Generalized symmetries of topological field theories

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(Dated: September 28, 2022)

We study generalized symmetries in a simplified arena in which the usual quantum field theories of physics are replaced with topological field theories and the smooth structure with which the symmetry groups of physics are usually endowed is forgotten. Doing so allows many questions of physical interest to be answered using the tools of homotopy theory. We study both global and gauge symmetries, as well as 't Hooft anomalies, which we show fall into one of two classes. Our approach also allows some insight into earlier work on symmetries (generalized or not) of topological field theories.

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I. INTRODUCTION

Generalized symmetries have come to play an important rôle in quantum field theory. Nevertheless, they retain an air of mystery. In [1], a q-form global symmetry in spacetime dimension d was defined as 'topological operators $U_g(M^{(d-q-1)})$ associated to codimension $q+1$ manifolds M^{d-q-1} that fuse according to a group law $U_g(M^{(d-q-1)})U_{g'}(M^{(d-q-1)}) = U_{g''}(M^{(d-q-1)})$' (where $g'' = gg'$), but this leaves us with several questions. Firstly, the definition makes sense for every $0 \leq q+1 \leq d$, so do the individual groups assemble themselves into some larger structure, and if so how? Secondly, we know that even in quantum mechanics, there exist ordinary symmetries in which the associated operators do not quite fuse according to a group law (or to put it more prosaically, do not form a representation). For example, a rotation of an electron is represented projectively and a reversal of time is represented antilinearly. Can such symmetries be generalized? Thirdly, what does 'topological' mean, exactly, and how does it relate to our usual understanding of ordinary symmetries in quantum mechanics as (unitary) operators that commute with the hamiltonian operator? Fourthly, can we give a meaning to gauging a generalized symmetry? If so, can it always be done, or are there possible 't Hooft anomalies? If it can be done, is the resulting theory unique?

Here, we wish to shed some light on such questions by studying generalized symmetries in a simplified arena in which we replace the usual dynamical quantum field theories of physics with topological field theories and we forget the smooth Lie group structure with which the symmetry groups of physics are usually endowed (though we make some remarks about Lie groups and the connection to Noether's theorem in §VIII).

Such theories are somewhat boring, dynamically speaking, in that the theories are few and far between and, in any given one of them, very little can actually happen. But, as we will see, this disadvantage is offset by the consequent advantage that they exhibit larger amounts of symmetry than theories with additional structures. Moreover, they can be formulated precisely using the language of category theory, and we can often calculate everything we desire. (Indeed, many of the mathematical constructions we describe are known to mathematicians, or at least will come as no surprise, but the interpretation in terms of generalized symmetries of physics is hopefully new.)

Generalized symmetries by their very nature require us to use higher categories, which in practice involves a great deal of faff. Fortunately, almost all of it can be avoided by going all the way to infinity and observing that both the generalized symmetries and the topological field theories on which they act form very special cases of ∞-categories, namely ∞-groupoids, in which all morphisms are invertible. As such, they can be replaced by topological spaces, or rather homotopy types, and the requisite mathematics can mostly be phrased in terms of homotopy theory. To translate back to physics requires us to return to higher categories, whose terminology we use in a colloquial sense, except when it comes to concrete examples.

So, for example, a point Z in a topological space Θ representing some ∞-groupoid of topological field theories corresponds to an object, *i.e.* a specific topological field theory, a path between two points corresponds to a 1-morphism between the corresponding topological field theories, a homotopy between two paths corresponds to a 2-morphism, and so on.

Our questions above are easily answered using this language. To give a sketch, it is convenient to distinguish the three mathematical notions of a *group*, an *action* of that group, and a *fixed point* of that action, and to generalize each of these.

In our simplified arena, a generalized group can be completely characterized by the homotopy type of a pointed connected topological space, or equivalently the classifying space BG of a single *topological* group G (which is not unique). The semi-infinite tower of homotopy groups $\pi_{q+1}(BG) \simeq \pi_q(G)$ encode the abstract groups of q-form symmetries for each non-negative integer q. The tower comes equipped with a rich structure. For example, for $q \geq 1$ the groups are abelian, in agreement with expectations for generalized symmetries. Furthermore, there is an action of the ordinary symmetry group $\pi_0(G)$ on each of the generalized symmetry groups $\pi_q(G)$, induced by the action of the topological group G on itself by conjugation. Thus, an answer to our first question is: a generalized group is the classifying space of a topological group (and an ordinary group corresponds to the special case of a discrete group).

A generalized action of a generalized group on some space Θ of topological field theories is then a fibration over BG, together with an identification of Θ with the fibre over the basepoint. (Many, but not all, of these arise via a continuous action of G on Θ as the bundle $EG \times_G \Theta \to BG$.) A generalized fixed point is a homotopy fixed point, namely a section of the fibration (in the case of the bundle $EG \times_G \Theta \to BG$, this is equivalently a G-equivariant map from EG to Θ).

Our second question can then be answered as follows. Every space Θ admits the trivial action of G, in which no points are moved. By passing back to the language of category theory, one sees that the corresponding homotopy fixed points (which are simply maps $BG \to \Theta$) correspond to true representations. But a specific Θ may also admit non-trivial actions of G, and we will see in examples how these reproduce projective and antilinear representations, and more besides. In category-theoretic language, these exotic possibilities arise because topological field theories (and presumably quantum field theories in general) form an ∞-groupoid, whose morphisms record (some of) their internal structure. So the right notion of a (generalized) group action is not one on a set, but on an ∞-groupoid, and the right notion of a fixed point is a limit in the sense of ∞-categories.

To answer the third question, consider again the homotopy fixed points of the trivial action, *i.e.* the maps $BG \to \Theta$. Looping such a map, we obtain a map from G to the space, $\Omega_Z\Theta$, of loops in Θ based at $Z \in \Theta$. On the category-theoretic side, these correspond to the automorphisms of the topological field theory Z, whose objects are invertible natural transformations from the topological field theory to itself. They thus commute with all possible dynamical evolutions. This is just the same as what happens for ordinary symmetries in quantum mechanics, except that in topoogical field theories the possible dynamical evolutions are different in nature, being evolutions along spacetimes with non-trivial topology or along spacetimes equipped with non-trivial geometric structures. There are induced homomorphisms $\pi_q(G) \to \pi_{q+1}(\Theta, Z)$, showing that the composition of natural transformations respects the group law on the nose. For non-trivial actions, we get transformations whose composition does not respect the group law on the nose, but is merely coherent with respect to it.

On the category theory side, a q-form symmetry corresponds to choices of $(q+1)$-transfors of TFTs (where a 0-transfor is a functor, a 1-transfor a natural transformation, etc.). Part of the data of such a $(q+1)$-transfor is the assignment of top-level morphisms in the target category to codimension $(q+1)$-manifolds. This is in accordance with the notion of a q-form symmetry given in [1]. In particular, 0-form symmetries assign top-level morphisms to codimension 1-manifolds. On looping, these morphisms descend down to linear maps between vector spaces.

To deal with the fourth question, of gauging generalized symmetries, we equip the bordism category underlying topological field theories with tangential structure. One can do this in a very general way, following Lurie [2], that allows for gauge symmetries that act non-trivially on spacetime. By introducing a notion of fibrations of tangential structures, we construct maps, in the language of homotopy theory, which we call *globalization maps*, that send spaces of theories with gauge symmetry to spaces of theories with global symmetry. This gives a convenient framework for discussing ‘t Hooft anomalies, which can be seen to be of one of two kinds. The first is an anomaly afflicting an entire space of theories with global symmetry, so we call it a *metaphysical ’t Hooft anomaly*, and arises when that space is not the image of any globalization map. An example familiar from quantum mechanics are theories with genuinely projective representations (meaning the associated 2-cocycle is not a coboundary) of an ordinary symmetry group.

The second kind of anomaly is that even if a suitable globalization map exists, it may fail to be surjective (on π_0); a theory with a global symmetry lying outside the image will then be anomalous. We call these *unphysical ‘t Hooft anomalies*, for reasons which will soon become clear.

These considerations also show that one can have what we call *'t Hooft ambiguities.* Namely, even if a theory is non-anomalous, so it is in the image of some globalization map, there is no guarantee that that map is unique, nor that it injects. So there may be many ways to gauge a global symmetry.

In the case of topological field theories that are fully local (or fully extended in the mathematicians' jargon), the cobordism hypothesis implies that the globalization maps are homotopy equivalences, so unphysical anomalies and ambiguities are necessarily absent. They thus arise purely in theories that fail to fully respect the sacred physics principle of locality, hence the moniker unphysical.

Our approach also allows us to shed further light on several earlier observations in the literature regarding symmetries (generalized or ordinary) of topological field theories.

To set the scene for this, consider the example of the orientable topological field theory in $d = 2$ obtained by quantizing a classical field theory with gauge symmetry $\mathbb{Z}_n^2$, as described in [3]. For now we will be deliberately vague regarding whether this theory is considered to be fully extended or not, as well as regarding what the target category is.

The classical action of this theory is specified by $p \in H^2(B(\mathbb{Z}_n^2), \mathbb{C}^*) \simeq \mathbb{Z}_n$. In [1], it was argued that this theory has 0- and 1-form symmetries given by $\mathbb{Z}_{n/k}^2$, where $k = \gcd(p, n)$. To make sense of this in our language requires us to consider the theories to be fully extended (with values in a certain bicategory of algebras over $\mathbb{C}$). We will then show that in fact $\mathbb{Z}_{n/k}^2$ is merely a subgroup of $\pi_1(\Theta, Z) \simeq S_{n^2/k^2}$ (*i.e.* the permutation group on n^2/k^2 elements) and of $\pi_2(\Theta, Z) \simeq (\mathbb{C}^*)^{n^2/k^2}$ for the corresponding theory Z. These larger symmetries cannot be seen by inspection of the classical action. More generally, we show how to compute $\pi_1(\Theta, Z)$ and $\pi_2(\Theta, Z)$ for every fully-extended topological field theory in $d = 2$ and show how they arise as subgroups of S_m and $(\Bbbk^*)^m$, for some $m \in \mathbb{N}$ (which are $\pi_1(\Theta, Z)$ and $\pi_2(\Theta, Z)$ for every possible Z for the base case of topological field theories defined on manifolds equipped with 2-framings) with values in algebras over any separably closed field $\Bbbk$. We also show how one may characterize all possible homotopy fixed points of all possible actions in such cases.

These results are suspiciously close to those of [4], which showed that 0-form symmetries of *un*extended oriented topological field theories whose corresponding commutative Frobenius algebras are semisimple act on a basis of idempotents by permutations preserving the trace map, while 1-form symmetries act by multiplication of idempotents by elements of $\mathbb{C}^*$. In fact this is no coincidence, because the semisimple commutative Frobenius algebras correspond precisely to oriented topological field theories which are extend*able*: by extending, one can make a proper definition of a 1-form symmetry, and show that the conditions obtained in [4] are not only necessary, but also sufficient. Our results thus not only generalize those of [4], but also place them in their proper context.

The same (unextended) $\mathbb{Z}_n^2$ gauge theory appears elsewhere [5] as an example of a theory with ordinary global symmetries that ostensibly suffer from a ‘t Hooft anomaly. This seems odd, given that the theory is extendable, and given that for extended theories the cobordism hypothesis implies that the globalization map is a homotopy equivalence. The resolution of this apparent paradox is as follows. Ref. [5] in fact describes a method for constructing theories with global symmetries that are free of ‘t Hooft anomalies (see the diagram in Eq. 17). The construction only works if a certain necessary condition is satisfied. Ref. [5] defines a theory to be ‘anomalous’ if that condition is violated, but that merely means that the construction cannot be carried out. As a result one does not even have a theory of which one can ask the question of whether it is anomalous or not, in the usual sense.

The outline of the paper is as follows. In §II, we give a brief introduction to ∞-categories and describe some examples relevant for topological field theory. In §III we define generalized groups and generalized global symmetries of topological field theories. In §IV we define generalized gauge symmetries, construct the globalization maps, and define ’t Hooft anomalies and ambiguities in that context. In §§V–VII we discuss examples of topological field theories in $d = 1$ and $d = 2$. In §VIII we discuss Lie group symmetries and in §IX we compare our results with earlier literature.

II. TOPOLOGICAL FIELD THEORIES

A. Unextended topological field theories

To set the scene, we begin with a review of unextended topological field theories, formulated using the category-theoretic approach pioneered by Atiyah, Kontsevich, and Segal. We need the notions of category, functor, natural transformation, and equivalence of categories, all of which are standard and may be found in [6]. To set the notation, a category[1] C is a collection of objects with a set of composable morphisms between each pair of objects; given a pair of categories C and D, the functors between them themselves form the objects of a category $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ whose morphisms are the natural transformations.

We also need extra structure on a category C, namely a symmetric monoidal structure, for which definitions may be found in [7]. Roughly, we have a functor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, a unit object $1 \in \mathsf{C}$, natural braiding isomorphisms $s_{a,b} : a \otimes b \xrightarrow{\sim} b \otimes a$ that square to the identity, and natural isomorphisms $a \otimes 1 \xrightarrow{\sim} a$ exhibiting 1 as a unit. A dual to a is an object $a^\vee$ along with an evaluation morphism $\mathrm{ev} : a^\vee \otimes a \to 1$ and a coevaluation morphism $\mathrm{coev} : 1 \to a \otimes a^\vee$ obeying certain familiar conditions; we say that a is invertible if these morphisms are, moreover, isomorphisms. Given a pair of symmetric monoidal categories, there is a notion of a (strong) symmetric monoidal functor; these form the objects of a symmetric monoidal category $\mathsf{Fun}^\otimes(\mathsf{C}, \mathsf{D})$, whose morphisms are the monoidal natural transformations.

For unextended topological field theory in spacetime dimension d, we start from the symmetric monoidal category $\mathsf{Bord}_{d,1}$. An object in $\mathsf{Bord}_{d,1}$ is a closed (*i.e.* compact without boundary) $(d-1)$-manifold M. (In general, we may wish to equip manifolds with additional structures, such as an orientation or spin structure, but since this will not play a significant rôle until § IV, we elide it for now.) A morphism from M to N in $\mathsf{Bord}_{d,1}$ is an equivalence class of compact d-manifolds W whose boundary is identified with the disjoint union $M \coprod N$, where two W's are considered equivalent if they are related by a diffeomorphism which is the identity on the boundary.[2] Composition of bordisms is defined by gluing manifolds along the appropriate boundary components. The product in the symmetric monoidal structure is given by the disjoint union of manifolds (so we denote it $\coprod$) and the unit is given by the empty manifold $\emptyset$. Physically, W represents the 'spacetime' of a euclidean quantum field theory evolving in euclidean time from space M to space N, but the evolution is allowed to be topologically non-trivial.

An unextended topological field theory is then a symmetric monoidal functor Z out of

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[1] A note on notation: since categories and higher categories can be thought of as special cases of infinity categories, we use a sans serif font for all of them (and even the same letter C to denote a generic one), except in the case of ∞-groupoids, which will later be treated as topological spaces.

[2] We remark that this unpleasantry is already a good motivation to go to $(\infty, 1)$-categories, in which a morphism is simply a bordism.

$\mathsf{Bord}_{d,1}$ to some target symmetric monoidal category, which we must now choose. Given that we are modelling a euclidean theory, and given that there is is no obvious available notion of a Wick rotation to a lorentzian theory, it seems artificial to demand the usual quantum-mechanical structure of complex Hilbert space and operators that are self-adjoint or isometries (or hermitian and unitary in the physics lingo), or even some Wick-rotated version thereof. We therefore choose the target to be $\mathsf{Vect}_{\Bbbk}$ whose objects are vector spaces over an arbitrary field $\Bbbk$, whose morphisms are $\Bbbk$-linear maps, and whose product is the usual tensor product of vector spaces (so the unit may be taken to be $\Bbbk$). Physically, the fact that Z is a functor, so preserves composition, encodes (partially) the fact that theories of physics should be local. Indeed, going backwards we see that we can recover the evolution along W from the evolution along bordisms obtained by cutting it along an arbitrary submanifold of codimension one.[3] The symmetric monoidal structure of Z allows for composite systems to be entangled in the usual way.

The correlation functions of the theory are encoded as follows. The usual local operators of quantum field theory are supported on points, and the effect of inserting such operators at points $w_1, \dots, w_i, \dots, w_n$ in a closed d-manifold W is found by deleting disjoint open neighbourhoods of each w_i in W, resulting in a compact d-manifold with boundary W', which in turn defines a bordism from $\coprod_i S^{d-1}$ to the empty d-manifold $\emptyset$. Applying the functor Z returns a linear map $Z(W') : \bigotimes_i Z(S^{d-1}) \to \Bbbk$. The vectors in $Z(S^{d-1})$ may thus be regarded as the local operators of the theory, and the linear map $Z(W')$, which returns a complex number given a choice of local operator at each w_i, may be regarded as the correlation function.

Key to our story will be the category $\mathrm{TFT}_{d,1} := \mathsf{Fun}^{\otimes}(\mathsf{Bord}_{d,1}, \mathsf{Vect}_{\Bbbk})$ of symmetric monoidal functors. Its objects are topological field theories and its morphisms (which are monoidal natural transformations) give us a way to compare topological field theories with one another and thus detect at least some of their structure. In particular, the presence of a morphism between two theories that is an isomorphism allows us to conclude that they are are physically equivalent, since they will lead to theories in which the correlation functions (and ultimately the observables) are related to one another in the same way.

[3] We remark that the fact that we are only allowed to cut along codimension one means that, locality is not fully manifest. This is already a good motivation to go to (∞, d)-categories.

In fact, every morphism in $\mathrm{TFT}_{d,1}$ is a isomorphism (so $\mathrm{TFT}_{d,1}$ is a groupoid). To show this, we need to show that given a natural transformation $\eta : Z \to Z'$ and any closed d-manifold M, the induced linear map $\eta_M : Z(M) \to Z'(M)$ is an isomorphism. Regarding the 'cylinder' $M \times I$ as a bordism $M \coprod M \to \emptyset$ or $\emptyset \to M \coprod M$ and applying Z furnishes us with evaluation and coevaluation maps that exhibit $Z(M)$ as a dual of itself and the dual map $\eta_M^\vee : Z'(M) \to Z(M)$ turns out to be the sought-after inverse to η_M.

Even though every morphism in a groupoid such as $\mathrm{TFT}_{d,1}$ is an isomorphism, the groupoid can tell us much more than just whether two theories are equivalent, because each theory (and indeed any object in any category) has associated to it a group of automorphisms, namely the isomorphisms from the theory to itself. It is natural to guess that this group is related to the global symmetry of the theory and this guess is confirmed by picking apart the definition of an automorphism of $Z \in \mathrm{TFT}_{d,1}$: it is, for each closed d-manifold M, a linear isomorphism $\eta_M : Z(M) \to Z(M)$ such that, for any bordism $W : M \to N$, the diagram

$$\begin{array}{ccc} Z(M) & \xrightarrow{Z(W)} & Z(N) \\ \downarrow{\scriptstyle \eta_M} & & \downarrow{\scriptstyle \eta_N} \\ Z(M) & \xrightarrow{Z(W)} & Z(N) \end{array} \tag{1}$$

commutes. So the components of η are linear maps (for each state space $Z(M)$) that commute with the dynamical evolutions $Z(W)$ along all possible euclidean spacetimes W. This looks very close to the usual quantum-mechanical notion of a unitary operator on the Hilbert space of states that commutes with the unitary time evolution operator, except that the notion of unitarity has gone and that the evolutions are now trivial (since the cylinder $M \times I$ is the identity bordism on M, it gets sent by the functor Z to the identity linear map on $Z(M)$), unless spacetime is topologically non-trivial. Moreover, the diagram shows that the components of η are compatible with locality, expressed in terms of cutting and pasting of bordisms.

We could, therefore, make an intrinsic definition of the global symmetry group of Z to be its group $\mathrm{Aut}(Z)$ of monoidal natural automorphisms, or alternatively make an extrinsic definition of a global symmetry of Z as a group G together with a homomorphism $G \to \mathrm{Aut}(Z)$, but we will see in the next Section that it pays to do something which is naïvely rather different, namely to consider fixed points (in a appropriate sense) of actions of G on the groupoid $\mathrm{TFT}_{d,1}$. In fact this turns out to generalize the notion of a homomorphism

$G \to \mathrm{Aut}(Z)$ (which is recovered as a fixed point of the trivial action of G on $\mathrm{TFT}_{d,1}$). Doing so allows us to capture the notion that an element of a physical symmetry group need not fix Z, but rather can send it to an isomorphic theory, without affecting physical observables. We will see in §VI, moreover, that this generalization is needed to describe well-known physical phenomena such as the behaviour of electrons under spatial rotations and time-reversal invariance.

Before doing that, we describe extended topological field theories. These will be needed not only to define generalized global symmetries in §III, but also to formulate physics in a way which is fully local. As we will argue in §VI, certain ‘t Hooft anomalies are best viewed as arising from a failure to define a theory in such a way and so should be regarded as unphysical.

B. Extending topological field theories downwards

We have already seen that defining topological field theories using ordinary categories only captures a part of the local structure of physics. To fully capture locality, the theory should be defined not only on closed $(d-1)$-manifolds and d-manifolds with boundary, but on manifolds with corners of all possible codimensions, so that dynamics can be reconstructed by pasting together simplices.

For that, we require higher categories. Roughly, these should consist of objects, morphisms, higher morphisms, and so on, which can be composed in multiple ways in a coherent fashion. Precise definitions are, however, somewhat involved. Since we will only go one step higher in our examples, and since we will anyway soon need the yet more general notion of an ∞-category, we will content ourselves here with sketching the simplest case, namely a bicategory. Full details are given in, *e.g.*, [8].

A *bicategory*, C, is a collection of objects with a category $\mathsf{C}(a,b)$ for each ordered pair (a,b) of objects in C. The objects of $\mathsf{C}(a,b)$ are called *1-morphisms* and the morphisms of $\mathsf{C}(a,b)$ are called *2-morphisms*. In addition, there is a functor $\mathsf{C}(b,c) \times \mathsf{C}(a,b) \to \mathsf{C}(a,c)$ known as *horizontal composition*, with a unit 1-morphism $1_a \in \mathsf{C}(a,a)$ whilst composition within $\mathsf{C}(a,b)$ is called *vertical composition*. An *equivalence* between objects a and b is a pair of 1-morphisms $f : a \leftrightarrow b : g$ and a pair of 2-morphisms $\alpha : 1_a \to g \circ f$ and $\beta : f \circ g \to 1_b$[4] that are isomorphisms in $\mathsf{C}(a, a)$ and $\mathsf{C}(b, b)$ respectively.

Given two bicategories, we have the notion of a functor between them; given two functors we have the notion of a transformation, and given two transformations we have the notion of a modification. The functors, transformations, and modifications assemble themselves respectively into the objects, 1-morphisms, and 2-morphisms of a functor bicategory.

We will also need a symmetric monoidal structure on bicategories and the corresponding bicategory of symmetric monoidal functors.

An example of a symmetric monoidal bicategory, which will play the rôle of the target bicategory in our examples, is $\mathsf{Alg}_{\Bbbk}$: an object is an algebra over a field $\Bbbk$, a 1-morphism from an algebra A to an algebra B is an (A, B)-bimodule, and a 2-morphism is an (A, B)-bilinear map. The horizontal composition of 1-morphisms is given by the tensor product of bimodules (over the algebra in the middle) and the symmetric monoidal structure is given by the tensor product over $\Bbbk$. The relevance of this bicategory to physics is as follows. Given any (symmetric) monoidal bicategory C, the endomorphisms of the unit object form a (symmetric) monoidal category, which we denote $\Omega\mathsf{C}$. This looping construction extends to higher monoidal categories and is adjoint to a delooping construction, which sends a monoidal higher category C to a monoidal category $B\mathsf{C}$ one level higher with a single object whose endomorphisms are C. The unit object of the bicategory $\mathsf{Alg}_{\Bbbk}$ is $\Bbbk$ and its endomorphism category consists of $\Bbbk$-vector spaces and $\Bbbk$-linear maps. Thus, $\mathsf{Alg}_{\Bbbk}$ may be regarded as an extension (not unique) of the usual target category $\mathsf{Vect}_{\Bbbk}$ of unextended topological field theories. Looping again, we obtain the symmetric monoidal 0-category (*i.e.* commutative monoid) of linear endomorphisms of $\Bbbk$, which is isomorphic to $\Bbbk$ itself. This provides a target for maximally *un*extended theories, in which the source 'category' contains only closed d-manifolds, which we discuss as a toy example in §V.

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[4] We reluctantly perpetuate the now-standard practice of denoting both horizontal composition of 1-morphisms and vertical composition of 2-morphisms with $\circ$, with $*$ being used for horizontal composition of 2-morphisms.

C. Extending topological field theories upwards

It is convenient, for a number of reasons, to extend topological field theories upwards as well, using the language of ∞-categories. One is that, as we have already hinted, it leads to a simplification of the domain. Another is that the connection with homotopy theory is more explicit. A third is that it then becomes obvious that symmetries of topological field theories should be described by homotopy fixed points of group actions, since homotopy limits are the only kind of limits in the ∞-categorical context.

As for higher categories, we shall content ourselves with a sketch of the relevant concepts and definitions. For more details, see [2].

An (∞, n)-category C has objects, and morphisms of all levels, where the morphisms at level greater than n are invertible, but now in a recursive sense. So a morphism f is invertible if there exists a morphism g in the other direction along with morphisms $\alpha : 1 \to g \circ f$ and $\beta : f \circ g \to 1$ at one level higher that are themselves invertible. An ∞-category is n-truncated if the morphisms at level greater than n are equivalent to identity morphisms. Evidently, a higher n-category can be identified with an n-truncated ∞-category (which we hope excuses the somewhat overloaded notation). Going in the other direction, we obtain the homotopy n-category of an ∞-category C by replacing the morphisms at level n with their equivalence classes under the equivalence described above.

Given two (∞, n)-categories C, D, there is an (∞, n)-category of functors $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ from one to the other, and if D is n-truncated then $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ is an n-truncated category. There is an $(\infty, n+1)$-category Cat_n, whose objects are (∞, n)-categories and whose endomorphisms are the (∞, n)-categories of functors.

An $(\infty, 0)$-category, in which all morphisms are invertible, is also called an ∞-groupoid and corresponds, via the homotopy hypothesis, to a homotopy type, *i.e.* a topological space up to weak homotopy type. In one direction, this correspondence is given by forming the fundamental ∞-groupoid of a topological space X: objects are points in X, 1-morphisms are continuous paths, 2-morphisms are homotopies between paths, and so on.

The (∞, p)-category $\mathsf{Bord}_{d,p}$ has objects given by $(d-p)$-manifolds (with suitable corners), 1-morphisms given by $(d-p-1)$-manifolds,. . . , p-morphisms given by d-manifolds, $(p+1)$-morphisms given by diffeomorphisms of d-manifolds, $(p+2)$-morphisms given by isotopies of diffeomorphisms, and so on.

In this picture, topological field theories are (∞, p)-functors from $\mathsf{Bord}_{d,p}$ to some target (∞, p)-category D. An argument similar to the one we gave for unextended topological field theories shows that the category $\mathsf{Fun}^{\otimes}(\mathsf{Bord}_{d,p}, \mathsf{D})$ is in fact an ∞-groupoid, or a homotopy type [2, Remark 2.4.7]. In our examples, we will take D to be p-truncated, so that a topological field theory factors through the homotopy p-category of $\mathsf{Bord}_{d,p}$. Moreover, $\mathsf{Fun}^{\otimes}(\mathsf{Bord}_{d,p}, \mathsf{D})$ is a homotopy p-type.

III. (GENERALIZED) GLOBAL SYMMETRIES

A. Ordinary global symmetries

Before discussing generalized global symmetries, let us consider how to describe ordinary global symmetries of topological field theories. As usual in mathematics, it is convenient to separate the notions of a group, an action of that group on something, and a fixed point of that action.

The most straightforward example is that of a group action on a set S, where a group action is a homomorphism from G to the group $\mathrm{Aut}(S)$ of bijections of S. A fixed point is an element of S that is fixed by each element of G and the set of all fixed points forms a subset S^G of S.

This can be formulated using the language of category theory as follows. A group G can be considered as a category (in fact, a groupoid) BG with a single object and an isomorphism for each element of G. A group action of G on S is then a functor from BG to the category Set, whose objects are sets and whose morphisms are functions, that sends the single object of BG to the set S. The fixed point set S^G along with its inclusion in S then arises as the limit (in the category theory sense) of this functor.

From here it is easy to see what an ordinary global symmetry of a topological field theory should be. Topological field theories do not form a set, but rather an ∞-groupoid, which we generically denote by Θ. To discuss a G-symmetry of a topological field theory $Z \in \Theta$ we should therefore first give a G-action on the ∞-groupoid Θ of topological field theories of interest, *i.e.* an ∞-functor $BG \to \mathsf{Gpd}_\infty$ sending the unique object of BG to Θ, and then the ∞-limit Θ^{hG} of this functor should be thought of as an ∞-groupoid of topological field theories equipped with G-symmetry. The ∞-functor $\Theta^{hG} \to \Theta$ sends a topological field theory equipped with a G-symmetry to the underlying topological field theory. It need not be either essentially surjective or injective (unlike the inclusion $S^G \hookrightarrow S$), reflecting the fact that not every theory need admit a symmetry for the given action, and that if it does the symmetry need not be unique.

Now the advantage of working exclusively with ∞-groupoids, as we have done, becomes clear: it is that the category of such is equivalent to the category of spaces, and the categorical constructions we have just described have down-to-earth interpretations in terms of homotopy theory of topological spaces and the familiar tools of algebraic topology can be used to study them.

The following, then, is a reformulation of the above in terms of topological spaces. Firstly, the ∞-groupoid BG corresponds to the classifying space BG of the group G, considered as a pointed space (this excuses the clash of notation). Then, giving an action of G on the ∞-groupoid Θ of topological field theories corrresponds to giving a commutative square of spaces

$$\begin{array}{ccc} \Theta & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \{*\} & \longrightarrow & BG \end{array}$$

which is a homotopy pull-back, *i.e.* giving a fibration $\pi : E \to BG$ of spaces, along with an identification $\pi^{-1}(*) = \Theta$ of the fibre over the basepoint $* \in BG$ with Θ. Finally, the ∞-groupoid Θ^{hG} corresponds to the space of sections $s : BG \to E$ of the fibration π, and the forgetful morphism $\Theta^{hG} \to \Theta$ corresponds to the map sending a section s to the point $s(*) \in \Theta = \pi^{-1}(*)$.

As we will see in §VI in the examples with $d = 1$, corresponding to quantum mechanics, these notions naturally give rise to global symmetries with the usual physical properties. In particular, the oriented topological field theories in $d = 1$ correspond to finite dimensional vector spaces, with equivalences given by linear isomorphisms. For the trivial action of G on this groupoid, the groupoid of homotopy fixed points has objects that are finite-dimensional representations of G and morphisms that are invertible G-equivariant linear maps. For non-trivial actions, we obtain both projective and semi-linear representations, and more besides.

An important subtlety is the following. In defining the notion of a global symmetry, we did not take the symmetric monoidal structure of topological field theories into account. We could instead have defined a group action to be an ∞-functor from BG to the $(\infty, 1)$-category of *symmetric monoidal* ∞-groupoids. This makes a difference when we try to take the limit, since Θ^{hG} must itself then be a symmetric monoidal ∞-groupoid. This would exclude, for example, projective representations, the category of which (for a specified cocycle) does not have the necessary monoidal structure. Since these are well-known to occur as global symmetries in Nature, we consider our construction to be the appropriate one.

B. Generalized global symmetries

Our formulation of ordinary global symmetries of topological field theories makes it easy to extend to generalized symmetries. Indeed, in the categorical language, the group G is considered as a 1-truncated ∞-groupoid with a single object. So the only non-identity morphisms are the 1-morphisms, and these correspond to the elements of G. The only change we need to make to consider generalized symmetries is to relax the requirement that our ∞-groupoid be 1-truncated. So it may now have invertible morphisms at all levels, and these give rise, albeit indirectly, to higher-form symmetries.

On the homotopy theory side, an ∞-groupoid with a single object corresponds to a pointed connected topological space. Every such space has the homotopy type of the classifying space BG of some topological group G, so we continue to refer to it as such. It is important to note, however, that G is not necessarily unique. For example, any connected Lie group has a maximal compact subgroup, and the embedding is both a homomorphism and a homotopy equivalence. Since the classifying space construction can be made functorial, we conclude that the classifying space of any connected Lie group has the same homotopy type of a maximal compact subgroup.[5]

With this change made, everything goes through as before. A generalized group action of G on Θ is again an ∞-functor $BG \to \mathrm{Gpd}_\infty$ sending the unique object of BG to Θ, or equivalently, on the homotopy theory side, a fibration $\pi : E \to BG$ with fibre Θ over the basepoint, and Θ^{hG} is the limit of this ∞-functor, or equivalently the space of sections of $\pi : E \to BG$.

Thus we have an extrinsic notion of generalized global symmetry. Let us now give, as we

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[5] We will use this fact later when we discuss generalized gauge symmetries to replace general linear groups by orthogonal groups.

did earlier for ordinary symmetries, an intrinsic notion, and connect the two. As we saw in the previous section, for any ordinary category, there is a natural notion of the symmetry of an object, given by the group of automorphisms of that object, with multiplication given by composition of morphisms. (For an object in a groupoid, such as a TFT, every morphism is an isomorphism, so we can equivalently consider the endomorphisms.) For an object Z in an ∞-groupoid Θ, the morphisms from that object to itself themselves form an ∞-groupoid. The corresponding space is the homotopy pullback of $\{Z\} \hookrightarrow \Theta \hookleftarrow \{Z\}$, *i.e.* the loop space $\Omega_Z(\Theta)$. To see that this is sensible, note that an object in the fundamental ∞-groupoid of the space Θ is a point $Z \in \Theta$ and a 1-morphism from Z to Z is a path from Z to Z in Θ, *i.e.* a loop at Z. Now, $\Omega_Z(\Theta)$ does not quite have the structure of a group,[6] but rather that of an H-group. That is, it has a multiplication (given by concatenation of loops) and an identity (given by the constant loop at Z), such that the usual group axioms are obeyed up to homotopy.

To see the relation between the extrinsic and intrinsic symmetries, consider the special case of a trivial fibration $E = \Theta \times BG$. A section of this is simply a map $BG \to \Theta$. It sends $* \in BG$ to some $Z \in \Theta$ and looping we get a map $\Omega BG \to \Omega_Z(\Theta)$ and consequently homomorphisms $\pi_q(G) \simeq \pi_{q+1}(BG) \to \pi_{q+1}(\Theta, Z)$, corresponding to actions of q-form symmetry groups $\pi_q(G)$ on the theory Z.

Passing back to the category theoretic side, we see that $\pi_0(G)$ acts by transformations, $\pi_1(G)$ acts by modifications, and so on. Now, part of the data of a transformation is a 1-morphism in the target for each object in the source, a 2-morphism in the target for each 1-morphism in the source, *&c*, whereas the data of a modification is a 2-morphism in the target for each object in the source, *&c*, *&c*. Spacetime evolutions are associated to d-morphisms, and we see that then the q-form symmetries act on what the topological field theory associates to manifolds of codimension $q + 1$, exactly as we expect for q-form symmetries.

Moreover, we see that a non-trivial q-form symmetry can only arise for a topological field theory that has been extended at least $q + 1$ times. (Our convention is that a maximally unextended theory is a functor out of a 0-category, *i.e.* a set, whose objects are closed d-manifolds.) So for a topological field theory formulated using ordinary category theory (*i.e.*

[6] It is, however, a theorem that every loop space has the homotopy type of a topological group.

once extended, according to the convention just given), we can have at most an ordinary symmetry. But maximally extended theories, in which locality is fully manifest, can have q-form symmetries for all $q \leq d-1$.[7]

IV. GENERALIZED GAUGE SYMMETRIES

We now wish to describe generalized gauge symmetries of topological field theories. An ordinary gauge theory corresponds to equipping spacetime M with a principal G-bundle, or equivalently a map from M to BG. Since we saw in the last Section that a generalized global symmetry can be obtained by replacing the abstract group G by a topological group, it is natural to suppose that the same is true in the gauge case.

As we shall see, it is fruitful to do something more general than equip manifolds with maps to BG. Indeed, as well as giving us a notion of gauge symmetries that act non-trivially on spacetime, it allows us to subsume the notion of spacetime structures, such as an orientation or a spin structure. We refer to gauge symmetries that act trivially on spacetime as internal gauge symmetries.

This construction closely follows [2], though the interpretation in terms of generalized symmetries is presumably new.

Letting X be a topological space, and ξ a rank d real-vector bundle over X, we define the p-category ${}^{(X,\xi)}\mathsf{Bord}_{d,p}$ as follows: A $(p-k)$-morphism, for $0 \leq k \leq p-1$ is a triple (M, f, s) consisting of:

- a $(d-k)$-dimensional manifold M, with boundary, corners, *&c*;
- a continuous map $f : M \to X$;
- an isomorphism of real vector bundles $s : TM \oplus \underline{\mathbb{R}^k} \to f^*\xi$, where TM is the tangent bundle of M, $\underline{\mathbb{R}^k}$ is the trivial rank-k real vector bundle over M, and $\oplus$ denotes the Whitney sum of bundles.

For p-morphisms, we take the equivalence class of such triples up to structure- and corner-preserving diffeomorphisms.

[7] As we shall see in an example in V, at the level of the action it makes sense to speak of q-form symmetries acting on q-times extended theories. But the resulting fixed points represent properties, rather than structures, of the corresponding theories.

One source of (X, ξ)'s is as follows: if G is a topological group and $\chi : G \to O(d)$ is a d-dimensional representation, we can take $X = BG$ and ξ to be the vector bundle $EG \times_G \mathbb{R}^d$ over BG.

Instead of writing ${}^{(BG,EG\times_G\mathbb{R}^d)}\mathsf{Bord}_{d,p}$, we denote this category by ${}^{G}\mathsf{Bord}_{d,p}$, with the homomorphism χ left implicit. Similarly, by ${}^{G}\mathrm{TFT}_{d,p}$ we denote the ∞-groupoid of symmetric monodial functors from ${}^{G}\mathsf{Bord}_{d,p}$ to some symmetric monodial p-category C.

Some relevant examples are: *(i)* ${}^{*}\mathrm{TFT}_{d,p}$ corresponds to framed TFTs (where $*$ denotes the group with one element); *(ii)* ${}^{SO(d)}\mathrm{TFT}_{d,p}$, with χ the obvious inclusion, corresponds to oriented TFTs, since our conditions correspond to a reduction of structure group from $SO(d) \to O(d)$; and *(iii)* ${}^{O(d)}\mathrm{TFT}_{d,p}$, with χ the identity map, corresponds to unoriented TFTs, which we earlier denoted simply $\mathrm{TFT}_{d,p}$.

A notion of equivalence of tangential structures is naturally built in as follows. A map $\xi \to \xi'$ of vector bundles which covers a homotopy equivalence $X \to X'$ and induces a linear isomorphism on fibres, gives an equivalence ${}^{(X,\xi)}\mathsf{Bord}_{d,p} \xrightarrow{\sim} {}^{(X',\xi')}\mathsf{Bord}_{d,p}$ of categories, and therefore an equivalence ${}^{(X',\xi')}\mathrm{TFT}_{d,p} \xrightarrow{\sim} {}^{(X,\xi)}\mathrm{TFT}_{d,p}$ of spaces of TFTs.

A. Globalization maps

We now wish to make a connection between generalized global symmetries and generalized gauge symmetries and to discuss possible generalized 't Hooft anomalies, *i.e.* obstructions to gauging generalized global symmetries.

As we will see, this question of physics has a natural mathematical formulation in terms of *globalization maps* relating spaces of theories with various combinations of gauged and global symmetries. These globalization maps formalize, in the topological field theory context, the physicist's notion (for Lie group gauge symmetries of theories on spacetime $\mathbb{R}^d$) of 'turning off the gauge field'. The issue of 't Hooft anomalies can then be broken down into whether a suitable globalization map exists and, if so, whether it surjects (on π_0). For maximally-extended theories, the cobordism hypothesis guarantees the latter.

Let us first try to develop some intuition for globalization maps by describing the simplest case in which we have an internal gauge symmetry G (*i.e.* the homomorphism χ maps G to the identity element in $O(d)$) which we wish to globalize. We can achieve this by restricting a topological field theory to spacetime manifolds equipped with the trivial G-bundle, which defines a functor ${}^G\mathrm{TFT}_{d,p} \to^* \mathrm{TFT}_{d,p}$, and then consider the effect of bundle automorphisms, which allows us to factor ${}^G\mathrm{TFT}_{d,p} \to^* \mathrm{TFT}_{d,p}$ through ${}^*\mathrm{TFT}^{hG}_{d,p}$, where the action of G on ${}^*\mathrm{TFT}_{d,p}$ is the trivial one. (In the physicist's lingo, we switch off the gauge field and do a constant gauge transformation.)

More generally, we might want to retain some some normal subgroup of G as a gauge symmetry (or as a spacetime structure), or preserve an existing global symmetry, or both. The following construction allows us to cover all of these possibilities, and more besides.

We consider the following data: a tangential structure (X, ξ) and a fibration $\Pi : X \to B$. For each point $b \in B$ we then have a space $X_b := \Pi^{-1}(b)$ with a vector bundle $\xi_b := \xi|_{X_b}$ on it, so we think of this data as a continuous family of tangential structures $\{(X_b, \xi_b)\}_{b\in B}$ parameterised by B. To this data we may associate the family of bordism (∞, p)-categories $\{{}^{(X_b,\xi_b)}\mathsf{Bord}_{d,p}\}_{b\in B}$, and by applying the functor $\mathrm{Fun}^{\otimes}(-, \mathsf{C})$ to each member of this family we obtain a family of ∞-groupoids $\{{}^{(X_b,\xi_b)}\mathrm{TFT}_{d,p}\}_{b\in B}$ parameterised by B. Equivalently, we have a space ${}^{(X,\xi;\Pi)}\mathrm{TFT}_{d,p}$ and a fibration

$$\tau : {}^{(X,\xi;\Pi)}\mathrm{TFT}_{d,p} \longrightarrow B$$

such that $\tau^{-1}(b) = {}^{(X_b,\xi_b)}\mathrm{TFT}_{d,p}$.

The inclusions $i_b : X_b \to X$ are by definition covered by bundle isomorphisms $\xi_b \to i_b^*\xi$, so we can canonically consider any manifold equipped with a (X_b, ξ_b)-structure as being equipped with a (X, ξ)-structure: this defines symmetric monoidal functors $(i_b)_* : {}^{(X_b,\xi_b)}\mathsf{Bord}_{d,p} \to {}^{(X,\xi)}\mathsf{Bord}_{d,p}$ and hence restriction functors

$$i_b^* : {}^{(X,\xi)}\mathrm{TFT}_{d,p} \longrightarrow {}^{(X_b,\xi_b)}\mathrm{TFT}_{d,p}.$$

Thus any topological field theory defined for (X, ξ)-manifolds provides a theory for (X_b, ξ_b)-manifolds, varying continuously with $b \in B$. That is, there is a map

$$\Pi_* : {}^{(X,\xi)}\mathrm{TFT}_{d,p} \longrightarrow \{\text{Sections of } \tau : {}^{(X,\xi;\Pi)}\mathrm{TFT}_{d,p} \to B\}.$$

This map is contravariantly functorial in the data $(\Pi : X \to B, \xi)$. In particular, if Γ is a group of symmetries of this data then it acts on source and target of this map, and we can further take the (∞-)fixed points for these Γ-actions.

The most important source of examples for us will arise from having a group extension $1 \to K \to G \xrightarrow{q} Q \to 1$ and a representation $\chi : G \to O(d)$, then taking $X = BG$, $B = BQ$,

$\Pi = Bq : BG \to BQ$, and $\xi = EG \times_G \mathbb{R}^d$. In this case the construction gives a homotopy pull-back square

$$\begin{array}{ccc} {}^K\mathrm{TFT}_{d,p} & \longrightarrow & {}^{(G;q)}\mathrm{TFT}_{d,p} \\ \big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ \{*\} & \longrightarrow & BQ, \end{array}$$

which as usual corrresponds to an (∞-)Q-action on ${}^K\mathrm{TFT}_{d,p}$, along with a map

$$\Pi_* : {}^G\mathrm{TFT}_{d,p} \longrightarrow \{\text{Sections of } \tau : {}^{(G;q)}\mathrm{TFT}_{d,p} \to BQ\},$$

where the latter corresponds to the (∞-)fixed points of the Q-action, and might equally well be denoted by ${}^K\mathrm{TFT}^{hQ}_{d,p}$. In physics terms, this corresponds to passing from a (generalized, not necessarily internal) gauge symmetry G to a normal subgroup K, such that the quotient group Q becomes a global symmetry.

A notable example comes from the degenerate extension $1 \to 1 \to G \xrightarrow{\mathrm{Id}} G \to 1$. In this case the homotopy pull-back square

$$\begin{array}{ccc} {}^*\mathrm{TFT}_{d,p} & \longrightarrow & {}^{(G;\Pi)}\mathrm{TFT}_{d,p} \\ \big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ \{*\} & \longrightarrow & BG \end{array}$$

can be identified with that given by the G-action via χ on ${}^*\mathrm{TFT}_{d,p}$ by the symmetries of the tangential structure $(*, \mathbb{R}^d)$, and so the map in question is

$$\mathrm{Id}_* : {}^G\mathrm{TFT}_{d,p} \longrightarrow \{\text{Sections of } \tau : {}^{(G;\mathrm{Id})}\mathrm{TFT}_{d,p} \to BG\} = {}^*\mathrm{TFT}^{hG}_{d,p}.$$

Physically, we have turned all of the gauge symmetry into a global symmetry (of d-framed TFTs).

B. Anomalies and the cobordism hypothesis

The globalization map allows us to discuss the notion of ‘t Hooft anomalies (*i.e.* global (generalized) symmetries that can’t be gauged) in a precise way. Indeed, we see that to be anomaly-free, a theory $Z \in \Theta$ must be in the image of some globalization map. This condition can be violated in two ways.

Firstly, the space Θ in which Z lives may not be the target of any globalization map; if so we say we have a *metaphysical anomaly*, since the anomaly afflicts the entire space of TFTs. Indeed, to be the target of a globalisation functor, the action of this global symmetry has to be of a special kind, namely it must act via symmetries of the tangential structure. An example of such an anomaly, as we will later see, is given by representations of topological field theories in $d = 1$ that are genuinely projective, *i.e.* those that correspond to cohomologically non-trivial group cocycles.[8]

Secondly, we may have a globalization map, but Z may not be in its image. We call this an *unphysical anomaly*, because it cannot arise in theories that are maximally extended, *ergo* fully local, as we believe theories of physics should be. This follows by the cobordism hypothesis, whose proof is sketched in [2], which implies that the map

$$\pi_* : {}^{(X,\xi)}\mathrm{TFT}_{d,d} \longrightarrow \{\text{Sections of } \tau : {}^{(X,\xi;\pi)}\mathrm{TFT}_{d,d} \to B\}$$

is a weak homotopy equivalence.

So (up to equivalence), any theory in the target corresponds to a unique theory in the source, meaning that neither ‘t Hooft anomalies nor ‘t Hooft ambiguities (by which we mean multiple gauge theories with the same image under globalization) can arise in this way. But for non-maximally extended theories where the cobordism hypothesis does not apply, we may also find that the globalization maps fail to be either surjective or injective (on π_0), leading to what we call *unphysical* ’t Hooft anomalies or ambiguities, respectively.

It is natural to ask whether one can also have *metaphysical ambiguities*, in the sense that there exist globalization maps from multiple sources to a given target. But to give this concept any teeth, one would first need to impose some coarse notion of equivalence on tangential structures, presumably based on considerations from physics. If not, then for underlying spaces of TFTs that are homotopy n-types, one will always find ambiguities between tangential structures whose Xs are equivalent as n-types.

V. MAXIMALLY-UNEXTENDED THEORIES

We begin our discussion of examples by considering a case which, although uninteresting as far as physics is concerned, nevertheless illustrates the mathematics well enough. To wit,

[8] More generally, any space of TFTs with global symmetry that does not admit a suitable symmetric monoidal structure must be anomalous.

we consider TFTs that are maximally unextended, in that the (truncated) bordism category is a 0-category, or set, whose objects are diffeomorphism classes of closed d-manifolds.

Such theories contain no physics, because they contain no relations (beyond those implied by the symmetric monoidal structure) between observables: a theory is specified by its values on diffeomorphism classes of connected manifolds, and those values are independent of one another. Nevertheless, the constructions described in previous sections can be carried out.

On the homotopy theory side, the space Θ has the homotopy type of a discrete space (so each of its connected components is contractible). We claim that an ∞-action of G then corresponds to the usual notion of a set-theoretic action of $\pi_0(G)$ on $\pi_0(\Theta)$, and a homotopy fixed point corresponds to a set-theoretic fixed point. To see this, note that as the notion of ∞-action is intrinsically homotopy-invariant, there is no loss of generality in replacing Θ with the homotopy equivalent discrete space $\pi_0(\Theta)$. Then a fibration $\pi : E \to BG$ with fibre $\pi_0(\Theta)$ is a covering space, so is determined by the monodromy action of $\pi_0(G) = \pi_1(BG, *)$ on $\pi^{-1}(*) = \pi_0(\Theta)$. A homotopy fixed point of this action is by definition a section s of π. As BG is path-connected, by the uniqueness of lifts to covering spaces such a section is uniquely determined by $s(*) \in \pi^{-1}(*) = \pi_0(\Theta)$. Considering s as a map of covering spaces from the trivial covering space $id : BG \to BG$ to $\pi : E \to BG$, we see that $s(*) \in \pi_0(\Theta)$ must be invariant under the monodromy of π, *i.e.* must be a G-fixed point. This identifies $\Theta^{hG} \simeq \pi_0(\Theta)^{hG} = \pi_0(\Theta)^G$, as claimed.

A. $d = 1$

Things are particularly simple when $d = 1$, where ${}^*\mathrm{TFT}_{1,0}$ is equivalent to the space $\Bbbk$, equipped with the discrete topology. To show this, observe that the source ${}^*\mathsf{Bord}_{1,0}$ is the symmetric monoidal ∞-groupoid consisting of finite disjoint unions of framed circles. The corresponding homotopy type is the free E_∞-algebra on the space

$$\{\text{framings of } S^1\} /\!\!/ \mathrm{Diff}(S^1).$$

The space of framings of S^1 is the same as the space of orientations of S^1, and consists of two contractible path components. The action of an orientation-reversing element of $\mathrm{Diff}(S^1)$ interchanges these components, so the resulting homotopy type is $B\mathrm{Diff}^+(S^1)$, the classifying space of the group of orientation-preserving diffeomorphisms of S^1, which in turn is equivalent to $BSO(2) \simeq \mathbb{CP}^\infty$. The target category is obtained by looping Vect, so it is the set of linear maps $\Bbbk \to \Bbbk$ with the symmetric monoidal structure given by tensor product, which is isomorphic to $\Bbbk$ itself with symmetric monoidal structure given by multiplication. It follows that ${}^*\mathrm{TFT}_{1,0}$ is the space of continuous maps $B\mathrm{Diff}^+(S^1) \to \Bbbk$, which as $B\mathrm{Diff}^+(S^1)$ is connected is simply isomorphic to the space $\Bbbk$ with the discrete topology. In other words, such a theory is determined by its value on any framed circle, and this value can be chosen freely.[9]

Now let us consider the possible gauge symmetries. For simplicity, we consider here only internal gauge symmetries, so we take a topological group G and the trivial representation $\chi : G \to O(1)$, and describe ${}^G\mathrm{TFT}_{1,0}$. Now the source is the symmetric monoidal ∞-groupoid ${}^G\mathsf{Bord}_{1,0}$, whose corrresponding homotopy type is the free E_∞-algebra on the space

$$\{\text{framings of } S^1,\ f : S^1 \to BG\} /\!\!/ \mathrm{Diff}(S^1).$$

Following the discussion above we only need to understand the set of path-components of this space, which is the same as $\{f : S^1 \to BG\} /\!\!/ \mathrm{Diff}^+(S^1)$, and as the group $\mathrm{Diff}^+(S^1)$ is connected the path components of this are identified with $\pi_0\{f : S^1 \to BG\}$, or in other words with the set $\mathrm{Conj}(\pi_0 G)$ of conjugacy classes of elements of $\pi_0 G$. Thus ${}^G\mathrm{TFT}_{1,0}$ is identified with the set of $\Bbbk$-valued functions on this set, *i.e.* the $\Bbbk$-valued class functions on $\pi_0 G$.[10] Thus we might as well take G to be discrete in what follows. By the general arguments already given, the possible global symmetries of ${}^K\mathrm{TFT}_{1,0}$ correspond to fixed points of some action of some discrete group Q on the set of class functions $K \to \Bbbk$, so let us now consider the globalization maps. Starting from an internal gauge symmetry based on discrete group G as above, every fibration Π of tangential structures is equivalent to a short exact sequence $* \to K \to G \to Q \to *$ of groups. For such a sequence, there is an action of $Q \simeq G/K$ on the set C_K of conjugacy classes of K given by $\rho : G/K \times C_K \to C_K : ([g],[k]) \mapsto [gkg^{-1}]$. The fibration τ corresponds to the Q action on $\mathrm{Map}(C_K, \Bbbk)$ that is induced by ρ, and Π_* corresponds to the map $\mathrm{Map}(C_G, \Bbbk) \to \mathrm{Map}(C_K, \Bbbk)^Q$ given by restriction. Since a normal subgroup of G is a union of conjugacy classes, it is easy to see that the map Π_*

[9] We shall later see that the extended theories in $d = 1$ have values on a framed circle given by the trace of the identity map on some finite-dimensional vector space, so the extendable theories are those which take the value $\sum_{i=1}^n 1 \in \Bbbk$, for some $n \in \{0, 1, 2, \ldots\}$.

[10] The extendable theories are given by the class functions that are characters of representations of $\pi_0 G$.

always surjects, but injects iff $K = G$. So there are no unphysical anomalies, but plenty of unphysical ambiguities.

In contrast, we certainly have metaphysical anomalies, for any Q-action on $\mathrm{Map}(C_K, \Bbbk)$ that is not of the form above (*e.g.* one that does not fix the image of all maps) cannot be gauged.

VI. MAXIMALLY-EXTENDED THEORIES

We now wish to focus on the case of maximally-extended theories, with $p = d$. Not only are these the ones of interest to physics (being fully local), but they also lead to a number of simplifications thanks to the cobordism hypothesis. Theories that are not maximally extended are less pleasant and will be studied in §VII.

At least in low dimension d, maximally-extended theories are fairly simple to classify and study. In the following subsections, we focus on the cases of $d = 1$, and $d = 2$, where we investigate symmetries and anomalies in more detail. Unfortunately, for $d > 2$, there is even less consensus on what a suitable target category for TFTs might be, let alone an description of the corresponding space of TFTs.

A. $d = 1$

Here we will classify group actions on framed (equivalently oriented), fully-extended TFTs for $d = 1$ valued in $\mathsf{Vect}_{\Bbbk}$. It is easily shown that ${}^*\mathrm{TFT}_{1,1}$ is equivalent to the groupoid (*i.e.* 1-truncated ∞-groupoid) of finite-dimensional vector spaces (including the zero-dimensional space) and linear isomorphisms.

Though this case of 'topological quantum mechanics' may seem rather boring from the dynamical perspective, we will see that it admits a rich variety of global symmetries.

As a topological space, we may take the 1-type

$$ {}^*\mathrm{TFT}_{1,1} = \coprod_{n \geq 0} BGL(n, \Bbbk), \tag{2} $$

the disjoint union of the classifying spaces of the groups $GL(n, \Bbbk)$ with the discrete topology. As we have discussed, an (∞-)action of G on ${}^*\mathrm{TFT}_{1,1}$ corresponds to the data of a homotopy pull-back square

$$\begin{array}{ccc} {}^*\mathrm{TFT}_{1,1} & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \{*\} & \longrightarrow & BG. \end{array}$$

As two $GL(n,\Bbbk)$ for different n cannot be isomorphic [9], we must have $E \simeq \coprod_{n\geq 0} E_n$, and assuming $E_n \to BG$ is splittable (if not, the space of homotopy fixed points will be empty) we must have $E_n \simeq B\tilde{G}_n$ for topological groups $\tilde{G}_n$ fitting into splittable extensions

$$0 \to GL(n,\Bbbk) \stackrel{\beta}{\to} \tilde{G}_n \stackrel{\alpha}{\to} G \to 0. \tag{3}$$

Having fixed a G-action on ${}^*\mathrm{TFT}_{1,1}$ as above, we find that

$${}^*\mathrm{TFT}_{1,1}^{hG} \simeq \coprod_{n\geq 0} \{\text{splittings } s \text{ of } \alpha : \tilde{G}_n \to G\} /\!\!/ GL(n,\Bbbk),$$

where $/\!\!/$ denotes the action groupoid (or homotopy quotient) for the $GL(n,\Bbbk)$-action on the set of splittings by conjugation. As $GL(n,\Bbbk)$ is discrete, splitting such an extension is the same as splitting $\pi_0\alpha : \pi_0\tilde{G}_n \to \pi_0 G$, so we may as well suppose that G is discrete.

Group extensions of G by $GL(n,\Bbbk)$ are classified by the non-abelian group cohomology $H^2(G, GL(n,\Bbbk))$ (see *e.g.* [10, 11]), in which a 2-cocycle is a pair $(\sigma : G \to \mathrm{Aut}(GL(n,\Bbbk)), \epsilon : G \times G \to GL(n,\Bbbk))$ of functions satisfying certain conditions.[11] Two pairs (σ,ϵ) and (σ',ϵ') represent the same cohomology class (which we denote $[\![\sigma,\epsilon]\!] \in H^2(G, GL(n,\Bbbk)))$ iff there exists a function $t : G \to GL(n,\Bbbk)$ such that

$$\begin{aligned} \sigma'(g)(N) &= t(g)\sigma(g)(N)t(g)^{-1}, \\ \epsilon'(g_1,g_2) &= t(g_1)\sigma(g_1)(t(g_2))\epsilon(g_1,g_2)t(g_1g_2)^{-1}, \end{aligned} \tag{4}$$

for all $N \in GL(n,\Bbbk)$ and $g, g_1, g_2 \in G$.

As described above, we are only interested in short exact sequences which admit a splitting, since otherwise there will be no homotopy fixed points. In terms of non-abelian group cohomology, a sequence splits iff its cohomology class has a representative of the form $[\![\sigma,\mathbb{I}]\!]$ where $\mathbb{I}(g_1,g_2) = 1$ is the constant map, in which case σ is a homomorphism. Indeed given a section s of the sequence, we can define $\sigma_s : G \to \mathrm{Aut}(GL(n,\Bbbk)) : g \mapsto (N \mapsto$

[11] To wit (letting 1 denote the identities in both G and $GL(n,\Bbbk)$): $\sigma(1) = \mathrm{id}_{GL(n,\Bbbk)}$, $\epsilon(1,1) = 1$, $\sigma(g_1g_2)(N) = \epsilon(g_1,g_2)^{-1}\sigma(g_1)(\sigma(g_2)(N))\epsilon(g_1,g_2)$, and $\epsilon(g_1,g_2)\epsilon(g_1g_2,g_3) = \sigma(g_1)(\epsilon(g_2,g_3))\epsilon(g_1,g_2g_3)$.

$\beta^{-1}(s(g)\beta(N)s(g)^{-1})$) and $[\![\sigma_s, \mathbb{I}]\!]$ is the cohomology class of the SES. This reproduces the well-known fact that a sequence splits iff it is equivalent to a semi-direct product.

The set of sections of a sequence with representative 2-cocycle (σ, ϵ) can be conveniently described as follows: it is in bijection with functions $r : G \to GL(n, \mathbb{k})$ that are twisted versions of representations, in the sense that

$$r(g_1)\sigma(g_1)(r(g_2))\epsilon(g_1, g_2) = r(g_1 g_2). \tag{5}$$

From the homotopy quotient, we see that two twisted representations r and r' are to be considered equivalent if there exists an $M \in GL(n, \mathbb{k})$ such that

$$Mr(g)\epsilon(1, g) = r'(g)\sigma(g)(M)\epsilon(g, 1) \tag{6}$$

for all $g \in G$.

In the special case when σ and ϵ are both trivial, Eq. 5 corresponds to the condition for standard representations, and Eq. 6 the usual equivalence of representations. More generally, if just σ is the trivial map, then Eq. 5 corresponds to the condition for projective representations with twisting ϵ, and, since $\epsilon(1, g) = \epsilon(g, 1)$ in this case, Eq. 6 corresponds to the usual 'linear equivalence' of projective representations.

It follows from the above discussion that all we need to describe a splittable short exact sequence is a homomorphism $\sigma : G \to \mathrm{Aut}(GL(n, \mathbb{k}))$. Thus we need to describe all possible automorphisms of $GL(n, \mathbb{k})$.

These are indeed known, though complicated [12]. In a nutshell, all automorphisms arise from: *(i)* inner automorphisms; *(ii)* field automorphisms of $\mathbb{k}$; *(iii)* the involution given by taking the inverse transpose; and *(iv)* homomorphisms $\chi : GL(n, \mathbb{k}) \to \mathbb{k}^*$.

Let us now spell out some examples of physical interest, corresponding to the different types of automorphism of $GL(n, \mathbb{k})$ described above. We will see that Nature makes use of all but the last one.

The usual representations of physics arise from the trivial action, which exists for every G. Here σ_s sends all of G to the identity automorphism and we may take $\tilde{G}_n = GL(n, \mathbb{k}) \times G$ for all n. For a given n, the space of sections are then in 1–1 correspondence with homomorphisms $r : G \to GL(n, \mathbb{k})$, *i.e.* representations of dimension n. A morphism between two homotopy fixed points with representations $r_1, r_2 : G \to GL(n, \mathbb{k})$ corresponds to an $M \in GL(n, \mathbb{k})$ such that $Mr_1(g) = r_2(g)M$, for all $g \in G$, *i.e.* the usual notion of equivalence of representations.

More generally, inner automorphisms of $GL(n, \Bbbk)$ give rise to projective representations, as described above. We remark that the possible occurrence of projective representations in physics is usually derived from the axiom that physical states in quantum mechanics correspond to rays in Hilbert space. Here we have no such axiom (the notion of a ray in a vector space certainly makes sense, but it is not clear what 'physical' should mean, given that we have no way of extracting real numbers that could be interpreted as predictions for physics measurements),[12] but it is nevertheless reassuring to see that projective representations are nevertheless allowed by the primitive requirements of locality and entanglement that the axioms of topological field theories encode.

When the homomorphism of $G \to \mathrm{Aut}(GL(n, \Bbbk))$ is induced by a homomorphism $G \to \mathrm{Aut}(\Bbbk)$, we get semi-linear representations. An important case for physics occurs when $\Bbbk = \mathbb{C}$ and we choose some involution of $\mathbb{C}$ (which defines a complex conjugation in $\mathbb{C}$ relative to the real line, defined as the fixed point subset). Then we get the antilinear representations, of which time reversal symmetry is an example. Here, though, we have no notion of time and no notion of unitarity.

Next consider the inverse transpose automorphism. This has a special rôle to play, because it corresponds to the $O(1)$-action on ${}^*\mathrm{TFT}_{1,1}$. By the cobordism hypothesis, its groupoid of homotopy fixed points should be equivalent to the groupoid of topological field theories on unoriented manifolds. To see this, let $G = O(1) = \{+1, -1\}$, and let $\tilde{G}_n = GL(n, \Bbbk) \rtimes O(1)$, where the semi-direct product is defined via the multiplication rule

$$(M_2, 1) \cdot (M_1, \pm 1) = (M_2 M_1, \pm 1), \quad (M_2, -1) \cdot (M_1, \pm 1) = (M_2 (M_1^{-1})^T, \mp 1). \tag{7}$$

A splitting s of the corresponding extension is specified by its value $A \in GL(n, \Bbbk)$ on $-1 \in O(1)$. By considering $s(-1 \cdot -1) = s(1)$, we find that $A^T = A$. So splittings correspond to non-degenerate symmetric bilinear forms. A morphism between the splittings corresponding to $A_1, A_2 \in GL(n, \Bbbk)$ is given by an $M \in GL(n, \Bbbk)$ such that $M A_1 M^T = A_2$, which corresponds to the usual notion of equivalence of non-degenerate symmetric bilinear forms (*i.e.* ${}^*\mathrm{TFT}_{1,1}^{hO(1)}$ is equivalent to the groupoid of finite-dimensional vector spaces equipped with non-degenerate symmetric bilinear forms and whose morphisms are linear isomorphisms which preserve the forms under pullback, which is indeed the same as ${}^{O(1)}\mathrm{TFT}_{1,1}$).

[12] Even for $\Bbbk = \mathbb{C}$, we have no way to identify $\mathbb{R} \subset \mathbb{C}$.

It is difficult to say much more for generic fields $\Bbbk$. However, for algebraically closed fields (such as $\mathbb{C}$) all A are equivalent to the identity matrix, and their automorphisms are isomorphic to $O(n, \Bbbk)$. Thus, in this case we have that

$$ {}^{O(1)}\mathrm{TFT}_{1,1} = \coprod_{n=0}^{\infty} BO(n, \Bbbk). \tag{8} $$

When $\Bbbk = \mathbb{R}$, Sylvester's law of inertia tells us that, up to equivalence, the A are given by diagonal matrices whose diagonal entries are ± 1, with automorphisms given by $O(p, q, \mathbb{R})$. Thus we have

$$ {}^{O(1)}\mathrm{TFT}_{1,1} = \coprod_{p=0}^{\infty} \coprod_{q=0}^{\infty} BO(p, q, \mathbb{R}). \tag{9} $$

In closing, it is perhaps of interest to speculate whether there might exist yet more ways of realizing symmetries, as yet unknown to physics. At least when $\Bbbk = \mathbb{C}$, this seems unlikely. All the field automorphisms of $\mathbb{C}$ are either of order two, so define a real line and a complex conjugation as needed to define the values of physical observables, or are of infinite order. But complex conjugation is the only automorphism of $\mathbb{C}$ considered as an $\mathbb{R}$-algebra. The only non-trivial automorphisms obtained from homomorphisms $\chi : GL(n, \mathbb{C}) \to \mathbb{C}^*$ correspond to powers of the determinant map. For a non-zero power, the resulting automorphism of $GL(n, \mathbb{C})$ generates a subgroup isomorphic to $\mathbb{Z}$, so doesn't admit a non-trivial automorphism from a finite group.

B. $d = 2$

We now carry out an analysis of the possible generalized global symmetries of framed TFTs in $d = 2$ valued in $\mathsf{Alg}_{\Bbbk}$. As we will see, this can be done in full, at least when the field $\Bbbk$ is separably closed, though the result is somewhat complicated.

Keeping $\Bbbk$ general to begin with, the cobordism hypothesis [2] states that the bicategory of framed TFTs valued in $\mathsf{Alg}_{\Bbbk}$ is given by the core of the fully-dualizable objects in $\mathsf{Alg}_{\Bbbk}$, where a $\Bbbk$-algebra A is fully-dualizable iff it is separable, meaning that $A \otimes_{\Bbbk} \mathbb{K}$ is finite dimensional and semisimple for every field extension $\mathbb{K} \supset \Bbbk$. Choosing $\mathbb{K} = \Bbbk$, we see that A itself is finite dimensional and semisimple, so we can apply the Artin–Wedderburn theorem. To do so, we need to know the finite-dimensional division algebras over $\Bbbk$. Since

A is separable, these must be too, so we are looking for finite-dimensional division algebras whose centres are finite-dimensional separable field extensions of the field $\Bbbk$.

In general, this is difficult, as can be seen by considering the extreme case in which $\Bbbk$ is perfect (meaning that every algebraic extension is separable), in which case every finite-dimensional division algebra over $\Bbbk$ is valid and being separable is equivalent to being finite dimensional and semisimple. This case includes all fields of characteristic zero and all finite fields, so probably every field that could be of interest to physicists. But finding the division algebras, even for a specific $\Bbbk$, is a hard (though well-studied) problem.

Instead we choose to focus here on the opposite extreme in which $\Bbbk$ is separably closed (meaning that no algebraic extension is separable), in which case the only division algebra is $\Bbbk$ itself. This case includes the one of most interest, namely $\Bbbk = \mathbb{C}$ (which is algebraically closed so separably closed).

Separably closed fields

For separably closed fields $\Bbbk$, the Artin–Wedderburn theorem tells us that every separable algebra is isomorphic as an algebra to a finite product of matrix algebras over $\Bbbk$, but every such algebra is Morita equivalent to $\Bbbk^n$ for some positive integer n. Generalizing the arguments in [13], one finds

$$ {}^*\mathrm{TFT}_{2,2} \cong \coprod_{n\geq 1} ES_n \times_{S_n} K(\Bbbk^*, 2)^{\times n} \tag{10} $$

where the permutation group S_n acts on $K(\Bbbk^*, 2)^{\times n}$ by permuting the factors. (We remark that unlike for theories in $d = 1$, the sum here starts from $n = 1$, since there is no zero-dimensional algebra.)

As usual, an action of G on ${}^*\mathrm{TFT}_{2,2}$ is described by a homotopy pull-back square

$$ \begin{array}{ccc} {}^*\mathrm{TFT}_{2,2} & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \{*\} & \longrightarrow & BG. \end{array} $$

As the path-components of ${}^*\mathrm{TFT}_{2,2}$ have non-isomorphic fundamental groups (namely the distinct symmetric groups), the G-action preserves path-components and so there is a corresponding decomposition $E = \coprod_{n\geq 1} E_n$. As we are only interested in G-actions which admit homotopy fixed points, writing $X_n = ES_n \times_{S_n} K(\Bbbk^*, 2)^{\times n}$ we are therefore looking for homotopy fibre sequences

$$X_n \longrightarrow E_n \xrightarrow{\pi} BG \tag{11}$$

which admit a section. Using that $\pi_1(X_n) = S_n$ and $\pi_2(X_n) = (\Bbbk^*)^{\oplus n}$ with S_n-module structure given by permuting the summands, we can understand such a homotopy fibre sequence by developing the diagram

$$\begin{array}{ccccc} K((\Bbbk^*)^{\oplus n}, 2) & = & K((\Bbbk^*)^{\oplus n}, 2) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ X_n & \longrightarrow & E_n & \underset{s}{\overset{\pi}{\rightleftarrows}} & BG \\ \downarrow & & \downarrow{\scriptstyle p} & & \| \\ BS_n & \longrightarrow & E'_n & \underset{s_0 = ps}{\overset{\pi'}{\rightleftarrows}} & BG \end{array} \tag{12}$$

in which all rows and columns are homotopy fibre sequences, by letting π' be the fibrewise 1-truncation of π. The bottom row, with a choice s_0 of section, is classified by the data of:

(i) a homomorphism $\pi_0 G \to \mathrm{Aut}(S_n)$.

Given such a choice, which in particular identifies $E'_n \simeq B(S_n \rtimes G)$, the middle row is classified by the data of:

(ii) a $S_n \rtimes \pi_0 G$-module structure on $(\Bbbk^*)^{\oplus n}$ extending the S_n-module structure,

(iii) a class $k \in H^3(B(S_n \rtimes G); (\Bbbk^*)^{\oplus n})$ which vanishes when restricted to $H^3(BS_n; (\Bbbk^*)^{\oplus n})$.

In order for the resulting π to admit a section, this should satisfy

(iii') $k \in H^3(B(S_n \rtimes G); (\Bbbk^*)^{\oplus n})$ vanishes when restricted to $H^3(BG; (\Bbbk^*)^{\oplus n})$.

In this case the homotopy classes of sections s lifting the given s_0 are given by the reasons this class vanishes, *i.e.* are a torsor for $H^2(BG; (\Bbbk^*)^{\oplus n})$.

The data in (i) and (ii) can be packaged together as follows. There is a group $\mathrm{Aut}(S_n, (\Bbbk^*)^{\oplus n})$ consisting of pairs of a group isomorphism $f_0 : S_n \to S_n$ and a f_0-linear module isomorphism $f_1 : (\Bbbk^*)^{\oplus n} \to (\Bbbk^*)^{\oplus n}$, and (i) and (ii) combined correspond to a homomorphism

$$\phi : \pi_0 G \to \mathrm{Aut}(S_n, (\Bbbk^*)^{\oplus n}).$$

To analyse this group, first note that for $n \neq 6$ all automorphisms f_0 of S_n are inner, and it is clear that these admit a canonical corresponding f_1. On the other hand, for $n = 6$ the outer automorphism f_0 of S_6 does *not* admit a corresponding f_1 (unless $\mathbb{k}^*$ is trivial), so for the data (ii) to exist, in (i) we must choose a homomorphism $\pi_0 G \to \mathrm{Inn}(S_n)$. This discussion gives us a split extension

$$1 \to \mathrm{Aut}_{\mathbb{Z}[S_n]}((\mathbb{k}^*)^{\oplus n}) \to \mathrm{Aut}(S_n, (\mathbb{k}^*)^{\oplus n}) \to \mathrm{Inn}(S_n) \to 1.$$

The kernel in this extension can be interpreted as the subgroup of $GL(n, \mathrm{End}_{\mathbb{Z}}(\mathbb{k}^*))$ consisting of those matrices which centralise the permutation matrices. It is easy to see that these are the invertible matrices which have a common entry at all diagonal positions and another common entry at all off-diagonal positions, *i.e.* invertible matrices of the form

$$\begin{pmatrix} a+b & b & \dots & b \\ b & a+b & \dots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \dots & a+b \end{pmatrix} \qquad a, b \in \mathrm{End}_{\mathbb{Z}}(\mathbb{k}^*) \tag{13}$$

Such a matrix is invertible for $n > 1$ if and only if both a and $a + n \cdot b$ are invertible in $\mathrm{End}_{\mathbb{Z}}(\mathbb{k}^*)$.

The scope of the data in (iii) (satisfying (iii')) can be analysed by considering the Serre spectral sequence for the bottom row of (12) with $(\mathbb{k}^*)^{\oplus n}$-coefficients. This describes the group K of all possible k's in terms of an exact sequence

$$\begin{array}{l} H^0(BG; H^2(S_n; (\mathbb{k}^*)^{\oplus n})) \xrightarrow{d_2} H^2(BG; H^1(S_n; (\mathbb{k}^*)^{\oplus n})) \longrightarrow K \\ \hookrightarrow H^1(BG; H^2(S_n; (\mathbb{k}^*)^{\oplus n})) \xrightarrow{d_2} H^3(BG; H^1(S_n; (\mathbb{k}^*)^{\oplus n})) \end{array} \tag{14}$$

As a S_n-module we have $(\mathbb{k}^*)^{\oplus n} = \mathrm{coInd}_{S_{n-1}}^{S_n} \mathbb{k}^*$, so by Shapiro's lemma we have $H^i(S_n; (\mathbb{k}^*)^{\oplus n}) \cong H^i(S_{n-1}; \mathbb{k}^*)$, which may be determined using the Universal Coefficient Theorem and the known low-degree homology of symmetric groups: the result is shown in Table I. The group $\mathrm{Aut}(S_n, (\mathbb{k}^*)^{\oplus n})$ acts on $H^i(S_n; (\mathbb{k}^*)^{\oplus n})$ by functoriality of group cohomology in the group and in the module. The subgroup $\mathrm{Inn}(S_n) \leq \mathrm{Aut}(S_n, (\mathbb{k}^*)^{\oplus n})$ given by the splitting acts trivially on $H^i(S_n; (\mathbb{k}^*)^{\oplus n})$, because inner automorphisms act trivially on cohomology [14, Proposition III.8.3]. A matrix of the form (13) in the subgroup

n	1	2	3	4	≥ 5
$H^2(S_n;(\Bbbk^*)^{\oplus n})$	0	0	$\Bbbk^*/(\Bbbk^*)^2$	$\Bbbk^*/(\Bbbk^*)^2$	$\Bbbk^*/(\Bbbk^*)^2 \oplus \Bbbk^*[2]$
$H^1(S_n;(\Bbbk^*)^{\oplus n})$	0	0	$\Bbbk^*[2]$	$\Bbbk^*[2]$	$\Bbbk^*[2]$

TABLE I: $\Bbbk^*[2]$ and $\Bbbk^*/(\Bbbk^*)^2$ denote the kernel and cokernel respectively of the squaring map $(-)^2 : \Bbbk^* \to \Bbbk^*$.

$\mathrm{Aut}_{\mathbb{Z}[S_n]}((\Bbbk^*)^{\oplus n}) \leq \mathrm{Aut}(S_n, (\Bbbk^*)^{\oplus n})$ acts as $a + n \cdot b \in \mathrm{End}_{\mathbb{Z}}(\Bbbk^*)$ on $H^i(S_{n-1};\Bbbk^*)$, so by the induced map on the $\Bbbk^*[2]$ and $\Bbbk^*/(\Bbbk^*)^2$ in Table I. This describes the $\pi_0 G$-action on $H^i(S_n;(\Bbbk^*)^{\oplus n})$ with which the cohomology groups in (14) are taken.

If $\Bbbk$ is a separably closed field of characteristic $\neq 2$ then it is closed under the formation of square roots, and so $\Bbbk^*/(\Bbbk^*)^2$ is trivial, and $\Bbbk^*[2] = \{\pm 1\}$, which has no automorphisms. Thus for $n \geq 5$ the group K fits into an exact sequence

$$H^0(BG;\{\pm 1\}) \overset{d_2}{\to} H^2(BG;\{\pm 1\}) \to K \to H^1(BG;\{\pm 1\}) \overset{d_2}{\to} H^3(BG;\{\pm 1\}).$$

(If the $\pi_0 G$-action on S_n is trivial then the d_2-differentials are zero, so K is determined up to an extension problem.) If instead $\Bbbk$ has characteristic 2 then $\Bbbk^*[2]$ is trivial, so for $n \geq 3$ there is an isomorphism

$$K \overset{\sim}{\to} H^1(BG;\Bbbk^*/(\Bbbk^*)^2),$$

where $\pi_0 G$ acts on $\Bbbk^*/(\Bbbk^*)^2$ as described in the previous paragraph.

Let us now discuss the space of sections of (11), *i.e.* the homotopy G-fixed points of the G-actions on X_n that we have just described. As we have classified such fibrations with a choice of section $s_0 : BG \to E'_n$ of $\pi' : E'_n = B(S_n \rtimes G) \to BG$, we may as well focus on the space $\Gamma(\pi)_{s_0}$ of sections of $\pi : E_n \to BG$ such that ps is in the path component of s_0. Composing with p gives a fibration

$$p_* : \Gamma(\pi)_{s_0} \to \Gamma(\pi')_{s_0}$$

to the space of sections of π' in the path component of s_0. Precisely as in the previous section, the space $\Gamma(\pi')$ is homotopy equivalent to $\{\text{splittings of } S_n \rtimes \pi_0 G \to \pi_0 G\}/\!\!/S_n$. Thus the path-component of s_0 is a classifying space for the stabiliser $\mathrm{St}_{S_n}(s_0) \leq S_n$, which may be seen to be the subgroup of elements which centralise $\mathrm{Im}(\pi_0 G \to \mathrm{Inn}(S_n))$. The fibre F of p_* over s_0 is the space of sections $s : BG \to E_n$ such that $ps = s_0$. This may be viewed as the space of trivialisations of the (trivial) class $(s_0)^*(k) \in H^3(BG; (\Bbbk^*)^{\oplus n})$, so its set of path components is a torsor for $H^2(BG; (\Bbbk^*)^{\oplus n})$, and

$$\pi_i(F, s) \cong H^{2-i}(BG; (\Bbbk^*)^{\oplus n}) \text{ for } i > 0.$$

The long exact sequence on homotopy groups for the fibration p_* then gives

$$\begin{aligned} 0 \to H^1(BG; (\Bbbk^*)^{\oplus n}) \to \pi_1(\Gamma(\pi), s) \to \mathrm{St}_{S_n}(s_0) \overset{\partial}{\to} H^2(BG; (\Bbbk^*)^{\oplus n}) \to \pi_0(\Gamma(\pi)_{s_0}) \to * \\ H^0(BG; (\Bbbk^*)^{\oplus n}) \overset{\sim}{\to} \pi_2(\Gamma(\pi), s) \end{aligned}$$

and all higher homotopy groups of $\Gamma(\pi)$ are trivial. The map ∂ is a crossed homomorphism, *i.e.* an element of $H^1(\mathrm{St}_{S_n}(s_0); H^2(BG; (\Bbbk^*)^{\oplus n}))$, and corresponds to restricting $k \in H^3(B(S_n \rtimes G); (\Bbbk^*)^{\oplus n})$ to the subgroup $\mathrm{St}_{S_n}(s_0) \times G \leq S_n \rtimes G$ and then applying the map

$$\mathrm{Ker}(H^3(B(\mathrm{St}_{S_n}(s_0) \times G); (\Bbbk^*)^{\oplus n}) \to H^3(BG; (\Bbbk^*)^{\oplus n})) \to H^1(\mathrm{St}_{S_n}(s_0); H^2(BG; (\Bbbk^*)^{\oplus n}))$$

coming from the Serre spectral sequence for $\mathrm{St}_{S_n}(s_0) \times G \to \mathrm{St}_{S_n}(s_0)$.

As an example, let us consider the trivial G-action on ${}^*\mathrm{TFT}_{2,2}$. A G-homotopy fixed point whose underlying topological field theory is an algebra Morita equivalent to $\Bbbk^n$ corresponds to a section s of the trivial fibration $BG \times X_n \to BG$, or in other words to a map $f : BG \to X_n$. In terms of our classification this corresponds to the homomorphism $\pi_0 G \overset{\pi_1 f}{\to} S_n \to \mathrm{Inn}(S_n)$ and the $S_n \rtimes \pi_0 G$-module structure on $(\Bbbk^*)^{\oplus n}$ induced by $(\sigma, g) \mapsto \sigma \cdot \pi_1 f(g) : S_n \rtimes \pi_0 G \to S_n$ and the usual S_n-module structure. As the underlying fibration is trivial, $k = 0$ and so the crossed homomorphism ∂ is trivial. We have $\mathrm{St}_{S_n}(s_0) = \{\sigma \in S_n \text{ centralising } \mathrm{Im}(\pi_1 f : \pi_0 G \to S_n)\}$. The discussion above then gives

$$\begin{aligned} H^2(BG; (\Bbbk^*)^{\oplus n})/\mathrm{St}_{S_n}(s_0) &\overset{\sim}{\to} \{\text{those elements of } \pi_0({}^*\mathrm{TFT}_{2,2}^{hG}) \text{ inducing the splitting } \pi_1 f\} \\ 0 \to H^1(BG; (\Bbbk^*)^{\oplus n}) &\to \pi_1({}^*\mathrm{TFT}_{2,2}^{hG}, s) \to \mathrm{St}_{S_n}(s_0) \to 0 \\ H^0(BG; (\Bbbk^*)^{\oplus n}) &\overset{\sim}{\to} \pi_2({}^*\mathrm{TFT}_{2,2}^{hG}, s). \end{aligned}$$

For example, taking $G = S_n$ and $f : BS_n \to X_n$ to be the map that acts on the algebra $\Bbbk^n$ by permuting the factors, then for $n \geq 3$ there are no elements centralising all of S_n so

$$\begin{aligned} \pi_1({}^*\mathrm{TFT}_{2,2}^{hG}, s) &\cong H^1(BS_n; (\Bbbk^*)^{\oplus n}) \cong H^1(BS_{n-1}; \Bbbk^*) \cong \Bbbk^*[2] \\ \pi_2({}^*\mathrm{TFT}_{2,2}^{hG}, s) &\cong H^0(BS_n; (\Bbbk^*)^{\oplus n}) \cong \Bbbk^*. \end{aligned}$$

On the other hand if $n = 2$ then $H^1(BS_2; (\Bbbk^*)^{\oplus 2}) \cong H^1(BS_1; \Bbbk^*) = 0$ so

$$\pi_1({}^*\mathrm{TFT}^{hG}_{2,2}, s) \cong S_2$$
$$\pi_2({}^*\mathrm{TFT}^{hG}_{2,2}, s) \cong H^0(BS_2; (\Bbbk^*)^{\oplus 2}) \cong \Bbbk^*.$$

These groups are abstractly isomorphic to the above if $\Bbbk$ does not have characteristic 2, but their origin, and presumably therefore the physical interpretation, is different.

As another example, let G be a connected group and $f : BG \to X_n$ be a map. This map must be trivial on π_1, so $\pi_0({}^*\mathrm{TFT}^{hG}_{2,2}) \cong H^2(BG; (\Bbbk^*)^{\oplus n})/S_n$, and the f corresponds to an S_n-orbit of an element $\xi \in H^2(BG; (\Bbbk^*)^{\oplus n}) = H^2(BG; \Bbbk^*)^n$. Such a theory has

$$\pi_1({}^*\mathrm{TFT}^{hG}_{2,2}, s) \cong \{\text{stabiliser of } S_n\text{-action on } \xi\}$$
$$\pi_2({}^*\mathrm{TFT}^{hG}_{2,2}, s) \cong H^0(BG; (\Bbbk^*)^{\oplus n}) \cong (\Bbbk^*)^{\oplus n}.$$

This generalises [13, Lemma 3.3.1]. In particular, letting $G = SO(2)$ act trivially on ${}^*\mathrm{TFT}_{2,2}$ we obtain a space with $\pi_0({}^*\mathrm{TFT}^{hSO(2)}_{2,2}) \cong H^2(BSO(2); (\Bbbk^*)^{\oplus n})/S_n \cong (\Bbbk^*)^{\oplus n})/S_n$. Since the action of $SO(2) \subset O(2)$ via tangential symmetries on ${}^*\mathrm{TFT}_{2,2}$ trivializes [13], and $Spin^r(2) \cong SO(2)$, this reproduces the classification of TFTs with r-spin structure for any $r \geq 1$ [8, 15], and shows that abstractly their classification is independent of r. Namely [8], since every algebra is Morita equivalent to $\Bbbk^n$, the components are given by a choice of n and a choice of Frobenius structure on $\Bbbk^n$. The latter is classified by the trace (*i.e.* identity) map on each factor of $\Bbbk$, each of which may be multiplied by a non-vanishing (to ensure non-degeneracy) element in $\Bbbk$, *i.e.* an element in $\Bbbk^*$, up to permutation. However, the map ${}^*\mathrm{TFT}^{hSO(2)}_{2,2} \to {}^*\mathrm{TFT}^{hSpin^r(2)}_{2,2}$ induced by the covering map $Spin^r(2) \to SO(2)$ is not an equivalence: the functoriality with respect to G of our arguments above allows us to see that on π_0 it sends each of the n elements of $\Bbbk^*$ to its rth power, while it induces an isomorphism on all higher homotopy groups. In particular, we note that if two oriented theories differ in their structure-constants by rth roots of unity, then they become isomorphic as r-spin theories.

VII. NON-MAXIMALLY-EXTENDED EXAMPLES

Here we discuss the example of TFTs in $d = 2$ based on ordinary categories rather than bicategories. We will see by means of an example that, even by taking just one step down the ladder compared to the maximally-extended case, unphysical anomalies can arise, in that a globalization map can fail to be π_0-surjective.

We consider the well-studied case of oriented topological field theories in $d = 2$ with a finite group internal gauge symmetry. So we consider the homomorphism $\chi : G \times SO(2) \to O(2) : (g, s) \mapsto s$, with G finite, along with the globalization map

$$\Pi_* : {}^{G\times SO(2)}\mathrm{TFT}_{2,1} \to \mathrm{Map}_G(EG, {}^{SO(2)}\mathrm{TFT}_{2,1}) \cong \mathrm{Map}(BG, {}^{SO(2)}\mathrm{TFT}_{2,1}). \quad (15)$$

To describe the category ${}^{G\times SO(2)}TFT_{2,1}$, let us introduce some definitions. A *Frobenius G-algebra over* $\Bbbk$ is a pair (A, η) consisting of a G-graded $\Bbbk$-algebra A (so $A = \oplus_{g\in G} A_g$ such that $A_g A_h \subseteq A_{gh}$), and $\eta : A \times A \to \Bbbk$ is a $\Bbbk$-bilinear form, or equivalently a $\Bbbk$-linear map $A \otimes A \to \Bbbk$, such that: $\eta(A_g \otimes A_h) = 0$ if $gh \neq 1$; η is non-degenerate when restricted to $A_g \otimes A_{g^{-1}}$; and $\eta(ab, c) = \eta(a, bc)$ [16]. A *Frobenius algebra* is a Frobenius G-algebra with G given by the trivial group.

A *crossed G-algebra over* $\Bbbk$ is a triple (A, η, ϕ), where (A, η) is a Frobenius G-algebra over $\Bbbk$ and $\phi : G \to \mathrm{Aut}(A)$ is a group homomorphism such that: ϕ_g preserves η; $\phi_g(A_g) \subseteq A_{ghg^{-1}}$; $\phi_g|_{A_g} = \mathrm{id}$; for $a \in A_g$ and $b \in A_h$ then $\phi_h(a)b = ba$; and for $g, h \in G$ and $c \in A_{ghg^{-1}h^{-1}}$ we have that $\mathrm{Tr}(c\phi_h : A_a \to A_a) = \mathrm{Tr}(\phi_{g^{-1}}c : A_b \to A_b)$ [16]. We remark that a crossed G-algebra for $G = *$ is a commutative Frobenius algebra.

The category ${}^{G\times SO(2)}TFT_{2,1}$ is equivalent to that whose objects are crossed G-algebras and whose morphisms are unital algebra maps $f : A \to B$ which are G-equivariant and preserve η [17]. In turn, ${}^{SO(2)}\mathrm{TFT}_{2,1}$ is equivalent to the category whose objects are commutative Frobenius algebras with unit and whose morphisms are Frobenius algebra maps.

On the right-hand side of our globalization map, we therefore have the category whose objects are commutative Frobenius algebras equipped with a G-action and whose morphisms are G-equivariant isomorphisms of commutative Frobenius algebras.

From this point of view, the globalization map takes (A, η) to its *principal component* $(A_e, \eta|_{A_e\otimes A_e})$ equipped with the homomorphism which sends $g \in G$ to $\phi_g|_{A_e}$ [17].

To exhibit an unphysical ‘t Hooft anomaly, observe that for an object in ${}^{SO(2)}\mathrm{TFT}^{hG}_{2,1}$ to be gaugeable, we require that for each $g \in G$ the corresponding algebra morphism ϕ_g have integer trace [16]. An example of an object in ${}^{SO(2)}\mathrm{TFT}^{hG}_{2,2}$ which fails this criterion may be given as follows: Let $A = \mathbb{C}[x, y]/(x^2, y^2)$ with trace $\eta(1, xy) = 1$ and $\eta(1, r) = 0$ for

$r = 1, x, y$. The automorphism

$$x \mapsto ux, \quad y \mapsto u^{-1}y$$

has trace given by $2 + u + u^{-1}$ which is generically non-integral. Thus if the action of G on A involves such an automorphism it is not gaugeable.

We remark that the algebra A is not semi-simple, since any semi-simple Frobenius algebra only has integral traced automorphisms. Thus we know that A cannot descend from a fully-extended theory by looping.

VIII. LIE GROUP SYMMETRIES AND NOETHER'S THEOREM

Up until now, we have treated the individual q-form symmetry groups for each q of topological field theories as abstract groups, without the smooth (*i.e.* Lie group) structure with which symmetries in physics are usually endowed. Doing so allowed us to package the tower of q-form symmetries into a single topological group, and so on. But it brings significant disadvantages. In particular, there is no possibility of deriving a generalized version of Noether's theorem, which associates conserved currents to a Lie group symmetry.

In this Section, we take a first step in the direction of extending our results to Lie groups by sketching a version of Noether's theorem for ordinary global Lie group symmetries of unextended oriented topological field theories.

Let us first ask roughly what form this 'theorem' might take. In physics, we have a Lie group of symmetries of a theory and Noether's theorem gives us, for each element in the corresponding Lie algebra, a conserved current or a conserved charge. The conserved current is usually thought of as a vector or a 1-form, but this requires a Hodge structure of some kind (*e.g.* from a metric), which is not available to us in topological field theory. In fact, the conserved current arises as a differential form whose degree is one less than the dimension of spacetime W. Current conservation is then simply the statement that the form is closed.

The degree of the form is such that one can integrate it on a closed oriented submanifold M of codimension one (or more generally a closed cycle). We call the value of the integral, which will vanish if the manifold bounds in W, the charge on M. This can be viewed as a vast generalization of the usual notion in physics that 'the charge is conserved', which amounts to the statement that the charge evaluated on one connected component of the boundary of $M \times I$ equals to the charge on the other connected component. Importantly, it allows for spacetime evolutions that are topologically non-trivial, which is just as well, since these are the only non-trivial evolutions in a TFT.

In the above, we tacitly assumed that our theory was a classical one, in which the conserved charge is a number (obtained from fields satisfying the equations of motion). In quantum field theory, we instead obtain the Ward identities for correlation functions involving the conserved charge. This is what we want to reproduce in TFT. We shall do so in the following way: a correlator is interpreted as the result of applying a functor to a bordism to the empty set obtained by cutting out tubular neighbourhoods of the supports of the operators appearing in the correlator. For a conserved charge, this means a submanifold M of codimension one, whose normal bundle will be trivial (since everything is oriented) and whose tubular neighbourhood has boundary $M \coprod \overline{M}$. We will see that we get a map $\mathfrak{g} \to Z(M) \otimes Z(M)^\vee$ which picks out the conserved charges. They are conserved in the sense that, if the bordism when cut contains a piece $M \to \emptyset$, then the correlation function vanishes when evaluated on elements of $Z(M) \otimes Z(M)^\vee$ in the image of the above map from $\mathfrak{g}$. The requirement that the bordism factors in this way corresponds to our earlier requirement that the conserved charge is to be evaluated on a cycle that bounds.

Let us see in more detail how this happens. We suppose that we are given a TFT whose automorphism group can be given a smooth structure, such that it acts smoothly on the vector space $Z(M)$ assigned to each object M in the bordism category. For example, for oriented theories in $d = 1$, we have seen that this group is $GL(n, \Bbbk)$ for some n, so this is certainly the case for $\mathbb{R}$ or $\mathbb{C}$. We take a Lie group G and a smooth homomorphism into the automorphism group (which corresponds to our earlier notion of the trivial action of the group on the space of TFTs).

Differentiating the map $G \to GL(Z(M))$ for each object then gives a map $\mathfrak{g} \to \mathrm{End}(Z(M))$, which, since $Z(M)$ is finite-dimensional, is canonically isomorphic to $Z(M) \otimes Z(M)^\vee$. The latter space can be interpreted as the vector space of operators associated to a codimension one submanifold M of some spacetime W. Indeed, when everything is oriented, the normal bundle of M in W is necessarily trivial and so the boundary of a tubular neighbourhood is given by two copies of M with opposite orientations, to which the TFT functor assigns $Z(M) \otimes Z(M)^\vee$.

Now let us consider the effect of inserting these operators into correlation functions. Let

$W : N \to N'$ be a cobordism with M embedded in the interior of W. Excising an open tubular neighbourhood of M in W, we obtain another cobordism $\tilde{W} : N \coprod M \coprod M^\vee \to N'$. Given $\xi \in \mathrm{End} Z(M)$, we form the linear map

$$Z(N) \xrightarrow{\mathrm{id}\otimes\tilde{\xi}} Z(N) \otimes Z(M) \otimes Z(M^\vee) \xrightarrow{Z(\tilde{W})} Z(N') \tag{16}$$

where $\tilde{\xi} : \Bbbk \to Z(M) \otimes Z(M^\vee)$ is the linear map such that $\tilde{\xi}(1)$ is the element corresponding to ξ. Our map $\mathfrak{g} \to Z(M) \otimes Z(M)^\vee$ picks out a distinguished subspace of such linear maps and gives it the structure of an algebra, just as we expect for conserved charges.

To see the sense in which charges are conserved, force N' to be the empty set. Combining $\mathfrak{g} \to \mathrm{End}(Z(M))$ and (16) we get a natural map $Z(N) \otimes \mathfrak{g} \to \Bbbk$, but it is easy to show that this map is in fact the zero map. Translating back to physics, we see that any correlation function with an insertion of a conserved charge on M vanishes, when M is nullbordant in W, as desired.

IX. DISCUSSION

In closing, we would like to show how our results shed some light on the previous literature. We begin with ‘t Hooft anomalies of ordinary symmetries of non-maximally extended orientable TFTs, which were studied in [5] (and formalized and generalized in [10, 11]). In our notation (omitting the $SO(2)$ factors corresponding to orientation), the idea can be described as follows. Letting K be an ordinary group acting trivially on $\mathrm{TFT}_{2,1}$, a TFT in $\mathrm{TFT}_{2,1}^K \simeq \mathrm{Map}(BK, \mathrm{TFT}_{2,1})$ may not have a preimage in ${}^K\mathrm{TFT}_{2,1}$ under the obvious globalization map. In [5], this problem was studied using an approach which is natural from the physicist's (if not the physics) point of view, namely to study K symmetries of quantum theories which arise by quantizing classical theories. We have not discussed quantization in the present work, but for the discussion that follows it suffices to know that there exists an *orbifoldization functor* [18] from the category of finite groups to the category of spaces that sends a group G to ${}^G\mathrm{TFT}$ and that for the homomorphism $G \to *$ to the trivial group this corresponds, when restricted to invertible theories in ${}^G\mathrm{TFT}$ to quantization à la Dijkgraaf–Witten [3].

Given a short exact sequence $* \to H \to G \to K \to *$ of finite groups, there is a commutative diagram

$$\begin{array}{ccc} {}^{G}\mathrm{TFT}_{2,1} & \longrightarrow & {}^{K}\mathrm{TFT}_{2,1} \\ {\scriptstyle \mathrm{Glob}}\downarrow & & \downarrow{\scriptstyle \mathrm{Glob}} \\ \mathrm{Map}_K(EK, {}^{H}\mathrm{TFT}_{2,1}) & \longrightarrow & \mathrm{Map}(BK, \mathrm{TFT}_{2,1}) \end{array} \tag{17}$$

The proposal of [5] is, given K, to start from an invertible theory in ${}^{H}\mathrm{TFT}_{2,1}$ (with the trivial H action) and to ask if there exists an extension G of K by H and a theory in the resulting ${}^{G}\mathrm{TFT}_{2,1}$ (with the trivial G action) that maps to it under the left-hand map followed by composition with the map that forgets the K symmetry. If so, one may say that the corresponding theory in $\mathrm{Map}_K(EK, {}^{H}\mathrm{TFT}_{2,1})$ is non-anomalous, since one can follow the diagram to find a theory in the top right-hand corner that is a gauging of the theory in the bottom right-hand corner that is the quantization of the theory in $\mathrm{Map}_K(EK, {}^{H}\mathrm{TFT}_{2,1})$. In [5] and [10, 11], obstructions to this were given, as well as sufficient conditions for the construction to go through.

Though this construction is well-motivated from the physics point of view, it seems unreasonable to us to describe failures of this construction as anomalies. When the construction fails, one cannot even find a quantum theory with global K symmetry in $\mathrm{Map}(BK, \mathrm{TFT}_{2,1})$ for which one can ask the question of whether K is gaugeable. Moreover, any theory in K which one actually can construct by quantizing a classical theory is automatically gaugeable, because the Dijkgraaf–Witten construction extends to fully-extended theories [19], where the cobordism hypothesis applies.

The same trick of extending theories can be applied to understanding the results of [4], where it is shown that 0- and 1-form symmetries of a sub-class of oriented non-maximally extended TFTs in $d = 2$ (*i.e.* ${}^{SO(2)}\mathrm{TFT}_{2,1}$) valued in $\mathsf{Vect}_{\mathbb{C}}$, to wit those corresponding to commutative Frobenius algebras that are additionally semisimple, are given by permutations and phasings, respectively. These are the symmetries in the intrinsic sense of being automorphisms of the TFTs and the result comes as no surprise once we observe that the semisimple algebras are the ones that arise from fully-extended TFTs upon looping. So the automorphism groups can be read off directly from (10) and correspond to permutations of the simple factors that preserve the trace map and rephasings (or rather elements in $\mathbb{C}^*$ given that we have no inner product structure).

Finally, generalized symmetries of certain extended, oriented topological field theories in $d = 2$ are described in [1], namely those obtained by quantizing classical $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ gauge theories. The resulting algebra is the twisted group algebra $\mathbb{C}_{\omega_p}[\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}]$ where $\omega_p((a_1, b_1), (a_2, b_2)) = \exp(2\pi \mathrm{i} p a_1 b_2/\mathrm{GCD}(n_1, n_2))$. The algebra $\mathbb{C}_{\omega_p}[\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}]$ is Morita equivalent (and therefore equivalent in $\mathrm{TFT}_{2,2}$) to the algebra $\Bbbk^{n_1 n_2/m^2}$ where $m = \mathrm{GCD}(n_1, n_2)/\mathrm{GCD}(p, n_1, n_2)$ is the dimension of the irreducible projective representation with twisting ω_p, and $n_1 n_2/m^2$ their number. Correspondingly the automorphism groups are $S_{n_1 n_2/m^2}$ and $(\mathbb{C}^*)^{n_1 n_2/m^2}$, which are substantially larger than the $\mathbb{Z}_{n_1/m} \times \mathbb{Z}_{n_2/m}$ subgroups obtained in [1] by inspection of the classical action. This example shows that, at least for spaces of topological field theories with their large number of equivalences, studying classical actions gives a rather poor guide to the resulting quantum physics. Happily, the power of the cobordism hypothesis for physical (*i.e.* fully-extended) theories suggests that we may one day no longer need to.

X. ACKNOWLEDGMENTS

BG is supported by STFC consolidated grant ST/T000694/1. JTS is supported in part by the U.S. National Science Foundation (NSF) grant PHY-2014071.

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