TWISTED COMPOSITION ALGEBRAS AND ARTHUR PACKETS FOR TRIALITY Spin(8)

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1. Introduction

The purpose of this paper is to construct and analyze certain square-integrable automorphic forms on the quasi-split simply-connected groups Spin_8 of type D_4 over a number eld F. Since the outer automorphism group of Spin_8 is S_3, these quasi-split groups are parametrised by etale cubic F-algebras E and we denote them by Spin_8^E (to indicate the dependence on E). We shall specialize to the case when E is a cubic eld: this gives the so-called triality Spin_8.

The square-integrable automorphic forms we construct are associated to a family of discrete Arthur parameters which are quite degenerate. Indeed, apart from the A-parameters of the trivial representation and the minimal representation of Spin_8^E, the A-parameters we consider here are the most degenerate among the rest. These A-parameters are analogs of the cubic unipotent A-parameters for the exceptional group G_2 studied in [GGJ]. In particular, the component groups associated to these A-parameters can be the non-abelian group S_3, leading to high multiplicities in the automorphic discrete spectrum, as in [GGJ].

For each such A-parameter, we shall give a construction of the local A-packets and establish the global Arthur multiplicity formula. Both the local and global constructions are achieved using exceptional theta correspondences for a family of dual pairs H_C Spin_8^E in an ambient adjoint group of type E_6 (considered with its outer automorphisms); these dual pairs are associated to E-twisted composition algebras of dimension 2 over E. We shall in particular determine the local and global theta lifting completely. The automorphic forms constructed via these theta correspondences, though quite degenerate, can be cuspidal and have some special properties. For example, when one considers their Fourier coecients along the Heisenberg maximal parabolic subgroup of Spin_8^E (corresponding to the branch vertex in the Dynkin diagram), one sees that these automorphic forms support only one orbit of generic Fourier coecients: they are distinguished in the sense of Piatetski-Shapiro. The relevant Fourier coecients are parametrised by E-twisted composition algebras of E-rank 2, as shown in our earlier work [GS2] on twisted Bhargava cubes. Such properties allow us to determine their multiplicity in the automorphic discrete spectrum completely.

Because the objects mentioned above may be unfamiliar to the typical reader, and the precise results require a substantial amount of notation and language to state, we will leave

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the precise formulation of the results to the main body of the paper and content ourselves with the rather cursory overview above.

We would however like to emphasize the pivotal role played by the notion of a twisted composition algebra (of rank 2) and its relation to embeddings of the cubic algebra E into a degree 3 Jordan algebra (of dimension 9). This algebraic theory was created and developed by T. Springer (see [SV, Chap. 4] and [KMRT, x36]). Its relation with Spin_8^E has been explored in our earlier paper [GS2] and we shall apply the algebraic results of [GS2] to the study of automorphic forms here. In addition, we also need arithmetic results about twisted composition algebras and their automorphism groups, such as local and global Tate dualities, weak approximation and Hasse principles. These arithmetic results are supplied by the papers of Tate [T], Voskresenskii [V1, V2] and Prasad-Rapinchuk [PR]. These algebraic and arithmetic results, together with the representation theoretic results from exceptional theta correspondence, combine in rather intricate and (to these authors) utterly amazing ways to give the elegant Arthur multiplicity formula.

Given the length of the paper, it will be pertinent to provide a brief summary as a roadmap for the reader:

- We introduce in x2 the group $G_E = \mathrm{Spin}_8^E$ and its relevant structures, and give a description of its A-parameters in x3, reviewing Arthur's conjecture in the process.
- The theory of twisted composition algebras is introduced in x4. Though this theory is due to Springer, we have needed to supplement it with some observations of our own. In particular, Proposition 4.20 plays an important role in the interpretation of our results in the framework of Arthur's conjecture. We then recall in x5 our results from [GS2] concerning nondegenerate twisted Bhargava cubes and supplement the discussion with results about degenerate cubes.
- x6 is devoted to the construction of the various dual pairs that will be studied in this paper. It is followed by a detailed description of the Levi subgroup (of type A_5) of the Heisenberg parabolic subgroup of the adjoint group of type E_6 in x7.
- The minimal representation of the adjoint group of type E_6 is introduced in x8 and its Jacquet module for the Heisenberg parabolic subgroup is determined in x9,
- In the spirit of the tower property of classical theta correspondence, we determine the mini-theta correspondence for the Heisenberg Levi subgroup in x10. This is based on relating it to a classical similitude theta correspondence for unitary groups. It is needed for the study of the theta correspondence in E_6 which is carried out in x12, after introducing some notations for representations of G_E in x11. In particular, Theorem 12.1 is the main local result of this paper in the nonarchimedean case. We recall in x13 the analogous result in the archimedean case, but the proofs of Theorems 13.1, 13.2 and 13.3 there will be deferred to a joint paper with J. Adams and A. Paul.
- After this, we move to the global setting, starting with x14 which is devoted to the study of global theta correspondence. Here, we rst need to understand the space of automorphic forms of the disconnected group $H_C = \mathrm{Aut}_E(C)$, where C is a twisted composition algebra of rank 2. Not surprisingly, the automorphic multiplicity for H_C can be 1 or 2. In x15, we relate the relevant A-parameters to the theory of twisted

composition algebras. The important ingredients here are the local-global principles in Lemma 15.5, the consequence of local Tate-Nakayama duality in Proposition 15.12 and the global Poitou-Tate duality in Proposition 15.16. After this preparation, we interpret the space of global theta liftings in the framework of Arthur's conjecture in x16. More precisely, we construct the local A-packets as well as their bijection with characters of the local component groups, and then establish the Arthur multiplicity formula (AMF) for the space of global theta liftings in Theorem16.6. Finally, we show in Theorem 16.8 that the number provided by the AMF is in fact the true discrete multiplicity of the relevant representation in the automorphic discrete spectrum of G_E. For the interest of the reader, the following are some examples of numbers which arise as such multiplicities:

$$2^n; \quad \frac{2^n + 2(\ \ 1)^n}{3}; \quad \frac{2^n + (\ \ 1)^{n+1}}{3} \quad \text{for } n\ \ 0.$$

In particular, the multiplicities in the automorphic discrete spectrum are unbounded. The main source of these high multiplicities comes from the failure of Hasse principle for twisted composition algebras of E-dimension 2, or alternatively, the failure of Hasse principle for Jordan algebras of dimension 9.

- We have included two appendices. In Appendix A, we consider an analogous theta correspondence for a dual pair $SL_2(E)=_2\ G_E$ in E_7, associated to a rank 4 twisted composition algebra. This theta correspondence can be used to construct another family of Arthur packets for G_E, but we do not pursue this here. Instead, we only determine the theta lift of the trivial representation of $SL_2(E)=_2$ in Corollary 17.6; this result is used in our paper [GS3]. The long Appendix B is devoted to the study of unramied degenerate pricipal series representations of G_E for the various maximal parabolic subgroups and the various possibilities of E. Our approach is via the Iwahori Hecke algebra, and in each case, we determine the points of reducibility and the module structure at each such point. This allows us to introduce various interesting representations of G_E with nonzero Iwahori-xed vectors which intervene in the theta correspondence studied in the paper. In particular, we shall refer to the terminology and results of Appendix B in the description of theta lifting, for example in Theorem 12.1.

We wrap up this introduction by mentioning some recent papers which are devoted to the (automorphic) representation theory of triality $Spin_8$:

- the paper [L] of C.H. Luo on determining the unitary dual of the adjoint form of G_E over p-adic elds;
- the papers [Se1] and [Se2] of A. Segal on the structure of degenerate principal series representations (which builds upon and complements our results in Appendix B) and poles of degenerate Eisenstein series of G_E;
- the paper [La] of J.F. Lau on the determination of the residual spectrum of G_E.

It is interesting to relate the local and global A-packets we construct here with the results of these other papers.

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The work for this paper was initiated in 2015 and we are relieved to finally complete it. It is a pleasure to dedicate this paper to Dick Gross on the occasion of his 70th birthday. It is through our collaboration with him that we have developed a deeper appreciation for the beauty of exceptional groups and exceptional algebraic structures. We are grateful to have the opportunity to learn from him over the years and hope that there will continue to be such opportunities in the years to come.

2. Structure Theory of Spin_8^E

2.1. Etale cubic algebras. Let F be a field of characteristic 0 and with absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. An etale cubic algebra is an F-algebra E such that $E \otimes_F \overline{F} \cong \overline{F}^3$. More concretely, an etale cubic F-algebra is of the form:

$$E = \begin{cases} F \times F \times F; \\ F \times K, \text{ where } K \text{ is a quadratic field extension of } F; \text{ a} \\ \text{cubic field.} \end{cases}$$

Since the split algebra $F \times F \times F$ has automorphism group S_3 (the symmetric group on 3 letters), the isomorphism classes of etale cubic algebras E over F are naturally classified by the set of conjugacy classes of homomorphisms

$$\rho_E : \mathrm{Gal}(\overline{F}/F) \longrightarrow S_3.$$

By composing the homomorphism ρ_E with the sign character of S_3, we obtain a quadratic character (possibly trivial) of $\mathrm{Gal}(\overline{F}/F)$ which corresponds to an etale quadratic algebra K_E. We call K_E the discriminant algebra of E. To be concrete,

$$K_E = \begin{cases} F \times F; \text{ if } E = F^3 \text{ or a cyclic cubic field;} \\ K; \text{ if } E = F \times K; \\ \text{the unique quadratic subfield in the Galois closure of } E \text{ otherwise.} \end{cases}$$

We shall let χ_{K_E} denote the quadratic idele class character associated to K_E.

The etale cubic F-algebra E possesses a natural cubic form $N_{E/F} : E \longrightarrow F$ known as its norm form: for $a \in E$, $N_{E/F}(a)$ is the determinant of the multiplication-by-a map on the F-vector space E. Then there is a natural quadratic map

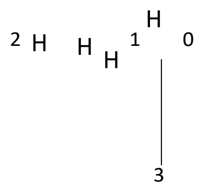
$$(2.1) \qquad (-)^{\#} : E \longrightarrow E$$

characterized by $a \cdot a^{\#} = N_{E/F}(a)$ for all $a \in E$.

2.2. Twisted form of S_3. Fix an etale cubic F-algebra E. Then, via the associated homomorphism ρ_E, $\mathrm{Gal}(\overline{F}/F)$ acts on S_3 (by inner automorphisms) and thus defines a twisted form S_E of the finite constant group scheme S_3. For any commutative F-algebra A, we have

$$S_E(A) = \mathrm{Aut}_A(E \otimes_F A).$$

2.3. D_4 root system. Let be a root system of type D_4 with a set of simple roots = f$_0$; $_1$; $_2$; $_3$g. The highest root is $_0$ = $_1$ + $_2$ + $_3$ + 2$_0$. The corresponding Dynkin diagram is



Hence the group Aut() of diagram automorphisms is identied with S_3 (the group of permutations of f1; 2; 3g).

2.4. Quasi-split groups of type D_4. Let G = $Spin_8$ be a split, simply connected Chevalley group of type D_4. We x a maximal torus T contained in a Borel subgroup B dened over F. The group G is then generated by root groups U = G_a, where 2 . Steinberg showed that one can pick the isomorphisms x : G_a ! U such that

$$[x(u); x_0(u^0)] = x_{+0}(uu^0)$$

whenever + ^{0} is a root. Fixing such a system of isomorphisms for 2 is xing an epinglage (or pinning) for G. By the discussion on page 40 in [FK], commutators signs can be specied by choosing an orientation of the Dynkin diagram. There is a short exact sequence:

$$1 \quad ! \quad G_{ad} = Inn(G) \quad ! \quad Aut(G) \quad ! \quad Aut() = S_3 \quad ! \quad 1:$$

As one can pick an orientation of the Dynkin diagram which is invariant under Aut(), one has a splitting S_3 = Aut() ! Aut(G), where the action of S_3 permutes the root subgroups U and the isomorphisms x.

Since S_3 is also the automorphism group of the split etale cubic F-algebra F^3, we see that every cubic etale algebra E denes a simply-connected quasi-split form G_E of G, whose outer automorphism group is the nite group scheme S_E. It comes equipped with a pair B_E T_E consisting of a Borel subgroup B_E containing a maximal torus T_E, both dened over F. Moreover, we inherit a Chevalley-Steinberg system of epinglage relative to this pair and a splitting of the outer automorphism group

$$S_E \ ,! \ Aut(G_E):$$

If E is a cubic eld, then Gal($\overline{F}$=F) permutes the roots $_1$, $_2$ and $_3$ transitively. If E = F K with K a quadratic eld, then without loss of generality, we assume that $_1$ is xed, whereas $_2$ and $_3$ are exchanged by the Galois action. If E is the split algebra, the Galois action on is trivial.

2.5. Center. The center of the split group G is

$$Z = \{(z_1; z_2; z_3) \in \mu_2 \times \mu_2 \times \mu_2 : z_1 z_2 z_3 = 1\}.$$

By Galois descent, we deduce that the center of G_E is

$$Z_E = \mathrm{Res}^1_{E/F}(\mu_2) = \mathrm{Ker}(N_{E/F} : \mathrm{Res}_{E/F}(\mu_2) \longrightarrow \mu_2).$$

In particular, from the short exact sequence

$$1 \longrightarrow Z_E \longrightarrow G_E \xrightarrow{p} G_E^{ad} \longrightarrow 1;$$

we deduce that

$$G_E^{ad}(F)/p(G_E(F)) = \mathrm{Ker}(H^1(F; Z_E) \longrightarrow H^1(F; G_E)). \qquad (2.2)$$

The finite group scheme Z_E will play an important role in this paper and we will see several other incarnations of it later on.

2.6. L-group. The Langlands dual group of G_E is the adjoint complex Lie group

$$G_E^\vee = PGSO_8(\mathbb{C}).$$

It inherits a pinning from that of G_E. The L-group ${}^L G_E$ is the semidirect product of $PGSO_8(\mathbb{C})$ with $\mathrm{Gal}(\overline{F}/F)$, where the action of $\mathrm{Gal}(\overline{F}/F)$ on $PGSO_8(\mathbb{C})$ is via the homomorphism ρ_E as pinned automorphisms. Thus there is a natural map

$${}^L G_E \longrightarrow PGSO_8(\mathbb{C}) \rtimes S_3;$$

whose restriction to $\mathrm{Gal}(\overline{F}/F)$ is ρ_E.

2.7. G_2 root system. The subgroup of G_E fixed pointwise by S_E is isomorphic to the split exceptional group of type G_2. Observe that $B_0 = G_2 \cap B_E$ is a Borel subgroup of G_2 and $T_0 = T_E \cap G_2$ is a maximal split torus of G_2. Via the adjoint action of T_0 on G_E, we obtain the root system Φ_{G_2} of G_2, so that

$$\Phi_{G_2} = \Phi|_{T_0}.$$

We denote the short simple root of this G_2 root system by α and the long simple root by β, so that

$$\alpha = \alpha_0|_{T_0} \quad \text{and} \quad \beta = \alpha_1|_{T_0} = \alpha_2|_{T_0} = \alpha_3|_{T_0}.$$

Thus, the short root spaces have dimension 3, whereas the long root spaces have dimension 1. For each root $\gamma \in \Phi_{G_2}$, the associated root subgroup U_γ is defined over F and the Chevalley-Steinberg system of epinglage gives isomorphisms:

$$U_\gamma \cong \begin{cases} \mathrm{Res}_{E/F}\, \mathbb{G}_a; & \text{if } \gamma \text{ is short;} \\ \mathbb{G}_a; & \text{if } \gamma \text{ is long.} \end{cases}$$

When E is a cubic field, T_0 is in fact the maximal F-split torus of G_E and Φ_{G_2} is the relative root system of G_E.

For each $\gamma \in \Phi_{G_2}$, we shall also let N_γ denote the root subgroup of G_2 corresponding to γ. In particular,

$$N_\gamma = U_\gamma \cap G_2.$$

Because the highest root β_0 of the D_4-root system restricts to that of the G_2-root system, we shall let β_0 denote the highest root of the G_2-root system also.

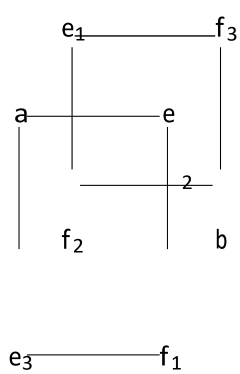
2.8. Two parabolic subgroups. The G_2 root system gives rise to 2 parabolic subgroups of G_E. One of these is a maximal parabolic $P_E = M_E N_E$ known as the Heisenberg parabolic. Its unipotent radical N_E is a Heisenberg group and its Levi subgroup M_E is spanned by the 3 satellite vertices in the Dynkin diagram. The other parabolic $Q_E = L_E U_E$ is a not-necessarily-maximal parabolic (it is not maximal over F); its Levi subgroup L_E is spanned by the branch vertex $_0$ and its unipotent radical U_E is a 3-step unipotent group. We shall need to examine the structure of these 2 parabolic subgroups more carefully.

2.9. The Heisenberg parabolic P_E. Let us begin with the Heisenberg parabolic $P = MN$ of G. Its unipotent radical N is a 2-step nilpotent group with the center $Z = [N; N] = U_0$. As we explained in [GS2], The Levi factor M can be identied with

$$GL_2(F^3)^{det} = fg = (g_1; g_2; g_3) \ j \ g_i \ 2 \ GL_2(F); \ det(g_1) = det(g_2) = det(g_3)g:$$

We may also identify $V = N=Z$ with F^2
F^2
F^2, so that the action of M on V corresponds
to the standard action of $GL_2(F^3)^{det}$ twisted by $det(g)^{\ 1} := det(g_i)^{\ 1}$ (for any i). Moreover, we can assume that the torus T M corresponds to the subgroup of $GL_2(F^3)^{det}$ consisting of g = $(g_1; g_2; g_3)$ where g_i are diagonal matrices, and the standard basis elements of F^2
F^2
F^2 correspond to the basis of $N=Z$ given by the xed pinning.

Thus, an element v 2 V can be conveniently represented by a cube



where $a; : : : ; b$ 2 F, and the vertices correspond to the standard basis in F^2
F^2
F^2. We shall assume that the vertex marked by a corresponds to $_0$, and that the vertex marked by b corresponds to $_0$ $_0$. The group Aut() acts as the group of symmetries of the cube xing these two vertices. We shall often write the cube as a quadruple

$$(a; e; f; b)$$

where $e = (e_1; e_2; e_3)$ and $f = (f_1; f_2; f_3)$ 2 F^3.

The quasi-split group G_E contains a maximal parabolic $P_E = M_E N_E$ which is a form of P. The structure of P_E can be determined by Galois descent. The highest root $_0$ is invariant under Aut(), hence the center Z_E is equal to the center Z of P. The Levi factor M_E can be identied with

$$GL_2(E)^{det} := fg \ 2 \ GL_2(E) : det(g) \ 2 \ Fg;$$

and

$$V_E := N_E = Z_E = U \ U_+ \ U_{+2} \ U_{+3} = F \ E \ E \ F$$

can be identied with the space of \twisted cubes" i.e. quadruples $(a; e; f; b)$ where $a; b$ 2 F and $e; f$ 2 E. The cube

$$v_E = (1; 0; 0; \quad 1)$$

is called the distinguished cube. Its stabilizer in M_E can be easily computed using Galois descent:

$$\mathrm{Stab}_{M_E}(v_E) = E^1 \text{ o } (Z=2Z)$$

where E^1 denotes the group of norm one elements in E. In this isomorphism, 2 E^1 corresponds to

$$_1 \quad 2\ GL_2(E)^{det}$$

and the nontrivial element in Z=2Z corresponds to w

$$= \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \ :$$

Note that $P_E \setminus G_2$ is the Heisenberg maximal parabolic $P_0 = M_0 N_0$ of G_2, with

$$M_0 = G_2 \setminus M_E = GL_2 \quad \text{and} \quad N_0 = G_2 \setminus N_E :$$

2.10. The 3-step parabolic Q_E. Now we come to the parabolic Q_E. The unipotent radical U_E has a ltration

$$f1g \ U_E^{(1)} \ U_E^{(2)} \ U_E$$

such that

$$U_E^{(1)} = U_0 \ U_0 \quad \text{is}$$

the center of U_E. Further,

$$U_E^{(2)} = [U_E; U_E] = U_0 \ U_0 \quad U_{2+}$$

is the commutator subgroup of U_E and is abelian. In particular, U_E is a 3-step unipotent group; hence we call Q_E the 3-step parabolic. Note that $Q_0 = G_2 \setminus Q_E = L_0 U_0$ is the 3-step maximal parabolic of G_2, with

$$L_0 = G_2 \setminus L_E = GL_2 \text{ and } U_0 = G_2 \setminus U_E :$$

One has an isomorphism

$$L_E = (GL_2 \ \mathrm{Res}_{E=F} G_m)^{det} = f(g; e) : \det(g) \ N_{E=F}(e) = 1g:$$

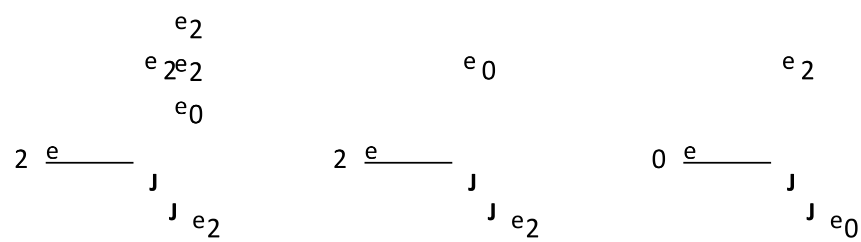
2.11. Nilpotent orbits. Assume that E is a eld. In this subsection, we shall describe the nilpotent orbits of $\mathrm{Lie}(G_E)(F) = g_E(F)$ and the centralizers of the nilpotent elements.

Let $t_E(F) = \mathrm{Lie}(T_E)(F)$ be the maximal toral subalgebra in $g_E(F)$. Let e be a nilpotent element in $g_E(F)$ belonging to a nilpotent $G_E(F)$-orbit . By the Jacobson-Morozov theorem, the element e is a member of an sl_2-triple $(f; h; e)$ dened over F, so that h is a semi-simple element such that $[h; e] = 2e$. We can assume that h 2 $t_E(F)$ and lies in the positive chamber. Then the values of the simple roots on h are nonnegative integers and give a marking of the Dynkin diagram of type D_4; this marking parameterizes the orbit . Note that the marking of the Dynkin diagram must necessarily be invariant under Aut(). In fact, this condition is necessary and sufficient (see [Dj]) for a nilpotent orbit in $\mathfrak{g}_E(F)$ to be defined over F and to have an F-rational point.

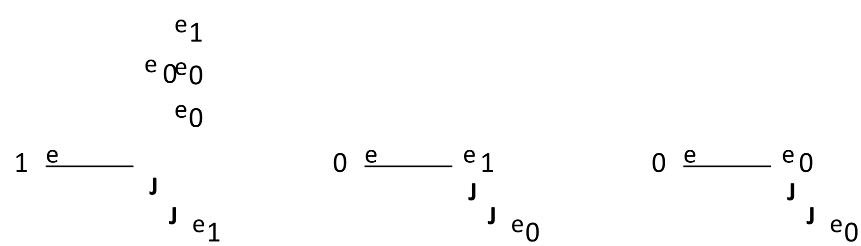
The semisimple element h gives a $\mathbb{Z}$-grading $\mathfrak{g}_E = \bigoplus \mathfrak{g}_{E,i}$, with $e \in \mathfrak{g}_{E,2}$. Let $P_e = M_e N_e$ be the parabolic group such that the Lie algebra of M_e is $\mathfrak{g}_{E,0}$. By a result of Kostant, the centralizer $Z_{M_e}(e)$ of e in M_e is the reductive part of $Z_{G_E}(e)$. Moreover, by Galois cohomology, the nilpotent $G_E(F)$-orbits contained in (F) are parametrized by

$$\mathrm{Ker}(H^1(F, Z_{M_e}(e)) \to H^1(F, G_E)).$$

We now list all nilpotent orbits defined over F and the corresponding $Z_{M_e}(e)$ (the re-ductive part of the centralizer $Z_{G_E}(e)$). First, we have three Richardson orbits corresponding to the following diagrams:



The first two diagrams correspond to the regular and the subregular orbit respectively, and the reductive part of the centralizer is the center of G_E in each case. The third case is the most interesting. In this case $Z_{M_e}(e)$ is generally disconnected and its identity component is a 2-dimensional torus. In fact, $Z_{M_e}(e) = \mathrm{Aut}_E(C)$ where C is an E-twisted composition algebra of E-dimension 2 (see later for this notion). We also have the three orbits corresponding to the following diagrams:



The first two orbits correspond to a short root $\varphi : \mathfrak{sl}_2(E) \to \mathfrak{g}_E(F)$ embedding and a long root embedding $\varphi : \mathfrak{sl}_2(F) \to \mathfrak{g}_E(F)$ respectively. The reductive part of the centralizer is isomorphic to $SL_2(F) \times Z$ and $SL_2(E)$, respectively. The last diagram corresponds to the trivial orbit.

Summarizing our findings, if F is a local field, then (F) consists of a single $G_E(F)$-orbit, except in one case when $G_E(F)$-orbits in (F) are parameterized by E-isomorphism classes of E-twisted composition algebras C of E-dimension 2.

2.12. Unipotent orbits of ${}^L G_E$. We also need a description of the conjugacy classes of maps

$$SL_2(\mathbb{C}) \to {}^L G_E \to G_E^\vee \rtimes S_3$$

which are invariant under the S_3-action. These correspond to unipotent conjugacy classes of $G_E^\vee = PGSO_8(\mathbb{C})$ which are stable under the action of S_3. As in the previous subsection, these unipotent conjugacy classes in turn correspond to markings of the D_4 Dynkin diagram which are invariant under the S_3-action. In particular, such markings have been enumerated in the previous subsection.

3. Arthur Parameters of Spin_8^E

In this section, we shall enumerate the (elliptic) Arthur parameters for G_E and single out a particularly interesting family of Arthur parameters. Thus, in this section, we assume that F is a number field and E is a cubic field extension of F.

3.1. A-parameters. An A-parameter for G_E is a $G_E^\vee$-conjugacy class of homomorphism

$$\psi : L_F \times SL_2(\mathbb{C}) \to {}^L G_E = G_E^\vee \rtimes_{\rho_E} \mathrm{Gal}(\overline{F}/F) \to G_E^\vee \rtimes S_3;$$

such that $pr_{S_3} \circ \psi|_{L_F} = \rho_E$, where pr_{S_3} stands for the projection

$$pr_{S_3} : G_E^\vee \rtimes S_3 \to S_3.$$

In particular, $\psi|_{SL_2(\mathbb{C})}$ is of the type considered in Section 2.12.

For each place v of F, we have a conjugacy class of embeddings $L_{F_v} \hookrightarrow L_F$, from which we obtain by restriction a local A-parameter

$$\psi_v : L_{F_v} \times SL_2(\mathbb{C}) \to G_E^\vee \rtimes S_3.$$

3.2. Component groups. For an A-parameter ψ, we set

$$S_\psi = \pi_0\left(Z_{G_E^\vee}(\mathrm{Im}(\psi))\right).$$

This is the global component group of ψ, and we say that ψ is elliptic if S_ψ is finite. Likewise, we have the local component group S_{ψ_v}. There is a natural diagonal map

$$\Delta : S_\psi \to S_{\psi,\mathbb{A}} := \prod_v S_{\psi_v}.$$

Hence there is an induced pullback map

$$\Delta^* : \mathrm{Irr}\, S_{\psi,\mathbb{A}} \to R(S_\psi);$$

where $R(S_\psi)$ denotes the (Grothendieck) representation ring of S_ψ.

3.3. Arthur's conjecture. We briey recall Arthur's conjecture. Associated to each elliptic A-parameter , one expects to have the following:

for each place v of F, a nite packet

$$_v = f_v : _v 2 \text{ IrrS }_v g$$

of unitary representations of nite length (possibly zero), indexed by the irreducible characters of the local component group S_v.

set

$$= f = \bigotimes_{v} {}_v : = {}_v{}_v 2 \text{ IrrS }_{;A} g; v$$

and

$$m = h(); i_S$$

where is a certain quadratic character of S (whose denition we won't recall here). Then the automorphic discrete spectrum L^2_{disc} of G_E contains a submodule isomorphic to

$$L^2 := \bigoplus_{2 \text{IrrS}_{;A}} m :$$

Moreover, we have:

$$L^2_{disc} = \bigoplus L^2$$

where the sum runs over equivalence classes of elliptic A-parameters .

3.4. Enumeration. In view of the above discussion, there are 6 families of A-parameters for G_E, according to the type of $j_{SL_2(C)}$. We list them below, together with the component group S :

(i) $j_{SL_2(C)}$ is the regular orbit: S is trivial and the resulting A-packet consists of the trivial representation (both locally and globally).

(ii) $j_{SL_2(C)}$ is the subregular orbit: S is trivial and the resulting local A-packet consists of the minimal representation.

(iii) $j_{SL_2(C)}$ is given by:

$$: SL_2(C) \; ! \; SO_3(C) \; SL_3(C) \; G_2(C) \; G_E : \quad -$$

This is the case of interest in this paper and we shall give a more detailed discussion in the next subsection.

(iv) $j_{SL_2(C)}$ is given by

$$: SL_2(C) \; ,! \; SL_2(C) \; SL_2(C) \; SL_2(C) \; ! \; M_E \; G_E ; \quad -$$

where the rst map is the diagonal embedding.

(v) $j_{SL_2(C)}$ is a root SL_2: we shall discuss this case briey as well.

(vi) $j_{SL_2(C)}$ is the trivial map: this is the tempered case.

3.5. The case of interest. Now we examine the case of interest (case (iii) above) in greater detail. The centralizer of (SL$_2$(C)) in G$_E$ is isomorphic to the subgroup

$$S \rtimes S_2 = f(a; b; c) \; 2 \; (C)^3 : abc = 1g \rtimes S_2;$$

where the nontrivial element of S$_2$ acts on S by inverting. Moreover, the group S$_3$ = Aut() commutes with (SL$_2$(C)) and S$_2$ and acts on S by permuting the coordinates. Thus we have an embedding

$$S \rtimes (S_2 \; S_3) \; ! \; G_E \rtimes S_3:$$

To give an A-parameter of this type is thus equivalent to giving a map

$$: L_F \; ! \; S \rtimes (S_2 \; S_3):$$

The composition of with the projection to S$_2$ S$_3$ gives a homomorphism L$_F$! S$_2$ S$_3$ and thus determine an etale quadratic algebra K and the xed etale cubic algebra E. We shall say that is of type (E; K).

To give an A-parameter of type (E; K) amounts to giving a L-homomorphism

$$L_F \; ! \; S \rtimes_{E\,K} W_F:$$

Now the group S $\rtimes_{E\,K}$ W$_F$ is the L-group of a torus

$$T_{E;K} = fx \; 2 \; (E \; {}_F \; K) : N_{E \; K=E}(x) \; 2 \; F g=K:$$

As shown in [GS2], this torus is the identity component of the E-automorphism group of any rank 2 E-twisted composition algebra C with quadratic invariant K$_C$ satisfying

$$[K_E] \; [K] \; [K_C] = 1 \; 2 \; F=F^2:$$

By an exceptional Hilbert Theorem 90 [GS2, Theorem 11.1], one has

$$T_{E;K} = T_{E;K_C} := fx \; 2 \; (E \; {}_F \; K_C) : N_{E \; K_C=E}(x) = 1 = N_{E \; K_C=K_C}(x)g:$$

Thus to give an A-parameter of type (E; K) is to give a L-parameter for the torus T$_{E;K}$, taken up to conjugation by S $\rtimes$ S$_2$. In other words, it is to give an automorphic character of T$_{E;K}$ up to inverse.

This suggests that the A-packet $_v$ or can be constructed as a \lifting" from T$_{E;K}$ to G$_E$. The goal of this paper is to carry out such a construction, using the fact that there is a dual pair

$$H_C \; G_E \; Aut(E_6)^J$$

where H$_C$ is the automorphism group of a rank 2 E-twisted composition algebra (whose identity component is T$_{E;K}$) and E$_6^J$ is an adjoint group of type E$_6$ (depending on a Freudenthal-Jordan algebra J with K$_J$ = K; see later).

3.6. An example. The simplest A-parameter of type $(E;K)$ is determined by the natural map

$$L_F \quad {}^K \quad {}^E ! \quad S_2 \ S_3 \quad ! \quad S \circ (S_2 \ S_3) \quad ! \quad G_E \circ S_3:$$

We denote this special A-parameter by $_{E;K}$. Its global component group is thus

$$S \quad = \begin{cases} {}_3 \circ S_2 = S_3 \text{ if } K = F \ F; {}^{E;K} \\ S_2 \text{ if } K \text{ is a eld.} \end{cases}$$

The local component groups $S_{E;K;v}$ are a bit more involved to describe, as they depend on the type of E_v and K_v. We list them in the following table.

E_v	K_v	$S_{E_v;K_v}$
eld	eld	S_2
eld	split	S_3
$F_v \ K_{E;v}$	K_v splits or $K_v = K_{E;v}$	S_2
$F_v \ K_{E;v}$	$K = K_{E;v}$ is a eld	$_2 \ S_2$
$F_v \ F_v \ F_v$	eld	$(_2 \ _2) \ S_2$
$F_v \ F_v \ F_v$	split	S_2

Let's see what Arthur's conjecture implies for this particular A-parameter, specialising to the case when $K = F \ F$ is split:

if E_v is a eld, then

$$_{E;K;v} = f_{1;v}; {}_{r;v}; {}_{;v}g$$

if $E_v = F_v \ K_{E;v}$ or F_v^3, then

$$_{E;K;v} = f_{1;v}; {}_{;v}g:$$

For appropriate disjoint nite subsets $_r$ and of the set of places of F, we thus have the representation

$$_{r;} = \left(\bigotimes_{v \in _r} {}_{r;v} \right) \left(\bigotimes_{v \in} {}_{;v} \right) \left(\bigotimes_{v \notin _r \cup} {}_1 \right)$$

in the global A-packet $_{E;K}$. The multiplicity attached to this representation is the multiplicity of the trivial representation of S_3 in $(r^{j_r j}) \ (\ ^{jj})$. A short computation using the character table of S_3 shows that this multiplicity is equal to

$$\begin{cases} \frac{1}{6} (2^{j_r j} + 2 (\ 1)^{j_r j}); & \text{if } _r \text{ is nonempty;} \\ \frac{1}{2} (1 + (\ 1)^{jj}); & \text{if } _r \text{ is empty.} \end{cases}$$

We shall see later how to construct this many automorphic realisations of $_{;\,,r}$ using exceptional theta correspondence.

3.7. Root SL_2. We consider briey the case when $j_{SL_2(C)}$ is a root SL_2. We may assume that $(SL_2(C))$ is the SL_2 corresponding to the highest root which is S_3-invariant. Then the centralizer of $(SL_2(C))$ in ${}^LG_E$ is

$$({}^LM_E)^{der} = (SL_2(C)\ SL_2(C)\ SL_2(C)) = f(a;b;c)\ 2\ _2 : abc^3 = 1g:$$

This is the L-group of

$$H = GL_2(E)^{det} = F:$$

Hence to give such an elliptic A-parameter is to give an L-parameter

$$: L_F \ ! \ {}^LH$$

which corresponds to an L-packet of $H = GL_2(E)^{det} = F$, or more simply to a cuspidal representation of $GL_2(E)$ (with trivial restriction to F).

As we shall see in x4.11, the group H is the E-automorphism group of a E-twisted composition algebra of E-rank 4. Indeed, given any E-twisted composition algebra C of E-rank 4, its automorphism group H_C is an inner form of H above and there is a dual pair (see x6.6)

$$H_C \ G_E \ E_7 ; \ ^B$$

where E_7^B is a group of type E_7 (associated to a quaternion algebra B). This suggests that the A-packets associated to as above can be constructed via exceptional theta lifting from H_C. We do not discuss this construction in this paper, but in Appendix A, we shall lay some algebraic and geometric groundwork to facilitate the further study of this case. In particular, we determine in Appendix A the theta lifting of the trivial representation of H to G_E. This is needed for our paper [GS3].

4. Twisted Composition and Freudenthal-Jordan Algebras

As we alluded to in the introduction and x3.5 above, the theory of twisted composition algebras plays a fundamental role in this paper. In this section, we shall briey recall this notion and its relation with Freudenthal-Jordan algebras. This theory is largely due to Springer, though we shall need to supplement it with some results and observations of our own needed for our application.

4.1. Twisted composition algebra. For a given etale cubic F-algebra E, an E-twisted composition algebra C is a vector space over E, equipped with a pair of tensors $(Q;\,)$ where

- $Q : C \ ! \ E$ is a non-degenerate quadratic form on C, and
- $: C \ ! \ C$ is a quadratic map

such that

$$(e\ x) = e^{\#}(x); \qquad Q((x)) = Q(x)^{\#} \qquad \text{and} \qquad N_C(x) := b_Q(x;(x))\ 2\ F;$$

for all $e\ 2\ E$ and $x\ 2\ C$, where $b_Q(x;y) = Q(x+y)\ \ Q(x)\ \ Q(y)$ and $e^{\#}$ is dened in (2.1).

Given two E-twisted composition algebras $(C; Q;)$ and $(C^0; Q^0; {}^0)$, an E-morphism between them is an E-linear map $: C \;\; ! \;\; C^0$ such that

$$Q^0 = Q \qquad \text{and} \qquad {}^0 = \;:$$

The automorphism group $\mathrm{Aut}_E(C; Q;)$ of a twisted composition algebra $(C; Q;)$ is an algebraic group over F.

These algebras were introduced by Springer and it is a fact that $\dim_E C = 1, 2, 4$ or 8. In this paper, we shall chiey be concerned with the case where $\dim_E C = 2$, though the case where $\dim_E C = 1$ or 4 will also be considered.

4.2. Rank 1 case. When $\dim_E C = 1$, we may write $C = E \;\; v_0$ for a basis vector $v_0 \; 2 \; C$. It is not dicult to see that the tensors $(Q;)$ are of the form

$$Q_a(x \;\; v_0) = a^\# \;\; x^2 \qquad \text{and} \qquad {}_a(x \;\; v_0) = a \;\; x^\# \;\; v_0$$

for some $a \; 2 \; E$. We shall denote this rank 1 E-twisted composition algebra by C_a. Its automorphism group is

$$\mathrm{Aut}(C_a) = \mathrm{Res}^1_{E=F}(\;_2) = \mathrm{Ker}(N_{E=F} \; : \mathrm{Res}_{E=F}(\;_2) \; ! \;\; _2):$$

We have encountered this group before in x2.5, as the center of the quasi-split group G_E, whence it was denoted by Z_E. The various interpretations of Z_E account for the intricate and sometimes surprising connections between dierent objects we will encounter later on.

Lemma 4.1. The E-isomorphism classes of rank 1, E-twisted composition algebras are parametrized by $E=F \; E^2$ under the construction $a \; ! \;\; C_a$.

Proof. For $a; b \; 2 \; E$, C_a is isomorphic to C_b if and only if there exists $\; 2 \; E$ such that

$$Q_b(v_0) = Q_a(v_0) \qquad \text{and} \qquad {}_b(v_0) = \;\; {}_a(v_0);$$

i.e.

$$a^\# = b^\# = \;^2 \qquad \text{and} \qquad a = b = \;^\# = :$$

In fact, the rst requirement above is implied by the second (on taking # on both sides). Now observe that

$$^\# = = N_{E=F}()=^2 \; 2 \; F \;\; E^2$$

and conversely, for any $e \; 2 \; E$ and $f \; 2 \; F$,

$$e^2 \;\; f = \frac{(e^\# f)^\#}{e^\# f}:$$

Hence, we deduce that

$$F \;\; E^2 = f^\# = : \; 2 \; Eg;$$

so that

$$C_a = C_b \;(\;)\;\; a = b \; 2 \; F \;\; E^2:$$

The lemma can also be shown via cohomological means. Namely, by considering the long exact sequence associated to the short exact sequence of algebraic groups

$$1 \longrightarrow Z_E = \mathrm{Res}^1_{E/F}\mu_2 \longrightarrow \mathrm{Res}_{E/F}\mu_2 \xrightarrow{N_{E/F}} \mu_2 \longrightarrow 1;$$

one sees that

$$H^1(F; Z_E) = \mathrm{Ker}(N_{E/F} : E^\times/E^{\times 2} \longrightarrow F^\times/F^{\times 2}).$$

Then [KMRT, Prop. 18.34] shows that the map $\#$ gives an isomorphism of $E^\times/F^\times E^{\times 2}$ with the kernel above.

4.3. **Rank 2 case.** Every twisted composition algebra (E, C, Q, β) has a cubic invariant: the etale cubic algebra E. On the other hand, when $\dim_E C = 2$, one can attach to it a quadratic invariant, i.e. an etale quadratic F-algebra K_C. Indeed, K_C is determined by the requirement that the discriminant quadratic algebra of Q is $E \otimes_F K_C$. In fact, C can be realized on $L := E \otimes K_C$ with Q and β given by

$$Q(x) = e \cdot N_{E \otimes K_C/E}(x) \quad \text{and} \quad \beta(x) = \bar{x}^{\#} \cdot e^{-1} \cdot \nu \quad \text{for}$$

some $e \in E^\times$ and $\nu \in K_C^\times$ satisfying

$$N_{E/F}(e) = N_{K_C/F}(\nu).$$

Here $\bar{x}$ and $\bar{\nu}$ refer to the action of the non-trivial automorphism of K_C on x and ν. We shall denote this rank 2 E-twisted composition algebra by $C_{e,\nu}$. For a more detailed discussion of this, see [GS2].

Given an E-twisted composition algebra $C = C_{e,\nu}$ as above, consider its automorphism group $H_C = \mathrm{Aut}_E(C) \subset GL_E(L)$. One has a short exact sequence

$$1 \longrightarrow (\mathrm{Aut}_E C)^0 \longrightarrow \mathrm{Aut}_E(C) \longrightarrow S_2 \longrightarrow 1$$

with

$$\mathrm{Aut}_E(C)^0(F) = T_{E,K_C}(F) := \{x \in L^\times : N_{L/E}(x) = 1 \quad \text{and} \quad N_{L/K_C}(x) = 1\}.$$

The identity component $H_C^0 = \mathrm{Aut}_E(C)^0$ is a 2-dimensional torus over F depending only on E and K_C and as (e, ν) varies, the algebraic subgroups $H^0_{C_{e,\nu}} \subset GL_E(L)$ are physically the same subgroup T_{E,K_C}. The conjugation action of S_2 on H_C^0 is by inversion. In particular, the center of H_C is

$$(H_C^0)^{S_2} = H_C^0[2] = \mathrm{Res}^1_{E/F}\mu_2 = Z_E. \tag{4.2}$$

Hence, we see yet another incarnation of the finite algebraic group Z_E; the consequence of this incarnation will be explained in §4.9 and §4.10.

The torus $H_C^0 = \mathrm{Aut}_E(C)^0$ can be interpreted as the group $\mathrm{Aut}_L(C)$ of L-linear automorphisms of C. It was observed in [GS2] that $C_{e,\nu}$ and $C_{e',\nu'}$ are L-linearly isomorphic if and only if there exists $x \in L^\times$ such that

$$e/e' = N_{L/E}(x) \quad \text{and} \quad \nu/\nu' = N_{L/K_C}(x); \tag{4.3}$$

in which case, multiplication-by-x gives an L-linear isomorphism $_x : C_{e;} \to C_{e';'}$. Moreover, the isomorphism $_x$ induces an isomorphism

$$(4.4) \qquad \mathrm{Ad}(_x) : \mathrm{Aut}_E(C_{e;}) \to \mathrm{Aut}_E(C_{e';'})$$

It is easy to check that the restriction of this isomorphism to the identity components is the identity map on $T_{E;K_C}$. In any case, we have shown:

Lemma 4.5. The L-isomorphism classes of E-twisted composition algebras of rank 2 and quadratic invariant K_C are parametrized by

$$(E \times K_C)^0/\mathrm{Im}(L)$$

where

$$(E \times K_C)^0 = \{(e;\,) \in E \times K_C : N_{E/F}(e) = N_{K_C/F}(\,)\}$$ and the

map $L \to E \times K_C$ is given by

$$x \to (N_{L/E}(x); N_{L/K_C}(x)):$$

This lemma can also be seen cohomologically. As was observed in [GS2], there is a short exact sequence of algebraic tori

$$1 \to T_{E;K_C} \to \mathrm{Res}_{L/F}\,\mathbb{G}_m \xrightarrow{N_{L/E}\,N_{L/K_C}} (\mathrm{Res}_{E/F}\,\mathbb{G}_m \times \mathrm{Res}_{K_C/F}\,\mathbb{G}_m)^0 \to 1$$

giving rise to an associated long exact sequence

$$1 \to T_{E;K_C}(F) \to L \to (E \times K_C)^0 \to H^1(F; T_{E;K_C}) \to 1:$$

$$(E \times K_C)^0/\mathrm{Im}(L)$$

There is a natural action of $\mathrm{Aut}(K_C/F)$ (as group automorphisms) on $(E \times K_C)/\mathrm{Im}(L)$ with the action of the nontrivial element given by $(e;\,) \to (e;\,)$. The orbits under this action parametrize the E-isomorphism classes of E-twisted composition algebras of rank 2 with quadratic invariant K_C. Observe that since $N_{E/F}(e) = $,

$$(e;\,) = (e^{-1};\;^{-1}) \in (E \times K_C)^0/\mathrm{Im}(L):$$

Hence, the action of $S_2 = \mathrm{Aut}(K_C/F)$ on $H^1(F; T_{E;K_C})$ is by inversion, and its fixed subgroup $H^1(F; T_{E;K_C})^{S_2}$ is the 2-torsion subgroup $H^1(F; T_{E;K_C})[2]$.

Finally, note that the map

$$H_C(F) := \mathrm{Aut}_E(C)(F) \to S_2$$

need not be surjective. Indeed,

$$H_C(F) = H_C^0(F) \rtimes (\;) \iff [C] \in H^1(F; T_{E;K_C})[2];$$

that is, the L-isomorphism class of C is fixed by $\mathrm{Aut}(K_C/F)$.

4.4. Freudenthal-Jordan algebras. Twisted composition algebras are closely related to Freudenthal-Jordan algebras; see [KMRT, Theorem 37.10] for a precise definition. Let J be a Freudenthal-Jordan algebra; it is a cubic Jordan algebra, so that every element $a \in J$ satisfies a characteristic polynomial

$$X^3 - T_J(a)X^2 + S_J(a)X - N_J(a) \in F[X].$$

The maps T_J and N_J are called the trace and norm maps of J respectively. The element

$$a^\# = a^2 - T_J(a)a + S_J(a)$$

is called the adjoint of a and satisfies $a \cdot a^\# = N_J(a)$. The cross product of two elements $a, b \in J$ is defined by

$$a \times b = (a + b)^\# - a^\# - b^\#.$$

The trace form T_J defines a nondegenerate bilinear form $\langle x, y\rangle = T_J(xy)$ on J. We shall identify J and J^* using this bilinear form. Let (x, y, z) be the symmetric trilinear form associated to the norm form N_J, so that $(x, x, x) = 6N_J(x)$. For any $x, y \in J$, one has

$$\langle x \times y, z\rangle = (x, y, z).$$

An etale cubic algebra E is an example of a Freudenthal-Jordan algebra. In general, it is a fact that $\dim_F J = 1$, 3, 6, 9, 15 or 27. In this paper, we shall largely be interested in the case where $\dim_F J = 9$, though the case where $\dim_F J = 15$ will also be considered.

The split Freudenthal-Jordan algebra of dimension 9 is simply the Jordan algebra M_3^+ of 3×3-matrices. Its automorphism group is

$$\mathrm{Aut}(M_3^+) = PGL_3 \rtimes S_2,$$

with the nontrivial element of S_2 acting by $a \mapsto a^t$. Hence, isomorphism classes of Freudenthal-Jordan algebras are classified by $H^1(F, \mathrm{Aut}(M_3^+))$. Since there is a natural homomorphism

$$H^1(F, \mathrm{Aut}(M_3)^+) \longrightarrow H^1(F, S_2),$$

one sees that to every Freudenthal-Jordan algebra J, one can attach an invariant which is an etale quadratic algebra K_J; this quadratic invariant determines the inner class of the group $\mathrm{Aut}(J)^0$ of type A_2. More generally, if J is a 9-dimensional Freudenthal-Jordan algebra, then $\mathrm{Aut}(J)$ sits in a short exact

$$1 \longrightarrow (\mathrm{Aut}J)^0 \longrightarrow \mathrm{Aut}J \longrightarrow S_2 \longrightarrow 1$$

where $\mathrm{Aut}(J)^0$ is an adjoint group of type A_2. Note that the map

$$H_J = \mathrm{Aut}(J)(F) \longrightarrow S_2$$

need not be surjective.

As explained in [KMRT, Prop. 37.6 and Theorem 37.12] and [GS2, §4.2], a Freudenthal-Jordan algebra J of dimension 9 over F is obtained from a pair (B, τ), where B is a central simple algebra over $K = K_J$ of dimension 9 and τ is an involution of second kind on B, as the subspace B^τ of τ-symmetric elements, equipped with the Jordan product $x \circ y = (xy + yx)/2$. For a fixed etale quadratic algebra K, this construction gives an essentially surjective faithful functor of groupoids:

$$\{K\text{-isomorphism classes of } (B, \tau)\} \longrightarrow \{F\text{-isomorphism classes of } J \text{ with } K_J = K\}$$

(where $\dim_K B = 9 = \dim_F J$); it is fully faithful and thus an equivalence if we allow F-linear isomorphisms on $(B;)$ and not just K-linear ones. Thus $\mathrm{Aut}(J) = {}^0\mathrm{Aut}_K(B;)$ and there is an S_2-action on the source given by

$$(B;) \to (B^{op};);$$

so that the fibers of the map are precisely the S_2-orbits (and hence have size 1 or 2). Further, $\mathrm{Aut}(J)^0(F) = \mathrm{Aut}(J)(F)$ if and only if the fiber of J has size 2, i.e. $(B;)\ (B^{op};)$.

4.5. Springer decomposition. Twisted composition algebras are related to Freudenthal-Jordan algebras by the Springer construction. Suppose we have an algebra embedding

$$i : E \hookrightarrow J.$$

Then, with respect to the trace form T_J, we have an orthogonal decomposition

$$J = E \quad C$$

where $C = E^{\perp}$. For $e \in E$ and $x \in C$, one can check that $e\ x \in C$. Thus, setting e

$$x := \quad e\ x$$

equips C with the structure of an E-vector space. Moreover, for every $x \in C$, write

$$x^{\#} = (\quad Q(x); (x)) \in E \quad C = J$$

where $Q(x) \in E$ and $(x) \in C$. In this way, we obtain a quadratic form Q on C and a quadratic map on C. Then, by [KMRT, Theorem 38.6], the triple $(C; Q;)$ is an E-twisted composition algebra over F. Conversely, given an E-twisted composition algebra C over F, the space $E \quad C$ can be given the structure of a Freudenthal-Jordan algebra over F, by [KMRT, Theorem 38.6] again. We recall in particular that for $(a; x) \in E \quad C$,

$$(a; x)^{\#} = (a^{\#} \quad Q(x); (x) \quad a\ x). \tag{4.6}$$

This construction gives a bijection

$$\{E\text{- isomorphism classes of } E\text{-twisted composition algebras}\}$$
$$|$$
$$\{H_J\text{-conjugacy classes of pairs } (J; i : E \hookrightarrow J)\}$$

where J is a Freudenthal-Jordan algebra of dimension 9 and $i : E \hookrightarrow J$ is an algebra embedding. Moreover, this bijection induces an isomorphism

$$H_C := \mathrm{Aut}_E(C) = \mathrm{Aut}(i : E \hookrightarrow J);$$

where the latter group is the pointwise stabilizer in $\mathrm{Aut}(J)$ of $i(E)\ J$. In other words, the Springer construction is an equivalence of groupoids. If an E-twisted composition algebra C corresponds to an embedding $i : E \hookrightarrow J$ under this equivalence, then one has:

$$[K_E]\ [K_C]\ [K_J] = 1 \in F^{\times}/F^{\times 2}. \tag{4.7}$$

One consequence of the Springer construction is that it gives us an alternative description of the torus $T_{E;K_C}$. It was shown in [GS2] that there is an isomorphism (an exceptional Hilbert Theorem 90),

$$\mathrm{Aut}_E(C_{e;})^0 = T_{E;K_C} = \tilde{T}_{E;K_J} = \{x \in (E \otimes_F K_J) : N_{E \otimes K_J / E}(x) \in F\} / K_J$$

when $J = E \quad C_{e;}$. We will next recall how this isomorphism arises.

4.6. An isomorphism of tori. Given an E-twisted composition algebra C corresponding to an embedding $: E \hookrightarrow J$, let us pick a pair $(B;)$ over K_J such that $J = B$. The embedding gives rise to an embedding of K_J-algebras compatible with involutions of second kind:

$$\tilde{} : E \otimes_F K_J \longrightarrow B;$$

where we have used the involution on $E \otimes K_J$ induced by the nontrivial automorphism of $K_J = F$. This induces an embedding of algebraic groups

$$\tilde{} : (E \otimes K_J) = K_J \longrightarrow PB = \mathrm{Aut}_{K_J}(B)$$

whose image is precisely the pointwise stabilizer of $\tilde{}$ in $\mathrm{Aut}_{K_J}(B)$. The map $\tilde{}$ restricts to give an isomorphism

$$\tilde{T}_{E;K_J} = \mathrm{Aut}_{K_J}(B; ; \tilde{}) \quad \mathrm{Aut}_{K_J}(B;):$$

where

$$\tilde{T}_{E;K_J} = \mathrm{Ker}\ N_{K_J=F} : (E \otimes_J K_J) = K \longrightarrow E = F \quad : \text{Since}$$

$$\mathrm{Aut}_{K_J}(B; ; \tilde{}) = \mathrm{Aut}_F(J;)^0 = \mathrm{Aut}_E(C)^0;$$

we see that the choice of a $(B;)$ with $J = B$ gives an isomorphism of algebraic groups

$$\tilde{T}_{E;K_J} \longrightarrow H_C^0 = \mathrm{Aut}_E(C)^0:$$

If one had chosen $(B^{op};)$ instead, the resulting isomorphism is the composite of the one for $(B;)$ with the inversion map. If it turns out that $(B;) = (B^{op};)$, then these two isomorphisms are conjugate by an element of $H_C(F) \setminus H_C^0(F)$. Thus, each E-twisted composition algebra C with quadratic invariant K_C comes equipped with a pair of isomorphisms of algebraic groups

$$_C; \ _C^{1} : H^0{}_C \longrightarrow T_{\tilde{E};K_J};$$

where $[K_E]\ [K_C]\ [K_J] = 1 \ 2 \ F = F^2$. This gives a canonical isomorphism

$$[_C] : H^0_C(F) = H^0_C(F)^2 = T_{\tilde{E};K_J}(F) = T_{\tilde{E};K_J}(F)^2:$$

In particular, if we consider $C = C_{e;}$ and $J = E \quad C_{e;}$, then we obtain a pair of isomorphisms of algebraic tori

(4.8) $$_{e;} \ _{e;}^{1} : T_{E;K_C} = T_{E;\tilde{K_J}} : \text{We}$$

have:

Lemma 4.9. The pair of isomorphisms in (4.8) is independent of the choice of $(e;)$.

Proof. Suppose first that $C_{e;}$ and $C_{e^0;^0}$ are L-isomorphic, with an L-isomorphism given by a multiplication-by-x map $_x$ as in (4.3) and (4.4). Then it follows by the functoriality of Springer's construction that

$$_{e;} = {}_{e^0;^0}^{1} \mathrm{Ad}(_x)|_{T_{E;K_C}} :$$

Here the sign arises because of the possibility of using a central simple algebra B or B^{op} in the construction of . We have observed after (4.4) that $\mathrm{Ad}(_x)$ is the identity map on $T_{E;K_C}$, so that $_{E;} = {}_{e^0;^0}^{1}$.

Now given any two $C_{e;}$ and $C_{e^0;^0}$, one knows that they become L_F F-isomorphic over a nite Galois extension $\tilde{F}$ of F. Hence the two pairs of isomorphisms $_{e;}$ and $_{e^0;^0}$ of algebraic tori become equal after a base change to $\tilde{F}$. But then they are already equal over F.

Thus we have a canonical pair of isomorphisms

$$(4.10) \qquad ; \ ^{1} : T_{E;K_C} = \tilde{T}_{E;K_J} :$$

This is the exceptional Hilbert 90 Theorem from [GS2]. It gives a canonical isomorphism

$$[] : T_{E;K_C}(F)/T_{E;K_C}(F)^2 = \tilde{T}_{E;K_J}(F)/\tilde{T}_{E;K_J}(F)^2 :$$

One consequence of this alternative description of H^0_C is that its gives an alternative computation of $H^1(F; H^0_C)$. In particular, it follows from [GS2, Prop. 11.2] that

$$(4.11) \qquad H^1(F; \tilde{T}_{E;K_J})[2] = E/F N_{E K_J/E}((E K_J)) :$$

This description of $H^1(F; T_{E;K_C})[2] = H^1(F; \tilde{T}_{E;K_J})[2]$ will be very helpful later on.

4.7. Examples. As an example, consider the case where $E = F^3$, and $J = M_3(F)$ is the Jordan algebra of 3 3 matrices. We have a natural embedding of F^3 into $M_3(F)$ where $(a_1; a_2; a_3) 2 F^3$ maps to the diagonal matrix with $a_1; a_2; a_3$ on the diagonal. If $x 2 M_3(F)$, then $x^\#$ is the adjoint matrix. Thus it is easy to describe the structure of the twisted composition algebra C in this case. An element x in C is given by a matrix

$$x = \begin{pmatrix} 0 & x_3 & y_2 \\ y_3 & 0 & x_1 \\ x_2 & y_1 & 0 \end{pmatrix} :$$

If we write $x = ((x_1; y_1); (x_2; y_2); (x_3; y_3))$ then the structure of F^3-space on C is given by

$$(a_1; a_2; a_3) \ ((x_1; y_1); (x_2; y_2); (x_3; y_3)) = ((a_1x_1; a_1y_1); (a_2x_2; a_2y_2); (a_3x_3; a_3y_3))$$

for all $(a_1; a_2; a_3) 2 F^3$. The structure of the twisted composition algebra on C is given by

$$Q((x_1; y_1); (x_2; y_2); (x_3; y_3)) = (x_1y_1; x_2y_2; x_3y_3)$$

and

$$((x_1; y_1); (x_2; y_2); (x_3; y_3)) = ((y_2y_3; x_2x_3); (y_3y_1; x_3x_1); (y_1y_2; x_1x_2)):$$

This twisted composition algebra $(C; Q;)$ has cubic invariant F^3 and quadratic invariant F^2.

Here is another example. Assume that E is a cyclic cubic eld extension of F, with Galois group generated by . Let D be a degree 3 central simple algebra over F containing E as a subalgebra. Then as a vector space over E, D has a basis $1; \$; \$^2$, for some element $\$ 2 D$ satisfying $\$x = (x)\$$, for all $x 2 E$, and $\$^3 = 2 F$. The corresponding E-twisted composition algebra is isomorphic to $C() = E E$, with

$$Q(x; y) = xy \text{ and } (x; y) = (^{1}y^\#; x^\#):$$

Moreover, C ( ) has cubic invariant E and quadratic invariant F^2 and is associated to (e;) = (1; (; ^{1})). The algebra D is split if and only if is a norm of an element in E. The group of E-automorphisms of C(1) is

$$Aut_E(C(1)) = E^1 \text{ o } (Z=2Z)$$

where 2 E^1 acts on C(1) by (x; y) ! (x; y), and the nontrivial element in Z=2Z by (x; y) ! (y; x), for all (x; y) 2 C(1).

4.8. When is J division? Following up on the last example above, one may consider the question: under what conditions on (e;) is $J_{e;}$ = E $C_{e;}$ associated to a division algebra? An answer for the general case is provided by [KMRT, Thm. 38.8], but we provide an alternative treatment adapted to the rank 2 case here.

Proposition 4.12. Fix (e;) 2 (E K)^0, so that $N_{E=F}$(e) = $N_{K_C=F}$( ). Then the following are equivalent:

(i) 2 $N_{L=K_J}$(L) (where L = E K_C); (ii) (e;) = (e^0; 1) 2 (E K_C)^0=Im(L);

(iii) (e;) = (e^0; ^{0}) 2 (E K_C)^0=Im(L), with ^{0} 2 F; (iv) [(e;)] 2 $H^1(F; T_{E;K_C})[2]$;

(v) J = E $C_{e;}$ is not a division Jordan algebra.

When these equivalent conditions hold for C, $H_C(F) = H^0_C(F)$ o Z=2Z. Indeed, for any C = $C_{e;}$ with 2 F,

$$Aut_E(C_{e;}) = T_{E;K_C} \text{ o } Aut(K_C=F) \; GL_E(L):$$

In other words, these automorphism groups are physically the same subgroup of $GL_E(L)$.

Proof. We rst show the equivalence of the rst four statements. The implications (i) =) (ii) =) (iii) =) (iv) are clear. Assume that (iv) holds, so that [(e;)] = [(e;)]. Then there exists x 2 L such that

$$N_{L=E}(x) = x \; x = 1 \quad \text{and} \quad = x \; x^{\#}:$$

Now the rst condition implies that x = z z for some z 2 L, which when substituted into the second gives $N_{L=K}$ (z) 2 F. Hence, replacing (e;) by an equivalent pair, we may assume that 2 F, so that $N_{E=F}$(e) = ^{2}. But then

$$(e;) = (e \; N_{L=E}(e); \; N_{L=K_C}(e)) = (e^3; ^3) \; 2 \; (E \; K_C)^0=Im(L): \text{Since } ^3 = N_{L=K_C}(),$$

we conclude that (i) holds.

We note that the equivalent conditions (i)-(iv) always hold when E is not a eld, for then the norm map $N_{L=K_C}$: L ! K_C is surjective.

Finally, to check the equivalence with (v), note that J = E $C_{e;}$ is not a division Jordan algebra if and only if there exists nonzero (a; x) 2 E $C_{e;}$ such that (a; x)$^\#$ = 0. By (4.6), this is equivalent to

$$a^{\#} = Q(x) = e \; N_{L=E}(x) \quad \text{and} \quad a \; x = (x) = e^{1} \; x^{\#}: \tag{4.13}$$

When E is not a eld, we can always take nonzero $(a; 0)$ with $a^{\#} = 0$, so that J is never a division algebra in this case.

We may henceforth assume that E is a eld. Suppose that (ii) holds, so that = 1 and $N_{E=F}$ (e) = 1. Then we may take $(a; x) = (e; e^{\#})$; one checks that this satises the two equations in (4.13) and hence J is not division. We have thus shown (ii) =) (v).

Conversely, we shall show (v) implies (i) (when E is a eld). Assume that there is a nonzero $(a; x)$ such that the two equations in (4.13) hold. Then x must be nonzero (otherwise, we deduce by the rst equation that $a^{\#} = 0$ and hence $a = 0$ since E is a eld). Multiplying the two equations in (4.13), we obtain

$$N_{E=F}(a) \ x = \ N_{L=K_C}(x) \ x; \text{ so}$$

that

(4.14) $$x \ (N_{E=F}(a) \quad N_{L=K_C}(x)) = 0:$$

Hence, if K_C is a eld, so that L is a eld also, then we may cancel x (noting that x = 0) to deduce that

$$= N_{E=F}(a) \ N_{L=K_C}(x) \ ^{1} \ 2 \ N_{L=K_C}(L): \text{On}$$

the other hand, if $K_C = F \ F$, then let

$$x = (x_1; x_2) \ 2 \ E \ E = L \qquad \text{and} \qquad = (_1; _2) \ 2 \ F \ F: \text{The}$$

two equations in (4.13) becomes:

$$a^{\#} = e \ x_1 x_2 \qquad \text{and} \qquad (ax_1; ax_2) = e^{\ 1} (_2 \ x_2^{\#}; _1 \ x_1^{\#}):$$

From this, we see that a = 0 (otherwise, the second equation would give $x_1 = x_2 = 0$ also), and hence $x_1; x_2 \ 2 \ E$. Hence, we may cancel x in (4.14) as before and conclude that 2 $N_{L=K_J}$ (L), as desired.

4.9. Embeddings. We record here some results that we will need later, concerning embeddings of rank 1 twisted composition algebras into rank 2 ones.

Lemma 4.15. Let us x

- a 2 E with corresponding rank 1 E-twisted composition algebra $C_a = E$ and
- an E-twisted composition algebra $C = C_{e;}$ of rank 2, corresponding to an embedding E ,! J, with resulting Springer decomposition J = E C.

There are natural equivariant bijections between the following three $\mathrm{Aut}_E(C)$-sets (possibly empty)

(a) the set of E-morphisms $f : C_a \ ! \ C$;
(b) the set of rank 1 elements x 2 J (i.e. $x^{\#} = 0$ but x = 0) of the form x = (a; v) 2 E C = J;
(c) the set

$$X_{a;C}(F) = X_{a;e;}(F) = fx \ 2 \ L := E$$
$$K_C : N_{L=E}(x) = e^{\ 1} a^{\#} \text{ and } N_{L=K_C}(x) = N_{E=F}(a)^{\ 1} g:$$

The bijection between (a) and (b) is given by f ! (a; f(1)), whereas that between (b) and (c) is given by x = (a; v) ! v.

Note that the 3 sets are possibly all empty. For example, if J is associated with a cubic division algebra, then there are no rank 1 elements in J, so that the set in (b) is empty, and hence so are the other 2 sets. On the other hand, we note:

Lemma 4.16. For any $a \in E$, there exists a unique $E \otimes K_C$-isomorphism class $[C]$ such that $X_{a;C}(F)$ is nonempty. This unique $E \otimes K_C$-isomorphism class is represented by $C_{a^{\#};N_{E/F}(a)}$. Hence we have a group homomorphism

$$f : E/F E^{2} \to (E \otimes K_C)^{0}/\mathrm{Im}(L)$$

given by

$$f(a) = (a^{\#}; N_{E/F}(a))$$

and characterized by the requirement that C_a embeds into $C_{e;}$ if and only if $(e;) = f(a) \in H^1(F; T_{E;K_C})$. The image of f is equal to $H^1(F; T_{E;K_C})[2]$, i.e. consists precisely of those twisted composition algebras C whose associated Jordan algebra is not division, whereas

$$\mathrm{Ker}(f) = \{x^{\#}/x : x \in L \text{ and } N_{L/K_C}(x) \in F\} / F E^{2}:$$

Proof. It is clear that if $C = C_{a^{\#};N_{E/F}(a)}$, then $1 \in X_{a;C}(F)$; this shows the existence of C and that it has the desired form. For the uniqueness, suppose that $X_{a;e;}(F)$ and $X_{a;e^0;{}^0}(F)$ are both nonempty. Then there exist $x; x \in L^0$ such that

$$N_{L/E}(x) = e^{-1}a^{\#} \quad \text{and} \quad N_{L/K}(x) = N_{E/F}(a)^{-1}$$

and

$$N_{L/E}(x^0) = e^{0\,-1}a^{\#} \quad \text{and} \quad N_{L/K}(x^0) = N_{E/F}(a)^{0\,-1}:$$

On dividing one equation by the other, we see that

$$N_{L/E}(x^0/x) = e/e^0 \quad \text{and} \quad N_{L/K}(x^0/x) = /^0:$$

This implies that $(e;) = (e^0;{}^0) \in H^1(F; T_{E;K_C})$, as desired.

By Proposition 4.12, the image of f consists of twisted composition algebras associated to non-division Jordan algebras J. On the other hand, to prove that any such C is in the image of f, it suces by Proposition 4.12 to consider $C = C_{e;1}$, with $N_{E/F}(e) = 1$. We claim that $f(e) = [(e;1)]$. Indeed,

$$f(e) = (e^{\#}; N_{E/F}(e)) = (e^{-1}; 1) = (e; 1) \in H^1(F; T_{E;K_C}):$$

We leave the statement about $\mathrm{Ker}(f)$ to the reader.

Since the image of the map f in the above lemma is $H^1(F; T_{E;K_C})[2]$, we deduce from (4.11) that f can be simply interpreted as the natural map

(4.17) $$f : E/F E^{2} \to E/F N_{E \otimes K_J/E}((E \otimes K_J)):$$

Finally, we note that $X_{a;C} = X_{a;e;}$ is an algebraic variety which is evidently a torsor for the torus $H_C^0 = T_{E;K_C}$. If $X_{a;e;}(F)$ is nonempty, then $H_C^0(F) = T_{E;K_C}(F)$ acts simply transitively on it. Thus, the action of $H_C(F)$ on $X_{a;e;}(F)$ is transitive and the stabilizer of a point $x \in X_{a;e;}(F)$ has order 2, with the nontrivial element $h_x \in H_C(F) \setminus H_C^0(F)$.

For example, the stabilizer of $1 \in X_{a;C_{a^{\#};N(a)}}(F)$ is $\mathrm{Aut}(K_C/F)$. Indeed, h_x is the map on $C_{e;} = E$ K_C given by

$$h_x : z \mapsto \frac{x}{\bar{x}}\,\bar{z}.$$

If $x' \in X_{a;e;}(F)$ is another element, then $x' = t\,x$ for a unique $t \in H^0_C(F)$ and

$$h_{x'} = t\,h_x\,t^{-1} = t^2\,h_x.$$

Thus the element h_x gives a well-defined class in $(H_C(F) \backslash H^0_C(F))/H^0_C(F)^2$ as $x \in X_{a;e;}(F)$ varies. We record this as a lemma.

Lemma 4.18. Suppose that $f(a) = [C] \in H^1(F; T_{E;K_C})[2]$ so that $X_{a;C}(F)$ is nonempty. Then one obtains a class

$$g_C(a) \in (H_C(F) \backslash H^0_C(F))/H^0_C(F)^2$$

consisting of elements which stabilize some points in $X_{a;C}(F)$.

4.10. Cohomological interpretation. The embedding problem studied in the previous subsection can be given a rather transparent cohomological treatment. The map f in Lemma 4.16 is a surjective homomorphism $H^1(F; Z_E) \to H^1(F; T_{E;K_C})[2]$. This map can be obtained from our observation in (4.2) that $T_{E;K_C}[2] = Z_E$. From the Kummer exact sequence

$$1 \to Z_E \to T_{E;K_C} \xrightarrow{2} T_{E;K_C} \to 1;$$

one deduces the following fundamental short exact sequence
(4.19)

$$1 \to T_{E;K_C}(F)^2 \backslash T_{E;K_C}(F) \xrightarrow{b} H^1(F; Z_E) \xrightarrow{f} H^1(F; T_{E;K_C})[2] \to 1.$$

The map f here is precisely the one described in Lemma 4.16. This cohomological discussion also gives us a more conceptual description of $\mathrm{Ker}(f)$:

$$\mathrm{Ker}(f) = T_{E;K_C}(F)^2 \backslash T_{E;K_C}(F).$$

The map b can be described explicitly as follows. Given $t \in T_{E;K_C}(F)$ L, since $N_{L/E}(t) = 1$, we can write

$$t = \bar{y}/y \qquad \text{with } N_{L/K_C}(y) \in F \ \ (\text{since } N_{L/K_C}(t) = 1).$$

Then

$$b(t) = y^{\#}/y \in E/F\,E^2;$$

The reader can easily verify that b(t) is independent of the choice of y and is trivial if $t \in T_{E;K_C}(F)^2$.

Here is another interesting observation arising from (4.19) and Lemma 4.18. Let us fix $[C] \in H^1(F; T_{E;K_C})[2]$ and consider the fiber $f^{-1}([C])$ which is a $T_{E;K_C}(F)/T_{E;K_C}(F)^2$-torsor. Then we have:

Proposition 4.20. The map $a \mapsto g_C(a)$ (with $g_C(a)$ defined in Lemma 4.18) gives an isomorphism

$$f^{-1}([C]) \to (H_C(F) \backslash H^0_C(F))/T_{E;K_C}(F)^2$$

of $T_{E;K_C}(F)/T_{E;K_C}(F)^2$-torsor.

Proof. Assume without loss of generality that $C = C_{e;\delta}$. Since both $f^{-1}([C])$ and $(H_C(F) \setminus H_C^0(F))/T_{E;K_C}(F)^2$ are torsors under $T_{E;K_C}(F)/T_{E;K_C}(F)^2$, it suces to show that if a'

$$= b(t) \cdot a \in f^{-1}([C]);$$

then

$$g_C(a') = t \cdot g_C(a) \in (H_C(F) \setminus H^0_C(F))/T_{E;K_C}(F)^2:$$

Write

$$t = \bar{y}/y \quad \text{with } N_{L/K_C}(y) \in F;$$

so that

$$b(t) = y^{\#}/y \quad \text{and hence} \quad a' = a \cdot y^{\#}/y$$

This implies in particular that

$$N_{E/F}(a') = N_{E/F}(a) \cdot N_{L/K_C}(y) \quad \text{and} \quad a'^{\#} = a^{\#} \cdot N_{L/E}(y):$$

Now suppose that $x \in X_{a;e;\delta}(F) \subset L$, so that

$$N_{L/E}(x) = e^{-1} a^{\#} \quad \text{and} \quad N_{L/K_C}(x) = N_{E/F}(a)^{-1}:$$

Then one checks that $x' := xy \in X_{a';C}(F)$. Hence, if h_x and $h_{x'}$ are the nontrivial elements stabilizing x and x' respectively, then for any $z \in C$,

$$h_{x'}(z) = \frac{x'}{\bar{x}'} \bar{z} = \frac{xy}{\bar{x}\bar{y}} \bar{z} = t^{-1} h_x(z):$$

Thus we have

$$h_{x'} = t \cdot h_x \in (H_C(F) \setminus H_C^0(F))/T_{E;K_C}(F)^2:$$

Indeed, if $[C]$ is a nontrivial element of $H^1(F; T_{E;K_C})[2]$, then $[C]$ generates a subgroup of order 2 and we have a short exact sequence of abelian groups

$$(4.21) \qquad 1 \longrightarrow T_{E;K_C}(F)/T_{E;K_C}(F)^2 \longrightarrow f^{-1}(\langle [C] \rangle) \longrightarrow \langle [C] \rangle \longrightarrow 1:$$

On the other hand, with $C = C_{e;1}$ (without loss of generality), one has another extension:

$$(4.22) \qquad 1 \longrightarrow T_{E;K_C}(F)/T_{E;K_C}(F)^2 \longrightarrow H_C(F)/T_{E;K_C}(F)^2 \longrightarrow S_2 \longrightarrow 1$$

Then the following is a consequence of Proposition 4.20:

Proposition 4.23. *The two extensions (4.21) and (4.22) are isomorphic via a canonical isomorphism of extensions dened as follows. For any $a \in E^\times/F^\times E^{\times 2} = H^1(F; Z_E)$ with $f(a) = [C]$, the isomorphism sends a to $g_C(a)$.*

4.11. Rank 4 and 8 cases. We conclude with a brief sketch of the rank 4 and 8 cases. The case $\dim_E C = 4$ corresponds to embeddings of Jordan algebras $E \; ! \; J$ with $\dim_F J = 15$. Examples of such J are of the form $H_3(B)$, the Jordan algebra of 3 3-Hermitian matrices with entries in a quaternion algebra B. This case is discussed in some detail in Appendix A below. We simply note here that the automorphism group of such a C is

$$\mathrm{Aut}_E(C) = \mathrm{Res}_{E=F}(B \;{}_F\; E)^{\det} = F$$

where the RHS consists of elements in $(B \; E)$ whose norm lies in F. See x6.6 below.

Finally, when $\dim_E C = 8$, one has $\dim_F J = 27$, so that J is an exceptional Jordan algebra. An example is $J = H_3(O)$, the Jordan algebra of 3 3-Hermitian matrices with entries in an octonion algebra O. When the octonion algebra is split, the automorphism group of such a C is isomorphic to the group

$$G_E = \mathrm{Spin}_8^E :$$

Moreover, the action of G_E on C is (the Galois descent of) the sum of the 3 irreducible 8-dimensional representations of Spin_8 over $\overline{F}$. It is no wonder that the structure of the group G_E is intimately connected with the theory of twisted composition algebras.

5. Twisted Bhargava Cubes

To connect the theory of twisted composition algebras with our earlier discussion on $G_E = \mathrm{Spin}_8^E$, let us recall the main result of [GS2].

5.1. Nondegenerate cubes. Recall the Heisenberg parabolic subgroup $P_E = M_E N_E \; G_E$ and the natural action of $M_E = GL_2(E) \; {}^{\det}$ on the space $V_E = N_E = [N_E; N_E]$ of E-twisted cubes. Now we have [GS2, Prop. 10.4]:

Proposition 5.1. The nondegenerate $M_E(F)$-orbits on $V_E(F)$ are in natural bijection with E-isomorphism classes of E-twisted composition algebras of rank 2. More precisely, to every nondegenerate E-twisted cube , we attached in [GS2] a pair $(Q;)$ giving a structure of E-twisted composition algebra on $E \; E$, with an isomorphism

$$\mathrm{Stab}_{M_E(F)}() = \mathrm{Aut}_E(Q;):$$

If $g \; 2 \; M_E(F) = GL_2(E)^{\det}$ and ${}^0 = g()$, then the pair $(Q_0; {}_0)$ attached to ^{0} is obtained from $(Q;)$ by the change of variables given by the matrix g, i.e.

$$Q_0 = Q \; {}^t g \qquad \text{and} \qquad {}_0 = {}^t g^{\;1} \; {}^t g:$$

Hence,

$$g \; 2 \; \mathrm{Stab}_{GL_2(E)^{\det}}() \; GL_2(E)^{\det} \; (\;) \quad {}^t g^{\;1} \; 2 \; \mathrm{Aut}_E(E^2; Q;):$$

In particular, if F is a local eld, then the $M_E(F)$-orbits of generic unitary characters of $N_E(F)$ are parametrized by E-twisted composition algebras (modulo E-isomorphisms). Likewise, when F is a number eld, the $M_E(F)$-orbits of (abelian) Fourier coecients along N_E are parametrised by E-twisted composition algebras (modulo E-isomorphisms).

We shall not need the general procedure to pass from to $(Q;)$, but only for the so-called reduced cubes:

Proposition 5.2. (i) If = (1; 0; f; b) 2 $V_E(F)$ (such a is called a reduced cube), then its associated pair (Q;) is given by:

$$Q(x; y) = \quad f x^2 \quad bxy + f^{\#} y^2$$

and

$$(x; y) = (\quad by^{\#} \quad (f x) \ y; x^{\#} + f y^{\#})$$

so that (1; 0) = (0; 1).

(ii) Conversely, let (C; Q;) be an E-twisted composition algebra of E-dimension 2. For v 2 C, set (v) := $N_C(v)^2 \quad 4N_{E=F}$ (Q(v)) 2 F. Then there exists v 2 C such that (v) = 0. Moreover, the set fv; (v)g is an E-basis of C if and only if (v) = 0. Given such a v 2 C and identifying C with E E using the basis fv; (v)g, the pair (Q;) corresponds to the reduced cube (1; 0; Q(v); $N_C(v)$) under the recipe in (i).

We record a corollary which will be used later, concerning isomorphisms between rank 2 twisted composition algebras:

Corollary 5.3. Let (C; Q;) be an E-twisted composition algebra of E-dimension 2. Let f 2 E and b 2 F, such that $b^2 + 4N_{E=F}(f) = 0$. Then the set of

$$_{C;f;b} := fv \ 2 \ C : Q(v) = \quad f \ \text{and} \ N_C(v) = \quad bg$$

is a principal homogeneous space for $\mathrm{Aut}_E(C)$, which contains an F-rational point if and only if (C; Q;) is isomorphic to the E-twisted composition algebra C = (E^2; Q;) dened by the reduced cube = (1; 0; f; b). Indeed, there is an $\mathrm{Aut}_E(C)$-equivariant isomorphism

$$\mathrm{Isom}_E(C; C) \quad ! \quad _{C;f;b}$$

dened by

$$! \quad (1; 0):$$

Proof. An E-linear isomorphism : C ! C is determined by v = (1; 0) (for (0; 1) has no choice but to be equal to (v)) and this v 2 C must satisfy

$$Q(v) = \quad f, \ \text{and} \ N_C(v) = \quad b.$$

Conversely, when v 2 C satises these two conditions, one checks using [GS2, x3.1 and Lemma 3.2, eqn. (3.4)] that the map given by (1; 0) = v and (0; 1) = (v) is an isomorphism of twisted composition algebras.

Observe that $\mathrm{Isom}_E(C; C)$ has an action of $\mathrm{Aut}_E(C)$ $\mathrm{Aut}_E(C)$ for which it is a torsor for each of the two factors. Hence, assuming $\mathrm{Isom}_E(C; C)$ is nonempty and after xing a base point $_0$ 2 $\mathrm{Isom}_E(C; C)$, one obtains an isomorphism

$$\mathrm{Ad}(_0) : \mathrm{Aut}_E(C) = \mathrm{Aut}_E(C):$$

By transport of structure, we also see that $_{C;f;b}$ carries an action of $\mathrm{Aut}_E(C)$ $\mathrm{Aut}_E(C)$. Let us describe the action of $\mathrm{Aut}_E(C) = \mathrm{Stab}_{GL_2(E)^{\det}}()$ on $_{C;f;b}$ concretely.

Lemma 5.4. Given

$$g = \begin{matrix} p & q \\ r & s \end{matrix} \quad 2 \ \mathrm{Stab}_{GL_2(E)^{\det}}();$$

so that ${}^{t}g\ {}^{1}\ 2\ \mathrm{Aut}_{E}(E^{2}; Q;)$, and v 2

$_{C;f;b}$ associated to 2 $\mathrm{Isom}_{E}(C; C)$, one has g v = ${}^{t}g\ (1; 0) = (p; q) =$ $pv + q(v)\ 2$ $_{C;f;b}$:

5.2. Degenerate cubes. It will be useful to have an understanding of the degenerate $M_E(F)$-orbits on $V_E(F) = N_E(F)=Z(F)$. The nontrivial degenerate orbits correspond to the nilpotent G_E-orbits which are denoted by A_1, $2A_1$ and $3A_1$ in the Bala-Carter classica-tion. Accordingly, we shall say that the corresponding elements in $V_E(F)$ are of rank 1, 2 or 3. We may refer to generic elements (non-degenerate cubes) as rank 4 elements. The set of elements in V_E of rank k is a Zariski closed subset. For example, the elements of rank 1 are precisely the highest weight vectors, and the set of elements of rank 1 can be described by a system of equations given in Proposition 8.1 below (see also [GS1, Prop. 11.2]).

We shall now describe the $M_E(F)$-orbits of elements of rank 2 and 3.

Proposition 5.5. (1) Every $M_E(F)$-orbit of rank 3 elements in $V_E = F\ E\ E\ F$ contains an element $(0; 0; e; 0)$ where e 2 E . Two rank 3 elements $(0; 0; e; 0)$ and $(0; 0; f; 0)$ belong to the same orbit if and only if e=f 2 F E^2.

(2) Every $M_E(F)$-orbit of rank 2 elements in $V_E = F\ E\ E\ F$ contains an element $(1; 0; e; 0)$ where e 2 E such that e = 0 and $e^{\#} = 0$. Two rank 2 elements $(1; 0; e; 0)$ and $(1; 0; f; 0)$ belong to the same orbit if and only if e=f 2 $(F)^2$.

Proof. (1) Consider = $(0; 0; 1; 0)$. This element has rank 3 since, over F, 1 = $(1; 1; 1)$ 2 F^3 sits across three orthogonal root spaces, hence the notation $3A_1$. A long but fascinating computation shows that the stabilizer $S_{M_E}()$ of in M_E consists of all elements

$$a\ 0\ \ b \qquad d$$

where ad 2 F, $d=d^{\#} = 1$ and $T_{E=F}(bd^{\#}) = 0$. Let T_E M_E be the maximal torus of diagonal matrices in M_E. The stabilizer $S_{T_E}()$ of in T_E consists of matrices as above with b = 0. Since

$$H^1(F; S_{M_E}(C)) = H^1(F; S_{T_E}(C))$$

it suces to classify the orbits of T_E on elements of the type $(0; 0; e; 0)$ where e 2 E. On these elements, the diagonal matrices act by multiplication by $d=d^{\#}$. Since the set of all $d=d^{\#}$ is F E^2, (1) follows. Statement (2) is proved in the same way, and we leave details to the reader.

Remark: If E is a eld, the set of e 2 E such that $e^{\#} = 0$ consists only of 0, so that there are no rank 2 elements in V_E. If E = F K with K a eld, the set of such e's is one F-line, and it consists of three F-lines if E = F^3. This reects the fact that $G_E(F)$ has three orbits with Bala-Carter notation $3A_1$, permuted by the group of outer automorphisms.

6. Dual Pairs

In this section, we introduce the various dual pairs which we will study in this paper. In particular, we shall see that given a E-twisted composition algebra C, with corresponding embedding i : E ,! J under the Springer decomposition, one may construct a dual pair:

$$H_C \quad G_E = \mathrm{Aut}_E(C) \quad \mathrm{Spin}_8^E \quad G_J;$$

where G_J is a group we shall introduce in due course. We shall rst construct this dual pair on the level of Lie algebras.

6.1. Lie algebras. Let us begin with an arbitrary Freudenthal-Jordan algebra J (not necessarily of dimension 9). Let l_J End(J) be the Lie subalgebra preserving the trilinear form (; ;) associated to the norm form N_J, i.e. a 2 End(J) lies in l_J if and only if

$$(a \quad x; y; z) + (x; a \quad y; z) + (x; y; a \quad z) = 0$$

for all x; y; z 2 J. The trace form denes an involution a ! $a^>$ on l_J by

$$ha \quad x; yi = hx; a^> \quad yi$$

for all x; y 2 J.

With h = sl(V) for V a 3-dimensional vector space, the space

$$g_J = h \quad l_J \quad (V \quad J) \quad (V \quad J)$$

has the structure of a simple Lie algebra, such that the above decomposition arises from a Z=3Z-grading. The brackets [h l_J ; V J] and [h l_J ; V J] are given by the natural action of h l_J on V J and V J, with the action of a 2 l_J on the second factor of V J is given by that of a$^>$. The brackets

[V J; V J] V J and [V J; V J] V J

are dened by respectively.

$$[v \quad x; u \quad y] = (v \wedge u) \quad (x \quad y) \qquad [v \quad x; u \quad y] = (v \wedge u) \quad (x \quad y)$$

The remaining bracket (between V J and V J) is determined by the invariant Killing form. More precisely, the Killing form on g_J is an extension of the Killing form on h l_J (we shall specify the normalization later), such that

$$\langle v \otimes x; u \otimes y\rangle = \langle v; u\rangle \langle x; y\rangle$$

if $v \otimes x \in V \otimes J$ and $u \otimes y \in V^* \otimes J$, where $\langle v; u\rangle$ is the evaluation of u on v and $\langle x; y\rangle$ is the trace pairing on J. Then the bracket $[V \otimes J; V^* \otimes J] \subset h \oplus l_J$ is completely determined by:

$$\langle [x; y]; z\rangle = \langle [z; x]; y\rangle$$

for any $x; y; z \in g_J$. We refer the reader to [Ru] for explicit formulae in this case. However, if $\langle v; u\rangle = 0$, the bracket of $v \otimes x \in V \otimes J$ and $u \otimes y \in V^* \otimes J$ is contained in h, and is given by

$$[v \otimes x; u \otimes y] = \langle x; y\rangle\, v \otimes u \in sl(V)$$

and/or

Explicitly, if i = j,

$[u \ y; v \ x] = \langle x; y \rangle \ u \ v \ 2 \ sl(V)$

$[e_i \ x; e_j \ y] = \langle x; y \rangle e_{ij}$

$[e_j \ y; e_i \ x] = \langle x; y \rangle e_{ji}$:

We highlight two cases here:

(a) If J = F, considered as a cubic algebra, so that 1 1 = 2 and $T_F(1) = 3$, then this construction returns the simple split algebra g of type G_2.

(b) If J = E is a cubic etale algebra, then $l_E = E^0$, the subspace of trace 0 elements in E. The action of x 2 E^0 on e 2 E is x e = 2xe. We x a symmetric bilinear form on l_E by $\langle x; x \rangle = 2 \ T_E(x^2)$. Then the Lie algebra g_E is of type D_4; it is the Lie algebra of the group G_E = $Spin_8^E$.

6.2. Groups. In order to explain the two appearances of 2 in (b) above, let J = E C, where C is E-twisted composition algebra (of arbitrary rank). For 2 E, let c : J ! J be dened by

$$c : (e; v) \ ! \ (^{\#} = e; \ v) \text{ for all}$$

(e; v) 2 E C. By (38.6) in [KMRT], one has

$$N_J((e; v)) = N_E(e) + N_C(v) \quad T_E(e \ Q(v));$$

and it readily follows that

$$N_J(c(e; v)) = N_E() \ N_J(e; v);$$

so that c is a similitude map of N_J with similitude factor $N_E()$. In particular, if has norm 1, then c preserves the norm N_J. Since $^{\#} = {}^1$ (if N () = $_E$1), we can write c(e; v) = (^{2}e; v). By passing to Lie algebras, we get an embedding $l_E = E^0$ l_J where x 2 E acts on J = E^0 C by

$$x \ (e; v) = (\ 2xe; v) + (e; xv):$$

By setting v = 0, we get the previously dened action of $l_E = E^0$ on E.

On the other hand, we x the Aut(l_J)-invariant form on l_J so that the restriction to l_E is 2 T_E(x). ^{2}For example, suppose that J = $M_3(F)$ and E = F ^{3} is diagonally embedded in $M_3(F)$. Then $l_J = sl_3 \ sl_3$, so that an element (x; z) 2 $sl_3 \ sl_3$ acts on y 2 $M_3(F)$ by xy yz, and l_E is the set of trace zero diagonal matrices x embedded in $sl_3 \ sl_3$ as (x; x).

We embed $Aut_E(C)$ Aut(J) so that it acts trivially on E, the rst summand in J = E C.

Proposition 6.1. Let J = E C. Every F-rational similitude map of N_J commuting with the algebraic group $Aut_E(C)$ is equal to c for some 2 E. Likewise, every F-rational similitude map of N_J commuting with the algebraic group Aut(J) is equal to c for 2 F.

Proof. Let g be a F-rational similitude of N_J commuting with $\mathrm{Aut}_E(C)$. Then g preserves both summands E and C of J. The algebra of F-rational endomorphisms of C commuting with the action of $\mathrm{Aut}_E(C)$ is E. Thus $g = c$ on C, for some $\in E$. Let $g^0 = c^{-1} g$. Clearly, g^0 belongs to the similitude group of N_J; however, since $g^0(0; v) = (0; v)$ for all $v \in C$, the similitude factor is 1, i.e. g^0 preserves N_J.

Now $x\ e \in E$. Then $g^0(e; v) = (e^0; v)$ for all $v \in C$ and some $e^0 \in E$. We want to show that $e = e^0$. It suces to do so over the algebraic closure $\bar{F}$. Since g^0 preserves N_J, use $v = 0$ to show rst that $N_E(e) = N_E(e^0)$, and then $T_E(e\,Q(v)) = T_E(e^0 Q(v))$ for all $v \in C$. Since Q is surjective over $\bar{F}$, $T_E(ee^{00}) = T_E(e^0 e^{00})$ for all $e^{00} \in E \otimes \bar{F}$. Hence $e = e^0$. Finally, if g is a similitude that commutes with $\mathrm{Aut}(J)$, then it commutes with $\mathrm{Aut}_E(C) \subset \mathrm{Aut}(J)$, so $g = c$. Since $\mathrm{Aut}(J)$ acts absolutely irreducibly on J^0, the space of trace 0 elements in J, $\in F$.

Let $G_J = \mathrm{Aut}(g_J)$. We note that G_J is not necessarily connected. From the construction of the Lie algebra g_J, it is evident that $\mathrm{Aut}(J) \subset G_J$. Assume, furthermore, that $J = E \oplus C$ and $J \neq E$. The natural action of $\mathrm{Aut}_E(C)$ on C, extended trivially to $E \subset J$ gives an embedding $\mathrm{Aut}_E(C) \subset \mathrm{Aut}(J)$. Hence we have a natural embeddings

$$\mathrm{Aut}_E(C) \subset \mathrm{Aut}(J) \subset G_J:$$

We have also constructed inclusions of $g \subset g_E \subset g_J$ of vector spaces.

Proposition 6.2. The inclusions $g \subset g_E \subset g_J$ are homomorphisms of Lie algebras, thus giving rise to inclusion of algebraic groups

$$G_2 \subset G_E = \mathrm{Spin}_8^E \subset G_J:$$

Proof. Let $x; y \in E$. The cross product xy, computed in J, is the same as the one computed in E. Hence the bracket $[V \otimes E; V \otimes E]$ in g_J coincides with the one in g_E. The bracket $[V \otimes E; V \otimes E]$, computed in g_J, is xed by $\mathrm{Aut}_E(C)(F)$ hence it is contained in $h \oplus l_E$. Since the Killing form on $h \oplus l_E$ is the restriction of the Killing form on $h \oplus l_J$ it follows, from the denition of the Lie brackets, that the two Lie brackets coincide. This shows that the inclusion $g_E \subset g_J$ is a homomorphism. A similar argument shows that the inclusion $g_E \subset g_J$. Indeed, the bracket $[V \otimes F; V \otimes F]$, computed in g_J, is xed by $\mathrm{Aut}(J)$ hence it is contained in h.

The inclusion of Lie algebras induce a corresponding inclusion of the corresponding connected algebraic subgroups of G_J, and we know what these algebraic subgroups are up to isogeny. It is clear that the algebraic subgroup associated to g is G_2. Over $\bar{F}$, under the adjoint action of g_E, the algebra g_J contains the three 8-dimensional fundamental representations of Spin_8, each occurring with multiplicity $\dim_E(C)$. This shows that the connected algebraic subgroup corresponding to g_E is simply-connected and is thus isomorphic to $G_E = \mathrm{Spin}_8^E$.

6.3. Relative root system. We x a basis $e_1; e_2; e_3$ of V and let $t \subset h$ be the Cartan subalgebra consisting of diagonal matrices, with respect to this basis of V. Under the adjoint action of t,

$$g_J = g_{J;0} \oplus \left(\bigoplus_{\in 2} g_{J;}\right)$$

where t is a root system of type G_2. Note that $g_{J;0}$

$$= t\ l_J:$$

The short root spaces are Fe J i or i Fe J, so we have canonical identications with J given by $x\ !\ e_2$ x and $x\ !\ e$ x respectively. The long root spaces are one-dimensional and contained in h. In particular, there are two choices for the basis vector : e_{ij} or e_{ji} (= e_{ij} under the identications h = sl(V) = sl(V)).

In particular, when J = E, $g_{E;0}$ = t l_E is a torus, and by choosing a set of positive roots in , we have constructed a Borel subalgebra in g_E, so that g_E is quasi-split. Indeed, we have mentioned before that g_E is the Lie algebra of Spin_8^E. What we have done here is to give a direct construction of this Lie algebra, recover some of the structure theory described in x2 from this construction and show that this Lie algebra ts into a family of such Lie algebras which is associated to a Freudenthal-Jordan algebra J.

6.4. Two step parabolic subalgebra. Let s 2 sl(V) be the diagonal matrix (1; 0; 1). The adjoint action of s on g_J gives a Z-grading

$$g_J = {}_{n2Z}\ g_J(n):$$

Then $g_J(n)$ = 0 only for n = 2; 1; 0; 1; 2. Let

$$m = g_J(0) \quad \text{and} \quad n = g_J(1)\ g_J(2):$$

Then p = m n is a maximal parabolic subalgebra, with Levi subalgebra m and nilpotent radical n. Let us examine the structure of each of these parts in turn.

The Levi subalgebra m has a decomposition

$$m = t\ l_J\ e_2\ J\ e_2\ J:$$

The derived algebra

$$[m; m] = l_J\ e_2\ J\ e_2\ J$$

is generated by short root spaces e_2 J and e $_2$ J. The above decomposition also exhibits a (Siegel-type) parabolic subalgebra

$$s = (t\ l_J)\ e_2\ J\ m$$

with abelian nilpotent radical e_2 J.

Considering now the nilradical n, the center of n is [n; n] = $g_J(2)$ = Fe_{13}. As an m-module, the quotient n=[n; n] is isomorphic to

$$g_J(1) = Fe_{21}\ Fe_1\ J\ Fe_3\ J\ Fe_{23} = F\ J\ J\ F:$$

Henceforth, an element in $g_J(1)$ is a quadruple (a; y; z; d) where a; d 2 F and y; z 2 J. Using our formulae, we can describe this m-module. One sees that the Lie bracket of e_2 x 2 e_2

J and (a; y; z; d) is

[e2

x; (a; y; z; d)] = (0; ax; x y; hx; zi) and the Lie bracket of e2

x 2 e2

J and (a; y; z; d) is

[e2

x; (a; y; z; d)] = (hx; yi; x z; dx; 0):

If $J = E$, a cubic etale algebra, then $\mathfrak{g}_E(1)$ is the space of E-twisted Bhargava cubes and $[\mathfrak{m}, \mathfrak{m}]$ is identified with $\mathfrak{sl}_2(E)$ by

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mapsto e_2 \otimes x \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \mapsto e_2 \otimes x$$

Let $P_J = M_J N_J$ be the parabolic subgroup associated to $\mathfrak{p}_J$. If we fix an embedding $E \hookrightarrow J$ of Jordan algebras, then we have a corresponding embedding $\mathfrak{p}_E \hookrightarrow \mathfrak{p}_J$ of parabolic subalgebras such that

$$G_E \cap P_J = P_E$$

on the level of groups.

6.5. **3-step parabolic subalgebra.** Now let $s \in \mathfrak{sl}(V)$ be the diagonal matrix $(1, 1, -2)$. As above, the adjoint action of s on $\mathfrak{g}_J$ gives a $\mathbb{Z}$-grading

$$\mathfrak{g}_J = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_J(n).$$

Then $\mathfrak{g}_J(n) \neq 0$ only for $n = -3, \dots, 3$. Let

$$\mathfrak{l} = \mathfrak{g}_J(0) \quad \text{and} \quad \mathfrak{u} = \mathfrak{g}_J(1) \oplus \mathfrak{g}_J(2) \oplus \mathfrak{g}_J(3).$$

Then $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a parabolic subalgebra whose nilradical $\mathfrak{u}$ is 3-step nilpotent. Note that

$$\mathfrak{g}_J(1) = Fe_1 \otimes J \oplus Fe_2 \otimes J; \quad \mathfrak{g}_J(2) = Fe_3 \otimes J \quad \text{and} \quad \mathfrak{g}_J(3) = Fe_{13} \oplus Fe_{23}.$$

Let $Q_J = L_J U_J$ be the corresponding parabolic subgroup in G_J. Thus, the unipotent radical U_J has a filtration

$$U = U_1 \supset U_2 \supset U_3 \quad \text{such that } U_i / U_{i+1} = \mathfrak{g}_J(i) \text{ for all } i.$$

If we fix an embedding $E \hookrightarrow J$, then we have a corresponding embedding $\mathfrak{q}_E \hookrightarrow \mathfrak{q}_J$ of parabolic subalgebras such that

$$G_E \cap Q_J = Q_E.$$

on the level of groups.

6.6. **See-saw dual pairs.** To summarise the discussion in this section, relative to an embedding $E \hookrightarrow J$, we have constructed the following see-saw of dual pairs in G_J:

$$\begin{array}{ccc} G_E & & H_J = \mathrm{Aut}(J) \\ & \times & \\ G_2 & & H_C = \mathrm{Aut}_E(C) \end{array}$$

We highlight two cases:

- The particular case of interest in this paper is the case when $\dim_E C = 2$ or equivalently $\dim_F J = 9$. In this case, G_J and $\mathrm{Aut}_E(C)$ are disconnected and we have a short exact sequence

$$1 \longrightarrow \underline{G}_J^0 \longrightarrow \underline{G}_J \longrightarrow S_2 \longrightarrow 1$$

where the identity component G_J^0 is an adjoint group of type E_6 and whose inner class correspond to the quadratic algebra K_J. Note that on taking F-points, we have a map

$$G_J = \underline{G}_J(F) \quad ! \quad S_2$$

which need not be surjective.

When $\dim_E C = 4$ (i.e. $\dim_F J = 15$), then G_J is an adjoint group of type E_7 associated to a quaternion F-algebra B. In this case,

$$\mathrm{Aut}_E(C) = \mathrm{Res}_{E=F}(B$$
$$_F\ E)^{det} = F$$

where the RHS consists of elements in $(B$ $E)$ whose norm lies in F.

7. Levi Factor

In this section, we investigate some further properties of the dual pair H_C G_E in G_J, with $J = E$ C and $\dim_E C = 2$. The group G_J has a (Heisenberg) maximal parabolic subgroup $P_J = M_J N_J$ $P_J = M_J$ N_J,^{0} whose^0 Levi factor M_J is of type A_5^0. Moreover,

$$(H_C\ G_E) \setminus P_J = H_C\ P_E;$$

so that

$$H_C\ M_E \quad ! \quad M_J$$

is itself a dual pair in M_J. Indeed, if we intersect the seesaw diagram in x6.6 with M_J, we obtain the following seesaw diagram in M_J:

$$GL_2(E)^{det} \qquad H_J = \mathrm{Aut}(J)$$

@
@
@@

$$GL_2(F) \qquad H_C = \mathrm{Aut}_E(C)$$

For our purposes, when J is not a division algebra, we need to describe the Levi subgroup M_J and the above embedding concretely. This is because of the need to relate the theta correspondence associated to H_C M_E to a classical similitude theta correspondence. We treat the various cases in turn.

7.1. Split case. Suppose rst that $J = M_3(F)$, so that G_J ^{0}is split. In this case,

$$M_J^0 = (GL_1\ SL_6) = {}_6$$

where $_6$ is viewed as a subgroup of GL_1 SL_6 by the map x

$$! \quad (x^3; x):$$

A more convenient description is:

$$M_J^0 = (GL_1\ GL_6) = GL_1$$

where GL_1 is viewed as a subgroup of GL_1 GL_6 by the map x ! $(x^3; x)$. The character

$$(x; g) = \det(g)=x^2$$

of $GL_1 \ GL_6$ descends to M^0_J and is a generator of $\mathrm{Hom}(M^0_J; G_m)$. The character arises naturally when M^0_J acts by conjugation on the center of N_J.

If we identify $F^6 = E^2$ (by choosing an F-basis of E), then $M_E = GL_2^{det}(E)$ is naturally a subgroup of GL_6. We dene an embedding $GL_2(E)^{det} \to M_J$ by the map

$$g \to (\det(g); g):$$

Note that $(\det(g); g) = \det(g)$ since the determinant of g, viewed as an element in GL_6 is $\det(g)^3$. On the other hand, since $K_J = F \ F$, one has $H^0_C = E=F$. The right-multiplication action of $e \in E$ on E^2 gives an embedding $E \to GL_6$, so that any element $e \in E$ can be viewed as an element of GL_6 denoted by the same letter. Thus we have a map $E \to GL_1 \ GL_6$ given by

$$e \to (N_{E=F}(e); e):$$

If $e \in F$, then the image is $(e^3; e)$. The map thus descends to an inclusion of $E=F \to M_J$ and we have dened an embedding

$$H^0_C \ M_E = E=F \ GL_2(E)^{det} \hookrightarrow M^0_J$$

when $J = M_3(F)$. Note that the character of M^0_J is trivial on $E=F$.

7.2. Quasi-split case. Consider now the case when $J = J_3(K)$, so that G_J is quasi-split but not split. In this case,

$$\underline{M}^0_J = (GL_1 \ SU^K_6)=\mathrm{Res}^1{}_{6;K}$$

where $\mathrm{Res}^1{}_{6;K} = \mathrm{Ker}(N_{K=F} : \mathrm{Res}_{K=F}\ {}_6 \to {}_6)$ is viewed as a subgroup of $GL_1 \ SU^K_6$ by the map $x \to (x^3; x)$.

Fix an involution $g \to g$ of $GL_6(K)$ that denes the quasi-split form U^K_6. In particular, $\det(g) = \det(g)^{1}$ and $x = x^{1}$ for any scalar matrix $x \in GL_6$. Consider the involution

$$: (x; g) \to (x \det(g)^{1}; g)$$

of $GL_1 \ GL_6$. Since $(x^3; x) = (x^{3}; x^{1})$, for every $x \in GL_1$, the involution descends to the quotient $(GL_1 \ GL_6)=GL_1$.

Now $\underline{M}^0_J$ is the subgroup of

$$\mathrm{Res}_{K=F}(\underline{M}^0_J \ {}_F \ K) = \mathrm{Res}_{K=F}(GL_1 \ GL_6=GL_1)$$

xed under the Galois action twisted by . From our knowledge in the split case, we deduce an exact sequence of algebraic groups,

$$1 \to U^K_1 \to (\mathrm{Res}_{K=F} G_m \ U^K_6)^y \to M^0_J \ 1$$

where $(\mathrm{Res}_{K=F} G_m \ U^K_6)^y$ is the subgroup consisting of pairs $(x; g)$ such that

$$x=(x) = \det(g) \quad \text{with } 1 = \ \in \mathrm{Aut}(K=F):$$

On the level of F-points, one has

$$1 \to K^1 \to (K \ U^K_6(F))^y \to M^0_J(F) \to H^1(F; U^K_1) = F=N_{K=F}(K):$$

Let

$$M^0_{J;K} = (K \ U^K_6(F))^y=K^1:$$

so that $M_J(F)^0/M^0_{J,K} \cong F^\times/N_{K/F}(K^\times)$. We claim that this is an isomorphism. The condition $x\bar{x} = \det(g)$ implies that $\lambda(x,g) \in N_{K/F}(K^\times)$, for all $(x,g) \in M^0_{J,K}$. On the other hand, the character $\lambda : M^0_J(F) \to F^\times$ is surjective, and the claim follows. Thus, we have an exact sequence of topological groups

$$1 \longrightarrow M^0_{J,K} \longrightarrow M^0_J(F) \longrightarrow F^\times/N_{K/F}(K^\times) \longrightarrow 1.$$

We would now like to describe the embedding of $\mathrm{Aut}_E(C) \times GL_2(E)^{\det}$ into M_J. While this can be done by writing down some explicit formulas, we would like to view this embedding through the lens of a see-saw pair in the classical similitude theta correspondence. For this, let us set up the relevant notation and recall the relevant background.

7.3. **Similitude dual pairs.** Here is the general setup. For $a \in E^\times$, let

$$W_a = Ee_1 \oplus Ee_2$$

be a 2-dimensional symplectic vector space over E equipped with the alternating form

$$\langle e_1, e_2\rangle_a = -\langle e_2, e_1\rangle_a = a.$$

With respect to the basis $\{e_1, e_2\}$, we have an identication of the symplectic similitude group $GSp(W_a)$ with $GL_2(E)$. The subgroup $GSp(W_a)^{\det}$ of elements whose similitude factor lies in $F^\times$ is then identied with $M_E = GL_2(E)^{\det}$. For $g \in GL_2(E)^{\det}$, the corresponding similitude factor is

$$\lambda(g) = \det_E(g);$$

where $\det_E(g)$ refers to the determinant of g considered an element of $GL_2(E)$. We write $GL_2(E)^{\det}_K$ for the index 2 subgroup of elements whose similitudes lie in $N_{K/F}(K^\times)$. Hence, we set

$$M_{E,K} = GL_2(E)^{\det}_K = \{g \in M_E = GL_2(E)^{\det} : \det_E(g) \in N_{K/F}(K^\times)\}.$$

From this symplectic space W_a, we deduce the following 3 other spaces and groups:

(a) By restriction of scalars from E to F, we obtain a 6-dimensional symplectic space $\mathrm{Res}_{E/F}(W_a)$ with alternating form $\mathrm{Tr}_{E/F} \circ \langle -, -\rangle_a$. One has a natural inclusion of similitude groups:

$$M_E = GL_2(E)^{\det} = GSp(W_a)^{\det} \hookrightarrow GSp(\mathrm{Res}_{E/F}(W_a)) = GSp_6(F).$$

We write $GSp(\mathrm{Res}_{E/F} W_a)_K$ for the index 2 subgroup of elements whose similitudes lie in $N_{K/F}(K^\times)$.

(b) With $L = E \otimes K$, the 2-dimensional L-vector space $V_a = W_a \otimes_E L$ is naturally equipped with a skew-Hermitian form induced by the alternating form on W_a, with $\langle -, -\rangle_a$ given by the same formula as above on the basis $\{e_1, e_2\}$. Then we have

$$GL_2(E)^{\det} = GSp(W_a)^{\det} \hookrightarrow GU(V_a)^{\det}$$

where the superscript det refers to those elements whose similitude (which a priori lies in $E^\times$) belongs to $F^\times$.

(c) As above, by considering restriction of scalars from L to K, we see that $\mathrm{Res}_{L/K}(V_a)$ is a 6-dimensional K-vector space equipped with the skew-Hermitian form $\mathrm{Tr}_{L/K} \circ \langle -;- \rangle_a$. This 6-dimensional skew-Hermitian space over K is also the one naturally induced from the symplectic space $\mathrm{Res}_{E/F}(W_a)$ over F, in the same way as V_a is obtained from W_a. One has a natural inclusion of unitary similitude groups:

$$GU(V_a)^{det} \hookrightarrow GU(\mathrm{Res}_{L/K}(V_a));$$

In fact, both similitude maps here have image equal to $F^\times$, but we shall consider the index 2 topological subgroups of elements whose similitude lies in $N_{K/F}(K^\times)$, denoted by:

$$GU(V_a)^{det}_K \hookrightarrow GU(\mathrm{Res}_{L/K}(V_a))_K.$$

Observe that

$$GU(\mathrm{Res}_{L/K}(V_a))_K = (K^\times \times U(\mathrm{Res}_{L/K}(V_a)))/\nabla K^1$$

with $\nabla K^1 = \{(z, z^{-1}) : z \in K^1\}$.

Summarizing, starting with W_a, we have the following containment diagram for the 4 groups we introduced:

(7.1)
$$\begin{array}{ccc} & GU(\mathrm{Res}_{L/K}(V_a))_K & \\ GU(V_a)^{det}_K & & GSp(\mathrm{Res}_{E/F}(W_a))_K \\ & GSp(W_a)^{det}_K = GL_2(E)^{det}_K & \end{array}$$

These groups appear in the classical similitude theta correspondence, and we proceed next to describe the other member of the relevant dual pairs, namely those lying on the other side of a seesaw diagram.

Regard K as a rank 1 Hermitian space (relative to K/F) with the form $(x;y) \mapsto x \cdot \sigma(y)$. Then $GU(K) = K^\times$ and $GU(\mathrm{Res}_{L/K}(V_a))_K$ form a similitude dual pair. Here it is neces-sary to consider the index 2 subgroup $GU(\mathrm{Res}_{L/K}(V_a))_K$ as opposed to $GU(\mathrm{Res}_{L/K}(V_a))$, because the similitude map on $GU(K)$ has image $N_{K/F}(K^\times)$. Starting from this rank 1 Hermitian space, one deduces the following 3 spaces and groups:

(a') By restriction of scalars from K to F, we regard K as a 2-dimensional F-vector space with quadratic form $N_{K/F}$, with similitude group

$$GO(K; N_{K/F}) = K^\times \rtimes \langle \sigma \rangle;$$

with σ acting on K as the unique nontrivial automorphism σ of K/F. Then $GO(K; N_{K/F}) \times GSp(\mathrm{Res}_{E/F}(W_a))_K$ is a similitude dual pair.

(b') By base change from F to E, we obtained a rank 1 Hermitian space (relative to L/E) over L, so that $GU(L)^{det} \times GU(V_a)^{det}_K$ forms a similitude dual pair.

(c') By restriction of scalars $\mathrm{Res}_{E=F}$ on the space in (b') or the base change from F to E of the space $(K; N_{K=F})$ in (a'), we obtain the quadratic space $(L; N_{L=E})$ of dimension 2 over E, with similitude group

$$GO(L; N_{L=E})^{det} := GSO(L; N_{L=E})^{det} \text{ o hi} = (L)^{det} \text{ o hi}:$$

This group form a similitude dual pair with $M_{E;K} = GL_2(E)^{det}_K = GSp(W_a)^{det}_K$.

Summarizing, starting from a rank 1 Hermitian space (relative to $K=F$), one have the following diagram

(7.2)

$$GO(L; N_{L=E})^{det} = (L)^{det} \text{ o hi}$$

$$GU(L) = L \qquad\qquad GO(K; N_{K=F}) = K \text{ o hi}$$

$$GU(K) = K$$

As mentioned above, the groups in (7.2) form a seesaw diagram of dual pairs with the corresponding group in (7.1). We shall only make use of the groups at the top and bottom of the diagrams, so that we have a similitude seesaw pair:

(7.3)

$$GO(L; N_{L=E})^{det} = (L)^{det} \text{ o hi} \qquad GU(\mathrm{Res}_{L=K}(V_a))_K = (K \;\; U(\mathrm{Res}_{L=K}(V_a)))=rK^1$$

$$GU(K) = K \qquad\qquad M_{E;K} = GL_2(E)^{det}_K$$

7.4. Embedding. We can now describe the embedding

$$\mathrm{Aut}_E(C)^0 \;\; GL_2(E)^{det} ,! \;\; M^0_{:J}$$

Recall that we are considering

$$[C] \; 2 \; H^1(F; \tilde{T}_{E;K})[2] = \; E=F \, N_{L=E}(L) \qquad\qquad \text{(by (4.11))}.$$

Take any a 2 E representing the class of $[C]$, so that we have the above constructions of similitude dual pairs using a 2 E. Recall further that one has a natural isomorphism of algebraic groups

$$M_J^0 = (\mathrm{Res}_{K=F}\, G_m \;\; U(\mathrm{Res}_{L=K}(V_a)))^y = U^K_{:1}$$

Now there is a natural map (with nite kernel) of algebraic groups

(7.4)

$$f : GU(\mathrm{Res}_{L=K}(V_a)) = (K \;\; U(\mathrm{Res}_{L=K}(V_a))_K) = rU^K_{\;1}! \;\; (K \;\; U(\mathrm{Res}_{L=K}(V_a))_K) = U^K; \quad {}_1$$

given by

$$(z; g) \; ! \;\; (z \;\; {}^3; g):$$

The restriction of this map to the subgroup M_E (see (7.3)) gives the embedding of algebraic groups

$$GL_2(E)^{\det} \hookrightarrow M_J^0.$$

When restricted to the topological subgroup $M_{E,K} = GL_2(E)^{\det}_K$, the map f is given by the formula

$$g \mapsto (z^{-3}, gz^{-1}),$$

where $\det_E(g) = N_{K/F}(z)$. Observe that this is clearly well defined, as z is unique up to K^1.

On the other hand, we have the natural isomorphism of algebraic groups

$$\mathrm{Aut}_E(C) = (L)^{\det} \rtimes \langle i\rangle / K = GO(L, N_{L/F})^{\det}/GU(K),$$

which is a quotient of the two algebraic groups appearing on the LHS of the seesaw diagram in (7.3). Hence

(7.5) $$\mathrm{Aut}_E(C)^0 = GU(L)^{\det}/GU(K) = U(L)/U(K).$$

The embedding

$$\mathrm{Aut}_E(C)^0 = U(L)/U(K) \hookrightarrow M_J^0$$

is given by

$$e \mapsto (N_{L/K}(e), e),$$

where $e \in U(L)$ acts on $\mathrm{Res}_{E/F}(V_a)$ through its scalar multiplication action on $V_a = Le_1 \oplus Le_2$.

It is useful to note the following lemma which says that the last isomorphism in (7.5) continues to hold on the level of F-rational points.

Lemma 7.6. The inclusion $L^1 \subset (L)^{\det}$ gives an isomorphism $L^1/K^1 = (L)^{\det}/K$.

Proof. We have a long exact sequence

$$1 \to K^1 \to L^1 \to (L)^{\det}/K \to H^1(F, U(K)) \to H^1(F, \mathrm{Res}_{E/F}U(L))$$ so

we need to show that the last arrow is injective. To that end, the map

$$N_{L/K} : \mathrm{Res}_{E/F}U(L) \to U(K)$$

gives

$$H^1(F, U(K)) \to H^1(F, \mathrm{Res}_{E/F}U(L)) \to H^1(F, U(K))$$

such that the composite is multiplication by 3. Since $H^1(F, U(K))$ is a 2-group, the composite is the identity. This proves the lemma.

The lemma implies that, for any $x \in (L)^{\det}$, $N_{L/E}(x) \in N_{K/F}(K)$. Thus, the embedding

$$(L)^{\det}/K \hookrightarrow M_J^0(F)$$

takes value in the index 2 subgroup $M^0_{J,K}$ and is given by the formula x

$$\mapsto (N_{L/K}(x/z), x/z), \quad \text{where } N_{L/E}(x) = N_{K/F}(z).$$

Again this is well-defined as z is determined up to an element of K^1.

We have thus described the embedding of algebraic groups

$$H_C^0 \subset M_E \hookrightarrow M_J^0.$$

This embedding depends only on a $2 E=F N_{L=E}(L) = H^1(F;T_{E;K})[2]$. On the level of points, it gives the embedding

$$H_C^0(F) \; M_{E;K} = (L)^{det}=K \; GL_2(E)^{det} ,! \; M_{J;K}: \quad {}^0$$

Though the embedding could have been written down via formulas, without mention of the framework of similitude dual pairs, this framework will help us in x10 to relate the mini-theta correspondence associated to this commuting pair of groups by reducing it to the classical similitude theta correspondence. So we shall have occasion to return to the material in x7.3 later on.

7.5. Siegel parabolic. Recall that the Lie algebra m has a Siegel parabolic subalgebra s. This gives rise to a Siegel parabolic subgroup

$$S_J \; M_J$$

whose Levi factor is of type $A_2 \; A_2$ and whose unipotent radical can be identied with J. Moreover, $H_C \; S_J$ and the intersection of M_E with S_J is a Borel subgroup of M_E. If we identify M_E with $GL_2(E)^{det}$, we may assume that $S_J \setminus M_E$ is the Borel subgroup of upper triangular matrices.

8. Minimal Representation

In this section, we assume that F is a non archimedean local eld. Let be the minimal representation of $G_J(F)$ (see [GS1]). In this section, we recall the relevant properties of that we need. We rst note that the algebraic group G_J is not connected, but the minimal representation in [GS1] is a representation of the subgroup $G^0_J(F)$ of $G_J(F)$. Thus there are two ways of extending to $G_J(F)$ and we shall rst need to specify the extension we use below.

8.1. Extending the minimal representation. Recall the Heisenberg parabolic subgroup $P_J = M_J N_J$ of G_J, with Z the center of N_J and and let

$$: M_J \; ! \; F$$

be the character of M_J given by the action of M_J on Z. By composition with , we may regard any character of F as a character of $M_J(F)$. Henceforth, we shall write in place of for a character of $M_J(F)$.

Now we consider the degenerate principal series representation of $G_J(F)$:

$$I_J(s_0) := Ind_{P_J}^{G_J} {}_J = Ind_{P_J}^{G_J^0} {}_J \qquad \text{(unnormalized induction)}$$

where

$${}_J = !_{K=F} \; j \quad j^{s_J}$$

with $!_{K=F}$ the quadratic character associated to $K = K_J$ by local class eld theory and s_J given by the following table:

G_J	E_6	E_7	E_8
s_J	2	3	5

The minimal representation of $G^0_J(F)$ is the unique irreducible subrepresentation of $I_J(s_0)$, regarded as a representation of $G^0_J(F)$. This unique irreducible submodule is thus stable under the action of $G_J(F)$ and this denes the extension of to $G_J(F)$. When we regard $I_J(s_0)$ as a space of functions on $G^0_J(F)$ transforming under $(P^0_J(F); {}_J)$ on the left, the action of $G_J(F)$ is by right translation whereas the action of p 2 $P_J(F)$ is given by:

$$(p\ f)(h) = {}_J(p)\ f(p^{-1}hp) \qquad \text{for } h\ 2\ G^0_J(F) \text{ and } f\ 2\ I_J(s_0).$$

This describes the action of $G_J(F) = P_J(F)\ G^0_J(F)$.

8.2. Restriction of to P_J. The restriction of to P_J sits in a short exact sequence 0

$$!\ C_c^1(\ )\ !\ {}_Z\ !\ {}_{N_J}\ !\ 0;$$

where

$N_J=Z$ is the minimal nontrivial (highest weight) M_J-orbit.

To describe the action of P_J on $C^1_c($), let $h;ni$ be the natural pairing of $N_J=Z$ and $N_J=Z$ and x a non-trivial additive character of F. Then the action is given as follows. For f 2 $C_c($),

n 2 $N_J=Z$ acts by

$$(n)f(n) = \ (h;ni)\ f(n):$$

m 2 M_J acts by

$$(m)f(n) = {}_J(m)\ f(m^{-1}nm):$$

8.3. The minimal orbit. Recall from 6.4 that we have an identication

$$W_J := N_J=Z_J = F\ J\ J\ F:$$

By [GS1, Proposition 11.2], we have the following description of :

Proposition 8.1. A non-zero element ! = (a; x; y; d) 2 $N_J=Z_J$ is in the minimal M_J-orbit if and only if

$$x^{\#} = ay;\ y^{\#} = dx \text{ and } l(x)\ l(y) = ad \text{ for all } l\ 2\ L_J$$

where x y is the product in J, L_J the group of linear transformations of J preserving the norm form, and l the dual action of L_J on J = J, with the identication given by the trace pairing. In particular, if a = 1, then ! = $(1; x; x^{\#}; N_J(x))$.

Erratum: In fact, [GS1, Proposition 11.2] asserts that it suces to use x y = ad in place of the family of equations obtained by the L_J-action. This is false. Writing $W_J = N_J=Z_J$, the M_J-module $S^2(W_J)$ is a direct sum of an irreducible module whose highest weight is equal to twice the highest weight of W_J and the adjoint representation of M_J. The quadratic equations given here span the latter summand and hence give a complete set of generators.

Note however that in [GS1, Proposition 11.2], only the proof of the \only if" statement was given, as the other direction was not used in [GS1]. Hence this error does not aect any result in [GS1].

8.4. The M_J-module $_N$ $\cdot_J$ A complete description of the Jacquet module $_N$ is $_J$given in [GS1]. We have

$$_J\ _N ! _{K=F}\ j \quad j^{\ 2}\ j \quad j^{\ 3=2}\ _{M_J}$$

for an M_J-module $_{M_J}$ which is 0 if J is a division algebra and is a unitary minimal representation of M_J otherwise. We will assume that J is not division henceforth and describe the M_J-module $_{M_J}$ in some detail.

Recall that $M_J^0(F)$ contains a subgroup $M_{J;K}^0$ of index 2. We rst describe a representation of $M_{J;K}^0$, using the classical theta correspondence for the pair

$$U(K)\ U(\mathrm{Res}_{K=F}(V_a)) = U_1(F)\ U_6(F)$$

constructed in x7.3.

To give a Weil representation for this dual pair, we need to choose a character of K whose restriction to F is the quadratic character $!_{K=F}$, which gives a splitting of the meta-plectic cover over $U_6(F)$. Then we may consider the Weil representation $!_{;}$ for U_1 U_6 associated to the pair of splitting characters (1;) and a nontrivial additive character of F. With respect to the choice of (1;) and , the associated Weil representation $!_{;}$ can be realised on $C^1_c(L)$, where L = Le_2 is a polarization of V_a = Le_1 Le_2. The action of U(K) = K^1 and the Siegel parabolic subgroup of $U(\mathrm{Res}_{K=F}(V_a))$ stabilizing Le_1 is given by the usual formulas in the Schrodinger model:

The group U_1 = K^1 acts geometrically on $C^1_c(Le_2)$: for z 2 K^1,

$$(z\ f)(v) = f(z^{\ 1}v):$$

If $GL_K(Le_2)$ is the Levi subgroup that preserves the decomposition V_a = Le_1 Le_2, the action of g 2 $GL_K(Le_2)$ is given by

$$(g\ f)(v) = (\det(g))\ jN_{K=F}\ \det(g)j^{\ \frac{1}{2}}\ f(g^{\ 1}v):$$

An element u in the unipotent radical of the Siegel parabolic subgroup stabilizing Le_2 acts by:

$$(n\ f)(v) = \quad (hn; vi_a)\ f(v):$$

In particular, we see the dependence on a 2 E in the last formula above. If we replace by , where is a character of K=F, then the splitting of $U_6(F)$ changes by det, where is a character of K^1,determined by via: (z=(z)) = (z). Moreover, for a xed , the Weil representation depends only on the orbit of under $N_{K=F}(K)$.

We can now consider the classical theta lift (1) of the trivial representation of U_1, which is an irreducible representation of $U_6(F)$ realized on the subspace

$$C_c^1(Le_2)^{K^1}\ C^1_c(Le_2):$$

Consider the representation of K $U^K(F_6)$ on $C^1_c(Le_2)^{K^1}$ given by

$$_{M_{J;K}} := \ ^1(1):$$

It is a simple check that the restriction of $_{M_{J;K}}$ to the subgroup

$$f(x; g)\ 2\ K\ \ U^K(F_6) : x=(x) = \det(g)g$$

is independent of and that it descends to a representation of $M^0_{J;K}$. We extend this representation to $M_{J;K}$ by letting act on $f \in C^1_c(L)^{K^1}$ via

$$(f)(v) = f((v)):$$

Thus we have a representation $_{M_{J;K}}$ of $M_{J;K} = M^0_{J;K} \rtimes \langle \rangle$ on $C^1_c(Le_2)^{K^1}$, which depends on the orbit of under $N_{K=F}(K)$. Now we have:

$$(8.2) \qquad _{M_J} = \mathrm{Ind}^{M_J}_{M_{J;K}} \, _{M_{J;K}} = \mathrm{Ind}^{M_J}_{M_{J;K}} \, {}^1 (1): \text{This}$$

representation is now independent of and .

8.5. Similitude theta lifting. It is in fact better to think of the representation $_{M_{J;K}}$ from the viewpoint of the similitude theta correspondence for the pair

$$GU(K) \quad GU(\mathrm{Res}_{K=F}(V_a))_K = K \quad GU_6(F)_K:$$

In particular, we may consider the similitude theta lift $\tilde{}(1)$ of the trivial representation of K; this representation is also realized on $C^1_c(Le_2)^{K^1}$, and is merely an extension of (1) to $GU_6(F)_K$ with the center K acting by the central character ^{3}. Recall from (7.4) the isogeny

$$f : GU(\mathrm{Res}_{L=K}(V_a))_K = (K \quad U_6(F))=rK^1 \quad ! \quad (K \quad U_6(F))=K^1$$

dened by

$$f(z;g) = (z \;\; ^3;g):$$

Then we have;

$$\tilde{}(1) = (\;\; ^1 (1)) \;\; f = \, _{M_{J;K}} \;\; f:$$

In other words, $\tilde{}(1)$ factors through f and when restricted to $(K \;\; U_6(F))^y$ is independent of .

From this viewpoint, the restriction of the $M_{J;K}$-module $_{M_{J;K}}$ to the commuting pair $H_C(F) \;\; GL_2(E)^{det}$ can be transparently described using the seesaw diagram (7.3). More precisely, we pick a $\in E$ so that its class in $E=F \, N_{L=E}(L) = H^1(F; T_{E;K})[2]$ (see (4.11) and (4.17)) corresponds to $[C]$. From the seesaw identity arising from (7.3), the representation $\tilde{}(1)$ is naturally a representation of

$$((L)^{det} \rtimes \langle \rangle)=K \quad GSp(W_a)^{det} = \mathrm{Aut}_E(C) \quad GL_2(E)^{det}:$$

This representation is precisely the restriction of $_{M_{J;K}}$ to $\mathrm{Aut}_E(C) \;\; GL_2(E)^{det}$.

8.6. Some formulas. We write down some formulas for $_{M_{J;K}}$ which are relevant to us.

An element $e \in L^1=K^1 = \mathrm{Aut}_E(C)$ acts on $f \in C^1_c(Le_2)^{K^1}$ by

$$(e \;\; f)(v) = f(e \;\; ^1v):$$

The element

$$= \begin{pmatrix} t(x) & 0 \\ 0 & x^1 \end{pmatrix} \in GL_2(E)^{det};$$

with $x = N_{K/F}(z)$ for some $z \in K^{\times}$, acts on $f \in C_c^{\infty}(Le_2)^{K^1}$ by

$$(t(x) \cdot f)(v) = |x|^{\frac{3}{2}} f(z^{-1}v).$$

The element

$$u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in GL_2(E)^{\det},$$

acts by

$$(u(b)f)(v) = \psi(Tr_{E/F}(a \cdot N_{L/E}(v) \cdot b)) \cdot f(v).$$

The dependence of the $H_C(F) \times GL_2(E)^{\det}_K$-module $\Omega_{M_{J,K}}$ on $a \in E^{\times}$ is thus evident from the action of the unipotent radical of the upper triangular matrices in $GL_2(E)^{\det}$. In particular, one sees that the Whittaker support (relative to ψ) of $\Omega_{M_{J,K}}$ as an $GL_2(E)^{\det}_K$-module is on the coset $a \cdot N_{L/E}(L^{\times}) \subset E^{\times}$. Thus, the Whittaker support of the $GL_2(E)^{\det}$-module

$$\Omega_{M_J} = \mathrm{Ind}_{GL_2(E)^{\det}_K}^{GL_2(E)^{\det}} \Omega_{M_{J,K}}$$

is on the coset $a \cdot F^{\times} N_{L/E}(L^{\times})$. This is the coset corresponding to $[C] \in H^1(F, T_{E,K})[2]_c$, by our choice of a.

8.7. Split case. If $K = F^2$. Then $K^1 = \{(x, y) \in F^2 \mid xy = 1\} = F^{\times}$, $L = E^2$ and $L^1 = E^{\times}$. In this case, we can simplify the description of Ω_{M_J}.

If we apply a partial Fourier transform to $C_c^{\infty}(L) = C_c^{\infty}(E^2)$ with respect to the second factor E of L, the action of $K^1 = F^{\times}$ on $C_c^{\infty}(L)$ becomes the action by homotheties. The representation $\Omega_{M_J} = $ – Ω_{M_J} is the maximal $F^{\times}$-invariant quotient of $C_c^{\infty}(L)$, and is isomorphic to the space of smooth functions f on $L \setminus \{0\}$ such that

$$f(xv) = |x|_F^{-3} f(v) \quad \text{for all } v \in L \setminus \{0\} \text{ and } x \in F^{\times}.$$

The restriction of Ω_{M_J} to $M_E \times Aut_E(C)$ is given as follows. If $g \in GL_2(E)^{\det}$ then

$$\Omega_{M_J}(g)f(v) = |\det(g)|^{-\frac{3}{2}} f(g^{-1}v),$$

where $g^{-1}v$ is the natural action of $g^{-1} \in GL_2(E)$ on $v \in E^2 = L$. If $e \in E^{\times}$ then

$$\Omega_{M_J}(e)f(v) = |N_{E/F}(e)|^{-1} \cdot f(e^{-1}v),$$

where $e^{-1}v$ is the product of the scalar $e^{-1} \in E$ and the vector $v \in E^2$. The involution acts by the Fourier transform, viewing f as a distribution on $C_c^{\infty}(L)$.

8.8. Schrödinger model of Ω_{M_J}. The description we have given above for Ω_{M_J} allows one to relate the theta correspondence arising from its restriction to the dual pair $H_C \times M_E$ in M_J to the classical theta correspondence. As a minimal representation, Ω_{M_J} also has a Schrödinger model adapted to the Siegel parabolic subgroup $S_J \subset M_J$, which we will describe next.

As a representation of S_J, Ω_{M_J} sits in a short exact sequence

$$0 \longrightarrow C_c^{\infty}(J_{rk=1}) \longrightarrow \Omega_{M_J} \longrightarrow r_{S_J}(\Omega_{M_J}) \longrightarrow 0$$

where $J_{rk=1}$ denotes the set of rank 1 elements in J and $r_{S_J}(\ )$ denotes the (normalized) Jacquet module with respect to S_J. The action of some elements of $H_C B_E = (H_C M_C) \cap S_J$ on $C_c^\infty(J_{rk=1})$ can be described as follows:

For $b \in E$, the upper triangular unipotent element $u(b) \in M_E(F) = GL_2(E)^{det}$ acts by

$$(u(b)\ f)(x) = \ (Tr_J(bx))\ f(x) = \ (Tr_{E/F}(b\ e))\ f(x)$$

where $x = (e; v) \in E \ C = J$ has rank 1 and is a fixed nontrivial additive character of F.

For $h \in H_C(F)$, h acts by

$$(h\ f)(x) = f(h^{-1}x)$$

where we have identified H_C with the pointwise stabilizer of $E \ J$, so that $H_C \ Aut(J)$.

Observe that by Lemma 4.15, and Lemma 4.16, $x = (e; v) \in E \ C$ has rank 1 if and only if the map f in Lemma 4.16 sends e to $[C] \in H^1(F; T_{E;K})[2] = E/F N_{L/E}(L)$. In view of (4.17), this is equivalent to the coset $e\ F N_{L/K}(L)$ being equal to that of $[C]$. So the Whittaker support of $_M$ as a $GL_2(E)^{det}$-module is as we had determined in §8.6 via the classical theta correspondence.

The description of the minimal representation $_M$ given here will be used for the study of the theta correspondence for $H_C \ M_E$ in §10. This is necessary for the study of the theta correspondence for $H_C \ G_E$, which will be carried out in §12.

9. Jacquet functors for E_6

In this section, we continue to assume that F is a nonarchimedean local field. The goal of this section is to describe the (un-normalized) Jacquet module $_{N_E}$ as a representation of $M_E \ Aut_E(C)$. Here, recall that $P_E = M_E N_E = P_J \cap G_E$ is the Heisenberg parabolic subgroup in G_E and N_J and N_E share the center Z. Let

$^? = \{x \in$

$: x$ is perpendicular to $N_E/Z\}$. Then we have an exact sequence

$$0 \to C_c^1(^?) \to _{N_E} \to _{N_J} \to 0$$

Thus, we need to:

- determine the set $^?$ and describe $C_c^1(^?)$ as a module for $M_E\ Aut_E(C)$; we shall do this in this section.
- study the theta correspondence for $M_E \ Aut_E(C)$ with respect to $_{M_J}$: we shall study this in the next section.

Now as a $GL_2^{det}(E)$-module, the orthogonal complement of N_E/Z in N/Z is given by the natural action of $GL_2^{det}(E)$ on $C \ C = E^2 \otimes_E C$ via its action on E^2. Thus, an $! \in$ $^?$ is of the form $(0; x; y; 0)$ where $x, y \in C$ such that

$$x^\# = (\ Q(x); (x)) = 0 = y^\#; \quad \text{and} \quad x \ y = 0 \in J:$$

Now we note the following proposition, which uses the structure theory of twisted composition algebras:

Proposition 9.1. If x 2 C is such that $Q(x) = 0$ and $N_C(x) = b_Q(x; (x)) = 0$, then $x = 0$ except when

(1) $E = F^3$ and $J = M_3(F)$.
(2) $E = F$ K, where K is a eld, and $J = M_3(F)$. (3) $E = F$ K, where K is a eld, and $J = J_3(K)$.

Hence ? is empty unless we are in the three cases above.

Proof. It suces to look at the cases when Q is isotropic. If K_C is a eld, then the norm N_{E} $_{K=E}$ $_C$ is isotropic only when $E = F$ K and $K_C = K$. Since $K_E = K$, it follows that $K_J = F^2$. Hence we are in the second case. If $K_C = F^2$, then Q is always isotropic. The cases $E = F^3$ and $E = F$ K correspond to the rst and third cases, respectively, in the statement of the proposition.

If E is a eld, then $C = E$ $F^2 = E$ E and, up to an invertible scalar, $Q(y; z) = yz$ and $(y; z) = (z^\#; y^\#)$, for $(y; z)$ 2 $C = E^2$. Here $Q(y; z) = 0$ implies $y = 0$ or $z = 0$. Assuming $z = 0$, we see that $b_Q((y; 0); (0; y^\#) = yy^\# = N_{E=F}(y) = 0$, which implies that $y = 0$.

Hence, to explicate C^1_0(?), we need to treat the 3 cases highlighted in the proposition, and we shall deal with them in turn.

9.1. Case 1: $E = F^3$ and $J = M_3(F)$. . In this case, C is a split twisted composition algebra. Write

$$x = ((x_1; y_1); (x_2; y_2); (x_3; y_3)); \qquad y = ((x_1^0; y_1^0); (x_2^0; y_2^0); (x_3^0; y_3^0))$$

and suppose that $(x; y)$ 2 ?. Let X_i, respectively Y_i, be the 2-dimensional F-subspace of C C consisting of all pairs $(x; y)$ such that all coordinates except x_i and x_i^0 are trivial, respectively, all coordinates except y_i and y_i^0 are trivial. On each X_i and Y_i, two of the three $SL_2(F)$ M_E act trivially, and the quotient group, isomorphic to $GL_2(F)$, acts via the standard representation.

The condition $x^\# = 0$ holds if and only if there exists a pair of indices $i = j$ such that all coordinates of x are 0 except possibly for x_i and y_j. An analogous statement holds for y: all coordinates are 0 except possibly for x_a^0 and y_b^0 for some $a = b$. The last condition, x $y = 0$, implies that $i = a$ and $j = b$. This can be easily seen by writing x and y as matrices, say

$$x = \begin{pmatrix} 0 & x_3 & y_2 \\ y_3 & 0 & x_1 \\ x_2 & y_1 & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & x_3^0 & y_2^0 \\ y_3^0 & 0 & x_1^0 \\ x_2^0 & y_1^0 & 0 \end{pmatrix}:$$

Hence, if $(0; x; y; 0)$ 2 ?, then $(x; y)$ 2 X_i Y_j for some $i = j$, and we have:

$$? \ [\ f0g = \bigcup_{i=j} X_i \ Y_j:$$

Let X_i and Y_i denote the corresponding punctured planes. As M_E-module, the space $C_c^1($? $)$ has a 2-step ltration with submodule

$$\bigoplus_{i=j} C_c^1(X_i \; Y_j)$$

and quotient (via restriction)

$$\bigoplus_i C_c^1(X_i) \quad \bigoplus_j C_c^1(Y_j):$$

The action of M_E is geometric, with the same twist $_J$ as the one-dimensional summand of N_J.

9.2. Case 2: $E = F \; K$ and $J = M_3(F)$. In this case $K_C = K$, so $C = E \; K = K^3$. The structure of E-module on C is given by

$$(f; e) \; (z_1; z_2; z_3) = (fz_1; ez_2; ez_3)$$

where $(f; e)$ 2 $F \; K$ and $z = (z_1; z_2; z_3)$ 2 K^3. The composition algebra structure is given by

$$Q(z) = (N_K(z_1); z_2 z_3)$$

and

$$(z_1; z_2; z_3) = (z_2 z_3; z_3 z_1; z_1 z_2):$$

This algebra C can be obtained from the split algebra C_s by Galois descent from $C_s \; K$ where the usual action of the Galois group of K over F is twisted by

$$((x_1; y_1); (x_2; y_2); (x_3; y_3)) = ((y_1; x_1); (y_3; x_3); (y_2; x_2)):$$

Note that $Q(z) = 0$ implies that $z_1 = z_2 = 0$ or $z_1 = z_3 = 0$. For $i = 2$ or 3, let Z_i be the two-dimensional K-plane in $C \; C$ consisting of pairs $(z; z^0)$ such that $z_j = z_j^0 = 0$ for all $j = i$. Now ? is the union of the punctured planes Z_2 and Z_3. This claim can be easily veried form the split case using Galois descent. The group $GL_2(E)^{det}$ acts on each plane via projection onto $GL_2(K)^{det}$, with $SL_2(F)$ as the kernel. As M_E-module, the space $C_c^1($? $)$ is a direct sum

$$C_c^1(Z_2) \; C_c^1(Z_3):$$

The action of M_E is geometric, with the same twist $_J$ as the one-dimensional summand of N_J.

9.3. Case 3: $E = F \; K$ and $J = J_3(K)$. In this case $K_C = F^2$, so $E \; C = F^2 \; K^2$. If $z = ((x_1; y_1); (x_2; y_2))$ 2 C, then

$$Q(z) = (x_1 x_2; y_1 y_2) \quad \text{and} \quad (z) = ((N_K(y_2); N_K(y_1)); (y_1 y_2; x_1 x_2)):$$

This algebra C can be obtained from the split algebra C_s by Galois descent from $C_s \; K$ where the usual action of the Galois group of K over F is twisted by

$$((x_1; y_1); (x_2; y_2); (x_3; y_3)) = ((x_1; y_1); (x_3; y_3); (x_2; y_2)):$$

In this case $Q(z) = 0$ and $(z) = 0$ imply that $x_2 = y_2 = 0$ and $x_1 = 0$ to $x_2 = 0$. Let X_1 (respectively Y_1) be the plane in $C \; C$ consisting of all elements $(z; z^0)$ such that all coordinates of z and z^0 are 0 except x_1 and x_1^0 (respectively, except y_1 and y_1^0). Then ? is the union of the punctured planes X_1 and Y_1. Again, this claim can be easily veried form the split case using Galois descent. The group $GL_2(E)^{det}$ acts on each plane via projection onto $GL_2(F)$, with $SL_2(K)$ as the kernel. As M_E-module, the space $C_c^1($

$$?)^1 \text{ is a direct sum } C_c(X_1) \; C_c(Y_1):$$

The action of M_E is geometric, with the same twist $_J$ as the one-dimensional summand of N_J.

10. Mini Theta Correspondence

In this section, we shall determine the local theta correspondence given by the M_E $Aut_E(C)$-module $_{M_J}$ when F is a nonarchimedean local eld. This is only relevant when J = E C is not a division algebra. Understanding this mini-theta correspondence is necessary for our main goal of understanding the theta correspondence for G_E $Aut_E(C)$ G_J. We begin by introducing notation for the irreducible representations of $H_C(F)$ and $M_E(F) = GL_2(E)^{det}$.

10.1. Representations of $Aut_E(C)$. Since J = E C is not a division algebra, we see by Proposition 4.12 that

$$H_C(F) = H_C^0(F) \rtimes Z=2Z$$

where the action of Z=2Z on $Aut_E(C)^0$ is by inverting. Note however that the above isomorphism is not canonical and amounts to choosing an element (necessarily of order 2) in $H_C(F) \smallsetminus H_C^0(F)$.

The irreducible representations of $H_C(F)$ are not hard to classify:

(a) For every character of the torus $H_C^0(F)$ such that $^2 = 1$, we have a two dimensional representation

$$() = \mathrm{Ind}^{H_C(F)}_{H^0_C(F)} = (^1):$$

Note that $() = (^0)$ if and only if $^1 = ^0$.

(b) For each character such that $^2 = 1$, there are two extensions of to $H_C(F)$. If = 1, these two representations are easily distinguishable from each other: one is trivial whereas the other is not. We denote them by 1 and $= _C$ (the sign character of $H_C(F)$) respectively.

(c) When $^2 = 1$ but = 1, we can use the xed isomorphism $H_C(F)$ $H_C^0(F) \rtimes Z=2Z$ to distinguish the two extensions. Namely, we may denote the two extensions by $()^+$ and $()^-$, where the sign denotes the action of the nontrivial element of Z=2Z.

Note however that the labelling in (c) above is not really canonical. We shall see much later that one has a better parametrization. This is based on the following canonical bijection of 2-element sets deduced from Proposition 4.20:

$$f^{-1}([C])=b(\mathrm{Ker}()) \; ! \; (H_C(F) \smallsetminus H_C^0(F))=\mathrm{Ker}():$$

and the observation that any extension of is a nonconstant 1-valued function on the RHS. For this section, the labelling provided by (c) above is sucient.

10.2. Induced representations of $GL_2(E)^{det}$. Writing E as a product $\prod_i E_i$ of elds E_i, we have a similar product $L = E \quad K = \prod_i L_i$ with $L_i = E_i \quad K$. Let $!_{L=E}$ be the quadratic character of E such that the restriction to each E_i is the quadrate character corresponding to the extension L_i.

Now let be a unitary character of E and consider the induced representation $!_{L=E}$ of $GL_2(E)$ in the notation of Bernstein and Zelevinski. We shall need some simple results on the restriction of $!_{L=E} = \quad ^1 \, !_{L=E}$ to $GL_2(E)^{det}$.

Proposition 10.1. Let be a unitary character of E=F. In the following, \ the restric-tion" refers to the restriction of $!_{L=E}$ to $GL_2(E)^{det}$.

(1) Assume that $K = F^2$, and is a character of E trivial on F. The restriction is irreducible unless $^2 = 1$ and $= 1$, in which case it is a direct sum of 2 non-isomorphic irreducible representations.

(2) Assume that K is a eld and E = F K. Let is a character of F K trivial on F K^1. The restriction is irreducible unless $^2 = 1$ and $= 1$, in which case it is a direct sum of 2 non-isomorphic irreducible representations.

(3) Assume that K is a eld, but E = F K. Let = 1. The restriction of 1 $!_{!_{L=E}}$ is a direct sum of $2^{n \quad 1}$ non-isomorphic irreducible representations where n is the number of factors of E.

Proof. These statements can be deduced from the well known facts about representations of $GL_2(E)$ and $SL_2(E)$. We provide the details in the case when E is a eld and $K = F^2$; the general case is treated by a similar argument.

The representation 1 is irreducible when restricted to $SL_2(E)$ (and hence to $GL_2(E)^{det}$) unless $^2 = 1$ and $= 1$. If $^2 = 1$ and $= 1$, then 1 reduces to two non-isomorphic summands on $SL_2(E)$ and also on the intermediate group consisting of elements g 2 $GL_2(E)$ such that det(g) is in the kernel of . Since, by our assumption, is trivial on F , the character is trivial on det(g) for g 2 $GL_2(E)^{det}$. Thus 1 is a sum of two non-isomorphic irreducible representations.

10.3. Theta lifting. For every irreducible representation of $H_C(F)$, let $_M()$ be a representation of M_E such that $_J \quad _M()$ is the maximal -isotypic quotient of $_M$. We shall now give a description of $_M()$ for unitary representations . The results are es-sentially a reformulation of the classical similitude theta correspondence for the dual pair $GO_2(E)$ $GL_2(E)$, together with an understanding of the restriction of representations from $GL_2(E)$ to $GL_2(E)^{det}$ (as we did in the previous proposition).

Recall from (8.2) that

$$_{M_J} = \mathrm{Ind}_{M_{J;K}}^{M_J} \, _{M_{J;K}};$$

with $_{M_{J;K}}$ equal to the restriction of the similitude theta lift of the trivial representation of $GU(K) = K$. From the seesaw diagram in (7.3), $_{M_{J;K}}$ is naturally a module for

$$GO(L; N_{L=E})^{det} \quad GSp(W)^{det} = ((L)^{det} \text{ o hi}) \quad GL_2(E)^{det} \qquad _K$$

which factors through to the quotient

$$H_C(F)\ \ GL_2(E)^{det}_K = ((L)^{det} \text{ o } hi)=K\ \ GL_2(E)^{det}: \text{Here}_K$$

we recall that (see Lemma 7.6) that

$$H^0_C(F) = (L)^{det}=K = L^1=K^1$$

and

$$GL_2(E)^{det}_K = fg\ 2\ GL_2(E)^{det} : \det(g)\ 2\ N_{K=F}(K)g:$$

Thus, we need to understand the theta correspondence for $H_C(F)\ \ GL_2(E)^{det}_K$ arising from $M_{J;K}$. Indeed, if we let $_{M_K}$ denote this theta correspondence, then for any 2 $Irr(H_C(F))$,

$$_M() = Ind^{GL_2(E)^{det}}_{GL_2(E)^{det}_K} {}_{M_K}():$$

We have thus explained the reduction of the determination of the mini-theta correspondence to the similitude theta correspondence for

$$GO(L; N_{L=E})\ \ GSp(W)^+$$

together with the understanding of the restriction of the theta lifts to the subgroup $GSp(W)^{det}$. With our knowledge of the theta correspondence for $GO_2\ \ GL^+_2$, this interpretation immediately gives us the following:

Lemma 10.2. (i) For any = (the sign character of $H_C(F)$), $_M()$ is nonzero, whereas $_M()$ = 0.

(ii) For an irreducible representation () of $H_C(F)$, where is a character of $H^0_C(F)$ = $(L)^{det}=K$, $(())$ is noncuspidal if and only if j_{L^1} is trivial on all the anisotropic factors of L^1 = $\prod_i L^1_i$.

In the context of (ii) of the Lemma, we note:

- if $K = F^2$, then $H^0_C(F) = E=F$ and there are no anisotropic factors of L^1, so that $_M()$ is noncuspidal (as long as =).
- if K is a eld and $E = F\ \ K$, then $H^0_C(F) = K^1\ \ K=K^1 = K$, and a character trivial on anisotropic factors can be identied with a character of $K=K^1$.
- if K is a eld and $E = F\ \ K$, only $_M(1)$ is noncuspidal.

It will turn out that the theta lifts in these cases are contained in the principal series representations we considered in Proposition 10.1.

The following proposition continues our study of the mini-theta correspondence by rening Lemma 10.2:

Proposition 10.3. For every irreducible unitary representation = of $H_C(F)$, $_M()$ is an irreducible nonzero representation of M_E, whereas $_M()$ = 0. Moreover, if $_M() = {}_M(\ ) = 0$, then $\cong {}^0$. More precisely:

(1) $_M(1)$ is an irreducible summand of $1\ !_{L=E}$.

(2) Let $K = F^2$ and be a character of $H^0_C(F) = E/F$. Then $^2 =$

$$1 =) \quad _M(()) = \quad 1;$$

whereas

$$^2 = 1 \text{ but } = 1 =) \quad _M(()^+) \quad _M(() \quad) = \quad 1:$$

(3) Let K be a eld, $E = F \quad K$ and a character of $H^0(F)_C = K$ trivial on K^1. Extend to a character ~ of $F \quad K$, so that it is trivial on the rst factor. Then: $^2 = 1$

$$=) \quad _M(()) = \sim \; !_{L=E};$$

whereas

$$^2 = 1 \text{ but } = 1 =) \quad _M(()^+) \quad _M(() \quad) = \sim \; !_{L=E}:$$

(4) For all other cases of the triple $(E; K;)$ not covered above, $_M(())$ is cuspidal.

Proof. In view of Lemma 10.2, the main issue here is the irreducibility of $_M$ () for 2 $Irr(H_C(F))$. We shall illustrate the argument in the case where K is a eld and $E = F^3$; the other cases are similar and sometimes easier.

For the case under consideration, we have

$$Aut_E(C)^0(F) = (K \quad K \quad K)^{det}=K \text{ o } hi;$$

where the superscript det refers to the subgroup of elements $(x; y; z)$ with $N_{K=F}(x) = N_{K=F}(y) = N_{K=F}(z)$. Ignoring the element for the moment, we are thus considering a triple similitude theta correspondence for $GSO^K_2(F) \quad GL_2(F)_K$. We record the following known results concerning this similitude theta correspondence:

(a) If is a unitary character of $GSO^K_2(F) = K$ such that j_{K^1} is not quadratic, or equivalently = does not factor through $N_{K=F}$), then

$$() = () \; 2 \; Irr(GL_2(E)^{det}) \text{ is}$$

supercuspidal. Indeed,

$$\tilde{()} := Ind^{GL_2(F)}_{GL_2(F)_K}()$$

is an irreducible supercuspidal representation which is dihedral with respect to $K=F$ and no other quadratic elds, so that () remains irreducible when restricted to $SL_2(F)$.

(b) if j_{K^1} is quadratic but nontrivial, or equivalently = is nontrivial but factors through $N_{K=F}$, then $() = (\;)$ is an irreducible supercuspidal representation of $GL_2(E)^{det}$. Indeed,

$$\tilde{()} := Ind^{GL_2(F)}_{GL_2(F)_K}()$$

is an irreducible supercuspidal representation which is dihedral with respect to $K=F$ and two other quadratic elds. Hence, () decomposes as the sum of two irreducible supercuspidal representations when restricted to $SL_2(F)$:

$$()j_{SL_2} = (j_{K^1}) \quad (\bar{j}_{K^1});$$

where the two summands are the theta lifts (to $SL_2(F)$) of the two extensions of j_{K^1} to O_2 (F). Indeed, if we consider the index 2 subgroup

$$GL_2(F)_K = fg\ 2\ GL_2(F)_K : \det(g) = N_{K=F}(z); (z=(z)) = 1g;$$

then each of the two summands (j^K_1) is an irreducible $GL_2(F)_K$-module.

(c) If j_{K^1} = 1, or equivalently = , then = $N_{K=F}$ for some (well-determined up to multiplication by $!_{K=F}$) and () is one of the two irreducible summands of the restriction of $!_{K=F}$ to $GL_2(F)_K$. Moreover, these two summands remain irreducible when restricted to $SL_2(F)$.

(d) () = (^{0}) if and only if ^{0}= or .

Now we are ready to analyze the triple similitude theta correspondence. Let = ($_1$; $_2$; $_3$) be a character of (K)^3 such that $_1$ $_2$ $_3$ = 1. We need to study the reduciblity of

($_1$; $_2$; $_3$) := ($_1$)

($_2$)

K

($_3$) when restricted to $GL_2(E)^{det}$. We shall consider several cases in

turn:

(i) If $_i j_{K^1}$ is not quadratic nontrivial for all i, then by (a) and (c) above, ($_i$) remains irreducible when restricted to $SL_2(F)$. Hence ($_1$; $_2$; $_3$) is irreducible when restricted to $GL_2(E)^{det}_K$.

(ii) Assume now that exactly one of the $_i j_{K^1}$ is quadratic nontrivial. Without loss of generality, suppose that $_3 j_{K^1}$ is quadratic nontrivial but the other two restrictions are not. Then $_K$($_1$) and $_K$($_2$) are irreducible as $SL_2(F)$-representations, while $_K$($_3$) is irreducible as $GL_2(F)_K$-representation. It follows readily that ($_1$; $_2$; $_3$) irreducible as an $GL_2(E)_K$-representation.

(iii) Assume next that exactly two of the $_i j_{K^1}$ is quadratic nontrivial. Without loss of generality, we may suppose

$$_1 j_{K^1} = {}_2 j_{K^1} = \qquad \text{and} \quad {}^3 j_{K^1} = 1$$

for some quadratic character of K^1. In this case, by (b) above, we have

$$(_1)j_{SL_2} = (_2)j_{SL_2} = (^+)\ (\ )$$

as $SL_2(F)$-modules. Now it is easy to check that

[($^+$)
($^+$) ( )
( )]
(1)

and

[($^+$)
( ) ( )
($^+$)]
(1)

are irreducible representations of $GL_2(E)_K$. In particular, ($_1$; $_2$; $_3$) is the sum of two irreducible representations as $GL_2(E)^{det}_K$-modules.

(iv) Finally, we consider the case when $_i$:= $_i jK^1$ is quadratic nontrivial for all i; this case can only occur when the residue characteristic of F is 2. In this case,

$$(_1; _2; _3) = [(_1\ )\ (_1\ )]\ ^+ \quad {}^+ \quad {}^+$$
$$[(_2\ )\ (_2\ )]$$
$$[(_3\ )\ (_3\ )]$$

as $SL_2(F)^3$-modules. The key observation here is that each $GL_2(F)^i_K$ acts irreducibly on $(_j) = ( ) ( )$ if $i = j$, and preserves each summand if $i = j$. Now it is easy to check that for every $= 1$,

$$_{123} = (_1 \ ) \quad ^1 \quad ^2 \quad ^3$$
$$(_2 \ )$$
$$(_3 \ ):$$

is an irreducible representation of $GL_2(F^3)_K$. In particular, $(_1;_2;_3)$ decomposes as the sum of two irreducible $GL_2(E)^{det}_K$-modules.

With the above results, we can now complete the proof of the proposition when $E = F^3$ and K is a eld. Note that we are only concerned with the restriction of $_1\ _2\ _3$ to the subgroup:

$$H_{C_0}(F) = ((K)^3)^{det} = K = (K^1)^3 = K^1: \text{ So for}$$

example, we have:

$_1\ _2\ _3$ restricts to the trivial character if and only if $_i j_{K^1} = 1$ for each i.

The restriction of $_1\ _2\ _3$ is a nontrivial quadratic character if and only if $_i j_{K^1}$ is quadratic for all i and is nontrivial for some i.

In particular, we see that the latter case corresponds precisely to the cases (iii) and (iv) analyzed above. In this case, there are thus two extensions of to $H_C(F)$ and (in view of Lemma 10.2) $_M$ () are both nonzero and hence are precisely the two irreducible summands of $(_1;_2;_3)j_{GL_2(E)_K}$ described in (iii) and (iv) above.

Finally, from the properties of the similitude theta correspondence, we deduce that

$$_0 = {}^1 \text{ on } (K^1)^3 \ ( ) \quad () = (^0) \text{ on } GL_2(E)^{det}_K.$$

This concludes the proof of the proposition, at least in the case when $E = F^3$ and K is a eld.

10.4. Whittaker models. For a xed C, with associated Springer decomposition $J = E\ C$, we have obtained a subset

$$Irr_C(M_E(F)) := f_{M;C}() \ 2 \ Irr(M_E(F)) : = \ 2 \ Irr_{unit}(H_C(F))g \ \ Irr(M_E(F)):$$

Moreover, the representations in $Irr_C(M_E(F))$ are innite-dimensional and hence generic. In this subsection, we investigate the Whittaker models supported by the representations in $Irr_C(M_E(F))$. This serves to complete our analysis of the mini-theta correspondence by specifying precisely the irreducible representations $_{M;C}()$.

We had briey alluded to the Whittaker support of $_M$ as an $GL_2(E)^{det}$-module in x8.6 and x8.8, but let us be more precise here. Fix a nontrivial additive character of F. Then a generic character for the unipotent radical of the upper triangular Borel subgroup B_E of $M_E(F) = GL_2(E)^{det}$ is of the form

$$u(b) \ ! \quad (Tr_{E=F}(ab)) \quad \text{for some } a \ 2 \ E.$$

We denote this generic character by $_a$. Two such generic characters $_a$ and $_{a^0}$ are equivalent if they are conjugate by the action of the diagonal torus and we call an equivalence class a Whittaker datum for $M_E(F)$. A short computation shows that the set of Whittaker data is parametrized by $E = F E^2 = H^1(F; Z_E)$. Hence we have yet another interpretation of $H^1(F; Z_E)$:

$$H^1(F; Z_E) \quad = \quad \text{fWhittaker datum for } GL_2(E)^{det}\text{g}$$

$$G_E^{ad}(F)=Im(G(E)) \quad = \quad \text{frank 1 E-twisted composition algebrasg}$$

We are interested in computing the twisted Jacquet module

$$(_{M_J})_{U_E; \ a} \qquad \text{as a } H_C(F)\text{-module,}$$

For this purpose, we shall make use of the Schrodinger model of $_{M_J}$ introduced in x8.8 and the results of x4.9. To formulate the result, let us recall from Lemma 4.15 the $H_C(F)$-set

$$X_{a;C}(F) = \text{f}x \ 2 \ C : Q(x) = a^{\#} \text{ and } (x) = ax\text{g}$$

which is in bijection with the set of embeddings $C_a \ ,! \ C$, where C_a is a rank 1 E-twisted composition algebra dened in x4.2. Moreover, if $X_{a;C}(F)$ is nonempty, then it is a principal homogeneous space for $H_C^0(F)$, so that the stabilizer in $H_C(F)$ of any point in $X_{a;C}(F)$ has order 2. Now we have:

Lemma 10.4. Fix an E-twisted composition algebra C of rank 2, with associated Springer decomposition $J = E \ C$. For each $a \ 2 \ E$, one has

$$(_{M_J})_{U_E; \ a} = 0 \ (\ ) \ \ X_{a;C}(F) = ;;$$

in which case

$$(_{M_J})_{U_E; \ a} \ \ Ind_{H_{C;x_a}(F)}^{H_C(F)} 1$$

where $x_a \ 2 \ X_{a;C}(F)$ and $H_{C;x_a}(F) \ \underline{=} Z=2Z$ is the stabilizer of x_a in $H_C(F)$.

Proof. From the Schrödinger model of $_{M_J}$ discussed in x8.8 and the results of Lemma 4.15, we see that

$$(_{M_J})_{U_E; \ a} = C_c \ (J_{rk=1})_{U_E; \ a} = C_c \ (X_{a;C}(F));$$

as $H_C(F)$-module. Since $X_{a;C}(F) = H_C(F) \ x_a \ \underline{=} H_C(F)=H_{C;x_a}(F)$, the result follows.

Recall the map

$$f : H^1(F; Z_E) = E=F E^2 \quad ! \quad H^1(F; T_{E;K_C})[2]: \text{For}$$

each $[C] \ 2 \ H^1(F; T_{E;K_C})[2]$, we have

$$f^{\ 1}([C]) = \text{f}a \ 2 \ E : X_{a;C}(F) = ;\text{g}:$$

Then the above lemma gives the following corollary:

Corollary 10.5. For any $= () \ 2 \ Irr(H_C(F))$, $_{M;C}()_{U_E; \ a} = 0$ if $f(a) = [C]$. On the other hand, if $f(a) = [C]$, then we have:

If $^2 = 1$, $\dim (())_{U_E; \ a} = 1$.

If $^2 = 1$, so that has two extensions ~ to $H_C(F)$, then

$$\dim \ _{M;C}(\tilde{\ })_{U_{E;\ a}} = \begin{cases} 1 \text{ if } \tilde{\ }(g_C(a)) = 1; \\ 0; \text{ if } \tilde{\ }(g_C(a)) = \ 1. \end{cases}$$

where $g_C(a)$ is the nontrivial element in $H_{C;x_a}(F)$ for some x_a 2 $X_{a;C}(F)$ (see Lemma 4.18).

10.5. As C varies. In this nal subsection, we allow [C] to vary over $H^1(F; T_{E;K_C})[2]$. Then by Lemma 4.16, we have a disjoint union

$$E{=}FE^2 = \bigsqcup_{[C]} f^{\ 1}([C])$$

where each $f^{\ 1}([C])$ is nonempty and is a $T_{E;K_C}(F){=}T_{E;K_C}(F)^2$-torsor. We deduce:

Corollary 10.6. The union

$$\bigcup_{[C]2H^1(F;T_{E;K_C})[2]} \mathrm{Irr}_C(M_E(F)) \ \mathrm{Irr}(M_E(F))$$

is disjoint, since the representations in dierent subsets have dierent Whittaker support.

We can in fact rene this corollary. A character of $T_{E;K_C}(F)$ or $T_{E;\tilde{K}_J}(F)$ gives rise to a character $_C$ of each $H^0_C(F)$. We then consider the $M_E(F)$-module

$$_{M_E}[] := \bigoplus_{[C]2H^1(F;T_{E;K_C})[2]} \ _{M;C}((_C)) \qquad \text{with } (_C) = \mathrm{Ind}^{H_C(F)}_{H^0_C(F)} \ _C.$$

Then we have:

Corollary 10.7. For each a 2 E,

$$\dim \ _{M_E}[]_{U_{E;\ a}} = 1:$$

In particular,

$$\bigoplus_{[C]2H^1(F;T_{E;K_C})[2]} \ _{M;C}(1) = 1 \quad !_{L=E}:$$

Indeed, one can show in general that $_{M_E}[]$ is the restriction to $M_E(F) = GL_2(E)^{\det}$ of an irreducible generic representation of $GL_2(E)$. Together with our knowledge of the Whittaker support of the mini-theta lifts. this has the following nice consequence. If M_E^{ad} denotes the Levi subgroup of the Heisenberg parabolic subgroup in the adjoint quotient G_E^{ad}, recall that

$$M_E^{ad}(F){=}\mathrm{Im}(M_E(F)) = G_E^{ad}(F){=}\mathrm{Im}(G_E(F)) = H^1(F; Z_E) = E{=}FE^2:$$

Hence, $H^1(F; Z_E)$ acts naturally on $\mathrm{Irr}(M_E(F))$ and also on $H^1(F; T_{E;K_C})[2]$ (via the projection $H^1(F; Z_E) \ H^1(F; T_{E;K_C})$). For an element 2 $H^1(F; Z_E)$ and a character of $T_{E;K_C}(F)$, we then have

$$_{M;C}((_C)) \ = \ _{M;C}((_C));$$

where the superscript denotes the two actions of on the relevant objects mentioned above.

11. Langlands quotients of D_4

The purpose of this section is to write down some representations of G_E that will appear in the theta lifting from $H_C = Aut_E(C)$ in terms of their Langlands data, and to give explicit realizations of these representations in some cases. It thus provides the language needed to express the answer for the theta correspondence treated in the next section. In fact, in Appendix B below, we consider the decomposition of unramied degenerate principal series representations of G_E and introduce notations for many irreducible representations with nonzero Iwahori-xed vectors, constructed via Hecke algebra considerations. These representations will also appear in this section and the next one.

11.1. Langlands quotient from P_E. As previously, let $P_E = M_E N_E$ be the Heisenberg maximal parabolic subgroup. The modular character $_{N_E}$ of M_E is

$$_{N_E} = j \det j^5 :$$

Let be a tempered representation of M_E. Using the normalized parabolic induction, we induce
$j \det j^s$ from P_E to G_E, giving a standard module if $s > 0$. Let $J_2(;s)$ be the corresponding Langlands quotient when $s > 0$. The representation $J_2(;s)$ is also the unique submodule of the representation obtained by inducing
$j \det j^s$ from the opposite parabolic $P_E = M_E N_E$. This point of view is more useful to us.

11.2. Langlands quotient from Q_E. We shall also need some Langlands quotients attached to the 3-step parabolic subgroup $Q_E = L_E U_E$ corresponding to the middle vertex of the Dynkin diagram. Then

$$L_E = (GL_2(F) \quad E)^{det} = f(g;e) \; j \; det(g) = N_{E=F}(e)g:$$

Let $N_{E=F}$ also denote the character of L_E obtained by projecting L_E to E followed by the norm on E. The modular character $_{U_E}$ of L_E is

$$_{U_E} = j N_{E=F} j^3 :$$

For a tempered irreducible representation of L_E, consider the normalized parabolic induc-tion of
$j N_{E=F} j^s$ from Q_E to G_E. If $s > 0$, this is a standard module and we let $J_1(;s)$ be the corresponding Langlands quotient.

We shall need this parabolically induced representation when is one of the following representations:

- $= St_E$ is the Steinberg representation of L_E obtained by projecting L_E to $GL_2(F)$ and pulling back the Steinberg representation of $GL_2(F)$.
- If $E = F \; K$, then we dene a character of E equal to $_K$ on the rst factor F and trivial on the second factor K. We can pull this character back to L_E, and abusing notation, denote it by $_K$. Note that $_K$ is of course a nontempered representation of L_E.

11.3. Degenerate principal series. We shall also need the structure and constituents of various unramed degenerate principal series representations induced from maximal parabolic subgroups. The necessary results are provided in Appendix B below. We provide here a roadmap for where the various results are located there:

- when E is a eld, the only maximal parabolic subgroups are P_E and Q_E. The degenerate principal series associated to P_E is denoted by

$$I(s) = \mathrm{Ind}_{P_E}^{G_E} j\det j^s \quad \text{(normalized induction).}$$

The points of reducibility and the module structure at those points are given in Theorem 18.1. On the other hand, the degenerate principal series associated to Q_E is denoted by

$$J(s) = \mathrm{Ind}_{Q_E}^{G_E} jN_{E=F} j^s \quad \text{(normalized induction).}$$

Its reducibility points and module structure is described in Theorem 18.2.

- when E = F K where K is a eld, there are 3 families of degenerate principal series: B(s) (associated with the B_2-maximal parabolic), A(s) (associated to the A_2-maximal parabolic) and I(s) (associated to the Heisenberg parabolic, which is the A_1 A_1-parabolic). The points of reducibility for these are given in Theorem 18.3, Proposition 18.4 and Proposition 18.5 respectively.

- when $E = F^3$ is split, the degenerate principal series has been studied to some extent in the literature, such as [BJ] and [We1]. We only need the results concerning I(s) (associated to Heisenberg parabolic) summarized in Proposition 18.6.

11.4. A_2-parabolic. We shall need an explicit description of the quotients $J_2(; s)$ in certain cases. Assume now that $E = F^3$. When writing $M_E^{der} = SL_2 \ SL_2 \ SL_2$, we shall assume that the three SL_2 correspond, respectively, to simple roots $_1$, $_2$ and $_3$. Let () (or ()) be a representation (or two) of M_E corresponding to = $(_1; _2; _3)$, a character of E , as x10.2. In particular, $_1 \ _2 \ _3$ = 1. We shall assume that is unitary, so that () is tempered. Consider the parabolic subgroup in standard position corresponding to the A_2 diagram, containing the vertex corresponding to $_1$. The character denes a unitary character (temporary notation) of the Levi subgroup given by

$$(_2(t)) = \ _3(t) \text{ and } (_3(t)) = \ _2(t):$$

Let D() be the unitary representation of G_E obtained by inducing (unitary induction) the character . Since D() is unitary, it is completely reducible. We now consider three cases:

- Suppose that ^{2} = 1. By working out exponents (there are 32 of these), one sees that D() has a unique irreducible subrepresentation and hence is irreducible. Using exponents again, one may determine the Langlands parameter of D(). It turns out that

$$D() = J_2((); 1):$$

- Suppose that ^{2} = 1 but = 1. Then D() has two irreducible summands: D()

$$= J_2(()^+; 1) \ J_2(() \ ; 1):$$

- Suppose that = 1. Then D(1) has two irreducible summands. The unique spherical summand is isomorphic to $J_2((1); 1)$. The exponents of the non-spherical summand can be determined. Indeed, the spherical summand of D(1) is also the quotient of I(1=2), and the exponents of this quotient are known by Prop 18.6. Then, using the exponents, one can determine the Langlands parameter of the non-sperhcial summand. It turns out that the non-spherical summand is isomorphic to $J_1(St_E; 1=2)$. Hence

$$D(1) = J_2((1); 1) \quad J_1(St_E; 1=2):$$

Remark: Despite the fact that D() is dened by an arbitrary choice of the A_2 parabolic, the Langlands parameter of D() is independent of this choice. Hence the isomorphism class of D() is, remarkably, independent of the choice, i.e. the isomorphism class of D() is invariant by the triality automorphism.

We need a similar discussion in the case E = F K. Let = (; $)_F$ be $_K$a character of E trivial on the diagonally embedded F. Consider the (unique) parabolic subgroup in standard position corresponding to the A_2 diagram. Now denes a character of the Levi subgroup given by

$$(-(t))_2 = {}_K(t) \text{ for all } t \; 2 \; K:$$

Let D() be the unitary representation of G_E obtained by parabolically inducing the charac-ter (unitary induction). The structure of D() is similar to that in the split case discussed above. The only dierence is that the non-spherical summand of D(1) is the representation V_1^0 (introduced in x18.5.1 and x18.5.3 of Appendix B below) with a one-dimensional space of Iwahori-xed vectors. It is a Langlands quotient of a standard module induced from B_2-parabolic.

We summarize both cases in the following proposition.

Proposition 11.1. Assume that E is not a eld. Let be a unitary character of E trivial on F and consider the representation D() induced from a parabolic subgroup of type A_2 as dened above. Then

(1) If $^2 = 1$, then D() = J_2 ((); 1).
(2) If $^2 = 1$ but = 1, then D() = $J_2(()^+; 1)$ $J_2(()$; 1).
(3) If E = F , then D(1) = $J_2((1); 1)$ $J_1(St_E; 1=2)$.
(4) If E = F K, then D(1) = $J_2((1); 1)$ V_1^0 (where V_1^0 is introduced in x18.5.1 and x18.5.3).

As we see from the above proposition, we shall need to refer to representations of $G_E(F)$ which are constructed in Appendix B below, where we study the decomposition of unramied degenerate principal series representations of G_E. Some of these representations will appear in the theta lifting from H_C which we shall consider next.

12. Theta correspondence for E_6

In this section, we will study the theta correspondence for H_C G_E G_J = Aut(J), where J = E C is a Freudenthal Jordan algebra of dimension 9. The main goal is the following theorem, whose proof will occupy the rest of this section.

Theorem 12.1. For every unitary irreducible representation of $H_C(F)$, () is non-zero and irreducible. If () = (^{0}), for two irreducible representations and ^{0} of $H_C(F)$, then = ^{0}. More precisely:

(1) If J = D (a cubic division algebra), so that E is a eld, then (1) = $V^0 = J_1$ (St_1 ; 1=2) (see x18.2.1 and x18.2.4 for the denition of V^0_1, as well as Theorem 18.1) and () is supercuspidal for all = 1.

(2) If J = D and = , then () = J_2(M (); 1=2).

(3) If J = D and H^0_C is anisotropic, then () is supercuspidal. Otherwise:

If E = F^3 and J = $M_3(F)$, then () = $J_1(St_E; 1=2)$.
If E = F K and J = M (F_3), then () = V_1 (see x18.5.1 and x18.5.3 for the denition of V^0_1).
If E = F K and J = $J_3(K)$, then () = $J_1(St_E$ $_K$; 1=2).

12.1. E-twisted cubes. Recall from x5 that if $P_E = M_E N_E$ is the Heisenberg parabolic subgroup of G_E, then the representation of $M_E = GL_2(E)^{det}$ on

$$N_E = Z_E = F \quad E \quad E \quad F$$

is the space of E-twisted Bhargava cubes. As we summarized in Proposition 5.1, the M_E-orbits of nondegenerate cubes are parametrized by E-isomorphism classes of E-twisted composition algebra of dimension 2 over E. Indeed, for any nondegenerate cube , one attaches a twisted composition algebra structure (Q;) on C = E^2, so that there is a natural isomorphism

$$\text{(12.2)} \qquad Stab_{M_E}() = Aut_E(C) \qquad \text{given by } g \,!\; {}^tg^{\;1}.$$

If we x a nontrivial additive character of F, then the natural pairing between $N_E = Z_E$ and $N_E = Z_E$ allows us to identify the unitary characters of N_E with elements of $N_E = Z_E$. In particular, an E-twisted cube determines a corresponding character of N_E.

12.2. Twisted Jacquet module. Let = $_J$ be the minimal representation of G_J. We have computed the Jacquet module $_N$ in x9. In this subsection, we determine the twisted Jacquet module $_{N;E}$ for the character of N_E attached to a nondegenerate E-twisted cube . Note that $_{N;E}$ is naturally a representation of $Stab_{M_E}()$ $Aut_E(C)$, and thus of $Aut_E(C)$ $Aut_E(C)$ in view of (12.2).

In x8.2, we have seen that

$$C^1_c(\;) \quad _E \quad C^1(\;) \quad Z$$

where is the minimal M_J-orbit on $N_J = Z_E$, which can be identied with a set of unitary characters of N_J. It follows from the description of $_{Z_E}$ given in (8.2) that

$$_{N_E;} = C_c(^1 \;)$$

where is the set of elements ! 2 such that ! restricted to N_E is . Based on our description of in x8.3, the following proposition determines the set concretely.

Proposition 12.3. Let J = E C be a Freudenthal Jordan algebra of dimension 9. Let be a nondegenerate E-twisted cube. Then $_{N;E}$ = 0 unless belongs to the M_E-orbit corresponding to C (i.e. C = C). If C = C, then

$N_{E;} = K^{1}$

$C_c\ (\mathrm{Isom}(C; C))$

where

- $_K$ is the restriction of $_J$ to $\mathrm{Stab}_{M_E}()$; in particular, $_K$ is either trivial or the sign character of $\mathrm{Stab}_{M_E}() = H_C(F)$ depending on whether $!_{K=F}(\ 1) = +1$ or $\ 1$;
- the action of $\mathrm{Stab}_{M_E}()\ \mathrm{Aut}_E(C)$ on $C^1(\mathrm{Isom}_C(C; C))$ is the regular representation (via (12.2)).

Proof. Since every nondegenerate M_E-orbit contains reduced cubes, we may assume without loss of generality that is reduced, i.e.

$$= (1; 0; f; b);$$

The associated twisted composition algebra C is then described in Proposition 5.2.

Now the projection map

$$N_J = Z_J = F\ J\ J\ F \quad ! \quad N_E = Z_E = F\ E\ E\ F \text{ induced}$$

by the restriction of characters is given by

$$(a; x; y; d)\ !\ \ (a;\ \ e_x; e_y;\ \ d)$$

where we have writtem

$$x = (e_x; c_x) \quad \text{and} \quad y = (e_y; c_y)\ 2\ E\ \ C = J:$$

Hence, if $! = (a; x; y; d)$ 2

, so that $_!$ restricts to , then $a = 1$, so that !

$= (1; x; x^{\#}; N_J(x))$ (by Proposition 8.1).

Writing $x = (e; v)\ 2\ E\ \ C = J$ and noting that $(0; v)^{\#} = (\ \ Q(v); (v))$, we then deduce that

$$e = 0 \quad \text{and} \quad Q(v) = \quad f:$$

Finally, since $N_J(x) = N_C(v)$, we also have

$$N_C(v) = \quad b:$$

Hence, we have a natural $\mathrm{Stab}_{M_E}()\ \ \mathrm{Aut}_E(C)$-equivariant identication

$$= f(v; (v))\ 2\ C^2 : Q(v) = \quad f \text{ and } N_C(v) = \quad bg\ \ C^2 = E^2\ {}_E\ C;$$

where the action of $\mathrm{Aut}_E(C)$ is componentwise, whereas that of $\mathrm{Stab}_{M_E}()\ \ GL_2(E)^{det}$ is via the standard representation on E^2. Thus, the $\mathrm{Stab}_{M_E}()\mathrm{Aut}_E(C)$-set is nothing but the $\mathrm{Stab}_{M_E}()\mathrm{Aut}_E(C)$-set $_{C;f;b}$ studied in Corollary 5.3 and Lemma 5.4. We thus deduce that $_{N_E;}$ = ; unless C is isomorphic to C, in which case is identied with $\mathrm{Isom}_C(C; C)$ and $_E$ = $C^1(\mathrm{Isom}(C; C))$ is the regular representation of $\mathrm{Stab}_{M_E}()\ \ \mathrm{Aut}_E(C)$ twisted by the quadratic character $_K$.

If we x a base point $_0$ 2 $\mathrm{Isom}(C; C)$, we get an isomorphism $\mathrm{Stab}_{M_E}() = \mathrm{Aut}_E(C)$ and with respect to this, $_{N_E;}$ is the regular representation of $\mathrm{Aut}_E(C)\ \ \mathrm{Aut}_E(C)$. We assume that this isomorphism has been xed henceforth. We remark also that the quadratic character $_K$ is trivial when K is not a eld. In any case, this extra twist will be quite innocuous for our purpose.

For later use, we shall now compute the twisted co-invariants for some degenerate cubes in the case when $Aut_E(C)$ is anisotropic. Consider

$$= (1; 0; f; 0) \qquad \text{with } f^{\#} = 0.$$

We have:

If $f = 0$, this cube belongs to the minimal G_E orbit (A_1).

If $f = 0$ and $f^{\#} = 0$, then E is not a eld. We consider the two cases:

{ If $E = F + K$ with K a eld, then $f = (a; 0)$ and belongs to a G_E-orbit denoted by $2A_1$.

{ If $E = F^3$ then $f = (a; 0; 0)$, $(0; a; 0)$ or $(0; 0; a)$, reecting the fact that G_E has three orbits of type $2A_1$ over the algebraic closure, permuted by the outer automorphism group S_3.

The rational orbits of these types are parameterized by classes of squares, and belongs to the class of a.

Proposition 12.4. Let $J = E \; C$ be a Freudenthal Jordan algebra of dimension 9. Assume that $Aut_E(C)$ is anisotropic. Let $= (1; 0; f; 0)$ be an E-twisted cube such that $f^{\#} = 0$. Then

(i) $_{N_E;} = C_c\,(^1$), with

$$= fv \; 2 \; C \;\; j \; Q(v) = \quad f \text{ and } b_Q(v; (v)) = 0g:$$

(ii) If $f = 0$, then $= f0g$.

(iii) If $f = 0$, then is compact (possibly empty) and $Aut_E(C)^0$ acts transitively on it.

Proof. The assertion (i) is clear. For (ii), since $Aut_E(C)$ is anisotropic, Proposition 9.1 implies that $= 0$ if $f = 0$.

The assertion (iii) can be checked by an explicit computation. There are two cases to consider, depending on whether $E = F^3$ or $E = F \; K$ with K a eld. We examine the case $E = F^3$ as an illustration.

When $E = F^3$, we have $C = K^3$ for a quadratic eld extension K of F. Moreover, Q and are of the form

$$Q(x; y; z) = (N_{K=F}(x); N_{K=F}(y); N_{K=F}(z)) \quad (\text{up to an element in } (F^3))$$

and $(x; y; z) = (yz; zx; xy)$. Then

$$Aut_E(C)^0 = f(x; y; z) \; 2 \; (K)^3 \; j \; N_{K=F}(x) = N_{K=F}(y) = N_{K=F}(z) = xyz = 1g:$$

If $f = (a; 0; 0) \; 2 \; F^3$, then $= f(x; 0; 0) \; 2 \; C \; j \; N_{K=F}(x) = ag$, which is a principal homogeneous variety for the group of norm one elements in K (possibly with no F-rational points).

12.3. Nonvanishing and injectivity of theta lifts. Using the above results, we can now begin our determination of the theta liftings from $Aut_E(C)$ to G_E.

Proposition 12.5. Fix an embedding E ! J, so J = E C. Let be an irreducible representation of $Aut_E(C)$. Then

(i) () = 0.

(ii) If ^{0} is another irreducible representation of $Aut_E(C)$, then

$$() = (^0) ( ) \quad ^0 = :$$

Proof. Proposition 12.3 shows that as a module for $Stab_{M_E}$ (),

(12.6) $$()_{N_E;} = \begin{cases} 0; \text{ f } C \quad C; \\ - {}_K; \text{ if } C = C, \end{cases}$$

Thus () = 0 and the second statement also follows.

12.4. Langlands parameters of theta lifts. We shall construct an explicit subquotient of (), for = 1 if J = D and all unitary = if J = D, using the mini theta correspondence. Recall that we have an exact sequence

$$0 ! \quad C_c^1(^?) ! \quad {}_{N_E} ! \quad {}_{N_J} ! \quad 0:$$

Furthermore, ${}_{N_J}$, as M_E $Aut_E(C)$-module decomposes as

(12.7) $${}_{N_J} = j\det j^{\,2} \quad {}^3 \quad !_{K=F} \; j\det j^{\,2} \; {}_{M_J}$$

where $!_{K=F}$ is the quadratic character corresponding to K = K_J, viewed as a character of M_E by precomposing det, and ${}_{M_J}$ is the minimal representation of M_J that has been described in x8.4. The summand ${}_{M_J}$ appears if and only if J = D . The action of $Aut_C(E)$ on the one-dimensional summand is trivial.

Assume rst that E is a eld and J = D, which is the easiest case. Then ${}_{N_E}$

$$= {}_{N_J} = j\det j^{\,2}$$

so $()_{N_E}$ = 0 for all = 1. We shall see later in x12.8 that this vanishing implies the cuspidality of (); for now, we shall deal with (1). By Frobenius reciprocity, we have a map from (1) into the degenerate principal series representation I(1=2) (see x18.3.1) induced from the Heisenberg parabolic subgroup. The image of this map must be V^0= J (St $_1$;1=2) since $(V^0)_{N_E}$ = $j\det j_1^{\,2}$ (and the other irreducible constituents of I(1=2) have 2- or 3-dimensional space of N_E-coinvariants, by Theorem 18.1). Thus, (1) contains V_1^C as an irreducible quotient and we shall see later that it is in fact irreducible.

Now assume J = D. We have seen in (12.7) that there is an M_E $Aut_E(C)$-equivariant surjection

$${}_{N_E} \quad ! \quad j\det j^{\,3=2} \; {}_{M_J}$$

where ${}_{M_J}$ is the minimal representation of M_J. We have also described in Proposition 10.3 the theta correspondence for the pair M_E $Aut_E(C)$ acting on ${}_{M_J}$. For any 2

Irr(Aut_E(C)) with = , its theta lift $_M$() on M_E is nonzero irreducible. Hence by Frobenius reciprocity, we obtain a nonzero equivariant map

$$() \ ! \ Ind^{G_E}_{P_E} j\det j \quad () \quad {}_M \qquad \text{(normalized induction)},$$

with $_M$() as described in Proposition 10.3. Now the induced representation is essentially the dual of a standard module and hence contains a unique irreducible submodule , which is the Langlands quotient J_2($_M$(); 1). This Langlands quotient is thus an irreducible subquotient of () when = .

12.5. Irreducibility of () I. We shall now complete the correspondence in the case when Aut $_E$(C)^0 is isotropic. In this case, there exists a non-trivial co-character : F ! $Aut_E(C)^0$. The centralizer of in G_J is a Levi subgroup. The restriction of the minimal representation on any (maximal) Levi subgroup is fairly easy to compute. Indeed, this is a standard technique in the theory of exceptional theta correspondences. With that in hand, () is easy to compute for every unitary character of $Aut_E(C)^0$.

We shall execute this strategy in detail in the split case, where E = F^3 and J = M_3(F), so that G_J is a split group and G_E is the derived group of the D_4-parabolic in E_6. Then $Aut_E(C)^0$ = (F^3)=F and we can x this isomorphism as follows. By extending the E_6 diagram, we see that D_4 sits in three Levi subgroups $G_1; G_2$ and G_3 in E_6 of type D_5. Let $_i$: F ! G_i be the co-character generating the center of G_i. (These co-characters are miniscule co-weights.) They are each unique up to inverse, but we can pick them so that $_1(t)_2(t)_3(t)$ = 1 for every t 2 F. Now the map ($t_1; t_2; t_3$) ! $_1(t_1)_2(t_2)_3(t_3)$ gives the claimed isomorphism.

The restriction of the minimal representation to a D_5 maximal parabolic has been determined in [MS]. In particular, the restriction to G_1 is given by an exact sequence

$$0 \,! \ C_c^1(!) \ ! \quad ! \ {}_1 \, C \ ! \ 0$$

where ! is the highest weight orbit in a 16-dimensional Spin module for G_1, the action of G_1 is geometric, and $_1$ is the minimal representation of G_1, twisted by an unramied character. More precisely, the action of $_1$(t) on $_1$ and C is given by jtj^s and jtj^r for two non-zero real numbers. In particular, since these characters are not unitary, the two terms will not contribute to () for unitary. Thus we can concentrate on C_c^1 (!).

The group G_E has three irreducible 8-dimensional representations V_1, V_2 and V_3. We pick this numbering so that the restriction of the 16-dimensional Spin module for G_1 containing ! decomposes as $V_2 V_3$. Let $!_i$ V_i be the G_E-orbit of highest weight vectors. Then it is a simple exercise, using the Bruhat decomposition for G_1, to see that ! decomposes into three G_E-orbits:

- an open G_E-orbit $!_0$!, such that the stabilizer of a point in $!_0$ is the derived group of an A_2 parabolic subgroup,
- $!_2$ V_2 and
- $!_3$ V_3.

Thus we have an exact sequence of G_E-modules:

$$0 \,! \ C_c^1(!_0) \ ! \ C_c^1(!) \ ! \ C_c^1(!_2) \ C_c^1(!_3) \ ! \ 0:$$

Of course, by the S_3-symmetry of the situation, $C^1_c(!_1)$ must also contribute in the restriction of . Indeed, it is contained in $_1$, where $_1(t)$ acts by the non-unitary character jtj . Hence $s_i(t)$ acts on C $(!_i)$ by the same character, and these terms will not contribute to () if is unitary. In particular, we have shown that for unitary, () arises from C $(!_0)$, whence it is clear that () = D().

It is now easy to nish the argument. For example, for two characters 1 and of $Aut_E(C)$, we have just proved that

$$D(1) = (1) \ ():$$

On the other hand, recall from Proposition 11.1(3), that

$$D(1) = J_1(St_E; 1=2) \ J_2((1); 1):$$

Since (1) $J_2((1);1)$ and () = 0, it follows that (1) = $J_2((1);1)$ and () = $J_1(St_E;1=2)$).

12.6. Subregular nilpotent orbit. Assume now that $Aut_E(C)$ is anisotropic. We shall prove the irreducibility of the theta lift () by studying its restriction to N_E in detail. However, in order to make this strategy work, we need to eliminate subregular nilpotent orbits as leading terms of the wave-front set of ().

The subregular nilpotent orbit is the Richardson orbit for the 3-step parabolic subgroup $Q_E = L_E U_E$ corresponding to the the middle vertex of the Dynkin diagram for D_4, with $[L_E; L_E] = SL_2(F)$. Recall from (6.5) that there is a parabolic subgroup $Q_J = L_J U_J$ of G_J whose intersection with G_E is Q_E. The unipotent radical of its Lie algebra has a decomposition

$$u_J = g_J(1) \ g_J(2) \ g_J(3)$$

with

$g_J(1) = Fe_1$ 3

J Fe_2

$J = J^2$; $g_J(2) = Fe$

$J = J$ and $g_J(3) = Fe_{13} \ Fe_{23} = F^2$ in the notation of (6.5). The unipotent radical U_J of Q_J has a ltration

$$U_J = U_1 \ U_2 \ U_3 \qquad \text{such that } U_i = U_{i+1} = g_J(i) \text{ for all i.}$$

Hence, the minimal representation has a ltration

$$_1 \ _2 ::: \qquad \text{such that } =_i = {}_{U_i}.$$

In particular, each quotient $_i = _{i+1}$ is naturally a $U_i = U_{i+1}$-module. The group $U_i = U_{i+1}$ is abelian and its characters are parameterized by $g_J(i)$. The characters of $U_i = U_{i+1}$ that appear as quotients of $_i = _{i+1}$ are in $_{min}(F)$ \ $g_J(i)$ where $_{min}$ is the minimal orbit in g_J.

The embedding E J gives rise to G_E G_J such that $Q_J \setminus G_E = Q_E = L_E \ U_E$. In particular, we have an analogue of the above sequence of inclusions

$g_E(1) = Fe_1$ 3

E Fe_2

$E = E^2$; $g_E(2) = Fe$

$E = E$ and $g_E(3) = Fe_{13} \ Fe_{23} = F^2$:

Thus a character of U_E is specied by a pair $(a;b) \ 2 \ E^2 = g_E(1)$. We say that the character is non degenerate if a and b are linearly independent over F. We now have:

Lemma 12.8. Let J = E C be a 9-dimensional Freudenthal Jordan algebra such that $Aut_E(C)$ is anisotropic. Let be the minimal representation of G_J and a non-degenerate character of U_E. Then $_{U_E;}$ = 0.

Proof. The rst step is to show that $_{[U\ ;U\ ]} = {}_{E}[U\ ;U\ ]$. To that end, for i = 3; 2, we need to show that there are no elements in $_{min}(F) \setminus g_J(i)$ perpendicular to $g_E($ i). If i = 3 there is nothing to prove, since $g_E($ 3) = $g_J($ 3).

If i = 2, then $g_J(2)$ = F e J 3 J and elements in $_{min}(F) \setminus g_J(2)$ perpendicular to $g_E($ 2) are given by x 2 C, x = 0, such that $x^{\#}$ = 0. But there are no such elements, since $Aut_E(C)$ is anisotropic.

As the next step, we need to show that no character of U_J in the minimal orbit restricts to a non-degenerate character of U_E. A character of U_J is specied by (x; y) 2 J ^{2} $g_J(1)$, and the restriction to U_E is given by projecting x and y on the rst summand in the decomposition J = E C. If (x; y) is in $_{min}(F) \setminus g_J(1)$ then x and y are linearly dependent over F, and hence so are their E-components. This completes the proof of the lemma.

12.7. Irreduciblity of () II. We assume that $Aut_E(C)$ is anisotropic and note the following consequence of Proposition 12.4 :

Lemma 12.9. Let J = E C be a Freudenthal Jordan algebra of dimension 9. Assume that $Aut_E(C)$ is anisotropic. Let = (1; 0; f; 0) be an E-twisted cube such that $f^{\#}$ = 0. Then

(i) If f = 0, then

$$()_{N_E;} = \begin{cases} C; & \text{if } = 1; \\ 0; & \text{if } = 1. \end{cases}$$

(ii) If f = 0, then $()_{N_E;}$ is nite-dimensional for any . Moreover, $()_{N_E;}$ = 0.

We can now prove that () is irreducible. The rst step is to show that () has its wave-front set supported on the orbit A_2, that is, the Richardson orbit for the parabolic P_E. There are three larger families of orbits: the regular orbit, the subregular orbit and the Richardson orbits for parabolic subgroups of the type $2A_1$ and we deal with each in turn:

The subregular orbits are eliminated by Lemma 12.8.

We now deal with the regular orbit. Assume that () is Whittaker generic, where we are using Whittaker characters of a maximal unipotent subgroup containing N_E. Observe that there are innitely many Whittaker characters which restrict to the character $_0$ of N_E, where $_0$ = (1; 0; 0; 0). This contradicts Lemma 12.9(i) which shows that $()_{N_E;_0}$ is nite-dimensional.

The last case, which concerns the Richardson orbit for parabolic subgroups of type $2A_1$ and thus does not occur if E is a eld, is treated similarly. In this case, there are innitely many characters of the unipotent radical of the $2A_1$ parabolic which restrict to , where = (1; 0; f; 0) with f = 0 but $f^{\#}$ = 0. This again contradicts the nite-dimensionality in Lemma 12.9(ii).

This completes the rst step of the argument.

The second step is to show that there are no irreducible subquotients of () supported on smaller orbits: $3A_1$, $2A_1$, A_1 and the trivial orbit. The orbit $3A_1$ is not special, so we can disregard it. We now consider the other possibilities in turn:

Lemma 12.9 and the nite-dimensionality of $()_{N\ ;\ \ E}$ for nondegenerate imply that () has nite length. Together with the unitarity of (), this implies that any irreducible subquotient of () is a summand of the minimal representation . Hence, by the theorem of Howe and Moore, the trivial representation of G_E can not be a summand.

The remaining possible small summands are eliminated using the Fourier-Jacobi functor [We1] for the Heisenberg parabolic P_E. The output of this functor is a $[M_E; M_E]$ = $SL_2(E)$-module. It is easy to check that the Fourier-Jacobi functor applied to gives the Weil representation $C_c^1(C)$ of $SL_2(E)$ O(Q), where O(Q) is the orthogonal group for the quadratic form Q on C. On the other hand, the Fourier-Jacobi functor applied to an irreducible representation of G_E with the wave-front set supported in $2A_1$ or A_1 gives a representation of $SL_2(E)$ with the trivial action of $SL_2(K)$ or $SL_2(E)$ respectively. Since the matrix coecients of the Weil representation decay, $SL_2(E)$ or any of its factors, cannot x a vector in $C_c^1(C)$.

Now we can complete the proof of the irreducibility of () when $\mathrm{Aut}_E(C)$ is anisotropic. The wave-front set of every irreducible subquotient of () is supported on orbits of the type A_2. However, we know that $()_{N\ ;\ \ E}$ is non-zero only for in a single M_E-orbit of non-degenerate cubes, in which case this space is an irreducible Stab_{M_E} ()-module. Thus there is room for only one irreducible representation in (). This proves the desired irreducibility of () in all cases.

12.8. Cuspidality. It remains to prove that () is supercuspidal if () $_{N_E}$ = 0. This follows from Lemma 12.9 combined with the following proposition.

Proposition 12.10. Let be an irreducible representation of G_E such that $_{N_E}$ = 0 and $_{N_E;}$ = 0 for all = (1; 0; f; 0) such that $f^{\#}$ = 0. Then is supercuspidal.

Proof. Consider the case E = F K. Let Q = L U be a maximal parabolic subgroup of G_E such that $_U$ = 0. Because $_{N_E}$ = 0, there are two other maximal parabolic subgroups to consider.

If [L; L] = SL_3, then $_U$ = 0 will admit a non-trivial functional for a character of U_L, the unipotent radical of a Borel subgroup of L. This character can be inated to U U_L and then restricted to N_E. The restriction is where = (a; 0; 0; 0) for some a 2 F . This contradicts the hypotheses of the proposition.

If [L; L] = $SU_{2;2}$, then we take U_L to be the unipotent radical of the maximal parabolic subgroup whose (derived) Levi subgroup is $SL_2(K)$. This is an abelian subgroup (it is the space of 2 2 hermitian matrices) and $_U$ will admit a non-trivial functional for a character of U_L. The rest of the argument goes in the same way as above, leading to with = (1; 0; f; 0) for an f such that $f^{\#}$ = 0.

We have thus dealt with the case E = F K. The cases when E a eld or F^3 are similar and easier. Indeed, for these cases, it suces to assume that $_{N_E}$ = 0 and $_{N_E;}$ = 0 for = (1; 0; 0; 0) to conclude the desired cuspidality.

We have now completed the proof of Theorem 12.1. The following corollary gives an alternative description of (1) and will be used in [GS3].

Corollary 12.11. Let be a quadratic character of F. Let $I(; s)$ be the degenerate principal series representation for G_E associated to the Heisenberg parabolic subgroup $P_E = M_E N_E$. Then the co-socle of $I(; 1=2)$ is a direct sum of the theta lifts $_C(1)$ over all isomorphism classes of twisted composition algebras C of E-dimension 2 with associated embedding E ! J such that K_J corresponds to by local class eld theory.

Proof. Consider any embedding E ,! J such that corresponds to K_J by local class eld theory and write $J = E + C$. Then we have the dual pair G_E $\mathrm{Aut}_E(C)$! G_J, and we may consider the big theta lift $_C(1)$ of the trivial representation of $\mathrm{Aut}_E(C)$. By Theorem 12.1, we know that $_C(1)$ is irreducible. On the other hand, observe that $_C(1)$ maps nontrivially to $I(; 1=2)$ (by using the one dimensional summand of $_{N_J}$), and thus it is an irreducible submodule of $I(; 1=2)$. Since N_E spectra of $_C(1)$ for non-conjugate embeddings E ! J are dierent, we thus have a submodule

$$\bigoplus_C {}_C(1) ,\! \; I(; 1=2);$$

with the sum running over isomorphism classes of C's considered here.

Now the corollary follows by counting: the number of classes of embeddings with E and K_J xed, given by [GS2, Prop. 12.1], is equal to the number of representations in the socle of $I(; 1=2)$, which is given by [Se2, Thm 4.1]. For example, if = 1, and $E = F + K$, where K is a eld, then we have one class of embeddings if $K = K_J$ and two otherwise. These two cases can be characterized by $N_{K=F}$ = 1 and $N_{K=F}$ = 1 respectively, and correspond to the cases (6) and (7) in [Se2, Thm. 4.1]. However, the conditions were mistakenly stated there as $N_{E=F}$ = 1 and $N_{E=F}$ = 1, when in fact it was what we wrote here.

13. Archimedean Theta Correspondence

In this section, we consider the theta correspondence for H_C G_E over archimedean local elds and formulate the analog of Theorem 12.1. The main theorems here are Theorems 13.1 and 13.3. The proofs of these theorems will appear in a separate paper, joint with Je Adams and Annegret Paul.

13.1. Real Freudenthal-Jordan algebras. Assume rst that $F = \mathbb{R}$; the case $F = \mathbb{C}$ will be dealt with at the end of this section. Firstly, we enumerate the real Freudenthal-Jordan algebra J of dimension 9:

- For $K_J = \mathbb{R}^2$, we have $J = M_3(\mathbb{R})$;
- For $K_J = \mathbb{C}$, J is given as the set of xed points of involutions of the second kind on $M_3(\mathbb{C})$. Involutions of the second kind on $M_3(\mathbb{C})$ arise from nondegenerate Hermitian forms h on $\mathbb{C}^3$, which we may assume to be given by:

$$h = {}_1 z_1 z_1 + {}_2 z_2 z_2 + {}_3 z_3 z_3; \quad \text{with } {}_i = 1.$$

There are 8 choices for signs, but we get only 4 dierent involutions, since h and h give the same involution. In this way, we get 4 Jordan algebras $J_{{}_1,{}_2,{}_3}$, but the 3 of them corresponding to $f_1; _2; _3 g = f+; ; g$ are isomorphic. Hence, up to isomorphism, there are two such J's:

{ $J = J_{3;0}(C) = J_{+++}$;
{ $J = J_{1;2}(C) = J_{+}$

We shall sometimes denote the last two cases of J collectively as $J_3(C)$. The group G_J depends only on K_J. It is the split group if $K_J = R^2$, and quasi-split if $K_J = C$ [LS15] .

13.2. Embeddings of cubic algebras. We shall next enumerate the E-twisted composition algebra of rank 2 over R by describing embeddings of cubic etale algebras E into J. Note that there are 2 cubic etale R-algebras:

$$E = R^3 \quad \text{or} \quad E = R \; C:$$

We consider the various cases in turn:

(a) $J = M_3(R)$: in this case, both R^3 and R C embeds into $M_3(R)$ and these embeddings are unique up to conjugation.

(b) $J = J_3(C)$ and $E = R^3$: in this case, we may work with the 4 Jordan algebras $J = J_{1;2;3}$ as described above. For each of these J's, there is an embedding of R^3 into J as diagonal matrices. Though 3 of these Jordan algebras are isomorphic (to $J_{1;2}(C)$), the three embeddings are not isomorphic. To conclude, we get 4 classes of embeddings in all.

(c) $J_3(C)$ and $E = R \; C$: in this case, E does not embed into $J_{3;0}(C)$ and there is a unique embedding of E into $J_{1;2}(C)$.

We take this opportunity to correct a typo at the very end of [GS2], where it was incorrectly asserted in [GS2, Pg. 1956] that in the context (b), there are only 2 embeddings of R^3 into $J_3(C)$, even though the table on [GS2, Pg 1954] clearly shows that this set of embeddings have 4 elements.

13.3. The torus $Aut_E(C)^0$. For each embedding E ,! J, we have a decomposition J = E C. The corresponding $H_C = Aut_E(C)$ is always a semi-direct product $Aut_E(C)^0$ o Z=2Z such that the conjugation action of the non-trivial element in Z=2Z on $Aut_E(C)^0$ is the inverse involution. The possible cases of the two-dimensional torus $Aut_E(C)^0$ are tabulated in the following table, where T is the group of complex numbers of norm one.

	$E = R^3$	E = R C (R
$K = R^2$	$(R)^3=(R)$ K =	C)=(R)
C	$(T)^3=(T)$	(T C)=(T)

13.4. Characters of $Aut_E(C)^0$. We introduce a rened notation for characters of these tori.

A character of $(R)^3=R$ is a triple of characters $(_1;_2;_3)$ of R such that $_1 \, _2 \, _3 = 1$.

A character of T is represented by an integer. Thus a character of $(T)^3=T$ is represented by a triple of integers $(n_1;n_2;n_3)$ such that $n_1 + n_2 + n_3 = 0$.

In the remaining two cases a character of the torus is identied with a pair of charac-ters $(_R;_C)$, such that $_R \, _C = 1$ on R, and with a pair $(m;_C)$, where m 2 Z, such that the restriction of $_C$ to T is given by z ! z^m.

13.5. Representations of $Aut_E(C)$. Let be a character of $Aut_E(C)$. If = ^{1}, let () = (^{1}) be the unique irreducible representation of $Aut_E(C)$ such that the restriction to $Aut_E(C)^0$ is ^{1}. If = ^{1}, then extends to a character of $Aut_E(C)$ in two ways, denoted by () . These two representations are indistinguishable unless = 1, in which case one extension is the trivial representation, denoted by (1), and the other the sign representation . Note that non-trivial quadratic characters appear only in the split case (where $E = R^3$ and $K_J = R^2$), since $Aut_E(C)^0(R)$ is connected as a real Lie group otherwise.

13.6. Some tempered representations of M_E. To every unitary character of $Aut_E(C)^0$, we shall attach a packet $P(E; K_J;) = P(E; K_J; \ ^1)$ of tempered representations of $M_E = GL_2(E)^{det}$, obtained by restricting an irreducible representation of $GL_2(E)$. We need additional notation.

- For a local eld F and a pair of characters ($_1$; $_2$) of F, let $_1$ $_2$ be the unique innite-dimensional subquotient of the principal series representation of $GL_2(F)$ ob-tained by normalized parabolic induction from the pair of characters.
- Let ! : R ! f1g be the sign character. It is the unique non-trivial quadratic character of R.
- Let : R ! R be the identity character (x) = x, for all x 2 R.
- For n 2 Z, the principal series representation ^{n} !, when restricted to $SL_2(R)$, contains a sum of two (limits of) discrete series representations with the lowest SO_2-types (jnj + 1).

We can now describe the packet $P(E; K_J;) = P(E; K_J; \ ^1)$ of tempered representations of $M_E = GL_2(E)^{det}$.

Case $E = R^3$ and $K_J = R^2$. Let = ($_1$; $_2$; $_3$) be a unitary character of $(R)^3$=R. The packet $P(E; K_J;)$ consists of representations appearing in the restriction to $GL_2(R^3)^{det}$ of

$$(_1 \ 1) \\ (_2 \ 1) \\ (_3 \ 1):$$

This representation is irreducible when restricted to $SL_2(R^3)$ unless $_i$ = ! for at least one i. The group $GL_2(R^3)^{det}$ is large enough so that the restriction is still irreducible if precisely one $_i$ is !. In view of the relation $_1$ $_2$ $_3$ = 1, at most two $_i$ can be !, and this is precisely when is a non-trivial quadratic character. Then and only then the packet consists of two elements. The standard intertwining operator provides an identication of $P(E; K_J;)$ and $P(E; K_J; \ ^1)$.

Case $E = R^3$ and $K_J = C$. Let = (n_1; n_2; n_3) be a character of T^3=T. The packet $P(E; K_J;)$ consists of representations appearing in the restriction to $GL_2(R^3)^{det}$ of

$$(^{n_1} \ !) \\ (^{n_2} \ !) \\ (^{n_3} \ !):$$

The restriction to $SL_2(R^3)$ consists of 8 summands, hence the packet $P(E; K_J;)$ consists of 4 representations.

Case $E = R \ C$ and $K_J = R^2$. The restriction from $GL_2(R C)$ to $GL_2(R C)^{det}$ is always irreducible, hence the packets are singletons. Let = ($_R$; $_C$) be a unitary character of

$(R\ C)=(R)$. The packet $P(E;K_J;)$ consists of the restriction to $GL_2(R\ C)^{det}$ of $(_R\ 1)$

$$(_C\ 1):$$

Case $E = R\ C$ and $K_J = C$. We are again restricting from $GL_2(R\,C)$ to $GL_2(R\,C)^{det}$ hence the packets are singletons. Let $= (m;\ )$ be a unitary character of $(T\ C)=T$. The packet $P(E;K_J;)$ consists of the restriction to $GL_2(R\ C)^{det}$ of

$$(^m\ !)$$
$$(_C\ 1):$$

Summarizing, we have 4 families of tempered packets $P(E;K_J;) = P(E;K_J;\ ^1)$ of $GL_2(E)^{det}$, parameterized by unitary characters of $Aut_E(C)^0$. If $E = R^3$ and $K_J = C$, then $jP(E;K_J;)j = 4$. As a part of our correspondence result, we will see that the 4 members of this packet are naturally parameterized by the 4 embeddings $R^3\ !\ J_3(C)$. If is a non-trivial quadratic character (this happens only if $E = R^3$ and $K_J = R^2$) then $jP(E;K_J;)j = 2$. Let $()$; $()$ be its constituents. Otherwise $jP(E;K_J;)j = 1$ and its unique element will be denoted by $()$.

13.7. Main result. Let V be the Harish-Chandra module of the minimal representation of G_J. Consider the dual pair $G_E\ Aut_E(C)$ corresponding to an embedding $E\ !\ J$. For every irreducible representation of $Aut_E(C)$ let

$$() = V=\backslash_{'2Hom(V;)}Ker(')$$

where $'$ are homomorphisms in the sense of Harish-Chandra modules. We note that $()$ is naturally a $(g_E;K_E)$-module, where K_E is the maximal compact subgroup of G_E. The following will be proved in a joint paper with Je Adams and Annegret Paul, though we note that the second bullet, when $Aut_E(C)$ is compact, is contained in Loke's thesis [Lo].

Theorem 13.1. Let $G_E\ Aut_E(C)$ be the dual pair arising from an embedding $E\ !\ J$. Let be a unitary character of $Aut_E(C)\ .^0$

- If $E\ !\ J$ is not one of the 4 embeddings $R^3\ !\ J_3(C)$, then $(())\ J_2((); 1)$, unless is quadratic and non-trivial, in which case we have $(\ ()) = J_2(\ (); 1)$.
- If $E\ !\ J$ is one of the 4 embeddings $R^3\ !\ J_3(C)$, then $(())\ J_2(; 1)$, where $2\ P(E;K_J;)$. As we run through all 4 embeddings $R\ !\ J_3(C)$, runs through the 4 representations in $P(E;K_J;)$.

The representation $()$ is always irreducible, and can be described as it sits in a degener-ate principal series representations, along with $((1))$. Let $I_E(s)$ denote the (normalized) degenerate principal series for G_E where we induce $j\det j^s$ from P_E. Let $I_E(!;s)$ be the qua-dratic twist of this series, i.e. we induce $!(\det)\ j\det j\ .^s$ (Recall that $!$ is the sign character of R.) The following result is due to Avner Segal [Se2, Appendix A], but formulated with our interpretation in terms of theta lifts.

Theorem 13.2. Let $_{E!J}()$ denote the theta lift of in the correspondence arising from the embedding $E\ !\ J$.

For every E, we have an exact sequence

$$0\ !\ _{E!J_3(C)}()\ !\ I_E(1=2)\ !\ _{E!M_3(R)}((1))\ !\ 0:$$

For every E, we have an exact sequence

$$0 \,!\ \ {}_{E!M_3(R)}()\ !\ \ I_E(!;1{=}2)\ !\ \ {}_{E!J_3(C)}((1))\ !\ \ 0:$$

Here $J_3(C) = J_{3;0}(C)$ or $J_{1;2}(C)$ is any Jordan algebra with $K_J = C$, and the sum in both sequences is over the isomorphism classes of embeddings of E into $J_{3;0}(C)$ or $J_{1;2}(C)$ (recall that there is one class if $E = R\ C$, and four if $E = R^3$).

13.8. Complex case. Assume now that $F = C$. In this case $E = C^3$ is the only possible case. We have:

Theorem 13.3. Let $= ({}_1;{}_2;{}_3)$ be a unitary character of $(C)^3{=}C$. Let () be the tempered representation of $M_E = GL_2(C^3)^{det}$ dened as in the real split case. Then (()) $= J_2(())$ if $= 1$ and (1) () = D(1) is the degenerate principal series for an A_2 parabolic subgroup,.

14. Global Theta Lifting

In this section, let $E{=}F$ be a cubic eld extension of number elds, so that G_E is a so-called triality $Spin_8$. We shall consider the global theta correspondence for the dual pair

$$H_C\ \ G_E = Aut_E(C)\ \ Spin_8\ \ ,!\ \ G_J$$

associated to a twisted composition algebra C over F with $\dim_E C = 2$, corresponding to an embedding of Jordan algebras $E\ ,!\ J$, for some Freudenthal-Jordan algebra J of dimension 9 over F.

14.1. Hecke characters of $\tilde{T}_{E;K}$. Recall from x4.6 that H^0_C is isomorphic to the 2-dimensional torus

$$\tilde{T}_{E;K} = Ker\ \ N_{K=F} : (Res_{E\ K=F}\, G_m)=(Res_{K=F}\, G_m)\ \ !\ \ (Res_{E=F}\, G_m)=G_m\ \ ;$$

so that

$$\tilde{T}_{E;K}(F) = Ker\ \ N_{K=F} : (E\ \ K)=K\ \ !\ \ E=F\ \ :$$

Before describing the automorphic representation theory of $H_C = Aut_E(C)$, let us record some relevant facts about automorphic characters of $\tilde{T}_{E;K}$.

Proposition 14.1. (i) The torus $\tilde{T}_{E;K}$ satisies the weak approximation property. As such, any two Hecke characters and ^{0} of $T_{E;\tilde{K}}$ such that ${}_v = {}^0_v$ for almost all v are equal.

(ii) Let and ^{0} be two unitary Hecke characters of $T_{E;\tilde{K}}$ such that for almost all v, either ${}^0_v = {}_v$ or ${}^0_v = {}_v^{1}$. Then ${}^0 =$ or ${}^0 = {}^{1}$.

Proof. (i) By a result of Voskresenskii [V2], any tori of dimension 2 over F satises the weak approximation property.

(ii) Assume rst that $K = F\ \ F$ is split. Then $T_{E;K} \tilde{=} (Res_{E=F}(G_m)=G_m$, so that $T(F) = \tilde{E}{=}F$. We may thus regard and ^{0} as Hecke characters of E. Consider now the principal series representations

$$:= (;\ \ {}^1) \text{ and }{}_0 := ({}^0;{}^0\ \ {}^1) \text{ of } PGL_2(A_E).$$

These are irreducible automorphic representations which are nearly equivalent to each other under our hypothesis. If these two principal series representations are locally equivalent for places of E outside a nite set S, then we have an equality of partial Rankin-Selberg L-functions:

$$L^S(s; \ ) = L^S(s; _0 \ );$$

which is more explicitly written as:

$$^S(s)^2 \ L^S(s; ^2) \ L^S(s; \ ^2) = L^S(s; ^0) \ L^S(^0 \ ^1) \ L^S(s; ^0 \ ^1) \ L^S(s; ^0 \ ^1 \ ^1):$$

Now the LHS has a pole at s = 1 and hence so must the RHS. This implies that $^0 =$ or ^{1}, as desired.

Assume now that K is a eld. We shall invoke the base change from F to K. We claim that the norm maps

$$\tilde{T}_{E;K}(K_v) \ ! \ \tilde{T}_{E;K}(F_v) \quad \text{and} \quad \tilde{T}_{E;K}(A_K) \ ! \ \tilde{T}_{E;K}(A_F)$$

are surjective. Since

$$\tilde{T}_{E;K \ F} \ K = (E \ K)=K;$$

this surjectivity claim allows one to reduce to the case of split K treated above, by composing and ^{0} with the norm map.

To show the surjectivity of the local norm map, we shall treat the most nondegenerate case where $L_v := E_v$ K_v is a eld; the other cases are easier. Then the norm map

$$\tilde{T}_{E;K}(K_v) = L_v = K_v \ ! \ \tilde{T}_{E;K}(F_v) = \text{Ker} \ N_{L_v=E_v} : L_v = K_v \ ! \ E_v = F_v$$

is given by

$$x \ ! \ x=(x) \qquad \text{where} \ 2 \ \text{Aut}(L_v=E_v) = \text{Aut}(K_v=F_v).$$

We thus need to show that

$$fy \ 2 \ L_v \ : N_{L_v=E_v}(y) \ 2 \ F_v \ g = K_v \ \ fz \ 2 \ L_v \ : N_{L_v=E_v}(z) = 1g:$$

For this, we need to observe that if y 2 L_v satises $N_{L_v=E_v}(y)$ 2 F_v , then in fact $N_{L_v=E_v}(y)$ 2 $N_{K_v=F_v}(K_v)$. This in turn follows from the fact that the natural map

$$F_v = N_{K_v=F_v}(K)_v \ ! \ E_v = N_{L_v=E_v}(L_v)$$

is an isomorphism (using the fact that E_v is an odd degree extension of F_v).

To deduce the surjectivity of the adelic norm map from the local ones, it suces to note that at places v of F unramied over L, the local norm map remains surjective when all the local elds are replaced by their ring of units.

14.2. Automorphic representations of $\text{Aut}_E(C)$. Recall that one has a short exact sequence of algebraic groups

$$1 \ ! \ H^0_C \ ! \ H_C \ ! \ _2 \ ! \ 1$$

From this, one obtains:

$$\begin{array}{ccccccc} 1 & \longrightarrow & H_C^0(F) & \longrightarrow & H_C(F) & \longrightarrow & \mu_2(F) \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ 1 & \longrightarrow & H_C^0(\mathbb{A}) & \longrightarrow & H_C(\mathbb{A}) & \longrightarrow & \mu_2(\mathbb{A}). \end{array}$$

Because E is a field, the torus H_C^0 is anisotropic so that

$$[H_C^0] := H_C^0(F)\backslash H_C^0(\mathbb{A}) \quad \text{and } [H_C] := H_C(F)\backslash H_C(\mathbb{A})$$

are compact. The automorphic representations of H_C^0 are unitary automorphic characters which are classified by global class field theory. We will need to discuss the automorphic representations of the disconnected algebraic group H_C.

Let $\mathcal{A}(H_C^0)$ denote the space of automorphic forms on H_C^0. Since $H_C(F)$ acts naturally on $H_C^0(\mathbb{A})$ by conjugation (preserving $H_C^0(F)$), we have a natural action of $H_C(F)$ on $\mathcal{A}(H_C^0)$ by

$$(\sigma \cdot f)(t) = f(\sigma^{-1} t \sigma) \qquad \text{for } \sigma \in H_C(F),\ t \in H_C^0(\mathbb{A}) \text{ and } f \in \mathcal{A}(H_C^0).$$

Since H_C^0 is abelian, this action factors through the quotient $H_C(F)/H_C^0(F) \hookrightarrow \mu_2(F)$. We now consider two cases, depending on whether this last injection is surjective or not.

(a) $H_C^0(F) = H_C(F)$. In this case, C corresponds to an embedding $E \hookrightarrow J$ with J a division algebra. At the nonempty finite set Σ_C of places v where $J \otimes_F F_v$ is division, we have $H_C^0(F_v) = H_C(F_v)$.

Let $\chi = \otimes_v \chi_v$ be a unitary automorphic character of the torus H_C^0, so that

$$\chi : [H_C^0] = H_C^0(F)\backslash H_C^0(\mathbb{A}) \longrightarrow S^1;$$

and hence $\mathbb{C} \cdot \chi \subset \mathcal{A}(H_C^0)$. Consider the induced representation

$$V_C(\chi) := \operatorname{ind}_{H_C(F) H_C^0(\mathbb{A})}^{H_C(\mathbb{A})} \chi = \operatorname{ind}_{H_C^0(\mathbb{A})}^{H_C(\mathbb{A})} \chi.$$

Then an element in $V(\chi)$ is a smooth function

$$f : H_C(F)\backslash H_C(\mathbb{A}) \longrightarrow \mathbb{C}$$

such that

$$f(tg) = \chi(t) \cdot f(g) \qquad \text{for any } t \in H_C^0(\mathbb{A}) \text{ and } g \in H_C(\mathbb{A}).$$

Hence we have:

$$V_C(\chi) \hookrightarrow \mathcal{A}(H_C).$$

As an abstract representation, $V_C(\chi)$ is the multiplicity-free direct sum of all irreducible representations of $H_C(\mathbb{A})$ whose abstract restriction to $H_C^0(\mathbb{A})$ contains χ. Indeed, if one considers the restrictions of functions from $H_C(\mathbb{A})$ to $H_C^0(\mathbb{A})$, the submodule $V_C(\chi)$ is characterizted as the subspace of functions whose restrictions are contained in $\mathbb{C} \cdot \chi \subset \mathcal{A}(H_C^0)$.

Thus one has the following description of $\mathcal{A}(H_C)$:

$$\mathcal{A}(H_C) = \bigoplus_{\chi} V_C(\chi);$$

which is an orthogonal direct sum with running over the automorphic characters of H_C. ^{0}

We note that $A(H_C)$ is not multiplicity-free as a representation of $H_C(A)$. Indeed, if and ^{0} are two distinct automorphic characters of H^0_C, then $V_C() = V_C(^0)$ as abstract representations if and only if the following two conditions hold:

{ for all $v \notin$ $_C$, $^0 = {}_v^{1}$, { for all $v \in$ $_C$, $_v =$ $_v$.

By Proposition 14.1(ii), the rst condition implies that $^0 =$ ^{1} and hence $^0 =$ ^{1} (since we are assuming that and ^{0} are distinct); this then implies by the second condition that $_v^2 = 1$ for all $v \in$ $_C$. Thus, if is an automorphic character of H^0_C = $T_{E;K}$, with the property that $_v^2 = 1$ for all $v \in$ $_C$, but $^2 = 1$, then $V_C() = V_C(\ ^1)$ as abstract representations, but $V_C()$ and $V_C(\ ^1)$ are orthogonal as subspaces of $A(H_C)$; alternatively, one distinguishes them by their restriction as functions to H^0_C. Thus, $A(H_C)$ has multiplicity-at-most 2, but fails to have multiplicity one. What is interesting, however, is that even if the multiplicity of an irreducible representation in $A(H_C)$ is 2, there is a canonical decomposition of the -isotypic submodule of $A(H_C)$ into two irreducible summands. These summands are characterized by their restriction (as functions) to H^0_C belonging to C or C ^{1} for a special as above.

(b) $H_C(F)=H^0_C(F) = {}_2(F)$. Then for every place v, $H_C(F_v)=H^0_C(F_v) = {}_2(F_v)$. In this case, the action of $H_C(F)=H^0_C(F) = {}_2(F)$ on $A(H^0_C)$ needs to be taken into account. As before, let $=$ $_v$ $_v$ be a unitary automorphic character of the torus H^0_C. The action of $H_C(F)=H^0_C(F)$ sends to its inverse ^{1}. Hence, we consider the equivalence relation on automorphic characters of H^0_C given by this action, i.e. modulo inversion. Denote the equivalence class of by [].

There are now two subcases to consider:

(i) $^2 = 1$, so that is xed by $H_C(F)$ as an abstract representation and the equivalence class [] is a singleton. In this case, is xed by $H_C(F)$ as a function on $H^0_C(A)$ and C $A(H^0_C)$ aords a representation of $H_C(F)\ H^0_C(A)$ extending , characterized by the requirement that is trivial on $H_C(F)$. Consider the induced representation

$$V_C[] := \mathrm{ind}_{H_C(\ )H_C(A)}^{H_C(A)} :$$

Then an element in $V_C[]$ is a smooth function

$$f : H_C(F) \backslash H_C(A) \ ! \ C$$

such that

$$f(tg) = (t)\ f(g) \qquad \text{for any } t \in H^0_C(A) \text{ and } g \in H_C(A).$$

Hence we have:

$$V_C[] \ ,\! \ A(H_C):$$

As an abstract representation, V [] is the multiplicity-free direct sum of all irreducible representations of $H_C(A)$ whose abstract restriction to $H_C(F)H^0_C(A)$ contains .

(ii) ^{2} = 1, so that is not xed by H $(F)_C$ as an abstract representation and [] = f; ^{1}g. In this case, the span of , for all 2 $H_C(F)$, is the 2-dimensional subspace

$$W_{[]} = C \quad C \quad ^1 \; A(H^0) \qquad C$$

such that

$$W_{[]} \cong \mathrm{ind}^{H_C(F)H_C(A)^0}_{H^0_C(A)}$$

as $H_C(F)$ $H^0_C(A)$-module. Consider the induced representation

$$V_C[] = \mathrm{ind}^{H_C(A)}_{H_C(F) \; H^0_C(A)} W_{[]} \cong \mathrm{ind}^{H_C(A)}_{H : H^0_C(A)}$$

An element of $V_C[]$ is thus a function

$$: H_C(A) \; ! \; W_{[]} = C \; + C \quad ^1 \; A(H_C):$$

Setting

$$f(h) = (h)(1);$$

so that f is the composition of with evaluation at 1 2 $H_C(A)$, we see that the map ! f denes an embedding

$$V_C[] \; ,! \; A(H_C):$$

In this way, we shall regard $V_C[]$ as a submodule of $A(H_C)$ henceforth. As an abstract representation, V_C [] is the multiplicity-free direct sum of all irreducible representations of $H_C(A)$ whose restriction to $H^0_C(A)$ contains and ^{1}.

Now we have:

$$A(H_C) = \bigoplus_{[]} V_C[]$$

as [] runs over equivalence classes of automorphic characters of H^0_C. The subspace $V_C[]$ is characterized as the subspace of functions whose restriction to H^0_C is contained in $W_{[]}$ = C + C ^{1}. We observe that in this case, the representation $A(H_C)$ is multiplicity-free.

14.3. Global minimal representation. To carry out the global theta correspondence, we need another ingredient: the global minimal representation of $G_J(A)$. For each place v of F, we have a local minimal representation $_v$ of $G_J(F_v)$ which is unramied for almost all v, so that we may set = $_v$$_v$. Using residues of Eisenstein series, it has been shown that there is an ($G_J(A)$-equivariant) automorphic realisation

$$: \; ,! \; A(G_J)^0:$$

As before, the group $G_J(F)$ acts on $A(G_J)^0$ via

$$(\;)(g) = (\; ^1 g) \qquad \text{for } 2 \; G_J(F) \text{ and } g \; 2 \; G_J(A)^0.$$

The embedding is easily checked to be $G_J(F)$ $G_J(A)^0$-equivariant.

We now recall the main properties of the global minimal representation we shall use. Recall the Heisenberg parabolic subgroup $P_J = M_J \; N_J$ of G_J with

$$V_J := N_J^{ab} = F + J + J + F:$$

Using a xed character of $F \backslash A$ and the natural pairing between N_J and its opposite $\overline{N}_J$, the elements of V_J parametrizes automorphic characters of $\overline{N}_J(A)$ (trivial on $\overline{N}_J(F)$). Let V_J be the minimal nonzero M_J-orbit in V_J. For 2 , one has the Fourier expansion

$$()_{Z_J}(g) = ()_{N_J}(g) + \sum ()_{N_J, x}(g); \; x \, 2$$

where Z_J is the 1-dimensional center of N_J. If $M_{J;x}$ denotes the stabilizer of x 2 in the Levi subgroup M_J, then the Fourier coecient $()_{N_J, x};$ is left-invariant under $M_{J;x}^{der}(A) := M_{J;x}(A) \backslash M_J^{der}(A)$. On the other hand, when restricted to $M_J(A)$, the constant term $()_{N_J}$ is an automorphic form on M_J. One has

$$()_{N_J} 2 \; !_{K_J = F} \; j \quad j^{\,2} \; j \quad j^{\,3=2} \; {}_{M_J};$$

where ${}_{M_J} = 0$ unless G_J (or equivalently M_J) is quasi split, in which case ${}_{M_J}$ is the global minimal representation of M_J.

14.4. Global theta lifts. For any automorphic form f on H_C, and 2 , we consider the associated global theta lift:

$$(; f)(g) = \int_{[H_C]} (h \;)(g) \; \overline{f(h)} \, dh; \qquad \text{with } g \; 2 \; G_E(A).$$

Note that we have written (h)(g) instead of ()(gh) in the integral because () is only dened as a function of $G_J(A)$. Observe however that for 2 $H_C(F)$,

$$(h \;)(g) = (h \;)(\;^{1}g) = (h \;)(g) \qquad \text{for } g \; 2 \; G_E(A).$$

In any case, (; f) 2 $A(G_E)$. For any irreducible summand V(), the global theta lift () of is dened as the span of all (; f) with 2 and f 2 , so that

$$() \; A(G_E):$$

14.5. Cuspidality. We rst show the following analog of the tower property in classical theta correspondence.

Proposition 14.2. *The global theta lift () is contained in the space $A_2(G_E)$ of square-integrable automorphic forms of G_E. Moreover, it is cuspidal if and only if the (mini-)theta lift (via ${}_{M_J}$) of to M_E is zero.*

Proof. To detect if () is square-integrable or cuspidal, we need to compute the constant terms of a global theta lift (; f) along the two maximal parabolic subgroups $P_E = M_E N_E$ and $Q_E = L_E \; U_E$ of G_E. Hence, we rst compute the constant term $(; f)_{N_E \backslash U_E}$ along the unipotent subgroup $N_E \backslash U_E$. We note that

$$N_E = Z_E = F \;\; E \;\; E \;\; F \qquad (N_E \backslash U_E) = Z_E = 0 \;\; E \;\; E \;\; F:$$

Recall that the Heisenberg parabolic subgroup $P_J = M_J \; N_J$ of G_J satises $P_J \setminus G_E = P_E$, with $N_E \; N_J$ such that

$$V_E := N_E/Z_E \quad V_J := N_J/Z_E = F \; J \; J \; F;$$

where the embedding $E \hookrightarrow J$ is such that $E^{?} = C$. There is a natural projection map

$$\mathrm{pr} : V_J \to V_E:$$

which corresponds to the restriction of (automorphic) characters from $\overline{N_J}(A)$ to $\overline{N_E}(A)$.

For

V_J the minimal M_J-orbit, let

$$_0 = \{x \in \; : \mathrm{pr}(x) = (; 0; 0; 0) \in V_E\}:$$

Then one has

$$(14.3) \qquad (; f)_{\overline{N_E} U_E}(g) = \int_{[H_C]} f(h) \left(()_{N_J}(hg) + \sum_{x \in _0} ()_{N_J ; x}(hg) \right) dh:$$

To proceed further, we need to understand the set

$_0$. Clearly, we have

$_0 =$

$_1 \cup$

$_2$ where

$_1 = \{x \in \; : \mathrm{pr}(x) = (0; 0; 0; 0) \in V_E\}$

and

$$_2 = \{x \in \; : \mathrm{pr}(x) = (t; 0; 0; 0); t = 0\}:$$

By Proposition 8.1, and using the fact that E is a eld, we see that $_1$ is empty whereas $_2 = \{(t; 0; 0; 0) : t \in F\}$.

Hence, we see that

$$(; f)_{\overline{E}}(g) = \int_{[H_C]} f(h) \; () \qquad (hg)\, dh:$$

Since

$$_{\overline{N_E}} = \; !_{K_J / F} \; j \quad j^{2} \; j \quad j^{3/2} \; _{M_J};$$

with $_{M_J}$ only present when J is not division, we deduce that the constant term of $()$ along $\overline{N_E}$ vanishes unless is the trivial representation or if the (mini-)theta lift of to M_E (via $_{M_J}$) is nonzero. One may check that if is trivial, then it does have nonzero (mini-)theta lift to M_E, so that we may subsume the condition that is trivial into the second condition.

On the other hand, if $_t$ is the automorphic character of $\overline{N_J}(A)$ corresponding to $(t; 0; 0; 0) \in \; _2(F)$ with $t = 0$, then $H_C(F)$ stabilizes $_t$. This implies that in (14.3),

$$(14.4) \qquad ()_{N_J ; t}(hg) = ()_{\overline{N_J} ; t}(g);$$

so that the contribution of

$_2$ to (14.3) vanishes if f is not a constant function. We have thus shown that if the mini-theta lift of to M_E vanishes (so that is nontrivial in particular), then the constant term of () along $N_E \setminus U_E$ given in (14.3) vanishes, so that () is cuspidal.

Conversely, it is clear from (14.3) and the above discussion that if the mini-theta lift of to M_E is nonzero, then the constant term of () along $\overline{N}_E$ is nonzero and hence () is noncuspidal. To summarise, we have shown that () is cuspidal if and only if the mini-theta lift of to M_E (via $_M$) vanishes. It remains to examine the case when () is noncuspidal and show that () is square-integrable nonetheless.

Suppose then that () is not cuspidal, so that has nonzero (mini-)theta lift to M_E. For each parabolic subgroup $R = \overline{P}_E$, $\overline{Q}_E$ or $\overline{B}_E = \overline{P}_E \setminus \overline{Q}_E$, we consider the normalized constant term of () along $\overline{R}$. Since the Levi subgroup of R is a product of groups of GL-type, the strong multiplicity one theorem for GL_n implies that each of these normalized constant terms is a direct sum of a cuspidal component and a noncuspidal component such that the two components are spectrally disjoint (i.e. the system of spherical Hecke eigenvalues supported by the two parts are dierent). By the standard square-integrability criterion, we need to show that the (real parts of the) central characters appearing in the cuspidal component lie in the interior of the cone spanned by the positive simple roots which occur in the unipotent radical of $\overline{R}$.

For the case $\overline{R} = \overline{P}_E$, the cuspidal component of the normalized constant term is contained in the mini-theta lift $_M$ () of to M_E. Since the center of M_E is equal to the center of M_J, and the central character of $_N$ $_J$ is of the form z ! jzj^2, this gives the desired positivity for the cuspidal component of $_M$ (). By the results of x9.3 and Proposition 9.2, $_M$ () $j\det j$ ^{1} is a summand of a tempered principal series representation of M_E. Thus, the noncuspidal component of $_M$ () $j\det j$ ^{1} has normalised constant term consisting of unitary characters. Since $j\det j$ corresponds to the highest root 3 + 2, we have the positivity of cuspidal exponents along the Borel subgroup $\overline{P}_E \setminus \overline{Q}_E$.

Finally, for the constant term along $\overline{Q}_E$ we claim that there are no cuspidal exponents. For if $(;f)_{\overline{U}_E}$ has nonzero projection to the space of cusp forms of L_E, then $(;f)_{\overline{U}_E}$ is in fact cuspidal and so has nonzero Whittaker-Fourier coecients. However, it follows from (14.4) that such Whittaker-Fourier coecients all vanish, unless f is a constant function. If f is constant, then (; f) has nonzero constant term along $\overline{B}_E$ (via our computation of the constant term along $\overline{P}_E$) and so $(;f)_{\overline{U}_E}$ cannot be nonzero cuspidal on L_E.

Hence, we have shown that (; f) is square-integrable. This completes the proof of Propo-sition 14.2.

14.6. **Nonvanishing and Disjointness.** We now consider the question of nonvanishing of the global theta lifting. We shall do this by computing the generic Fourier coecients of (; f) along the unipotent radical $\overline{N}_E$ of the Heisenberg parabolic subgroup $\overline{P}_E$. These Fourier coecients are parametrised by generic cubes in $V_E(F) = N_E(F)^{ab}$. Recall that the $M_E(F)$-orbits of generic elements in $V_E(F)$ are parametrised by E-isomorphism classes of E-twisted composition algebras A. For each such A, we let $_A$ denote a character of $\overline{N}_E(A)$ trivial on $\overline{N}_E(F)$ in the corresponding orbit; there is no loss of generality in assuming that $_A$ corresponds to a reduced cube in $V_E(F)$, and note that the stabilizer of $_A$ in M_E is isomorphic to $H_A = \mathrm{Aut}_E(A)$.

Recall that if N_J denotes the unipotent radical of the Heisenberg parabolic subgroup of G_J, then there is a natural projection map pr : $V_J = N_J^{ab}$! V_E. This projection map corresponds to the restriction of characters from $\overline{N}_J(A)$ to $N_E(A)$. Let V_J be the minimal nonzero M_J-orbit in V_J. Set

$$_A = pr^{\ 1}(\ _A) \setminus \ :$$

Then Corollary 5.3 says that
$_A(F)$ is empty unless A is E-isomorphic to C, in which case,

$^A(F)$ is a principal homogeneous space of $H^C(F)$. Thus, when $A = C$, we may x an element $\tilde{}_C$ 2

$_C(F)$, so that $\tilde{}_C$ restricts to $_N$ $_C$ on $N_E(A)$. $_{N_J;\ C}$

Now we have:

Proposition 14.5. For 2 and f 2 $A(H_C)$, $(;f)_{;\ N_E\ A}$ vanishes (as a function on $G_E(A)$) unless $A = C$, in which case

$$(;f)_{E;\ C}(g) = \int_{H_C(A)} (h)_{\sim}(g)\ f(h)\,dh:$$

Moreover, there exist and f such that $(;f)_{N_E;\ C}(1) = 0$.

Proof. We have:

$$(;f)_{EN\ A}(g) = \int_{[V_E]} \overline{_A(n)} \int_{[H_C]} (h)_{Z_J}(ng)\ \overline{f(h)\,dh} \ ! \ dn$$

$$= \int_{[H_C]} \left(\int_{[V_E]} \overline{_A(n)} \sum_{\tilde{}2} (h)_{N_J;\ \tilde{}}(ng)\,dn \right) \overline{f(h)\,dh}$$

$$= \int_{[H_C]} \left(\sum_{\tilde{}2\ _A(F)} (h)_{N_J;\ \tilde{}}(g) \right) \overline{f(h)\,dh}:$$

This gives the vanishing of $(;f)_{N_E;\ A}$ when $A\ \ C$ since $_A(F)$ is empty in that case. When $A \cong C$ and $_C$ 2 $_C(F)$, then we have an identication $H_C(F)\ _C = \ _C(F)$, in which case

$$(;f)_{N_E;\ C}(g) = \int_{[H_C]} \sum_{2H_C(F)} (h)\ _{N_J;\ \tilde{}_C}(g)\ \overline{f(h)}\,dh$$

$$= \int_{H_C(A)} (h)\ _{N_J;\ \tilde{}_C}(g)\ \overline{f(h)}\,dh;$$

as desired. This proves the rst statement.

To show the second statement, we need to understand the function $h\ !\ (h)_{N_E;\ \tilde{}_C}(1)$ as a function on $H_C(A)$. For a nonarchimedean place v of F, a property of the local minimal representation $_v$ is that

$$\dim \mathrm{Hom}_{N_J(F_v)}(_v;\ \tilde{}_{C;v}) = 1:$$

Moreover, a nonzero element of this 1-dimensional space can be constructed as follows. Recall that, at a nonarchimedean place v, one has [KP, Thm. 6.1.1]

$C_c^1($
$(F_v)) \hookrightarrow Z_E(F_v) \hookrightarrow \bar{C}^1($
$(F_v)):$

Thus elements of $_v$ gives rise to functions on the cone ($\tilde{F}_v$). Then the evaluation map at

$_C$ 2 $\overline{}_J$ $_v$ ~

(F_v) denes a nonzero element of $\mathrm{Hom}_{N\ (F\ )}(v;\ _{C;v})$. For v outside some suciently large set S of places of F, $_v$ is the unramied vector in $_v$, in which case the corresponding function $f_{0;v}$ on the cone (F_v) has the following properties. The function $f_{0;v}$ is supported on the subset

$$\bigcup_{n0} \$^n_v (O_v);$$

is constant on each annulus $\$^n$ (O_v), and takes value 1 on (O_v). Indeed, [KP] gives an explicit formula for the value taken by $f_{0;v}$ on each annulus, but we won't need this here.

We need to understand the restriction of $f_{0;v}$ to the subset $_C(F_v)$. Since $_C$ V_J is a Zariski closed subset of V_J, we see that for v 2 S (with S containing all archimedean places and enlarged if necessary),

$$@ \bigcup_{n0} \$^n_v (O_v) \overline{A} \setminus {}_C(F_v) = {}_C(O_v) \quad \overline{N}_{J;\ C} \quad (O_v):$$

Hence, for v 2 S, the restriction of $f_{0;v}$ to $_C(F_v) = H_C(F_v)\ \tilde{}_{C;v}$ is the characteristic function of $H_C(O_v)$.

By the above discussion, we deduce that for S suciently large and with $F_S := \prod_{v2S} F_v$,

$$(;f)_{N_E;\ C}(1) = \int_{H_C(F_S)} (h\ ) \quad \tilde{}\ (1)\ f(h)\, dh:$$

We need to show that we can nd some f and such that the above integral is nonzero.

To this end, we start with a xed pair of f and such that the integrand in the above integral is nonzero as a function of h. Now we consider an arbitrary Schwarz function ' on $\overline{N}_J(F_S)$ and replace by the convolution ' in the above formula. This gives:

$$(' \ ;f)_{N_E;\ C}(1) = \int_{H_C(F_S)} \widehat{'}_Z(h^{\ 1}\ \tilde{}_C)\ (h\ )_{\overline{N}_J;\ C}(1)\ f(h)\, dh;$$

where $'_Z$ is the constant term of ' along $\overline{Z}$ $\overline{N}_J$ (which is a Schwarz function on $V_J(F_S) = N_J(F_S)/Z(F_S)$) and $\widehat{'}_Z$ is its Fourier transform. Since $H_C(F_S)\ \tilde{}_C = {}_C(F_S)$ $V_J(F_S)$ is a Zariski-closed subset, and $\widehat{'}_Z$ can be an arbitrary Schwarz function (as ' varies), we see that the above integral is nonzero for some choice of '.

This completes the proof of the second statement.

Corollary 14.6. (i) If $A(H\ )_C$, then () $A_2(G_E\ )$ is a nonzero irreducible square-integrable automorphic representation of G_E. Moreover, () = ^{abs}() := $_v(_v)$, where $(_v)$ denotes the local theta lift of $_v$ to $G_E(F_v)$ (which is nonzero irreducible).

(ii) For an abstract irreducible representation of $H_C(A)$, we have

$$\dim \mathrm{Hom}_{H_C}(; A_2(H_C)) = \dim \mathrm{Hom}_{G_E}(^{abs}( ); (A(H_C)))$$

where

$$(A(H_C)) = h(; f) : \ 2 \ _J ; f \ 2 \ A(H_C)i \ \ A_2(G_E):$$

(iii) If $A(H_C)$ and ^{0} $A(H_{C^0})$ satisfy () = (^{0}) as submodules of $A_2(G_E)$, then C is E-isomorphic to C (so that $H_C = H_{C^0}$) and = as subspaces of $A(H_C)$.

Proof. (i) This follows from Proposition 14.2 and Proposition 14.5.

(ii) This statement is often called the multiplicity-preservation of theta correspondence and in fact follows from (i) and the local Howe duality theorem we established in our local study, which says that:

$$\dim \mathrm{Hom}_{H_C G_E}(_J ; {}^{abs}()) \; 1$$

and

$$\dim \mathrm{Hom}_{G_E}({}^{abs}(); {}^{abs}({}^0)) \; \dim \mathrm{Hom}_{H_C}(; {}^0) \; 1:$$

In view of (i) and the local Howe duality theorem, the statement here is only interesting when $A(H_C)$ is not multiplicity-free. To prove (ii), we dene a pairing of nite-dimensional vector spaces:

$$B : \mathrm{Hom}_{H_C}(; A_2(H_C)) \; \mathrm{Hom}_{G_E}({}^{abs}(); (A(H_C))) \; ! \; \mathrm{Hom}_{H_C G_E}(_J \; {}^{abs}(); C)$$

by

$$B(f;)(; v; w) = \int (; f(v))(g) \; (w)\overline{(g)}\, dg_{[G_E]}$$

for 2 $_J$, v 2 and w 2 $^{abs}()$. The local Howe duality theorem says that the target space is 1-dimensional (so we may identify it with C). Now (i) and the local Howe duality theorem imply that this C-valued pairing is nondegenerate, giving us the desired equality of dimensions of the two Hom spaces on the left.

(iii) It follows from Proposition 14.5 that for $A(H_C)$, () supports only one orbit of generic Fourier coecients along N_E, namely the orbit associated to C. Thus, if () = (^{0}), then we must have $C = C^0$. The equality of and ^{0} now follows by (ii).

14.7. Canonical decomposition. To nish this section, let us examine the case when $H^0_C(F) = H_C(F)$: this is the case when $A(H_C)$ has multiplicity 2. In this case, we have an orthogonal decomposition

$$A(H_C) = \bigoplus V_C()$$

as runs over automorphic characters of $H^0_C = T_{E;K}$ and $V_C()$ is characterised as the subspace of functions whose restriction to H^0_C is contained in C. Each $V_C()$ is multiplicity-free and the occurrence of multiplicity 2 is due to isomorphisms $V_C() = V_C(\;^{1})$ for those satisfying

- ^{2} = 1 but
- ^2_v = 1 for the nitely many places v where $H_C(F_v) = H^0_C(F_v)$.

For satisfying these two conditions, and an abstract irreducible representation of $H_C(A)$ which occurs in $V_C()$ and $V_C(\;^{1})$ and write for the corresponding submodule $V_C()$. Then the -isotypic summand of $A(H_C)$ has the canonical decomposition:

$$A(H_C)() = \quad _1:$$

On considering the global theta lifting, Corollary 14.6 gives a direct sum

$$() \ (\ _1) \ A_2(G_E)$$

of two irreducible summands. This gives a canonical decomposition of the $^{abs}()$-isotypic summand $(A(H_C))[^{abs}()]$. One may ask how decomposition can be characterized directly on the side of G_E, i.e. without reference to H_C. We shall address this question in the remainder of this section.

We have seen in Proposition 14.5 the Fourier coecient formula

$$(; f)_{E N C}(g) = \int_{H_C(A)} (h)_{N_J; \tilde{C}}(g) \ \overline{f(h) \, dh}:$$

for 2_J and f 2 , where we recall that $\tilde{}_C$ 2 $_C$. Let $S_C = \mathrm{Stab}_{M_E}(_C)$ be the stabilizer of $_C$ in M_E. Then we have an action of $S_C \ H_C$ on $_C$ for which $_C$ is a torsor for each of the two factors. This gives an isomorphism

$$: S_C = H_C;$$

characterized by

$$(t) \ \tilde{_C} = t^{1} \ \tilde{}_C:$$

Now we may regard $(; f)_{N_E; C}$ as a function on $S^0_C(F) \backslash S^0_C(A)$. The following proposition, which strengthens Proposition 14.5 and is the global analog of (12.6), describes this function explicitly.

Proposition 14.7. For t 2 $S^0_C(A) = H^0_C(A)$ and f 2 , we have

$$(; f)_{E N C}(t) = ((t))^{1} \ (; f)_{N_E; C}(1):$$

In other words,

$$(; f)_{E N C} \ ^1 2 \ C \ ^1 \ A(H_C):$$

Proof. Write $= _1 \ ^1_J \ 2_{J;1} \ ^1$. With $_1$ xed, we consider the Fourier coecient map

$$_J^1 \ ! \ C$$

given by

$$^1 \ ! \ (_1 \ ^1)_{N_J; \tilde{C}}(1):$$

As we have noted in the proof of Proposition 14.5, there is a $P_J(A^1)$-equivariant map

$$q: ^1_J \ ! \ C^1(_{A^1})$$

so that

$$()_{N_J; \tilde{C}}(1) = (_1) \ q(^1)(\tilde{}_C):$$

for some $(_1)$ 2 C. Then for t 2 $S^0_C(A^1)$, we have:

$$()_{N_J; \tilde{C}}(t) = (_1) \ q(^1)(t^{1} \ \tilde{}_C) = (_1) \ q(^1)((t) \ \tilde{}_C) = ((t)^{1})^{N}_{J; \tilde{C}}(1):$$

Hence,

$$\begin{aligned}\theta(\phi; f)_{N_E;\psi_C}(t) &= \int_{H_C(\mathbb{A})} \phi(h)\cdot\overline{\omega_{N_J;\tilde\psi_C}(t)\, f(h)}\,dh\\
&= \int_{H_C(\mathbb{A})} \phi(\nu(t)^{-1}h)\cdot\overline{\omega_{N_J;\tilde\psi_C}(1)\, f(h)}\,dh\\
&= \int_{H_C(\mathbb{A})} \phi(h)\cdot\overline{\omega_{N_J;\tilde\psi_C}(1)\, f(\nu(t)h)}\,dh\\
&= \chi(\nu(t))^{-1}\cdot\int_{H_C(\mathbb{A})} \phi(h)\cdot\overline{\omega_{N_J;\tilde\psi_C}(1)\, f(h)}\,dh.\end{aligned}$$

This proves the desired identity for $t \in S^0_C(\mathbb{A}^1)$. However, both sides of the desired identity are automorphic functions of $S^0_C = H^0_C = \tilde{T}_{E;K}$. The desired identity then follows by the weak approximation theorem (Proposition 14.1(i)) for $T_{E;K}$.

What the lemma says is that the consideration of the ψ_C-Fourier coefficient gives an $(N_E;\psi_C)\times S^0_C$-equivariant map

$$\mathcal{A}(H_C)[\pi^{abs}(\chi)] \longrightarrow \mathbb{C}\chi\oplus\mathbb{C}\chi^{-1}\subset\mathcal{A}(S^0_C)$$

The canonical decomposition of the codomain is given by the irreducible summands whose image is contained in $\mathbb{C}\chi$ or $\mathbb{C}\chi^{-1}$.

15. A-parameters and Twisted Composition Algebras

In the next two sections, we relate the square-integrable automorphic representations constructed in the previous section to Arthur's conjecture for G_E. We begin by explicating the connections between twisted composition algebras and the relevant class of A-parameters in this section.

15.1. A-parameters. We shall consider A-parameters

$$\psi : W_F\times SL_2(\mathbb{C}) \longrightarrow PGSO_8(\mathbb{C})\rtimes S_3:$$

such that the centralizer of $\psi(SL_2(\mathbb{C}))$ is isomorphic to the group

$$S\rtimes(S_2\times S_3) = (\mathbb{C}^\times\times\mathbb{C}^\times\times\mathbb{C}^\times)^1\rtimes(S_2\times S_3).$$

We fix the isomorphism

$$Z_{PGSO_8\rtimes S_3}(\psi(SL_2(\mathbb{C}))) = S\rtimes(S_2\times S_3)$$

throughout. Associated to such a ψ is thus a map

$$\rho = \rho_E\times\rho_K : W_F \longrightarrow S_2\times S_3;$$

i.e. a pair $(E;K)$ consisting of an etale cubic F-algebra E and an etale quadratic algebra K; we shall say that ψ is of type $(E;K)$. With the etale cubic algebra E fixed, ψ is an A-parameter for the group G_E.

If we let W_F act on S through the map ρ, then $S\rtimes W_F$ is the L-group of the torus

$$\tilde{T}_{E;K} = \{x\in(E\otimes_F K)^\times : N_{E\otimes K/E}(x)\in F^\times\}/K^\times.$$

Hence, to give an A-parameter of type (E, K) is equivalent to giving an L-parameter

$$\phi : W_F \to {}^L\tilde{T}_{E,K} \subset S \rtimes (S_2 \times S_3)$$

modulo conjugacy by $S \rtimes S_2$, or equivalently an automorphic character of the torus $\tilde{T}_{E,K}$ up to inverse, i.e. a pair of automorphic characters $[\chi] = \{\chi, \chi^{-1}\}$.

To summarize, the A-parameters we are considering are determined by the triple $(E, K, [\chi])$. We had already highlighted and discussed these A-parameters in §3.5.

15.2. Component groups. An important structure associated to an A-parameter $\psi = \psi_{E,K,[\chi]}$ as above is its global and local component groups. The global component group is

$$S_\psi = \pi_0(Z_{PGSO_8}(\psi)) = \pi_0(Z_{S\rtimes S_2}(\phi)).$$

On the other hand, for each place v of F, one has the restriction ψ_v of ψ to $W_{F_v} \times SL_2(\mathbb{C})$ (the associated local A-parameter), and one has likewise the local component group

$$S_{\psi_v} = \pi_0(Z_{PGSO_8}(\psi_v)) = \pi_0(Z_{S\rtimes S_2}(\phi_v)).$$

There is a natural diagonal map

$$\Delta : S_\psi \to \prod_v S_{\psi_v} =: S_{\psi,\mathbb{A}}.$$

The following lemma gives a description of these component groups.

Lemma 15.1. Fix an A-parameter $\psi = \psi_{E,K,[\chi]}$ as above, with associated ϕ. For each place v of F, one has an exact sequence

$$1 \to Z_S(\phi_v) \to Z_{S\rtimes S_2}(\phi_v) \to S_2$$

and this sequence is exact at the right if and only if the character χ_v associated to ϕ_v satisfies $\chi_v^2 = 1$. Moreover, the abelian group $Z_S(\phi_v)$ depends only on (E_v, K_v) (i.e. is independent of $[\chi_v]$) and is given by

$$Z_S(\phi_v) = S^{W_{F_v}} = (\tilde{T}^\vee_{E,K})^{W_{F_v}}:$$

where the action of W_{F_v} on $S = \tilde{T}^\vee_{E,K}$ is via the map $\rho : W_{F_v} \to S_2 \times S_3$. Hence, one has

$$1 \to \pi_0(S^{W_{F_v}}) \to S_{\psi_v} = \pi_0(Z_{S\rtimes S_2}(\phi_v)) \to S_2$$

with exactness on the right if and only if $\chi_v^2 = 1$, in which case

$$S_{\psi_v} = \pi_0(S^{W_{F_v}}) \rtimes S_2.$$

The analogous result holds for the global parameter ψ. In §3.6, we had considered an example of a family of such ψ's and tabulated the corresponding groups S_{ψ_v}. To simplify notations, we will henceforth set

$$S^0_\psi := \pi_0(S^{W_F}) \quad \text{and} \quad S^0_{\psi_v} := \pi_0(S^{W_{F_v}}).$$

15.3. From A-parameters to twisted composition algebras. As we observed in x4.6, the group $\tilde{T}_{E;K}$ is (canonically up to inverse) isomorphic to the identity component of the automorphism group of any E-twisted composition algebra C with $\dim_E(C) = 2$ and quadratic invariant K_C such that $[K_E]$ $[K_C]$ $[K] = 1$. This motivates the following denition:

Denition 15.2. (i) Let $_{E;K}$ denote the set of E-isomorphism classes of rank 2 E-twisted composition algebras with quadratic invariant $K_C = [K_E]$ $[K]$.

(ii) Let $\tilde{}_{E;K}$ denote the set of E $_F$ K_C-isomorphism classes of rank 2 E-twisted compo-sition algebras with quadratic invariant $[K_C] = [K_E]$ $[K]$.

Then any C 2 $_{E;K}$ corresponds under the Springer decomposition to an algebra embedding E ,! J for some 9-dimensional Freudenthal-Jordan algebra J with $K_J = K$.

The following long lemma summarizes the discussion in x4, especially x4.3, x4.5, x4.6 and x4.8 (see also [GS2, x11.5 and x11.6]).

Lemma 15.3. (i) There is a natural commutative diagram

$$\begin{array}{ccccc} H^1(F;\tilde{T}_{E;K}) & = & \tilde{}_{E;K} & = & \{\text{isomorphism classes of triples } (B;\,;\,)\} \\ \downarrow & & \downarrow & & \downarrow \\ H^1(F;\tilde{T}_{E;K})/S_2 & = & _{E;K} & = & \{\text{equivalence classes of } : E \hookrightarrow J\} \end{array}$$

where the horizontal arrows are natural bijections (and hence written as equal signs). Moreover,

- in the rst row, for the triple (B; ;),
  - { B is a central simple K-algebra of degree 3;
  - { is an involution of second kind on B (relative to K=F)
  - { : E ! B is a Jordan algebra embedding.

  Two such triples $(B_1; _1; _1)$ and $(B_2; _2; _2)$ are equivalent if there is a K-algebra isomorphism $f : B_1 = B_2$ such that $_2$ f = f $_1$ and f $_1$ = $_2$.
- the group S_2 acts on $H^1(F; T_{\tilde{E};K})$ by inverting; this action is described in terms of the other two sets in the row by

  $$C \to C \quad \text{on } \tilde{}_{E;K}$$
  K_C; K_C

  where is the nontrivial element in $\mathrm{Aut}(K_C/F)$, and

  $$(B;\,;\,) \to (B^{op};\,;\,) \quad \text{on the last set.}$$
- in the second row, the second bijection is via the Springer decomposition, so : E ,! J refers to an embedding of Jordan algebras;
- the rst two vertical arrows are the natural ones whereas the last vertical arrow is the forgetful map given by

  $$(B;\,;\,) \to \,:$$

(ii) For any C 2 $_{E;K}$, its preimage in $_{E;\tilde{K}}$ is an S_2-orbit and thus has 1 or 2 elements. Moreover, one has:

$$\text{Fiber over } C \text{ has 2 elements} \iff H_C(F) = H_C^0(F):$$

Thus, the restriction of the rst vertical arrow gives a bijection from $H^1(F; T_{E;K})[2]$ onto its image.

(iii) If we pick any triple $(B; ;)$ in the preimage of C, we obtain an isomorphism of algebraic tori over F:

$$_{B;} : H_C{}^0 \; ! \; \tilde{T}_{E;K}:$$

Hence, we have the following canonical bijection which gives another interpretation of $\tilde{}_{E;K}$:

$$\tilde{}_{E;K} \quad ! \quad f\text{equivalence classes of } (C; i)g$$

where

C is an E-twisted composition algebra with quadratic invariant $K_C = [K_E]\ [K]$ and automorphism group H_C;
$i : H^0_C \; ! \; T_{E;K}$ is an isomorphism of F-tori, arising in the manner above;
two pairs $(C; i)$ and $(C^0; i^0)$ are equivalent if and only if there is an isomorphism $j : C \; ! \; C^0$ of E-twisted composition algebras, inducing an isomorphism $Ad(j) : H_C \; ! \; {}^0H^0{}_{C^0}$, so that $i^0\ Ad(j) = i$.

15.4. Local elds. In particular, the above results apply to the case where F is a number eld, as well as the local completions F_v. In [GS2, x12], we have examined the case of a local eld F_v as an explicit example. Summarizing the results there, we note:

Lemma 15.4. Assume that F_v is a local eld. We have two cases:

(i) If $(E_v; K_v)$ = (eld, split), then $H^1(F_v; T_{\tilde{E}_v;K_v})$ is an elementary abelian 2-group and the action of S_2 on $H^1(F_v; T_{E_v;K_v})$ is trivial, so that

$$_{E_v;K_v} \quad ! \quad \tilde{}_{E_v;K_v} \quad ! \quad H^1(F_v; \tilde{T}_{E_v;K_v}):$$

Hence, for any C 2 $_{E_v;K_v}$, its ber in $\tilde{}_{E_v;K_v}$ has 1 element and $H_C(F_v) = H^0_C(F_v)$ o $Z=2Z$.

(ii) If E_v is a eld and K_v is split (so that F_v is nonarchimedean), one has isomorphisms

$$\tilde{}_{E_v;K_v} = H^1(F_v; T_{\tilde{E}_v;K_v}) = Ker(H^2(F_v; G_m) \; ! \; H^2(E_v; G_m)) = Z=3Z$$

via

$$(B; ;) \; ! \; inv(B) \qquad (\text{the invariant of } B)$$

and the action of S_2 on $Z=3Z$ is by inverting. Hence $_{E_v;K_v}$ has 2 elements, corresponding to

$$C_v^+ = (E_v\ ,!\ M_3(F_v)) \quad \text{and} \quad C_v = (E_v\ ,!\ D_v^+)$$

where D_v^+ denotes the Jordan algebra attached to a cubic division algebra D_v over F_v. The preimage of C_v in $\tilde{}_{E_v;K_v}$ has two elements (associated to D_v and D_v^{op}) and in this case, $H_{C_v}(F_v) = H^0_{C_v}(F_v)$. However, the choice of D_v gives an isomorphism

$$_{D_v} : H_{C_v} \quad ! \quad \tilde{T}_{E_v;K_v};$$

with $_{D_v^{op}}(\ ) = {}_{D_v}(\ )^{1}$.

Hence, we have:

$$H^1(F_v; \tilde{T}_{E_v;K_v})[3] = 1 \text{ or } \mathbb{Z}/3\mathbb{Z}$$

and $H^1(F_v; \tilde{T}_{E_v;K_v})/H^1(F_v; \tilde{T}_{E_v;K_v})[3]$ is an elementary abelian 2-group.

15.5. Local-global principles. When F is a number field, there is a commutative diagram of localisation maps

$$\begin{array}{ccc} \tilde{\Phi}_{E;K} & \xrightarrow{\widetilde{loc}} & \prod_v \tilde{\Phi}_{E_v;K_v} \\ \big\downarrow & & \big\downarrow \\ \Phi_{E;K} & \xrightarrow{loc} & \prod_v \Phi_{E_v;K_v}. \end{array}$$

It will be necessary to explicate the image of loc and to determine the size of its fibers.

Lemma 15.5. (1) Assume that $K = F \times F$ is split.

(i) One has a short exact sequence of abelian groups

$$0 \to \tilde{\Phi}_{E;K} \xrightarrow{\widetilde{loc}} \bigoplus_v \tilde{\Phi}_{E;v;K_v} \xrightarrow{\sum inv_v} \mathbb{Z}/3\mathbb{Z} \to 0$$

(ii) Let $C = \{C_v\}$ be a collection of local twisted composition algebras, with $C_v = (E_v \to B_v)$, where B_v is a central simple algebra of degree 3 over F_v which is split for almost all v, and let S_C denote the set of places where B_v is a cubic division algebra. Then we have:

$$\#loc^{-1}(C) = \begin{cases} 1 & \text{if } S_C \text{ is empty;} \\ (2^{\#S_C} + 2\cdot(-1)^{\#S_C})/6, & \text{if } S_C \text{ is nonempty.} \end{cases}$$

In particular, C lies in the image of loc if and only if $\#S_C \neq 1$.

(2) Assume that K is a field.

(i) The map $\widetilde{loc}$ is bijective and the map loc is surjective.

(ii) Given a collection of local twisted composition algebras $C = \{C_v\}$, let S_C denote the finite set of places of F where E_v is a field, K_v is split and $C_v = (E_v \to D_v)$ with D_v a division algebra of degree 3 over F_v. Then we have:

$$\#loc^{-1}(C) = \begin{cases} 1; & \text{if } S_C \text{ is empty;} \\ 2^{\#S_C - 1}; & \text{if } S_C \text{ is nonempty.} \end{cases}$$

In both cases, the restriction of $\widetilde{loc}$ gives an isomorphism

$$H^1(F; \tilde{T}_{E;K})[2] = \prod_v H^1(F_v; \tilde{T}_{E;K})[2].$$

Proof. (1i) Recalling that

$$\tilde{\Phi}_{E;K} = H^1(F; \tilde{T}_{E;K}) = Ker(H^2(F; G_m) \to H^2(E; G_m)),$$

the short exact sequence in (1i) is a consequence of global class field theory.

(1ii) Given a set S of places of F, there are

$$\frac{2^{\#S} + 2\,(\;\;1)^{\#S}}{3}$$

central simple F-algebras of degree 3 which are ramied precisely at S; this is an interesting exercise which we leave to the reader. This number is thus the cardinality of the ber of loc over a collection C with $S_C = S$. The action of S_2 on $_{E;K}$ preserves this ber and its action there is free, unless S is empty (in which case the ber is a singleton set and S_2 acts trivially). This proves (1ii).

(2i) The map lõc is injective by the Hasse principle for 2-dimensional tori, proved by Voskresenskii [V1]. To show the surjectivity, we make use of the moduli interpretation of $_{E;K}$ as the set of tuples (B; ;) provided by Lemma 15.3. One has the local-global principle for odd degree division algebras equipped with involutions of second kind, which says that any collection f(B_v; $_v$)g of local pairs comes from a unique global pair (B;). Equivalently, the natural map

$$H^1(F; PU_3^K) \;\; ! \;\; \bigoplus_v H^1(F_v; PU_3^{K_v})$$

is an isomorphism. In addition, for a xed (B;) and a collection of local embeddings

$$_v : (E_v \;\; K_v; \;_v) \;\; ! \;\; (B_v; \;_v); \qquad\qquad \text{with } 1 = \;_v\; 2\; \mathrm{Aut}(K_v=F_v),$$

a local-global principle of Prasad-Rapinchuk [PR] shows that there exists

$$: (E \;\; K; \;) \;\; ! \;\; (B; \;);$$

which localizes to $_v$ for all v. This shows the surjectivity of lõc.

The surjectivity of loc follows by that of lõc and the surjectivity of the two vertical arrows.

(2ii) Given a nite set S of nite places of F which split over K, there are $2^{\#S}$ pairs (B;) of central simple K-algebras with an involution of the second kind, with B ramied precisely at places of K lying over S. The S_2 action on these is free unless S is empty (in which case the action is trivial). This proves (ii).

In particular, the map loc is not injective: this is the failure of the Hasse principle for twisted composition algebras which is ultimately responsible for the high multiplicities in the automorphic discrete spectrum of G_E.

15.6. Local Tate dualities. The connection between our A-parameters and twisted composition algebras is provided by the local and global Tate duality theorems. We rst note the local Tate-Nakayama duality theorem (see [K1, x2] and [Mi, Cor. 2.4]).

Lemma 15.6. Let T be a torus over a local eld F_v with character group $X(T) = \mathrm{Hom}(T; G_m)$. Then one has a commutative diagram:

$$\begin{array}{ccc}
H^1(F_v;T) & \longrightarrow & \mathrm{Irr}(H^1(F_v;X(T))) \\
\Big\uparrow {\scriptstyle inj.} & & \Big\uparrow {\scriptstyle inj.} \\
H^1(F_v;T)[2] & \longrightarrow & \mathrm{Irr}(H^1(F_v;X(T))/2H^1(F_v;X(T))) \\
\Big\uparrow {\scriptstyle surj.\, f} & & \Big\uparrow {\scriptstyle surj.} \\
H^1(F_v;T[2]) & \longrightarrow & \mathrm{Irr}(H^1(F_v;X(T)/2X(T))) \\
\Big\uparrow {\scriptstyle inj.\, \delta} & & \Big\uparrow {\scriptstyle inj.} \\
T(F_v)/T(F_v)^2 & \longrightarrow & \mathrm{Irr}(H^2(F_v;X(T))[2]);
\end{array}$$

whose horizontal arrows are isomorphisms. Here, in the left column, the maps f and δ form a short exact sequence

$$1 \longrightarrow T(F_v)/T(F_v)^2 \stackrel{\delta}{\longrightarrow} H^1(F_v;T[2]) \stackrel{f}{\longrightarrow} H^1(F_v;;T)[2] \longrightarrow 1$$

arising from the Kummer sequence

$$1 \longrightarrow T[2] \longrightarrow T \stackrel{2}{\longrightarrow} T \longrightarrow 1;$$

and the corresponding terms in the right column arises from the dual short exact sequence

$$1 \longrightarrow X(T) \stackrel{2}{\longrightarrow} X(T) \longrightarrow X(T)/2X(T) \longrightarrow 1.$$

We apply the above to our particular situation at hand. Fix an A-parameter $\psi = \psi_{E;K;[\,]}$ as above and let $T = \tilde{T}_{E;K}$ for ease of notation. Then for each place v, we have the following canonical isomorphism [K2, §1]:

$$H^1(F_v;X(T)) = \pi_0((T^\vee)^{W_{F_v}}) = S^0_{\psi_v};$$

where $T^\vee$ is the complex dual torus of T. Hence, by Lemma 15.4, $S^0_{\psi_v}[3] = 1$ or μ_3. Let us set

$$\bar{S}^0_{\psi_v} = S^0_{\psi_v}/S^0_{\psi_v}[3] \quad \text{and} \quad \bar{S}_{\psi_v} = S_{\psi_v}/S_{\psi_v}[3].$$

These are elementary abelian 2-groups, and we have

$$H^1(F_v;X(T))/2H^1(F_v;X(T)) = \bar{S}^0_{\psi_v}.$$

Further,

$$T[2] = Z_E; \quad \text{and} \quad H^1(F_v;X(T)/2X(T)) = H^1(F_v;Z(G_{\bar{E}}^{sc}));$$

where $Z(G_{\bar{E}}^{sc})$ is the center of $G_{\bar{E}}^{sc} = \mathrm{Spin}_8(\mathbb{C})$. Replacing these terms, the diagram in Lemma 15.6 now becomes:

(15.7)

$$\begin{array}{ccc}
H^1(F_v; T) & = & \mathrm{Irr}(S^0_{v}) \\
\text{inj:}\uparrow & & \uparrow\text{inj:} \\
H^1(F_v; T)[2] & = & \mathrm{Irr}(S^0_{v}) \\
\text{surj:}\uparrow f & & \uparrow\text{surj:} \\
H^1(F_v; Z_E) & = & \mathrm{Irr}(H^1(F_v; Z(G_{E}^{sc}))) \\
\text{inj:}\uparrow b & & \uparrow\text{inj:} \\
T(F_v)/T(F_v)^2 & = & \mathrm{Irr}(H^2(F_v; X(T))[2]);
\end{array}$$

Now, if $_v \neq 1$, then $S_{v} = S^0_{v}$ and the first row of (15.7) already gives a bijection

$$\mathrm{Irr}(S_{v}) \longrightarrow H^1(F_v; T_{E_v;K_v}).$$

Assume now that $_v \neq 1$. In this case, $S_{v} = S^0_{v} \rtimes S_2$ and we shall try to understand $\mathrm{Irr}(S_{v})$, or rather the subset $\mathrm{Irr}(S_{v})$, in terms of Lemma 15.6 and (15.7).

To bring the component group S_{v} into the picture, consider the projection

$$p : G_{E}^{sc} = \mathrm{Spin}_8(\mathbb{C}) \longrightarrow G_{E} = \mathrm{PGSO}_8(\mathbb{C})$$

Taking the preimage of $S \rtimes S_2 \subset \mathrm{PGSO}_8(\mathbb{C})$, we obtain the following commutative diagram of short exact sequences of W_{F_v}-modules:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Z(G_{E}^{sc}) & \longrightarrow & p^{-1}(S) & \longrightarrow & S & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z(G_{E}^{sc}) & \longrightarrow & p^{-1}(S \rtimes S_2) & \stackrel{p}{\longrightarrow} & S \rtimes S_2 & \longrightarrow & 1
\end{array}$$

where the action of W_{F_v} is by conjugation via the map $_v : W_{F_v} \longrightarrow S \rtimes (S_2 \times S_3)$ associated to $_v$. The coboundary map in the long exact sequence then gives:

$$\begin{array}{ccc}
S^0_{v} & \longrightarrow & H^1(F_v; Z(G_{E}^{sc})) \\
\downarrow & & \| \\
S_{v} & \stackrel{v}{\longrightarrow} & H^1(F_v; Z(G_{E}^{sc}))
\end{array}$$

Because the target of the map $_v$ is an elementary abelian 2-group (since $H^1(F_v; Z_E)$ is so), the map $_v$ factors through the quotient S_{v} of S_{v}. Moreover, $_v$ is injective on the index 2 subgroup S^0_{v}; indeed, the map $_v : S^0_{v} \longrightarrow H^1(F_v; Z(G_{E}^{sc}))$ is dual to the surjective map in the right column of (15.7). Hence $\mathrm{Ker}(_v) \subset S_{v}$ is either trivial or has order 2 and we would like to determine precisely what it is.

Together with (15.7), the above gives rise to a group homomorphism

(15.8) $$_v : H^1(F_v; Z_E) = \mathrm{Irr}(H^1(F_v; Z(G_{E}^{sc}))) \longrightarrow \mathrm{Irr}(S_{v}) \subset \mathrm{Irr}(S_{v}).$$

Thus, the diagram (15.7) can now be enhanced to:

$$(15.9)\qquad \begin{array}{ccccc}
H^1(F_v;T) & = & \mathrm{Irr}(S^0_v) & & \\
\text{inj:}\uparrow & & \uparrow\text{inj:} & & \\
H^1(F_v;T)[2] & = & \mathrm{Irr}(S^0_v) & = & \mathrm{Irr}(S^0_v) \\
\text{surj:}\uparrow f & & \uparrow\text{surj:} & & \uparrow\text{surj:} \\
H^1(F_v;Z_E) & = & \mathrm{Irr}(H^1(F_v;Z(G_{\overline{E}}^{sc}))) & \xrightarrow{v} & \mathrm{Irr}(S_v) \\
\text{inj:}\uparrow b & & \uparrow\text{inj:} & & \\
T(F_v)/T(F_v)^2 & = & \mathrm{Irr}(H^2(F_v;X(T))[2]) & & ;
\end{array}$$

What is the kernel of $?_v$ Consider the fundamental short exact sequence in the left column of (15.9):

$$(15.10)\qquad 1 \to T(F_v)/T(F_v)^2 \xrightarrow{b} H^1(F_v;Z_E) \xrightarrow{f} H^1(F_v;T)[2] \to 1:$$

We had rst encountered this sequence in (4.19). Now $_v$ is a character of the rst term in the short exact sequence. Pushing out this sequence by $_v$, one obtains:

$$(15.11)\qquad 1 \to {}_2 \to H^1(F_v;Z_E)/b(\mathrm{Ker}(_v)) \xrightarrow{f_v} H^1(F_v;T)[2] \to$$

1 when $_v = 1$. Now we have:

Proposition 15.12. Fix a local A-parameter $_v = {}_{E_v;K_v;[v]}$.

(i) There is a natural bijection

$$\mathrm{Irr}\ S^0_v \to H^1(F_v;\tilde{T}_{E;K}):$$

(ii) Assume that $_v^2 = 1$, but $_v = 1$. The natural map

$$_v : S_v \to H^1(F_v;Z(G_{\overline{E}}^{sc}))$$

is injective and the dual map $_v$ in (15.8) is surjective with kernel $b(\mathrm{Ker}(_v))$, so that it induces an isomorphism

$$H^1(F_v;Z_E)/b(\mathrm{Ker}(_v)) = \mathrm{Irr}(S_v):$$

Moreover, one has a commutative diagram of short exact sequence:

$$\begin{array}{ccccccccc}
1 & \to & {}_2 & \to & H^1(F_v;Z_E)/b(\mathrm{Ker}(_v)) & \xrightarrow{f_v} & H^1(F_v;\tilde{T}_{E;K})[2] & \to & 1 \\
 & & & & \downarrow{}_v & & & & \\
1 & \to & {}_2 & \to & \mathrm{Irr}(S_v) & \xrightarrow{\text{rest}} & \mathrm{Irr}(S^0_v) & \to &
\end{array}$$

1; where the third vertical arrow is that given by (i).

(iii) If $_v = 1$, then $\mathrm{Ker}(_v) = \langle s_0\rangle$ has order 2 and hence one has a canonical element $s_0 \in S_v \setminus S_v^0$. In this case, $_v$ induces an injection

$$_v : H^1(F_v;Z_E)/b(\tilde{T}_{E;K}(F_v)) = H^1(F_v;\tilde{T}_{E;K})[2] = \mathrm{Irr}(S^0_v) \to \mathrm{Irr}(S_v)$$

which is a section to the restriction map $\mathrm{Irr}(S_{v}) \to \mathrm{Irr}(S^0_{v})$ and whose image consists of those characters of S_{v} which are trivial on s_0.

15.7. Global Tate duality. We now consider the global analog of the above discussion. We shall fix a global A-parameter $\psi = \psi_{E;K;[]}$ with global component group S containing $S^0 = \pi_0(S^{W_F})$ of index 2. Because E is a field, we have

$$S^0 = \begin{cases} \mu_3; & \text{if } K = F \times F; 1; \\ & \text{if } K \text{ is a field.} \end{cases}$$

So $S[3] = S^0[3] = S^0 = 1$ or μ_3, and as in the local case, we set

$$\bar{S} = S/S[3]$$

which is an elementary abelian 2-group.

Our discussion of local Tate duality allows us to reformulate the results of Lemma 15.5 in terms of characters of S^0:

Lemma 15.13. Writing $T = T_{E;K}$ for ease of notation, we have the short exact sequence:

$$1 \to H^1(F;T) \to \prod_v H^1(F_v;T) \to \mathrm{Irr}(\pi_0(S^{W_F})) \to 1:$$

$$\mathrm{Irr}(S^0_{;A}) \qquad \mathrm{Irr}(S^0)$$

In particular,

$$H^1(F;T)[2] = \prod_v{}' H^1(F_v;T)[2] = \mathrm{Irr}(S^0_{;A}):$$

After this recollection, we consider the following commutative diagram of short exact sequences.

$$\begin{array}{ccccccccc} 1 & \to & T(A)/T(A)^2 & \xrightarrow{b} & \prod'_v H^1(F_v;Z_E) & \xrightarrow{f} & \prod'_v H^1(F_v;T)[2] & \to & \\ & & \uparrow & & \uparrow s & & & & \\ 1 & \to & T(F)/T(F)^2 & \to & H^1(F;Z_E) & \to & H^1(F;T)[2] & \to & 1 \end{array}$$

This diagram gives rise to the short exact sequence:

$$1 \to T(F)\backslash T(A)/T(A)^2 \xrightarrow{b} b(T(F))\backslash \prod'_v H^1(F_v;Z_E) \xrightarrow{f} H^1(F;T)[2] \to 1:$$

This is the global analog of the fundamental short exact sequence (15.10) in the local setting. Moreover, it is equipped with a canonical section: the map s descends to give a section to f

$$s: H^1(F;T)[2] \to b(T(F))\backslash \prod_v{}' H^1(F_v;Z_E):$$

Now suppose we have a global A-parameter = $_{E;K;[]}$ as above. We shall assume that ^{2} = 1 but = 1, so that is a quadratic character of $T(F)\backslash T(A)/T(A)^2$. Pushing out the last short exact sequence by , we get a short exact sequence

(15.14) $$1 \;!\; _2 \;!\; b(\mathrm{Ker}())\backslash \textstyle\prod_v H^1(F_v; Z_E) \;\overset{f}{!}\; H^1(F;T)[2] \;!$$

1: Moreover, the above short exact sequence is equipped with a section s of f.

We can also arrive at the above short exact sequence by using our local discussion in the previous subsection. We have:

$$1 \;!\; _{v2} \;!\; \textstyle\prod_v b_v(\mathrm{Ker}(_v))\backslash H^1(F_v; Z_E) \;!\; H^1(F;T)[2] \;!\; 1:$$

Pushing this out by the sum map $_{v2}$! $_2$ and denoting its kernel by $(_{v2})^1$, we obtain 1 ! $_2$

$$! \;\; \textstyle\prod_v b_v(\mathrm{Ker}(_v))\backslash H^1(F_v; Z_E) \;/(_{v2})^1 \;!\; H^1(F;T)[2] \;!\; 1;$$

which is the exact sequence in (15.14).

To reformulate the above discussion in the language of characters of component groups, let us introduce the following notions.

Denition 15.15. Fix a global A-parameter = $_{E;K;[]}$ with ^{2} = 1.

(i) For each place v, the sign character of S_v is the nontrivial character $_v$ of S_v/S^0_v.

(ii) For any nite subset of places of F, we set

$$= \prod_{v} \prod_{v2} 1_v$$

and call a global sign character of $S_{;A}$. We say that is automorphic if it is trivial on S . This holds if and only if jj is even. The set of automorphic sign characters is a subgroup of $\mathrm{Irr}(S_{;A})$.

(iii) Set

$$[\mathrm{Irr}(S_{;A})] = \mathrm{Irr}(S_{;A})/\text{fautomorphic sign charactersg}:$$

Summarizing the above discussion and applying global Poitou-Tate duality [Mi, Thm. 4.10], we obtain:

Proposition 15.16. Fix a global A-parameter = $_{E;K;[]}$ as above with ^{2} = 1. (i) If = 1, one has the following commutative diagram of short exact sequences:

$$1 \;!\; _2 \;!\; b(\mathrm{Ker}())\backslash \textstyle\prod_v H^1(F_v; Z_E) \;\overset{f}{!}\; H^1(F;T)[2] \;!$$

1 ? y = $\prod_v$ $_v$

$$1 \;!\; _2 \;!\; [\mathrm{Irr}(S_{;A})] \;!\; \mathrm{Irr}(S^0_{;A}) \;!$$

1 y rest

Irr(S)

which are equipped with a canonical section s for f given by the image of $H^1(F;Z_E)$. Finally,

$$\mathrm{Ker}(\mathrm{rest}\ ) = \mathrm{Im}(s) = \text{the image of } H^1(F;Z_E):$$

Equivalently,

$$\mathrm{Ker}(\mathrm{rest}) = \mathrm{Im}(\ s):$$

(ii) If = 1, the map = $\prod_v$ $_v$ descends to give a section

$$: H^1(F;T)[2] \ ! \ \mathrm{Irr}(S_{;A}) \ ! \ [\mathrm{Irr}(S_{;A})]$$

Then

$$\mathrm{Ker}(\mathrm{rest}) = \mathrm{Im}()$$

where rest : $[\mathrm{Irr}(S_{;A})] \ ! \ \mathrm{Irr}(S\ )$.

It is interesting to observe the following subtlety. When = 1 in the above lemma. it is of course possible that $_v$ = 1 for some places v. Let be the set of places where $_v$ = 1. Then for places v 2 , recall by Proposition 15.12(iii) that the map

$$_v : H^1(F_v;Z_E)=b(\mathrm{Ker}(_v)) = H^1(F_v;T)[2] \ ! \ \mathrm{Irr}(S_{\ v})$$

is only injective but not surjective: its image is a subgroup of index 2. Hence, we only have an injection

$$\prod_v \ _{;v} \prod_v{}^{C} H^1(F_v;Z_E)=b_v(\mathrm{Ker}(_v)) \ ,! \ \mathrm{Irr}(S_{;A}):$$

However, the composite of this injection with the projection to $[\mathrm{Irr}(S_{;A})]$ is surjective. This amounts to seeing that given any 2 $\mathrm{Irr}(S_{;A})$, one can twist by an automorphic sign character to ensure that at all places v 2 , $_v$ belongs to the image of $_v$.

16. A-packets and Multiplicity Formula

After this long preparation, we are nally ready to dene local and global Arthur packets and establish the Arthur multiplicity formula for the A-parameters = $_{E;K;[]}$ considered above.

16.1. Near equivalence classes and Arthur's conjectures. A global A-parameter = $_{E;K;[]}$ as above (with E xed) gives rise to a near equivalence class of representations of $G_E(A)$. Namely, for almost all places, $_v$ is unramied and

$$_v \quad \mathrm{Frob}_v; \quad \begin{pmatrix} q_v^{1=2} & \\ & q_v^{\ 1=2} \end{pmatrix} \ 2 \ PGSO_8(C) \rtimes_E W_F$$

gives a semisimple conjugacy class in $PGSO_8(C)\,\mathrm{Frob}_v$, which in turns determines an unram-ied representation of $G_E(F_v)$. We denote the associated near equivalence class in $A_2(G_E)$ by $A_{2;}(G_E)$.

To a rst approximation, Arthur's conjectures describe the structure of this submodule $A_{2;}(G_E)$. Though we have already discussed these conjectures in x3.3, we highlight the two key points here for the convenience of the reader:

(Local) One expects to have a local A-packet $_v$, which is a nite multi-set over $Irr(G_E(F_v))$ equipped with a map

$$_v \; ! \; Irr(S_{\;v}):$$

We may thus view $_v$ as a nite length representation of $S_{\;v} \; G_E(F_v)$:

$$_v = \bigoplus_{v 2 Irr(S_{\;v})} \; v \quad _v:$$

(Global) One has:

$$A_{2;}(G_E) = \bigoplus_{2 Irr S_{\;;A}} \dim Hom_S(\;;C)$$

where

$$= \quad _{vv} \; 2 \; Irr(S_{\;;A}) = {\prod_v}^0 Irr(S_{\;v}) \quad \text{and} \quad := \quad _{vv}:v$$

We shall see that the square-integrable automorphic representations we have constructed by theta lifting in x14 verify the above conjectures of Arthur.

16.2. Theta lifts and near equivalence class. Given a global A-parameter $= _{E;K;[]}$, we have the pair f; ^{1}g of automorphic characters of $T_{E;K}$.~For any C 2 $_{E;K}$, we have noted in x4.6 that there is a pair of isomorphisms

$$(16.1) \qquad _C; \; _C^{\;1} : H_C \overset{0}{=} T^{\sim}_{E;K}$$

of algebraic tori over F (associated to the two choices of (B; ;) with C corresponding to E ,! B). Pulling back and ^{1} via $_C$, we obtain a pair of automorphic characters ^{1} of $H^0 = _C$ $Aut_E(C)^0$. Set

$$V_C[] \;\; A(H_C)$$

to be the submodule spanned by all irreducible summands whose restriction to H^0_C contains $_C$ or ^{1} $_C$; this submodule is thus independent of the isomorphism $_C$. In earlier sections, we have studied the theta lifting from $A(H_C)$ to $A_2(G_E)$. From our local results, one sees that the theta lift of the submodule $V_C[]$ is contained in the near equivalence class $A_{2;}(G_E)$. More precisely, Corollary 14.6 gives

Proposition 16.2. Given $= _{E;K;[]}$,

$$V[\;] := \bigoplus_{C 2 _{E;K}} (V_C[]) \;\; A_{2;}(G_E):$$

Moreover, if $V_C([]) = m_C()$, then

$$(V_C[]) = \bigoplus m_C() \;\; ^{abs}():$$

16.3. Local A-packets. Our goal in the remainder of this section is to show that the submodule V [ ] in the above proposition can be described in the form dictated by Arthur's conjectures. Let us rst collect together all the local components of the constituents of V ($_{E;K;[]}$).

Denition 16.3. Given = $_{E;K;[]}$, set

$$_{E_v;K_v;[v]} = f(C_v; _v) \; 2 \; _{E_v;K_v} \; Irr(H_{C_v}(F_v)) : _v j_{H^0_v(F_v)} \; _v \; _C C_v \text{ or } _v{}^1 \; C_v \, g$$

and

$$_v = f_{C_v}(_v) : (C_v; _v) \; 2 \; _{E_v;K_v;[v]} g \; Irr(G_{E_v}(F_v)):$$

We have shown in Theorems 12.1, 13.1, 13.2 and 13.3 that for $(C_v; _v) \; 2 \; _{E_v;K_v;[v]}$, the theta lift $_C(_v)$ is nonzero irreducible. Moreover, $_v$ is a set (rather than a multiset). It is clear that the set $_v$ contains all possible local component at v of the constituents of V ($_{E;K;[]}$); this will be our denition of the local A-packet associated to $_v$. Observe that, by denition, there is a natural bijection

$$_v \; ! \; _{E_v;K_v;[v]}:$$

16.4. The bijection j_{v}. Our next task is to construct a natural bijection

$$_v \; ! \; Irr(S_{v})$$

or equivalently a bijection

$$j_{v} : Irr(S_{v}) \; ! \; _{E_v;K_v;[v]};$$

which then induces the deisred bijection with $_v$. To do this, we shall exploit Lemma 15.1, Lemma 15.4, Proposition 15.12 as well as Proposition 4.20.

Let us begin with some general observations:

(a) By restriction, one obtains (by Lemma 15.1 and Proposition 15.12(i)) a natural map

$$Irr(S_{v}) \; ! \; (Irr(S^0_{v}))=S_2 = H^1(F_v; T_{E;K})=S_2 = _{E_v;K_v}:$$

Hence, each $_v$ 2 Irr(S $_v$) gives rise to a C_v 2 $_{E_v;K_v}$.

(b) Suppose that $_v \neq 1$ but $_v = 1$. Then by Proposition 15.12(ii), we have:

$$\begin{array}{ccc} Irr(S_{v}) & = & H^1(F_v; Z_E)=b(Ker(_v)): \\ \downarrow & & \downarrow f_v \\ Irr(S^0_{v}) & = & H^1(F_v; T_{E_v;K_v})[2] \end{array}$$

For any given $[C_v]$ 2 $H^1(F_v; T_{E_v;K_v})[2]$, write

$$Irr_{C_v}(S_{v}) \; ! \; f_v^{-1}([C_v]):$$

These are sets of size 2.

Now Proposition 4.20 gives a natural isomorphism of $T_{E_v;K_v}(F_v)=T_{E_v;K_v}(F_v)^2$-torsors

$$g_{C_v} : f^{-1}([C_v]) \; ! \; (H_{C_v}(F_v) \; r \; H^0_{C_v}(F_v))=T_{E_v;K_v}(F_v)^2;$$

which induces a bijection

$$g_{C_v;v} : f^{\ 1}([C_v])=b(\mathrm{Ker}(_v)) \quad ! \quad (H_{C_v}(F_v) \; r \; H^0_{C_v}(F_v))=\mathrm{Ker}(_v):$$

Taken together, we thus have a canonical bijection

$$\mathrm{Irr}_{C_v}(S_{\ v}) \quad ! \quad (H_{C_v}(F_v) \; r \; H^0_{C_v}(F_v))=\mathrm{Ker}(_v):$$

Hence, given $_v$ 2 $\mathrm{Irr}_{C_v}(S_{\ v})$ (so that $C_v = C_v$), $_v$ corresponds to an element a_v

$$2 \; f^{\ 1}([C_v])=b(\mathrm{Ker}(_v))$$

and then an element

$$g_{C_v;v}(a_v) \; 2 \; (H_{C_v}(F_v) \; r \; H^0_{C_v}(F_v))=\mathrm{Ker}(_v):$$

On the other hand, the character $_{v\,C_v}$ of $H^0_{C_v}(F_v)$ has two extensions to $H_{C_v}(F_v)$, which are distinguished by the value 1 they take on $g_{C_v}(a_v)$. We dene

$$_v = \text{ the extension of } _v\ _{C_v} \text{ which takes value } +1 \text{ on } g_{C_v}(a_v)$$

and set

$$j_{\ v}(_v) = (C_v;_v) \; 2 \; _{E_v;K_v;[_v]}:$$

By Corollary 10.5, $_v$ is also characterized as the unique extension of $_{v\,C_v}$ whose mini-theta lift to $GL_2(E_v)^{det}$ is supported on the Whittaker data in $a_v \; b(\mathrm{Ker}(_v))$.

(c) If $_v = 1$, then by Proposition 15.12(iii), there is a canonical section :

$$H^1_v(F_v;T_{E_v;\tilde{K}_v})[2] = \mathrm{Irr}(S^0_{\ v}) \quad ! \quad \mathrm{Irr}(S_{\ v}):$$

So for the two extensions of a character ^0_v of $S^0_{\ v}$, there is a distinguished one contained in the image of $_v$. On the other hand, for any $[C_v]$ 2 $H^1(F_v;T_{E_v;\tilde{K}_v})[2]$, there is a distinguished extension of the trivial character $_{v\,C_v}$ from $H^0_{C_v}(F_v)$ to $H_{C_v}(F_v)$, namely the trivial character. Hence if $_v = {}_v(C_v)$, we set

$$_v = 1_{C_v} \qquad \text{and} \qquad _{v\,v} = {}_{C_v}$$

where $_v$ is the sign character of $S_{\ v}$ and $_{C_v}$ is the nontrivial (sign) character of $H_{C_v}(F_v)=H^0_{C_v}(F_v)$.

Hence, when $_v \overset{2}{=} 1$, we have dened in (b) and (c) above a canonical bijection

(16.4) $$\mathrm{Irr}(S_{\ v}) \quad ! \quad _{E_v;K_v;[_v]} = f(C_v;_v) \; 2 \; _{E_v;K_v;[_v]} : [C_v]^2 = 1g:$$

To complete the construction of $j_{\ v}$, it will now be convenient to consider dierent cases, depending on whether $(E_v;K_v) = $ (eld; split) or not, and whether $_v =^2 1$ or not.

(1) Suppose rst that $(E_v;K_v) = $ (eld; split). Then $S_{\ v} = S_{\ v}$ is an elementary abelian 2-group. If $_v \overset{2}{=} 1$, the (16.4) already gives the construction of $j_{\ v}$. On the other hand, when $^2_v = 1$, then $S_{\ v} = S^0_{\ v}$. For $_v$ 2 $\mathrm{Irr}(S_{\ v})$, we set

$$_v = \mathrm{Ind}^{H_{C_v}(F_v)}_{H^0\ \ )} 1_C \; ;_{C_v(F_v)}\ _v$$

recalling that $H_{C_v}(F_v) = H^0_{C_v}(F_v)$ for any $[C_v]$ 2 $_{E_v;K_v}$.

(2) Suppose now that $(E_v; K_v)$ = (eld; split), so that v is necessarily a non-archimedean place of F. We x the map $_v$ (as opposed to considering it as a conjugacy class of maps) and suppose that $_v j_{W_{F_v}}$ corresponds to the character $_v$ (as opposed to $_v{}^{1}$) of $T_{E_v;K_v}$. Then Proposition 15.12 and Lemma 15.4 give

$$Irr(S^0_{\ v}) = H^1(F_v; T_{E_v;K_v}) = {}_{E_v;K_v} = Br_3(F_v) = Z{=}3Z:$$

Thus, an element $_v$ 2 $Irr(S^0_{\ v})$ gives rise to an E_v-twisted composition algebra C_v and then a central simple algebra D_v 2 $Br_3(F_v)$ with an isomorphism

$$i_v = i_{D_v} : H_{C_v}(F_v) \quad ! \quad T_{E_v;K_v}:$$

Explicitly, we have two possible twisted composition algebras

$$C_v^+ = (E_v ,!\ M_3(F_v)) \quad \text{and} \quad C_v = (E_v ,!\ D_v^+);$$

where D_v is any of the two cubic division F-algebras. Moreover, the two isomorphisms i_{D_v} and $i_{D_v^{op}}$ dier from each other by composition with inversion. We recall also that

$$[H_{C_v^+}(F_v) : H^0_{C_v^+}(F_v)] = 2, \text{ but } H_{C_v}(F_v) = H^0_{C_v}(F_v).$$

We now consider two cases:

(a) ^2_v = 1. In this case, one has $S_{\ v} = S^0_{\ v} = {}_3$, so (16.4) tells us nothing in this case. To speciy the bijection

$$j_{\ v} : Irr(S_{\ v}) = Z{=}3Z \quad ! \quad {}_{E_v;K_v;[_v]};$$

the trivial character of $S_{\ v}$ is sent to the element $(C_v^+; {}_v[_v])$ 2 ${}_{E_v;K_v;[_v]}$, where $[_v]$ is dened as in case (1a) above. For a nontrivial character $_v$ of $S_{\ v}$, we set

$$j_{\ v}(_v) = (C_v ;_v\ i_v):$$

We note that the above recipe is independent of the choice of the representative $_v$ in its conjugacy class. Indeed, if we had used the map $_v{}^{1}$ (which corresponds to $_v{}^{1}$), then one has an equality of the component groups $S_{\ v} = S_{\ _v^{1}}$ as subsets of S o (S_2 S_3). However, an element of the latter which conjugates $_v$ to $_v{}^{1}$ induces not the identity automorphism of $S_{\ v}$ but the inverse automorphism. This implies that

$$j_{\ v}(_v) = j_{\ _v^{1}}(_v{}^{1});$$

so that the above recipe is independent of the choice of the representative map $_v$ in its conjugacy class. A better language to express this is to work with the projective systems of $[_v]$ and $[S_{\ v}]$, as we did in [GS4, Prop. 3.2], where a similar situation arises.

(b) $_v^2$ = 1. In this case, we have the short exact sequence

$$1 \quad ! \quad S^0_{\ v} = {}_3 \quad ! \quad S_{\ v} = S_3 \quad ! \quad S_2 \quad ! \quad 1;$$

so that $S_{\ v}$ is the nonabelian group S_3. Let us denote the irreducible representations of S_3 by 1, (the sign character) and r (the unique 2-dimensional irreducible representation). Because S $_v$ = S $_v$=S^0_v = S_2, (16.4) already determines for us j $_v$(1) and j $_v$(). Hence we have no choice for j $_v$(r):

$$j\ _v(r) = (C_v ;_v\ _{C_v}):$$

This completes our construction of a canonical bijection

$$j\ _v : Irr(S\ _v)\ \ !\ \ _{E_v;K_v;[_v]};$$

For any $_v$ 2 Irr(S $_v$), if j $_v$($_v$) = (C_v;$_v$), we write

$$_v := _{C_v}(_v)\ 2\ Irr(G_E(F_v)):$$

16.5. Global A-packets. We come now to the global setting. Without loss of generality, x a global A-parameter, or more precisely a map

$$= \ _{E;K;[]} : W_F\ \ !\ \ S\ o\ (S_2\ \ S_3)\ \ PGSO_8(C)\ o\ S_3$$

and suppose that its restriction to W_F corresponds to the Hecke character of the torus $\tilde{T}_{E;K}$. The $PGSO_8$(C)-conjugacy class of then corresponds to the pair [] = f; ^{1}g of Hecke characters of the torus $T_{E;K}$. ~

As we explained in x16.4, the local A-packets $_v$ are equipped with canonical bijections j

$$_v : Irr(S\ _v)\ \ \ !\ \ _{E_v;K_v:[_v]}\ \ \ !\ \ \ _v$$

The global A-packet associated to is simply the restricted tensor product of the local ones, so that

= f = ⊗

$_v$$_v$: =

$_v$$_v$ 2 Irr(S $_{;A}$)g:

The irreducible summands of V[ ] $A_{2;}$ (G_E) are isomorphic to elements of .

16.6. Multiplicity formula. Our remaining task is to verify that the Arthur multiplicity formula holds for V[ ]. In other words, for each = $_v$$_v$, we need to determine the multiplicity of in V[ ]. Now

=

$_v$$_{C_v}$($_v$) where j $_v$($_v$) = (C_v;$_v$) for each v. To

determine the multiplicity of in V[ ], we consider the subset

$$_{E;K;[];}\ \ _{E;K}$$

consisting of those C's satisfying:

for each place v of F, there is an isomorphism

$$_v : C_v := C$$
$$_F\ F_v = C_v:$$

Note that the isomorphism $_v$ is unique up to composition by elements of $H_C(F_v)$, and so induces an isomorphism

$$_v : H_{C_v} = H_{C_v}$$

which is well-determined up to conjugation. Hence, $_v$ is a well-dened element of Irr($H_{C_v}(F_v)$). In particular, we have a well-dened abstract irreducible representation

$$_{;C} := ⊗$$

$_v$$_v$ $_v$ of $H_C(A)$

such that

$$^{abs}(_{;C}) = \qquad \text{as abstract representations.}$$

the representation $_{;C}$ is automorphic and hence occurs in $V_C[]$.

To decide if $_{;C}$ is automorphic, an important role is played by the following diagram:

$$H^0_{C_v} \xrightarrow{\ C;v\ } \tilde{T}_{E_v;K_v}, \qquad H^0_{C_v} \,\Big|\, v \,\Big|\, H_{C_v} \xrightarrow{\ v\ } \tilde{T}_{E_v;K_v}$$

Here $_{C;v}$ is the localization of $_C$ at the place v and we recall that $_C$ is well-determined up to conjugacy by $H_C(F)$, and likewise $_v$ is well-determined up to conjugacy by $H_{C_v}(F_v)$. It is natural to ask if this diagram is commutative, or can be rendered such. We have:

Lemma 16.5. The above diagram commutes up to inverting, i.e.

$$_v\ v = {}_{C;v} \qquad \text{or} \qquad {}_{C;v}^{1}:$$

Hence, if $H_C(F_v) = H^0_C(F_v)$, then the above diagram is commutative by replacing $_v$ by $_v^1$ if necessary. In particular, if $H_C(F) = H^0_C(F)$, then the above diagram can be made commutative at all places v (by appropriate choices of $_v$ at each v).

For C 2 $_{E;K;[];}$, the multiplicity $m_C(_{;C})$ of $_{;C}$ in $V_C[]$ is in fact independent of C, by our discussion in x14.2. We thus denote this multiplicity by m() > 0. Given this, we see that

$$\text{Multiplicity of in } V[\ ] = m()\ \#_{E;K;[];}:$$

To establish the multiplicity formula, we need to show that the above number is equal to

$$m := h\ ; 1i_S = \frac{1}{\# S} \sum_{S 2 S} tr(((s))):$$

We consider the dierent cases of $= {}_{E;K;[]}$ in turn in the subsequent subsections.

16.7. K is a eld and $^2 = 1$. This is in some sense the most nondegenerate case, as all possible local scenarios we discussed in x16.4 can occur. However, it is also the least subtle case because

$$S = f1g \quad \text{so that} \quad m = \dim:$$

Let S denote the nite set of places v of F where C_v is associated with a cubic division algebra; at these places, we have $(E_v; K_v)$ = (eld, split). We have a decomposition

$$S = S^{0} t S^{00}$$

where S^0 consists of those places v where $_v^2 = 1$. Then

$$\dim_v = \begin{cases} 1; & \text{if } v \not 2 S^0 \\ 2; & \text{if } v 2 S^0 \end{cases}$$

so that

$$m = \dim = 2^{\# S} :$$

We now need to determine the size of $_{E;K;[];}$. For C 2 $_{E;K;[];}$ corresponding to E ,! B (for a central simple algebra B over K of degree 3, equipped with an involution of the second kind), B is ramied precisely at v 2 S. The number of possible C's is, at this point,

$$\begin{cases} 2^{\# S \; 1} & \text{if } S \text{ is nonempty; } 1; \\ & \text{if } S \text{ is empty.} \end{cases}$$

However, we also need to impose the condition that $_{;C}$ is automorphic.

Assume rst that S is nonempty. For any C 2 $_{E;K;[];}$, we have $H_C(F) = H^0(F)$. $_C$ From our discussion in x14.2, the abstract representation $_{;C}$ is automorphic if and only if its abstract restriction to $H^0_C(A)$ contains $_C$ or $^1{}_C = {}^1$. In other words, we need

$$_v j_{H^0_{vC}} \; v \; v \; C;v \qquad \text{for all places } v:$$

for one of the two choices of $_C$.

Now

$$j_{H^0_v} \; v = \begin{cases} v \; v + v^1 \; v & \text{if } v \not\in S \text{ and } v = 1; \\ v ; & \text{otherwise.} \end{cases}$$

From this and Lemma 16.5, we see that the desired containment holds for any v $\not\in$ S^{00} for both choices of $_C$.

It remains to consider the places in S^{00}, where we need the following to hold:

$$v \; v \; v = v \; C;v :$$

This identity xes $_{C;v}$ for every v 2 S^{00}. In other words, if $_C$ is associated to E ,! B for a pair (B;), then the invariant of B_v for every v 2 S^{00} is xed, and we only have the freedom to dictate the invariant of B_v at v 2 S.

Hence, the number of possible (B;)'s is $2^{\# S}$ and

$$\# \; E;K;[]; = \begin{cases} 2^{\# S^0}; & \text{if } S^{00} \text{ is nonempty;} \\ 2^{\# S^0 \; 1}; & \text{if } S^{00} \text{ is empty.} \end{cases}$$

On the other hand, by our discussion in x14.2,

$$m(\;) = m\,(\; ;C\;) = \begin{cases} 1; & \text{if } S^{00} \text{ is nonempty;} \\ 2; & \text{if } S^{00} \text{ is empty.} \end{cases}$$

Taken together, we see that

$$m() \; \#_{E;K;[];} = 2^{\# S} = m; \text{ as}$$

desired.

The case when S is empty is dealt with similarly, with both quantities equal to 1; we omit the details.

16.8. K is a eld and $^2 = 1$. In this case

$$S = S_2:$$

Given $= _{v v} 2 \text{Irr}(S_{;A})$, let S be the nite set of places v of F where C_v is associated with a cubic division algebra. Then $_v$ is the 2-dimensional representation r of $S^{\vee} = S_3$ if v 2 S, and $_v$ is 1-dimensional otherwise. Then

$$m = \dim \text{Hom}_{S_2}(r^{\# S};$$

$_{v \not\in S\, v}) = 2^{\# S \; 1}$ if S is nonempty. On the other hand, if S is empty, then

$$m = \frac{1}{2}(1 + (\; 1)^b) = \begin{cases} 1 \text{ if } b \text{ is even;} \\ 0 \text{ if } b \text{ is odd.} \end{cases}$$

where b is the nite number of places v of F where $_v$ is nontrivial on S .

Assume rst that S is nonempty. For any C 2 $_{E;K;[];}$, $H_C(F) = H^0_C(F)$, and if C is associated with E ,! B , then B is ramied precisely at places in S. Further, for $_{;C}$ to be automorphic, we need to verify that, for one of the two choices of $_C$, one has

$$_v \; _v j_{H_C(F_v)} = _v \; _{C;v} \qquad \text{for all places v.}$$

In fact, since $^2 = 1$, it is immaterial which of the two $_C$'s we use. Now $_v$

$$j_{H_{C_v}(F_v)} = _v \; _v \qquad \text{for all v.}$$

Hence the desired equality follows from Lemma 16.5 and the hypothesis that $^2 = 1$. In other words, $_{;C}$ is necessarily automorphic for any C 2 $_{E;K;[];}$, with $m_C(_{;C}) = 1$. Hence,

$$\#_{E;K;[];} = 2^{\# S \; 1}$$

so that

$$m() \; \#_{E;K;[];} = 2^{\# S \; 1} = m$$

as desired.

Consider now the case when S is empty, so that is a character of $S_{;A}$. In this case, C_v 2 $H^1(F_v; T_{E_v;K_v})[2]$ for all v, and so by Lemma 15.5, there is a unique C 2 $H^1(F; T_{E;K})[2]$ so that $C_{;v} = C_v$ for all v, and we need to determine if is automorphic for H_C . For this, we shall appeal to Proposition 15.16 and Proposition 4.20.

By Proposition 15.16, we see that [Irr($S_{;A}$)] is divided into two equivalence classes, depending on whether the restriction to $S = _2$ is trivial or not. The distinguished class, with trivial restriction to S , is thus the one for which m = 1 (instead of 0). Proposi-tion 15.16 says that this distinguished class is precisely the one which contains the image of a section $H^1(F; T_{E;K}) \; ! \; \text{Irr}(S_{;A})$. Equivalently, it is the image of the natural map $H^1(F; Z_E) \; ! \; \prod_v H^1(F_v; Z_E) \; ! \; [\text{Irr}(S_{;A})]$.

For with m = 1, there is thus an element a 2 $H^1(F; Z_E)$ and an automorphic sign character such that for all places v, $_v \; _{;v}$ corresponds to a under the bijection

$$H^1(F_v; Z_E)=b(\text{Ker}(_v)) \; f^{\; 1}([C_v]) \qquad ! \; \text{Irr}(S_v)$$

in Proposition 15.12. Observe that m = 1 as well, and = $_{C\,;}$, where $_{C\,;}$ is the automorphic sign character of H_C nontrivial precisely at places in . Hence in deciding the automorphy of , there is no harm in assuming that is empty, by replacing by if necessary.

By Proposition 4.20, the element a 2 $H^1(F; Z_E)$ gives rise to an element

$$g(a) \; 2 \; H_C(F_v) \; r \; H_C(F_v)^0 \qquad \text{for each place } v.$$

Now is automorphic if and only if (g(a)) = 1. But its local component $_v$ is charac-terized by the property that

$$_v(g(a)) = 1 \qquad \text{for all } v.$$

In particular, is automorphic when m = 1, as desired.

On the other hand, if m = 0, it is clear that is not automorphic, since diers from an automorphic $_0$ by a twist of a global sign character of H_C which is the local sign character at an odd number of places.

16.9. K is split and ^{2} = 1. In this case,

$$S = {}_3;$$

For a given 2 Irr($S_{;A}$), let S be the nite set of places where C_v is associated with a cubic division algebra. For v 2 S, E_v is necessarily a eld. We have a decomposition

$$S = S^0 \; t \; S^{00}$$

where S^0 consists of those v where ^2_v = 1. Hence, for v 2 S^0, S_v = S_3 and $_v$ is the 2-dimensional irreducible representation r of S_3; at all other places, $_v$ is 1-dimensional. For places v 2 S^{00}, S_v = $_3$ and we further decomposes

$$S^{00} = S^{00}_{;1} \; t \; S^{00}_{;2}$$

where $S^{00}_{;1}$ consists of those v such that $_v$ corresponds to the element 1=3 2 Z=3Z = Irr($_3$) and $S^{00}_{;2}$ those v such that $_v$ corresponds to 2=3. For ease of notation, let us set

$$a = \#S^0_{;} \qquad b_1 = \#S^{00}_{;1} \qquad \text{and} \qquad b_2 = \#S^{00}_{;2}.$$

Considering the pullback of $_v$ to S, we have:

$$= \begin{cases} 1 & \text{if } v \not 2 \; S; \; {}_v j_S \\ {}_v; & \text{if } v \; 2 \; S^{00}; \\ \text{the sum of the two nontrivial characters of } {}_3, & \text{if } v \; 2 \; S^0. \end{cases}$$

Hence,

$$m = \frac{1}{3} \quad 2^a + (\;\; 1)^a \;\; {}^{b_1} \;\; {}^{b_2} + (\;\; 1)^a \;\; {}^{b_2} \;\; {}^{b_1}$$

where 2 C is a primitive cube root of 1. To further explicate the above formula, we have

$$m = \begin{cases} (2^a + 2(\;\; 1)^a)=3; & \text{if } b_1 \;\; b_2 = 0 \mod 3; \\ (2^a + (\;\; 1)^{a+1})=3; & \text{if } b_1 \;\; b_2 = 0 \mod 3. \end{cases}$$

In particular, if S is empty (so that a = b_1 = b_2 = 0), we see that m = 1.

We now enumerate the set $_{E;K;[];}$. Any $C \in {}_{E;K;[];}$ corresponds to $E \hookrightarrow B^{+}$ for a central simple F-algebra B ramied precisely at S. Assume rst that S is nonempty, so that $H_C(F) = H^0_C(F)$ for any $C \in {}_{E;K;[];}$. To check if ${}_{;C}$ is automorphic, we need to veify that, for one of the two choices of ${}_C$, we have

$$_v \; {}_v j_{H_C(F_v)} \supseteq {}_v \; {}_{C;v} \qquad \text{for all places } v.$$

Now

$$j_{{}_v \; {}_v} = \begin{cases} {}_v + {}_v^{1}; & \text{if } v \notin S \text{ and } {}^{2}_v = 1 \quad {}_v \; H^0_{{}_v}(F_v) \\ {}_v; & \text{otherwise.} \end{cases}$$

So the desired containment holds at all places outside S^{00}.

It remains to consider the places $v \in S^{00}$. For such a v, we need to verify if

$$_v \; {}_v \; {}_v = {}_v \; {}_{C;v}:$$

This holds if and only if

$$_{C;v} = {}_v \; {}_v:$$

In other words, if ${}_C$ is associated to the associative algebra embedding $E \hookrightarrow B$, then the invariants of B at $v \in S^{00}$ are constrained by ${}_v$ as follows:

$$\mathrm{inv}(B_v) = \begin{cases} 1/3; & \text{if } v \in S_{;1}; \\ 2/3; & \text{if } v \in S_{;2}; \\ 1/3; & \text{if } v \in S^{0}; \\ 0; & \text{otherwise.} \end{cases}$$

We leave it as an amusing exercise to verify that the number of B's satisfying these requirements is equal to m (with m computed above). It follows that

$$\# \; {}_{E;K;[];} = \begin{cases} m; & \text{if } S^{00} \text{ is nonempty;} \\ m/2; & \text{if } S^{00} \text{ is empty.} \end{cases}$$

However, from the discussion in §14.2, we have:

$$m(\;) = \begin{cases} 1; & \text{if } S^{00} \text{ is nonempty;} \\ 2; & \text{if } S^{00} \text{ is empty.} \end{cases}$$

Taken together, we thus conclude that, when S is nonempty,

$$m() \; \#_{E;K;[];} = m;$$

as desired.

Now consider the case when S is empty. In this case, the only possible $C \in {}_{E;K;[];}$ is C^{+} corresponding to $E \hookrightarrow M_3(F)$, and $H_{C^+}(F) = H^0_{C^+}(F)$. By our discussion in §14.2, we see easily that ${}_{;C^+}$ is automorphic with $m_{C^+}({}_{;C^+}) = 1$. Hence

$$m() \; \#_{E;K;[];} = 1 = m \text{ as}$$

desired.

16.10. K is split and $^2 = 1$. In this case,

$$S = S_3$$

and we x an element s_0 in S_3 n $_3$, so that S = $_3$ o S_2 and S = hs_0i. For all places v, we then have S_v = $_0(S^{W_{F_v}})$ o $_2$.

Given an , let S be the nite set of places where C_v is associated to a cubic division algebra. Then for v 2 S, S_v = S_3 and $_v$ is the 2-dimensional irreducible representation r of S_3. For all other v, $_v$ is 1-dimensional. On pulling back to S = S_3, we have

$$= \begin{cases} 1; & \text{if } v \not\in S \text{ and } _v(s_0) = 1; \\ ; & \text{if } v \not\in S \text{ and } _v(s_0) = 1; \\ r; & \text{if } v \in S. \end{cases}$$

Hence,

$$m = \frac{1}{6}\, 2^{\#S} + 2\,(\;\; 1)^{\#S} \qquad \text{if } S \text{ is nonempty,}$$

and if S is empty,

$$m = \frac{1}{2}\,\left(1 + (\;\; 1)^{b}\right) = \begin{cases} 1; & \text{if } b \text{ is even;} \\ 0; & \text{if } b \text{ is odd.} \end{cases}$$

where b is the cardinality of the set of places v where $_v(s_0)$ = 1.

We now consider the set $_{E;K;[];}$. For C 2 $_{E;K;[];}$, associated to E ,! B^+ say, we see that B is ramied precisely at S . We know that

$$\# f B \in Br_3(F) : B \text{ is ramied precisely at } S g = \frac{1}{3}\, 2^{\#S} + 2\,(\;\; 1)^{\#S}$$

if S is nonempty, and is 1 if S is empty.

Assume rst that S is nonempty, so that $H_C(F) = H^0(F_C)$ for any C 2 $_{E;K;[];}$. Then for $_{;C}$ to be automorphic, we need

$$_v \;\; _v j_{H_C(F_v)} \;\; _v \;\; _{C;v} \qquad \text{for all } v$$

for one of the two choices of $_C$. Now

$$_v j_{H_{C_v}(F_v)} = \;_v \;\; _v; \text{ so that}$$

$_{;C}$ is automorphic if and only if

$$_v \;\; _v \;\; _v = \;_v \;\; _{C;v} \qquad \text{for all } v.$$

By Lemma 16.5, this holds automatically since $^2 = 1$. Hence $_{;C}$ is always automorphic, with $m_C(_{;C}) = 1$ (by the discussion in x14.2), and

$$\#_{E;K;[];} = = \frac{1}{6}\, 2^{\#S} + 2\,(\;\; 1)^{\#S} = m$$

as desired.

On the other hand, if S is empty, then the only possible C 2 $_{E;K;[];}$ is C^+ corresponding to E ,! $M_3(F)$. This is treated in exactly the same way as the corresponding case when K is a eld, using the global Poitou-Tate duality summarized in Proposition 15.16. We omit the details.

To summarize, we have shown the following result which is one of the main global theorems of this paper:

Theorem 16.6. Let $= {}_{E;K:;[]}$ be a given global A-parameter of G_E over a number eld F. Let 2 Irr($S_{;A}$) be an irreducible character of its adelic component group with associated representation in the global A-packet . Then the multiplicity of in the submodule $V[\ ]$ $A_{2;}$ (G_E) is equal to

$$m = \dim \mathrm{Hom}_S\ (\ ; C):$$

16.11. Main global theorem. If $m_{disc}()$ denotes the multiplicity of an irreducible representation in the automorphic discrete spectrum $A_2(G_E)$, then the last theorem shows that

$$m_{disc}()\ \ m \qquad \text{for any } 2\ \mathrm{Irr}(S_{;A}).$$

In this nal subsection, we shall show the reverse inequality and hence strengthen this inequality to an equality.

The argument is analogous to that for the cubic unipotent A-packets of G_2 given in [G]. The proof will require two ingredients: one local and the other global in nature. We begin by describing these two ingredients. Hence, we x a global A-parameter $= {}_{E;K;[]}$ and $=$ $_{vv}$ 2 Irr($S_{;A}$), $^{C}{}_{v}$ so that $=$ $_{v\,v}$ $=$ $_{v}^{abs}(_{v})$.

(Local) For each place v of F, and for each nondegenerate E_v-twisted Bhargava cube $_v$ with associated character $_v$ of $N_{E_v}(F_v)$, we have

$$(16.7) \qquad \mathrm{Hom}_{N_{E_v}(F_v)}(_v;\ _v) = \begin{cases} _v\ \ _{K_v}; & \text{if } C_v = C_v; \\ 0; & \text{otherwise,} \end{cases}$$

as a module for the stabilizer $M_{E_v;}(F_v)$ of $_v$. Here, $_{K}\ _{v}$ is either the trivial character or the sign character of $M_{E_v;}(F_v) = H_{C_v}(F_v)$ depending on whether $!_{K_v=F_v}(\ 1) = +1$ or 1.

This result is Proposition 12.3 in the nonarchimedean case. For archimedean v, note that the Hom space here refers to the space of continuous linear functionals of $_v$ (as a Casselman-Wallach representation). The result for archimedean v will be shown in a paper with J. Adams and A. Paul, where we studied the archimedean theta correspondence and prove the results in x13.

(Global) Let

$$_{E;K;} = fC\ 2\ _{E;K} : C_v = C_v \text{ for all places } vg:$$

For any embedding $f :\ ,!\ A(G_E)$, there exists $C\ 2\ _{E;K;}$ such that the $_C$-Fourier coecient of $f()$ is nonzero. We shall show this as a consequence of Proposition 16.9 and Corollary 16.10 below.

Taking these two ingredients for granted, we proceed to show the reverse inequality. By the consideration of Fourier coecients, we have a natural map

$$\mathrm{Hom}_{G_E(\mathbb{A})}(\ ; \mathcal{A}(G_E)) \quad ! \quad \bigoplus_{C\, 2\ E;K;} \mathrm{Hom}_{N_E(\mathbb{A})}(\ ; \ _C)^{M_{E;C}(F)}$$

The global ingredient shows that this map is injective, so that one has an upper bound

$$m_{disc}() \qquad \sum_{C\, 2\ E;K;} \dim \mathrm{Hom}_{N_E(\mathbb{A})}(\ ; \ _C)^{M_{E;C}(F)}:$$

Here, we have used the fact that $\prod_v K_v$ is an automorphic character and hence is trivial on $H_C(F)$. The local ingredient, on the other hand, shows that for each C,

$$\dim \mathrm{Hom}_{N_E(\mathbb{A})}(\ ; \ _C)^{M_{E;C}(F)} = \dim \ ^{H_C(F)} = \dim \mathrm{Hom}_{H_C(F)}(-; \mathbb{C}):$$

The latter dimension is simply the automorphic multiplicity of – in $\mathcal{A}(H_C)$. We have seen that this automorphic multiplicity is independent of $C\ 2\ \ _{E;K;}$ and have denoted it by $m(\ ) = m()$. Hence, we obtain

$$m_{disc}() \ \ m(\ ) \ \#\ _{E;K;} = \ m;$$

where the second equality is precisely what we showed when we veried the Arthur multiplic-ity formula for the space of global theta liftings. Summarizing, we have the following theorem which strengthens Theorem 16.6 and which is the main global theorem of this paper.

Theorem 16.8. Let $\ = \ \ _{E;K:;[]}$ be a given global A-parameter of G_E over a number eld F. Let $\ 2\ \mathrm{Irr}(S\ _{;\mathbb{A}})$ be an irreducible character of its adelic component group with associated representation in the global A-packet . Then

$$m_{disc}() = \dim \mathrm{Hom}_S(\ ;\mathbb{C}):$$

It remains to establish the global ingredient above. For this, we recall the following notion from [GS1]: when $F_v = \mathbb{R}$ or $\mathbb{C}$, we say that a representation $_v$ of $G_E(F_v)$ is weakly minimal if the associated variety of its annihilator in the universal enveloping algebra is the minimal nilpotent orbit. Now we note:

Proposition 16.9. Let $\ = \ C$ be an irreducible automorphic subrepresentation of G_E such that $_v$ is not weakly minimal for at least one archimedean place v. Then there exists a nondegenerate cube $C\ 2\ V_E(F)$ and $f\ 2$ such that $f_{\overline{N_E};\ C} = 0$.

Proof. Let $f\ 2$ and consider the Fourier expansion of the constant term $f_{\overline{Z}}$ along $\overline{V}_E = \overline{N}_E/\overline{Z}$. If this expansion is supported on cubes of rank one, then is weakly minimal in the sense of Denition 4.6 in [GS1]. Then, by [GS1, Thm. 5.4], $_v$ is weakly minimal at all archimedean places, which contradicts our assumption. Moreover, since E is a eld, $V\ (F_E)$ has no rank 2 elements. Thus, $f_{\overline{Z}}$ has a non-trivial Fourier coecient for a cube C^0 of rank 3 or 4.

If C^0 is rank 3, then by Proposition 5.5, we can assume that $C^0 = (0; 0; e; 0)$ with $e\ 2\ E$, Let U_E be the unipotent radical of the 3-step maximal parabolic subgroup Q_E in G_E, with N_E and U_E in standard position, such that $\ _{C^0}$ restricts to a non-trivial character of $[\overline{U}_E; \overline{U}_E]$. The character of $[\overline{U}_E; \overline{U}_E]$ thus obtained is associated to an sl_2-triple corresponding to the non-special nilpotent orbit $3A_1$ (see the introduction to [JLS]). By [JLS, Cor. 6.6] (the conditions of Lemma 4.3 there are satised since the orbit $3A_1$ is not special) there exists $x \in F$ such that , with $C = (x; 0; e; 0)$, $f_{N_{E;C}} \neq 0$ for some $f \in$. This proves the proposition.

Corollary 16.10. For any embedding $f : \hookrightarrow A(G_E)$, there exists $C \in _{E;K;}$ such that the $_C$-Fourier coecient of $f()$ is nonzero.

Proof. By the local ingredient (16.7), we see that the only possible nonzero nondegenerate Fourier coecients supported by $f()$ correspond to the nitely many $C \in _{E;K;}$. Hence the corollary follows from Proposition 16.9.

17. Appendix A: A theta correspondence for E_7

In this section, we consider a dual pair $G_E \times H_C$ in the split adjoint group of type E_7, where $H_C = \mathrm{Aut}_E(C)$ for a 4-dimensional E-twisted composition algebra C. This theta correspondence (and its version for inner forms) can be used to construct the A-packets corresponding to a root SL_2, as we discussed briey in §3.7. We will not launch into this detailed study in this paper. The main purpose of this appendix is simply to compute the theta lift of the trivial representation of $H_C = SL_2(E)/\mu_2$; this result is needed in our paper [GS3].

17.1. Twisted composition. Assume that B is a composition algebra over F. Let $N(x) = x\bar{x}$ and $Tr(x) = x + \bar{x}$ be the norm and the trace on B. Then $C_B = B \oplus B \oplus B$ has a structure of an F^3-twisted composition algebra, given by

$$Q(x_1; x_2; x_3) = (N(x_1); N(x_2); N(x_3))$$

$$(x_1; x_2; x_3) = (\bar{x}_2\bar{x}_3; \bar{x}_3\bar{x}_1; \bar{x}_1\bar{x}_2)$$

$$N_C(x_1; x_2; x_3) = Tr(x_3x_2x_1):$$

The symmetric group S_3 acts on C_B as F-automorphisms by permuting the three summands of C_B, with the action of odd permutations twisted by the map $(x_1; x_2; x_3) \mapsto (\bar{x}_1; \bar{x}_2; \bar{x}_3)$. Let E be a cubic etale algebra over F. Since $\mathrm{Aut}(E/F)$ is isomorphic to a subgroup of S_3, by xing an embedding of $\mathrm{Aut}(E/F)$ into S_3, we obtain an E-twisted composition algebra C_B^E by Galois descent.

We shall now describe the group $\mathrm{Aut}(C_B^E)$ of automorphisms of C_B^E for $B = M_2(F)$. In this case $\bar{x}$ is dened as the adjoint of the matrix x,

$$\bar{x} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{if } x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}:$$

Assume rst that $E = F^3$. Let

$$GL_2(F^3)^{det} = \{(g_1; g_2; g_3) \mid \det g_1 = \det g_2 = \det g_3\}$$

This group acts on $C_B = B \oplus B \oplus B$ by

$$g(x_1; x_2; x_3) = (g_3x_1g_2^{-1}; g_1x_2g_3^{-1}; g_2x_3g_1^{-1}):$$

It is fairly straightforward to check that this action preserves Q and . An element g acts trivially if and only if it belongs to F. The group $GL(F^3)^{\det}_2 = F$ is the group of F-points of the algebraic group $SL_2(F^3)=_2$. The action of S_3 on C_B normalizes that of $SL_2(F\ )=_2$, on which it acts by permuting the 3 factors. Hence, for a general cubic etale algebra E over F, the group of F-automorphisms of C_B (with $B = M_2(F)$) is

$$Aut_F(C^E_B) = SL_2(E)=_2 \text{ o } S_E;$$

and the group of E-automorphisms is its identity component

$$Aut_E(C^E_B) = SL_2(E)=_2:$$

Since

$$H^1(F; SL_2(E)=_2) = H^2(F;_2) = Br_2(F)$$

we see that the E-isomorphism classes of E-twisted composition algebras C of E-dimension 4 correspond to isomorphism classes of quaternion algebras. In particular, as B varies over quaternion F-algebras, the algebras C^E_B exhaust all E-isomorphism classes of E-twisted composition algebras of E-dimension 4.

Via the Springer decomposition, we may connect the above discussion with the theory of Freudenthal-Jordan algebras of dimension 15. The split Jordan algebra of dimension 15 is $J_s = F^3\ C_{M_2(F)}$ and its automorphism group is $PGSp_6 = Sp_6(F)=_2$. Since

$$H^1(F; Sp_6(F)=_2) = H^2(F;_2) = Br_2(F);$$

we see that the isomorphism classes of Freudenthal Jordan algebras of dimension 15 are parametrized by isomorphism classes of quaternion algebras as well. If J is a form of J_s, let $[J]$ 2 $Br_2(F)$ denote the corresponding Brauer class. Similarly, for B 2 $Br_2(F)$, let J_B be the corresponding Freudenthal-Jordan algebra. It is clear that $[J] = B$ if $J = E\ C_B$.

17.2. Some embedding problems. Let C_B be an E-twisted composition algebra of E-dimension 4. Every element x in C_B satises the quadratic equation

$$^2(x) + Q(x)(x) \qquad N_{C_B}(x)x = 0:$$

If we x $e = Q(x)$ and $d = N_{C_B}(x)$, such that the cube = (1; 0; e; d) is non-degenerate, then x and (x) span an E-twisted subalgebra of E-dimension 2, corresponding to the cube . Thus, in order to understand embeddings of the E-twisted composition algebras of E-dimension 2 into C_B, it suces to understand solutions of the above equation.

Proposition 17.1. Assume that $E = F^3$ and consider C_B with $B = M_2(F)$. The group $Aut_E(C_B) = GL_2(F^3)^{\det} = F$ acts transitively on the set of elements x 2 C_B such that $Q(x) = 0$, and $N_C(x) = 1$. The stabilizer $Stab_{Aut_E(C_B)}(x_0)$ of

$$x_0 = (\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right); \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right); \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right))$$

is the quotient by F of the subgroup of $GL_2(F^3)$ consisting of elements $^{a\ 0}$

$$\left(\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right) \left(\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right) \begin{smallmatrix}\\0&d\end{smallmatrix} :$$

In particular, $Stab_{Aut_E(C_B)}(x_0)$ $\underline{E}$. The stabilizer of x_0 in $Aut_F(C_B) = Aut_E(C_B) \text{ o } S_3$ is a semi-direct product F o S_3^{ϵ}, where S_3^{ϵ} is a \quadratic twist" of S_3: we multiply any transposition in S_3 by

$$w = \left(\begin{pmatrix}0&1\\1&0\end{pmatrix};\begin{pmatrix}0&1\\1&0\end{pmatrix};\begin{pmatrix}0&1\\1&0\end{pmatrix}\right):$$

Proof. Let $x = (x_1; x_2; x_3) \in C_B$ such that $Q(x) = 0$, and $N_{C_B}(x) = 1$. We want to show that x is conjugated to x_0 by an element in $GL_2(F^3)^{det}$. Since $Q(x) = 0$, we have $\det x_i = 0$ for all i. Hence, we can write

$$x_1 = v_3 \, w_2^{>}; x_2 = v_1 \, w_3^{>}; x_3 = v_2 \, w_1^{>}$$

for some column vectors v_i and w_i. Note that

$$N_{C_B}(x) = Tr(x_3x_2x_1) = (w_1 \, v_1^{>}) \, (w_2^{>} v_2) \, (w_3 \, v_3^{>}) = 1:$$

Hence, all vectors are non-zero, and we can pick $g_1; g_2; g_3 \in SL_2(F)$, so that $g_i(v_i) = (1; 0)^{>}$ for all i. Thus, we can assume that $v_1 = (1; 0)^{>}$ for all i. Since $(w_2 \, v_2^{>}) = 0$, $w_i = (a_i; b_i)$ with $a_i = 0$. Hence, using the unipotent g_i stabilizing $(1; 0)^{>}$, we can arrange all $b_i = 0$. Thus x is conjugate to

$$\begin{pmatrix}a_2&0\\0&0\end{pmatrix};\begin{pmatrix}a_3&0\\0&0\end{pmatrix};\begin{pmatrix}a_1&0\\0&0\end{pmatrix}$$

such that $a_1a_2a_3 = 1$. But this element is conjugated to x_0 by a triple of diagonal matrices. The stabilizers can be computed directly.

Let C_B be an E-twisted composition algebra of E-dimension 4. For a nondegenerate E-twisted cube $= (1; 0 \quad f; \quad b)$, consider the set

$$= \{v \in C \mid Q(v) = f; N_C(v) = b\}:$$

Recall that to , we attach an E-twisted algebra C of E-dimension 2, equipped with a reduced basis $\{v; (v)\}$. Any element $x \in (F)$ denes an E-embedding of C into C_B, where v is sent to x. Hence is in bijection with the set of embeddings $C \hookrightarrow C_B$.

Corollary 17.2. Assume that F is a local eld, and C_B is an E-twisted composition algebra of E-dimension 4. If (F) is nonempty, then $Aut_E(C_B)$ acts transitively on (F).

Proof. Fix a point $v_0 \in (F)$. By Proposition 17.1, $GL_2(E)^{det}$ acts transitively on (F) (through its quotient $GL_2(E)^{det}/F = H_{C_B}(F)$) and the stabilizer of $v_0 \in (F)$ is a maximal torus T_{v_0} in $GL_2(F) \quad GL_2(E)^{det}$. Hence the F-rational orbits under $GL_2(E)^{det}$ is parametrized by $H^1(F; T_{v_0})$, which is trivial since $T_{v_0} = Res_{K/F} G_m$ for some quadratic etale algebra K over F. The corollary follows.

Next, we need to understand when (F) is nonempty:

Proposition 17.3. Let $B = M_2(F)$. Let C be an E-twisted composition algebra of E-dimension 2. Then C embeds into C_B^E if and only if $J = E \quad C$ is not a division algebra.

Proof. If $J = E \quad C$ is not a division algebra, then by [KMRT, Thm 38.8], $J = J_3(K)$ for a quadratic etale algebra K over F. Since K embeds into $B = M_2(F)$, we deduce that J embeds into $J_3(B)$ and hence C into C_B^E, where $J_3(B) = E \quad C_B^E$.

Now assume that $J = E \quad C$ is a division algebra. By tensoring with K_E if necessary, we can assume without loss of generality that E is a cyclic eld, with the Galois group generated by of order 3. Then $C_B = M_2(E)$, and we have

$$Q(x) = \det(x); \quad (x) = \overset{2}{x}; \qquad \text{and} \quad N_{C_B^E}(x) = Tr(x^2 xx):$$

On the other hand, there exists $\alpha \in F$ such that $C = C(\alpha) = E \oplus E$, with

$$Q(a;b) = ab; \quad (a;b) = (\alpha^{-1} b^{\#}; a^{\#}) \qquad \text{and} \quad N_C(a;b) = N_E(a) + \alpha^{-1} N_E(b):$$

Moreover, since $E \oplus C$ is a division algebra, $\alpha \notin N_{E/F}(E)$.

Assume, for the sake of contradiction, that $C(\alpha)$ embeds into C^E_B. Let x be the image of $(1;0)$. Since $Q_{C^E_B}(x) = Q_{C(\alpha)}(1;0) = 0$, the determinant of x is 0. Hence $x = v \cdot w^{\top}$ for two 2×1 column vectors v and w, with coefficients in E. One checks that

$$N_{C^E_B}(x) = N_{E/F}(w^{\top} v):$$

This implies that $N_{C(\alpha)}(1;0) = \alpha = N_{C^E_B}(x)$ is the norm of an element in E, a contradiction.

17.3. D_4 geometry. Now let O be the 8-dimensional composition algebra of split octonions. The automorphisms group of C_O is a semi-direct product of the split simply connected group G of type D_4 with S_3. We remind the reader that S_3 acts on $C_O = O \oplus O \oplus O$ is by permuting the three summands of C_O, with a twist by the map $(x_1;x_2;x_3) \mapsto (\bar{x}_1;\bar{x}_2;\bar{x}_3)$ for odd permutations. Tits [Ti] has given a beautiful description of the flag varieties for G in terms of geometry of C_O. We follow the exposition of Weissman [We2].

Fix a triple $(i;j;k)$ of integers $0 \le i;j;k \le 2$. Let F_{ijk} be the set of subspaces X

$$\oplus Y \oplus Z \subset C_O$$

where $X;Y;Z$ are subspaces of O of dimensions $i;j;k$, respectively, such that $N_O(X) = N_O(Y) = N_O(Z) = 0$ and $XY = YZ = ZX = 0$. Then F_{ijk} is a flag variety for G with respect to a parabolic $P = MN$, as indicated in the following table, where Δ_M is the subset of simple roots "contained" in M.

$i;j;k$	Δ_M
$0;0;0$	$\{\alpha_0;\alpha_1;\alpha_2;\alpha_3\}$
$1;0;0$	$\{\alpha_0;\alpha_2;\alpha_3\}$
$1;1;0$	$\{\alpha_0;\alpha_3\}$
$1;1;1$	$\{\alpha_0\}$
$2;1;1$	$\{\alpha_1\}$
$2;2;1$	$\{\alpha_1;\alpha_2\}$
$2;2;2$	$\{\alpha_1;\alpha_2;\alpha_3\}$

Consider now, C^E_O, the E-twisted version of C_O. As we noted in x4.11,

$$G_E = \mathrm{Aut}_E(C^E_O):$$

For $i = 1$ or 2, we define F_i to be the set of E-subspaces $V_i \subset C^E_O$ of dimension i such that $V_i \otimes_F$ ... belongs to F_{iii} for $C^E_O \otimes_F \bar{F} = C_O$. A pair $V_1 \subset V_2$ is a full flag if E is a field. Let $P_i = M_i N_i$ be the stabilizer of V_i in G_E. Then

$$M_1^{der} \cong SL_2(F) \text{ (long root)}, \qquad M_1/M_1^{der} \cong GL(V_1) = E$$

and

$$M_2^{der} = SL_2(E) \text{ (short root)}, \quad M_2 = GL(V_2)^{det}.$$

These claims can be easily checked over F. The modular characters are

$$_{U_1} = jN_E j^3 \text{ and } _{U_2} = j\det j^5$$

We have degenerate principal series J(s) and I(s) corresponding to P_1 and P_2, respectively.

17.4. Dual pair. Now let F be a nonarchimedean local eld and E a cubic etale algebra over F. Let C_B be the E-twisted composition algebra of dimension 4 associated to B = M_2(F), with corresponding Springer decomposition J_B = E C_B. By our discussion in x6, this data gives rise to a dual pair

$$G_E \; H_{C_B} \; ! \; G_B := G_{J_B}$$

where G_E = $Spin^E_8$, H_{C_B} = $Aut_E(C_B)$ = $SL_2(E)=_2$ and G_B is the split adjoint group of type E_7. Our goal is to determine the theta lift (1), where 1 is the trivial representation of H_{C_B}(F).

For this purpose, it will be more convenient to work with an alternative construction or description of the above dual pair which is adapted to the Siegel maximal parabolic subgroup in G_B and which makes use of the interpretation of G_E as the automorphism group of an 8-dimensional E-twisted composition algebra. We give this alternative description next.

Let S = G_m be a maximal split torus in $SL_2(F)=_2$ H_C. The torus S gives a short Z-grading of g_B and h_E:

$$g_B = n \; m \; n \text{ and } h_E = u \; l \; u$$

Let P = MN and Q = LU be the corresponding maximal parabolic subgroups in G_B and H_C respectively. The unipotent radical N is commutative and can be identied with an exceptional Jordan algebra J. The Levi subgroup M can be identied with with the similitude group of the cubic form N_J, with corresponding similitude character

$$i_J : M ! \; F :$$

Now the group G_E is contained in M and J, and under its adjoint action on N, one has the decomposition N = J = E C^E_O where C^E_O is the E-twisted composition algebra of E-dimension 8.

Note that the M-module N is dual to N and hence can be identied with J. Since J is identied with J using the trace form T_J, we can identify both N and N with J. Under this identication, both U and U are identied with E J. The Levi factor L is the centralizer of G_E in M. By Proposition 6.1, L can be identied with E. Indeed, for every 2 E, let c : J ! J be dened by

$$c : (e; v) ! \; (^{\#}= e; \; v)$$

for all (e; v) 2 E C^E_O. Then c is a similitude of N_J with $i_J(c) = N_E()$. Henceforth, we x an isomorphism L = E such that 2 E acts on N as c . Using this identication, $i_J() = jN_E()j^{\;1}$ and the center of M consists of 2 F.

17.5. Theta lift. Let be the minimal representation of G_B. Let N = J be the set of elements of rank 1, i.e. x 2 if and only if x = 0 and $x^{\#}$ = 0. As P-modules, we have an exact sequence [MS]

$$0 ! \; C^1_c(\,) ! \; ! \; _N ! \; 0$$

where n 2 N acts on f 2 C_c^1(
) by

$$(n)f(n) = (hn; ni)f(n)$$

where hn; ni is the natural pairing of N and N , and m 2 M by

$$(m)f(n) = ji_J(m)j^{2}f(m^{1}nm):$$

Moreover, = M
N
ji$_J$ j^1 ji$_J$ j^2, where $_M$ is a minimal representation of M , trivial on the center. It follows that a central element 2 F acts as multiplication by jj^3 and jj^6 on the two summands.

Considering U-coinvariants, we have a short exact sequence of G_E L-modules: 0

! C_c^1(
$^?$) ! $_U$! $_N$! 0;

where O
$^?$ is the set of elements x 2
perpendicular to E , i.e. the set of x = (0; v) 2 E C ^{E} such that

$$v = 0 \quad \text{and} \quad x^{\#} = (Q(v); (v)) = 0:$$

Assume, for simplicity, that E is a eld. Then
$^?$ is the set of v 2 C_E^E spanning an E-line in F_1 . Recall that G_1 acts transitively on F_1 O. Fix a line V_1 2 F_1 , and let P_1 be the stabilizer of V_1. Then P_1^{der} acts trivially on the line, and we identify P_1=P^{der} with GL(V_1) = E . Summarizing, we have an isomorphism of G_E L-modules,

$$C_c^1(^?) = \mathrm{Ind}_{P_1}^{G_E} C_c^1(E) \quad jN_Ej^2$$

where the induction is not normalized and C_c^1(E) is the regular representation of E E P_1=P^{der} = L, twisted by the character jN_Ej^2 of L = E$_1$ as indicated.

Proposition 17.4. Let (1) be the theta lift of the trivial representation of H_{C_B}(F) = $GL_2(E)^{det}$=F . Then (1) is a quotient J(1=2), the degenerate principal series represen-tation associated to the parabolic P_1.

Proof. Assume, for simplicity, that E is a eld. Since (1) $1_{H_{C_B}}$ is a quotient of , one sees by passing to U_E-coinvariants that (1) 1 is a quotient of $_U$. Let 2 F be in the center of M. Then acts trivially on 1_E, and as jj^3 and jj^6 on the two summands of $_N$. Hence (1)$_{G_E}$ 1_E is a quotient of $\mathrm{Ind}_{P_1}^{G_E}$ C_c^1(E) jN_Ej^2. Hence (1) is a quotient of Ind_{P_1} jN_Ej^2 . Since $_{N_1}$ = $jN_Ej^{3=2}$, it follows that Ind_{P_1} jN_Ej^2 = J(1=2).

Proposition 17.5. Let (1) be the theta lift of the trivial representation of H_{C_B}(F) = $GL_2(E)^{det}$=F . Let be a non-degenerate E-twisted cube, with associated E-twisted com-position algebra C. Then

(1)$_{N_2;}$ = 0 if E C is a division algebra;
(1)$_{N_2;}$ = C otherwise.

Proof. The space of twisted coinvariants $_{N_2;}$ is computed exactly as in Proposition 12.3, giving

$_{N_2;}$ $= C^1(_c$
$(F))$

where is as in Corollary 17.2. By the same corollary, if (F) is nonempty, then it is a single $H_C(F)$-orbit, in which case $(1)_{N_2;}$ is one dimensional. On the other hand, when

$N_2;$ (F) is empty, (1) = 0. By Proposition 17.3, (F) is empty precisely when E C is a division algebra.

Theorem 17.6. Let (1) be the theta lift of the trivial representation of $GL_2(E)^{det}=F$. Then (1) embeds as a submodule of the degenerate principal series I(1=2). If E is a eld, then I(1=2)=(1) = V_1 in the notation of Theorem 18.1. Otherwise (1) = I(1=2).

Proof. The minimal representation of G_B is a submodule of a degenerate principal series representation induced from the Heisenberg parabolic subgroup of G_B. Via restriction of functions to G_E, one obtains a nonzero H_C-invariant and G_E-equivariant map

$$_{H_E} = (1) \ ! \ I(1=2):$$

Since the spherical function restricts to a spherical function, the image must contain the submodule generated by the non-zero spherical vector in I(1=2). This is the whole I(1=2) unless E is a eld, by Propositions 18.5 and 18.6. If E is a eld, the spherical vector generates the submodule whose quotient is V_1. Next, we need to use the fact that

$$I(1=2)_{N_2; \ C} = C$$

for all nondegenerate cubes , which is a simple consequence of the Bruhat decomposition. Moreover, recall that $V_1^0 = {}_D(1)$ is the theta lift via the minimal representation of G_D (the rank 2 E_6). Hence $(V_1)_{N_2; \ C}$ = C precisely when E C is a division algebra. Combining with Proposition 17.5, we see that the image of the map (1) ! I(1=2) is exactly as predicted and the kernel consists of small representations, i.e. those for which (N_2; C) co-invariants vanish for all nondegenerate cubes . Since we know that (1) is a quotient of J(1=2), to nish the proof, it suces to show that any irreducible constituent of J(1=2) satises $_{N_2; \ C}$ = 0, for some nondegenerate .

To that end, we claim that it suces to check one of the following two conditions:

(a) The Jacquet functor of for any parabolic subgroup with Levi subgroup of type A_2 is Whittaker generic;
(b) The Jacquet functor of with respect to N_2 is a Whittaker generic representation of the Levi subgroup M_2.

Indeed, if (a) holds, then $_{N_2; \ C}$ = 0 by [GGS, Thm. A], interpreted in our setting for the nilpotent orbit A_2. By the same result of [GGS], the condition (b) implies that $_{[N_1;N_1];}$ = 0 for a generic character of of $[N_1; N_1]$, which in turn implies the existence of a nondegenerate such that $_{N_2; \ C}$ = 0, by the main result in [JLS] and the fact that the nilpotent orbit $3A_1$ is not special.

If E is a eld, we have only one additional constituent V_1^{00} in J(1=2) (see Theorem 18.2). Its Jacquet functor with respect to N_2 is a twist of the Steinberg representation of M_2, hence the condition (b) holds and we are done in this case.

If E is not a eld, then we have not analyzed J(1=2). In these remaining cases, we shall treat all representations whose exponents lie in the Weyl group orbit of the leading exponent of the spherical quotient of I(1=2), namely (1; 1; 0; 0) if E = F^3 or (1; 1; 0) if E = F K for

K a eld. In both cases, we have two tempered representations,

(17.7) $$D(St) = D(St)_{gen} \; D(St)_{deg};$$

which are the generic and non-generic summands of the unitary representation D(St) obtained by parabolic induction from the Steinberg representation of the Levi group of type A_2. There are three such parabolic groups if E = F^3, but the resulting representation does not depend on this choice, just as in the case of D(1), which is the Aubert involute of D(St). Observe that these tempered representations satisfy the condition (a).

In order to tabulate all possible standard modules, let us recall their properties, working with a general root system = $f_1; : : : ; _n g$. Let $f_1; : : : ; _n g$ be the corresponding funda-mental weights. A parabolic subgroup in standard position corresponds to a subset S .
A standard module associated to the parabolic subgroup has leading exponents

$$= (\sum x_i \; _i) + (\sum_{i \not\in S} y_i \; _i) \; _{i 2 S}$$

where x_i 0, $y_i > 0$ and the rst summand is an exponent of the tempered representation dening the standard module. Now it is easy to determine all leading exponents in the cases at hand, and thus determine all irreducible Langlands quotients in both cases:

<u>Case E = F^3</u>:

We have three Langlands quotients of G_E for the three maximal parabolic subgroups whose Levi subgroups are of type A_3. The tempered representation on the Levi subgroup is obtained by inducing the Steinberg representation of the Levi subgroup of the type A_1 A_1, that is, whose derived group is $SL_2(F)$ $SL_2(F)$. These Langlands quotients clearly satisfy the condition (a).

There are three remaining representation: the spherical quotient of I(1=2), the Langlands quotient $J_2(St_E; 1=2)$ and the Langlands quotient $J_1(St; 1=2)$. For these representations we have complete control of their (N_2; $_C$)-coinvariants, since the spherical representation and $J_1(St; 1=2)$ are the theta lifts $_{M_3(F)}(1)$ and $_{M_3(F)}()$ respectively, and $J_2(St_E; 1=2)$ is a submodule and the only other constituent of I(1=2). This settles the case E = F^3.

<u>Case E = F K</u>:

Here we have an interesting twist, when compared to the split case: there are two Langlands quotients of G_E forming an L-packet which prove especially challenging.

More precisely, instead of the three A_3 maximal parabolic subgroups considered in the split case, we have a maximal parabolic subgroup in the standard position with Levi subgroup of the type B_2, so that its derived group is a quasi-split $SU_4(K)$. Inducing the Steinberg representation of the Levi subgroup of $SU_4(K)$ whose derived group is $SL_2(K)$, gives a rep-resentation of $SU_4(K)$ with two irreducible summands. They in turn give two Langlands quotients of G_E with the leading exponent (1; 0; 1). One of these two representation is the summand of D(1), denoted by V_1^0, with (1; 0; 1) as its only exponent. The other represen-tation V is the potentially troublesome one.

Finally, we have three additional representations: the spherical quotient of I(1=2) (which is the other summand of D(1) besides V_1^{ζ}, by Proposition 18.5(4)), the Langlands quotient

$J_2(St_E; 1=2)$ and the Langlands quotient $J_1(St; 1=2)$. Clearly, $J_2(St_E; 1=2)$ satises the condition (b) above. Now $J_1(St; 1=2)$ is a submodule of $I(1=2)$, while the spherical representation and V_1^0 are the theta lifts $_{M_3(F)}(1)$ and $_{M_3(F)}()$. For these representations, we have a similar situation as in the split case, with complete control of their $(N_2; \ _C\ )$-coinvariants, and in particular non-vanishing for some .

It remains to deal with the other representation V with leading exponent (1; 0; 1). Recall that, counting two tempered representations in (17.7), we have seven representations in all. Let us examine the eect of the Aubert involution on this set of representations:

- The Aubert involution takes the two summands of D(St) to the two summands of D(1).
- It takes the degenerate series $I(1=2)$ to the generalized principal series $I(St_E; 1=2)$. It follows that the Aubert involution takes $J_1(St; 1=2)$ $I(1=2)$ to $J_2(St_E; 1=2)$ $I(St_E; 1=2)$.

From this, one deduces that the involution xes the remaining representation V, and hence (1; 0; 1) is also an exponent of V. But with respect to the A_2 Levi subgroup, this is the exponent of the Steinberg representation and hence condition (a) holds for V. This completes the proof in the case E = F K.

This theorem is used in our paper [GS3].

18. Appendix B: Degenerate principal series

In this section, we analyze unramied degenerate principal series representations for G_E (the quasi-split simply connected reductive group of absolute type D_4 determined by E). The results here are new if E is a eld and a mixture of new and known results if E = F K. We have used the results and language introduced in this appendix for the description of theta lifting in the main body of the paper.

18.1. Ane Weyl groups, when E a eld. Let A = f(x; y; z) 2 R^3 j x + y + z = 0g be the 2-dimensional euclidean space equipped with the usual dot product. Let A (we identify A with A using the dot product) be the root space of type G_2 such that $_1$ = (1; 1; 0) and $_2$ = ($\frac{1}{3}$; $\frac{2}{3}$; $\frac{1}{3}$) are the simple roots. Let W be the corresponding Weyl group. It is generated by the simple reections s_1 and s_2 corresponding to the simple roots.

Assume rst that E is unramied.

Ane roots are the ane functions + k on A where 2 and k 2 Z. The ane Weyl group W_a is generated by reections about the lines where the ane roots vanish. Let $_l$ 2 be the highest root. The fundamental cell in A for W_a is given by the inequalities

$$0 < \ _1; \quad 0 < \ _2 \quad \text{and} \quad _l < 1:$$

In particular, W_a is generated by s_1, s_2 and s_0, the reections about the three lines bounding the fundamental cell. Let X A be the lattice spanned by

$$!_1 = (1; 0; 1) \text{ and } !_2 = (1; 1; 2):$$

Then W_a is a semi direct product of W and the group of translations t_ω where $\omega \in X$. We note the following relations in W_a:

$$t_{\omega_1} = s_0s_1s_2s_1s_2s_1 \text{ and } t_{\omega_2} = (s_0s_1s_2s_1s_0)(s_2s_1s_2s_1s_2).$$

<u>Assume now that E is ramified</u>.

Affine roots are the affine functions $\alpha + k$ on A where $\alpha \in \Phi$ and $k \in \mathbb{Z}$, if α is long, and $k \in \frac{1}{3}\mathbb{Z}$, if α is short. The affine Weyl group W_a is generated by reflections about the lines where the affine roots vanish. Let $\alpha_s \in \Phi$ be the highest short root. The fundamental cell in A for W_a is given by the inequalities

$$0 < \alpha_1; \quad 0 < \alpha_2 \quad \text{and} \quad \alpha_s < 1/3.$$

In particular, W_a is generated by s_1, s_2 and s_0, the reflections about the three lines bounding the fundamental cell. Let $X \subset A$ be the lattice spanned by

$$\omega_1 = (1; 0; -1) \text{ and } \omega_2 = \left(\frac{1}{3}; \frac{1}{3}; -\frac{2}{3}\right).$$

Then W_a is a semi direct product of W and the group of translations t_ω where $\omega \in X$. We note the following relations in W_a:

$$t_{\omega_2} = s_0s_2s_1s_2s_1s_2 \text{ and } t_{\omega_1} = (s_0s_2s_1s_2s_0)(s_1s_2s_1s_2s_1).$$

Let G_E be the simply connected quasi-split group of type D_4 corresponding to the cubic field E. Let I be the Iwahori subgroup corresponding to the fundamental cell. Let $\ell : W_a \to \mathbb{Z}$ be the length function such that, for every $w \in W_a$,

$$[IwI : I] = q^{\ell(w)}.$$

Let H be the Iwahori Hecke algebra. It is spanned by T_w, the characteristic functions of IwI for all $w \in W_a$. As an abstract algebra, H is generated by T_0, T_1 and T_2 corresponding to simple reflections, modulo braid and quadratic relations given by the diagrams in the above picture, where the left diagram corresponds to the case when E is unramified. Let $\hat{T}_w = q^{-\frac{\ell(w)}{2}} T_w$. The elements $\hat{T}_\omega$ for dominant $\omega = n_1\omega_1 + n_2\omega_2$ (i.e. $n_1, n_2 \geq 0$) form a commutative semi-group

$$\hat{T}_\omega \hat{T}_{\omega'} = \hat{T}_{\omega+\omega'}.$$

Let V be a nite-dimensional H-module. Since $T_{!}^{\wedge}$, for dominant $!$, commute and are invertible, we can decompose

$$V = V$$

where, for every $2 A$

$_R$ C,

$$V = fv\ 2\ V\ j\ (T_{!}^{\wedge} \quad q^{(!)})^{1} v = 0 \text{ for all dominant } !g:$$

Note that $V = V_{+\frac{2i}{\ln q}}$ for any $2 X$, the lattice dual to X. Thus, we say that $;^{0}$ are congruent if $^{0}2 \frac{2i}{\ln q}X$. The congruence class of such that $V = 0$ is called an exponent of V. A representation V is a discrete series if

$$<((!_i)) < 0$$

for $i = 1; 2$ for all exponents of V. Exponents represented by $2 A$ are called real. The exponent of the trivial representation (i.e. $T_w ! \quad q^{`(w)}$ for all $w\ 2\ W_a$) is

$$(2; 1; \quad 3):$$

The Iwahori-Matsumoto (IM) involution changes the exponents by the sign. In particular, the exponent of the Steinberg representation is $(\quad 2; \quad 1; 3)$. It is a discrete series representation.

18.2. Some representations, when E is a eld. We shall now construct small dimensional representations of the Hecke algebra H that will appear in the description of degenerate principal series.

Assume rst that E is unramied.

18.2.1. One dimensional representations. Let V be a one dimensional complex vector space spanned by e. Let $V_1^{(}$ be the representation of H on V dened by

$$T_0 e = \quad e; \quad T_1 e = \quad e \quad \text{and} \quad T_2 e = q^3 e:$$

The exponent of $V_1^{(}$ is

$$(0; 1; \quad 1):$$

Let V_1^{00} be be the representation of H on V dened by

$$T_0 e = qe; \quad T_1 e = qe \quad \text{and} \quad T_2 e = \quad e:$$

Then V_1^{00} is the IM-involute of V_1^{0} and is a discrete series representation.

18.2.2. Two dimensional representations. The subalgebra generated by T_0 and T_1 is isomorphic to the group algebra of S_3. It is not too dicult to see that any irreducible two dimensional representation of H, when restricted to this sub algebra, must be isomorphic to the reection representation of S_3. Thus let V be a two dimensional complex vector space spanned by e_0 and e_1 on which T_0 and T_1 act as matrices

$$\begin{matrix} 1 & q^{\frac{1}{2}} \\ 0 & q \end{matrix} \quad \text{and} \quad \begin{matrix} q & 0 \\ q^{\frac{1}{2}} & 1 \end{matrix} \quad ;$$

respectively. We can extend this representation to H in three dierent ways. Two of these extensions are easy to construct. Let V_2^0 be the representation of H on V such that T_2 acts the scalar q^3. The exponents of V_2^0 are

$$(1\ \ \frac{2i}{3\ln q}; 1+\frac{2i}{3\ln q};\ \ 2) \text{ and } (1+\frac{2i}{3\ln q}; 1\ \ \frac{2i}{3\ln q};\ \ 2):$$

This is the minimal representation. Let V_2^{00} be the representation of H on V such that T_2 acts the scalar 1. Then V_2^{00} is the IM-involute of V_2^{0} and is a discrete series representation.

These two representations do not have real exponents, however. We shall be interested in the third extension such that T_2 acts as the matrix

$$\begin{pmatrix} 1 & q^{\frac{1}{2}}{}_6(q) \\ 0 & q^3 \end{pmatrix}$$

where ${}_6$ is the characteristic polynomial (over $\mathbb{Q}$) of the primitive 6-th roots of unity. This representation, henceforth denoted by V_2, is invariant under the involution. Its exponents are real and given by:

$$(1;\ \ 1;0) \text{ and } (\ \ 1;1;0):$$

18.2.3. Three dimensional representations. Let V be a three dimensional complex vector space spanned by e_0, e_1 and e_2. Let V_3^0 be a representation of H on V such that T_0, T_1 and T_2 act as matrices

$$\begin{pmatrix} 1 & q^{\frac{1}{2}} & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix}; \begin{pmatrix} q & 0 & 0 \\ q^{\frac{1}{2}} & 1 & q^{\frac{1}{2}} \\ 0 & 0 & q \end{pmatrix} \text{ and } \begin{pmatrix} q^3 & 0 & 0 \\ 0 & q_3 & 0 \\ 0 & q^{\frac{1}{2}}{}_3(q) & 1 \end{pmatrix}$$

respectively. This is the reection representation. The exponents of V_3^0, counted with multiplicities, are

$$(0;1;\ \ 1);(1;0;\ \ 1) \text{ and } (1;0;\ \ 1):$$

Let V_3^{00} be the IM-involute of V_3^{0}. It is a discrete series representation.

<u>Assume now that E is ramied</u>.

18.2.4. One dimensional representations. Let V be a one dimensional complex vector space spanned by e. Let V_1^{0} be a representation of H on V dened by

$$T_0e = qe; \quad T_1e = \ \ e \quad \text{and} \quad T_2e = qe:$$

The exponent of V_1^{0} is

$$(0;1;\ \ 1):$$

Let V_1^{00} be be the representation of H on V dened by

$$T_0e = \ \ e; \quad T_1e = qe \quad \text{and} \quad T_2e = \ \ e:$$

Then V_1^{00} is the IM-involute of V_1^{0} and is a discrete series representation.

18.2.5. Two dimensional representations. The subalgebra generated by T_0 and T_2 is isomorphic to the group algebra of S_3. It is not too dicult to see that any irreducible two dimensional representation of H, when restricted to the subalgebra, must be isomorphic to the reection representation of S_3. Thus let V be a two dimensional complex vector space spanned by e_0 and e_2. Then T_0 and T_2 act on V as matrices

$$\begin{matrix} 1 & q^{\frac{1}{2}} \\ 0 & q \end{matrix} \quad \text{and} \quad \begin{matrix} q & 0 \\ q^{\frac{1}{2}} & 1 \end{matrix} ;$$

respectively. We can extend this representation to H in three dierent ways. Two of these extensions are easy to construct. Let V_2^0 be the representation of H on V such that T_1 acts as the scalar q. The exponents of V_2 are

$$(1 \quad \frac{2\ i}{3}; \frac{4i}{3\ln q}; \quad 1 \quad \frac{2i}{3\ln q}) \text{ and } (1 + \frac{2\ i}{3}; \quad \frac{4i}{3\ln q}; \quad 1 + \frac{2i}{3\ln q}):$$

This is not the minimal representation. Let V_2^{00} be the representation of H on V such that T_1 acts as the scalar 1. Then V_2^{00} is the IM-involute of V_2^0 and is a discrete series representation. Again, these representations do not have real exponents.

We shall be interested in the third extension such that T_1 acts as T_0. This representation, henceforth denoted by V_2, is invariant under the involution. Its exponents are real and given by

$$(1; \quad 1; 0) \text{ and } (\quad 1; 1; 0):$$

18.2.6. Three dimensional representations. Let V be a three dimensional complex vector space spanned by e_1, e_2 and e_0. Let V_3^0 be a representation of H on V such that T_0, T_1 and T_2 act as matrices

$$\begin{pmatrix} 1 & q^{\frac{1}{2}} & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix}; \begin{pmatrix} q & 0 & 0 \\ 3q^{\frac{1}{2}} & 1 & q^{\frac{1}{2}} \\ 0 & 0 & q \end{pmatrix} \text{ and } \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & q^{\frac{1}{2}} & 1 \end{pmatrix}$$

respectively. This is the reection representation. The exponents of V_3^0, counted with multiplicities, are

$$(0; 1; \quad 1); (1; 0; \quad 1) \text{ and } (1; 0; \quad 1):$$

Let V_3^{00} be the involute of V_3^0. It is a discrete series representation.

18.3. Degenerate principal series, when E is a eld. We now study the unramied degenerate principal series representation of G_E associated to the Heisenberg parabolic subgroup P_E. Let e and f be the ramication and inertia indices of E over F, so that $e\ f = 3$. The simple coroots are

$$\bar{}_1 = (1; \quad 1; 0) \text{ and } _2 = \bar{}(\quad \frac{1}{e}; \frac{1}{e}; \frac{2}{e}):$$

Let V be an irreducible representation of H. Let $2\ A$
C such that $V = 0$ i.e. the class of is an exponent of V. Then, from the representation theory of $SL_2(F)$ and $SL_2(E)$,

If $(\bar{}_1) = 1 + \frac{2i}{\ln q}Z$ then $s_1()$ is an exponent of V.
If $(\bar{}_2) = f + \frac{2i}{\ln q}Z$ then $s_2()$ is an exponent of V.
If $s_i()$ is congruent to and $(_i) = 0$ then V is at least two dimensional.

Two exponents are equivalent if one is obtained from another by a repeated use of the rst two bullets.

In the following, we shall consider the decomposition of various unramied degenerate principal series representations of G_E. The representations V of the ane Hecke algebra that we constructed above will occur in the subspace of Iwahori-xed vectors in these principal series representations So as not to introduce more notation, we will use V to denote the corresponding representation of $G_E(F)$ (whose space of Iwahori-xed vectors is V) as well.

18.3.1. Degenerate series I(s). Let

$$_s = (s \quad \frac{1}{2}; 1; \quad s \quad \frac{1}{2})$$

where s 2 C. Note that $_s$ and $_{s^0}$ are congruent if s s^0 2 $\frac{2i}{\ln q}Z$. Since $_s(-) = f_2$ the equivalence class of $_s$, for a generic s, contains the following six elements

$$(s \quad \frac{1}{2}; 1; \quad s \quad \frac{1}{2});$$
$$(1; s \quad \frac{1}{2}; \quad s \quad \frac{1}{2});$$
$$(s + \frac{1}{2}; \quad s + \frac{1}{2}; \quad 1);$$
$$(\quad s + \frac{1}{2}; s + \frac{1}{2}; \quad 1);$$
$$(1; \quad s \quad \frac{1}{2}; s \quad \frac{1}{2});$$
$$(\quad s \quad \frac{1}{2}; 1; s \quad \frac{1}{2}):$$

These are the exponents of a degenerate principal series I(s), attached to the Heisenberg maximal parabolic subgroup P_E. Since the representations I(s) form an algebraic family, these are the exponents for any s. The rst exponent ($_s$) is a leading exponent of I(s). The last exponents is a trailing exponent of I(s). (It is a leading exponent of I(s).) If V is a quotient of I(s) then the leading exponent is an exponent of V. If V is a submodule of I(s), then the trailing exponent of I(s) is also an exponent of V. We would like to determine the points of reducibility of I(s).

We say that an exponent is regular, if the stabilizer of in the Weyl group is trivial. A representation V of H is regular if the exponents of V are regular. It is well known that irreducible regular representations correspond to equivalence classes of regular exponents. One checks that I(s) is regular if

$$s = \frac{3}{2}; \frac{1}{2}; 0; \frac{i}{\ln q} \text{ and } 2 \quad \frac{1}{3} \frac{2i}{\ln q}$$

where the last possibility occurs only if E is unramied. If I(s) is regular, one checks that all exponents are equivalent, and hence I(s) is irreducible, if

$$s = \frac{5}{2}; \frac{1}{2} + \frac{i}{\ln q} \frac{}{2} \text{ and } \frac{3}{2} \quad \frac{2i}{3 \ln q}$$

and reducibility in the last case occurs only when E is unramied. In particular, I(s) is irreducible unless s is on one of the two lists.

Theorem 18.1. The representation I(s) = I(s) is reducible only if

$$s = \frac{5}{2}; \frac{1}{2}; \frac{1}{2} + \frac{i}{\ln q} \text{ and } \frac{3}{2} \quad \frac{2i}{3 \ln q}$$

and the last case occurs only if E is unramied. At the points of reducibility, we have:

(1) $I(\frac{5}{2})$ has length 2. The trivial representation is the unique irreducible quotient.
(2) $I(\frac{1}{2})$ has length 3. The representation V_2 is the unique irreducible submodule. The representations $V_1^{\complement}$ and $V_3^{\complement}$ are irreducible quotients.
(3) $I(\frac{1}{2} + \frac{i}{\ln q})$ has length 2. There is a unique irreducible submodule and a unique irreducible quotient.
(4) $I(\frac{3}{2} \quad \frac{2i}{3 \ln q})$ has length 2. The minimal representation V^0 is the unique irreducible quotient.

Proof. It remains to analyze the nite set of cases. We do so by considering the space of Iwahori-xed vectors in I(s), which is a H-module.

Case $s = \frac{5}{2}$. The exponents are

$$(2; 1; \quad 3);$$
$$(1; 2; \quad 3);$$
$$(3; \quad 2; \quad 1);$$
$$(\quad 2; 3; \quad 1);$$
$$(1; \quad 3; 2);$$
$$(\quad 3; 1; 2):$$

The leading exponent belongs to the trivial representation, the unique irreducible quotient of $I(\frac{5}{2})$. The other ve exponents are equivalent to the trailing exponent. Thus $I(\frac{5}{2})$ has length 2.

Case $s = \frac{3}{2}$. The exponents are

$$(1; 1; \quad 2);$$
$$(1; 1; \quad 2);$$
$$(2; \quad 1; \quad 1);$$
$$(\quad 1; 2; \quad 1);$$
$$(1; \quad 2; 1);$$
$$(\quad 2; 1; 1):$$

The last four exponents are equivalent. Let V be an irreducible subquotient such that $V_{(1;1;\ 2)} = 0$. The third bullet implies that this space is 2 dimensional. Thus, either $I(\frac{3}{2})$ is irreducible or it has a 2 dimensional irreducible quotient. But the exponents of $I(\frac{3}{2})$ are dierent from the exponents of irreducible 2 dimensional representations of H. Thus $I(\frac{3}{2})$ is irreducible.

Case $s = \frac{3}{2} + \frac{\pi i}{3\ln(q)}$. We assume that E is unramied. The exponents are

$$(1 + \frac{2i}{3\ln(q)}; 1; \quad 2 \quad \frac{2i}{3\ln(q)});$$
$$(1; 1 + \frac{2i}{3\ln(q)}; \quad 2 \quad \frac{2i}{3\ln(q)});$$
$$(2 + \frac{2i}{3\ln(q)}; \quad 1 \quad \frac{2i}{3\ln(q)}; \quad 1);$$
$$(\quad 1 \quad \frac{2i}{3\ln(q)}; 2 + \frac{2i}{3\ln(q)}; \quad 1);$$
$$(1; \quad 2 \quad \frac{2i}{3\ln(q)}; 1 + \frac{2i}{3\ln(q)});$$
$$(\quad 2 \quad \frac{2i}{3\ln(q)}; 1; 1 + \frac{2i}{3\ln(q)}):$$

All exponents are dierent. The rst two are equivalent and so are the last four. Since

$$(1 + \frac{2i}{3\ln(q)}; 1; \quad 2 \quad \frac{2i}{3\ln(q)}) \quad (1 \quad \frac{2i}{3\ln q}; 1 + \frac{2i}{3\ln q}; \quad 2) = \frac{2i}{\ln q}(\frac{2}{3}; \frac{1}{3}; \frac{1}{3})$$

the rst two are the exponents of the minimal representation V^0_2. The induced representation has length 2, with unique irreducible quotient V_2^ζ.

Case $s = \frac{1}{2}$. The exponents are

$$(0; 1; \quad 1);$$
$$(1; 0; \quad 1);$$
$$(1; 0; \quad 1);$$
$$(0; 1; \quad 1);$$
$$(1; \quad 1; 0);$$
$$(\quad 1; 1; 0):$$

In this case, V_2 is a unique irreducible submodule. The quotient is isomorphic to a direct sum of V_1^ζ and V_3^ζ.

Case $s = \frac{1}{2} + \frac{\pi i}{\ln(q)}$. The exponents are

$$(\frac{i}{\ln(q)}; 1; \quad 1 \quad \frac{i}{\ln(q)});$$
$$(1; \frac{i}{\ln(q)}; \quad 1 \quad \frac{i}{\ln(q)});$$
$$(1 + \frac{i}{\ln(q)}; \quad \frac{i}{\ln(q)}; \quad 1);$$
$$(\quad \frac{i}{\ln(q)}; 1 + \frac{i}{\ln(q)}; \quad 1);$$
$$(1; \quad 1 \quad \frac{i}{\ln(q)}; \frac{i}{\ln(q)});$$

$$(\ 1\ \ \frac{i}{\ln(q)}; 1; \frac{i}{\ln(q)}):$$

All exponents are dierent. The rst three are equivalent and so are the last three exponents. In particular, $I(\frac{}{2}+\frac{1}{\ln(q)}\frac{i}{})$ has length 2.

Case $s = \frac{1}{2} + \frac{i\,2}{3\ln q)}$. We assume that E is unramied. This representation is irreducible. The argument is similar to the argument for $s = \frac{3}{2}$. We omit details.

Case s = 0. The exponents are

$$(\ \ \frac{1}{2}; 1;\ \ \frac{1}{2}):$$
$$(1;\ \ \frac{1}{2};\ \ \frac{1}{2});$$
$$(\frac{1}{2}; \frac{1}{2};\ \ 1);$$
$$(\frac{1}{2}; \frac{1}{2};\ \ 1);$$
$$(1;\ \ \frac{1}{2};\ \ \frac{1}{2});$$
$$(\ \ \frac{1}{2}; 1;\ \ \frac{1}{2}):$$

We have three equivalent exponents each with multiplicity 2. Thus, either $I(0)$ is irreducible or it is a sum of two three dimensional representations with the same exponents. However, if $V_{(\frac{1}{2};\frac{1}{2};\ 1)_2} = 0$, then the third bullet implies that this space is 2 dimensional. Thus $I(0)$ is irreducible.

Case $s = \frac{i}{\ln q}$. This representation is irreducible. The argument is the same as for s = 0. We omit details.

18.3.2. Degenerate series J(s). We now study the unramied degenerate principal series associated to the 3-step parabolic subgroup Q_E of G_E. Let

$$_s = (s + \frac{1}{2}; s\ \ \frac{1}{2};\ \ 2s)$$

where $s \, 2 \, \mathbb{C}$. Note that $_s$ and $_{s^0}$ are congruent if $s\ \ s^0\ 2\ \frac{2i}{f\ln q}\mathbb{Z}$. Since $_s(_1) = 1$, the equivalence class of $_s$, for a generic s, contains the following six elements

$$(s + \frac{1}{2}; s\ \ \frac{1}{2};\ \ 2s);$$
$$(2s;\ \ s + \frac{1}{2};\ \ s\ \ \frac{1}{2});$$
$$(\ \ s + \frac{1}{2}; 2s;\ \ s\ \ \frac{1}{2});$$
$$(s + \frac{1}{2};\ \ 2s; s\ \ \frac{1}{2});$$
$$(\ \ 2s; s + \frac{1}{2}; s\ \ \frac{1}{2});$$

$$(\quad s + \frac{1}{2}; \quad s \quad \frac{1}{2}; 2s):$$

These are the exponents of a degenerate principal series $J(s)$, attached to the 3-step maximal parabolic subgroup Q_E. Since the representations $J(s)$ form an algebraic family, these are the exponents for any s. The rst exponent $(_s)$ is a leading exponent of $J(s)$. The last exponents is a trailing exponent of $J(s)$. (It is a leading exponent of $J(\quad s)$.) If V is a quotient of $J(s)$ then the leading exponent is an exponent of V. If V is a submodule of $J(s)$ then the trailing exponent of $J(s)$ is also an exponent of V.

We would like to determine points of reducibility of $J(s)$. One checks that $J(s)$ is regular if

$$s = \frac{1}{2}; \frac{1}{6}; 0; \frac{i}{\ln q} \text{ and } \frac{1}{6} \quad \frac{1}{3} \frac{2i}{\ln q}$$

where the last possibility occurs only if E is ramied. If $J(s)$ is regular, one checks that all exponents are equivalent, and hence $J(s)$ is irreducible, if

$$s = \frac{3}{2}; \frac{1}{2} + \frac{i}{\ln q} \text{ and } \frac{1}{2} \quad \frac{1}{3} \frac{2i}{\ln q}$$

and reducibility in the last case occurs only when E is ramied. Hence, again, $J(s)$ is irreducible unless s is on the two nite lists.

Theorem 18.2. The representation $J(s) = J(\quad s)$ is reducible only if 3

$$s = \frac{1}{2}; \frac{1}{2}; \frac{1}{2} + \frac{i}{\ln q} \text{ and } \frac{1}{2} \quad \frac{2i}{3 \ln q}$$

and the last case occurs occurs only if E is ramied. At the points of reducibility, we have:

(1) $J(\frac{3}{2})$ has length 2. The trivial representation is the unique irreducible quotient.
(2) $J(\frac{1}{2})$ has length 3. The representation V_1^{∞} is the unique irreducible submodule. The representation V_3^0 is the unique irreducible quotient. The remaining subquotient is V_2.
(3) $J(\frac{1}{2} + \frac{i}{\ln q})$ has length 2. There is a unique irreducible submodule and a unique irreducible quotient.
(4) $J(\frac{1}{2} \quad \frac{2i}{3 \ln q})$ has length 2. The representation V_2^0 is the unique irreducible quotient.

Proof. We shall provide details for $s = \quad 1{=}2$, which is the only case used in the paper.

Case $s = \frac{1}{2}$. The exponents are

$$(1; 0; \quad 1);$$
$$(1; 0; \quad 1);$$
$$(0; 1; \quad 1);$$
$$(1; \quad 1; 0);$$
$$(\quad 1; 1; 0);$$
$$(0; \quad 1; 1):$$

We see that V_1^{∞} is the unique irreducible submodule, V_3^0 is the unique irreducible quotient, and V_2 is the remaining subquotient.

18.4. Affine Weyl group, when K is a field. We now discuss the quasi split G_E where $E = F \times K$ with K a quadratic field. Let e and f be the ramification and inertia indices, so that $e f = 2$.

Let $A = \mathbb{R}^3$ equipped with the usual dot product. Let $\Phi \subset A$ (we identify A with A^* using the dot product) be the root space of type B_2 such that

$$\alpha_1 = (1; -1; 0);\ \alpha_2 = (0; 1; -1) \text{ and } \alpha_3 = (0; 0; 1) \text{ are}$$

the simple roots. The co-roots are

$$\alpha_1^\vee = (1; -1; 0);\ \alpha_2^\vee = (0; 1; -1) \text{ and } \alpha_3^\vee = (0; 0; \tfrac{2}{e}).$$

Let W be the corresponding Weyl group. It is generated by the simple reflections s_1, s_2 and s_3 corresponding to the simple roots.

Assume first that K is unramified.

Affine roots are the affine functions $\alpha + k$ on A where $\alpha \in \Phi$ and $k \in \mathbb{Z}$. The affine Weyl group W_a is generated by reflections about the lines where the affine roots vanish. Let $\alpha_l = (1; 1; 0) \in \Phi$ be the highest root. The fundamental cell in A for W_a is given by the inequalities $0 < \alpha_1$, $0 < \alpha_2$, $0 < \alpha_3$ and $\alpha_l < 1$. In particular, W_a is generated by s_1, s_2, s_3 and s_0, the reflections about the three planes bounding the fundamental cell.

Let $X \subset A$ be the lattice consisting of $(x; y; z) \in \mathbb{Z}^3$ such that $x + y + z$ is even. Then W_a is a semi direct product of W and the group of translations t_ω where $\omega \in X$. It will be convenient to work with the extended affine Weyl group $\tilde{W}_a = W_a \cup \tau W_a$ where τ is the involution defined by $\tau(x; y; z) = (1 - x; y; z)$. Note that $s_0 = \tau s_1 \tau$ and τ commutes with s_2 and s_3. The extended affine Weyl group is a semi direct product of W and $\tilde{X} = \mathbb{Z}^3$. Let

$$\omega_1 = (1; 0; 0);\ \omega_2 = (1; 1; 0) \text{ and } \omega_3 = (1; 1; 1).$$

We note the following relations in $\tilde{W}_a$:

$$t_{\omega_1} = s_1 s_2 s_3 s_2 s_1;\ t_{\omega_2} = s_0 s_2 s_3 s_2 s_1 s_2 s_3 s_2 \text{ and } t_{\omega_3} = s_0 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_3.$$

Assume now that K is ramified.

Affine roots are the affine functions $\alpha + k$ on A where $\alpha \in \Phi$ and $k \in \frac{1}{2}\mathbb{Z}$, but integral if α is long. The affine Weyl group W_a is generated by reflections about the lines where the affine roots vanish. Let $\alpha_s = (1; 0; 0) \in \Phi$ be the highest short root. The fundamental cell in A for W_a is given by the inequalities $0 < \alpha_1$, $0 < \alpha_2$, $0 < \alpha_3$ and $\alpha_s < 1/2$. In particular, W_a is generated by s_1, s_2, s_3 and s_0, the reflections about the three planes bounding the fundamental cell.

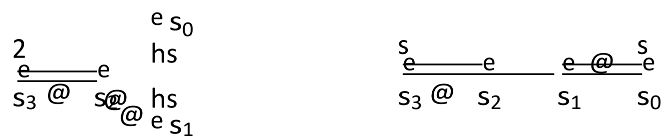
Let $X = \mathbb{Z}^3 \subset A$. Then W_a is a semi direct product of W and the group of translations t_ω where $\omega \in X$. The extended affine Weyl group is $\tilde{W}_a = W_a \cup \tau W_a$ where τ is the involution defined by $\tau(x; y; z) = (1/2 - z; 1/2 - y; 1/2 - x)$. Note that $s_0 = \tau s_1 \tau$ and $s_2 = \tau s_3 \tau$. The extended affine Weyl group is a semi direct product of W and $\tilde{X}$ generated by X and $(1/2; 1/2; 1/2)$. Let

$$\omega_1 = (1; 0; 0);\ \omega_2 = (1; 1; 0) \text{ and } \omega_3 = (1/2; 1/2; 1/2).$$

We note the following relations in $\tilde{W}_a$:

$$t_{\omega_1} = s_0s_1s_2s_3s_2s_1;\ t_{\omega_2} = s_0s_1s_2s_3s_2s_0s_1s_2s_3s_2 \text{ and } t_{\omega_3} = s_3s_2s_1s_3s_2s_3:$$

For any $E = F \otimes K$, the Iwahori Hecke algebra H of G_E is generated by the elements T_0, T_1, T_2 and T_3 corresponding to the simple reflections, modulo braid and quadratic relations given by the following diagrams, with the one on the left for the case of unramified K and the one on the right for the case of ramified K.



18.5. Some representations, when K is a field. We shall now construct some small dimensional representations of the Hecke algebra H that will appear in the description of the degenerate principal series representations.

Assume that K is unramified.

18.5.1. One dimensional representations. Let V be a one dimensional complex vector space spanned by e. There are four representations of H on V. We shall firstly describe two representations where

$$T_0e = qe; \quad T_1e = qe \quad \text{and} \quad T_2e = qe:$$

The remaining two are obtained by applying the IM-involution. If $T_3e = q^2e$, this is the trivial representation. Its exponent is

$$(3; 2; 1):$$

Let V_1^{ζ} be the representation of H on V such that $T_3e = e$. The exponent of V_1^{ζ} is

$$(1; 0; \ 1):$$

Let V_1^{ϖ} be the IM-involute of V_1^{0}. It is a tempered representation.

18.5.2. Two dimensional representations. Let V be a two dimensional complex vector space spanned by e_0 and e_1 on which T_0, T_1 and T_2 act by

$$T_0 = T_1 = \begin{pmatrix} 1 & q^{\frac{1}{2}} \\ 0 & q \end{pmatrix} \text{ and } T_2 = \begin{pmatrix} q & 0 \\ q^{\frac{1}{2}} & 1 \end{pmatrix}:$$

We can extend this representation to H in two ways. Let V_2^{0} be the representation of H on V such that T_3 acts the scalar q^2. The exponents of V_2^{0} are

$$(2; 0; 1) \text{ and } (0; 2; 1):$$

Let V_2^{ϖ} be the representation of H on V such that T_3 acts the scalar 1. Then V_2^{ϖ} is the IM-involute of V_2^{ζ} and is a discrete series representation.

Assume that K is ramified.

18.5.3. One dimensional representations. Let V be a one dimensional complex vector space spanned by e. There are eight representations of H on V. We shall rstly describe four representations where $T_1e = qe$, $T_2e = qe$. The remaining four representations are obtain by the IM-involution. The trivial representation is the one where $T_0e = qe$ and $T_3e = qe$. Its exponent is

$$(3; 2; 1):$$

Next, we have two representations where T_0 and T_3 act by dierent eigenvalues. These two representations occur in a restriction of a 2-dimensional representation of the extended ane Hecke algebra $\tilde{H}$. Their exponents are the same,

$$(2 + \frac{i}{\ln q}; 1 + \frac{i}{\ln q}; \frac{i}{\ln q}):$$

Let V_1^0 be the representation of H on V such that $T_0e = e$ and $T_3e = e$. The exponent of V_1^0 is

$$(1; 0; 1):$$

Let V_1^{00} be the IM-involute of V_1^0. It is a tempered representation.

18.5.4. Two dimensional representations. Let V be a two dimensional complex vector space spanned by e_0 and e_1 on which T_1 and T_2 act as matrices

$$T_1 = \quad 0 \; 1 \quad q \; q^{\frac{1}{2}} \qquad \text{and } T_2 = \quad q \; q^{\frac{1}{2}} \quad 0 \; 1 \quad ;$$

respectively. We can extend this representation to H in four ways. Let V_2^0 be the representation of H on V such that T_0 and T_3 act as the scalar q. The exponents of V_2^0 are

$$(2; 0; 1) \text{ and } (0; 2; 1):$$

Let V_2^{00} be the representation of H on V such that T_0 and T_3 act as the scalar 1. Then V_2^{00} is the IM-involute of V_2^0. It is a tempered representation. Finally, we have two additional representations, one where T_0 and T_3 act by dierent scalars. These two representations occur in a restriction of a 4-dimensional representation of the extended ane Hecke algebra $\tilde{H}$. Their exponents are the same and given by:

$$(1 + \frac{i}{\ln q}; \; 1 + \frac{i}{\ln q}; \frac{i}{\ln q}) \text{ and } (\; 1 + \frac{i}{\ln q}; 1 + \frac{i}{\ln q}; \frac{i}{\ln q}):$$

The sum of these two representation is an irreducible representation of $\tilde{H}$, the extended ane Hecke algebra.

18.6. Degenerate principal series, when K is a eld.

18.6.1. B_2 parabolic. Let $_s = (s; 2; 1)$. We have a degenerate principal series $B(s)$ (associated to the B_2-parabolic) whose exponents are

$$(s; 2; 1); \; (2; s; 1); \; (2; 1; s); \; (2; 1; \; s); \; (2; \; s; 1) \; (\; s; 2; 1):$$

Here $_s$ is a leading exponent and $_s$ is the trailing exponent. In particular, the trivial representation is the unique irreducible quotient of $B(3)$.

Proposition 18.3. The representation $B(s) = B(\ s)$ is reducible only if

$$s = 3; 1 + \frac{i}{\ln q}; 0; \text{ and } \frac{i}{\ln q}$$

where $s = 1 + \frac{i}{\ln q}$ occurs if K is unramied and $s = \frac{i}{\ln q}$ if K is ramied. At the points of reducibility, we have

1. (1) $B(3)$ has length 2. The trivial representation is the unique irreducible quotient.
2. (2) $B(1+\frac{i}{\ln q})$ has length 2. The minimal representation is the unique irreducible quotient.
3. (3) $B(0)$ is a direct sum of two non-isomorphic representations where one is V_2^{C}.
4. (4) $B(\frac{i}{\ln q})$ is a direct sum of two non-isomorphic representations.

Proof. This can be proved as in [We1]. Roughly speaking, o the unitary axis, i.e. $<(s) = 0$, reducibility happens only if the trivial or the minimal representations appear as subquotients. The case $s = 1 + \frac{i}{\ln q}$ merits a special discussion, as it illustrates a dierence between ramied and unramied cases. In both cases, $B(1 + \frac{i}{\ln q})$ is regular; however, the number of equivalence classes is one, if K is unramied, and 2 otherwise. This is due to the fact that $= (2; 1; 1 + \frac{i}{\ln q})$ is equivalent to $s_3() = (2; 1; \ 1 \ \frac{i}{\ln q})$ if and only if K is ramied.

On the unitary axis, all exponents are equivalent and $B(s)$ is irreducible, unless $s = 0$ or $s = \frac{i}{\ln q}$ and K ramied. By the Frobenius reciprocity, V_2 is a summand of $B(0)$, so (3) follows. Finally, $B(\frac{i}{\ln q})$ must reduce, otherwise $B(s)$ with $<(s) = \frac{i}{\ln q}$ would be all unitary, a contradiction.

18.6.2. A_2 parabolic. Let $_s = (s + 1; s; s \ \ 1)$. We have a degenerate principal series $A(s)$ (associated to the A_2-parabolic) whose exponents are

$$(s + 1; s; s \ \ 1); (s + 1; s; \ \ s + 1); (s + 1; \ \ s + 1; s); (\ \ s + 1; s + 1; s);$$

$$(s + 1; \ \ s + 1; \ \ s); (\ \ s + 1; s + 1; \ \ s); (\ \ s + 1; \ \ s; s + 1); (\ \ s + 1; \ \ s; \ \ s \ \ 1):$$

Here $_s$ is a leading exponent and $_s$ is the trailing exponent. In particular, the trivial representation is the unique quotient of $A(2)$. Note that $_s$ is congruent to $_{s+\frac{i}{\ln q}}$ if K is unramied.

Proposition 18.4. The degenerate principal series representation $A(s)$ (with $Re(s)\ 0$) is irreducible except in the following cases:

1. (1) $A(2)$ has length 2. The unique irreducible quotient is the trivial representation.
2. (2) $A(1)$ has length 2. The unique irreducible quotient is the orthogonal complement of V_2^0 in $B(0)$.
3. (3) when K is ramied, $A(1 + \frac{i}{\ln q})$ has length 3. It has two irreducible quotients, corresponding to two one-dimensional representations of H with the exponent

$$(2 + \frac{i}{\ln q}; 1 + \frac{i}{\ln q}; \frac{i}{\ln q}):$$

4. (4) $A(0)$ is a direct sum of two non-isomorphic representations where one of them is V_1^{C}.

Proof. (1) is trivial. For (2), observe that the spherical summand of B(0) is a unique irreducible quotient of A(1). The remaining sub quotients of A(1) have four exponents. As these exponents are equivalent, the length of A(1) is 2, as claimed. The statement (3) is proved similarly. For (4), observe that A(0) is semi-simple, and has at most two summands, since any summand contributes the exponent (1; 0; 1). Since V_1^0 is a summand of A(0) by the Frobenius reciprocity, we have two summands as claimed.

Note that the complement of V_1^0 in A(0) is spherical, and has seven exponents. We shall use this fact shortly.

18.6.3. A_1 A_1 parabolic. Let $_s$ = $(s + \frac{1}{2}; s \;\; \frac{1}{2}; 1)$. We have a degenerate principal series I(s) (associated to the A_1 A_1-parabolic, which is the Heisenberg parabolic), whose exponents are

$$(s + \frac{1}{2}; s \quad \frac{1}{2}; 1); \; (s + \frac{1}{2}; 1; s \quad \frac{1}{2}); \; (1; s + \frac{1}{2}; s \quad \frac{1}{2});$$
$$(s + \frac{1}{2}; 1; \quad s + \frac{1}{2}); \; (s + \frac{1}{2}; \quad s + \frac{1}{2}; 1); \; (\quad s + \frac{1}{2}; s + \frac{1}{2}; 1);$$
$$(1; s + \frac{1}{2}; \quad s + \frac{1}{2}); \; (1; \quad s + \frac{1}{2}; s + \frac{1}{2}); \; (\quad s + \frac{1}{2}; 1; s + \frac{1}{2});$$
$$(1; \quad s + \frac{1}{2}; \quad s \quad \frac{1}{2}); \; (\quad s + \frac{1}{2}; 1; \quad s \quad \frac{1}{2}); \; (\quad s + \frac{1}{2}; \quad s \quad \frac{1}{2}; 1):$$

Here $_s$ is a leading exponent and $_s$ is the trailing exponent. In particular, the trivial representation is the unique quotient of I(5=2). Points of reducibility of I(s) and its co-socle if Re(s) 0 was determined by Segal, Theorem 4.1 in [Se2]. Here we determine the complete composition series.

Proposition 18.5. The points of reducibility of I(s) (with Re(s) 0) are given as follows:

(1) I(5=2) has length 2. The unique irreducible quotient is the trivial representation.
(2) I(3=2) has length 2. The unique irreducible quotient is B(1).
(3) I(3=2 + $\frac{i}{\ln q}$) has length 2 when K is unramed, with the minimal representation as its unique irreducible quotient.
(4) I(1=2) has length 2. The unique irreducible quotient is the orthogonal complement of V_1^0 in A(0).
(5) I(1=2 + $\frac{i}{\ln q}$) has length 2 when K is unramed, and 3 with two irreducible quotients if K is ramied.

Proof. (1) is trivial. For (2), we observe that B(1) is the unique irreducible quotient of I(3=2). Since the remaining six exponents are equivalent, I(3=2) has length 2. The case (3) is regular, so the irreducible subquotients are easily determined by working out the equivalence classes of exponents. For (4), the spherical summand of A(0) is the unique quotient of I(1=2). The remaining subquotients of I(1=2) have ve exponents in total. Hence, if there are more than two irreducible subquotients in I(1=2), there would be one with one or two exponents. But, by inspection, these ve exponents are not among the exponents of one and two-dimensional H-modules. Hence, I(1=2) has length 2, as asserted in (4). For the last case, by the result of A. Segal, the representation has one, respectively two irreducible quotients. By working out equivalence classes of exponents, it is seen that there are no more irreducible subquotients than as stated.

18.7. Split D_4. Assume now that $E = F^3$ is split, so that G_E is the split Spin_8. Let $A = R^4$ and we identify A with A using the usual dot product. Let A be the root system of type D_4, so that the simple roots are

$$_1 = (1; \ 1; 0; 0); \quad _2 = (0; 1; \ 1; 0); \quad _3 = (0; 0; 1; \ 1); \quad _4 = (0; 0; 1; 1):$$

Let W be the corresponding Weyl group. For every k 2 Z and 2 , we have an ane root + k. Let W_a be the corresponding ane Weyl group. It is a semi-direct direct product of W and X = f(x; y; z; w) 2 Z^4 j x + y + z + w 0 (mod 2)g.

In this case, degenerate principal series representations have been well studied, and there are references in the literature, such as [BJ] and [We1]. So we shall be brief and put an emphasis on explaining, rather than giving the details.

Let T_i, i = 0; : : : ; 4 be the standard generators of the ane Hecke algebra H, such that T_2 corresponds to the branching point of the extended Dynkin diagram. The algebra H has a 2-dimensional irreducible representation V_2 such that

$$T_0 = T_1 = T_3 = T_4 = \begin{matrix} 1 & q^{\frac{1}{2}} \\ 0 & q \end{matrix} \quad \text{and } T_2 = \begin{matrix} q & 0 \\ q^{\frac{1}{2}} & 1 \end{matrix} :$$

The exponents of this representations are

$$(0; 1; \ 1; 0) \text{ and } (0; \ 1; 1; 0):$$

The minimal representation corresponds to the reection representation of H and its exponents are (the superscript 2 means that the exponent appears with multiplicity 2)

$$(2; 1; 1; 0)^2; (1; 2; 1; 0); (2; 1; 0; 1); (2; 1; 0; \ 1):$$

There are 3 maximal parabolic subgroups in standard position, of the type A_3, permuted by the group of outer automorphisms. Let A(s), B(s) and C(s) be the degenerate principal series, corresponding to these parabolic subgroups, normalized so that the trivial represen-tation occurs as the unique irreducible quotient for s = 3. For example, assuming that A(s) corresponds to the maximal parabolic whose Levi does not have $_1$ as a root, the leading exponent of A(s) is (s; 2; 1; 0). There are eight exponents:

$$(s; 2; 1; 0); (2; s; 1; 0); (2; 1; s; 0); (2; 1; 0; s);$$

$$(2; 1; 0; \ s); (2; 1; \ s; 0); (2; \ s; 1; 0); (\ s; 2; 1; 0):$$

By a result of Weissman [We1], A(1), B(1) and C(1) have length 2, and the minimal representation is the unique irreducible quotient. Let V_3^A A(1), V_3^B B(1) and V_3^C C(1) be the unique irreducible submodules. These representations are non-isomorphic, as they have dierent exponents.

Let I(s) be the principal series corresponding to the Heisenberg maximal parabolic (i.e. the Levi factor is A_1^3), normalized so that the trivial representation is the unique irreducible quotient for s = 5=2. The leading exponent is (s + $\frac{1}{2}$; s $\frac{1}{2}$; $\frac{1}{2}$; 0). There are 24 exponents in all. They are in 4 groups of 6 exponents

$$(1; 0; x; y); (1; x; 0; y); (1; x; y; 0); (x; 1; 0; y); (x; 1; y; 0); (x; y; 1; 0)$$

where

$$(x; y) = (s + \frac{1}{2}; s \quad \frac{1}{2}); (s + \frac{1}{2}; \quad s + \frac{1}{2}); (\quad s + \frac{1}{2}; s + \frac{1}{2}); (\quad s + \frac{1}{2}; \quad s \quad \frac{1}{2}):$$

The only other reducibility points are s = 1=2 and s = 3=2, which we examine in turn:

s = 3=2: the minimal representation is the unique irreducible quotient of I(3=2). Moreover, we have an intertwining map I(3=2) ! A(1), obtained by composing standard intertwining operators, which are non-trivial on the spherical vector. Hence A(1) (and analogously B(1) and C(1)) is a quotient of I(3=2). By removing these quotients, we are left with an irreducible submodule since its 10 exponents are equivalent.

s = 1=2: By the Frobenius reciprocity, V_2 is the unique irreducible submodule of I(1=2). The quotient is an irreducible spherical representation that appears as a summand of the representation induced from the trivial representation of (any) parabolic subgroup of the type A_2.

Summarizing, we have:

Proposition 18.6. (Theorems 5.3 and 5.5 in [BJ])

I(3=2) has a ltration of length 3, consisting of a unique irreducible submodule and a unique irreducible quotient (the minimal representation). The intermediate subquotient is isomorphic to $V_3^A \; V_3^B \; V_3^C$.

I(1=2) has length 2, and V_2 is the unique irreducible submodule.

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