On the Geometry of Stable Steiner Tree Instances

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Abstract

In this work we consider the Steiner tree problem under Bilu-Linial stability. We give strong geometric structural properties that need to be satisfied by stable instances. We then make use of, and strengthen, these geometric properties to show that 1.59-stable instances of Euclidean Steiner trees are polynomial-time solvable by showing it reduces to the minimum spanning tree problem. We also provide a connection between certain approximation algorithms and Bilu-Linial stability for Steiner trees.

1 Introduction and previous work

In this work, we initiate the study of Steiner tree instances that are stable to multiplicative perturbations to the distances in the underlying metric. Our analysis lies in the Bilu-Linial stability [9] setting, which provides a way to study tractable instances of NP-hard problems.

Instances that are γ-stable in the Bilu-Linial model have the property that the structure of the optimal solution is not only unique, but also does not change even when the underlying distances among the input points are perturbed by a multiplicative factor $\gamma > 1$. In their original paper, Bilu and Linial analyzed MAX-CUT clustering, and since their seminal work, other problems have been analyzed including center-based clustering [4, 6, 7], multi-way cut problems [15], and metric TSP [17].[1]

Here, we look at the metric Steiner tree problemand also the more restricted Euclidean version. For general metrics, the Steiner tree problem is known to be APX-hard in the worst case [10]. For the Euclidean metric, a PTAS is known [3].

In this paper we begin by providing strong geometric structural properties that need to be satisfied by stable instances. These point to the existence of algorithms for non-trivial families. We then make use of, and strengthen, these geometric properties to show that 1.59-stable instances of Euclidean Steiner trees are polynomial-time solvable. Finally, we discuss the connections between certain approximation algorithms and Bilu-Linial stability for Steiner trees.

2 Model and definitions

In this section, we recall the relevant definitions. Fist we define the Steiner tree problem, which is among Karp's 21 original NP-hard problems [13]. It has various applications including in network design, circuit layouts, and phylogenetic tree reconstruction.

Definition 1 (the Steiner tree problem). *Consider an undirected graph $G = (V, E)$ with edge weights $w_e \in \mathbb{R}_0^+$ for every edge $e \in E$, and a set $T \subseteq V$ of terminals. A Steiner tree S is a tree in the graph G that spans all terminal vertices T and may contain some of the non-terminals (also called Steiner points). The goal is to find such a tree of lowest weight, which we call* OPT,

$$\mathrm{OPT} = \arg\min_{S} \sum_{e \in S} w_e.$$

We can assume without loss of generality[2] that the vertices are points in a metric space and the weights of the edges are given by the distance function – when the input is in the form of a metric, we call this the **metric Steiner tree problem**. Our results use properties of metric spaces, but move freely between the metric space and graph representations of the problem. When the metric is Euclidean, this is called the **Euclidean Steiner tree problem**.

Now we move on to defining Bilu-Linial stability for the Steiner tree problem on metrics.

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[1] Bilu-Linial stability is one among other notions of data stability studied in the literature [1, 5]. This is in contrast to notions of algorithmic stability, which focus on properties algorithms as opposed to data, see e.g. [2, 8, 11, 14, 16].

[2] For any graph with distances specified on edges, a metric can be formed by taking the vertices to be points and considering the shortest path distances in the graph between pairs of vertices. Solving (or approximating) the Steiner tree problem on a metric formed in this matter solves (or approximates) the problem on the original graph. See Vazirani [18] for further discussion of this issue.

Definition 2 (**Bilu-Linial γ-stabile instances).** *Let $I = (G, w)$ be an instance of a metric Steiner tree problem and $\gamma > 1$. I is γ-stable if for any function $w' : V \times V \to \mathbb{R}_0^+$ such that $\forall u, v \in V$,*

$$w_{uv} \le w'_{uv} \le \gamma w_{uv},$$

the optimal Steiner tree OPT' under w' is equal to the optimal Steiner tree OPT under w.

We note that the perturbations can be such that instances originally satisfying the metric or Euclidean properties no longer have to satisfy these properties after perturbation. We also note that due to the triangle inequality, no instances have stability 2 or greater in the metric setting.

Notation: For a graph G, w^G_{ab} is the weight of edge ab in G. We abbreviate $w_{ab} = w^G_{ab}$ and $w'_{ab} = w^{G'}_{ab}$. Let $\mathrm{OPT} \subseteq E(G)$ denote the minimum weight Steiner tree of G, let $w(\mathrm{OPT}) = w^G(\mathrm{OPT})$ denote the weight of the Steiner tree.

3 Structural properties in general metrics

In this section, we work in the context of a general metric space, and we develop interesting restrictions on the types of problems with γ-stable solutions, for various values of γ.

The techniques of this section *do not* give, in complete generality, an efficient algorithm for finding the optimal Steiner tree for any value of γ less than 2, a problem we leave open. However, when more information about the metric space is available, one can use the structural results here to give restrictions on the arrangements of Steiner points which does yield a definitive solution. In particular,

1. In Section 4, we use Lemma 3 to give an algorithm for the Euclidean metric when $\gamma > 2^{2/3}$.

2. More generally, in the case that no two Steiner points are adjacent in the optimal solution, Lemma 10 together with the other results of the section can be used to give an efficient and very simple algorithm to find the minimal weight Steiner tree. Other more general situations can be efficiently handled via only slightly more elaborate arguments - e.g. if one has a bound on the length of the longest path of Steiner points in the optimal solution.

Lemma 3. *The degree of any Steiner point in the optimal solution is greater than $\frac{2}{2-\gamma}$.*

Proof. Consider a Steiner node s in the optimal solution, that is connected to $(m \neq n)$ other points, $a_1, ..., a_m$. Let $\overline{w} = \sum_{i=1}^{m} \frac{w_{sa_i}}{m}$, and let w_{sa_1} and w_{sa_m} be such that $w_{sa_1} + w_{sa_m} \ge 2\overline{w}$. Let G' be obtained by perturbing each edge sa_i by a factor of γ. Let

$$\mathrm{T}' := (\mathrm{OPT} \setminus \{sa_1, \dots, sa_m\}) \cup \{a_1a_2, \dots, a_{m-1}a_m\}.$$

Clearly, T' is also a Steiner tree. Triangle inequality gives us $w_{a_ia_{i+1}} \le w_{sa_i} + w_{s_{a_{i+1}}}$. So, we have

$$\begin{aligned} w'(\mathrm{T}') &\le w'(\mathrm{OPT}) - \sum_{i=1}^{m} \gamma w_{sa_i} \\ &\quad + \sum_{i=1}^{m-1} \left(w_{sa_i} + w_{sa_{i+1}}\right) \\ &= w'(\mathrm{OPT}) - \sum_{i=1}^{m} \gamma w_{sa_i} + \sum_{i=2}^{m-1} 2w_{sa_i} \\ &\quad + w_{sa_1} + w_{sa_m}. \end{aligned}$$

Using the fact that $w'(\mathrm{T}') > w'(\mathrm{OPT})$, we have

$$\sum_{i=1}^{m} \gamma w_{sa_i} < \sum_{i=2}^{m-1} 2w_{sa_i} + w_{sa_1} + w_{sa_m}$$

or

$$\gamma \cdot \overline{w}m < (2m-2)\overline{w}.$$

Rearranging, we have

$$\frac{2}{2-\gamma} < m.$$

□

Now we state some additional structural properties of optimal Steiner trees in γ-stable instances. These are not used in Section 4. Nevertheless, we hope that they are of independent interest.

Proposition 4. *If $a, b \in V(\mathrm{OPT})$ are nearest neighbors in the graph, then the edge ab is in the optimal solution.*

Lemma 5. *Suppose $ab, bc \in \mathrm{OPT}$, then*

1. $w_{ac} > \gamma \cdot \max\{w_{ab}, w_{bc}\}$.
2. $\frac{2}{\gamma} \cdot w_{ac} > w_{ab} + w_{bc}$.
3. $(\gamma - 1) \cdot w_{ab} < w_{bc}$, $(\gamma - 1) \cdot w_{bc} < w_{ab}$.

Proof. We handle the three parts in turn:

1. Assume w.l.o.g. $w_{ab} \ge w_{bc}$. Suppose that $w_{ac} \le \gamma \cdot \max\{w_{ab}, w_{bc}\}$, let G' be obtained by perturbing ab by a factor of γ. Then $(\mathrm{OPT} \setminus \{ab\}) \cup \{ac\}$ is also a Steiner tree in G' of weight $w'(\mathrm{OPT})$ contradicting stability. This completes the proof of 1.

2. The proof of 2. follows from 1. and the fact that $\max\{w_{ab}, w_{bc}\} \ge \frac{w_{ab}+w_{bc}}{2}$.

3. Let G' be obtained by perturbing bc by a factor of γ. Then $\mathrm{T}' := \mathrm{OPT} \setminus \{bc\} \cup \{ac\}$ is also a Steiner tree of weight

$$w'(\mathrm{T}') = w(\mathrm{OPT}) - w_{bc} + w_{ac} \leq w(\mathrm{OPT}) + w_{ab}. \quad (1)$$

On the other hand, stability gives us that

$$w'(\mathrm{T}') > w'(\mathrm{OPT}) = w(\mathrm{OPT}) + (\gamma - 1)w_{bc}. \quad (2)$$

Putting (1) and (2) together gives us that $(\gamma - 1) \cdot w_{bc} \leq w_{ab}$.

Repeating the same argument but swapping bc for ab gives us $(\gamma - 1) \cdot w_{ab} \leq w_{bc}$.

□

Lemma 6. *Let H be a subgraph of* OPT *with at least one edge. Let $ab \in H$. Fix any vertex $c \in V(\mathrm{OPT}) \setminus V(H)$ satisfying $w_{ca} \leq \gamma(\gamma - 1) \cdot w_{ab}$; then we have $ca \in \mathrm{OPT}$.*

Proof. If $ca \notin \mathrm{OPT}$, then adding the edge ac to OPT produces a cycle which includes edge ac. Suppose that the cycle also includes ab. Let G' be obtained by perturbing ab by a factor of γ. Then $(\mathrm{OPT} \setminus \{ab\}) \cup \{ac\}$ is a Steiner tree of weight at most $w'(\mathrm{OPT})$, contradicting stability.

If the cycle does not include ab, it includes some edge other than ac which has an endpoint at a. This edge, call it ad, is in OPT. By Lemma 5, $w_{ad} > (\gamma - 1)w_{ba}$. Let G' be obtained by perturbing ad by a factor of γ. We have $w'_{ad} > \gamma(\gamma - 1)w_{ba} \geq w_{ac}$. Then $(\mathrm{OPT} \setminus \{ad\}) \cup \{ac\}$ is a Steiner tree of weight less than $w'(\mathrm{OPT})$, again contradicting stability. □

Lemma 7. *Let $\gamma > \frac{1+\sqrt{5}}{2}$. Let $ab \in H$, a subgraph of* OPT. *Suppose that c is a vertex with $w_{ca} \geq \gamma \cdot w_{ab}$, then $ca \notin$* OPT.

Proof. Let $\gamma' = \frac{w_{ca}}{w_{ab}}$. Note that $\gamma' \geq \gamma$ is some real number larger than $\frac{1+\sqrt{5}}{2}$. If $ac \in \mathrm{OPT}$, then by part 1. of Lemma 5, we must have

$$\frac{w_{bc}}{w_{ac}} > \gamma.$$

On the other hand,

$$\begin{aligned} \frac{w_{bc}}{w_{ac}} &\leq \frac{w_{ab} + w_{ac}}{w_{ac}} \\ &\leq \frac{w_{ab} + \gamma' w_{ab}}{\gamma' w_{ab}} \\ &\leq \frac{1 + \gamma'}{\gamma'}. \end{aligned}$$

We now have a contradiction as long as $\frac{1+\gamma'}{\gamma'} < \gamma$. The function $f(x) = \frac{1+x}{x}$ is decreasing for $x > 0$ and $f(x) < x$ for any $x \geq \frac{1+\sqrt{5}}{2}$. So, we have that

$$\frac{1 + \gamma'}{\gamma'} < \frac{1 + \gamma}{\gamma} < \gamma$$

as desired. □

Proposition 8. *Let H be a subgraph of* OPT *with at least one edge. Suppose that $ab \in H$ and suppose that $c \in V(\mathrm{OPT}) \setminus V(H)$ with $w_{bc} < \gamma(\gamma - 1)w_{ab}$. Then we must have $w_{bc} < \frac{w_{ab}}{\gamma - 1}$ and $w_{ab} < \frac{w_{bc}}{\gamma - 1}$.*

Proof. By Lemma 6, we must have that $bc \in \mathrm{OPT}$. Therefore, property 3. of Lemma 5 gives us the desired inequalities. □

When $\gamma(\gamma - 1)^2 > 1$ Proposition 8 strengthens the bounds of Lemma 6. This holds, for instance, when $\gamma > 1.755$. In this case, we obtain:

Proposition 9. *Assume that $\gamma(\gamma - 1)^2 > 1$. Assume that H is a subgraph of* OPT *with at least one edge. Let $ab \in H$. Fix any vertex $c \in V(\mathrm{OPT}) \setminus V(H)$. Then we have $w_{ca} < \frac{1}{\gamma - 1} \cdot w_{ab}$ if and only if $ca \in \mathrm{OPT}$.*

Proof. By Lemma 6 and the assumption that $\gamma(\gamma - 1) > \frac{1}{\gamma - 1}$, we must have that $ac \in \mathrm{OPT}$. If $w_{ca} \geq \frac{1}{\gamma - 1} \cdot w_{ab}$, we can not have edge ac in OPT by Lemma 5 part 3.

□

Let $B = \{b_1, \ldots, b_m\}$ be vertices (either terminal or Steiner points). For a vertex a, we denote by $T(a, B)$ the tree on vertex set a, B in which a is connected to each element of B. Let the *average weight of* $T(a, B)$ be

$$\frac{\sum_{i=1}^{m} w_{ab_i}}{m}.$$

Suppose that H is a subgraph of OPT. We call $T(a, B)$ a *terminal component fan* relative to H if a is a Steiner point and B are all terminals or vertices in distinct connected components of H each with at least two vertices. We call the collection of components of H together with the terminals not in H the *terminal components of H.*

Lemma 10. *Let $\gamma > 1.755$ and suppose that H is a subgraph of* OPT *and in the optimal solution, no two Steiner points are adjacent. Suppose that $T(a, B)$ with $B = \{b_1, \ldots, b_m\}$ is a terminal component fan such that:*

- *The average weight of $T(a, B)$ is less than all edges not in H which connect two terminal components of H,*
- *the average weight of $T(a, B)$ is minimal among all terminal component fans, and*
- *the weights of the edges of $T(a, B)$ are all within a factor of $\frac{1}{\gamma - 1}$ of each other.*

Then $T(a, B)$ is a subgraph of OPT.

Proof. Suppose that the fan $T(a, B)$ is not in OPT. Specifically, if there are $k < m$ edges of $T(a, B)$ which are not in OPT, then there are at least k edges of OPT$\setminus H$ such that in OPT $\cup T(a, B)$ we may remove these k edges and still have a Steiner tree.[3] Moreover, since no two Steiner points are adjacent, these edges are either

- terminal to terminal edges, or
- part of a terminal component fan.

In the first case, the terminal to terminal edges have weight at least $\frac{\sum_{i=1}^{m} w_{ab_i}}{m}$. In this case perturb this edge by a factor of γ, and swap it with one edge of the terminal component fan $T(a, B)$. Since the weights of edges of T are within a factor of $\frac{1}{\gamma-1}$ of each other and their average weight is $\frac{\sum_{i=1}^{m} w_{ab_i}}{m}$, this swap decreases the weight of the resulting Steiner tree after the perturbation.

Similarly in the case that one of the k edges is in another terminal component fan, T_1, the average weight of edges in that fan is at least $\frac{\sum_{i=1}^{m} w_{ab_i}}{m}$, and applying part 3. of Lemma 5, the minimal weight edge in T_1 is at least $(\gamma - 1) \cdot \frac{\sum_{i=1}^{m} w_{ab_i}}{m}$. Now, perturb such an edge by a factor of γ to make the weight at least $\gamma \cdot (\gamma - 1) \cdot \frac{\sum_{i=1}^{m} w_{ab_i}}{m}$, which is larger than the weight of the largest weight edge of $T(a, B)$, which is a most $\frac{1}{\gamma-1} \cdot \frac{\sum_{i=1}^{m} w_{ab_i}}{m}$ because $\gamma > 1.755$.

Performing any of these k swaps yields a lower weight Steiner tree than OPT under the above perturbations, contradicting γ-stability. □

4 Euclidean Steiner trees

In this section, we consider the restriction of the Steiner tree problem to the Euclidean metric.

Definition 11 (angle). *Let a_1, a_2, b be points on a Euclidean metric. Then we call $\angle a_1 b a_2$ the* angle *between a_1, a_2 at b.*

Under the assumption of γ-stability the minimum angle between two terminal points at their common Steiner neighbor can be bounded from below as a function of γ.

Lemma 12. *For a γ-stable instance of a Euclidean Steiner tree, the angle between two terminal points at their common Steiner neighbor in the tree should be greater than $2\sin^{-1}(\gamma/2)$.*

[3] In the case that $k = m$, there may be only $m - 1$ such edges, as a may not be in OPT, but the argument works identically in that case.

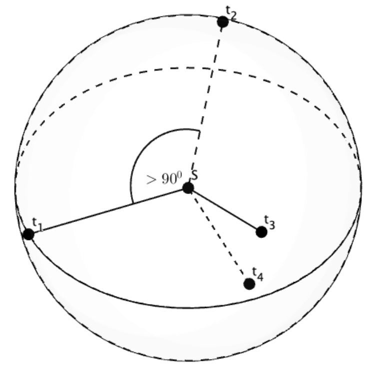

Figure 1: An example of points t_1, t_2, t_3, and t_4 surrounding Steiner point s at angles over $\theta > 90$ degrees. No more than $\frac{-1}{\cos\theta}$ can fit, independent of the dimension.

Proof. Let's assume, for a γ-stable instance of Steiner tree, the angle between two terminal points a_1, and a_2 at a Steiner point b is θ. Without loss of generality, let $w_{a_1 b} =: w \geq w_{a_2 b}$. Clearly $w_{a_1 a_2} > \gamma w$, since otherwise, perturbing edge $a_1 b$ by a factor of γ allows one to replace $a_1 b$ by $a_1 a_2$ in a minimal Steiner tree, contradicting stability. Let us use α to denote the angle $\angle a_1 a_2 b$. Clearly, $\alpha \geq \pi/2 - \theta/2$. Thus by the sine rule, we have

$$\frac{\gamma w}{\sin\theta} < \frac{w_{a_1 a_2}}{\sin\theta} = \frac{w}{\sin\alpha} \leq \frac{w}{\sin(\pi/2 - \theta/2)}.$$

Rearranging, we have

$$\begin{aligned} \gamma &< \frac{\sin\theta}{\sin(\pi/2 - \theta/2)} \\ &= \frac{2\sin(\theta/2)\cos(\theta/2)}{\cos(\theta/2)} \\ &= 2\sin(\theta/2) \end{aligned}$$

as desired. □

Thus we immediately get the following Corollary.

Corollary 13. *For a γ-stable instances of Steiner tree where $\gamma > \sqrt{2}$, the angle between two terminal points at their common neighbor in the optimal Steiner tree is greater than $\pi/2$.*

We say that a matrix $M \in \mathbb{R}^{d \times d}$ is *positive semidefinite* if for every $v \in \mathbb{R}^d$, it holds that $v^T R v \geq 0$.

Lemma 14. *If there are N points in $\mathbb{R}^d$ such that the angle between every pair with respect to a point u is at least $\theta > (\pi/2)$, then $N \leq 1 - \frac{1}{\cos\theta}$.*

Proof. Let $\theta > \pi/2$ and let $v_1, \ldots, v_N \in \mathbb{R}^d$ be unit vectors in $\mathbb{R}^d$ such that $\langle v_i, v_j \rangle \leq \cos\theta$. Consider the matrix V whose columns are the v_i's. By construction

$V^T V$ is positive semi-definite. Indeed, for any $u \in \mathbb{R}^d$, we have $u^T(V^T V)u = \langle Vu, Vu\rangle \geq 0$.

If $N - 1 > \frac{-1}{\cos\theta}$, then the sum of every row is negative. This is because each diagonal entry of $V^T V$ is 1, and every non-diagonal entry is at most $\cos\theta$. So we have that $\mathbf{1}^T(V^T V)\mathbf{1} < 0$ where $\mathbf{1} = (1, 1, \ldots, 1)$. This contradicts the positive semidefiniteness of $V^T V$. So it must be the case that $N \leq 1 - \frac{1}{\cos\theta}$. □

Corollary 15. *For $\gamma > \sqrt{2}$ the degree of a Steiner node in the optimal solution is at most $\frac{\gamma^2}{\gamma^2-2}$.*

Proof. Consider any two neighbors u, w of a given vertex v, and assume that $\angle uvw = \theta$. From Lemma 12 we have

$$\gamma < 2\sin(\theta/2).$$

So

$$\gamma^2 < 4\sin^2(\theta/2)$$

and so $\gamma^2/2 < 2\sin^2(\theta/2)$ or $1 - \gamma^2/2 > 1 - 2\sin^2(\theta/2)$. Since $\cos(\theta) = 1 - 2\sin^2(\theta/2)$, we have

$$\cos(\theta) < 1 - \gamma^2/2$$

or

$$\theta > \cos^{-1}(1 - \gamma^2/2).$$

Since the angle between any two neighbors of v is at least $\cos^{-1}(1-\gamma^2/2)$, Lemma 14 gives us that there are at most $1 - \frac{2}{2-\gamma^2} = \frac{\gamma^2}{\gamma^2-2}$ of them. □

Corollary 16. *When $\gamma > 1.59$, the optimal Steiner tree for a γ-stable instance does not have Steiner nodes.*

Proof. This happens when the min degree imposed by stability is larger than the max degree imposed by the packing bound. By Lemma 3 and Corollary 15, this happens when we have the following:

$$\frac{\gamma^2}{\gamma^2 - 2} \leq \frac{2}{2 - \gamma}$$

By solving the above equation for γ we get $\gamma \geq 2^{2/3}$, which is bounded from above by 1.59. □

This geometric property implies that for 1.59-stable instances, Steiner points will not be used in the optimal solution. Hence, an MST algorithm on just the terminal points will give the answer in polynomial time.

Finally, we point to the existence of Gilbert and Pollak's the Steiner ratio conjecture [12], which states that in the Euclidean plane, there always exists an MST within a cost of $2/\sqrt{3}$ of the minimum Steiner tree,and the behavior of this ratio for higher dimensions is yet unknown. Assuming this conjecture, in certain cases it may imply some limitations on the stability of Euclidean instances, especially in low dimensions, using the idea that even if the Steiner tree distances are "blown up" by more than the Steiner ratio, one could instead use the MST instead and get a cheaper solution. Unfortunately, because the MST may overlap with the Steiner tree, we cannot give a concrete statement.

5 Using approximation algorithms to solve stable instances

In this section we give a general argument about how strong approximation algorithms for Steiner tree problems give stability guarantees. We note that it is known that an FPTAS for the Steiner tree would imply P=NP [10], so there is no hope to use the result below in the general metric case. But if at some future point an FPTAS for the Euclidean variant of the Steiner tree problem is developed (currently, only a PTAS is known to exist [3]), then this would immediately imply the existence of polynomial-time algorithms for stable instances for any constant $\gamma > 1$.

Theorem 17. *An FPTAS for the Steiner tree problem gives a polynomial time algorithm for optimally solving any γ-stable Steiner tree problem in time poly$(n, (\gamma - 1)^{-1})$. In particular, this gives a polynomial-time algorithm for any constant $\gamma > 1$.*

Proof. Assume we are given an FPTAS for the Steiner tree problem. This means that we have an algorithm that runs in time poly$(n, 1/\epsilon)$ on instances of size n to give $(1+\epsilon)$-approximations to the optimum Steiner tree. Now consider a γ-stable instance for constant $\gamma > 1$. We run our FPTAS on that instance with $\epsilon = \frac{\gamma-1}{2n}$ to get a Steiner tree S' with weight within $\mathrm{OPT}(1+(\gamma-1)/2n)$. We now claim that every edge in the optimal solution whose weight is at least $\frac{\mathrm{OPT}}{n}$ must be in S'. Suppose it isn't – then we could perturb such an edge by γ and increase the weight of the optimal solution to $\mathrm{OPT}(1 + (\gamma - 1)/n)$ without increasing the weight of S', and S' would become cheaper than OPT, thereby violating γ-stability.

By the fractional pigeonhole principle, the most expensive edge of the FPTAS satisfies the desired property above and is therefore in OPT. Hence, we can contract this edge into a new vertex and get a new instance with $n - 1$ vertices at γ-stability. We can continue this process, getting one new edge of the optimal in each iteration, until we have a constant-size problem that we can brute-force. □

We note that the above technique could be used to convert even slightly weaker (than FPTAS) approximation algorithms to nontrivial stability guarantees.

Acknowledgments

We thank Xing Gao for useful discussions, and in particular, for pointing out an error in a previous version of Lemma 14.

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