HOWE DUALITY AND DICHOTOMY FOR EXCEPTIONAL THETA CORRESPONDENCES

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Abstract. We study three exceptional theta correspondences for p-adic groups, where one member of the dual pair is the exceptional group G_2. We prove the Howe duality conjecture for these dual pairs and a dichotomy theorem, and determine explicitly the theta lifts of all non-cuspidal representations.

1. Introduction

Let F be a non-archimedean local eld of characteristic 0 and residue characteristic p. In this paper, we study the local theta correspondence furnished by the following diagram of dual pairs:

$$
\begin{array}{ccccc}
 & & & & PGSp_6 \\
 & & G_2 & & \\
PD & & & & PGL_3 \text{ o } Z=2Z
\end{array}
$$

where D denotes a cubic division F-algebra, so that P D is the unique inner form of PGL_3. More precisely, one has the dual pairs

8
<(PGL$_3$ o Z=2Z) G_2 E_6 o Z=2Z
P D G_2 E^D_6
: G_2 PGSp$_6$ E_7

where the exceptional groups of type E are all of adjoint type. In each of the three cases, the centralizer of G_2 is a group H_J = Aut(J), where J is a Freudenthal-Jordan algebra of degree 3. One can thus consider the restriction of the minimal representation (see [GS05] or [LS]) of E to the relevant dual pair and obtain a local theta correspondence.

More precisely, if 2 Irr(G_2) is an irreducible smooth representation of G_2, then the maximal -isotypic quotient of

$$= \backslash_{2 \mathrm{Hom}_{G_2}(;)} \ker():$$

can be expressed as () for some smooth representation () of H_J [MVW, Lemme 2.III.4]. The representation () is called the big theta lift of , and its maximal semi-simple quotient (cosocle) is denoted (). We say that has nonzero theta lift to H_J if () = 0, or equivalently Hom_{G_2} (;) = 0. Similarly, one can consider the theta lift from H_J to G_2 and have the analogous notions.

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The rst main result of this paper is the following dichotomy theorem (for the statement in the theory of classical theta correspondences, see [KR], [HKS] and [SZ]):

Theorem 1.1. Let 2 $Irr(G_2(F))$. Then has nonzero theta lift to exactly one of $P D$ or $PGSp_6(F)$.

The group PGL_3 o $Z=2Z$ is not featured in the dichotomy theorem, but it is needed for some ner aspects of the theta correspondences. For example, every irreducible discrete series representation of G_2 lifts to a discrete series representation of precisely one of the three groups. After the above dichotomy theorem, we consider the problem of understanding these theta correspondences more precisely. These local theta correspondences have all been studied to some extent by Maggard-Savin [MS], Gross-Savin [GrS2], Gan [G99], Savin [Sa], Gan-Savin [GS99, GS04] and Savin-Weissman [SWe]. Though various neat results were obtained in the various cases, they fall short of determining the theta correspondences completely. One of the main results of this paper is the completion of the analysis begun in these papers.

The main diculty in studying these exceptional theta correspondences is that, unlike the classical theta correspondence, one does not know a priori the analog of the Howe duality conjecture. Namely, one does not know that () has nite length with unique irreducible quotient (that is, () is irreducible if () is nonzero). In this paper, we show that the analog of the Howe duality conjecture holds for these dual pairs. To summarize, we have:

Theorem 1.2. The Howe duality conjecture holds for the three dual pairs considered here. Namely, for $_1;_2$ 2 $Irr(G_2(F))$, $(_i)$ has nite length and

$$\dim Hom_{H_J}((_1);(_2)) \quad \dim Hom_{G_2}(_1;_2):$$

Likewise, for 2 $Irr(H_J)$, () has nite length with unique irreducible quotient (if nonzero). More precisely, we have:

(i) The theta correspondence for $P D$ G_2 denes an injective map

$$_D : \tilde{Irr}(P D) \hookrightarrow Irr(G_2(F));$$

where $\tilde{Irr}(P D)$ $Irr(P D)$ is the subset of representations which have nonzero theta lift to G_2. If p = 3, then $\tilde{Irr}(P D) = Irr(P D)$, so that one has an injective map:

$$_D : Irr(P D) \hookrightarrow Irr(G_2(F))$$

(ii) The theta correspondence for $(PGL_3(F)$ o $Z=2Z)$ G_2 denes an injective map $_B$:

$$\tilde{Irr}(PGL_3(F) \text{ o } Z=2Z) \hookrightarrow Irr(G_2(F));$$

where $\tilde{Irr}(PGL_3(F)$ o $Z=2Z)$ $Irr(PGL_3(F)$ o $Z=2Z)$ is the subset of representations which have nonzero theta lift to G_2. Moreover, one can determine the subset $\tilde{Irr}(PGL_3(F)$ o $Z=2Z)$ explicitly, and the image of $_B$ is disjoint from that of $_D$ by the dichotomy theorem.

(iii) The theta correspondence for G_2 $PGSp_6$ denes an injection :

$$Irr(G_2(F)) \smallsetminus Im(_D) \hookrightarrow Irr(PGSp_6(F)):$$

For an irreducible representation of P D, the non-vanishing of $()_D$ is equivalent to the existence of non-zero vectors in xed by a maximal torus in P D. The existence of such vectors has been checked by Lonka and Tandon [LT, Thm. 2.4] in the tame case, where p = 3. Thus, if p = 3, we do not know that all irreducible representations of P D lift to G_2 (though one certainly expects this to hold), but the lift is still one-to-one on the subset of those representations that have nonzero lift.

In fact, for the three dual pairs, we determine the theta lift of <u>all</u> non-supercuspidal representations of G_2, and the lift of supercuspidal representations whose lift is not supercuspidal. The detailed statements are in the main text, and we simply state the following qualitative result here:

Theorem 1.3. The three theta correspondences satisfy the following properties:

(i) The correspondences preserve tempered representations.

(ii) Any discrete series representation of G_2 lifts to a discrete series representation of precisely one of the three groups.

(iii) The correspondences are functorial for non-tempered representations.

The main motivation for showing the results of this paper is the application to the local Langlands correspondence for the exceptional group G_2. A proof of the local Langlands conjecture for G_2 is given in our followup work [GS22].

We would now like to explain the general idea and strategy for proving the Howe duality theorem. We begin with a discussion of the statement:

(a) () has nite length.

This niteness result is fundamental and it was shown by Kudla [K] for the classical theta correspondence. The main tools used are his computation of the Jacquet modules of the Weil representation (relative to maximal parabolic subgroups of the two members of the dual pair) and his exploitation of the doubling see-saw identity. One key consequence of the nite length of () is

(b) If () = 0, then it has an irreducible quotient.

For the dual pairs considered in this paper, we will in fact rst prove statement (b) and then use it with other inputs to show (a).

Let us elaborate on this slightly subtle point and our strategy of proof. By Bernstein's decomposition, we may decompose

$$() = ()_c \ ()_{nc}$$

as the sum of its cuspidal part and non-cuspidal part. If $()_c$ is nonzero, then it certainly has an irreducible quotient, since it is semisimple. On the other hand, we shall show using Jacquet module computations that

(c) $()_{nc}$ has nite length and hence has an irreducible quotient if it is nonzero.

The necessary Jacquet module computations are already available in the literature [MS, Sa] when H_J = P D or PGL_3 o Z=2Z and are partially available [MS, GrS2] for H_J = $PGSp_6$. In x13, we complete the remaining Jacquet module computations. We stress that the material in x13 is independent of the rest of the paper and could have been discussed earlier in the paper; we have refrained from doing so, as the computations are rather technical. Consequences of the results of x13 are then discussed in x14.

In any case, we show statement (b) by showing (c) via Jacquet modules; the proof is written in x14.2. The statement (b) is used in the proof of the dichotomy theorem (i.e. Theorem 1.1) for tempered representations in x6. For nontempered representations, the Jacquet module computations (of [MS, GrS2] and x13-14) will tell us everything about their theta lifts.

For the statement (a) (nite length of ()), it remains to show that $()_c$ is of nite length. We shall show this together with the Howe duality conjecture, by showing that $()_c$ is either irreducible or 0. This part of the argument may be considered the analog of the doubling see-saw argument, though one would legitimately question what that means in the setting of exceptional dual pairs.

It will be instructive to rst recall the argument for a classical dual pair Sp(W) O(V), where W is a symplectic space and V a quadratic space (see [Mi] and [GT16]). The Howe duality theorem was shown by examining the so-called doubling see-saw diagram:

$$\begin{array}{ccc} O(V) & & Sp(W)\ Sp(W) \\ | & \times & | \\ O(V)\ O(V) & & Sp(W) \end{array}$$

where V = V + V is the doubled quadratic space. Starting from ;^{0} 2 Irr(O(V)), the resulting see-saw identity gives

$$\dim \mathrm{Hom}_{Sp(W)}((^0); ()) \ \dim \mathrm{Hom}_{O(V)\,O(V)}((1); ^0 -):$$

where (1) is the big theta lift of the trivial representation of Sp(W) to O(V). By the local analog of the Siegel-Weil formula, one identies (1) with a submodule of a certain degenerate principal series representation I on O(V). This implies that, for outside a small family of representations,

$$\dim \mathrm{Hom}_{O(V)\,O(V)}((1); ^0 -) \ \dim \mathrm{Hom}_{O(V)\,O(V)}(I; ^0 -):$$

Using Mackey theory, one can analyze the latter space and show that, for outside another small family of representations,

$$\dim \mathrm{Hom}_{O(V)\,O(V)}(I; ^0 -) \ \dim \mathrm{Hom}_{O(V)}(^0;):$$

Taken together, one obtains the desired inequality

$$\dim \mathrm{Hom}_{Sp(W)}((^0); ()) \ \dim \mathrm{Hom}_{O(V)}(^0;)$$

for outside a small family of representations. For this small family of representations, one needs to do a separate argument.

Now for the exceptional dual pairs G H studied in this paper, there is no analog of the doubling see-saw; this is ultimately tied to the sporadic nature of the geometry underlying exceptional groups. There is thus no direct analog of the above argument. However, the above argument is a particular manifestation of a general principle:

Theta correspondence typically relates or transfers a period on G to a period on H.

More precisely, given a subgroup G_1 G and 2 $\mathrm{Irr}(G)$, we may consider the space Hom_G (;) for$_1$ some one-dimensional character of G_1. Let us call this Hom space the G_1-period for . Now assume that is a quotient of () for some 2 $\mathrm{Irr}(H)$. Then one typically obtains a statement of the form

$$G_1\text{-period of} \quad G_1\text{-period of } () = H_1\text{-period of } -$$

for some subgroup H_1 of H.

Now one can turn the table around. For an irreducible quotient of (), one can consider the G_1-period of (–), which has – as an irreducible quotient. One typically gets a statement

$$H_1\text{-period of } - \quad H_1\text{-period of } (-) = G_2\text{-period of for}$$

some subgroup G_2 of G.

Iterating this process, one obtains a family of periods relative to subgroups G_i G and H_i H such that

$$G_i\text{-period of} \quad G_i\text{-period of } () = H_i\text{-period of } -;$$

and

$$H_i\text{-period of } - \quad H_i\text{-period of } (-) = G_{i+1}\text{-period of };$$

thus leading to a chain of containment of periods of and . One may call this a game of ping-pong with periods. Now an (empirical) observation is that the subgroups G_i and H_i become more and more reductive (as i increases) and one ultimately obtains a reductive period. When that happens, the next iteration will result in a seesaw diagram analogous to that in the classical case above and the consideration of an appropriate degenerate principal series representation.

Now the miracle is that a Mackey theory argument with this degenerate principal series representation then returns us the initial G_1-period! In other words, for some $i > 1$, one has $G_i = G_1$, and this allows one to complete the chain of containment of periods into a cycle. In particular, if one of these period spaces is nite-dimensional, then this cycle of containment is a cycle of equalities. This is the key step in our proof of the Howe duality theorem for the dual pairs treated here. We shall play this game of ping-pong with periods on two occasions, in x6 and x12. This seems to us to be a rather robust method for proving the Howe duality conjecture and should be applicable to other exceptional dual pairs, though the precise details will undoubtedly be dierent in each case.

We nish the introduction by presenting the key case of this period ping-pong for this paper. Let G be the exceptional group of type G_2, and $H = \mathrm{Aut}(J)$, for a Freudenthal-Jordan algebra J. The group G has two conjugacy classes of maximal parabolic subgroups, one of which is the Heisenberg parabolic subgroup whose unipotent radical N is a 5-dimensional Heisenberg group. The conjugacy classes of generic characters $_E : N \; ! \; \mathbb{C}$ are parameterized by cubic etale algebras E over F. For such an E, x an embedding $i : E \; ! \; J$ (if it exists). Then we have a see-saw of dual pairs (in E_6 or E_7)

$$\begin{array}{ccc} G_E := \mathrm{Spin}_8^E & & \mathrm{Aut}(J) = H \\ | & \times & | \\ G & & \mathrm{Aut}(i : E \ ! \ J) =: H_{J;E} \end{array}$$

With the minimal representation, a twisted Jacquet module computation gives $_{N; \ E}$ = $\mathrm{ind}^{H}_{H_{E;J}}(1)$. This implies a chain of containments

$$\mathrm{Hom}_N(; \ _E) \ \mathrm{Hom}_N(( ); \ _E) = \mathrm{Hom}_{NH}(; \ _E$$
$$) = \mathrm{Hom}_{H_{E;J}}(^{-}; 1)$$

where the rst is a natural inclusion, since is a quotient of ( ), and the last follows by the twisted Jacquet module computation and Frobenius reciprocity. Now, in order to do the next step, we need to compute $H_{E;J}$-coinvariants of . This was done in our paper [GS21], where it was shown that $_{H_{E;J}}$ is a submodule of a degenerate principal series representation I_E of Spin_8^E. The miracle here is that, as a G-module, I_E contains $\mathrm{ind}^G_N(\ _E)$ as a large G-submodule. Thus, the seesaw identity associated to the above seesaw diagram gives the next chain of containments is

$$\mathrm{Hom}_{H_{E;J}}(^{-}; 1) \ \mathrm{Hom}_{H_{E;J}}((^{-}); 1) = \mathrm{Hom}_N(; \ _E);$$

that is, we arrive where we started.

To be honest, just as in the classical case, this last step will hold for representations of G outside a small family, roughly those that are in the quotient of I_E by $\mathrm{ind}^G_N(\ _E)$. One can characterize this exceptional family precisely, but we prefer not to do it, and simply observe that tempered irreducible representations of G do not lie in this exceptional family. (As mentioned earlier, the theta lifts of nontempered representations can be explicitly determined using the Jacquet module computations of [MS, GrS2] and x13-14.) Thus, with that caveat in mind, we conclude that

$$\mathrm{Hom}_N(; \ _E) = \mathrm{Hom}_N(( ); \ _E)$$

for all E, and this implies that = ( ). This is the argument which replaces the doubling seesaw argument in classical theta correspondence.

Finally, let us remark that many of the arguments in our paper work over nonarchimedean local elds of characteristic $p > 0$ as well, at least when p is not too small (say $p = 2; 3$). However, many of the prior results we rely on were only written in the context of characteristic 0 local elds. An example is the construction of the minimal representation itself. Hence, though our arguments should in principle work for positive characteristic local elds, many details require careful verication.

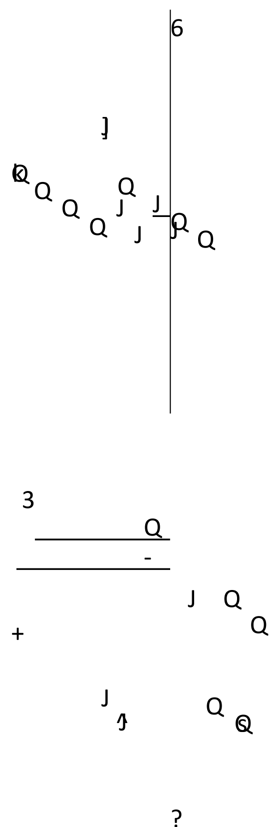
2. The Group G_2

We begin by introducing the algebraic group G_2 over F.

2.1. Octonion algebra. Let O be the split octonion algebra over F. Thus, O is an 8-dimensional non-associative and non-commutative F-algebra. It comes equipped with a conjugation map x ! $\overline{x}$ with associated norm N(x) = x $\overline{x}$ = $\overline{x}$ x and trace Tr(x) = x + $\overline{x}$. Moreover, N : O ! F is a nondegenerate quadratic form.

Every element x of O is a zero of its characteristic polynomial t^2 Tr(x)t + N(x). A nonzero element x 2 O is said to be of rank 1 if N(x) = 0. Otherwise it is of rank 2, in which case the subalgebra F[x] of O generated by x over F is isomorphic to the separable quadratic F-algebra F[t]=(t^2 Tr(x)t + N(x)). We denote by O_0 the 7-dimensional subspace of trace 0 elements in O.

2.2. Automorphism group. The group G_2 is the automorphism group of the F-algebra O. It is a split simple linear algebraic group of rank 2 which is both simply connected and adjoint. If we x a maximal torus T contained in a Borel subgroup B, then we obtain a system of simple roots f; g of G_2 relative to (T; B), with short and long. The resulting root system is given by the following diagram.



The highest root is $_0$ = 3 + 2.

2.3. Maximal torus. Following Muic, we will x the isomorphism T = $G_m{}^2$ by t

$$! \quad ((2+\ )(t);(+\ )(t)):$$

Any pair of characters ($_1$; $_2$) of F thus dene a character $_1$ $_2$ of T by composition with the above isomorphism.

2.4. Parabolic subgroups. Up to conjugation, G_2 has 2 maximal parabolic subgroups which may be described as follows. Let V_1 V_2 O_0 be subspaces of dimension 1 and 2 respectively on which the octonion multiplication is identically zero. Let P and Q be the stabilizers of V_2 and V_1 respectively. Then $P = MN$ and $Q = LU$ are the two maximal parabolic subgroups of G_2. Moreover, their intersection $B = P \backslash Q$ is a Borel subgroup of G_2.

The Levi factor M of P is given by

$$M = GL(V_2) = GL_2:$$

The isomorphism of M with GL_2 can be xed so that the modulus character of M is $_P$ = $j\det j^3$. Its unipotent radical N is a 5-dimensional Heisenberg group with 1-dimensional center $Z = {}_0 U$. The action of M on $N=Z$ is isomorphic to $Sym^3(F^2)$ $\det^{\ 1}$. Moreover, the generic $M(F)$-orbits on $N(F)=Z(F)$ is naturally parametrized by the set of isomorphism classes of separable cubic F-algebras.

The Levi factor L of Q is given by

$$L = GL(V_3=V_1) = GL_2$$

where

$$V_3 = fx\ 2\ O_0 : x \ \ y = 0 \text{ for all } y\ 2\ V_1g:$$

The isomorphism of L with GL_2 can be xed so that the modulus character of L is $_Q$ = $j\det j^5$. The unipotent radical U is a 5-dimensional 3-step nilpotent group:

$$U = U_0 \ \ U_1 \ \ U_2 \ \ U_3 = f1g;$$

such that

$$U_0=U_1 = U \ \ U_+; \qquad U_1=U_2 = U_{2+} \qquad U_2=U_3 = U_{_0} \ U_{_0} \ : \text{As}$$

representations of L, one has

$$U_0=U_1 = F^2; \quad U_1=U_2 = \det; \quad U_2=U_3 = F^2 \ \det:$$

2.5. The subgroup SL_3. The subgroup of G_2 generated by the long root subgroups is isomorphic to SL_3. The normaliser of SL_3 in G_2 is a semidirect product $SL_3 \text{ o } Z=2Z$, with the nontrivial element of $Z=2Z$ acting on SL_3 as a pinned outer automorphism. The subgroup SL_3 is the pointwise stabilizer of a quadratic subalgebra of O which is isomorphic to $F \ \ F$, whereas the setwise stabilizer of such a subalgebra is $SL_3 \text{ o } Z=2Z$.

More generally, given a subalgebra of O which is isomorphic to a quadratic eld extension of F, the pointwise stabilizer of this subalgebra is isomorphic to the quasi-split special unitary group SU_3^K; the setwise stabilizer of this subalgebra is $SU_3^K \text{ o } Z=2Z$.

2.6. The dual group. The Langlands dual group of G_2 is the complex Lie group $G_2(C)$. In particular, one has the subgroups

$$SO_3(C) \ \ SL_3(C) \ \ SL_3(C) \text{ o } Z=2Z \ \ G_2(C):$$

The centralizer of $SL_3(C)$ in $G_2(C)$ is $_3$, and the centralizer of $SO_3(C)$ in $G_2(C)$ is $S_3 = {}_3 \text{ o } Z=2Z$. Let $SL_{2;l}(C)$ be a long root SL_2. Then the centralizer of $SL_{2;l}(C)$ in $G_2(C)$ is $SL_{2;s}(C)$, a short root SL_2, and vice versa. Thus we also have the subgroup

$$SL_{2;l}(C) \ {}_2 \ SL_{2;s}(C) = SO_4(C) \ \ G_2(C):$$

2.7. Nilpotent orbits. Recall that the geometric nilpotent orbits (i.e. nilpotent orbits over $\overline{F}$) of a simple group G dened over F form a partially ordered set, where O_1 O_2 if O_1 is contained in the Zariski closure of O_2. Determining the $G(F)$-orbits in the set of F-points of each of these orbits is an exercise in Galois cohomology. More precisely, if $' : SL_2 ! G$ is a map that corresponds to a nilpotent orbit O by the Jacobson-Morozov theorem, then the $G(F)$-orbits of F-points in O are parametrized by

$$\mathrm{Ker} \quad H^1(F; C_{'}) ! \quad H^1(F; G)$$

where $C_{'}$ is the centralizer of $'(SL_2)$ in G.

For the group G_2, the geometric nilpotent orbits form a chain

$$f0g \quad O_l \quad O_s \quad O_{sr} \quad O_{reg}$$

where O_l and O_s are orbits of non-zero elements in long and short root spaces, that is $'(SL_2) = SL_{2;l}$ and $'(SL_2) = SL_{2;s}$ respectively, while O_{sr} is the subregular orbit, with $'(SL_2) = SO_3 \quad SL_3$, and O_{reg} is the regular nilpotent orbit. The centralizers of the respective $'(SL_2)$ are

$$G_2; \; SL_{2;s}; \; SL_{2;l}; \; S_3; \; \text{and } 1:$$

Since the Galois cohomology of p-adic simply connected groups is trivial, it follows at once that the set of F-points is a single $G_2(F)$-orbit except for O_{sr} where the $G_2(F)$-orbits are parameterized by cubic etale F-algebras E,

$$O_{sr}(F) = [_E O_E :$$

3. Representations of G_2

In this section, we state some facts for the representations of $G_2(F)$. In particular, we shall describe all non-supercuspidal representations. The results in this section are sourced from Muic [Mu, Thm. 3.1, Props 4.1, 4.2, 4.3 and 4.4, Thm 5.3] and organized for our purpose.

3.1. Representations of GL_2. Since the maximal parabolic subgroups of G_2 have GL_2 as Levi factors and we will be considering parabolic induction, let us begin by setting up some notations for representations of $GL_2(F)$. If $_1$ and $_2$ are two characters of F, then $_1 \; _2$ denotes the parabolically induced representation of $GL_2(F)$ constructed from the character $_1 \; _2$ of the diagonal split torus. This induced representation is irreducible unless $_1 = _2 = j \; j^{1}$, in which case it is non-semisimple of length 2. In particular, for a character of F, one has

$$0 \quad ! \quad st \quad ! \quad j \quad j^{1=2} \; j \quad j^{\;1=2} \quad ! \quad det \quad ! \quad 0$$

where det is a 1-dimensional character of $GL_2(F)$ and st is a discrete series representation. If $=$ 1 is trivial, we will simply write st as st: this is the Steinberg representation. For nontrivial, one has st = st (det) and we call st a twisted Steinberg representation.

3.2. Principal series representations for P. We rst consider the principal series representations for the Heisenberg parabolic subgroup $P = MN$, where $M = GL_2$. Let be an irreducible representation of M with central character $!$ and set

$$I_P() = \mathrm{Ind}_P^{G_2} \quad \text{and} \quad I_P(s;) = \mathrm{Ind}_P^{G_2}(j\det j^s \;)$$

if we need to consider a family of induced representations. If I P (s;) is a standard module, we will denote its unique Langlands quotient by J P (s;). Now we have:

Proposition 3.1. (i) If is a unitary supercuspidal representation, then I P (s;) is reducible if and only if – = (so ! = 1)[2] and one of the following holds:

! = 1 and s = 1=2, in which case there is a non-split short exact sequence of length 2,

0 ! P () ! I P (1=2;) ! J P (1=2;) !

0; where P () is a generic discrete series representation.

! = 1 and s = 0, in which case

I P () = I P ()gen I P ()deg where

I P ()gen is generic.

(ii) If = st is a twisted Steinberg representation, then I P (s;) is irreducible except for the following cases:

= 1 and s = 3=2 or 1=2, in which case one has:

0 ! St$_{G_2}$! I P (3=2; st) ! J P (3=2; st) ! 0;

with St$_{G_2}$ the Steinberg representation. On the other hand, I P (1=2; st) has length 3, with a unique irreducible submodule gen[1] which is a generic discrete series representation, a unique irreducible Langlands quotient J P (1=2; st) and a subquotient J$_Q$(1=2; st).

^{2} = 1 but = 1 and s = 1=2, in which case one has:

0 ! gen[] ! I P (1=2; st) ! J P (1=2; st) ! 0

where gen[] is a generic discrete series representation. [3]

= 1 but = 1 and s = 1=2, in which case one has:

0 ! gen[] ! I P (1=2; st) ! J P (1=2; st) !

0: where gen[] = gen[^{1}] is a generic discrete series representation.

(iii) If = is 1-dimensional unitary, then I P (s;) is irreducible except in the following cases:

= 1 and s = 1=2 or 3=2, in which case one has:

0 ! J$_Q$(5=2; st) ! I P (3=2; 1) ! 1$_{G_2}$! 0;

whereas I P (1=2; 1) is of length 3, with a unique irreducible submodule deg[1] which is a nongeneric discrete series representation, a unique irreducible quotient J$_Q$(1; (1; 1)) and a subquotient J$_Q$(1=2; st).

^{2} = 1 but = 1 and s = 1=2, in which case one has:

0 ! J$_Q$(1=2; st) ! I P (1=2;) ! J$_Q$(1; (1;)) ! 0:

^{3} = 1 but = 1 and s = 1=2, in which case one has:

0 ! J_P (1=2; st $_1$) ! I_P (1=2;) ! J_Q(1; (; ^{1})) ! 0:

3.3. Principal series representations for Q. Now we consider the principal series representations for the 3-step parabolic subgroup Q = LU , where L = GL_2. Let be an irreducible unitary representation of L with L-parameter and set

$$I_Q() = \mathrm{Ind}_Q^G \text{ and } I_Q(s;) = \mathrm{Ind}_Q^G \, j \det j^s$$

if we need to consider a family of induced representations. As before, we let J_Q(s;) denote the unique Langlands quotient of I_Q(s;) if the latter is a standard module. Then we have:

Proposition 3.2. (i) If is unitary supercuspidal, then I_Q(s;) is reducible if and only if – = (so !^{2} = 1) and one of the following holds:

! = 1 and s = 1=2, in which case one has:

0 ! $_Q$() ! I_Q(1=2;) ! J_Q(1=2;) !

0; where $_Q$() is a generic discrete series representation.

! = 1 (so is dihedral), Im( ) = S_3 (the symmetric group on 3 letters, regarded as a subgroup of GL_2(C)) and s = 1,in which case one has:

0 ! $_{gen}$[] ! I_Q(1;) ! J_Q(1;) !

0; where $_{gen}$[] is a generic discrete series representation.

! = 1, Im( ) = S_3 (the symmetric group on 3 letters, regarded as a subgroup of GL_2(C)) and s = 0, in which case one has:

$$I_Q() = I_Q()_{gen} \; I_Q()_{deg} \text{ where}$$

$I_Q()_{gen}$ is generic.

(ii) If = st is a twisted Steinberg representation, the I_Q(s;) is irreducible except for the following cases:

= 1 and s = 5=2 or 1=2, in which case one has

0 ! St_{G_2} ! I_Q(5=2; st) ! J_Q(5=2; st) !

0; and

0 ! $_{gen}$[1] $_{deg}$[1] ! I_Q(1=2; st) ! J_Q(1=2; st) ! 0:

Here $_{gen}$[1] is the generic discrete series representation already dened in Proposition 3.1(ii) (rst bullet point) and $_{deg}$[1] is the nongeneric discrete series representation already dened in Proposition 3.1(iii) (rst bullet point).

^{2} = 1 but = 1 and s = 1=2, in which case one has:

0 ! $_{gen}$[] ! I_Q(1=2; st) ! J_Q(1=2; st) ! 0:

Here, $_{gen}$[] is the generic discrete series representation dened in Proposition 3.1(ii) (second bullet point).

(iii) If = is 1-dimensional unitary, then $I_Q(s;)$ is irreducible except in the following cases:

= 1 and s = 1=2 or 5=2, in which case one has:

$$0 \quad ! \quad J_P(3=2; st) \quad ! \quad I_Q(5=2; 1) \quad ! \quad 1_{G_2} \quad ! \quad 0;$$

whereas $I_Q(1=2; 1)$ is of length 3, with unique irreducible submodule $J_Q(1=2; st)$, a unique irreducible quotient $J_Q(1; (1; 1))$ and subquotient $J_P(1=2; st)$.

^{2} = 1 but = 1 and s = 1=2, in which case one has:

$$0 \quad ! \quad J_P(1=2; st) \quad ! \quad I_Q(1=2;) \quad ! \quad J_Q(1; (1;)) \quad ! \quad 0:$$

3.4. Principal series representations for B. We now consider the principal series representations induced from the Borel subgroup B. More precisely, suppose that is a Langlands quotient of a standard module

$$I(s_1; s_2; {}_1; {}_2) \quad \text{with}$$

$$s_1 \; s_2 \; 0$$

and $_i$ unitary characters of F. Here, recall the convention about characters of T which we have xed in x2.3. Then

$$,! \; I(\; s_1; \; s_2; {}_1{}^1; {}_2{}^1) = I_P(({}_1{}^1 j \qquad j^{\,s_1}; {}_2{}^1 j \quad j^{\,s_2})):$$

Now the representation $({}_1{}^1 j \quad j^{\,s_1}; {}_2{}^1 j \quad j^{\,s_2})$ of M = GL_2 is reducible if and only if

$${}_2={}_1 \; j \qquad j^{s_2 \;\; s_1} = j \quad j^{\;1}; \quad \text{i.e. } {}_1 = {}_2 \text{ and } s_1 = s_2 + 1 \;\; 1,$$

in which case one has

$$,! \; I_P(\; s + \frac{1}{2}; {}_1{}^1); \qquad \text{with } s \;\; 1:$$

There is another, convenient, way to bookkeep the principal series $\mathrm{Ind}_B^{G_2}()$. Let $_1; _2; _3$ be three long roots such that $_1 + _2 + _3 = 0$. This triple is unique up to the action of the Weyl group of G_2. Then the corresponding co-roots : F ! T generate T, in particular, the character denes three characters of F by $_i$ = (and is determined by them). Clearly, these characters satisfy $_1 \; _2 \; _3 = 1$.

Proposition 3.3. The induced representation $\mathrm{Ind}_B^{G_2}()$ is irreducible unless one of the following two conditions hold:

$_i$ = j j^1 for some i or $_i=_j$ = j j^1 for a pair i = j.

The three characters $_i$ are quadratic, non-trivial and pairwise dierent. Then

$$\mathrm{Ind}_B^{G_2}() = \mathrm{Ind}_B^{G_2}()_{gen} \quad \mathrm{Ind}_B^{G_2}()_{deg} \text{ where}$$

$\mathrm{Ind}_B^{G_2}()_{gen}$ is generic.

3.5. Conjectural L-packets of G_2. The above results allow one to give an enumeration of the non-cuspidal representations of G_2. Using the desiderata of the conjectural local Langlands correspondence (LLC) for G_2, we explain how one can assign L-parameters to the noncuspidal representations of G_2, and hence partition them into L-packets. Recall that an L-parameter of G_2 is an admissible homomorphism

$$\varphi : WD_F \longrightarrow \widehat{G}_2 = G_2(\mathbb{C})$$

of the Weil-Deligne group $WD_F = W_F \times SL_2(\mathbb{C})$ to the dual group $G_2(\mathbb{C})$, taken up to conjugacy by $G_2(\mathbb{C})$. Let

$$A_\varphi = \pi_0(Z_{G_2}(\varphi))$$

be the associated component group of φ. Then one expects that there should be an L-packet

$$\Pi_\varphi = \{\pi(\eta) : \eta \in \widehat{A_\varphi}\} \subset \mathrm{Irr}(G_2)$$

associated to each φ, whose members are indexed by the characters of A_φ, such that

$$\mathrm{Irr}(G_2) = \bigcup_\varphi \Pi_\varphi:$$

The non-tempered irreducible representations of G_2 are uniquely realized as Langlands quotients of standard modules, so have the form $J_P(\pi)$, $J_Q(\pi)$ or $J_B(\pi)$. The Levi factors of the parabolic subgroups P, Q aand B are isomorphic to GL_2 and $GL_1 \times GL_1$. Since the LLC for these groups are known, one can assign L-parameters to the nontempered representations. For example, if $\pi = J_P(\tau)$, and $\varphi_\tau : WD_F \longrightarrow \widehat{M} = GL_2(\mathbb{C})$ is the L-parameter of τ, then the L-parametrer of $\pi = J_P(\tau)$ is the composite

$$\varphi : WD_F \longrightarrow \widehat{M} \hookrightarrow \widehat{G}_2 = G_2(\mathbb{C}):$$

Since the L-packets on the Levi subgroups are singletons, we see also that the nontempered L-packets of G_2 are singletons, and A_φ is correspondingly trivial.

In other words, the non-tempered irreducible representations of G_2 are naturally parametrized by the nontempered L-parameters of G_2; these are the L-parameters φ such that $\varphi(W_F)$ is unbounded. In the following, we will use this partial LLC to describe the eect of the various theta correspondences on nontempered representations.

By the same token, since irreducible tempered representations which are not square-integrable are uniquely realized as summands of principal series representations induced from unitary square-integrable representations of Levi factors, one can attach L-parameters to these tempered (but not square-integrable) representations of G_2. The resulting L-parameters φ have the property that $\varphi(W_F)$ is bounded but $\varphi(WD_F)$ is contained in a proper Levi subgroup. The size of such a tempered L-packet now depends on the number of irreducible summands in the corresponding parabolically induced representations. From the results recalled in this section, one sees that the size of a tempered L-packet Π_φ is 1 or 2. One can verify that this is the same as the size of A_φ. Moreover, in each tempered L-packet, there is a unique generic representation, and this is assigned to the trivial character of A_φ. Thus, the LLC for tempered non-discrete series representations of G_2 is also known, and we may refer to this partial LLC for describing these representations.

Hence, the main issue with the LLC for G_2 comes down to the classification of the square-integrable or discrete series representations by discrete series L-parameters; these are the L-parameters φ which do not factor through any proper Levi subgroup, or equivalently whose centralizer $C_\varphi = Z_{G_2^\vee}(\varphi)$ is finite. Guided by the desiderata of the LLC, we can now describe the various families of discrete series L-parameters, according to $\varphi(SL_2)$, and list all non-supercuspidal members.

(1) $\varphi(SL_2)$ is the principal SL_2. Then $A_\varphi = 1$ and the packet consists of the Steinberg representation:

$$\Pi_\varphi = \{St_{G_2}\}$$

(2) $\varphi(SL_2) = SO_3 \subset SL_3 \subset G_2$; this is the subregular SL_2. The centralizer of SO_3 in G_2 is the finite symmetric group S_3, so that φ gives by restriction a map $\rho : W_F \to S_3$. There are four cases to discuss:

- $\rho(W_F) = 1$. Then $A_\varphi = S_3$. Let $1; r; \varepsilon$ be the three irreducible representations of S_3: the trivial, 2-dimensional and the sign character respectively. Then

$$\Pi_\varphi = \{\pi(1) = \pi_{gen}[1];\ \pi(r) = \pi_{deg}[1];\ \pi(\varepsilon) = \pi_{sc}[1]\}$$

where $\pi_{gen}[1]$ is defined in Proposition 3.1(ii) (first bullet point) and $\pi_{deg}[1]$ is given in Proposition 3.1(iii) (first b.p.). The representation $\pi(\varepsilon)$ is a depth 0 supercuspidal representation induced from a cuspidal unipotent representation of $G_2(\mathbb{F}_q)$, inflated to a hyperspecial maximal compact group [HMS]. The cuspidal unipotent representation is denoted in the literature by $G_2[1]$ and hence our notation $\pi_{sc}[1]$.

- $\rho(W_F) \cong \mathbb{Z}/2\mathbb{Z}$. Then, by the local class field theory, ρ defines a quadratic character χ of $F^\times$. Let 1 and -1 denote the trivial and non-trivial characters of $A_\varphi = \mu_2$. Then

$$\Pi_\varphi = \{\pi(1) = \pi_{gen}[\chi];\ \pi(-1)\};$$

where $\pi_{gen}[\chi]$ is as defined in Proposition 3.1(ii) (second b.p.). If the character χ is unramified, then $\pi(-1) = \pi_{sc}[-1]$ is a depth 0 supercuspidal representation. It is induced from a cuspidal unipotent representation of $G_2(\mathbb{F}_q)$, denoted by $G_2[-1]$, inflated to a hyperspecial maximal compact group.

- $\rho(W_F) \cong \mathbb{Z}/3\mathbb{Z}$. Then, by local class field theory, ρ defines a cubic character χ of $F^\times$. Let 1, ω and ω^2 denote the characters of $A_\varphi = \mu_3$. Then

$$\Pi_\varphi = \{\pi(1) = \pi_{gen}[\chi];\ \pi(\omega);\ \pi(\omega^2)\};$$

where $\pi_{gen}[\chi]$ is as defined in Proposition 3.1(ii) (third b.p.). If the character χ is unramified, then $\pi(\omega) = \pi_{sc}[\omega]$ and $\pi(\omega^2) = \pi_{sc}[\omega^2]$ are induced from a cuspidal unipotent representations of $G_2(\mathbb{F}_q)$, denoted by $G_2[\omega]$ and $G_2[\omega^2]$, inflated to a hyperspecial maximal compact group.

- $\rho(W_F) = S_3$. Then $r \circ \rho$ corresponds to a supercuspidal representation τ of GL_2 (where we recall that r denotes the two-dimensional irreducible representation of S_3). In this case A_φ is trivial and

$$\Pi_\varphi = \{\pi_{gen}[\tau]\};$$

where $_{gen}[]$ is as dened in Proposition 3.2(i) (second b.p.).

(3) $'(SL_2) = SL_{2;s}$, a short root SL_2. The centralizer of $SL_{2;s}$ in G_2 is $SL_{2;l}$, a long root SL_2. Then $'$ gives, by restriction, a map form the Weil group $: W_F ! SL_{2;l}$, that corresponds to supercuspidal representation of GL_2 with the trivial central character (and hence $= -$). In this case $A_{'} = {}_2$, and

$$' = f(1) = {}_P(); (1)g;$$

where $_P()$ is as dened in Proposition 3.1(i) (rst b.p.) and (1) is supercuspidal.

(4) $'(SL_2) = SL_{2;l}$, a long root SL_2. The centralizer of $SL_{2;l}$ in G_2 is $SL_{2;s}$, a short root SL_2. Then $'$ gives, by restriction, a map from the Weil group $: W_F ! SL_{2;s}$, that corresponds to supercuspidal representation of GL_2 with the trivial central character (and hence $= -$). In this case $A_{'} = {}_2$, and

$$' = f(1) = {}_Q(); (1)g;$$

where $_Q()$ is as dened in Proposition 3.2(i) (rst b.p.) and (1) is supercuspidal.

(5) $'(SL_2) = 1$. Then $' : W_F ! G_2(C)$ gives rise to an L-packet consisting entirely of supercuspidal representations of G_2.

There has been some work towards the above conjectural LLC for G_2, most notably [SWe] and [HKT]. At the moment, we simply wish to point out that all the noncuspidal discrete series representations are fully accounted for by the above classication scheme.

3.6. Local Fourier coecients. It will be useful to consider the twisted Jacquet modules of a representation of G_2 along the unipotent radical N of P. The M-orbits of 1-dimensional characters of N are naturally indexed by cubic F-algebras, with the generic orbits corresponding to etale cubic F-algebras. For any such cubic F-algebra E, we shall write $_E$ for a character of N in the corresponding M-orbit. Then one may consider $_{N;\,_E}$. In particular, we note:

Proposition 3.4. For any irreducible, innite dimensional representation of G_2, there exists an etale cubic F-algebra E such that $_{N;\,_E} = 0$. Moreover, if is degenerate, then $_{N;\,_E}$ is nite-dimensional for any etale E.

Proof. Wave front sets of irreducible representations of G_2 are supported on special orbits, that is, $f0g$, O_{sr} and O_{reg}, see [LS] and [JLS]. Thus, if is degenerate (not Whittaker generic), and not the trivial representation, its wave-front set is supported on subregular nilpotent orbits. If O_E is in the wave-front set of then $_{N;\,_E}$ is non-zero and nite-dimensional, by the main result of [MW] and [Va].

Assume now that is generic. The restriction of a Whittaker character to U is a character supported on the simple root space U. Hence $_{U;} = 0$. Let N^0 be obtained by adding U to U and removing U_+, so that N is conjugate to N (by the simple Weyl reection w). Abusing notation, let be the character of N supported on the simple root space U N^0. Now we claim that there is an isomorphism (a root exchange)

$$\mathrm{Hom}_U(;\ ) = \mathrm{Hom}_{N^0}(;\ ):$$

which sends ‘ on the LHS to an element $‘^0$ on the RHS dened by the convergent integral Z

$$‘^0(v) = \quad ‘((u)\ v)\ du_U$$

Conversely, we can recover ‘ from $‘^0$ by integrating over U_+.

Assuming the claim, let U^0 be the conjugate of U by w_+ . Let Z0 be two-dimensional center of U^0. Observe that $[U0\ ; U^0]=Z^0 = U$. Since $[U^0;U^0]\ N^0$, it follows that $[U^0;U^0];$ = 0. But this means that the Fourier-Jacobi functor of with respect to the 3-step unipotent U is non-trivial, and $_{N;\ E}$ = 0 for some E = F + K, by [JLS, Proposition 6.1].

To justify the root exchange argument in the claim, we observe that U and U_+ generate a Heisenberg group with center U, modulo higher order commutators. More precisely, consider the group

$$V^0 = U\ U \ = N^0\ U_+;$$

which is a maximal unipotent subgroup of G_2 and hence conjugate to V (by the simple reection w). If we consider the lower central series of the unipotent group V^0:

$$V^0\ [V^0;V^0] = V^0_1 V^0 = {}_2[V^0;V^0]\ V^0_1\ f1g_3;$$

then $V=V^0_2$ is the Heisenberg group in question with center $V^0_1=V^0_2 = U$. Note moreover that the elements ‘ and $‘^0$ in the two Hom spaces in the claim both factors through $_{V^0_2}$ (which is a module for the Heisenberg group $V^0=V^0_2$). With this observation, the justication of the claim is given by the following lemma, included as a convenience to the reader.

Lemma 3.5. Let H be a Heisenberg group. Let be a smooth H-module. Let X and Y be two maximal abelian subgroups of H. Let $_X$ and $_Y$ be characters of X and Y, agreeing on the intersection $X \cap Y$, and non-trivial on the center of H. Then we have an isomorphism $\mathrm{Hom}_X(;\ _X) = \mathrm{Hom}_Y(;\ _Y)$, $‘ \mapsto ‘^0$, dened by

$$‘^0(v) = \int_{Y=X\cap Y} ‘((y)v)\ dy:$$

Proof. By the Frobenius reciprocity, we have

$$\mathrm{Hom}_X(;\ _X) = \mathrm{Hom}_H(;\mathrm{Ind}_X^H\ _X) \text{ and } \mathrm{Hom}_Y(;\ _Y) = \mathrm{Hom}_H(;\mathrm{Ind}_Y^H\ _Y):$$

We also have an isomorphism $\mathrm{Ind}_X^H\ _X = \mathrm{Ind}_Y^H\ _Y$, where every $f \in \mathrm{Ind}_X^H\ _X$ goes to $f0 \in \mathrm{Ind}_Y^H\ _Y$ dened by

$$f^0(h) = \int_{Y=X\cap Y} f(yh)\ _Y(y)\ dy:$$

This integral is convergent, in fact, it is a nite sum. The lemma follows by combining this isomorphism with the two Frobenius reciprocity isomorphisms.

4. Exceptional Dual Pairs

In this section, we briefly describe the dual pairs which intervene in this paper and some structural results which will be important in the study of the associated theta correspondences.

If A is an associative algebra over F, then A^+ will denote the underlying Jordan algebra, that is, A with the Jordan multiplication $a \circ b = \frac{1}{2}(ab + ba)$.

4.1. The group M_J. Let J be a Freudenthal-Jordan F-algebra [KMRT, §37 and §38]. The algebra J comes equipped with a cubic norm form N_J, and we let

$$M_J = \{g \in GL(J) : N_J \circ g = N_J\}.$$

It contains the automorphism group $\mathrm{Aut}(J)$ as a subgroup. Now we consider the F-vector space

$$\mathfrak{g}_J = \mathfrak{sl}_3 \oplus \mathrm{Lie}(M_J) \oplus (F^3 \otimes J) \oplus (F^3 \otimes J)$$

Then $\mathfrak{g}_J$ can be given the structure of a simple exceptional Lie algebra (see, for example, [GS05]). We have the following cases of interest:

$\dim J$	1	3	9	15
$\mathfrak{g}_J$	G_2	D_4	E_6	E_7

We observe:

- If $\dim J = 3$, then J is a cubic étale F-algebra E.
- If $\dim J = 9$, then J corresponds to a pair (B_K, σ) where B_K is a central simple algebra over an étale quadratic F-algebra K and σ is an involution of the second kind. Thus, $J = B_K^{\sigma}$ is the subspace of σ-symmetric elements. If $K = F^2$, then $J = B \times B$ for a central simple algebra B over F, σ permutes two summands, and $J = B^+$. The split version is when $B = M_3$, the algebra of 3×3 matrices, and $\mathrm{Aut}(M_3^+) = PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z}$.
- If $\dim J = 15$, then J is $H_3(B_F)$ is the space of all 3×3 hermitian-symmetric matrices, where B_F is a quaternion algebra over F. The split version is when $B_F = M_2$.

Let G_J be the identity component of $\mathrm{Aut}(\mathfrak{g}_J)$. If $\dim J = 9$ then

$$1 \longrightarrow G_J \longrightarrow \mathrm{Aut}(\mathfrak{g}_J) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

This short exact sequence may not be split in general.

4.2. Dual pair $G_2 \times \mathrm{Aut}(J)$. We can now describe some dual pairs in G_J or rather in $\mathrm{Aut}(\mathfrak{g}_J)$. It will be easier to do this on the level of Lie algebras.

The centralizer of $\mathrm{Aut}(J)$ in $\mathfrak{g}_J$ is

$$\mathfrak{sl}_3 \oplus (F^3 \otimes 1_J) \oplus (F^3 \otimes 1_J)$$

which one recognizes to be $\mathfrak{g}_F$ (i.e. taking $J = F$). Thus this is a Lie subalgebra of type G_2, and we have a dual pair

$$G_2 \times \mathrm{Aut}(J) \subset \mathrm{Aut}(G_J).$$

If $\dim J = 9$, we recall that $\mathrm{Aut}(J)$ sits in a short exact sequence

$$1 \longrightarrow \mathrm{Aut}(J)^0 \longrightarrow \mathrm{Aut}(J) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

If J is associated to a pair $(B_K;)$, then $\mathrm{Aut}(J)^0 = PGU(B_K;)$ is an adjoint group of type A_2.

4.3. Dual pair $\mathrm{Aut}(i : E \,!\, J)$ G_E. Now we x an embedding $i : E \,!\, J$, where E is a cubic etale F-algebra. We have the subgroup

$$\mathrm{Aut}(i : E \,!\, J) \quad \mathrm{Aut}(J):$$

If $\dim J = 9$, a detailed description of this group is in [GS14]. Its identity component is a 2-dimensional torus. The centralizer of $\mathrm{Aut}(i : E \,!\, J)$ in g_J contains

$$g_E = sl_3 \quad t_E \quad F^3 \quad E \quad (F^3 \quad E)$$

where $E \,,\!\, J$ via i and $t_E = E^0$ is the toral Lie subalgebra of trace 0 elements in E. This Lie algebra is isomorphic to $\mathrm{Lie}(G_E)$ (where G_E is the simply connected quasi-split group Spin_8^E), and we have the dual pair

$$\mathrm{Aut}(i : E \,!\, J) \quad G_E \quad ! \quad \mathrm{Aut}(G_J):$$

Note that this map is not injective.

4.4. A see-saw diagram. The two dual pairs we described above t together into a see-saw diagram:

$$\begin{array}{ccc} G_E := \left|\mathrm{Spin}_8^E\right. & & \left.\mathrm{Aut}(J)\right| =: H_J \\ & \times & \\ G_2 & & \mathrm{Aut}(i : E \,!\, J) =: H_{J;E} \end{array} \tag{4.1}$$

in $\mathrm{Aut}(G_J)$. The various J's of interest in this paper, and the corresponding groups $H_J = \mathrm{Aut}(J)$ and $H_{J;E} = \mathrm{Aut}(i : E \,!\, J)$ are given in the table below.

J	D^+	M_3^+	$H_3(M_2)$
H_J	PD	$PGL_3 \text{ o } Z=2Z$	$PGSp_6$
$H_{J;E}$	PE	$PE \text{ o } Z=2Z$	$\mathrm{Res}_{E=F}\, SL_2=_2$

Here, note that D^+ denotes the Jordan algebra associated to a cubic division F-algebra D, in which case E is necessarily a eld.

5. The See-Saw Argument

In this section, we shall consider the see-saw identity arising from the seesaw diagram (4.1) and pursue some of its consequences.

5.1. See-saw identity. Suppose that $2\ \mathrm{Irr}(G_2)$. Then we have the see-saw identity associated with the seesaw (4.1):

$$\mathrm{Hom}_{H_{J;E}}((); C) = \mathrm{Hom}_{G_2}(R_J(E);) \tag{5.1}$$

where

$$R_J(E) := (1)$$

is the big theta lift of the trivial representation of $H_{J;E}$. To make use of this see-saw identity, we need to understand the representation $R_J(E)$ of Spin^E_8. This has been studied in [GS21] and we recall the relevant results there.

5.2. Degenerate principal series of Spin^E_8. Let $P_E = M_E N_E \quad \mathrm{Spin}^E_8$ be the Heisenberg parabolic subgroup, so that its Levi factor is

$$M_E = GL_2(E)^{\det} = \{g \in GL_2(E) : \det(g) \in F\}:$$

Then the determinant map denes an algebraic character $M_E \to \mathbb{G}_m$ which is a basis element of $\mathrm{Hom}(M_E; \mathbb{G}_m)$. We may consider the degenerate principal series representation

$$I_E(s) = \mathrm{Ind}_{P_E}^{\mathrm{Spin}^E_8} |\det|^s:$$

In [S] and [GS21, Cor. 12.11, Thm. 17.6, Thm 18.1, Prop. 18.5 and Prop. 18.6], the module structure of this family of degenerate principal series representations has been determined. In particular, we have:

Proposition 5.2.

$$R_J(E) \hookrightarrow I_E(s_J)$$

where

$$s_J = \begin{cases} 1/2; & \text{if } J = D^+ \text{ or } M_3^+; \\ 1/2; & \text{if } J = H_3(M_2). \end{cases}$$

The representation $I_E(1/2)$ has length 3 when E is a eld and has length 2 otherwise. More precisely, it has a unique irreducible submodule V with quotient isomorphic to $R_{M_3}(E)$ $R_D(E)$ (where $R_D(E)$ is interpreted to be 0 when E is not a eld). Indeed, one has the short exact sequence:

$$0 \to R_{H_3(M_2)}(E) \to I_E(1/2) \to R_D(E) \to 0:$$

and

$$0 \to V \to R_{H_3(M_2)}(E) \to R_{M_3}(E) \to 0:$$

In particular, when E is not a eld, $I_E(1/2) = R_{H_3(M_2)}(E)$.

As a consequence of the above discussion, we see that it is useful to understand the Hom space

$$\mathrm{Hom}_{G_2}(I_E(s); \ ) \qquad \text{for } \ \in \mathrm{Irr}(G_2).$$

We shall study this in two ways.

5.3. Vanishing of an Ext^1. In view of the proposition, we see that there is an exact sequence

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Hom}_{G_2}(R_D(E); \ ) & \to & \mathrm{Hom}_{G_2}(I_E(1/2); \ ) & \to & \mathrm{Hom}_{G_2}(R_{H_3(M_2)}(E); \ ) \\ & & & & & & \big\downarrow \\ & & & & & & \mathrm{Ext}^1_{G_2}(R_D(E); \ ): \end{array}$$

Now we have the following useful fact:

Proposition 5.3. Assume that E is a eld. If 2 $\mathrm{Irr}(G_2)$ is tempered or has cuspidal support dierent from $_{\deg}[1]$, then

$$\mathrm{Ext}^1_{G_2}(R_D(E);) = 0;$$

so that one has a short exact sequence

$$0 \;!\; \mathrm{Hom}_{G_2}(R_D(E);) \;!\; \mathrm{Hom}_{G_2}(I_E(1=2);) \;!\; \mathrm{Hom}_{G_2}(R_{H_3(M_2)}(E);) \;!\; 0:$$

Proof. One needs to understand $R_D(E)$ as a representation of G_2, and this is essentially done in [Sa, Conj. 4.1 and x6], where the dual pair correspondence for $P D$ G_2 was studied. We shall recall the results of [Sa] in greater detail later on. At this point, we simply note that as a representation of G_2, $R_D(E)$ is the direct sum of a supercuspidal representation (of innite length) and the irreducible discrete series representation $_{\deg}[1]$, which is a constituent of $I_Q(1=2; st)$. From this, the vanishing of $\mathrm{Ext}^1_{G_2}(R_D(E);)$ for those with dierent cuspidal support from $_{\deg}[1]$ follows immediately. On the other hand, if is tempered, then one also has $\mathrm{Ext}^1(_{\deg}[1];) = 0$ since discrete series representations are projective in the category of tempered representations.

5.4. $I_E(s)$ **as G_2-module.** On the other hand, we may understand the restriction of $I_E(s)$ to G_2 using Mackey theory. The following is a key technical result:

Proposition 5.4. As a representation of G_2, $I_E(s)$ admits an equivariant ltration 0

$$I_0 \;\; I_1 \;\; I_2 \;\; I_3 \;\; I_4$$

with successive quotients described as follows:

$I_0 = \mathrm{ind}_N^{G} \; _E;$

$J_1 := I_1=I_0 \;\; I_P(\frac{}{2} + \frac{s}{4}; C^1_c(PGL_2)).$

$J_2 := I_2=I_1 \;\; m_E \;\; I_P(\frac{s}{2} + \frac{1}{4}; \mathrm{ind}_N^{PGL_2} \;)$

$J_3 := I_3=I_2 \;\; m_E \;\; I_Q(s + 1).$

$J_4 := I_4=I_3 \;\; I_P(s + 1).$

Here,

$$m_E = \begin{cases} 3; & \text{if } E = F^3; \\ 1; & \text{if } E = F \;\; K; \\ 0; & \text{if } E \text{ is a eld.} \end{cases}$$

The proposition implies that one has a short exact sequence

$$0 \;!\; \mathrm{ind}_N^{G_2} \; _E \;!\; I_E(s) \;!\; _E(s) \;!\; 0;$$

from which one deduces an exact sequence:

$$0 \;!\; \mathrm{Hom}_{G_2}(_E(s);) \;!\; \mathrm{Hom}_{G_2}(I_E(s);) \;!\; \mathrm{Hom}_N(-; \;_E) \;!\; \mathrm{Ext}^1_{G_2}(_E(s);):$$

We now specialize to $s = 1=2$, where we need to be more precise.

Proposition 5.5. Suppose that 2 $\mathrm{Irr}(G_2)$ is tempered or has cuspidal support along Q. Then

$$\mathrm{Hom}_{G_2}(I_E(1=2);) = \mathrm{Hom}_N(^-; \quad _E):$$

Proof. We need to prove

$$\mathrm{Hom}_{G_2}(_E(1=2);) = 0 = \mathrm{Ext}^1_{G_2}(_E(1=2);): \text{To}$$

that end, it suces to prove the following lemma:

Lemma 5.6. For all i and j, $\mathrm{Ext}^i_{G_2}(J_j;) = 0$, with as in Proposition 5.5.

Proof. Consider J_1 rstly. By the Frobenius reciprocity,

$$\mathrm{Ext}^i_{G_2}(J_1;) = \mathrm{Ext}^i_M(j\det j^{1=2} \; C^1_c(PGL_2); \overline{r}_{\overline{P}}()):$$

Since is tempered, the center of $M = GL_2$ acts on $\overline{R}_{\overline{P}}()$) by characters such that $j(z)j = jzj^t$ where t 0. On the other hand, the center of M acts on $j\det j^{1=2} \; C^1_c(PGL_2)$ by jzj. Thus the right hand side is 0. The other cases are dealt with in the same way.

This completes the proof of Proposition 5.5.

6. Dichotomy

The goal of this section is to prove the following theorem:

Theorem 6.1. For any representation 2 $\mathrm{Irr}(G_2)$ has nonzero theta lift to exactly one of P D or $PGSp_6$.

To prove this dichotomy theorem, we need some preliminary results which are consequences of the computation of the Jacquet modules of the minimal representation $_J$ with respect to the various maximal parabolic subgroups of G_2 and H_J. The required Jacquet module computations were carried out in [Sa, Prop. 5.1] when H_J = P D and in [MS, Thm. 4.3 and Thm. 7.6] when $H_J = PGL_3$ (see also [GS04, Prop. 4 and Prop. 6]). For $H_J = PGSp_6$, the Jacquet module computations for some parabolic subgroups were carried out in [MS, Thm. 5.3 and Thm 7.6]. The remaining ones will be done in x13 and some implications of these computations are discussed in x14. We note that x13 is a self-contained section independent of the rest of this paper. Hence, we rst record some results from x13-14 and the earlier references [Sa, MS, GS04] that we will use.

6.1. Consequences of Jacquet module computations.

We rst note:

Lemma 6.2. Consider the theta correspondence for G_2 H_J for the 3 cases of J.

(i) Let 2 $\mathrm{Irr}(G_2)$ and write

$$_J() = {}_J()_c \; {}_J()_{nc}$$

as a sum of its cuspidal and noncuspidal components. Then $_J()_{nc}$ has nite length. In particular, if $_J() = 0$, then it has an irreducible quotient.

(ii) Likewise, let 2 $\mathrm{Irr}(H_J)$ and write

$$_J() = {}_J()_c \; {}_J()_{nc}:$$

Then $_J()_{nc}$ has nite length. In particular, if $_J()$ = 0, then it has an irreducible quotient.

Proof. We consider the 3 cases of H_J in turn:

The case of $J = H_3(M_2)$ is shown in x14, based on the results of x13. As we remarked above, the results of x13 and x14 are independent of the rest of the paper.

The case of $J = D^+$ follows from results of [Sa, x6], proving [Sa, Conjecture 4.1(3)].

For $J = M^+_3$ (ii) follows from [GS04, Prop. 7, Cor. 9(i), rst paragraph of proof of Thm. 14 and last paragraph of x9]. The proof of (i) is analogous to that for the case $J = H_3(M_2)$, which we describe in x14.2, and uses the Jacquet module computation for PGL_3 given in [GS04, Prop. 4] and Proposition 14.2.

In fact, the Jacquet module computations allow one to determine the theta lift of non-tempered representations explicitly (see Theorem 14.1). We simply note the following here:

Lemma 6.3. (i) Let 2 $\mathrm{Irr}(G_2)$ and 2 $\mathrm{Irr}(H_J)$ be such that is a quotient of $_J$. Then

is tempered ( ) is tempered:

(ii) Let 2 $\mathrm{Irr}(G_2)$ be non-tempered. Then has nonzero theta lifting to $PGSp_6$.

Proof. For H_J = P D or PGL_3 o $Z=2Z$, the desired results have been veried in [Sa, x6] and [GS04, Cor. 9(i) and proof of Prop. 10] respectively. For the case when $H_J = PGSp_6$, this is shown in Theorem 14.1 in x14.

6.2. Reduction to non-generic tempered case. With the above inputs in place, we can now begin the proof of the dichotomy theorem. We note:

The dichotmy theorem holds for nontempered . Indeed, if is non-tempered, then Lemma 6.3(ii) says that has nonzero theta lift to $PGSp_6$, whereas [Sa] shows that has zero theta lift to P D.

The dichotmy theorem holds for generic . Indeed, it was shown in [GS04, Cor. 20] that a generic has nonzero theta lift to $PGSp_6$ (see also Cor. 11.2 below) and it was shown in [Sa] that has zero theta lift to P D.

Thus, to prove the dichotomy theorem, it remains to deal with non-generic tempered .

6.3. Weak dichotomy. We rst prove that a non-generic tempered has nonzero theta lift to one of P D or $PGSp_6$. Since is non-generic and innite-dimensional, there exists an etale cubic F-algebra E such that $\mathrm{Hom}_N(-;\ _E) = 0$. By Proposition 5.5, we have an isomorphism

$$\mathrm{Hom}_{G_2}(I_E(1=2);) \quad \mathrm{Hom}_N(-;\ _E) = 0:$$

This implies, by Proposition 5.3, that

$$\mathrm{Hom}_{G_2}(R_D(E);) = 0 \quad \text{or} \quad \mathrm{Hom}_{G_2}(R_{H_3(M_2)}(E);) = 0:$$

By the see-saw identity (5.1), we deduce that

$$\mathrm{Hom}_{H_E}({}_J(); C) = 0$$

for $J = D^+$ or $H_3(M_2)$. In particular, ${}_J() = 0$ for $J = D^+$ or $H_3(M_2)$. We have thus veried that has nonzero theta lift to at least one of P D or $PGSp_6$.

6.4. Curious chain of containments. It remains to show that a nongeneric tempered cannot lift to both P D and $PGSp_6$. Let be the complex conjugate of . If is unitarizable (e.g. if is tempered), then = –. Thus

$$_E = \ _E = (-) \ _E$$

where, in the second isomorphism, we assume that is unitarizable. Since the minimal representation $_J$ used in this paper is dened over R, we have a canonical isomorphism $_J = \ _J$. It follows that $_J()$ is the complex conjugate of $_J()$.

We shall make use of the curious chain of containment given in the following lemma; this is the rst instance of the game of ping-pong with periods discussed at the end of the introduction.

Lemma 6.4. Let 2 $\mathrm{Irr}(G_2)$ be tempered. For $J = D^+$, M_3^+ or $H_3(M_2)$, let 2 $\mathrm{Irr}(H_J)$ be tempered and such that

$$\mathrm{Hom}_{G_2 H_J}(_J;) = 0: \text{Then}$$

we have the following natural inclusions

$$\mathrm{Hom}_N(; \ _E) \ \mathrm{Hom}_N((); \ _E) = \mathrm{Hom}_{H_{J;E}}(-; C) \ \mathrm{Hom}_{H_{J;E}}((-); C) = \mathrm{Hom}_{G_2}(R_J(E); -):$$ If one of these spaces is nite-dimensional, then all inclusions are isomorphisms.

Proof. The rst inclusion arises from () . The second follows from

$$\mathrm{Hom}_N((); \ _E) = \mathrm{Hom}_{H_J}((_J)_{N; \ E};)$$

combined with (see [GrS2, Lemma 2.9, Pg 213])

$$(_J)_{N; \ E} = \mathrm{ind}_{H_{J;E}}^{H_J}(1)$$

and the Frobenius reciprocity. For the third, observe that $_J()$ is the complex conjugate of $_J()$. Since = – and = – and we have (–) –. The fourth is the see-saw identity (5.1).

If any of the spaces is nite-dimensional, then $\mathrm{Hom}_N(; \ _E)$ is nite-dimensional. If this space is nite dimensional then, since is tempered, by Propositions 5.3 and 5.5, one has

(6.5) $$\dim \mathrm{Hom}_{G_2}(R_J(E); -) \ \dim \mathrm{Hom}_{G_2}(I_E(1=2); -) = \dim \mathrm{Hom}_N(; \ _E):$$

It follows that all spaces have the same dimension and the lemma is proved.

6.5. Conclusion of proof. Using the lemma, we can now conclude the proof of Theorem 6.1.

Assume is tempered nongeneric and has nonzero theta lift to P D. Since P D is compact, one can nd 2 Irr(P D) such that is an irreducible quotient of $_D$(). Moreover is tempered. Choose E so that Hom_N(; $_E$) = 0. We may now apply Lemma 6.4 with the chosen and E to deduce that

$$d := \dim \mathrm{Hom}_{G_2}(I_E(1=2);) = \dim \mathrm{Hom}_{G_2}(R_D(E);) = \dim \mathrm{Hom}_N(; \ _E) = 0$$

Similarly, if has nonzero theta lift to PGSp$_6$, then we may nd a tempered 2 Irr(PGSp$_6$) such that is an irreducible quotient of () (by Lemma 6.2 and Lemma 6.3(i)). With E as above, an application of Lemma 6.4 shows that

$$d = \dim \mathrm{Hom}_{G_2}(I_E(1=2);) = \dim \mathrm{Hom}_{G_2}(R_{H_3(M_2)}(E);) = \dim \mathrm{Hom}_N(; \ _E) = 0$$

Moreover, since all these dimensions are nite, one deduces by Proposition 5.3 that

$$d = \dim \mathrm{Hom}_{G_2}(I_E(1=2);) = \dim \mathrm{Hom}_{G_2}(R_D(E);) + \dim \mathrm{Hom}_{G_2}(R_{H_3(M_2)}(E);) = 2d:$$

This gives the desired contradiction and completes the proof of Theorem 6.1.

6.6. Uniqueness results. As further applications of Lemma 6.4, we may now derive two multiplicity one statements which will play a key role in the reminder of the paper. These statements are the rst steps towards the proof of the Howe duality theorem.

Proposition 6.6. Let 2 Irr(H_J) be tempered. Let 2 Irr(G_2) be a tempered, non-generic quotient of $_J$(). Then $_J$() = .

Proof. Since is non-generic, for every E, the space Hom_N(; $_E$) is nite-dimensional. By Lemma 6.4, Hom_N($_J$(); $_E$) = Hom_N(; $_E$), for every E. Thus, by Proposition 3.4, the kernel of the projection $_J$() ! has trivial action of G_2. But this submodule would split o, giving a trivial representation as a quotient of $_J$(). This contradicts Lemma 6.3.

Proposition 6.7. Let 2 Irr(G_2) be tempered and non-generic. Then $_J$() cannot have two tempered irreducible quotients. In particular, the cuspidal representation $_J()_c$ is irre-ducible or 0.

Proof. Let $_1$; $_2$ 2 Irr(H_J), irreducible tempered, such that $_J$() $_1$ $_2$. Since is non-generic, there exists E such that d = dim Hom$_N$ (; $_E$) is nite and non-zero. By Lemma 6.4, applied to –, – $_1$ and then to –, –, $_2$

$$d = \dim \mathrm{Hom}_{H_{J;E}}(_1; C) = \dim \mathrm{Hom}_{H_{J;E}}(_J(); C) = \dim \mathrm{Hom}_{H_{J;E}}(_2; C):$$ Since

$_1$ $_2$ is a quotient of (),

$$d = \dim \mathrm{Hom}_{H_{J;E}}(_J(); C) \ \dim \mathrm{Hom}_{H_{J;E}}(_1; C) + \dim \mathrm{Hom}_{H_{J;E}}(_2; C) = 2d;$$

a contradiction.

Combining Propositions 6.6 and 6.7 with Lemmas 6.2(i) and 6.3(i), we deduce the following corollary which may be considered as a rst step towards the Howe duality theorem.

Corollary 6.8. Let 2 Irr(G_2) be tempered and non-generic. Then $_J$() has nite length. If $_J$() = 0, then it has a unique irreducible quotient () and () is tempered. Moreover, for $_1$; $_2$ 2 Irr(G_2) tempered and non-generic,

$$0 = (_1) = (_2) =) \quad _1 = \, _2:$$

Proof. Writing $_J$() = $_J$()$_c$ $_J$()$_{nc}$, Proposition 6.7 tells us that $_J$()$_c$ is irreducible or 0, whereas Lemma 6.2(i) tells us that $_J$()$_{nc}$ has nite length. Hence $_J$() has nite length, so that its cosocle $_J$() is a nite sum of irreducible representations. Moreover, Lemma 6.3(i) says that $_J$() is tempered, and Proposition 6.7 then shows the irreducibility of $_J$() if it is nonzero. The nal implication now follows by Proposition 6.6.

In the rest of the paper, we shall examine each of the 3 dual pairs G_2 H_J in turn and complete the proof of the Howe duality conjecture.

7. Theta Correspondence for PD G_2

In this section, we discuss the theta correspondence for the dual pair PD G_2. A preliminary study of this dual pair correspondence has been carried out by the second author in [Sa]. We rst recall the results established there.

Let $_D$ be the minimal representation of PD G_2. Then we have

$$_D = \bigoplus_{2 \, Irr(PD)} ():$$

The following was shown in [Sa, x6]:

Proposition 7.1. (i) If = 1 is the trivial representation of PD, then

$$(1) = \, _{deg}[1];$$

the unipotent discrete series representation introduced in Proposition 3.1(iii) (rst bullet point).

(ii) If is not the trivial representation, then () is nongeneric supercuspidal of nite length (possibly zero).

(iii) If = is a nontrivial unramied cubic character, then () = $_{sc}$[]

$$\text{and} \quad (^2) = \, _{sc}[^2]$$

the two depth 0 supercuspidal representations introduced in x3.5 (2) (third bullet point).

(iv) For each cubic eld extension E=F,

$$Hom_N((); \, _E) = Hom_{PE}(; C):$$

We can now easily extend the above results. More precisely,

Theorem 7.2. (i) For any 2 Irr(PD), () is an irreducible representation of G_2 if it is nonzero.

(ii) If $_1$; $_2$ 2 Irr(PD) are such that ($_1$) = ($_2$) = 0, then $_1$ = $_2$.

(iii) If $p = 3$, then the map ! () denes an injection

$$\mathrm{Irr}(PD) \,,\!\; \mathrm{Irr}(G_2):$$

Hence, the Howe duality theorem holds for $P D \; G_2$, so that

$$\dim \mathrm{Hom}_{G_2}((_1); (_2)) \;\; \dim \mathrm{Hom}_{PD}(_1; _2)$$

for any $_1; _2$ 2 $\mathrm{Irr}(PD)$. In particular, for any 2 $\mathrm{Irr}(G_2)$, the representation () of $P D$ is irreducible or zero.

Proof. The rst two parts follow from Propositions 6.6 and 6.7. As for (iii) we use

$$\mathrm{Hom}_N((); \;\; _E) = \mathrm{Hom}_{PE}(; C);$$

so it suces to show that there exists a eld E such that $\mathrm{Hom}_{P_E}(; C) = 0$. If $p = 3$, this was proved for all irreducible by [LT, Thm. 2.4].

8. Theta Correspondence for $(PGL_3 \rtimes Z=2Z) \; G_2$

In this section, we consider the theta correspondence for the dual pair $(PGL_3 \rtimes Z=2Z) \, G_2$ and prove various results analogous to those in the last section. In fact, the theta correspon-dence for $PGL_3 \;\; G_2$ was almost completely studied in [GS04]. But the treatment there ignores the outer automorphism group of PGL_3; this is akin to working with special orthog-onal groups instead of orthogonal groups in classical theta correspondence and is of course undesirable. Thus, we shall complete the results of [GS04] in their natural setting here. We extend the minimal representation of E_6 to $E_6 \rtimes Z=2Z$ so that $Z=2Z$ xes the spherical vector.

8.1. Representations of $H = PGL_3 \rtimes Z=2Z$. We realize $Z=2Z$, acting on GL_3 as a pinned automorphism preserving the standard pinning, i.e. acting via

$$A \; ! \;\; w_0 \, {}^{>}A^{\;\;1} \, w_0^{\;1} \quad \text{with} \quad @ \quad 1 \quad A$$

Let $U \;\; GL_3$ be the maximal unipotent subgroup of upper triangular matrices and let be a $Z=2Z$-invariant Whittaker character of U. Then extends to two characters of $U \rtimes Z=2Z$. Let $_1$ be the extension such that $Z=2Z$ acts trivially, and let $_{sign}$ be the other extension.

If 2 $\mathrm{Irr}(PGL_3)$, then there are two possibilities:

- if $\neq$, then

  $$^{+} := \mathrm{Ind}_{PGL_3}^{H} \;=\; \mathrm{Ind}_{PGL_3}^{H} \text{ is}$$

  irreducible. If is generic then

  $\dim \mathrm{Hom}_{U \rtimes Z=2Z}(^{+};$
  $_1) = \dim \mathrm{Hom}_{U \rtimes Z=2Z}(^{+};$
  $_{sign}) = 1:$

- if $=$, then has two extensions to H, which dier from each other by twisting with the unique quadratic character $\mathrm{sign} : H \;\; ! \;\; h1i$ of H. When is generic (for example when is tempered), we let $^{+}$ denote the unique extension of such that

  $\dim \mathrm{Hom}_{U \rtimes Z=2Z}(^{+};$
  $_1) = 1;$

and let denote the other extension.

The only nongeneric and self-dual representations of PGL_3 are the trivial representation and Langlands quotients $J_B()$, where $B = TU$ is the normalizer of U and

$$= j \quad j^{1=2}$$
$$1$$
$$j \quad j^{1=2}$$

is a character of T such that $^2 = 1$. In this case, $()$ is irreducible by [GS04, Thm. 11 and Cor. 13], and we dene $^+$ by setting $(^+) = ()$ and $( ) = 0$. Observe that 1^+ is the trivial representation of H.

It follows from the above discussion that any irreducible representation of H is self-dual.

8.2. Whittaker models. The following lemma summarizes some basic computations.

Lemma 8.1. Let be the minimal representation of split E_6 o $Z=2Z$.

(i) Let $_V : V$! C be a Whitaker character for G_2 (so V is a maximal unipotent subgroup of G_2). Then

$$_{V;\ _V} = \mathrm{ind}_{U\,oZ=2Z}^{H}$$

1: In particular, for any 2 $Irr(H)$,

$$\mathrm{Hom}_V((); \ _V) = \mathrm{Hom}_{U\,o\,Z=2Z}(;$$
$$1)$$

(ii) For any etale cubic F-algebra, we have:

$$\mathrm{Hom}_N((); \ _E) = \mathrm{Hom}_{P_E\,o\,Z=2Z}(; C):$$

8.3. Our earlier results. The following is a simple combination of the results of [GS04, Thms. 11, 14, 15 and 21] and the previous discussion:

Theorem 8.2. For 2 $Irr(PGL_3)$, let $: WD_F$! $SL_3(C)$ denote the L-parameter of . If is non-supercuspidal, then $()$ has nite length. If $_{-}$ is supercuspidal, then $()$ is irreducible supercuspidal. (This covers all 2 $Irr(PGL_3)$ if $p = 2$). In these cases, set $()$ to be the maximal semisimple quotient of $()$ for = .

More precisely, we have:

(i) If = –, then $(^+)$ is irreducible and nonzero. If is generic, or supercuspidal, or a discrete series representation, or tempered, so is $(^+)$. When is not supercuspidal, then $(^+)$ is not supercuspidal and its L-parameter is obtained by composing with the inclusion $SL_3(C)$ $G_2(C)$.

(ii) If = – and the parameter contains the trivial representation, then $( ) = 0$ and $(^+)$ is nonzero irreducible, non-discrete-series and its L-parameter is obtained by composing with the inclusion $SL_3(C)$ $G_2(C)$.

(iii) If = – and the parameter does not contain the trivial representation, then we have the following cases:

= St, the Steinberg representation. Then

$$(St^+) \ (St \ ) = {}_{gen}[1] \ {}_{sc}[1]$$

where $_{gen}[1]$ is the generic discrete series representation introduced in Proposition 3.1(ii) and $_{sc}[1]$ is the depth 0 supercuspidal representation introduced in x3.5 (2).

is a tempered representation induced from a supercuspidal representation = – of GL_2 with a non-trivial central character. Then

$$(^+)\ (\ ) = \mathrm{Ind}_P{}^2() = \mathrm{Ind}_P{}^2()_{gen}\ \mathrm{Ind}_P{}^2()_{deg} \qquad G$$

is a tempered principal series induced from a triple of non-trivial quadratic characters $(_1;_2;_3)$ such that $_1\ _2\ _3 = 1$ then

$$(^+)\ (\ ) = \mathrm{Ind}_B{}^2() = \mathrm{Ind}_B{}^2()_{gen}\ \mathrm{Ind}_B{}^2()_{deg} \qquad G$$

where is the quadratic character of T determined by $(_1;_2;_3)$ as in x3.4.

is a self-dual supercuspidal representation (so p = 2). Then () is supercuspidal and

$$(^+)\ (\ ) = {}_{gen}\ {}_{deg}$$

where $_{gen}$ is a generic irreducible supercuspidal representation, while $_{deg}$ is a non-generic supercuspidal representation of unknown length.

Observe that the only case for which we do not know that () has nite length (and hence () is dened) is when is a self-dual supercuspidal representation (so p = 2). In this case, however, the last bullet point states that () is supercuspidal and hence semisimple. Hence, even in this exceptional case, we may set () = (). Moreover, observe that if is nontempered, then () is irreducible nontempered and is completely determined by Theorem 8.2. On the other hand, when is tempered, then so is every irreducible summand of (). In particular, the results highlighted in Lemma 6.2 and 6.3 hold in this case.

In the rest of the section, we shall complete the results above by completing the unresolved parts of the theorem.

8.4. A miracle of Oberwolfach. Let 2 Irr(PGL_3) be a self-dual supercuspidal representation. The goal here is to show that ( ) = 0. Let Q = LU be the 3-step maximal parabolic subgroup of G_2. Recall that the group U has the 3-step ltration

$$U\ [U;U]\ Z_U$$

where Z_U is the 2-dimensional center of U and $U=Z_U$ is a 3-dimensional Heisenberg group with the center $[U;U]=Z_U$. Let be a non-trivial character of [U;U], trivial on Z_U. Then $_{[U;U];}$ is naturally a (PGL_3 o Z=2Z) SL_2-module, where SL_2 = [L; L]. In order to describe $_{[U;U];}$, we need some additional notation.

Consider the action of the group GL_2 o Z=2Z on M_2, the space of 2 2 matrices, with elements in $GL_2(F)$ acting by conjugation and the nontrivial element of Z=2Z acting via:

$$X\ !\ \begin{matrix}0 & 1\\ 1 & 0\end{matrix}\ X^{>}\ \begin{matrix}0 & 1\\ 1 & 0\end{matrix}\ :$$

This action preserves the determinant (quadratic) form on $M_2(F)$ and descends to the quotient group

$$PGL_2(F)\ Z=2Z = f1g\ SO_3 = O_3:$$

On the space $C_c(M_2(F))$, we have a Weil representation of O_3 SL_2, which we may regard as a representation of $GL_2(F) \rtimes Z/2Z$. Then the following lemma follows by a standard computation:

Lemma 8.3. We have an isomorphism of $(PGL_3 \rtimes Z/2Z)$ SL_2-modules:

$$_{[U;U];} \quad = \mathrm{ind}_{GL_2 \rtimes Z/2Z}^{PGL_3 \rtimes Z/2Z}(C_c(M_2(F))$$

where GL_2 is embedded in PGL_3 via

$$\begin{matrix} a & c \\ b & d \end{matrix} \; ! \; @ \begin{matrix} 0 & & \\ a & & b \\ & 1 & \\ c & & d \end{matrix} \; \begin{matrix} 1 \\ A \end{matrix} :$$

Using the lemma, we can now prove:

Proposition 8.4. Let (2 $\mathrm{Irr}(PGL_3)$ be a self-dual supercuspidal representation. Then) = 0.

Proof. It suces to show that is a quotient of $_{[U;U];}$, in fact we shall show that is a quotient of SL_2-coinvaraints of $_{[U;U];}$. Decompose $M_2(F) = M (F) F$, where $M_2(F)$ is the subspace of trace 0 elements and F is the center. Accordingly, the Weil representation of O_3 $\widetilde{SL}_2$ on $C_c(M_2(F))$ decomposes as a tensor product

$$C_c(M_2(F)) = C_c(M_2(F))$$

$$C_c(F) \quad f$$

where O_3 acts trivially on $C_c(F)$ and SL_2 acts by the Weil representations . Recall that as an $\widetilde{SL}_2$-module, decomposes as a sum

$$= {}^{+}$$

of even and odd Weil representations. Let $^{+}$() and () be the theta lifts of their contragredients to O_3, via the Weil representation on $C_c(M_2(F))$ with respect to $_2$. Thus the SL_2-coinvariant of $_{[U;U];}$ is given by

$$\mathrm{ind}_{GL_2 \rtimes Z/2Z}^{PGL_3 \rtimes Z/2Z}(()) \; \mathrm{ind}_{GL_2 \rtimes Z/2Z}^{PGL_3 \rtimes Z/2Z}(()):$$

Let st be the Steinberg representation of $SO(3) = PGL_2$. We extend st to two representations st^{+} and st of $O(3)$ by letting 1 2 $O(3)$ act by 1 and 1 respectively. Then () = st while () is^{+}the principal series representation with the trivial representation as a quotient and st^{+} as a submodule. Since is a supercuspidal representation, it suces to show that it is a quotient of

$$\mathrm{ind}_{GL_2 \rtimes Z/2Z}^{PGL_3 \rtimes Z/2Z}(st^{+}) \; \mathrm{ind}_{GL_2 \rtimes Z/2Z}^{PGL_3 \rtimes Z/2Z}(st \quad):$$

It is known that any generic representation of GL_2 , in particular st, is a quotient of . Hence either st^{+} or st is a quotient of . Now the proposition follows from Frobenius reciprocity.

8.5. Main result. We shall now strengthen Theorem 8.2.

Theorem 8.5. For any 2 Irr(PGL_3), let : W D_F ! SL_3(C) denote the L-parameter of .

(i) The representation () is zero if and only if contains the trivial representation (so = −) and = .

(ii) For any = , () has nite length with unique irreducible quotient () (if it is nonzero).

(iii) () is generic if and only if is generic and = +.

(iv) Suppose that () = 0. If is a discrete series (resp. tempered) representation, so is (). Moreover, () is supercuspidal if and only if is supercuspidal or = St .

(v) If (^{1}) = (^{2}) = 0, then $_1{}^1$ = $_2{}^2$.

In particular, the Howe duality theorem holds for the dual pair (PGL_3 o Z=2Z) G_2:

$$\dim \mathrm{Hom}_{G_2}((_1{}^1); (_2{}^2)) \quad \dim \mathrm{Hom}_{PGL_3 o Z=2Z}(_1{}^1; _2{}^2)$$

for any $^1_1$; $^2_2$ 2 Irr(PGL_3 o Z=2Z). Moreover, for 2 Irr(G_2), () is a nite length representation of PGL_3 with a unique irreducible quotient (if nonzero).

Proof. (i) From Theorem 8.2, it remains to show that () is nonzero for those represen-tations as in Theorem 8.2(iii) and any = . Consider rst the Steinberg representation. Recall that $_{gen}$[1] is generic while $_{sc}$[1] is not. It follows, from Lemma 8.1 part (ii), that $_{gen}$[1] is a summand of (St). Furthermore, by Proposition 6.6, $_{sc}$[1] cannot be a summand of (St). Hence

$$(St^+) = {}_{gen}[1] \text{ and } (St \ ) = {}_{sc}[1]:$$

The same argument works in the other three cases to show that ($^+$) is the generic G_2 summand and ( ) is the degenerate summand. Moreover, in the last case of Theorem 8.2, where is a self-dual supercuspidal representation (so p = 2), we deduce by Proposition 6.6 again that $_{deg}$ is nonzero irreducible.

(ii) This follows from Theorem 8.2 and the irreducibility of $_{deg}$ in the proof of (i) above.

(iii) and (iv): These summarize what we already know from Theorem 8.2.

(v) Suppose that

$$:= (_1{}^1) = (_2{}^2) = 0:$$

If is non-supercuspidal, then $_1$ and $_2$ are both non-supercuspidal. The desired equality $_1{}^1$ = $_2{}^2$ follows from the results of [GS04, Thms. 11, 14 and 15] and our new understanding in (i) (which determines () for those in Theorem 8.2(iii))

Suppose that is supercuspidal. Then $_i{}^i$ is either supercuspidal or St , in which case both $_1$ and $_2$ are generic discrete series representations. By (iii), we deduce that $_1$ = $_2$. Hence, it remains to show that $_1$ = $_2$ or $_2$. We now consider the following two cases:

(a) Suppose $_1$ − and $_2$ −. Then $_1$ = $_2$ = + and is an irreducible generic supercuspidal representation.

Consider, for i = 1 or 2, the induced representation $\mathrm{Ind}_{P_3}^{PGSp_6}(_i)$, where P_3 is the Siegel parabolic subgroup. Its normalized Jacquet functor with respect to P_3 is –. Since $_i$ $_i$ = –, it follows that $\mathrm{Ind}_{P_3}^{PGSp_6}(_i)$ is an irreducible generic tempered representation.

By the computation of the Jacquet module of the minimal representation ^{0} of G_2 $PGSp_6$ along P_3 given in [MS, Thm. 5.3], we deduce that $\mathrm{Ind}_{P_3}^{PGSp_6}(_i)$ is an irreducible quotient of . By [GS04, Prop. 19 and Cor. 20], a generic representation of G_2 cannot lift to two dierent generic representations of $PGSp_6$. Hence, we must have

$$\mathrm{Ind}_{P_3}^{PGSp_6}(_1) \cong \mathrm{Ind}_{P_3}^{PGSp_6}(_2):$$

By consideration of the Jacquet modules with respect to P_3, we see that $_2$ $_1$ or $_1$, as desired.

(b) Assume now that $_1$ = $_1$. In this case, we know that ($_1$)

$$= {}^{+}_{gen} \qquad \text{and} \qquad (_1) = {}_{deg}:$$

Moreover, the tempered representation $\mathrm{Ind}_{P_3}^{PGSp_6}(_1)$ is the sum of two representations, one of which is generic and the other degenerate (see Proposition 10.3(i)). By the Jacquet module of ^{0} again, we see that both $_{gen}$ and $_{deg}$ lifts to irreducible summands of $\mathrm{Ind}_{P_3}^{PGSp_6}(_1)$. Moreover, $_{deg}$ cannot lift to a generic representation of $PGSp_6$ and hence must lift to the degenerate summand [GS04, Prop. 19]. By Proposition 6.6, it follows that $_{gen}$ cannot lift to the degenerate summand and thus must lift to the generic summand.

Now suppose that $_1$ = $_2$ = +, so that = ($^+$) = ($^+$) is generic. Then as before, we see that lifts to the generic summand of $\mathrm{Ind}_{P_3}^{PGSp_6}(_i)$ (regardless of whether $_2$ is self-dual or not). By Jacquet module consideration, we see that $_1$ = $_2$. On the other hand, if $_1$ = $_2$ = , so that = ( ) = ( ) is nongeneric, then Proposition 6.7 implies that the nongeneric summand of $\mathrm{Ind}_{P_3}^{PGSp_6}(_1)$ is contained in $\mathrm{Ind}_{P_3}^{PGSp_6}(_2)$. Again, Jacquet module considerations show that $_1$ = $_2$.

The inequality at the end of the theorem is simply a restatement of (v). Finally, given 2 $\mathrm{Irr}(G_2)$, we write

$$() = ()_c \; ()_{nc}$$

as a sum of its cuspidal and noncuspidal component. As we noted in Lemma 6.2, the results of [GS04] imply that $()_{nc}$ has nite length. The result in (v) shows that () has a unique irreducible quotient if it is nonzero, implying in particular that $()_c$ is either 0 or irreducible, and hence () has nite length.

9. The group $PGSp_6$

Before discussing the last dual pair G_2 $PGSp_6$, we need to devote the next few sections to a discussion of the structure and representations of $PGSp_6$, as well as certain particular periods on G_2 and $PGSp_6$.

Let $e_1, \dots, e_6$ be the standard basis of F^6, where we have a symplectic form defined by

$$\omega(e_1, e_6) = \omega(e_2, e_5) = \omega(e_3, e_4) = 1$$

and all other $\omega(e_i, e_j) = 0$ with $i < j$. Let GSp_6 be the group of linear transformations g of F^6, such that for some $\lambda(g) \in F^\times$

$$\omega(gv, gw) = \lambda(g) \cdot \omega(v, w)$$

for all $v, w \in F^6$. Then $\lambda : GSp_6 \to F^\times$ is the similitude character.

Let $\tilde{P}_1$, $\tilde{P}_2$ and $\tilde{P}_3$ be maximal parabolic subgroups of GSp_6 defined as the stabilizers of subspaces

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle$$

respectively. For $i = 1, 2, 3$, let $P_i \subset PGSp_6$ be the quotient of $\tilde{P}_i$ by the center of GSp_6. The group $PGSp_6$ acts faithfully on $J = \wedge^2 F^6 \otimes \lambda^{-1}$, and we shall (partially) describe how the parabolic subgroups act on this module.

The group $PGSp_6$ can be explicitly described in terms of its action on J as follows. Let $x_{ij} = e_i \wedge e_j \in J$ for $i \neq j$. On J, we have a natural trilinear form (x, y, z)

$$\wedge^2 F^6 \times \wedge^2 F^6 \times \wedge^2 F^6 \to \wedge^6 F^6 = F.$$

The group of linear transformations of J preserving this form is SL_6/μ_2 and $PGSp_6 \cong Sp_6/\mu_2$ is the subgroup fixing

$$e = x_{16} + x_{25} + x_{34}.$$

The Levi factor M_3 of P_3, as an algebraic group, is isomorphic to GL_3/μ_2. Observe that group acts faithfully on $\wedge^2 F^3$, and since the latter is a three dimensional vector space, this action gives an isomorphism $GL_3/\mu_2 \cong GL_3$. Thus we have an identification

$$M_3 \cong GL(\langle x_{12}, x_{13}, x_{23} \rangle).$$

Under this identification, the maximal torus is given by diagonal matrices (t_1, t_2, t_3). The three simple co-roots of $PGSp_6$ are, respectively,

$$\alpha_1^\vee(t) = (1, t, t^{-1}); \quad \alpha_2^\vee(t) = (t, t^{-1}, 1); \quad \alpha_3^\vee(t) = (1, t, t).$$

An unramified character χ of the maximal torus is given by a triple of complex numbers (s_1, s_2, s_3)

$$\chi(t_1, t_2, t_3) = |t_1|^{s_1} |t_2|^{s_2} |t_3|^{s_3}.$$

The Weyl group action on the characters is somewhat different in this picture. The simple reflections corresponding to the first two roots α_1 and α_2 are the usual permutations of entries of (s_1, s_2, s_3), however, the simple reflection corresponding to the third simple root α_3 is given by

$$(s_1, s_2, s_3) \mapsto (s_1 + s_2 + s_3, -s_3, -s_2).$$

Thus the root hyperplanes are $s_i - s_j = 0$ and $s_i + s_j = 0$ for short and long roots, respectively. This looks like a D_3 root system; however, the Weyl-invariant quadratic form in this case is

$$q(s_1, s_2, s_3) = s_1^2 + s_2^2 + s_3^3 - \frac{1}{4}(s_1 + s_2 + s_3)^2$$

rather than the usual dot product, and with this form, we have a realization of the C_3 root system with simple roots

$$_1 = (0;1; \quad 1); \ _2 = (1; \quad 1;0); \ _3 = (0;2;2):$$

This somewhat unconventional description of the C_3 root system is a source of potential confusion, as one has the tendency to confound it with the more familiar description of the root system of Sp_6, but what we have done here is denitely the natural way to set things up for $PGSp_6$.

The character is in the positive chamber if for every positive root , (–(t)) = jtj^s for some s 2 C such that <(s) > 0 (the real part). One checks that is positive if

$$<(s_1) > <(s_2) > j<(s_3)j:$$

The modulus character of $M_3 = GL_3$ is

$$_{P_3}(m) = j\det(m)j^2:$$

It follows that the Levi factor M_{13} of $P_{13} = P_1 \setminus P_3$ is

$$M_{13} = GL(hx_{12};x_{13}i) \ GL(hx_{23}i):$$

The group P_{13} is the stabilizer of the space $V_2 = hx_{12};x_{13}i$.

Consider now the group P_2 and its Levi factor M_2. The standard Levi factor of P_2 is $\widetilde{G}L_2$ GL_2 where the rst GL_2 acts on $he_1;e_2i$ in the standard way, xes $he_3;e_4i$ and acts by transpose-inverse on $he_5;e_6i$. The second GL_2 acts on $he_3;e_4i$ in the standard way, by det on $he_1;e_2i$ and xes $he_5;e_6i$. The group P_2 is the stabilizer of the singular line $V_1 = hx_{12}i$, and the Levi factor M_2 acts faithfully on the 4-dimensional subspace

$$V_4 = hx_{13};x_{23};x_{14};x_{24}i$$

preserving the quadratic form x ! (x;x;x$_{56}$). If we identify $x = ax_{14} + bx_{13} + cx_{24} + d_{23}$ with the matrix

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

then $(x;x;x_{56}) = 2\det(x)$. Thus, with V_4 identied with the set of 2 2 matrices, we have M_2

$$= GL_2 \ GL_2 = GL^r_1 \quad \text{where } GL^r_1 = f(t;t^{\ 1}) : t\ 2\ GL_1g;$$

so that (;) 2 M_2 acts on x 2 V_4 by x ! x where is the transpose of . The element (;) acts on the line $hx_{12}i$ by det(). The modulus character is

$$_{P_2}((;)) = j\det()j^5:$$

This sets up the necessary notation to discuss the representations of $PGSp_6$.

10. Representations of $PGSp_6$

In this section, we list some irreducible non-supercuspidal representations of $PGSp_6(F)$, relevant to this work. Observe that the local Langlands correspondence is known for the Levi factors of all proper parabolic subgroups of $PGSp_6(F)$ (by Gan and Takeda [GT11] for $M_1 = GSp_4$). Thus, following Shahidi [Sh1], reducibility points of generalized principal series can be computed using L-functions of Langlands parameters.

10.1. **Principal series representations for P_2.** We rst consider certain principal series representations for the parabolic subgroup $P_2 = M_2N_2$, where $M_2 = GL_2 \;\; GL_2{=}GL^r_1$. Let be an irreducible representation of GL_2 with L-parameter and central character ! . Set

$I_2($
$) = \mathrm{Ind}^{PGSp_6}_{P_2}$ and $I_2(s;$
$) = \mathrm{Ind}^{PGSp_6}_{P_2}(j\det j^s)$
$(j\det j^s)$

if we need to consider a family of induced representations. Then we have:

Proposition 10.1. (i) If 2 $\mathrm{Irr}(GL_2(F))$ is unitary supercuspidal, then $I_2(s;$) is reducible if and only if = (so^2! = 1) and one of the following holds:

! = 1 and s = 1=2, in which case one has:

$0 \;\; ! \;\; _2() \;\; ! \;\; I_2(1{=}2;$
$) \;\; ! \;\; J_2(1{=}2;$
$) \;\; ! \;\; 0;$

where $_2()$ is a generic discrete series representation.

! = 1 (so is dihedral), Im( ) = S_3 (the symmetric group on 3 letters, regarded as a subgroup of $GL_2(\mathbb{C})$) and s = 1, in which case one has:

$0 \;\; ! \;\; _{gen}[] \;\; ! \;\; I_2(1;$
$) \;\; ! \;\; J_2(1;$
$) \;\; ! \;\; 0;$

where $_{gen}[]$ is a generic discrete series representation.

! = 1, Im( ) = S_3 (the symmetric group on 3 letters, regarded as a subgroup of $GL_2(\mathbb{C})$) and s = 0, in which case one has:

$I_2($
$) = I_2($
$)_{gen} \;\; I_2($
$)_{deg}$

where $I_2($
$)_{gen}$ is generic.

(ii) If = st is a twisted Steinberg representation, then $I_2(s;$) is irreducible except for the following cases:

= 1 and s = 5=2 or 1=2, in which case one has

$0 \;\; ! \;\; St_{PGSp_6} \;\; ! \;\; I_2(5{=}2; st$
$st) \;\; ! \;\; J_2(5{=}2; st$
$st) \;\; ! \;\; 0;$

and

$0 \;\; ! \;\; \mathrm{Ind}^{PGSp_6}_{P_3}(St)_{gen} \;\; ! \;\; I_2(1{=}2; st$

st) ! $J_2(1=2;$ st
st) ! 0;

where $\mathrm{Ind}_{P_3}^{PGSp_6}(\mathrm{St})_{gen}$ is the generic summand of $\mathrm{Ind}_{P_3}^{PGSp_6}(\mathrm{St})$.

^{2} = 1 but = 1 and s = 1=2, in which case one has:

0 ! $_{gen}[]$! $I_2(1=2;$ st
st) ! $J_2(1=2;$ st
st) ! 0;

where $_{gen}[]$ is a generic discrete series representation.

10.2. Principal series representations for P_{13}. Now we consider certain principal series representations for the parabolic subgroup P_{13} = $M_{13}N_{13}$, where M_{13} = GL_2 GL_1. Let be an irreducible representation of GL_2 with the central character ! and L-parameter .
Set

$I_{13}($

$1) = \mathrm{Ind}^{PGSp_6}$ P_{13} P_{13}

1 and $I_{13}(s;$

$1) = \mathrm{Ind}^{PGSp_6}(j \det j^s)$

1

if we need to consider a family of induced representations. In the more familiar language of representations of $_6$ Sp , the restriction of I $_{13}$ $_6$ (s; 1) to Sp is a principal series induced from j j^{2s}! j det j^s. In particular, if is unitary tempered and s > 0, this is a standard module. We have:

Proposition 10.2. If 2 $Irr(GL_2(F))$ is unitary supercuspidal, then $I_{13}(s;$
1) is reducible if and only if ^{2} = (so ! = 1) and one of the following holds:

! = 1 and s = 1=2, in which case $I_{13}(1=2;$
1) has length 4 and has a unique irreducible submodule $_{13}()$, which is a generic discrete series representation.

! = 1 and s = 0, in which case one has:

$I_{13}($

$1) = I_{13}($

$1)_{gen}$ $I_{13}($

$1)_{deg}$ where $I_{13}($

$1)_{gen}$ is generic.

10.3. Principal series representations for P_3. Now we consider certain principal series representations for the parabolic subgroup P_3 = M_3N_3, where M_3 = GL_3. Let be an irreducible representation of GL_3. We set

$$I_3() = \mathrm{Ind}^{PGSp_6}_{P_3}:$$

Proposition 10.3. (i) Assume that is discrete series representation with trivial central character. Then we have two cases:

If = – then

$$I_3() = I_3(-)$$

is irreducible.

If = – then

$$I_3() = I_3()_{gen} \quad I_3()_{deg}$$

where $I_3()_{gen}$ is generic.

(ii) Let $_1;_2;_3$ be three characters of F such that $_1$ $_2$ $_3$ = 1, and let = $(_1;_2;_3)$ be the associated principal series representation of $GL_3(F)$ (which is possibly reducible). Then the induced representation $I_3()$ is irreducible unless one of the following two conditions hold:

$_i$ = j j^1 for some i or $_i$=$_j$ = j j^1 for a pair i = j.

The three characters $_i$ are quadratic, non-trivial and pairwise dierent. Then $I_3()$

$$= I_3()_{gen} \quad I_3()_{deg}$$

where $I_3()_{gen}$ is generic.

Proof. These are some old results for representations of $GSp_6(F)$ translated to our setting. Following [Ta, x3], the Levi subgroup of the Siegel parabolic subgroup in $GSp_6(F)$ is isomorphic to $GL_3(F)$ $GL_1(F)$ such that, under this isomorphism, the center of $GSp_6(F)$ corresponds to the image of the map : $GL_1(F)$! $GL_3(F)$ $GL_1(F)$ dened by ! (;) where 2 F . It follows that

$$M_3(F) = (GL_3(F) \ GL_1(F))=GL_1(F);$$

a rather awkward description of a group isomorphic to $GL_3(F)$. However, by our identication M_3 = GL_3, the above isomorphism is given by the map g ! (g; det g). Keeping in mind that our have the trivial central character, it follows that our $I_3()$ is given by o 1 in the notation of Tadic.

With this translation, we now treat each part of the proposition in turn:

(i) By [Sh2], the representation o 1 reduces if and only if – and the exterior square L-function of has no pole at s = 0. In [Sh2], however, the result is stated for representations of $Sp_6(F)$. The variant of that result for representations o of $GSp_6(F)$ involves a twist by the character . Since, in our case, = 1 we have what we stated. Now observe that the exterior square L-function of is the same as the standard L-function of – (a simple observation that the exterior square of the standard representation of $SL_3(C)$ is the dual of the standard representation). Since the standard L-function of a square integrable representation of $PGL_3(F)$ has no pole at 0, it follows that $I_3()$ reduces if and only if = –.

(ii) This is [Ta, Example 7.7 and Theorem 7.9]. Observe that the condition $_{ij}$ = j j^1 is redundant since $_{123}$ = 1.

Remark: Observe that the description of reducibility points in Proposition 10.3 (ii) matches perfectly those for G_2 in Proposition 3.3. In fact these, and many other reducibility results for induced representations of $PGSp_6(F)$ stated in this section, can be derived using the theta lifting from G_2, see [GrS1] for an illustration of this idea.

10.4. Principal series representations for P_1. Now we consider certain principal series representations for the parabolic subgroup $P_1 = M_1N_1$, where $M_1 = GSp_4$. Let be an irreducible representation of GSp_4. We set

$$I_1() = \mathrm{Ind}_{P_1}^{PGSp_6} \quad \text{and} \quad I_1(s;) = \mathrm{Ind}_{P_1}^{PGSp_6} jj^s$$

where is the similitude character of GSp_4. Let be an irreducible supercuspidal representation of $GSp_4(F)$ with trivial central character. Let ’ : WD_F ! $Spin_5 = Sp(4)$ be its Langlands parameter [GT11].

Proposition 10.4. Assume that is a supercuspidal representation of $GSp_4(F)$ with trivial central character such that the parameter std ’ contains the trivial representation, where std denotes the 5-dimensional standard representation of $Spin_5$. Then $I_1(s;)$ is reducible if and only if s = 1=2, in which case one has:

$$0 \ ! \ _1() \ ! \ I_1(1=2;) \ ! \ J_1(1=2;) \ !$$

0; where $_1()$ is a discrete series representation.

11. Fourier-Jacobi and Shalika periods

In this section, we introduce and study a Fourier-Jacobi-type model for the group G_2 and a Shalika period for $PGSp_6$. These are some of the periods that will appear when we consider a game of ping-pong with periods for the dual pair G_2 $PGSp_6$, as discussed at the end of the introduction.

11.1. Whittaker periods. We begin by recalling the following results about Whittaker periods from [GS04, Prop. 19 and Cor. 20], see also the appendix of [HKT].

Proposition 11.1. Let be the minimal representation of E_7 and let $(V^0; \ {}_{V^0})$ be a Whittaker datum for $PGSp_6$ (so V^0 is a maximal unipotent subgroup and ${}_{V^0}$ a generic character of V^0). Then we have an isomorphism of G_2-modules:

$$_{V^0; \ {}_{V^0}} = \mathrm{ind}_V^{\ 2G} \ {}_V$$

where $(V; \ {}_V)$ is a Whittaker datum for G_2.

Corollary 11.2. (i) If 2 $\mathrm{Irr}(G_2)$ is generic and () is its big theta lift to $PGSp_6$, then

$$\dim \mathrm{Hom}_{V^0}((); \ {}_{V^0}) = 1$$

so that () contains a unique irreducible generic subquotient and thus is nonzero. (ii)

If 2 $\mathrm{Irr}(G_2)$ is non-generic and 2 $\mathrm{Irr}(PGSp_6)$ is generic, then

$$\mathrm{Hom}_{G_2 PGSp_6}(; \ ) = 0:$$

11.2. Fourier-Jacobi period of G_2. Let $Q = LU$ be the 3-step maximal parabolic subgroup of G_2. Recall that $[L; L] = SL_2$ corresponds to the long simple root . Thus $V = UU$ is the unipotent radical of the standard Borel subgroup of G_2. If we set $J = [L; L]U$, then the quotient of J by the two-dimensional center Z_U of U is the Jacobi group with one-dimensional central subgroup $[U; U]=Z_U = U_{2+}$. Fix a non-trivial additive character of $U_{2+} = F$. Let be the unique irreducible representation of the 2-fold cover $\tilde{J}$, trivial on Z_U, such that U_{2+} acts by . If is a genuine representation of $\widetilde{SL}_2$, we have a representation of J on . For 2 $\mathrm{Irr}(G_2)$, the Fourier-Jacobi period of with respect to is the space

$$\mathrm{Hom}_J(; \ ) = \mathrm{Hom}_{G_2}(; \mathrm{Ind}_J^{\ 2G} \ ):$$

The character denes a Weil index, that is, a function $: F \ ! \ C$ with values in roots of one such that (a) (b) = (a; b) (ab) where (a; b) is the Hilbert symbol. Let be a smooth character of F. Let I (s) be a principal series representation of $\widetilde{SL}_2$ obtained by inducing jj^s, via normalized induction. We can x our data so that, for $s = 1=2$, there is a short exact sequence:

$$0 \ ! \ St \ ! \ I \ (1=2) \ ! \ {}^+ \ ! \ 0;$$

where $^+$ is the even Weil representation, i.e. a summand of . The contragredient of I (s) is I (s).

Proposition 11.3. As a representation of G_2, $\mathrm{ind}_J^{G_2}(I\ \ (s)$
ltration) admits an equivariant

$$0 \quad I_0 \quad I_1$$

with successive quotients described as follows:

$I_0 = \mathrm{ind}_V^{G_2}\ \ _V;$

$J_1 := I_1 = I_0 \quad I_{\rho}(\frac{s}{2} + \frac{1}{4};\ \mathrm{ind}_N^{PGL_2}\ \ )$

Proof. Let $B = TU\ \ [L;L]\ \ SL_2$ be a Borel subgroup, where T is the one-dimensional torus, the image of the simple coroot $\ :GL_1\ \ !\ \ T$. Observe that we have an isomorphism of J-modules

$$I\ \ (s) = \mathrm{ind}^J_U(\ j\ j^{s+1}\ ^B\ )$$

where f

$v\ 2\ I\ \ (s)$ f $\sim$

is mapped to a function on J given by $g\ !\ \ f(g)\ \ (g)(v)$. (Here f is inated from SL_2 to J.) The later induction is not normalized.

Let N be the unipotent radical of the maximal parabolic P. Let $_{2+}$ be a character of N nontrivial only on the root space $U_{2+}\ \ N$. Then , restricted to BU, is induced from a character of TN equal to $j\,j^{1=2}$ on T and $_{2+}$ on N. Using transitivity of induction, and $= 1$, it follows that

$$I\ \ (s) = \mathrm{ind}^J_N(j\ \ j^{s+3=2}\ ^T\ \ _{2+});$$

and hence

$$\mathrm{ind}_J^{G_2}(I\ \ (s)) = \mathrm{ind}_{T}^{G_2}{}_N(j\ \ j^{s+3=2}\ \ _{2+}):$$

The next step requires the technique of root exchange, as in the proof of Proposition 3.4. Let U^0, a conjugate of U, be obtained by adding U to N and removing U_{3+} from N. The root exchange is an isomorphism

$$\mathrm{ind}^{G_2}_{TN}(j\ \ j^{s+3=2}\ \ _{2+}) = \mathrm{ind}^{T}{}_{U^0}(j\ {}^{G_2}j^{s+5=2}\ \ _{2+});$$

$f\ \ !\ \ f^0$ given by

$$f^0(g) = \int_U f(ug)\,du$$

where, abusing notation, $_{2+}$ is also viewed a character of U^0 supported on the root space U_{2+}.

As the last step, let $V^0 = U^0 U\ _3$. Then V^0 is the unipotent radical of a Borel subgroup of G_2 such that the simple roots are $2+$ and 3 , and the highest root is . Consider

$$\mathrm{ind}^{T\ V^c}_{U^0}(j\ \ j^{s+5=2}\ \ _{2+}) = C_c(U\ \ _3\ ):$$

We can analyze this module using the Fourier transform on $C_c(U\ \ _3\ )$. This gives an exact sequence of TV^c-modules

$$0\ !\ \ \mathrm{ind}^{T\ V^0}_{V^0}(\ \ _{V^0})\ !\ \ \mathrm{ind}^{T\ V^c}_{U^0}(j\ \ j^{s+5=2}\ \ _{2+})\ !\ \ j\ \ j^{s+7=2}\ \ _{2+}\ !\ \ 0$$

where $_{V^0}$ is a Whittaker character of V^0 and, in the last term, $_{2+}$ a character of V^0 supported on the root space U_{2+}. The lemma follows by induction in stages, giving $\mathrm{ind}^{G_2}_{V^0}\ \ _{V^0}$

as a submodule and the claimed quotient, after taking into account the relevant normalization for the parabolic P^0 with unipotent radical N^0 such that $V^0 = N^0 U_{2+}$.

Corollary 11.4. Let 2 $Irr(G_2)$ be generic and tempered. If $s >$ 1=2, then

$$Hom_{G_2}(ind_J^{G_2}(I\ \ (s)$$

$$);) = C:$$

Proof. We need to show that we can avoid the top piece of the ltration in Proposition 11.3. By the Frobenius reciprocity,

$$Ext^i{}_{G_2}(I_P(\frac{s}{2} + \frac{1}{4}; ind_N^{PGL_2}\ \ );) = Ext^i{}_M(j \det j^{s+1=2}\ ind_N^{PGL_2}\ \ ; r_P()):$$

Since is tempered, the center of M GL_2 acts on $r_P()$ by characters such that $jj = jj^t$ with t 0. Since $s >$ 1=2, all Ext groups vanish and is a quotient of $Hom_{G_2}(ind^{G_2}(J\ \ (s)$) since it is generic.

11.3. Shalika period on $PGSp_6$. We shall now discuss a Shalika period on $PGSp_6$.

Recall the maximal parabolic subgroup $P_2 = M_2N_2$ of $PGSp_6$, with identications of the Levi factor $M_2 = (GL_2\ \ GL_2)=GL_1^r$ and of the maximal abelian quotient $N_2=[N_2; N_2]$ of the unipotent radical N_2 with M_2, the space of 2 2 matrices. With these identications, let $_2$ be a character of $N_2(F)$ obtained by composing the trace on $M_2(F)$ with a non-trivial additive character of F. Then the stabilizer of $_2$ in the Levi group M_2 is the diagonally embedded PGL_2. The Shalika subgroup of $PGSp_6$ is the semi-direct product

$$S = PGL_2\ n\ N_2$$

and the Shalika character $_S$ is the character $_2$ extended to $S(F)$ (trivially on $PGL_2(F)$). For any smooth representation of $PGSp_6(F)$, the Shalika period of is the coinvariant space $_{S;\ \ S}$.

This Shalika period has already been exploited in [SWe]. Indeed, the following was shown in [SWe, Lemma 4.5]:

Proposition 11.5. Let be the minimal representation of E_7 and $(V;\ \ _V)$ a Whittaker datum for G_2. Then

$$_{V;\ \ V} = ind_S^{PGSp_6}\ \ _S$$

as $PGSp_6$-modules.

11.4. Shalika period of . We now consider the minimal representation of the dual pair $G_2\ PGSp_6$ and determine its Shalika period $_{S;\ \ S}$ as a representation of $G_2(F)$. To describe the answer, we need to introduce some more notations.

The group PGL_2 acts by conjugation on M_2 preserving the determinant (quadratic) form. As we saw in x8.4, there is a Weil representation of $PGL_2\ SL_2$ on $C_c(M_2(F))$ which decom-poses as a tensor product

$$C_c(M_2(F)) = C_c(M_2(F))$$

where $M_2(F)$ is the space of trace zero matrices. We view $C_c(M_2(F))$ as a representation of the group $J = [L; L]U$ G_2 introduced in x11.2, where the rst factor is a representation of SL_2 and is the irreducible representation of J introduced in x11.2. With the group PGL_2 acting trivially on , we see that $C_c(M_2(F))$ becomes a representation of PGL_2 J.

We are now ready to compute $_{S;}$ $_{S}$. Firstly, we need $_{N\ ;_2\ _2}$. This is a twisted variant of $_{N\ ,_2}$ given by Proposition 13.7, and computed along the same lines. In fact, since the character $_2$ is generic, instead of a ltration we end up with a single term:

$$_{N_2;\ _2} = \mathrm{ind}_J^{G_2}(C_c(M_2(F)) \quad)$$

as G_2 PGL_2-modules. It remains to compute the PGL_2-coinvariants of the right hand side. We need the following:

Lemma 11.6. Let H G and L be three p-adic groups. Let W be a smooth H L-module, and an irreducible representation of L. Let $()$ be the maximal -isotypic quotient of W. If $\mathrm{Ext}^1_L(;) = 0$ then

$$\mathrm{ind}_H^G()$$

is the maximal -isotypic quotient of $\mathrm{ind}_H^G W$. Here ind stands for induction with compact support.

Proof. Since $\mathrm{Ext}^1_L(;) = 0$, the kernel of the projection of W on $()$ does not have as a quotient. Thus, it suces to prove that if $\mathrm{Hom}_L(W;) = 0$, then $\mathrm{Hom}_L(\mathrm{ind}_H^G W;) = 0$. We shall prove that

$$\mathrm{Hom}_L((\mathrm{ind}_H^G W)^K;) = 0$$

for any open compact subgroup K of G. Write $G = [_{i 2 I} H g_i K$ where I is an index set, and set $K_i = H \setminus g_i K g_i^{\ 1}$ for every i 2 I. Then, as an L-module, $(\mathrm{ind}_H^G W)^K$ is a direct sum of W^{K_i}. Since W^{K_i} is a direct summand of the L-module W, it follows that $\mathrm{Hom}_L(W^{K_i};) = 0$, and this proves the lemma.

We apply Lemma 11.6 taking H G to be J G_2 and $L = PGL_2$. Since $\mathrm{Ext}^1_{PGL_2}(1;1) = 0$, the lemma implies that computing PGL_2-coinvariants of $_{N\ ;_2\ _2}$ boils down to computing the PGL_2-coinvariant of $C_c(M_2(F))$, where it is the full degenerate principal series $I(1{=}2)$. We have shown:

Proposition 11.7. As a representation of $G_2(F)$, one has:

$$_{S;\ _S} \quad (_{N_2;\ _2})_{PGL_2} \quad \mathrm{ind}_J^{G_2}(I(1{=}2) \quad):$$

12. Howe Duality for G_2 $PGSp_6$: Tempered Case

After the preparation of the previous 3 sections, we are now in a position to begin our study of the theta correspondence for the dual pair G_2 $PGSp_6$. In this section, we shall show the Howe duality theorem for tempered representations of G_2. The key is to show the analog of Propositions 6.6 and 6.7 for generic representations of G_2. This will rely on another curious chain of containments given in the following lemma, which comes from a consideration of a game of ping-pong with periods.

Lemma 12.1. Let be the minimal representation of E_7. Let 2 Irr(G_2) be tempered and let $_V$: V ! C be a Whittaker character for G_2. Let H = PGSp$_6$ and 2 Irr(H) be tempered such that

$$\mathrm{Hom}_{G_2 H}(;\ ) = 0:$$

Then we have the following natural inclusions

$$\mathrm{Hom}_V(;\ _V)\ \mathrm{Hom}_V((); \ _V) = \mathrm{Hom}_S(-;\ _S)$$

$$\mathrm{Hom}_S((-);\ _S) = \mathrm{Hom}_{G_2}(\mathrm{ind}_J{}^{2}I\ (1=2)^{G}$$

;−): If is generic, then all of these spaces are one-dimensional.

Proof. We examine each containment in turn:

The rst inclusion arises from the surjection () . The second follows from the identity

$$\mathrm{Hom}_V((); \ _V) = \mathrm{Hom}_{V H}(;\ _V\ ) = \mathrm{Hom}_H(_{V;\ _V};)$$ combined

with Proposition 11.5 (i.e. [SWe, Lemma 4.5]):

$$_{V;\ _V} = \mathrm{ind}_S{}^H\ _S$$

and the Frobenius reciprocity.

For the third, observe that () is the complex conjugate of (). Since = − and = −, we have (−) −.

The fourth follows from the identity,

$$\mathrm{Hom}_S((-);\ _S) = \mathrm{Hom}_{S G_2}(;\ _S\ -) = \mathrm{Hom}_{G_2}(_{S;\ _S};-)$$ combined with

Proposition 11.7:

$$_{S;\ _S} = \mathrm{ind}_J{}^{2}I^{G}\ (1=2)$$

and Frobenius reciprocity.

If is generic, then the rst and the last spaces are one-dimensional, with the latter by Corollary 11.4 applied to s = 1=2. Hence, all spaces in the chain are one-dimensional.

We can now obtain the following two propositions as consequences of Lemma 12.1.

Proposition 12.2. Let 2 Irr(PGSp$_6$) be tempered. Then () cannot have two irreducible tempered and generic quotients.

Proof. Let $_1;_2$ 2 Irr(G_2) be tempered and generic such that () $_1$ $_2$. Then dim $()_{V;\ _V}$

$$2:$$

On the other hand, dim $()_{V;\ _V}$ = 1 by Lemma 12.1, which is a contradiction.

Remark: This proposition is proved in [SWe] for generic supercuspidal representations using the uniqueness of Shalika functional (shown in [SWe] for supercuspidal representations), but the proof of this uniqueness given there is dicult. The proof here is based on Corollary 11.4.

Proposition 12.3. Let 2 Irr(G_2) be tempered and generic. Then () cannot have two tempered irreducible quotients. In particular, its cuspidal component $()_c$ is irreducible or 0.

Proof. Let $_1; _2$ 2 Irr(PGSp $)_6$ be irreducible tempered and such that () $_1$ $_2$. By Lemma 12.1, applied to , $_1$ and then to , $_2$, one has:

$$1 = \dim \mathrm{Hom}_S(_1; \quad _S) = \dim \mathrm{Hom}_S((); \quad _S) = \dim \mathrm{Hom}_S(_2; \quad _S): \text{Since}$$

$_1$ $_2$ is a quotient of (),

$$1 = \dim \mathrm{Hom}_S((); \quad _S) \quad \dim \mathrm{Hom}_S(_1; \quad _S) + \dim \mathrm{Hom}_S(_2; \quad _S) = 2;$$

which is a contradiction.

Combining Propositions 12.2 and 12.3 with the results of x6, we can now show the Howe duality theorem for tempered representations of G_2:

Theorem 12.4. Let 2 Irr(G_2) be tempered and consider its big theta lift () on PGSp . $_6$ Then

(i) () has nite length and a unique irreducible quotient () (if nonzero), which is tempered.

(ii) Moereover, for tempered $_1; _2$ 2 Irr(G_2),

$$0 = (_1) = (_2) =) \quad _1 = \ _2:$$

Proof. (i) We have seen (i) for non-generic in Corollary 6.8. The proof for generic is the same, using Lemmas 6.2(i) and 6.3(ii), as well as Proposition 12.3.

(ii) If one of $_1$ or $_2$ is nongeneric, then the desired result follows by Proposition 6.6. If $_1$ and $_2$ are both generic, then the desired result follows by Proposition 12.2.

We also point out the following corollary:

Corollary 12.5. Let 2 Irr(G_2) be generic, supercuspidal and not a theta lift from PGL_3. Then () is generic, supercuspidal and irreducible.

Proof. By [SWe], we have known that () is generic and supercuspidal (hence tempered and semisimple), but now we know by Proposition 12.3 that it is also irreducible.

13. Jacquet Modules

The purpose of this section is to compute various Jacquet modules of the minimal representation of E_7 with respect to the maximal parabolic subgroups of G_2 and $PGSp_6$. We note that the results of this section are entirely self-contained, and do not depend on any prior results in this paper. As consequences of the results here, we deduce Lemmas 6.2 and 6.3 for the dual pair G_2 $PGSp_6$. Indeed, we shall determine in Theorem 14.1 the theta lifts of all non-tempered representations of G_2 and $PGSp_6$ precisely.

13.1. Jacquet functors for G_2. Recall that $P = MN$ and $Q = LU$ are the two maximal parabolic subgroups of G_2 as before, in standard position relative to a maximal split torus T in G_2 and a choice of positive roots, so that $P \setminus Q$ is a Borel subgroup. In particular, their Lie algebras arise from Z-gradings given by two fundamental co-characters. Since G_2 is contained in E_7, as a memeber of the dual pair, the two co-characters give Z-gradings of the Lie algebra of E_7, dening parabolic subgroups $P = MN$ and $Q = LU$ of E_7, whose intersections with G_2 are P and Q, respectively. The Lie types of the Levi factors M and L are D_6 and A_1 A_5, as explained in [GS99]. In the rest of the paper, we shall x the following:

- The group P acts on $C_c(GL_2)$ and $C_c(GL_1)$ by left translation by g and det(g), respectively, via the identication $M = GL_2$.
- The group Q acts on $C_c(GL_2)$ and $C_c(GL_1)$ by left translation by g and det(g), respectively, via the identication $L = GL_2$.
- Let B be the group of lower-triangular matrices in GL_2. Then B acts on $C_c(GL_1)$ by right translation by the (1; 1) matrix entry of g 2 B.

We identify $M = GL_2$ such that the action of M on N=[N; N] is the symmetric cube of the standard representation of GL_2 twisted by determinant inverse. In particular, a scalar matrix (z; z) in GL_2 acts by z. We have [MS, Theorem 6.1],

Proposition 13.1. Let $H = PGSp_6$. As a GL_2 H-module, r_P() (the normalized Jacquet functor) has a ltration with three successive subquotients (top to bottom):

(1) $_P^{1=2}$ $_N$ = $_D$ $_6$ j det j$^{1=2}$; j det j$^{3=2}$.
(2) $\mathrm{Ind}_{B P_2}^{GL_2 H}$($C_c(GL_1)$).
(3) $\mathrm{Ind}_{P_{13}}^{H} C_c(GL_2)$.

Here, note that:

- In (1), the center of $M = GL_2$ acts trivially on both $_{D_6}$ and $_;$, the minimal and the trivial representation of the Levi M.
- In (2), = j j $^{1=2}$ j j is a character of the group B of lower triangular matrices in GL_2.

For the computation of r_Q(), we rst make some preparations. Let W be the Weil representation for the similitude dual pair GL_2 GSO_4; see [Ro] where theta correspondences for similitude groups are treated in detail. Observe that $GSO_4 = (GL_2\ \ GL_2)=GL_1$, with the isomorphism realized by latter acting on the space $M_2(F)$ of 2 2 matrices by left and right multiplication and the quadratic form given by the determinant. We identify the rst factor GL_2 with L so that the action of L on U=[U; U] is the standard representation of GL_2. The irreducible quotients _ of W are

, where is an irreducible representation of GL_2. We need a slight renement of this to obtain the big theta lifts.

Lemma 13.2. Consider the similitude theta correspondence for the dual pair GL_2 GSO_4 on W. Let be an irreducible generic representation of GL_2. Then ( ) = and ( ) = _.

Proof. Let V = F be a maximal unipotent subgroup of GL_2 and a non-trivial character of V. We shall use the fact that $W_{V;} = C_c(GL_2)$, the regular representation of GL_2, where V is in any of the three GL_2. Also, when we view $C_c(GL_2)$ as giving a theta correspondence between two GL_2, the big theta lift of any irreducible is either or $-$, depending on identications or convention [APS, Lemma 2.4].

We have (
)
(
) as a quotient of W. Applying the functor of (V;)-coinvariants, with V sitting in one of GL_2 factors of GSO_4, we conclude that (
)
is a quotient of the regular representation of GL_2. This implies that (
) = , as desired.

In the other direction, let $_1$ $_2$ be a quotient (if any) of the kernel of (−) ! . If $_1$ or $_2$ is generic, then we can take (V;) twisted co-invariants of − (−), for the corresponding GL_2, and obtain a contradiction to the fact that, for the regular representation, the big theta lift of − is . Thus both $_1$ and $_2$ are one-dimensional. By the Kunneth formula [Ra]

$$\mathrm{Ext}^1_{GL_2 GL_2}(_1 \; _2; \;) = \;_{i+j=1} \mathrm{Ext}^i_{GL_2}(_1; \;) \; \mathrm{Ext}^j_{GL_2}(_2; \;)$$

and this clearly vanishes since $\mathrm{Hom}_{GL_2}(_i; \;) = 0$, for $i = 1; 2$. Thus $_1$ $_2$ − is − a quotient of ( ). Hence $_1$ $_2$ is an irreducible quotient of W, contradicting the fact that all irreducible quotients are of the form

.

Proposition 13.3. Let $H = PGSp_6$. As a GL_2 H-module, $r_Q()$ (the normalized Jacquet functor) has a ltration with three successive subquotients (top to bottom):

(1) $^{1=2}_Q$ $_U$ = $_{A_5}$ $j\det j^{3=2}$ $_{A_1}$ $j\det j^2$. (2) $\mathrm{Ind}^{GL_2 H}_{B P_2}(\; C_c(GL_1))$.
(3) $\mathrm{Ind}^H_{P_2} W$.

- In (1), the center of $L = GL_2$ acts trivially on both $_{A_5}$ and $_{A_1}$, the minimal and a principal series representation of the two factors of L.
- In (2) = $j \; j^{1=2} \; j \; j$ is a character of the group B of lower triangular matrices in GL_2.

Proof. This proposition is entirely similar to [GS99, Prop. 6.8], which treated the case of non-split form of H, except the character was not determined there. This is done as follows. For a generic character of GL_2, representations $I_Q()$ and $I_2($) are both irreducible and $I_Q()$ $I_2($) is a quotient of ; this follows from the bottom factor (3) of the ltration. Hence $r_Q(I_Q())$ $I_2($) is a quotient of r_P (). Now determining is an easy exercise using $r_Q(I_Q())$.

13.2. Non-tempered representations. We enumerate the nontempered irreducible representations of G_2 using the discussion from Section 3. Let $P = MN$ and $Q = LU$ be the two maximal parabolic subgroups in G_2 as before. Their Levi groups are isomorphic to GL_2. Let be a representation of GL_2, and let I_P () and I_Q() be the corresponding normalized induced representations of G_2. Irreducible, non-tempered representations of G_2 are described as follows, where is irreducible, and ! is the central character of .

(a) Unique irreducible quotient of I_Q() where is an unramied twist of a tempered representation such that j ! j = j j^s for some $s > 0$.

(b) Unique irreducible quotient of $I_P()$ where is an unramied twist of a tempered representation such that $j ! j = j j^s$ for some $s > 0$.

(c) Unique quotient of $I_P()$ where is the unique quotient of a representation induced from an ordered pair of characters $_1;_2$ such that $j_1 j = j j^{s_1}$, $j_2 j = j j^{s_2}$ where $s_1 > s_2 > 0$.

In (a) and (b), $I_Q()$ and $I_P()$ are standard modules, while in (c), $I_P()$ is a quotient of a standard module associated to the minimal parabolic $P \setminus Q$. In any case, each of these induced representations has a unique irreducible quotient which we denote by $J_Q()$ in (a) and by $J_P()$ in (b) and (c). These representations $J_Q()$ and $J_P()$ exhaust the irreducible nontempered representations of G_2.

We also enumerate some relevant nontempered representations of $PGSp_6$. Let $P_i = M_i N_i$, $i = 1;2;3$ be the three maximal parabolic subgroups of $PGSp_6$. Let $I_i()$ denote the representation of $PGSp_6$ obtained by normalized parabolic induction from P_i, and let $I_{jk}()$ denote the representation of $PGSp_6$ obtained by normalized parabolic induction from $P_j \setminus P_k$. We shall consider the following non tempered representations of $PGSp_6$, corresponding to the cases (a), (b) and (c) above:

(a') If is an irreducible representation of $L = GL_2$ satisfying the conditions of (a) above, let =
be a representation of $M_2 = GL_2 \quad GL_2 = GL^r_1 = GSO_4$. Then $I_2()$ is a standard module, with unique irreducible quotient $J_2() = J_2($
).

(b') If is an irreducible representation of $M = GL_2$, satisfying the conditions of (b) above, let =
1 be a representation of $M_1 \setminus M_3 = GL_2 \quad GL_1$. Then $I_{13}()$ is a standard module with unique irreducible quotient $J_{13}() = J_{13}($
1).

(c') If is an irreducible representation of $M = GL_2$, satisfying the conditions of (c) above, let =
1 be a representation of $M_1 \setminus M_3 = GL_2 \quad GL_1$. Then $I_{13}()$ is a quotient of a standard module associated to the Borel subgroup, Hence, it has a unique irreducible quotient which we denote by $J_{13}() = J_{13}($
1).

13.3. Theta lifts from G_2. Now the following lemma attempts to compute the theta lifts of the above non-tempered representations of G_2 to $PGSp_6$.

Lemma 13.4. Let $2 \; Irr(G_2)$ be non-tempered.

If $I_Q(-)$ where is as in (a) above, then () is a quotient of $I_2($
) and hence has nite length. Moreover, $(J_2($
)) $=_2$ 0 where $J_2($
) is the unique irreducible quotient of I (
).

If $I_P(-)$ where is as in (b) and (c) above, then () is a quotient of $I_{13}($
1) and hence has nite length. Moreover, $(J_{13}($
1)) = 0 where $J_{13}($
1) is the unique irreducible quotient of $I_{13}($
1).

Proof. Let be the minimal representation, and $2 \; Irr(G_2)$. We shall use the fact that ()

$$= \mathrm{Hom}_{G_2}(;)$$

as non-smooth $H = PGSp_6$-modules, where the former is the linear dual of (). Assume that $I_Q(-)$. Then

$$() = \mathrm{Hom}_{G_2}(;) \ \ \mathrm{Hom}_{G_2}(; I_Q(-)) = \mathrm{Hom}_L(r_Q(); -):$$

Now we shall use the ltration of $r_Q()$ from Proposition 13.3.

Let $_1$, $_2$ and $_3$ denote the three subquotients in the same order. Observe that $\mathrm{Ext}^i_L(;_1)$ are trivial from the central character considerations, since the central char-acter of – is a negative power of jzj. Hence we have a long exact sequence

$$0 ! \ \mathrm{Hom}_L(_2; -) ! \ \mathrm{Hom}_L(r_Q(); -) ! \ \mathrm{Hom}_L(_3; -) ! \ \mathrm{Ext}^1_L(_2; -)$$

Since $_2$ is induced from B, by the second adjointness,

$$\mathrm{Ext}^i_L(_2; -) = \mathrm{Ext}^i_T(\mathrm{Ind}^H_{P_2}(C_c(GL_1)); r_B(-))$$

where $T = GL_1 \ GL_1$, the maximal torus in B. Observe that the action of the second GL_1 on $\mathrm{Ind}^H_{P_2}(C_c(GL_1))$ is jj, and this is dierent from the action on $r_B(\ )$ by our assumption on . Hence $\mathrm{Ext}^i_L(_2; -) = 0$ for all i, and we can conclude that

$$\mathrm{Hom}_L(r_Q(); -) = \mathrm{Hom}_L(_3; -) = \mathrm{Hom}_L(\mathrm{Ind}^H_{P_2} W; -);$$

where, for the second isomorphism, we have simply substituted the explicit expression for given in Proposition 13.3. By [GG06, Lemma 9.4], the maximal – isotypic quotient of $\mathrm{Ind}^H_{P_2}$ W is $(\mathrm{Ind}^H_{P_2}(-))$ – where (–) is the big theta lift for the similitude theta correspondence on W. Since is generic, Lemma 13.2 shows that (–) = and it follows that

$$\mathrm{Hom}_L(_3; -) = I_2( ):$$

Hence () $I_2($), and ()– $I_2($)– by taking smooth vectors. Thus () is a quotient of $I_2($). Observe that we have proved in the process that $I_2($) is a quotient of $r_Q()$, so that $(J_2($)) = 0. This establishes the rst bullet. The proof of the second is completely analogous.

13.4. Jacquet functors for $PGSp_6$. Recall that in $PGSp_6$, we have xed three standard maximal parabolic subgroups P_1, P_2 and P_3. They correspond to Z-gradings of the Lie algebra of $PGSp_6$ given by three fundamental co-characters. The action of each of these three co-characters gives a Z-grading of the Lie algebra of E_7, and these gradings dene three parabolic subgroups P_1, P_2 and P_3 of E_7. To recognize these parabolic subgroups, perhaps it is easiest to proceed as follows. Observe that the E_7 Dynkin diagram contains a unique D_4 subdiagram. We embed G_2 into D_4. The centralizer of G_2 in the split, adjoint E_7 is $PGSp_6$. Let P be the parabolic subgroup of E_7, whose Levi factor has the type D_4. This parabolic is contained in precisely three maximal parabolic subgroups denoted by P_1, P_2 and P_3, whose Levi factor types are, D_6, $A_1 D_5$ and E_6, respectively. The intersection of P_i and $PGSp_6$ is P_i, for each i. We write $P_i = M_i N_i$ and $P_i = M_i N_i$ for the Levi decompositions for these parabolic subgroups.

Case P_3: This is treated in [MS, x5], and we summarize the results as follows. The unipotent subgroups of P_3 and P_3 are abelian, $M_3 = GL_3$ and the modular character is

$$_{P_3}(m) = j\det(m)j^2:$$

Let O_0 denote the space of trace 0 elements in the octonion algebra O. On the space O_0^3, we have the natural diagonal action $(x; y; z) ! \ (gx; gy; gz)$ of $g \ 2 \ G_2$ and the row-vector action $(x; y; z) \ ! \ (x; y; z)m^{1}$ of $m \ 2 \ GL_3$. Let O^3_0 be the set of all nonzero $(x; y; z)$ such that the linear subspace $\langle x, y, z\rangle$ O_0 is a null-space for octonion multiplication, i.e. the product of any two elements in the space is 0. Such non-zero null-spaces in O_0 are of dimension 1 or 2.

We have an exact sequence of G_2 GL_3-modules

$$0 \to C_c($$

$) \to {}_{N_3} \to {}_{N_3} \to 0$ where $(g; m) \in G_2$ GL_3 acts on $f \in C_c($

$)$ by

$$((g; m) f)(x; y; z) = |\det(m)|^2 f((g^{-1}x; g^{-1}y; g^{-1}z)m):$$

The group G_2 GL_3 acts on with two orbits $_1$ and $_2$, where $_i$ is the subset of triples $(x; y; z)$ such that $\langle x; y; z\rangle$ has dimension i. Thus $C_c($ $)$ has a ltration with $C_c($ $_2)$ as a submodule and $C_c($ $_1)$ as a quotient. Each of these can be explicitly described as G_2 GL_3-modules.

In order to state the result, let Q_1 and Q_2 be the maximal parabolic subgroups of GL_3 stabilizing subspaces consisting of row vectors $(; 0; 0)$ and $(; ; 0)$, respectively. Observe that these are block lower-triangular groups with Levi factors isomorphic to GL_1 GL_2 and GL_2 GL_1, respectively. Their modular characters are

$$_{Q_1}(g_1; g_2) = |g_1|^{2} |\det(g_2)| \quad \text{and} \quad _{Q_2}(g_2; g_1) = |\det(g_2)|^{1} |g_1|^2:$$

Recall that $r_{P_3}() = {}^{1=2}_{P_3} {}_{N_3}$ is the normalized Jacquet module. Then:

Proposition 13.5. As a G_2 GL_3-module, $r_{P_3}()$ has a ltration with three successive subquotients (from top to bottom):

(1) ${}^{1=2}_{P_3} {}_{N_3} = {}_{E_6} \; ; \; |\det|$.
(2) $\mathrm{Ind}^{G_2 GL_3}_{Q Q_1}(C_c(GL_1))$.
(3) $\mathrm{Ind}^{G_2 GL_3}_{P Q_2}(C_c(GL_2))$.

Here, note that:

- In (1), the center of $M_3 = GL_3$ acts trivially on both $_{E_6}$ and $_{;}$, the minimal and the trivial representation of the Levi M_3.
- In (2), $(g_1; g_2) = |g_1|^{1=2} |\det(g_2)|^{1=2}$ is a character of Q_1.
- For $i = 1; 2$, Q_i acts on $C_c(GL_i)$ by right translations via the factor GL_i.

Case P_1: This case is not in the literature; however, it is similar to the computation of the Jacquet module of the minimal representation of E_8 with respect to a maximal parabolic subgroup of F_4 in [SWo, x5]. The unipotent radical subgroups of P_1 and P_1 are Heisenberg groups with $M_1 = GSp_4$. Let be the similitude character of GSp_4. The modulus character of M_1 is

$$_{P_1}(m) = |(m)|^{3}:$$

Recall that O_0 is the space of trace 0 octonions. On O_0^4, we have the row-vector action

$$(x; y; x^0; y^0) \to (x; y; x^0; y^0)m^{-1} \quad \text{of } m \in GSp_4$$

preserving the form $O_0^4 \to \wedge^2 O_0$ given by

$$(x; y; x^0; y^0) \to x \wedge x^0 + y \wedge y^0:$$

Let O^4_0 be the set of all nonzero $(x;y;x^0;y^0)$ such that the linear subspace $hx;y;x^0;y^0i$ O_0 is a null-space for octonion multiplication and $x \wedge x^0 + y \wedge y^0 = 0$. We have an exact sequence of G_2 GSp_4-modules

$$0 \,!\, C_c(\;) \,!\, {}_{N_1} \,!\, {}_{N_1} \,!\, 0$$

where $(g;m)$ 2 G_2 GSp_4 acts on f 2 $C_c(\;)$ by

$$((g;m)\ f)(x;y;x^0;y^0)) = j(m)j^3\ f((g^{\ 1}x;g^{\ 1}y;g^{\ 1}x^0;g^{\ 1}y^0)m):$$

Now the group G_2 GSp_4 acts on with two orbits $_1$ and $_2$, where $_i$ is the subset of quadruples $(x;y;x^0;y^0)$ such that $hx;y;x^0;y^0i$ has dimension i. Thus $C_c(\;)$ has a ltration with $C_c(\,_2)$ as a submodule and $C_c(\,_1)$ as a quotient. Each of these can be explicitly described as G_2 GSp_4-modules.

In order to state the result, let Q_1 and Q_2 be the maximal parabolic subgroups of GSp_4 stabilizing subspaces consisting of row vectors $(;0;0;0)$ and $(;;0;0)$, respectively. Let $L_1 = GL_1\ GL_2$ be the Levi subgroup of Q_1 such that $(g_1;g_2)$ 2 $GL_1\ GL_2$ acts on the quadruples, after rearranging the order, by

$$(x;x^0;y;y^0)\ !\ \ (xg_1^{\ 1};x^0g_1\det(g_2)^{\ 1};(y;y^0)g_2^{\ 1}):$$

Let $L_2 = GL_2\ GL_1$ be the Levi subgroup of Q_2 such that $(g_2;g_1)$ 2 $GL_2\ GL_1$ acts on the quadruples by

$$(x;y;x^0;y^0)\ !\ \ ((x;y)g_2^{\ 1};(x^0;y^0)g_1^{\ 1}g_2^{>}):$$

The similitude character , restricted to L_1 and L_2, is given by

$$(g_1;g_2) = \det g_2 \quad \text{and} \quad (g_2;g_1) = g_1$$

respectively, and the modulus characters are

$$_{Q_1}(g_1;g_2) = jg_1j^{\ 4}\ j\det(g_2)j^2 \qquad \text{and} \qquad _{Q_2}(g_2;g_1) = j\det(g_2)j^{\ 3}\ jg_1j^3:$$

Recalling that $r_{P_1}(\,) = {}^{1=2}_{P_1}\ {}_{N_1}$ is a normalized Jacquet module, we have:

Proposition 13.6. As a G_2 GSp_4-module, $r_{P_1}(\,)$ has a ltration with three successive subquotients (from top to bottom):

(1) ${}^{1=2}_{P_1}\ {}_{N_1} = {}_{D_6}\ jj^{1=2}\ ;\ \ jj^{3=2}$.

(2) $\mathrm{Ind}^{G_2\,GSp_4}_{Q\,Q_1}(\ C_c(GL_1))$.

(3) $\mathrm{Ind}^{G_2\,GSp_4}_{P\,Q_2}(C_c(GL_2))$.

Here, note that

- In (1), the center of $M_1 = GSp_4$ acts trivially on both $_{D_6}$ and $_;$, the minimal and the trivial representation of the Levi M_1.
- In (2), $(g_1;g_2) = jg_1j^{\ 1=2}\ j\det(g_2)j^{1=2}$, a character of Q_1.
- For $i = 1;2$, Q_i acts on $C_c(GL_i)$ by right translations via the factor GL_i.

Case P_2: A variant of this case can be found in [GS99] for the non-split form of $PGSp_6$. However, for the split case considered in this paper, the Jacquet module ltration contains an additional \middle" term.

The unipotent radical subgroups N_2 P_2 and N_2 P_2 are two-step nilpotent subgroups. Let Z_2 N_2 be the center of N_2. We now explain how the kernel of the natural projection $_{Z_2}$! $_{N_2}$ contributes to $_{N_2}$. We have

$$0 \,!\; C_c(!) \,!\; {}_{Z_2} \,!\; {}_{N_2} \,!\; 0$$

where ! is the M_2-highest weight orbit in

$$N_2 = Z_2 = O$$
$$M_2(F) = M_2(O)$$

where N_2 is the unipotent group opposite to N_2, and $M_2(F)$ is the set of two-by-two matrices. (In the non-split case $M_2(F)$ is replaced by a division algebra, so ! is empty; see the discussion on [GS99, Pg. 137].) Recall that the type of M_2 is D_5 A_1 and $N_2 = Z_2 = F^{16}$ F^{16} where F is a spin-module of D_5. In the above isomorphism, we assume that A_1 acts from the right on $M_2(O)$, and columns are vectors in the spin-module. Thus ! is the set of non-zero matrices

$$\begin{matrix} x & x^0 \\ y & y^0 \end{matrix}$$

where the two columns are linearly dependent over F and each column (if non-zero) is a highest weight vector in the spin-module. Let be the subset of ! such that $x; x^0; y; y^0$ are traceless octonions. We have an exact sequence of G_2 M_2-modules

$$0 \,!\; C_c(\;) \,!\; ({}_{Z_2})_{N_2} \,!\; {}_{N_2} \,!\; 0$$

where $(g; (;))$ 2 G_2 $(GL_2\ GL_2^r) = GL_1$ acts on f 2 $C_c(\;)$ by $((g; (;))\ f)\begin{matrix} x & x^0 \\ y & y^0 \end{matrix} = {}^0()\ f\begin{matrix} g^{\ 1}x & g^{\ 1}x^0 \\ g^{\ 1}y & g^{\ 1}y^0 \end{matrix}$

for some (unknown) character ^{0} and where = det(). The highest weight orbit in the 16-dimensional spin module is described in [MS]. That result, applied to each column of $M_2(O)$, implies that $x; x^0; y; y^0$ (of an element in) generate a nil-subalgebra. The group G_2 M_2 acts on with two orbits $_1$ and $_2$, where $_i$ consists of elements such that $\langle x; x\ ; y; y\ \rangle$ has dimension i. Thus $C_c(\;)$, as a G_2 M_2-module, has $C_c(\,_2)$ as a submodule and $C_c(\,_1)$ as quotient.

Proposition 13.7. As a G_2 $(GL_2\ GL_2) = GL_1^r$-module, $r_P\ ()$ has a ltration with four successive sub quotients:

(1) $^{1=2}_{P_2}\ {}_{N_2} = {}_{D_5}\ j\det j^{1=2}\ {}_{A_1}\ j\det j^{3=2}$.
(2) $\mathrm{Ind}^{G_2(GL_2 GL_2)=GL_1}_{Q(BB)=GL_1^r}(\ C_c(GL_1^r))$.
(3) $\mathrm{Ind}^{G_2 GL_2}_{P\ B}(C_c(GL_2))$.
(4) $\mathrm{Ind}^{G_2}_{Q} W$.

Here, note that:

- In (1), the second SL_2 M_2 acts trivially on the rst summand, and the rst SL_2 M_2 acts trivially on the second summand. The center of M_2 acts trivially on both $_{D_5}$ and $_{A_1}$, the minimal and a principal series representation of the two factors of M_3.
- In (2) $= j\ j^{1=2}\ j\ j$ on each B.

- In (3), B is the subgroup of the second factor GL_2 of M_2. It acts on $C_c(GL_2)$ by right translation by the scalar given by the (1; 1) matrix entry. The rst factor GL_2 of M_2 acts by right translations on $C_c(GL_2)$.
- In (4), W is the Weil representation of GL_2 $(GL_2$ $GL_2)=GL_1$ $={}^rGL_2$ GSO_4.

This proposition is a combination of [GS99, Proposition 8.1], which accounts for the bottom piece of the ltration (4), and the above discussion. The pieces (2) and (3) are the spaces of functions $C_c($ $_1)$ (and $C_c($ $_2)$, respectively. This also assumes that we have explicated the character $_2$ appearing in the action on $C_c($ $_6$ $)$. To that end, observe that (3) (or any unknown twist) gives a correspondence of generic principal series representations of G and $PGSp$ that has to be compatible with the one in Lemma 13.4, and this determines ^{0} uniquely.

13.5. Theta lifts from $PGSp_6$. Using Propositions 13.5, 13.6 and 13.7, we can now prove the following analog of Lemma 13.4.

Lemma 13.8. Let 2 $Irr(PGSp_6)$ be non-tempered. Then () = 0 unless is as described in Lemma 13.4. More precisely,

If $I_2(-$ $-)$, then () is a quotient of $I_Q()$ and hence has nite length. Moreover, $(J_Q()) = 0$ where $J_Q()$ is the unique irreducible quotient of $I_Q()$.

If $I_{13}($ $1)$, then () is a quotient of $I_P()$, and hence has nite length. Moreover, $(J_P()) = 0$ where $J_P()$ is the unique irreducible quotient of $I_P()$.

Proof. We set $H = PGSp_6$ Assume that is a Langlands quotient of a standard module for the maximal parabolic $_1P_2$. $_2$ Then $I_2(-$ $-)$ where $_1$ and $_2$ have the same central character and are both tempered representations of GL_2 twisted by a positive power of $j\det j$. Then

$$\mathrm{Hom}_H(;)\ \mathrm{Hom}_H(;I_2(- \quad _1 \quad _2 \quad _1 \quad _2 \quad -) = \mathrm{Hom}_{M_2}(r_{P_2}();- \quad -):$$

Let $_i$, $i = 1;2;3;4$ be the subquotients of $r_{P_2}()$ as in Proposition 13.7, in the same order. We claim that

$$\mathrm{Hom}_{M_2}(r_{P_2}();- \quad _1 \quad _2 \quad _1 \quad _2 \quad -) = \mathrm{Hom}_{M_2}(_4;- \quad -):$$

Assume this claim for a moment. Then

$$\mathrm{Hom}_{M_2}(r_{P_2}();_1 \quad - \quad - \quad _2) = \mathrm{Hom}_{M_2}(_4;_1 \quad - \quad - \quad _2) = \mathrm{Hom}_{M_2}(\mathrm{Ind}^{G}_{Q}{}^2 W;_1 \quad - \quad - \quad _2)$$

where W is the Weil representation of GL_2^3. This implies that () = 0 unless $_1 = _2$, and if we denote this representation as , then () is a non-zero quotient of the standard module $I_Q()$. In order to prove the claim, we need to show that $\mathrm{Ext}^n_{M_2}$ $_1$ $_2$ $(_i;-$ $-) = 0$ for all n and $i < 4$. Consider $i = 3$. Then, using the (second) Frobenius reciprocity for induction from B to GL_2, the rst factor of M_2, we have

$$\mathrm{Ext}^i_{M_2}(\mathrm{Ind}^{G_2\,GL_2}_{P\ B}(C_c(GL_2));_1 \quad - \quad - \quad _2) = \mathrm{Ext}^i_{T\,GL_2}(\mathrm{Ind}^{G}_{P}{}^2(C_c(GL_2));r_B(_1)$$

$_2$) where T GL_1 GL_1 is the torus of diagonal matrices in GL_2. Now recall that the second GL_1 acts trivially on Ind $^2(C_c(GL_2))$. On the other hand, since $_1$ is tempered with a positive twist of j det j, the second GL_1 acts on $r_B($ $)$ with characters such that jj is a negative power of absolute value. This proves the vanishing for $i = 3$. The other two cases are just as easy or even easier: for $i = 1$ vanishing follows from central character considerations, and for $i = 2$ using Frobenius reciprocity where it suces that either $_1$ or $_2$ is twist of a tempered representation by a positive power of j det j.

Now assume that is a Langlands quotient of a standard module for the parabolic $P_{23} = P_2 \cap P_3$. Then, by induction in stages, we get that $I_2(\bar{1} \; \bar{2}$) where $_1$ is a twist of a tempered representation by a positive power of $j \det j$. This is enough $_2$ $_1$ $_2$ to show that $_2$ Hom_{M} $_1$ $_2$ ($_i$; − −) = 0 for i < 4. Thus, if () = 0 then Hom_M ($_4$; − −) = 0. This implies that $_1$ = $_2$, contradicting that is a Langlands quotient of a standard module for the parabolic P_{23}. Hence () = 0.

If is a Langlands quotient of a standard module for the parabolic $P_{12} = P_1 \cap P_2$ then, by induction in stages, we get that $I_2(\bar{1} \; \bar{2}$) where now $_2$ is a twist of a tempered representation by a positive power of $j \det j$. In this case, $_2$ $_1$ $_2$ Hom_M ($_i$;) = 0 for i = 3 by repeating the above arguments. For i = 3 we have

$$\mathrm{Hom}_{M_2}(\mathrm{Ind}_{P\,B}^{G_2 GL_2}(C_c(GL_2)); - \;_1 \;_2 = \quad G \quad _1 \;_2$$

$$-) \ \mathrm{Hom}_{T\,GL_2}(\mathrm{Ind}_P{}^2(C_c(GL_2)); r_B(-)$$

−) and the last space is isomorphic to

$$\mathrm{Hom}_{T\,GL_2}(\mathrm{Ind}_P{}^{2\,G}(_2) \quad - \quad - \quad - \quad _2; r_B(_1) \quad _2):$$

Recall that $T = GL_1 \;\; GL_1$ and the second GL_1 acts trivially on $\mathrm{Ind}_P{}^{2\,G}(C_c(GL_2))$ and hence on its ^G_P quotient $_2$ Ind $^2(_2)$. The rst GL_1 acts on this space by the central character of $_2$, which is equal to the central character of $_1$, hence it is a nontrivial character, say . Hence the above Hom space, if non-zero, is non-trivial if and only if 1 is an exponent of $_1$, and then it is isomorphic to

$$\mathrm{Hom}_{GL_2}(\mathrm{Ind}_P^{G\,2}(_2) \quad _2 \;_2 \quad G \quad -;-) = \mathrm{Hom}(\mathrm{Ind}_P{}^2(_2); C) = I_P(_2):$$

Summarizing, () = 0 implies that () is a quotient of $I_P(_2)$. It follows that $J_P(_2)$ is a quotient of , where $J_P(_2)$ is the unique irreducible quotient of $I_P(_2)$. But, by Lemma 13.4, $J_P(_2)$ does not lift to . This is a contradiction, hence () = 0.

The remaining non-tempered representations of H (associated to standard modules induced from P_{123}, P_{13}, P_1 or P_3) are easily dealt with using $r_{P_1}()$ and $r_{P_3}()$. We leave details to the reader.

14. Consequences of Jacquet Module Computations

We can now draw some denitive consequences of the Jacquet module computations of the previous section. In particular, we shall determine the theta lift of nontempered representations explicitly, and also complete the proofs of Lemmas 6.2 and 6.3 for the dual pair G_2 $PGSp_6$.

14.1. Lift of nontempered representations. Taken together, Lemmas 13.4 and 13.8 allow us to determine the theta lift of nontempered representations explicitly:

Theorem 14.1. We have:

(a) (J_Q()) is a nonzero quotient of I_2( ) and hence has nite length with unique irreducible quotient J_2(

). Likewise, (J$_2$( )) is a nonzero quotient of I_Q() and hence has nite length with unique irreducible quotient J_Q().

(b) $(J_P\ ())$ is a nonzero quotient of $I_{13}($ 1) and hence has nite length with unique irreducible quotient $J_{13}($ 1). Likewise, $(J_{13}($ 1)) is a nonzero quotient of $I_P\ ()$ and hence has nite length with unique irreducible quotient $J_P\ ()$.

(c) For all other nontempered 2 $\mathrm{Irr}(PGSp_6)$ dierent from those in (a) and (b), () = 0.

In particular, if 2 $\mathrm{Irr}(G_2\ PGSp_6)$ is such that is a quotient of the minimal representation , then

$$\text{nontempered}\ (\ )\quad \text{nontempered}:$$

Hence, we have shown Lemma 6.2 for nontempered representations and also Lemma 6.3.

14.2. Finiteness of $()_{nc}$. To complete the proof of Lemma 6.2, we need to show that for tempered 2 $\mathrm{Irr}(G_2)$ and 2 $\mathrm{Irr}(PGSp_6)$, the noncuspidal components $()_{nc}$ and $()_{nc}$ are of nite length.

To show that $()_{nc}$ has nite length, it suces to show that for each maximal parabolic subgroup $P_i = M_i N_i$ (with 1 i 3) of $PGSp_6$, the Jacquet module $J_{P_i}\ (())$ has nite length as an M_i-module. In other words, we need to show that the multiplicity space of the maximal -isotypic quotient of $r_{P_i}()$ has nite length as an M_i-module.

We have described in Propositions 13.5, 13.6 and 13.7 an equivariant ltration of $r_{P_i}\ ()$ as an $G_2\ M_i$-module and described the successive quotients. It suces to show that, for each of these successive quotients , the multiplicity space of the -isotypic quotient of has nite length. We shall explain how this can be shown, depending on whether is a top piece of the ltration or not. The dierence lies in the fact that the top piece of the ltration involves a minimal representation of a smaller group M_i and hence one needs to consider theta correspondence in lower rank situations. When is not the top piece of the ltration, the nite length of the multiplicity space of the maximal -isotypic quotient of as an M_i-module follows readily from the explicit description of . We give two examples as illustration:

Consider the case of $P_3 = GL_3\ N_3$. The bottom piece of the ltration in Proposition 13.5 is

$$= \mathrm{Ind}_{P\ Q_2}^{G_2\ GL_3} C_c(GL_2):$$

Then for 2 $\mathrm{Irr}(G_2)$,

$$() := \mathrm{Hom}_{G_2}(;)\ \ \mathrm{Hom}_M\ \ \mathrm{Ind}_{M\ Q_2}^{M\ GL_3} C_c(GL_2);\ r_P()$$

where $M = GL_2$. Now $r_P()$ is a nite length M-module and for any of its irreducible subquotient ,

$$\mathrm{Hom}_M\ \ \mathrm{Ind}_{M\ Q_2}^{M\ GL_3} C_c(GL_2);\qquad \mathrm{Ind}_{Q_2}^{GL_3} -$$

using the fact that the maximal -isotypic quotient of the regular representation $C_c(GL_2)$ is of the form – . On taking smooth vectors (which is a left exact functor), we see that () has a nite ltration whose successive quotients are submodules of $\mathrm{Ind}_{Q_2}^{GL_3} -$ for some irreducible . In particular, () has nite length.

Consider the case of $P_2 = M_2 N_2$ with $M_2 = (GL_2 \; GL_2)/GL^r =_1 GSO_4$. The bottom piece of the ltration in Proposition 13.7 is

$$= \mathrm{Ind}_{QM_2}^{G_2 M_2} W$$

where W is the Weil representation for $GL_2 \; GSO_4$. Then for 2 $\mathrm{Irr}(G_2)$, ()

$$:= \mathrm{Hom}_{G_2}(;) = \mathrm{Hom}_{LM_2}(W; r_Q())$$

Now $r_Q()$ has nite length as L-module (where $L = GL_2$) and if is an irreducible subquotient, $\mathrm{Hom}_{LM_2}(W;) = {}_W()$ where ${}_W()$ is the big theta lift of 2 $\mathrm{Irr}(GL_2)$ to GSO_4, which has nite length by the Howe duality theorem for classical (similitude) theta correspondence. From this, one deduces as above that () has nite length as an M_2-module.

Now let's consider the case when is the top piece of the ltration. From Propositions 13.5, 13.6 and 13.7, we see that we need to consider the following theta correspondences in lower rank:

- $G_2 \; PGL_3$ in E_6: for this case, the nite length of the big theta lift has been veried in Theorem 8.5.
- $G_2 \; SO_3 \; SO_{10}$ or $G_2 \; SO_5 \; SO_{12}$; we shall now treat these two cases together in the following proposition.

Proposition 14.2. Let ${}_n$ be the minimal representation of SO(2n) for n = 5 or 6. Then for tempered 2 $\mathrm{Irr}(G_2)$, ${}_n()$ is a nite length H_n-module where $H_n = SO_{2n \; 7}$.

Proof. We shall use the fact that the minimal representation of SO_{2n} (n = 5 or 6) is the big theta lift of the trivial representation of SL_2 (see [Y, Prop. 8.4] for the irreducibility of this big theta lift) and then appeal to the seesaw identity arising from the seesaw diagram:

$$\begin{array}{ccc} \widetilde{SL}_2 \; \widetilde{SL}_2 & & SO_{2n} \\ | & \times & | \\ SL_2 & & G_2 \; SO_{2n\;7} \end{array}$$

From this, we see that

(14.3) ${}_n() = \mathrm{Hom}_{G_2}({}_n;) = \mathrm{Hom}_{SL_2}($ ~ f ~ 2n 7; (); C) $= \mathrm{Hom}_{SL_2}($ f 2n 7; ; ()) as H_n-modules, where

- 2n 7; is the Weil representation of $SL_2 \; SO_{2n \; 7}$ (with respect to a nontrivial additive character of F); f
- () denotes the big f -theta lift of to SL_2, with respect to the Weil representation of $SL_2 \; SO_7 \; SL_2 \; G_2$.

We see in particular that if ${}_n()$ is nonzero, then has nonzero -theta lift to $\widetilde{SL}_2$. Moreover, it remains now to show that () has nite length as an $\widetilde{SL}_2$-module; the desired result would then follow from this and the Howe duality theorem for $SL_2 \; \widetilde{H}_n$.

Now the theta correspondence for $\widetilde{SL}_2$ G_2 has been completely determined in [GG06], though the niteness of ( ) was not formally stated there. Let us see how this niteness can be deduced from [GG06].

As before, let us write ( ) = ( )$_c$ ( )$_{nc}$ as a sum of its cuspidal and noncuspidal component. To show that ( )$_{nc}$ has nite length as a $\widetilde{SL}_2$-module, it suces to show that the Jacquet module of ( ) with respect to a Borel subgroup B = T N of $\widetilde{SL}_2$ has nite length as a $\widetilde{T}$-module. Now [GG06, Prop. 8.1] gives a short exact sequence of G_2-modules:

$$0 \quad ! \quad \mathrm{Ind}^{G_2}_{Q} C_c(GL_1) \quad ! \quad {}_N \quad ! \quad C \quad ! \quad 0$$

where the action of L = GL_2 on GL_1 is via det. The nite length of ( )$_N$ follows from this via a similar argument as above, by examining Hom_{G_2}($_N$;).

It remains to show that ( )$_c$ has nite length. In fact, it was shown in [GG06, Thm. 9.1(c) and (d)] that for genuine supercuspidal representations $_1$ $_2$ of $\widetilde{SL}_2$, one has ($_1$) ($_2$). In other words, ( )$_c$ is irreducible or 0. This shows that ( ) has nite length.

14.3. Finiteness of ( )$_{nc}$. For tempered 2 Irr($PGSp_6$), the nite length of ( )$_{nc}$ as a G_2-module is shown in the same way, using Propositions 13.1 and 13.3. We leave the details to the reader and only consider the top pieces in the ltration of the two Jacquet modules.

For the maximal parabolic subgroup P , we have to consider the theta correspondence for $PGSp_6$ PGL_2 with respect to the minimal representation $_6$ of $PGSO_{12}$. For the purpose of showing niteness, there is no harm in working with Sp_6 SL_2. Hence, the theta correspondence in question arises as follows. If V_2 and V_6 denote the 2-dimensional and 6-dimensional symplectic vector spaces, then we are considering the map

$$Sp(V_2)\ Sp(V_6) \quad ! \quad SO(V_2\ V_6)$$

and pulling back the minimal representation $_6$ of $SO(V_2\ V_6)$. As before, we shall use the fact that this minimal representation is the big theta lift of the trivial repre-sentation of SL_2. More precisely, let V'_2 be another symplectic space of dimension 2, then we have the map

$$Sp(V_2^0)\ Sp(V_2)\ Sp(V_6) \quad ! \quad Sp(V_2)^0\ SO(V_2\ V_6) \quad ! \quad Sp(V_2^0\ V_2\ V_6):$$

Given the Weil representation of $Sp(V_2^0)$ $SO(V_2\ V_6)$ and 2 Irr($Sp(V_6)$), we have

$$( ) = \mathrm{Hom}_{Sp(V_6)}(_6;) = \mathrm{Hom}_{Sp(V_2^0)\,Sp(V_6)}(; 1_{Sp(V_2^0)}) = \mathrm{Hom}(^0( ); 1_{Sp(V_2^0)})$$

where ^{0}( ) is the big theta lift of to $SO(V_2^0\ V_2)$. Note that there is a natural isogeny

$$Sp(V_2^0)\ Sp(V_2) \quad ! \quad SO(V_2^0$$

$V_2)$

whose image is of nite index. Hence, by the classical Howe duality theorem, $^0()$ is a nite length representation of $Sp(V^0)\,Sp(V_2)$. This implies that $()$ has nite length.

For the maximal parabolic subgroup Q, we need to consider the restriction of $_{A_5}$, a minimal representation of SL_6 to Sp_6. Note that $_{A_5}$ is a degenerate principal series representation induced from a maximal parabolic subgroup which stabilizes a line in the standard representation. Since Sp_6 acts transitively on such lines, we see that the restriction of $_{A_5}$ to Sp_6 is simply a degenerate principal series representation of Sp_6. This implies the desired niteness.

We have thus completed the proofs of Lemmas 6.2 and 6.3.

15. Howe Duality for G_2 $PGSp_6$: General Case

Finally, by combining Theorem 12.4 and Theorem 14.1, we can establish the Howe duality theorem for G_2 $PGSp_6$.

Theorem 15.1. Let 2 Irr(G_2).

(i) () is nonzero if and only if has zero theta lift to P D.

(ii) If () = 0, then () is a nite length representation of $PGSp_6$ with a unique irreducible quotient ().

(iii) For $_1$; $_2$ 2 Irr(G_2),

$$(_1) = (_2) = 0 =) \quad _1 = _2:$$

(iv) If () = 0, then () is tempered if and only if is tempered.

(v) If is non-tempered, then () is nonzero and the L-parameter of () is obtained from that of by composing with the natural inclusion $G_2(C)$ $Spin_7(C)$.

15.1. Explicit correspondence.

We can in fact determine the theta lift () explicitly if is tempered and noncuspidal. Indeed, we may also determine () for those tempered which has nonzero theta lift to PGL_3. To achieve this, we shall use the following four facts:

- If does not appear in the correspondence with P D, then () = 0 (Theorem 15.1(i)).
- If is tempered and () = 0, then () is irreducible and tempered (Theorem 15.1(iv)).
- If is nongeneric, then () is nongeneric (Corollary 11.2(ii)).
- The cuspidal support of () can be computed (from the Jacquet module computations of x13).

More precisely we have:

Theorem 15.2. Let be an irreducible tempered representation of G_2. Assume that is a lift of a (necessarily tempered) representation of PGL_3, that is, = $_B$() for some = . Then we have the following:

(i) If – then () = I_3() = I_3(–) 2 Irr($PGSp_6$).

(ii) If = – and the parameter of contains a trivial summand, then () = I_3().

(iii) If = – and the parameter of does not contain a trivial summand, then is one of the two representations $_{gen}$ = $_{B}(^{+})$ and $_{deg}$ = $_{B}($). In this case $I_3()$ = $I_3()_{gen}$ $I_3()_{deg}$,

$$(_{gen}) = I_3()_{gen} \text{ and } (_{deg}) = I_3()_{deg}:$$

We now deal with the remaining tempered representations of G_2. Non-supercuspidal representations are mostly constituents of the principal series $I_Q()$ where is a discrete series representation. These representations lift to constituents of the principal series $I_2($). More precisely, we have:

Theorem 15.3. Let be an irreducible tempered representation of G_2 which is not a lift from PGL_3. Then we have the following:

(i) Let be a unitary discrete series representation of GL_2. Then $I_Q()$ is irreducible if and only if $I_2($) is irreducible. We have:

If $I_Q()$ is irreducible, then

$$(I_Q()) = I_2( ):$$

If $I_Q()$ is reducible, then

$$(I_Q()_{gen}) = I_2( )_{gen} \text{ and } (I_Q()_{deg}) = I_2( )_{deg}:$$

(ii) Assume that = – is a supercuspidal representation of GL_2 with the trivial central char-acter. Let $_Q()$ and $_P()$ be the square integrable constituents of $I_Q(1=2;)$ and $I_P(1=2;)$. Then

$$(_Q()) = {}_2() \text{ and } (_P()) = {}_{13}()$$

where $_2()$ and $_{13}()$ are the square integrable constituents of $I_2(1=2;$) and $I_{13}(1=2;$ 1).

(iii) Assume that = – is a supercuspidal representation of GL_2 whose Langlands parameter has the image S_3. Recall that $I_Q(1;)$ has a square integrable constituent denoted by $_{gen}[]$. Then

$$(_{gen}[]) = {}_{gen}[]$$

where $_{gen}[]$ is the square integrable constituent of $I_2(1;$).

(iv) Assume that ^{2} = 1 and = 1. Recall that $I_Q(1=2; st)$ has a square integrable constituent denoted by $_{gen}[]$. Then

$$(_{gen}[]) = {}_{gen}[]$$

where $_{gen}[]$ is the square integrable constituent of $I_2(1=2; st$ st).

(v) Steinberg lifts to Steinberg:

$$(St_{G_2}) = St_{PGSp_6}:$$

Finally we need to deal with supercuspidal representations. In view of Theorem 15.1(i) and Theorem 15.2, we only need to consider those supercuspidal representations which do not lift to PGL_3 or PD. We rst introduce a thin family of supercuspidal representatons of G_2, namely those which participate in the theta correspondence for $\widetilde{SL}_2$ G_2. We have already encountered this theta correspondence in the proof of Proposition 14.2. As mentioned there, this theta correspondence has been studied in detailed in [GG06].

We rst introduce some notation. For each cuspidal representation of $PGL_2 = SO_3$, let $JL()$ be its Jacquet-Langlands lift to the anisotropic inner form $PB = SO_3$ (where B is the quaternion division algebra) and let be the -theta lift of $JL()$ to $\widetilde{SL}_2$ (where is a xed nontrivial additive character of F and the theta lift is induced by the Weil representation ! associated to). Then is an irreducible supercuspidal genuine representation of $\widetilde{SL}_2$. Consider now the -theta lift

$$:= () \ 2 \ \mathrm{Irr}(G_2)$$

of from $\widetilde{SL}_2$ to G_2. Now we recall some results from [GG06, Thm. 9.1]:

Lemma 15.4. With the above notations, we have:

(i) The representation is nonzero irreducible supercuspidal. Moreover, $() =$ under the theta correspondence for $\widetilde{SL}_2 \ G_2$.

(ii) The map ! is an injective map from the set $\mathrm{Irr}_{sc}(PGL_2)$ of supercuspidal representations of PGL_2 to $\mathrm{Irr}_{sc}(G_2)$.

(iii) Any 2 $\mathrm{Irr}_{sc}(G_2)$ which lifts to $\widetilde{SL}_2$ but not PGL_3 or PD is of the form for some 2 $\mathrm{Irr}_{sc}(PGL_2)$.

For 2 $\mathrm{Irr}(\widetilde{SL}_2)$ as above, we may also consider its -theta lift from $\widetilde{SL}_2$ to SO_5 and set

$$= () = () \ 2 \ \mathrm{Irr}(SO_5):$$

Then is a nongeneric supercuspidal representation of SO_5 belonging to a so-called Saito-Kurokawa A-packet. The representations and are related as follows:

Lemma 15.5. Consider the restriction of the minimal representation of SO_{12} to $G_2 \ SO_5$. Then for 2 $\mathrm{Irr}_{sc}(PGL_2)$,

$$() = :$$

Proof. We shall use the seesaw diagram in the proof of Proposition 14.2. The ensuing seesaw identity (14.3) and Lemma 15.4(i) give:

$$() = \mathrm{Hom}_{\widetilde{SL}_2}(\ _5;) = :$$

Hence $() =$.

Now we have:

Proposition 15.6. Let be an irreducible supercuspidal representation of G_2 that is not a lift from PGL_3 or PD. Then we have the following two possibilities:

- If = for some 2 $\mathrm{Irr}_{sc}(PGL_2)$ (as in Lemma 15.4), then $() =$

$$_1()$$

  where $_1()$ is the (nongeneric) square integrable subquotient of $I_1(1=2;)$ given in Proposition 10.4.

- If is not of the above form, then $()$ is supercuspidal.

Proof. Let be the minimal representation of E_7. Recall that r_{P_i} is the normalized Jacquet functor with respect to the maximal parabolic P_i in $_i$ $PGSp_6$. Then r_P (()) is a quotient of r_{P_i}().

By the assumption that does not lift to PGL_3, it follows that r_{P_i} (()) = 0 for i = 2; 3. Thus either r_{P_1} (()) = 0, in which case () is supercuspidal, or r_{P_1} (()) is a supercuspidal representation of the Levi factor L_1 = GSp_4. In fact, from Proposition 13.6, it follows that r_{P_1} (()) = jj$^{1=2}$ where is a (possibly reducible) supercuspidal representation of $PGSp_4$ = SO_5 such that appears as a quotient of the minimal representation of SO_{12}. By the seesaw in the proof of Proposition 14.2, we see that must have nonzero theta lift to

$SL_2 and hence is of the form for some 2 Irr(PGL_2) by Lemma 15.4(iii). Then Lemma 15.5 implies that = . By Frobenius reciprocity and the fact that () is tempered, we see that () = $_1$(), as desired.

As a consequence of the explicit results in this section, we have:

Corollary 15.7. If 2 Irr(G_2) is a discrete series representation which does not lift to PGL_3 or P D , then () is an irreducible discrete series representation of $PGSp_6$. As a result, any discrete series representation of G_2 lifts to a discrete series of exactly one of P D, PGL_3 or $PGSp_6$. That lift is Whittaker generic if and only if is.

Finally, we have the following consequence, proving a case of a conjecture of Prasad [Pr, Remark 4, page 624].

Corollary 15.8. Every 2 Irr(G_2) that lifts to $PGSp_6$ is self dual. In particular, every Whittaker generic irreducible representation of $G_2(F)$ is self-dual.

Proof. By inspection, it suces to prove for tempered . Then () is also tempered. Recall that the complex conjugate of a tempered irreducible representation is isomorphic to its dual. Furthermore, since theta lift commutes with taking complex conjugates, we have

$$(-) = () = () = \overline{()-}:$$

As irreducible representations of $PGSp_6$ are self dual [MVW, page 91], it follows that ()– = () and = –, since the theta correspondence is one to one.

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