

## THE DILOGARITHM AND ABELIAN CHERN-SIMONS

DANIEL S. FREED &amp; ANDREW NEITZKE

**Abstract**

We construct the (enhanced Rogers) dilogarithm function from the spin Chern-Simons invariant of  $\mathbb{C}^\times$ -connections. This leads to geometric proofs of basic dilogarithm identities and a geometric context for other properties, such as the branching structure.

**1. Introduction**

Investigations of the dilogarithm function, in its simplest form defined via the power series

$$(1.1) \quad \text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2},$$

date back to Leibniz, Bernoulli, and Euler; one early reference is the 1809 essay of William Spence [S]. In recent times the dilogarithm makes appearances in hyperbolic geometry, algebraic  $K$ -theory, conformal field theory, and beyond. It and its relatives are the subject of survey articles, such as [G, K, Z] which provide a wealth of references to the literature. In their study of the scissors congruence problem, Dupont-Sah [DS] and subsequently Dupont [D] relate a variant of the dilogarithm function (1.1) to a Chern-Cheeger-Simons [CS, ChS] characteristic class of flat principal  $\text{SL}_2 \mathbb{C}$ -bundles. As part of our study [FN] of “stratified abelianization” of flat  $\text{SL}_2 \mathbb{C}$ -connections on 3-manifolds, we discovered a geometric construction of this *enhanced Rogers dilogarithm* using *abelian* Chern-Simons theory of flat  $\mathbb{C}^\times$ -connections on a 2-dimensional torus. In this paper we present our construction, and we use it to give geometric proofs of the basic dilogarithm identities.

We begin in §2 with an exposition of Chern-Simons invariants of  $\mathbb{C}^\times$ -connections. Our work uses a square root for spin manifolds, which we outline in §3 and develop with proofs in Appendix A. The construction

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of the dilogarithm is carried out in §4 and the identities are proved in §5. An inspiration for our construction, independent of stratified abelianization, is a heuristic computation motivated by topological string theory, as we explain in Appendix B.

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## 2. Classical Chern-Simons theory

The Chern-Simons invariant is defined for pairs  $(G, \lambda)$  consisting of a real Lie group  $G$  with finitely many components and a class  $\lambda \in H^4(BG; \mathbb{Z})$ . In this section we focus on the special case  $(\mathbb{C}^\times, c_1^2)$ , where  $c_1 \in H^2(B\mathbb{C}^\times; \mathbb{Z})$  is the universal first Chern class. Here we briefly indicate the constructions for trivializable  $\mathbb{C}^\times$ -bundles. In Appendix A we provide a general construction based on differential cohomology which applies to all principal  $\mathbb{C}^\times$ -bundles. The treatment in Appendix A emphasizes the spin refinement of the Chern-Simons invariant, though it can be adapted to the non-spin case discussed in this section. The references [F1, F2] contain more exposition and details. The basic properties of classical Chern-Simons theory are compactly expressed in the language of field theory (Theorem 2.13).

Let  $W$  be a closed oriented 4-manifold and  $P \rightarrow W$  a principal  $\mathbb{C}^\times$ -bundle. Then

$$(2.1) \quad \langle c_1(P)^2, [W] \rangle \in \mathbb{Z}$$

is a *primary topological invariant* of  $P \rightarrow W$ . Chern-Simons [CS] construct a *secondary geometric invariant* of  $\mathbb{C}^\times$ -bundles with connection as follows. Let  $M$  be an arbitrary smooth manifold and  $\pi: P \rightarrow M$  a principal  $\mathbb{C}^\times$ -bundle. Let  $\Theta \in \Omega_P^1(\mathbb{C})$  be a connection and  $\Omega \in \Omega_M^2(\mathbb{C})$  its curvature, i.e.,  $\pi^*\Omega = d\Theta$ . Define

$$(2.2) \quad \omega(\Theta) = -\frac{1}{4\pi^2} \Omega \wedge \Omega \in \Omega_M^4(\mathbb{C})$$

$$(2.3) \quad \alpha(\Theta) = -\frac{1}{4\pi^2} \Theta \wedge \Omega \in \Omega_P^3(\mathbb{C}).$$

Then  $d\omega = 0$  and  $d\alpha = \pi^*\omega$ . These are the *Chern-Weil* and *Chern-Simons* forms, respectively. For  $M = W$  a closed oriented 4-manifold, we have

$$(2.4) \quad \int_W \omega(\Theta) = \langle c_1(P)^2, [W] \rangle;$$

in particular, the left-hand side is independent of the connection  $\Theta$ .

Suppose  $M = X$  is a closed oriented 3-manifold and  $\pi: P \rightarrow X$  is trivializable. For each section  $s$  of  $\pi$  define

$$(2.5) \quad \Gamma(\Theta, s) = \int_X s^* \alpha(\Theta).$$

If  $s' = s \cdot g$  for  $g: X \rightarrow \mathbb{C}^\times$ , then

$$(2.6) \quad (s \cdot g)^* \alpha(\Theta) = s^* \alpha(\Theta) - \frac{1}{4\pi^2} d \left( s^* \Theta \wedge \frac{dg}{g} \right).$$

Therefore, by Stokes' theorem  $\Gamma(\Theta, s)$  is independent of  $s$ . The Chern-Simons invariant is

$$(2.7) \quad \mathcal{F}(X; \Theta) = \exp(2\pi\sqrt{-1}\Gamma(\Theta, s)) \in \mathbb{C}^\times.$$

**Remark 2.8.** An alternative definition uses the fact that  $P \rightarrow X$  can be written as the boundary of a principal  $\mathbb{C}^\times$ -bundle  $\tilde{P} \rightarrow W$  over a compact oriented 4-manifold  $W$  with  $\partial W = X$ . The connection  $\Theta$  extends to a connection  $\tilde{\Theta}$  on  $\tilde{P} \rightarrow W$ , and by Stokes' theorem the right hand side of (2.5) equals

$$(2.9) \quad \int_W \omega(\tilde{\Theta}).$$

This definition works for any (possibly nontrivializable)  $P \rightarrow X$  with connection, but (2.9) is only independent of the choice of the extension  $\tilde{P} \rightarrow W$  modulo integers; see (2.4).

If  $X$  is a compact oriented 3-manifold with  $\partial X = Y$ , then  $\Gamma(\Theta, s)$  is not independent of  $s$ ; the exact term in (2.6) produces a boundary correction. The dependence of  $\exp(2\pi\sqrt{-1}\Gamma(\Theta, s))$  on  $s$  is encoded as follows. Set  $\rho: Q = P|_Y \rightarrow Y$  and  $\eta = \Theta|_Q$ . Under our hypothesis the space  $\text{Sect}(\rho)$  of sections of  $\rho: Q \rightarrow Y$  is nonempty; it is a torsor over  $\text{Map}(Y, \mathbb{C}^\times)$ . Define the complex line

$$(2.10) \quad \mathcal{F}(Y; \eta) = \left\{ f: \text{Sect}(\rho) \rightarrow \mathbb{C} : \right.$$

$$f(t \cdot h) = \exp \left( -\frac{\sqrt{-1}}{2\pi} \int_Y t^* \eta \wedge \frac{dh}{h} \right) f(t)$$

$$\left. \text{for all } t \in \text{Sect}(\rho), h \in \text{Map}(Y, \mathbb{C}^\times) \right\}.$$

Then

$$(2.11) \quad \mathcal{F}(X; \Theta) := \exp(2\pi\sqrt{-1}\Gamma(\Theta, -)) \in \mathcal{F}(Y; \eta)$$

is a well-defined nonzero element of the line  $\mathcal{F}(Y; \eta)$ .

**Remark 2.12.** By construction, a trivialization  $t$  of  $\rho: Q \rightarrow Y$  induces a trivialization  $\mathbb{C} \xrightarrow{\cong} \mathcal{F}(Y; \eta)$ , i.e., a nonzero element  $\tau_t \in \mathcal{F}(Y; \eta)$ .

The formal properties are best summarized in field theory language. Let  $\text{Bord}_{\langle 2,3 \rangle}(\text{SO}_3 \times (\mathbb{C}^\times)^\nabla)$  denote the category<sup>1</sup> whose objects are

<sup>1</sup>For many reasons, among others to construct smooth gluings, the objects are germs of 3-dimensional data over the 2-dimensional data; see [Se]. Our language

closed oriented 2-manifolds  $Y$  equipped with a  $\mathbb{C}^\times$ -connection  $\Theta_Y$ ; a morphism  $(Y_0, \Theta_0) \rightarrow (Y_1, \Theta_1)$  is a compact oriented 3-manifold  $X$  equipped with a  $\mathbb{C}^\times$ -connection  $\Theta_X$ , a diffeomorphism  $-Y_0 \amalg Y_1 \xrightarrow{\cong} \partial X$  (an oriented manifold  $M$  has a canonical reflection  $-M$  with the opposite orientation), and a compatible isomorphism  $\Theta_0 \amalg \Theta_1 \xrightarrow{\cong} \partial \Theta_X$ . Composition of morphisms is gluing of bordisms, and disjoint union provides a symmetric monoidal structure. Let  $\text{Line}_{\mathbb{C}}$  denote the groupoid whose objects are 1-dimensional complex vector spaces and morphisms are invertible linear maps; tensor product provides a symmetric monoidal structure.

**Theorem 2.13.** *The exponentiated Chern-Simons invariant is a symmetric monoidal functor*

$$(2.14) \quad \mathcal{F}: \text{Bord}_{\langle 2,3 \rangle}(\text{SO}_3 \times (\mathbb{C}^\times)^\nabla) \longrightarrow \text{Line}_{\mathbb{C}}.$$

In other words,  $\mathcal{F}$  is an *invertible field theory*, often called *classical Chern-Simons theory*.

As mentioned in footnote <sup>1</sup>, one can extend  $\mathcal{F}$  to a theory defined on smooth families. Thus if  $\mathcal{Y} \rightarrow S$  is a fiber bundle with fibers closed oriented 2-manifolds, and  $\mathcal{Q} \rightarrow Y$  is a principal  $\mathbb{C}^\times$ -bundle with connection  $\eta$ , then the Chern-Simons theory produces  $\mathcal{F}(\mathcal{Y}/S; \eta) \rightarrow S$ , a complex line bundle with covariant derivative. Its curvature is

$$(2.15) \quad \text{curv}(\mathcal{F}(\mathcal{Y}/S; \eta) \rightarrow S) = \frac{\sqrt{-1}}{2\pi} \int_{\mathcal{Y}/S} \Omega(\eta) \wedge \Omega(\eta),$$

where  $\Omega(\eta) \in \Omega^2_{\mathcal{Y}}(\mathbb{C})$  is the curvature of the  $\mathbb{C}^\times$ -connection  $\eta$ . Parallel transport along a path  $\gamma: [0, 1] \rightarrow S$  is the value of  $\mathcal{F}(\gamma^*\mathcal{Y}; \gamma^*\eta)$  on the pullback connection  $\gamma^*\eta$  on  $\gamma^*\mathcal{Q} \rightarrow \gamma^*\mathcal{Y}$ . In particular, holonomies of  $\mathcal{F}(\mathcal{Y}/S; \eta) \rightarrow S$  are computed as values of  $\mathcal{F}$  on mapping tori.

**Remark 2.16.** Cheeger-Simons [ChS] introduce differential characters and write the Chern-Simons invariant of a closed oriented 3-manifold in those terms. The entire theory  $\mathcal{F}$  fits into the theory of differential cohomology [HS, BNV], beginning with a differential lift of  $c_1^2 \in H^4(B\mathbb{C}^\times; \mathbb{Z})$ . We develop this idea in Appendix A.

Our application of  $\mathcal{F}$  in §4 is to families of *flat*  $\mathbb{C}^\times$ -connections. In the remainder of this section we observe some properties which reflect the *topological* nature of  $\mathcal{F}$  on flat connections.

First, a principal  $\mathbb{C}^\times$ -bundle  $P \rightarrow M$  over any smooth manifold  $M$  admits a flat connection if and only if  $c_1(P) \in H^2(M; \mathbb{Z})$  has finite order. In particular, if  $H_1(M)$  is torsionfree, then only trivializable principal

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in the text is a simplification. Also, it is crucial to extend to parametrized families of  $\mathbb{C}^\times$ -connections, for example to capture the variation formula for Chern-Simons invariants, as we recount below. See [ST] for field theories sheafified over smooth manifolds.

$\mathbb{C}^\times$ -bundles on  $M$  admit flat connections. The equivalence class of a flat connection on  $P \rightarrow M$  is a lift  $[\Theta] \in H^1(M; \mathbb{C}/\mathbb{Z})$  of  $c_1(P) \in H^2(M; \mathbb{Z})$  in the long exact sequence of cohomology groups induced from the short exact sequence  $\mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$  of coefficients.

**Theorem 2.17.** (i) *Let  $\Theta$  be a flat connection on a principal  $\mathbb{C}^\times$ -bundle  $\pi: P \rightarrow X$  over a closed oriented 3-manifold. Then its Chern-Simons invariant is*

$$(2.18) \quad \mathcal{F}(X; \Theta) = \exp\left(2\pi\sqrt{-1}\langle [\Theta] \smile c_1(P), [X] \rangle\right),$$

$$[\Theta] \in H^1(M; \mathbb{C}/\mathbb{Z}).$$

- (ii) *Let  $\eta$  be a flat connection on a principal  $\mathbb{C}^\times$ -bundle  $\rho: Q \rightarrow Y$  over a closed oriented 2-manifold. Then the trivialization  $\tau_t \in \mathcal{F}(Y; \eta)$  in Remark 2.12 depends only on the homotopy class of the section  $t$  of  $\rho$ .*
- (iii) *Let  $\Theta$  be a flat connection on a principal  $\mathbb{C}^\times$ -bundle  $\pi: P \rightarrow X$  over a compact oriented 3-manifold with boundary. Suppose  $s$  is a section of  $\pi$ . Then  $\mathcal{F}(X; \Theta) = \tau_{\partial s}$  in  $\mathcal{F}(\partial X; \partial\Theta)$ .*

*Proof.* Part (i) is easy unless  $\pi$  does not admit a section, in which case the techniques here do not apply; we supply a proof at the end of Appendix A. For (ii) observe that the integrand in (2.10) is the product of two closed 1-forms, so only depends on their de Rham cohomology classes. The generalization of (iii) to arbitrary connections  $\Theta$  is

$$(2.19) \quad \mathcal{F}(X; \Theta) = \exp\left(2\pi\sqrt{-1}\int_X s^* \alpha(\Theta)\right) \tau_{\partial s},$$

which is essentially the construction of (2.11). For flat  $\Theta$  the Chern-Simons form  $\alpha(\Theta)$  vanishes. q.e.d.

### 3. The spin refinement

On spin manifolds Chern-Simons theory refines to a theory  $\mathcal{S}$  with  $\mathcal{S}^{\otimes 2} \cong \mathcal{F}$ . The extra factor of 2 flows from that in the primary invariant (2.1) on spin manifolds: if  $P \rightarrow W$  is a principal  $\mathbb{C}^\times$ -bundle over a closed spin 4-manifold  $W$ , then

$$(3.1) \quad \frac{1}{2} \langle c_1(P)^2, [W] \rangle \in \mathbb{Z}.$$

In this section we state the properties of  $\mathcal{S}$  we need; proofs are deferred to Appendix A.

**Remark 3.2.** The function

$$(3.3) \quad \begin{aligned} H^2(W; \mathbb{Z}) &\longrightarrow \mathbb{Z} \\ x &\longmapsto \frac{1}{2} \langle x \smile x, [W] \rangle \end{aligned}$$

is a quadratic refinement of the bihomomorphism

$$(3.4) \quad \begin{aligned} H^2(W; \mathbb{Z}) \times H^2(W; \mathbb{Z}) &\longrightarrow \mathbb{Z} \\ x \quad , \quad y &\longmapsto \langle x \smile y, [W] \rangle \end{aligned}$$

There is a compatible quadratic form on  $H^1(X; \mathbb{C}/\mathbb{Z})$  for  $X$  a closed spin 3-manifold [MS, LL]; it computes the value of  $\mathcal{S}(X; \Theta)$  for flat connections  $\Theta$ .

As in Theorem 2.13 the formal properties are summarized in the statement that  $\mathcal{S}$  is an invertible field theory (which can be evaluated on fiber bundles). Manifolds in the domain bordism category of  $\mathcal{S}$  carry a spin structure. The codomain of  $\mathcal{S}$  is the Picard groupoid  $s\text{Line}_{\mathbb{C}}$  whose objects are  $\mathbb{Z}/2\mathbb{Z}$ -graded (super) lines and whose morphisms are even isomorphisms of super lines. The monoidal structure is tensor product, and the symmetry implements the Koszul sign rule.

**Theorem 3.5.** *Spin Chern-Simons theory is a symmetric monoidal functor*

$$(3.6) \quad \mathcal{S}: \text{Bord}_{\langle 2,3 \rangle}(\text{Spin}_3 \times (\mathbb{C}^{\times})^{\nabla}) \longrightarrow s\text{Line}_{\mathbb{C}}.$$

*There is an isomorphism  $\mathcal{S}^{\otimes 2} \cong \mathcal{F}$ .*

**Remark 3.7.** The invariant of a  $\mathbb{C}^{\times}$ -connection over a closed spin 3-manifold has a description analogous to that in Remark 2.8. In that case we must bound  $X$  by a compact *spin* 4-manifold and put the factor  $1/2$  in the integral (2.9).

For convenience we state further properties of  $\mathcal{S}$  simultaneously for families of 2- and 3-manifolds. Consider the sequence

$$(3.8) \quad P \xrightarrow{\pi} M_{\sigma} \xrightarrow{p} S$$

of smooth maps in which  $p$  is a fiber bundle of smooth manifolds, or of bordisms;  $\pi$  is a principal  $\mathbb{C}^{\times}$ -bundle with connection  $\Theta \in \Omega_P^1(\mathbb{C})$ ; and  $\sigma$  is a spin structure on the relative tangent bundle  $T(M/S) \rightarrow S$ . Let  $\dim(p): M \rightarrow \mathbb{Z}^{\geq 0}$  be the locally constant function whose value at  $m \in M$  is the dimension of the relative tangent space  $T_m(M/S)$ . Let  $\Omega(\Theta) \in \Omega_M^2(\mathbb{C})$  be the curvature of  $\Theta$ . The following is proved in Appendix A.

**Theorem 3.9.** (i) *If  $\dim(p) = 3$  and the fibers of  $p$  are closed, then  $\mathcal{S}(M_{\sigma}/S; \Theta): S \rightarrow \mathbb{C}^{\times}$  satisfies*

$$(3.10) \quad \frac{d\mathcal{S}(M_{\sigma}/S; \Theta)}{\mathcal{S}(M_{\sigma}/S; \Theta)} = -\frac{\sqrt{-1}}{4\pi} \int_{M/S} \Omega(\Theta) \wedge \Omega(\Theta).$$

(ii) Continuing, if  $\sigma'$  is a spin structure whose difference with  $\sigma$  represents a class  $\delta \in H^1(M; \mathbb{Z}/2\mathbb{Z})$ , then the ratio of spin Chern-Simons invariants is

$$(3.11) \quad \frac{\mathcal{S}(M_{\sigma'}/S; \Theta)}{\mathcal{S}(M_{\sigma}/S; \Theta)} = (-1)^{p_*(\delta \cup \bar{c}_1(P))},$$

where  $\bar{c}_1$  is the mod 2 reduction of the first Chern class.

(iii) If  $\dim(p) = 3$  and the fibers of  $p$  are compact with boundary, then  $\mathcal{S}(M_{\sigma}/S; \Theta)$  is a section of the even complex line bundle  $\mathcal{S}(\partial M_{\sigma}/S; \Theta) \rightarrow S$ ; its covariant derivative is

$$(3.12) \quad \nabla \mathcal{S}(M_{\sigma}/S; \Theta) = \left[ -\frac{\sqrt{-1}}{4\pi} \int_{M/S} \Omega(\Theta) \wedge \Omega(\Theta) \right] \mathcal{S}(M_{\sigma}/S; \Theta).$$

(iv) Continuing, suppose  $s: M \rightarrow P$  is a section of  $\pi$ . Its restriction  $\partial s$  to  $\partial M$  induces a trivialization  $\tau_{\partial s}$  of  $\mathcal{S}(\partial M_{\sigma}/S; \Theta) \rightarrow S$ . Then

$$(3.13) \quad \mathcal{S}(M_{\sigma}/S; \Theta) = \exp \left( -\frac{\sqrt{-1}}{4\pi} \int_{M/S} s^* \Theta \wedge \Omega(\Theta) \right) \tau_{\partial s}.$$

(v) If  $\dim(p) = 2$  and the fibers of  $p$  are closed, then  $\mathcal{S}(M_{\sigma}/S; \Theta) \rightarrow S$  is a complex super line bundle with covariant derivative; its curvature is

$$(3.14) \quad \text{curv}(\mathcal{S}(M_{\sigma}/S; \Theta) \rightarrow S) = \frac{\sqrt{-1}}{4\pi} \int_{M/S} \Omega(\Theta) \wedge \Omega(\Theta).$$

Its  $\mathbb{Z}/2\mathbb{Z}$ -grading is  $p_*[\bar{c}_1(P)]: S \rightarrow \mathbb{Z}/2\mathbb{Z}$ , where  $\bar{c}_1$  is the mod 2 reduced first Chern class.

(vi) Continuing, a section  $t: M \rightarrow P$  of  $\pi$  induces a trivialization  $\tau_t: S \rightarrow L$  of the Chern-Simons line bundle, relative to which the connection form is

$$(3.15) \quad \frac{\nabla \tau_t}{\tau_t} = \frac{\sqrt{-1}}{4\pi} \int_{M/S} t^* \Theta \wedge \Omega(\Theta).$$

(vii) Continuing, given  $h: M \rightarrow \mathbb{C}^\times$  set  $t' = t \cdot h: M \rightarrow P$ . Then

$$(3.16) \quad \tau_{t'} = \epsilon_{t,h} \exp \left( -\frac{\sqrt{-1}}{4\pi} \int_{M/S} t^* \Theta \wedge \frac{dh}{h} \right) \tau_t,$$

where

$$(3.17) \quad \epsilon_{t,h}(s) = (-1)^{\sigma_s([h_s])}, \quad s \in S.$$

Here  $\sigma_s: H^1(p^{-1}(s); \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}$  is the quadratic refinement of the intersection pairing given by the spin structure on the fiber  $p^{-1}(s)$ , and  $[h_s] \in H^1(p^{-1}(s); \mathbb{Z}/2\mathbb{Z})$  is the reduction modulo two of the homotopy class of  $h|_{p^{-1}(s)}$ .

**Corollary 3.18.** *In the situation of Theorem 3.9 (vii), if  $\Theta$  is flat then (3.16) only depends on the homotopy class of  $h$ .*

*Proof.* The factor  $\epsilon_{t,h}$  in (3.17) depends only on the homotopy class of  $h$  modulo two. If  $\Theta$  is flat, then  $t^*\Theta$  is a closed 1-form, and the integral in (3.16) reduces to the cohomology pushforward  $p_*: H^2(M; \mathbb{C}) \rightarrow H^0(S; \mathbb{C})$  applied to  $[t^*\Theta] \smile [h]$ , as in Theorem 2.17(ii).  $\square$

In the setup of Theorem 3.9, suppose given two families over the same base  $S$  and maps

$$(3.19) \quad \begin{array}{ccc} P & \xrightarrow{\tilde{\psi}} & P' \\ \pi \downarrow & & \downarrow \pi' \\ M_\sigma & \xrightarrow{\psi} & M'_{\sigma'} \\ & \searrow & \swarrow \\ & S & \end{array}$$

such that  $\psi$  is a diffeomorphism of spin manifolds,  $\tilde{\psi}$  is an isomorphism of principal  $\mathbb{C}^\times$ -bundles, and  $\tilde{\psi}$  preserves the connections:  $\tilde{\psi}^*\Theta' = \Theta$ .

**Theorem 3.20.** (i) *If  $\dim(p) = 3$  and the fibers of  $p$  are closed, then  $\mathcal{S}(M'_{\sigma'}/S; \Theta') = \mathcal{S}(M_\sigma/S; \Theta)$ .*  
(ii) *If  $\dim(p) = 2$  and the fibers of  $p$  are closed, then there is a flat isomorphism of spin Chern-Simons line bundles*

$$(3.21) \quad \begin{array}{ccc} \mathcal{S}(M_\sigma/S; \Theta) & \xrightarrow{\Psi} & \mathcal{S}(M'_{\sigma'}/S; \Theta') \\ & \searrow & \swarrow \\ & S & \end{array}$$

(iii) *Continuing, if  $t: M \rightarrow P$  and  $t': M' \rightarrow P'$  are sections of  $\pi, \pi'$  such that  $\tilde{\psi} \circ t = t' \circ \psi$ , then the induced trivializations  $\tau_t, \tau_{t'}$  of the line bundles in (3.21) satisfy  $\tau_{t'} = \Psi \circ \tau_t$ .*

We leave the reader to formulate and prove (1) functoriality for the case  $\dim(p) = 3$  and the fibers of  $p$  are compact manifolds with boundary, and (2) the behavior of  $\mathcal{S}$  under reversal of orientation. Theorem 3.20 follows from the constructions in Appendix A.

#### 4. The dilogarithm from abelian Chern-Simons

In §4.1 we use the spin Chern-Simons theory of §3 to construct a holomorphic function  $L$  on an abelian cover of the thrice punctured complex projective line. We identify it with the enhanced Rogers dilogarithm in §4.2.

#### 4.1. Construction of $L$ .

Fix the standard torus

$$(4.1) \quad T = \mathbb{R}^2 / \mathbb{Z}^2$$

and standard coordinates  $\theta^1, \theta^2$  on  $\mathbb{R}^2$ . The first homology group  $H_1(T)$  has generators the coordinate loops  $t \mapsto (t, 0)$  and  $t \mapsto (0, t)$ ,  $0 \leq t \leq 1$ , in  $(\theta^1, \theta^2)$  coordinates. Let  $\sigma$  be the spin structure on  $T$  characterized by the property that the coordinate loops inherit the bounding spin structure. Introduce

$$(4.2) \quad \begin{aligned} \hat{\mathcal{M}}_T &= \mathbb{C}^2 \\ \mathcal{M}_T &= (\mathbb{C}^\times)^2 \end{aligned}$$

with standard coordinates  $u_1, u_2$  and  $\mu_1, \mu_2$ , respectively. Define the principal  $\mathbb{Z}^2$ -bundle

$$(4.3) \quad \begin{aligned} e: \hat{\mathcal{M}}_T &\longrightarrow \mathcal{M}_T \\ (u_1, u_2) &\longmapsto (e^{u_1}, e^{u_2}) \end{aligned}$$

The notation (4.2) is deliberately evocative of moduli spaces; see Remark 4.9 below.

The trivial  $\mathbb{C}^\times$ -bundle

$$(4.4) \quad \hat{\rho}: \hat{\mathcal{M}}_T \times T \times \mathbb{C}^\times \longrightarrow \hat{\mathcal{M}}_T \times T$$

with section  $\hat{t}_0$  carries a complex connection form  $\hat{\eta}$  characterized by

$$(4.5) \quad \hat{t}_0^* \hat{\eta} = -u_1 d\theta^1 - u_2 d\theta^2 \in \Omega^1_{\hat{\mathcal{M}}_T \times T}(\mathbb{C}).$$

The action of  $\mathbb{Z}^2$  on the base of (4.4) lifts to the total space:

$$(4.6) \quad \begin{aligned} &(n_1, n_2) \cdot (u_1, u_2, \theta^1, \theta^2, \lambda) \\ &= \left( u_1 + 2\pi\sqrt{-1}n_1, u_2 + 2\pi\sqrt{-1}n_2, \theta^1, \theta^2, \exp[2\pi\sqrt{-1}(n_1\theta^1 + n_2\theta^2)]\lambda \right), \end{aligned}$$

where  $(n_1, n_2) \in \mathbb{Z}^2$  and  $\lambda \in \mathbb{C}^\times$ ; the connection form  $\hat{\eta}$  is preserved. Hence the  $\mathbb{C}^\times$ -bundle (4.4) with connection descends to a  $\mathbb{C}^\times$ -bundle

$$(4.7) \quad \rho: \mathcal{Q} \longrightarrow \mathcal{M}_T \times T$$

with connection  $\eta$ . Its curvature is the differential of (4.5):

$$(4.8) \quad \Omega(\eta) = -\frac{d\mu_1}{\mu_1} \wedge d\theta^1 - \frac{d\mu_2}{\mu_2} \wedge d\theta^2.$$

This formula shows that the bundle  $\rho$  in (4.7) is topologically nontrivial. (The base  $\mathcal{M}_T \times T$  deformation retracts to a 4-torus; the restrictions to two sub 2-tori have nonzero first Chern class.)

**Remark 4.9.** The connection  $\eta$  on  $\rho$  in (4.7) defines a *universal* family of *flat*  $\mathbb{C}^\times$ -connections over  $T$ . Namely,  $\eta$  is flat on the fibers of the projection to  $\mathcal{M}_T$ , and its holonomies at  $(\mu_1, \mu_2) \in \mathcal{M}_T$  about the

standard cycles on  $T$  are  $\mu_1, \mu_2$ , as follows from (4.5).<sup>2</sup> Furthermore, the pullback under (4.3) is a universal family of flat  $\mathbb{C}^\times$ -connections over  $T$  equipped with a homotopy class of trivializations of the underlying  $\mathbb{C}^\times$ -bundle. This explains the moduli space notation (4.2). Also,  $\mathcal{M}_T$  is a holomorphic symplectic manifold; see (4.16) below.

Define the submanifolds

$$(4.10) \quad \begin{aligned} \hat{\mathcal{M}}'_T &= \{(u_1, u_2) \in \hat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\} \\ \mathcal{M}'_T &= \{(\mu_1, \mu_2) \in \mathcal{M}_T : \mu_1 + \mu_2 = 1\}. \end{aligned}$$

Then

$$(4.11) \quad \begin{aligned} \mathcal{M}'_T &\longrightarrow \mathbb{CP}^1 \setminus \{0, 1, \infty\} \\ (\mu_1, \mu_2) &\longmapsto \mu_1 \end{aligned}$$

is a diffeomorphism. Also, (4.3) restricts to a principal  $\mathbb{Z}^2$ -bundle  $e' : \hat{\mathcal{M}}'_T \rightarrow \mathcal{M}'_T$ . Then  $\rho$  in (4.7) and its pullbacks and restrictions fit into a diagram of principal  $\mathbb{C}^\times$ -bundles with connection:

$$(4.12) \quad \begin{array}{ccc} & (\epsilon')^* \mathcal{Q}' & \epsilon^* \mathcal{Q} \\ \hat{\rho}' \swarrow & \text{---} & \swarrow \hat{\rho} \\ \hat{\mathcal{M}}'_T \times T & \xhookrightarrow{\quad} & \hat{\mathcal{M}}_T \times T \\ \downarrow \epsilon' & \rho' \swarrow & \downarrow \epsilon \\ \mathcal{M}'_T \times T & \xhookrightarrow{\quad} & \mathcal{M}_T \times T \end{array}$$

Here  $\epsilon = e \times \text{id}_T$  and  $\epsilon' = e' \times \text{id}_T$ .

Apply the spin Chern-Simons theory  $\mathcal{S}$  of §3 to the four  $\mathbb{C}^\times$ -bundles with connection (two of them with trivialization) in (4.12). Let

$$(4.13) \quad \mathcal{L} \longrightarrow \mathcal{M}_T$$

be the spin Chern-Simons line bundle  $\mathcal{S}((\mathcal{M}_T \times T)/\mathcal{M}_T; \eta) \rightarrow \mathcal{M}_T$  with its covariant derivative. Observe that the  $\mathbb{C}^\times$ -bundle  $\rho$  restricted to  $\mu_1 \times \mu_2 \times T$  is topologically trivial for all  $(\mu_1, \mu_2) \in \mathcal{M}_T$ , since the restriction carries a flat connection, from which it follows that (4.13) is an *even* line bundle (see Theorem 3.9(v)). The Chern-Simons line bundle (4.13)

<sup>2</sup>The holonomy is the exponential of *minus* the integral of the connection form. The signs are chosen so that (4.28) matches the differential of the dilogarithm.

and its various pullbacks fit into the diagram

$$(4.14) \quad \begin{array}{ccccc} & & (e')^* \mathcal{L}' & & \\ & \swarrow & \downarrow \hat{\tau}'_0 & \searrow & \\ \hat{\mathcal{M}}'_T & \xleftarrow{\quad} & \hat{\mathcal{M}}_T & \xleftarrow{\quad} & e^* \mathcal{L} \\ \downarrow e' & \swarrow \mathcal{L}' & \downarrow e & \searrow \mathcal{L} & \\ \mathcal{M}'_T & \xleftarrow{\quad} & \mathcal{M}_T & \xleftarrow{\quad} & \end{array}$$

We explain the section  $\tau'$  of  $\mathcal{L}' \rightarrow \mathcal{M}'_T$  after the proof of the following.

**Proposition 4.15.** (i) *The curvature of the covariant derivative on (4.13) is*

$$(4.16) \quad \frac{1}{2\pi\sqrt{-1}} \frac{d\mu_1}{\mu_1} \wedge \frac{d\mu_2}{\mu_2}.$$

*The line bundle*

$$(4.17) \quad \mathcal{L}' \longrightarrow \mathcal{M}'_T$$

*is flat.*

(ii) *The line bundle (4.17) has trivial holonomy.*

*Proof.* The curvature statement (i) follows from (3.14) and (4.8). We compute the holonomy about  $(0, 1) \in \mathcal{M}'_T$  using the family of loops (set  $i = \sqrt{-1}$ )

$$(4.18) \quad \begin{aligned} \Gamma_\epsilon: [0, 2\pi] &\longrightarrow \mathcal{M}'_T \\ t &\longmapsto (\epsilon e^{it}, 1 - \epsilon e^{it}) \end{aligned}$$

Since the curvature vanishes, the result is independent of  $\epsilon \in (0, 1/2)$ . The computation for the holonomy about  $(1, 0) \in \mathcal{M}'_T$  follows by symmetry  $\mu_1 \leftrightarrow \mu_2$ .

The loop (4.18) lifts to the path  $\hat{\Gamma}_\epsilon: [0, 2\pi] \rightarrow \hat{\mathcal{M}}'_T$  defined by

$$(4.19) \quad \begin{aligned} u_1 &= \log \epsilon + it \\ u_2 &= \log(1 - \epsilon e^{it}), \end{aligned}$$

where for  $u_2$  choose the branch of the logarithm with  $\log 1 = 0$ . Use (3.15) to compute the connection form of  $e^* \mathcal{L} \rightarrow \hat{\mathcal{M}}_T$  relative to the trivialization  $\hat{\tau}_0$  as

$$(4.20) \quad \frac{1}{4\pi\sqrt{-1}} (u_1 du_2 - u_2 du_1).$$

To compute the holonomy integrate (4.20) along  $\hat{\Gamma}_\epsilon$ , and then use (3.16) to correct for the change of trivialization between the two endpoints  $(\log \epsilon, \log(1 - \epsilon))$  and  $(\log \epsilon + 2\pi i, \log(1 - \epsilon))$ . Set  $z = \mu_1 = e^{u_1}$  and compute

$$(4.21) \quad \int_{\hat{\Gamma}_\epsilon} u_1 du_2 - u_2 du_1 = - \int_{\Gamma_\epsilon} \frac{\log \epsilon}{1 - z} dz + \int_0^{2\pi} \frac{\epsilon t e^{it}}{1 - \epsilon e^{it}} dt - \int_{\Gamma_\epsilon} \frac{\log(1 - \epsilon z)}{z} dz.$$

The first and last terms vanish by Cauchy's theorem. The integrand in the second term has norm bounded above by  $4\pi\epsilon$ , hence the integral converges to zero as  $\epsilon \rightarrow 0$ .

Turning to the change of trivialization, the gauge transformation is multiplication by  $h = e^{2\pi i \theta^1}$  and so by (4.5) the integrand in (3.16) is the 2-form

$$(4.22) \quad (\log \epsilon d\theta^1 + \log(1 - \epsilon) d\theta^2) \wedge (2\pi\sqrt{-1} d\theta^1) = -2\pi\sqrt{-1} \log(1 - \epsilon) d\theta^1 \wedge d\theta^2;$$

both it and its integral over  $T$  vanish in the limit  $\epsilon \rightarrow 0$ . Finally, the quadratic function (3.17) is 1 on the coordinate loop in the  $\theta^1$ -direction, by our choice<sup>3</sup> of spin structure on  $T$ . Since the holonomy is independent of  $\epsilon$ , and the computation converges as  $\epsilon \rightarrow 0$ , that limit suffices to prove that the holonomy is trivial. q.e.d.

It follows that (4.17) admits a  $\mathbb{C}^\times$ -torsor  $\mathcal{T}'$  of flat nonzero sections  $\tau'$ . The pullback  $(e')^* \mathcal{L} \rightarrow \hat{\mathcal{M}}'_T$  has a canonical nonzero section  $\hat{\tau}'_0$  which is not flat; see (4.20). For each  $\tau' \in \mathcal{T}'$  define

$$(4.23) \quad \varphi = \frac{\hat{\tau}'_0}{\tau'}: \hat{\mathcal{M}}'_T \longrightarrow \mathbb{C}^\times.$$

Varying  $\tau' \in \mathcal{T}'$  changes  $\varphi$  by a multiplicative constant. From (4.20) compute

$$(4.24) \quad \frac{d\varphi}{\varphi} = \frac{1}{4\pi\sqrt{-1}} (u_1 du_2 - u_2 du_1)$$

for all  $\tau' \in \mathcal{T}'$ , where recall  $e^{u_1} + e^{u_2} = 1$ . Write

$$(4.25) \quad \varphi = \exp\left(\frac{L}{2\pi\sqrt{-1}}\right)$$

to define a function

$$(4.26) \quad L: \hat{\mathcal{M}}'_T \rightarrow \mathbb{C}/\mathbb{Z}(2).$$

Here we use the Tate twists

$$(4.27) \quad \begin{aligned} \mathbb{Z}(1) &= 2\pi\sqrt{-1}\mathbb{Z} \\ \mathbb{Z}(2) &= \mathbb{Z}(1)^{\otimes 2} = 4\pi^2\mathbb{Z}. \end{aligned}$$

<sup>3</sup>The value of the quadratic function on the  $\theta^2$ -coordinate loop enters the computation of holonomy around  $(0, 1) \in \hat{\mathcal{M}}'_T$ . The holonomy of (4.17) is not trivial for any other spin structure on  $T$ , but see Remark 4.29.

The function  $L$  is determined up to an additive constant: the choice of  $\tau' \in \mathcal{T}'$ . Any choice satisfies

$$(4.28) \quad dL = \frac{u_1 du_2 - u_2 du_1}{2}.$$

**Remark 4.29.** Our construction uses a particular spin structure on the torus  $T$  and the particular choice of lagrangian submanifold  $\mathcal{M}'_T \subset \mathcal{M}_T$ . There are three other spin structures and correlated choices of lagrangians which lead to variations of the enhanced Rogers dilogarithm. We use all four functions in [FN].

**4.2. Dilogarithms.** We refer to [G, K, Z] and the references therein for details about the various dilogarithm functions.

The theory begins with the power series

$$(4.30) \quad -\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

$$(4.31) \quad \text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2},$$

convergent for  $z \in \mathbb{C}$  satisfying  $|z| < 1$ , and analytically continued to  $z \in \mathbb{C} \setminus [1, \infty)$ . Equation (4.30) is an identity; equation (4.31) defines the *Spence dilogarithm*  $\text{Li}_2$ . Differentiate the power series:

$$(4.32) \quad d\text{Li}_2(z) = -\frac{\log(1-z)}{z} dz = -u_2 du_1 = -\frac{u_2 du_2}{1-e^{-u_2}},$$

using notation from (4.3), (4.11) and setting  $z = \mu_1$ . The last expression is a meromorphic 1-form on the  $u_2$ -line with simple poles at  $\mathbb{Z}(1) \subset \mathbb{C}$  and residues in  $\mathbb{Z}(1)$ . It follows that

$$(4.33) \quad \begin{aligned} F: \mathbb{C} \setminus \mathbb{Z}(1) &\longrightarrow \mathbb{C} / \mathbb{Z}(2) \\ u_2 &\longmapsto \text{Li}_2(1 - e^{u_2}) \end{aligned}$$

is a well-defined function. We lift it under the  $\mathbb{Z}$ -covering map

$$(4.34) \quad \begin{aligned} \hat{\mathcal{M}}'_T &\longrightarrow \mathbb{C} \setminus \mathbb{Z}(1) \\ (u_1, u_2) &\longmapsto u_2 \end{aligned}$$

Then

$$(4.35) \quad F(u_2) + \frac{1}{2}u_1 u_2 = \text{Li}_2(z) + \frac{1}{2}\log(z)\log(1-z) \quad \text{mod } \mathbb{Z}(2)$$

is a well defined function  $\hat{\mathcal{M}}'_T \rightarrow \mathbb{C} / \mathbb{Z}(2)$ , and from (4.32) its differential equals  $dL$  in (4.28). Therefore, (4.35) equals  $L$  up to an additive constant. (Recall that in any event  $L$  is only defined up to an additive constant.) The function (4.35) is called the *enhanced Rogers dilogarithm*. It is denoted ' $\hat{D}$ ' in [Z].

This discussion proves the following.

**Theorem 4.36.** *The function  $L$  in (4.26), defined up to a constant using spin Chern-Simons theory with gauge group  $\mathbb{C}^\times$ , equals the enhanced Rogers dilogarithm.*

## 5. Properties of the dilogarithm

In this section we use the geometric construction of the enhanced Rogers dilogarithm (§4.1) to derive a few of its standard properties directly from spin Chern-Simons theory.

Consider first the transformation law under a deck transformation of the  $\mathbb{Z}^2$ -covering map  $e': \hat{\mathcal{M}}'_T \rightarrow \mathcal{M}'_T$ . Fix  $\tau' \in \mathcal{T}'$  and so a choice of  $\varphi, L$ ; see (4.23) and (4.25).

**Theorem 5.1.** *For all  $(u_1, u_2) \in \hat{\mathcal{M}}'_T$  and  $(n_1, n_2) \in \mathbb{Z}^2$ ,*

$$(5.2) \quad \begin{aligned} L(u_1 + 2\pi i n_1, u_2 + 2\pi i n_2) = \\ L(u_1, u_2) + \pi i(n_1 u_2 - n_2 u_1) + 2\pi^2 n_1 n_2 \quad \text{mod } \mathbb{Z}(2). \end{aligned}$$

Equation (5.2) appears in [Z, p. 25] and as [APP, (A.18)].

*Proof.* Since the section  $\tau'$  of  $(e')^* \mathcal{L}' \rightarrow \hat{\mathcal{M}}'_T$  is pulled back via  $e': \hat{\mathcal{M}}'_T \rightarrow \mathcal{M}'_T$ , the change in  $\varphi = \hat{\tau}'_0/\tau'$  under a deck transformation of  $e'$  is due to the change in the trivialization  $\hat{\tau}'_0$ . Use the general formula (3.16) to compute. From (4.5) the connection form relative to the trivialization at  $(u_1, u_2) \in \hat{\mathcal{M}}'_T$  is

$$(5.3) \quad u_1 d\theta^1 + u_2 d\theta^2,$$

and the relevant gauge transformation is multiplication by the function

$$(5.4) \quad h = \exp(2\pi i(n_1 \theta^1 + n_2 \theta^2)).$$

The integral in (3.16) is then  $\frac{1}{2}(n_1 u_2 - n_2 u_1)$ ; with the correct prefactor it contributes the second term in (5.2). The third term derives from the factor (3.17) in (3.16). The quadratic form  $\sigma$  for our choice of spin structure on  $T$  is

$$(5.5) \quad \sigma(n_1, n_2) = n_1 n_2 \pmod{2},$$

where  $(n_1, n_2) \in \mathbb{Z}^2$  is the homotopy class of  $h$  in  $H^1(T; \mathbb{Z})$ , relative to the standard basis. q.e.d.

Next, we prove a reflection identity, which appears as [APP, (A.21)] and, for a restriction of  $L$ , in [Z, p. 23].

**Theorem 5.6.** *For the diffeomorphism*

$$(5.7) \quad \begin{aligned} \Psi: \hat{\mathcal{M}}'_T &\longrightarrow \hat{\mathcal{M}}'_T \\ (u_1, u_2) &\longmapsto (u_2, u_1) \end{aligned}$$

*the sum  $L + \Psi^* L$  is a constant function.*

Of course, the result holds for all  $\tau' \in \mathcal{T}'$ .

*Proof.* Consider the reflection diffeomorphism

$$(5.8) \quad \begin{aligned} \psi: T &\longrightarrow T \\ (\theta^1, \theta^2) &\longmapsto (\theta^2, \theta^1) \end{aligned}$$

on the torus  $T$ . It reverses orientation, maps our chosen spin structure  $\sigma$  to its opposite,<sup>4</sup> induces the reflection  $(\mu_1, \mu_2) \mapsto (\mu_2, \mu_1)$  on the moduli space  $\mathcal{M}_T$  of flat  $\mathbb{C}^\times$ -connections, which lifts to the reflection  $(u_1, u_2) \mapsto (u_2, u_1)$  on  $\hat{\mathcal{M}}_T$ , which in turn restricts to (5.7) on  $\hat{\mathcal{M}}'_T$ . Use formulas (4.5) and (4.6) to compatibly lift the involution to the bundles in (4.12); the lift preserves the universal connections. The functoriality of Chern-Simons (Theorem 3.20) implies that the involution lifts to the Chern-Simons line bundles with covariant derivative in (4.14), except because orientation on  $T$  is reversed the bundle  $\mathcal{L}$  is mapped to  $\mathcal{L}^{-1}$ . It follows that  $\Psi^* \varphi = \varphi^{-1}$ , up to a multiplicative constant. q.e.d.

**Remark 5.11.** A related identity states that  $L(z) + L(1/z)$  is constant; see [APP, (A.24)] for the precise form on the cover  $\hat{\mathcal{M}}'_T$ . It can be proved by a similar method, but the relevant diffeomorphism of  $T$  does not preserve the spin structure, so one needs to expand the theory as indicated in Remark 4.29.

Finally, we prove the 5-term relation satisfied by the dilogarithm [Z], [APP, (A.8)]. For that we change notation and use  $u, v$  in place of  $u_1, u_2$  as the standard coordinates on  $\hat{\mathcal{M}}_T = \mathbb{C}^2$ .

**Theorem 5.12.** *The sum  $L(u_1, v_1) + \dots + L(u_5, v_5)$  is independent of  $(u_i, v_i) \in \hat{\mathcal{M}}'_T$ ,  $i \in \mathbb{Z}/5\mathbb{Z}$ , which satisfy*

$$(5.13) \quad v_i = u_{i-1} + u_{i+1} \quad \text{for all } i.$$

**Remark 5.14.** Write  $z_i = e^{u_i}$  and assume  $e^{u_i} + e^{v_i} = 1$ . Then equation (5.13) implies

$$(5.15) \quad 1 - z_i = z_{i-1} z_{i+1}.$$

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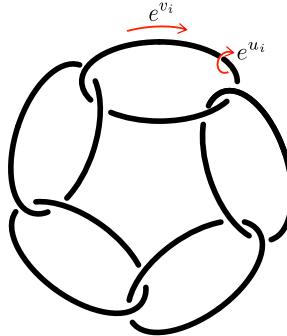
<sup>4</sup>A spin structure on an  $n$ -dimensional oriented Riemannian manifold  $M$  is encoded in a sequence

$$(5.9) \quad P \xrightarrow{\rho} \mathrm{SO}(M) \xrightarrow{\pi} M$$

in which  $\pi$  is the principal  $\mathrm{SO}_n$ -bundle of oriented orthonormal frames and  $\pi \circ \rho$  is a principal  $\mathrm{Spin}_n$ -bundle. The *opposite spin structure*  $P' \xrightarrow{\rho'} \mathrm{SO}'(M) \xrightarrow{\pi'} M$  is constructed by taking the complement of (5.9) in

$$(5.10) \quad P \times_{\mathrm{Spin}_n} \mathrm{Pin}_n \longrightarrow \mathrm{O}(M) \longrightarrow M,$$

where either pin group  $\mathrm{Pin}_n^\pm$  can be used. Here  $\mathrm{O}(M) \rightarrow M$  is the principal  $\mathrm{O}_n$ -bundle of orthonormal frames.



**Figure 1.**  $X$  is  $S^3$  with these five solid tori removed.

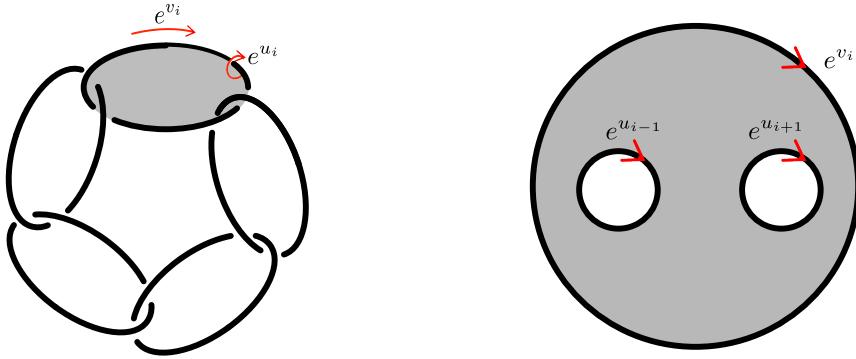
There is a connected complex 2-manifold  $\mathcal{M}'_X$  of solutions

$$(5.16) \quad x, \frac{1-x}{1-xy}, \frac{1-y}{1-xy}, y, 1-xy$$

to (5.15), parametrized by  $x, y \in \mathbb{C}$  satisfying  $xy \neq 0$ ,  $x \neq 1$ ,  $y \neq 1$ , and  $xy \neq 1$ .

*Proof.* Let  $X$  be the compact spin 3-manifold with boundary formed from  $S^3$  by removing a tubular neighborhood of the 5-component link depicted<sup>5</sup> in Figure 1. Fix a diffeomorphism  $\partial X \approx T^{115}$  which induces the basis of first homology indicated in the figure. Alexander duality implies  $H_1(X)$  is torsionfree of rank 5, so if a principal flat  $\mathbb{C}^\times$ -bundle over  $X$  admits a flat connection then it is trivializable. Let  $\mathcal{M}_X$  be the moduli space of flat  $\mathbb{C}^\times$ -connections on  $X$  and  $r: \mathcal{M}_X \rightarrow (\mathcal{M}_T)^5$  the restriction map to the boundary  $\partial X$ . Define  $\mathcal{M}'_X = r^{-1}[(\mathcal{M}'_T)^5]$ . (Remark 5.14 gives an explicit parametrization of  $\mathcal{M}'_X$ .) Similarly, let  $\widehat{\mathcal{M}}_X$  be the moduli space of flat  $\mathbb{C}^\times$ -connections with a homotopy class of trivialization,  $\widehat{r}: \widehat{\mathcal{M}}_X \rightarrow (\widehat{\mathcal{M}}_T)^5$  restriction to the boundary, and  $\widehat{\mathcal{M}}'_X = \widehat{r}^{-1}[(\widehat{\mathcal{M}}'_T)^5]$ . Each component of the link is the outer boundary of a neatly embedded disk  $D_i \subset X$  with two subdisks removed, as in Figure 2. For any collection  $(u_i, v_i) \in \widehat{\mathcal{M}}_T$  in the image of  $\widehat{r}$ , apply Stokes' theorem to the closed 1-form on  $D_i$  which is the flat connection form relative to the trivialization. Its integral over a boundary component is minus the log holonomy. Therefore, the relation (5.13) holds on the image of  $\widehat{r}$ .

<sup>5</sup>This link is a close cousin to the “minimally twisted 5-chain link” in [DT, §2.6]. (We thank Ian Agol for bringing this reference to our attention.)



**Figure 2.** A 2-manifold neatly embedded in  $X$ .

Spin Chern-Simons theory  $\mathcal{S}$  produces the diagram

$$\begin{array}{ccccc}
 & & (\widehat{\mathcal{L}}')^{\boxtimes 5} & & \\
 & \swarrow & & \searrow & \\
 \widehat{\mathcal{M}}'_X & \xrightarrow{\quad \widehat{r}' \quad} & (\widehat{\mathcal{M}}'_T)^5 & & \\
 \downarrow e'_X & \nearrow (r')^*(\mathcal{L}')^{\boxtimes 5} & \downarrow (e'_T)^5 & \nearrow (\mathcal{L}')^{\boxtimes 5} & \\
 \mathcal{M}'_X & \xrightarrow{\quad r' \quad} & (\mathcal{M}'_T)^5 & \nearrow (\tau')^{\boxtimes 5} & \\
 \end{array} \tag{5.17}$$

where  $\widehat{\mathcal{L}}' = (e'_T)^* \mathcal{L}'$  is the pullback of the spin Chern-Simons line bundle under  $e'_T: \widehat{\mathcal{M}}'_T \rightarrow \mathcal{M}'_T$ . Apply Theorem 3.9(iii) to obtain a section  $\mathcal{S}$  of  $(r')^*(\mathcal{L}')^{\boxtimes 5} \rightarrow \mathcal{M}'_X$ ; by (3.12) it is flat. There exists  $\tau' \in \mathcal{T}'$  a flat section of  $\mathcal{L}' \rightarrow \mathcal{M}'_T$ , unique up to a 5<sup>th</sup> root of unity, such that  $\mathcal{S} = (r')^*[(\tau')^{\boxtimes 5}]$ . Then (3.13) implies that the pullbacks of  $\mathcal{S}$  and  $(\widehat{\tau}_0)^{\boxtimes 5}$  to  $\widehat{\mathcal{M}}'_X$  agree. Recalling (4.23), we see that the ratio  $(\widehat{\tau}_0/\tau')^{\boxtimes 5}: (\widehat{\mathcal{M}}'_T)^5 \rightarrow \mathbb{C}^\times$  is the product of five exponentiated enhanced Rogers dilogarithms, and the fact that its pullback to  $\widehat{\mathcal{M}}'_X$  is identically one is the 5-term relation.  $\text{q.e.d.}$

**Remark 5.18.** The choice of  $\tau' \in \mathcal{T}'$  in the proof makes the 5-term sum in the theorem vanish.

## Appendix A. Classical spin Chern-Simons theory

In this appendix we sketch proofs of Theorem 3.9 and Theorem 3.20, which set out basic properties of spin Chern-Simons theory with gauge

group  $\mathbb{C}^\times$ . Theorem 2.17 can be proved by similar methods, or may be deduced as a corollary of the spin case; we give an argument for part (i) at the end. In this appendix we freely use generalized differential cohomology, as developed in [HS, BNV] and other references. To begin we work with the universal family of  $\mathbb{C}^\times$ -connections as described in [FH]; the parameter “space”<sup>6</sup> is  $B_\nabla \mathbb{C}^\times$ .

**Remark A.1.** An alternative approach to spin Chern-Simons for *compact* gauge groups is available using  $\eta$ -invariants and pfaffian lines of Dirac operators [J]. For a noncompact group one encounters Dirac operators coupled to non-unitary connections, for which the relevant parts of geometric index theory are not in the literature.

Let  $E$  be the spectrum (in the sense of stable homotopy theory) defined as the extension

$$(A.2) \quad H\mathbb{Z} \longrightarrow E \longrightarrow \Sigma^{-2}H\mathbb{Z}/2\mathbb{Z}$$

of Eilenberg-MacLane spectra with nonzero  $k$ -invariant. Its properties are stated and proved in [F3, §1]. The techniques used there, particularly around (1.13), imply that the short exact sequence

$$(A.3) \quad 0 \longrightarrow H^4(B\mathbb{C}^\times; \mathbb{Z}) \xrightarrow{i} E^4(B\mathbb{C}^\times) \xrightarrow{j} H^2(B\mathbb{C}^\times; \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

does not split. Hence there is a unique class  $\lambda \in E^4(B\mathbb{C}^\times)$  such that  $2\lambda = i(c_1^2)$  and  $j(\lambda) = \bar{c}_1$ , where  $\bar{c}_1$  is the mod 2 reduction of the universal first Chern class  $c_1 \in H^2(B\mathbb{C}^\times; \mathbb{Z})$ . Let  $\check{E}_\mathbb{C}$  be the complex differential refinement of  $E$ , defined as a homotopy fiber product as in [HS, (4.12)] (with  $\mathcal{V} = \mathbb{C}$  concentrated in degree zero); see also [BNV, §4.4] (with  $C = \mathbb{C}$  concentrated in degree zero). The differential cohomology group fits into an exact sequence

(A.4)

$$0 \longrightarrow \check{E}_\mathbb{C}^4(B_\nabla \mathbb{C}^\times) \longrightarrow E^4(B\mathbb{C}^\times) \times \Omega_{\text{cl}}^4(B_\nabla \mathbb{C}^\times; \mathbb{C}) \xrightarrow{-} H^4(B\mathbb{C}^\times; \mathbb{C}),$$

part of the Mayer-Vietoris sequence derived from the homotopy fiber product.<sup>7</sup> Here  $\Omega_{\text{cl}}^4(B_\nabla \mathbb{C}^\times; \mathbb{C})$  is the vector space of closed complex differential forms; by the main theorem in [FH] it is isomorphic to the three-dimensional complex vector space of symmetric functions  $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$  which are real bilinear. The final map in (A.4) is the difference between the homomorphism  $k: E^4(B\mathbb{C}^\times) \rightarrow H^4(B\mathbb{C}^\times; \mathbb{C})$  defined in [F3, (1.4)] and the Chern-Weil homomorphism. It follows that the generator  $\lambda \in E^4(B\mathbb{C}^\times)$  has a unique differential refinement  $\check{\lambda} \in \check{E}_\mathbb{C}^4(B_\nabla \mathbb{C}^\times)$  with

<sup>6</sup>It is, rather, a simplicial sheaf on the site of smooth manifolds. Computations may be carried out on a smooth “test” manifold  $M$  equipped with a  $\mathbb{C}^\times$ -connection, that is, equipped with a map  $M \rightarrow B_\nabla \mathbb{C}^\times$ .

<sup>7</sup>The short exact sequences one usually derives from it for smooth manifolds depend on the de Rham theorem, which does not necessarily hold for  $B_\nabla G$  if  $G$  is noncompact.

associated symmetric function  $(z_1, z_2) \mapsto -z_1 z_2/8\pi^2$ . This is the universal class which defines spin Chern-Simons theory. The constructions which follow use a geometric representative of this class, say a *differential E-function* as defined in [HS, §4.1]. Its “curvature”, the image under the homomorphism

$$(A.5) \quad \check{E}_{\mathbb{C}}^4(B_{\nabla}\mathbb{C}^{\times}) \longrightarrow \Omega_{\text{cl}}^4(B_{\nabla}\mathbb{C}^{\times}; \mathbb{C}),$$

is

$$(A.6) \quad \omega_{\text{spin}}(\Theta^{\text{univ}}) = -\frac{1}{8\pi^2} \Omega(\Theta^{\text{univ}}) \wedge \Omega(\Theta^{\text{univ}}),$$

where  $\Theta^{\text{univ}} \in \Omega^1(E_{\nabla}\mathbb{C}^{\times}; \mathbb{C})$  is the universal  $\mathbb{C}^{\times}$ -connection [FH, (5.25)] on the universal  $\mathbb{C}^{\times}$ -bundle

$$(A.7) \quad \pi: E_{\nabla}\mathbb{C}^{\times} \longrightarrow B_{\nabla}\mathbb{C}^{\times}.$$

Recall [FH, Example 5.14] that  $E_{\nabla}\mathbb{C}^{\times}$  is the classifying sheaf for triples  $(p, \Theta, s)$  consisting of a principal  $\mathbb{C}^{\times}$ -bundle  $p: P \rightarrow M$  with connection  $\Theta$  and section  $s$ . The pullback of (A.7) by  $\pi$  is canonically trivialized by the section, and this induces a “nonflat trivialization”<sup>8</sup> of  $\pi^*\check{\lambda}$ . The trivialization represents  $\pi^*\check{\lambda}$  by a complex 3-form, the universal Chern-Simons form

$$(A.8) \quad \alpha_{\text{spin}}(\Theta^{\text{univ}}) = -\frac{1}{8\pi^2} \Theta^{\text{univ}} \wedge \pi^*\Omega(\Theta^{\text{univ}})$$

whose de Rham differential is (A.6). Below in (A.17) we give a formula for this trivialization as an integral of  $\pi^*\check{\lambda}$ .

Turning now to a family  $P \xrightarrow{\pi} M_{\sigma} \xrightarrow{p} S$  with  $\mathbb{C}^{\times}$ -connection  $\Theta$ , as in (3.8), we obtain from these universal constructions: a differential class (rather, a geometric representative)  $\check{\lambda}(\Theta) \in \check{E}_{\mathbb{C}}^4(M)$  with curvature  $\omega_{\text{spin}}(\Theta)$ , as in (A.6); the Chern-Simons form  $\alpha_{\text{spin}}(\Theta) \in \Omega^3(P; \mathbb{C})$ ; and an isomorphism of  $\pi^*\check{\lambda}(\Theta)$  with the image of  $\alpha_{\text{spin}}(\Theta)$  in  $\check{E}_{\mathbb{C}}^4(P)$ . The spin Chern-Simons invariant is the pushforward in differential  $E$ -theory:

$$(A.9) \quad \mathcal{S}(M_{\sigma}/S; \Theta) = \int_{M_{\sigma}/S} \check{\lambda}(\Theta).$$

Integration in  $E$ , and so in  $\check{E}_{\mathbb{C}}$ , uses the spin structure  $\sigma$ ; see [F3, (1.6)]. If the fibers of  $p$  are closed of dimension  $k$ , then (A.9) lives in  $\check{E}_{\mathbb{C}}^{4-k}(S)$ . In low dimensions we identify

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<sup>8</sup>An explicit model for a nonflat trivialization of a geometric representative of an  $\check{E}$ -cohomology class—a *coned differential E-function*—is spelled out in [F3, Definition 5.12]. The data of a 3-form, such as (A.8), is part of that definition.

$$\begin{aligned}
& \check{E}_{\mathbb{C}}^1(S) \cong \text{Map}(S, \mathbb{C}^\times) \\
(\text{A.10}) \quad & \check{E}_{\mathbb{C}}^2(S) \cong \{ \text{iso classes of complex super line} \\
& \quad \text{bundles } \mathcal{L} \rightarrow S \text{ with connection} \}
\end{aligned}$$

Parts (i), (v), and (vi) of Theorem 3.9 follow immediately from the foregoing and the compatibility of the exact sequence (A.4) and integration. The  $\mathbb{Z}/2\mathbb{Z}$ -grading in (v) is determined by its restriction to each  $s \in S$ , and since the grading is a discrete invariant it only depends on  $\lambda(\Theta) \in E^4(M)$ . Writing  $M_s = p^{-1}(s)$ , the desired formula follows from the commutative diagram

$$\begin{array}{ccc}
E^4(M_s) & \xrightarrow{j} & H^2(M_s; \mathbb{Z}/2\mathbb{Z}) \\
p_* \downarrow & & \downarrow p_* \\
E^2(\{s\}) & \xrightarrow{j} & H^0(\{s\}; \mathbb{Z}/2\mathbb{Z})
\end{array}
\tag{A.11}$$

in which both maps  $j$  are isomorphisms and  $j(\lambda(\Theta)) = \bar{c}_1(P)$ . The dependence on spin structure in (ii) is an immediate consequence of the proof of [F3, Proposition 4.4]. For parts (iii) and (iv) use the notion of a nonflat trivialization of a geometric representative of an  $\check{E}_{\mathbb{C}}$ -cohomology class; see footnote <sup>8</sup>. If the fibers of  $p: M \rightarrow S$  are compact manifolds with boundary, a Stokes' theorem holds (see [HS, §3.4] for one version):

$$\begin{aligned}
(\text{A.12}) \quad & \int_{M_\sigma/S} \check{\lambda}(\Theta) \text{ is a nonflat trivialization of } \int_{\partial M_\sigma/S} \check{\lambda}(\Theta) \\
& \text{with "covariant derivative" } \int_{M/S} \omega_{\text{spin}}(\Theta).
\end{aligned}$$

For  $\dim(p) = 3$  the nonflat trivialization is a nonzero section of a (necessarily even) complex line bundle and (iii) follows. For (iv) we use the isomorphism of  $\check{\lambda}(\Theta)$  with the image of  $s^* \alpha_{\text{spin}}(\Theta)$  in  $\check{E}_{\mathbb{C}}^4(M)$  and apply (A.12).

We turn to Theorem 3.9(vii), and for that we prove a formula valid for any finite dimensional real Lie group  $G$  with finitely many components. Analogous to (A.4) is the exact sequence

$$\tag{A.13} \quad 0 \longrightarrow \check{E}_{\mathbb{C}}^4(B_\nabla G) \longrightarrow E^4(BG) \oplus \Omega_{\text{cl}}^4(B_\nabla G) \xrightarrow{\quad} H^4(BG; \mathbb{C}),$$

and the main theorem in [FH] identifies  $\Omega_{\text{cl}}^4(B_\nabla G)$  with the complex vector space of  $G$ -invariant symmetric bilinear forms  $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$  on the Lie algebra of  $G$ . It follows that  $\check{E}_{\mathbb{C}}^4(B_\nabla G)$  is isomorphic to the group of compatible pairs  $\check{\lambda} = (\lambda, \langle -, - \rangle)$  of a class in  $E^4(BG)$  and a symmetric bilinear form. Fix such a pair. Let  $\pi: E_\nabla G \rightarrow B_\nabla G$  be the universal  $G$ -bundle with universal  $G$ -connection  $\Theta^{\text{univ}} \in \Omega^1(E_\nabla G; \mathfrak{g})$ . The total space  $\mathcal{F}$  of the fiber product of  $\pi$  with itself classifies quartets

$(p, \Theta, s_0, s_1)$  consisting of a principal  $G$ -bundle  $P \rightarrow M$  over a smooth manifold  $M$  equipped with connection  $\Theta$  and sections  $s_0, s_1$ . Use  $s_0$  to trivialize the (universal) bundle, and so construct an isomorphism

$$(A.14) \quad \mathcal{F} \xrightarrow{\cong} E_{\nabla}G \times G.$$

The sections  $s_0, s_1$  each induce a trivialization of the pullback of  $\check{\lambda}$  to  $\mathcal{F}$ ; the difference of these trivializations represents an element  $\check{\eta} \in \check{E}_{\mathbb{C}}^3(\mathcal{F})$ .

**Proposition A.15.** *Under the isomorphism (A.14) the differential class  $\check{\eta}$  is the sum*

$$(A.16) \quad \check{\eta} = \check{\omega} + \langle \Theta^{\text{univ}} \wedge \theta \rangle,$$

where  $\check{\omega} \in \check{E}_{\mathbb{C}}^3(G)$  is the integral of  $\check{\lambda}(\tilde{\Theta})$  over  $S^1$  for a certain  $G$ -connection  $\tilde{\Theta}$  over  $S^1 \times G$ , and  $\theta \in \Omega_G^1(\mathfrak{g})$  is the Maurer-Cartan form.

$\tilde{\Theta}$  is a universal pointed connection: its holonomy around  $S^1 \times \{h\}$  equals<sup>9</sup>  $h^{-1}$  for all  $h \in G$ .

*Proof.* The pullback of the universal  $G$ -bundle  $\pi: E_{\nabla}G \rightarrow B_{\nabla}G$  via  $\pi$  is a principal  $G$ -bundle  $\varpi: \mathcal{F} \rightarrow E_{\nabla}G$  equipped with a canonical section  $s$ . The latter induces the canonical trivialization (A.14) as well as a canonical trivial connection  $\Theta_s$ , which under (A.14) maps to the Maurer-Cartan form  $\theta$ . Let  $\Delta^1 \rightarrow \mathcal{A}_{\varpi}$  be the affine map of the 1-simplex into the affine space of connections on  $\varpi$  which sends the endpoints to  $\Theta_s$  and  $\pi^*\Theta^{\text{univ}}$ . There results a connection  $\tilde{\Theta}_s$  on the trivial  $G$ -bundle over  $\Delta^1 \times E_{\nabla}G$ . By Stokes' theorem (A.12) the nonflat trivialization of  $\pi^*\check{\lambda}(\Theta^{\text{univ}})$  is

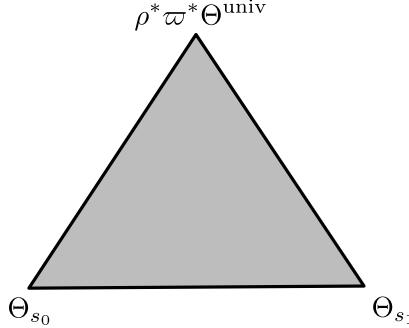
$$(A.17) \quad \int_{\Delta^1} \check{\lambda}(\tilde{\Theta}_s) = \int_{\Theta_s}^{\pi^*\Theta^{\text{univ}}} \check{\lambda}.$$

(The right hand side is shorthand notation for the left hand side.) The universal Chern-Simons form (A.8) is the same integral with the integrand replaced by its curvature  $\langle \Omega(\tilde{\Theta}_s) \wedge \Omega(\tilde{\Theta}_s) \rangle$ .

Working now on the universal bundle  $\rho: \mathcal{P} \rightarrow \mathcal{F}$ , which is equipped with two sections  $s_0$  and  $s_1$ , there is an affine map  $\Delta^2 \rightarrow \mathcal{A}_{\rho}$  as depicted in Figure 3. The class  $\check{\eta}$  is obtained by integrating  $\check{\lambda}$  over the path which begins at the lower left vertex, moves to the top vertex, and then down to the lower right vertex; we use a gauge transformation to identify the trivial connections at the initial and final vertices. A version of Stokes' theorem identifies the difference of the trivializations of the pullback of  $\check{\lambda}$  to  $\mathcal{F}$  induced by  $s_1$  and  $s_0$  as

$$(A.18) \quad \check{\eta} = \int_{\Theta_{s_0}}^{\Theta_{s_1}} \check{\lambda} - \int_{\Delta^2} \langle \Omega(\tilde{\Theta}) \wedge \Omega(\tilde{\Theta}) \rangle,$$

<sup>9</sup>The signs work out so that the holonomy is  $h^{-1}$ , not  $h$ .



**Figure 3.** The 2-simplex of connections  $\Delta^2 \rightarrow \mathcal{A}_\rho$ .

where  $\tilde{\Theta}$  is the  $G$ -connection over  $\Delta^2 \times \mathcal{F}$  constructed by affine interpolation from the vertices in Figure 3. As before, we identify the connections  $\Theta_{s_0}$  and  $\Theta_{s_1}$  by a gauge transformation. In terms of barycentric coordinates on  $\Delta^2$ , the map  $\Delta^2 \rightarrow \mathcal{A}_\rho$  is

$$(A.19) \quad (t^0, t^1, t^2) \longmapsto t^0 \Theta_{s_0} + t^1 \Theta_{s_1} + t^2 \rho^* \varphi^* \Theta^{\text{univ}}.$$

Write  $s_1 = s_0 \cdot h$  for  $h: \mathcal{F} \rightarrow G$ . Relative to the trivialization  $s_0$  and isomorphism (A.14) we rewrite<sup>10</sup> (A.19) as a map to  $\Omega^1_{E_\nabla G \times G}(\mathfrak{g})$ :

$$(A.20) \quad (t^0, t^1, t^2) \longmapsto -t^1 \text{Ad}_h \theta + t^2 \Theta^{\text{univ}}.$$

The last expression, interpreted as an element of  $\Omega^1_{\Delta^2 \times E_\nabla G \times G}(\mathfrak{g})$ , is precisely  $\tilde{\Theta}$ . Its curvature is

$$(A.21) \quad \Omega(\tilde{\Theta}) = -dt^1 \wedge \text{Ad}_h \theta + dt^2 \wedge \Theta^{\text{univ}} + \text{terms not involving } dt^1 \text{ or } dt^2,$$

from which

$$(A.22) \quad - \int_{\Delta^2} \langle \Omega(\tilde{\Theta}) \wedge \Omega(\tilde{\Theta}) \rangle = \langle \Theta^{\text{univ}} \wedge \theta \rangle.$$

To identify the first term in (A.18) in terms of a  $G$ -connection over  $S^1 \times G$ , restrict (A.20) to the 1-simplex  $t^2 = 0$  to obtain the connection form  $-t^1 \text{Ad}_h \theta$ . Compare with [F1, (4.14)].<sup>11</sup> q.e.d.

Apply Proposition A.15 with  $G = \mathbb{C}^\times$  to Theorem 3.9(vii). First, the universal  $\mathbb{C}^\times$ -bundle  $Q \rightarrow S^1 \times \mathbb{C}^\times$  has first Chern class a generator of  $H^2(S^1 \times \mathbb{C}^\times; \mathbb{Z})$  (as follows from (4.8), for example), so from (A.3) and (A.4)—applied to  $S^1 \times \mathbb{C}^\times$ —we see that  $\check{\lambda}(Q)$  is the generator of

<sup>10</sup>Since  $s_1^* \Theta_{s_1} = 0$ , the usual formula for the gauge transform of a connection implies  $0 = \theta + \text{Ad}_{h^{-1}}(s_0^* \Theta_{s_1})$ .

<sup>11</sup>Set  $g = 1$  in [F1, (4.14)] to compare; the sign discrepancy is explained by the appearance of the inverse ‘ $h^{-1}$ ’ in the gluing formula [F1, (4.13)].

$\check{E}_{\mathbb{C}}^4(S^1 \times \mathbb{C}^\times) \cong H^2(S^1 \times \mathbb{C}^\times; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ . Its integral over  $S^1$  is the nonzero element

$$(A.23) \quad \check{\omega} \in \check{E}_{\mathbb{C}}^3(\mathbb{C}^\times) \cong H^1(\mathbb{C}^\times; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Turning to Theorem 3.9(vii), the integrand in (3.16) equals the second term of (A.16), so it remains to identify the sign (3.17) with the integral of the first term in (A.16). Since the curvature of  $\check{\omega}$  vanishes, it suffices to compute the pushforward of its image  $\omega \in E^3(\mathbb{C}^\times)$ . Since  $E$  is a truncation of connective  $ko$ -theory, there is an isomorphism

$$(A.24) \quad E^3(\mathbb{C}^\times) \cong ko^{-1}(\mathbb{C}^\times) \cong ko^{-2}(\mathbb{C}^\times; \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Set  $\beta \in ko^{-2}(\mathbb{C}^\times; \mathbb{R}/\mathbb{Z})$  the image of  $\omega$  under (A.24).

Let  $M$  be a closed 2-manifold with spin structure  $\sigma$  and a map  $h: M \rightarrow \mathbb{C}^\times$ . Our task is to equate (3.17) with  $\pi_*^M h^* \beta \in ko^{-4}(\text{pt}; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$ , where  $\pi^M: M \rightarrow \text{pt}$ . First, deformation retract  $\mathbb{C}^\times$  to the circle group  $\mathbb{T} \subset \mathbb{C}^\times$  and so homotope  $h$  to a map with image  $\mathbb{T}$ . Then the generator  $\beta \in ko^{-2}(\mathbb{T}; \mathbb{R}/\mathbb{Z})$  is pushed forward from the generator  $\alpha \in ko^{-3}(\text{pt}; \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$  via the inclusion  $e: \text{pt} \hookrightarrow \mathbb{T}$ . By a further homotopy make  $h$  transverse to  $e \in \mathbb{T}$ ; then its inverse image  $S \subset M$  is a finite union of disjoint embedded circles, and  $S$  inherits a spin structure from  $M$ . Arguing from the diagram

$$(A.25) \quad \begin{array}{ccc} S & \xrightarrow{i} & M \\ q \downarrow & \nearrow \pi^M & \downarrow h \\ \text{pt} & \xrightarrow{e} & \mathbb{T} \end{array}$$

we have

$$(A.26) \quad \pi_*^M h^* \beta = \pi_*^M h^* e_* \alpha = \pi_*^M i_* q^* \alpha = q_* a^* \alpha = \alpha \cdot q_*(1).$$

Restricted to a component of  $S$ , the pushforward  $q_*(1) \in KO^{-1}(\text{pt}) \cong \mathbb{Z}/2\mathbb{Z}$  is 0 or 1 according as the spin structure on the component bounds or not. Since the homology class of  $S$  is Poincaré dual to  $[h] \in H^1(M; \mathbb{Z})$ , we conclude that  $q_*(1)$  maps to  $\sigma([h])$  under the isomorphism  $KO^{-1}(\text{pt}) \cong \mathbb{Z}/2\mathbb{Z}$ . It remains to observe that multiplication induces a nonzero pairing

$$(A.27) \quad KO^{-3}(\text{pt}; \mathbb{R}/\mathbb{Z}) \otimes KO^{-1}(\text{pt}) \longrightarrow KO^{-4}(\text{pt}; \mathbb{R}/\mathbb{Z}),$$

as proved for example in [FMS, (B.10)].

The functoriality properties in Theorem 3.20 follow from (A.9) and the functoriality of the differential characteristic class  $\check{\lambda} \in \check{E}_{\mathbb{C}}^4(B_{\nabla} \mathbb{C}^\times)$ .

Finally, we sketch a proof of (2.18). Let  $\check{c}_1 \in \check{H}_{\mathbb{C}}^2(B_{\nabla} \mathbb{C}^\times)$  be the universal differential first Chern class for principal  $\mathbb{C}^\times$ -bundles; it is the differential lift of  $c_1 \in H^2(B \mathbb{C}^\times; \mathbb{Z})$  with associated linear function

$z \mapsto \sqrt{-1}z/2\pi$ ,  $z \in \mathbb{C}$ . Its square  $\check{c}_1 \cdot \check{c}_1 \in \check{H}_{\mathbb{C}}^4(B_{\nabla} \mathbb{C}^{\times})$  is the universal Chern-Simons class. Let  $\Theta$  be a flat connection on a principal  $\mathbb{C}^{\times}$ -bundle  $\pi: P \rightarrow X$ , where  $X$  is a closed oriented 3-manifold. Then since  $\check{c}_1(\Theta)$  is flat, the product  $\check{c}_1(\Theta) \cdot \check{c}_1(\Theta)$  is computed by a cup product in cohomology with  $\mathbb{C}/\mathbb{Z}$  coefficients, precisely as in (2.18).

## Appendix B. A motivating Euler-Lagrange equation

In our construction (§4.1) of the dilogarithm, we introduce (4.10) the submanifold  $\mathcal{M}'_T \subset \mathcal{M}_T$  of flat  $\mathbb{C}^{\times}$ -connections on the torus  $T = \mathbb{R}^2/\mathbb{Z}^2$  such that the holonomies around the standard cycles sum to one. This condition arises naturally in the stratified abelianization of flat  $\mathrm{SL}_2 \mathbb{C}$ -connections [FN]. In this appendix we briefly indicate a formal computation motivated by the topological string [W], [OV] that produces this condition on holonomies.

Let  $M$  be a closed spin 3-manifold and  $S \subset M$  an oriented embedded circle. For  $\alpha \in \Omega_M^1(\mathbb{C})$  a connection on the trivial  $\mathbb{C}^{\times}$ -bundle over  $M$ , introduce

$$(B.1) \quad F(\alpha) = -\frac{1}{8\pi^2} \int_M \alpha \wedge d\alpha - \frac{1}{4\pi^2} \mathrm{Li}_2 \left[ \exp \left( - \int_S \alpha \right) \right].$$

In this expression  $\mathrm{Li}_2$  is the Spence dilogarithm (4.31) evaluated at the holonomy of  $\alpha$  about  $S$ . Since  $\mathrm{Li}_2$  is not a global function on  $\mathbb{C}$ —see (4.33)—this is ill-defined, so our computation based on (B.1) is heuristic. The first term in (B.1) is the spin Chern-Simons invariant (see (A.8)), and the normalization of the second term matches that of the first: we should view  $F$  as defined modulo integers. The differential of  $F$  is

$$(B.2) \quad dF_{\alpha}(\dot{\alpha}) = -\frac{1}{4\pi^2} \int_M \dot{\alpha} \wedge [d\alpha + \log(1-z)\delta_S], \quad \dot{\alpha} \in \Omega_M^1(\mathbb{C}),$$

where  $z = \exp(-\int_S \alpha)$  is the holonomy of  $\alpha$  about  $S$  and  $\delta_S$  is the distributional 2-form Poincaré dual to  $S$ . The critical point (Euler-Lagrange) equation is

$$(B.3) \quad d\alpha = -\log(1-z)\delta_S.$$

A critical  $\alpha$  is flat on the complement of  $S$  and the holonomy around a small loop linking  $S$  is  $\exp\{-[-\log(1-z)]\} = 1-z$ . Therefore, on the torus boundary of a tubular neighborhood of  $S$ , a critical connection  $\alpha$  is flat and the sum of holonomies about generating cycles is one.

**Remark B.4.** We can define  $F$  on a cover of the space of connections whose holonomy  $z$  about  $S$  is not equal to one, namely the cover on which we choose logarithms for  $z$  and  $1-z$ . But the critical point equation (B.3) takes us to a space of singular connections. In particular, the holonomy about  $S$  is no longer defined. To rectify this, we can from

the beginning choose an embedding of  $[0, 1) \times S^1 \hookrightarrow M$  such that the image of  $\{0\} \times S^1$  is  $S$ , and then replace the argument of  $\text{Li}_2$  in (B.1) with the limit of holonomies around loops  $\{t\} \times S^1$  as  $t \rightarrow 0$ . This produces a basis of the first homology of the torus boundary of a tubular neighborhood of  $S$ , so pins down the cycles on which the holonomies sum to one.

**Remark B.5.** One possible origin of the action (B.1) is as follows. We suppose  $M$  is a Lagrangian submanifold inside a Calabi-Yau threefold  $X$ , and we view  $(M, \alpha)$  as defining a Lagrangian boundary condition in the A type topological string theory of maps  $\varphi : (\Sigma, \partial\Sigma) \rightarrow (X, M)$ . Then  $F(\alpha)$  can be interpreted as the target space effective action of that string theory. This prediction comes from combining [W] and [OV]: in [W, (4.50)] it was explained that the effective action should be the Chern-Simons action plus additional contributions from holomorphic maps (if any), and in [OV, (3.22)] it was shown that the contribution from an isolated holomorphic disc  $\varphi$  is  $\text{Li}_2(e^{-x})/4\pi^2$ , where  $x = \int_{\partial D} \phi^* \alpha$ .

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DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF TEXAS  
AUSTIN, TX 78712  
USA

*E-mail address:* dafr@math.utexas.edu

DEPARTMENT OF MATHEMATICS  
YALE UNIVERSITY  
10 HILLHOUSE AVENUE  
NEW HAVEN, CT 06511  
USA

*E-mail address:* andrew.neitzke@yale.edu