necessity of having the same fraction field). Let $A = k[x^3, x^2y, xy^2, y^3]$ be the third Veronese of $B = k[x, y]$. The ring B is regular, A is (R_1), and $A \to B$ is an integral, but $\mathrm{Frac}(A) \neq \mathrm{Frac}(B)$. We have $\mathrm{Cl}(B) = 0$, but $\mathrm{Cl}(A) \cong \mathbb{Z}/3\mathbb{Z}$ is nonzero.

EXAMPLE 4.4 (Necessity of integrality). Let $A = k[x, y, xt, yt]$ be the Rees algebra of (x, y) in $k[x, y]$, and let $B = k[x, y, t]$. The ring B is regular and hence a UFD, A satisfies Serre's (R_1) property, and $\mathrm{Frac}(A) = \mathrm{Frac}(B)$, but $A \subseteq B$ is not an integral extension. Thus all assumptions of Corollary 4.2 apply except integrality.

Clearly, $\mathrm{Cl}(B) = 0$, but $\mathrm{Cl}(A) \neq 0$ as A is an integrally closed non-UFD. In fact, $\mathrm{Cl}(A) \cong \mathbb{Z}$. Thus we cannot remove the integral extension hypothesis in Corollary 4.2.

EXAMPLE 4.5 (Necessity of (R_1)). Let $A = k[x, xt, t^2]$ be the Rees-like algebra of (x) in $K[x]$, and let $B = k[x, t]$ be its integral closure. Then the ring B is regular, $A \to B$ is an integral extension with $\mathrm{Frac}(A) = \mathrm{Frac}(B)$, but A is not (R_1). Also, A is the coordinate ring of a Whitney Umbrella variety, which is a seminormal hypersurface that is not normal and thus not (R_1). Here we verify that $\mathrm{Cl}(A) \neq 0$.

Since A is not (R_1), some care must be taken with computing the class group. Consider the height one prime ideal $P = (x, xt) = xB \cap A$ of A. If $\mathrm{Cl}(A) = 0$, then $\mathrm{ord}_P(f) = 1$ for some $f \in \mathrm{Frac}(A) = \mathrm{Frac}(B)$. Writing $f = \frac{a}{a'}$ for $a, a' \in A$, we have $1 = \mathrm{ord}_P(f) = \lambda(A_P/aA_P) - \lambda(A_P/a'A_P)$.

We now find a contradiction by proving that $\lambda(A_P/cA_P)$ is an even integer for all $c \in A$.

Observe that $A \cong k[u, v, w]/(v^2 - u^2w)$, where we identify $x \leftrightarrow u$, $xt \leftrightarrow v$, and $t^2 \leftrightarrow w$. It is easy to see that the multiplicity of the one-dimensional ring A_P is 2 and A_P has Hilbert–Samuel function $\lambda(A_P/P^{i+1}A_P) = 1 + 2i$. Let $0 \neq c \in A_P$, and write $c = \alpha u^d + \beta u^{d-1}v +$ higher-order terms in u and v for some integer $d \geq 0$ and some α, β units in A_P not both 0. Then $\lambda(A_P/cA_P) = e(A_P/cA_P) = e(\mathrm{gr}_{PA_P}(A_P/cA_P))$. Since $\mathrm{gr}_{PA_P}(A_P/cA_P) \cong K(W)[U,V]/(V^2 - U^2W, \alpha U^d + \beta U^{d-1}V)$ is defined by a complete intersection of degrees 2 and d, it follows that $\lambda(A_P/cA_P) = 2d$. This gives a contradiction, so $\mathrm{Cl}(A) \neq 0$.

We now prove that Rees-like algebras have trivial class groups.

COROLLARY 4.6. *Let* $S = k[x_1, \ldots, x_n]$. *If* $I \subseteq S$ *is an ideal of height at least two, then* $\mathrm{Cl}(S[It, t^2]) = 0$.

Proof. By [15, Thm. 6], $A := S[It, t^2]$ satisfies Serre's condition (R_1) if $\mathrm{char}(k) \neq 2$. However, this hypothesis is not necessary. Indeed, the nonnormal locus of A is given by $V(I + It)$, where $I + It$ is the conductor ideal of A in $\overline{A} = S[t]$, and since $\mathrm{ht}(I + It) \geq 2$, A satisfies (R_1). Since the integral closure of A is the UFD $\overline{A} = S[t]$ and $\mathrm{Cl}(\overline{A}) = 0$, Theorem 4.1 yields $\mathrm{Cl}(S[It, t^2]) = 0$. □

4.2. *Picard Group*

Finally, we consider the Picard group, that is, the group $\mathrm{Pic}(R)$ of invertible fractional ideals modulo principal fractional ideals of R. In the normal case the Picard group is a subgroup of the divisor class group, and so if we were in a normal setting, it would be reasonable to expect the Picard group of a Rees-like algebra $S[It, t^2]$ to be also trivial. However, in our setting the situation is more interesting, as we show that $\mathrm{Pic}(S[It, t^2]) = 0$ if and only if I is radical.

Our approach uses Milnor squares; see, e.g., [27, Ex. 2.6]. We use the following setup. Suppose $A \to B$ is an inclusion of rings, and let $\mathfrak{c} := \mathrm{Ann}_A(B/A)$ be the conductor ideal. The ideal $\mathfrak{c}$ is the largest ideal of A that is also an ideal of B. In this situation, A is the pullback of the diagram

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ A/\mathfrak{c} & \hookrightarrow & B/\mathfrak{c} \end{array}$$

It is easy to verify that the conductor ideal for the Rees-like algebra $S[It, t^2]$ is $\mathrm{Ann}_{S[It,t^2]}(S[t]/S[It, t^2]) = I + It$ and the corresponding Milnor square is

$$\begin{array}{ccc} S[It, t^2] & \hookrightarrow & S[t] \\ \downarrow & & \downarrow \\ (S/I)[t^2] & \hookrightarrow & (S/I)[t]. \end{array}$$

Dual to this diagram is a pushout of affine varieties, which provides some perspective on the geometry of Rees-like algebras. Fix an ideal I in $S = k[x_1, \ldots, x_n]$ and consider the cylinder $V(I) \times \mathbb{A}^1_k$ inside $\mathbb{A}^{n+1}_k$. Identifying $(a, b) \sim (a, -b) \in V(I) \times \mathbb{A}^1_K$ creates a $2:1$ gluing which, when extended to all of $\mathbb{A}^{n+1}_k$, yields the affine variety $\mathrm{Spec}(S[It, t^2])$. The pinch point, or Whitney umbrella, is the

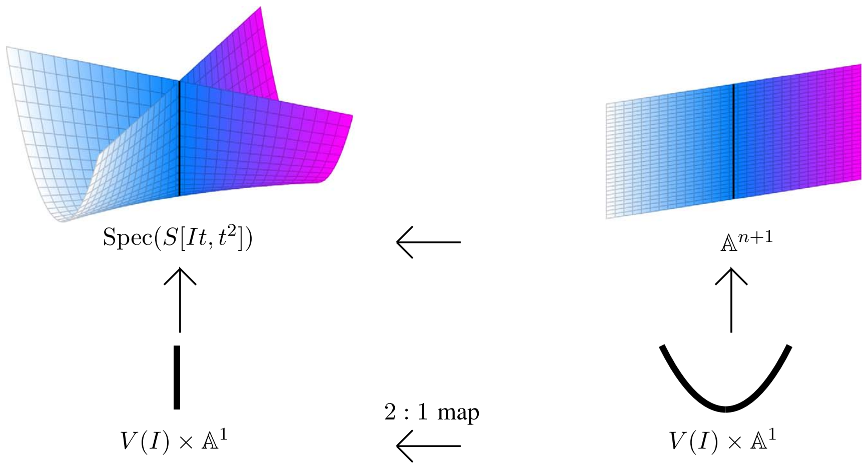

Figure 1 Pushout diagram for Spec(S[It, t^2]). Pictured with $I = (x) \subset k[x]$.

variety associated with the Rees-like algebra of the ideal $(x) \subseteq k[x]$. Thus we can view varieties defined by Rees-like algebras as higher-dimensional analogues of the pinch point surface. See Figure 1.

To compute the Picard group of a Rees-like algebra, we apply the Units-Pic exact sequence associated with the Milnor square for Rees-like algebra.

THEOREM 4.7. *Let $I \subseteq S = k[x_1, \dots, x_n]$ be a homogeneous ideal. The Picard group* $\mathrm{Pic}(S[It, t^2]) = 0$ *if and only if I is radical.*

Proof. Given the Milnor square above, the Units-Pic exact sequence [27, Thm. 3.10] is as follows:

$$1 \longrightarrow S[It, t^2]^\times \longrightarrow S[t]^\times \times (S/I)[t^2]^\times \longrightarrow (S/I)[t]^\times \xrightarrow{\partial}$$
$$\longrightarrow \mathrm{Pic}(S[It, t^2]) \longrightarrow \mathrm{Pic}(S[t]) \times \mathrm{Pic}((S/I)[t^2]) \longrightarrow \mathrm{Pic}((S/I)[t]).$$

As $S[It, t^2]$ and $S[t]$ are positively graded domains, both have units groups isomorphic to $k^\times$. If I is not radical, then S/I has a nonzero nilpotent element, say $\eta \in S/I$. Since $1 + \eta t \in (S/I)[t]^\times \setminus (S/I)[t^2]^\times$, it follows that $\mathrm{coker}(\partial) \neq 0$, whence $\mathrm{Pic}(S[It, t^2]) \neq 0$.

If, on the other hand, I is radical, then $(S/I)[t]^\times = (S/I)[t^2]^\times = k^\times$, and so $\partial = 0$. As S is regular, $\mathrm{Pic}(S[t]) = 0$. The inclusion $(S/I)[t^2] \to (S/I)[t]$ is a free extension, and hence the natural map $\mathrm{Pic}((S/I)[t^2]) \to \mathrm{Pic}((S/I)[t])$ is injective. It follows from the above sequence that $\mathrm{Pic}(S[It, t^2]) = 0$. □

REMARK 4.8. The Rees-like algebra $S[It, t^2]$ is seminormal if and only if I is radical, by [15, Cor. 4]. In general, it is not true that the Picard group of every seminormal ring is trivial. Clearly, any number ring with class number greater than 1 is a counterexample. Specifically, the Dedekind domain $R = \mathbb{Z}[\sqrt{-5}]$ R is of course normal, whence seminormal, but $\mathrm{Pic}(R) = \mathrm{Cl}(R) \cong \mathbb{Z}/2\mathbb{Z} \neq 0$.

REMARK 4.9. When I is not radical, $\mathrm{Pic}(S[It, t^2])$ is not just nonzero but infinite. Here we again consider the case $I = (x^2) \subset k[x]$. By the proof of Theorem 4.7, $\mathrm{Pic}(S[It, t^2]) \cong \mathrm{Im}(\partial) = \mathrm{coker}((S/I)[t^2]^\times \to (S/I)[t]^\times)$. The units group of $(S/I)[t^2]$ decomposes as $k^\times \oplus \bigoplus_{i\geq 1} k$ with $(\alpha_0, \alpha_1, \alpha_2, \ldots) \in k^\times \oplus \bigoplus_{i\geq 1} k$ corresponding to $\alpha_0(1 + \alpha_1 xt + \alpha_2 xt^2 + \cdots) \in (S/I)[t]^\times$. A similar calculation works for $(S/I)[t^2]^\times$, with the copies of k appearing in even degrees only. It follows that $\mathrm{Pic}(S[It, t^2]) \cong \bigoplus_{i\in\mathbb{N}} k$.

The same computation works for $I = (x^2, y) \subset k[x, y]$ when $\mathrm{ht}(I) = 2$ and $\mathrm{Cl}(S[It, t^2]) = 0$.

ACKNOWLEDGMENT. The authors thank the anonymous referee for helpful suggestions.

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