Howe duality for a quasi-split exceptional dual pair

Petar Bakic, Gordan Savin

Abstract

We prove Howe duality for the theta correspondence arising from the p-adic dual pair G_2 $(PU_3 \rtimes \mathbf{Z}{=}2\mathbf{Z})$ inside the adjoint quasi-split group of type E_6.

Introduction

Let F be a p-adic eld, that is, a non-archimedean local eld of characteristic 0. Simple exceptional Lie algebras over F can be constructed from pairs $(\mathbf{O}; J)$ where $\mathbf{O}$ is an octonion algebra over F, and J a Freudenthal Jordan algebra. Let $G = \mathrm{Aut}(\mathbf{O})$ and $G^0 = \mathrm{Aut}(J)$. Let g and g^0 be the Lie algebras of G and G^0, respectively. Then, by a construction of Tits [16],

$$h = g \quad g^0 \quad \mathbf{O} \quad J$$

has a structure of a simple exceptional Lie algebra over F, where $\mathbf{O}$ and J denote trace 0 elements in $\mathbf{O}$ and J, respectively. Let $H = \mathrm{Aut}(h)$. It is evident from the construction that there is an inclusion

$$G \quad G^0 \quad H:$$

The group G is a split exceptional group of type G_2, whereas G^0 and H depend on J. A Freudenthal Jordan algebra is a form of $J_3(C)$, the algebra of 33 Hermitian symmetric matrices with coecients in a composition F-algebra C, see Chapter IX in [17]. A composition algebra, roughly speaking, is a non-associative algebra with an anti-involution $x \,!\, x$such that $N_C(x) = xx$is a quadratic form satisfying composition, that is, $N_C(xy) = N_C(x)N_C(y)$ for all $x; y \; 2 \; C$. The case treated in this paper is $C = K$, a quadratic eld extension of F. Then

$$G^0 = PU_3(K) \rtimes \mathrm{Gal}(K{=}F);$$

where $PU_3(K)$ is the quotient of the unitary group $U_3(K)$ in three variables by its center, and $\mathrm{Gal}(K{=}F)$ acts on coecients of $U_3(K)$ naturally. The group H is quasi-split of absolute type E_6.

Let be the minimal representation of H. The goal of this paper is to understand the restriction of to the dual pair $G \quad G^0$. More precisely, let be a smooth, irreducible representation of G. Then there exists a smooth representation () of G^0 such that () is the maximal -isotypic quotient of . If () is non-zero, we prove that it is a nite length G^0-module, and that it has a unique irreducible quotient (). Conversely, if is an irreducible representation of G^0 then () is a nite length G-module and, if it is non-zero, then it has a unique irreducible module (). The results are summarized in Theorem 4.1.

These results are proved by a period ping-pong, introduced in [10], that can be viewed as a generalization of the doubling method for classical theta correspondences [14], [18]. Here, just as for classical theta correspondences, one needs the following ingredient: If is an irreducible quotient of (), then – is a quotient of (–), where – denotes the smooth dual of . For classical theta correspondences this statement can be obtained using the Mglin{Vigneras{Waldspurger involution [21]. Existence of such an involution is a non-trivial matter; however, if is tempered then – is isomorphic to the complex conjugate . Since = , it follows at once that () is the complex conjugate of (). Thus the method of period ping-pong works well for tempered representations; however, separate treatment is needed for non-tempered representations. These representations are realized as Langlands quotients of principal series representations and here the method of Jacquet functors works well. Thus a principal contribution of this paper is a computation of the Jacquet functors of with respect to maximal parabolic subgroups of G and G^0.

For non-tempered representations we obtain the following explicit result. The group G^0 is quasi-split of rank one. The Levi factor of a Borel subgroup is isomorphic to K o Gal(K=F). Let be a character of K. Let i() be the two-dimensional representation of K o Gal(K=F) obtained by inducing . Assume, for simplicity, that is not Gal(K=F)-invariant. Then i() is irreducible and i() i(^{0}) if and only if ^{0} is in the Gal(K=F)-orbit of . Now, i() denes a principal series representation of G^0. We now describe its theta lift to to G. The group G has two conjugacy classes of maximal parabolic subgroups; we shall use the letters Q_1 and Q_2 for parabolic subgroups in the two classes, where the unipotent radical of Q_2 is a two step nilpotent group, and the unipotent radical of Q_1 is a three step nilpotent group. The Levi factors of both parabolic groups are isomorphic to $GL_2(F)$. Let W_F W_K denote the Weil groups of F and K. Recall, by local class eld theory, that W^{ab}_K = K. Thus can be viewed as a character of W_K. We induce to W_F and obtain a parameter of a supercuspidal representation of $GL_2(F)$. The theta lift of is a representation of G obtained by inducing from the maximal parabolic Q_1.

The authors would like to thank Wee Teck Gan for initiating this project and for his continued interest. One of his letters to the authors has been adapted to form Section 6 in the present paper. We would also like to thank the referee for a number of useful comments and suggestions. G. Savin is partially supported by a grant from the National Science Foundation, DMS-1901745.

1 Preliminaries

1.1 Basic number theory, stealing title from Weil

Let F be a non-Archimedean local eld with the absolute value j j normalized as usual, and let K=F be a quadratic extension. We let z denote the Galois conjugate of an element z 2 K; we set $N_{K=F}(z)$ = z z and Tr(z) = z + z. We let $!_{K=F}$ denote the character of F that corresponds to K by local class eld theory.

Let W_K W_F be the Weil groups of K and F respectively. The quotient of W_F by the commutator subgroup of W_K is the relative Weil group $W_{K=F}$

$$1 ! K ! W_{K=F} ! Gal(K=F) ! 1:$$

Let D be the unique quaternion algebra over F. By Appendix III in [29] $W_{K=F}$ can be realized as the normalizer of K in D. Thus

$$W_{K=F} = K [Kj$$

where jz = z̄j for all z 2 K and j^2 is in F, but not in the index two subgroup $N_{K=F}(K)$. Now any character of K denes a two dimensional representation of W_F

$$() = \mathrm{Ind}_K^{W_F}().$$

Lemma 1.1. Let be a non-trivial element in Gal(K=F).

1. () = (^{0}) if and only if ^{0}= or . 2. () is irreducible if and only if = .
3. If = then () = $_{12}$ where $_i$ are two characters of F such that $_i(N_{K=F}(z))$ = (z) for all z 2 K .

Proof. This is all a simple consequence of the explicit description of $W_{K=F}$. For the last, observe that the condition = , by Hilbert 90, implies that is trivial on norm one elements in K, thus the formula $_i(N_{K=F}(z))$ = (z) denes $_1$ and $_2$ on the index two subgroup of F . The two characters dier by the character $!_{K=F}$. □

The determinant of () is an Asai character of W_F denoted by As (). A character is called conjugate dual if ^{1} = . Note that this implies that is trivial on $N_{K=F}(K)$. Thus the restriction of to F is either trivial or $!_{K=F}$. Respectively, we say that is conjugate-orthogonal or conjugate-symplectic. The following lemma is now again a simple exercise, using the explicit description of $W_{K=F}$.

Lemma 1.2. Assume that is a conjugate dual character of K. Then As () = 1 if is conjugate-symplectic and As () = $!_{K=F}$ if is conjugate-orthogonal.

1.2 Representations of p-adic groups

Let G denote the group of F-points of a reductive algebraic group G. We denote the category of smooth (complex) G-representations by R(G); the set of (equivalence classes of) irreducible representations of G will be denoted by Irr(G).

We recall the various functors which play a role in the representation theory of p-adic groups. Let P be a parabolic subgroup of G with Levi decomposition P = MN. We then have the parabolic induction functor Ind_P^G, as well its normalized version, i_P^G. If is a smooth representation of G, we may consider the Jacquet functor ! $_N$, where $_N$ denotes the space of N-coinvariants of . The normalized version of the Jacquet functor will be denoted by r_P (). Recall that parabolic induction is adjoint to the Jacquet functor. First, we have the (standard) Frobenius reciprocity, which states that there is a natural isomorphism

$$\mathrm{Hom}_G(; i_P^G()) \cong \mathrm{Hom}_M(r_P(););$$

here and are representations of G and M, respectively. Equally useful is the second (Bernstein) form of Frobenius reciprocity:

$$\mathrm{Hom}_G(i_P^G();) = \mathrm{Hom}_M(; \overline{r}_P());$$

here $\overline{P}$ denotes the parabolic subgroup opposite to P. Finally, we will occasionally use the compact induction functor, which we denote by c-ind.

1.3 Cubic Jordan algebras

The space J of Hermitian symmetric 3×3 matrices over K is a Jordan algebra with multiplication

$$x \circ y = \frac{1}{2}(xy + yx) = \frac{1}{2}[(x+y)^2 - x^2 - y^2]$$

and identity 1. A typical element of J is

$$\begin{pmatrix} a & x & \overline{y} \\ \overline{x} & b & z \\ y & \overline{z} & c \end{pmatrix};$$

where $x, y, z \in K$ and $a, b, c \in F$. We let J_{ij} (for $1 \le i \le j \le 3$) denote the subspace of J consisting matrices whose entries are 0 except on the positions (i,j) and (j,i). For more details on the subject of cubic Jordan algebras, the reader can consult Chapter 38 in [17] and Chapter 4 in [20].

Let N and T denote the norm (determinant) and the (usual) trace of 3×3 matrices. Recall that $N(x) = x x^{\#}$ where $x^{\#}$ is the usual adjoint matrix to x, i.e. made of 2×2 minors of x. Let (x, y, z) be the symmetric trilinear form on J such that $(x, x, x) = 6N(x)$, that is,

$$(x, y, z) = N(x + y + z) - N(y + z) - N(x + z) - N(x + y) + N(x) + N(y) + N(z):$$

Then $T(x) = \frac{1}{2}(x, 1, 1)$ and the adjoint $x^{\#}$ can be defined as the unique element in J such that

$$(x, x, y) = (x^{\#}, y, 1)$$

for all $y \in J$. A basic fact of linear algebra is that any $x \in J$ satisfies the characteristic polynomial

$$x^3 - T(x)x^2 + T(x^{\#})x - N(x) = 0:$$

Multiplying this equation by $x^{\#}$ and then factoring out $N(x)$ gives

$$x^2 - T(x)x + T(x^{\#}) - x^{\#} = 0$$

for all $x \in J$. This implies that x^2 and thus the Jordan multiplication is completely determined by the cubic form N and the identity 1. It follows at once that the group of automorphisms of the Jordan algebra is equal to the group of automorphisms of the cubic pointed space $(N, 1)$.

More generally, if $e \in J$ such that $N(e) = ee^{\#} = 1$, we can define a Jordan multiplication

$$x \circ y = \frac{1}{2}(xe^{\#}y + ye^{\#}x):$$

This is a Jordan algebra J_e with identity e, and trace

$$T_e(x) := \frac{1}{2}(x, e, e) = \frac{1}{2}(e^{\#}, x, 1)$$

The trace pairing is given by

$$(x,y)\mapsto T_e(x\circ y)=(e,e,x)(e,e,y)/4-(e,x,y)=T_e(x)T_e(y)-(e,x,y).$$

For every $x\in J$ we can define $x^{\#_e}$ by

$$(x,x,y)=(x^{\#_e},y,e)$$

for all $y\in J$. Although we shall not need it, we record that $x^{\#_e}=ex^{\#}e$. Again, J_e is determined by N and e, thus the automorphism group of J_e is the group of automorphisms of the cubic pointed space (N,e).

Elements of J are Hermitian symmetric matrices, in particular, any e such that $N(e)=1$ defines a symmetric Hermitian form on K^3 of discriminant one. Since F is p-adic, any two such Hermitian spaces are isomorphic. It follows that the Jordan algebras J_e are isomorphic. In the rest of this article, it will be convenient to fix

$$e=\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \end{pmatrix}.$$

To simplify notation, we will write J instead of J_e.

Finally, we let L_J denote the group of linear transformations of J which preserve N. Then

$$L_J=(\{g\in GL_3(K):\det(g)\in U(1)\}/U(1))\rtimes \mathbf{Z}/2\mathbf{Z}.$$

Here $U(1)$ is embedded into $GL_3(K)$ diagonally. The non-trivial element of $\mathbf{Z}/2\mathbf{Z}$ acts by transposition, and the action of $GL_3(K)$ on J is given by

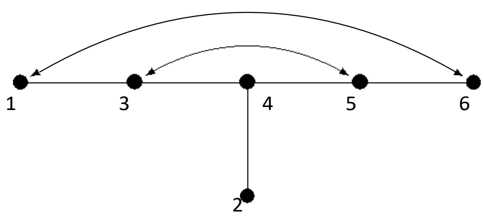
$$(g,X)\mapsto gXg^*,\quad \text{for } g\in L_J \text{ and } X\in J.$$

Here, g^* denotes the conjugate-transpose of g.

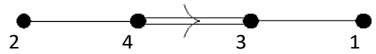
1.4 Groups

Here we describe the various groups that appear in this paper, including the quasi-split group H of type $E_{6,4}$ and the dual pair $G\times G'$ we wish to study.

Let H denote the adjoint quasi-split group of type $E_{6,4}$ defined over F, with splitting field K; its Dynkin diagram is given by



The relative Dynkin diagram is



The groups we consider are best described on the level of Lie algebras. Here we follow the construction in [26]. Let $\mathfrak{h}$ denote the Lie algebra of H. By covering the vertex α_4 in the above diagram, we see that $\mathfrak{h}$ contains a subalgebra $\mathfrak{h}_0 = \mathfrak{sl}_3 \oplus \mathfrak{l}$, where $\mathfrak{l} = \mathfrak{sl}_3(K)$ is the Lie algebra of L_J. Under the adjoint action of $\mathfrak{h}_0$, $\mathfrak{h}$ decomposes as

$$\mathfrak{h} = \mathfrak{sl}_3 \oplus \mathfrak{l} \oplus W \otimes J \oplus W^* \otimes J^*.$$

Here $W = \langle e_1, e_2, e_3 \rangle$ denotes the standard representation of $\mathfrak{sl}_3$, and $W^* = \langle e_1^*, e_2^*, e_3^* \rangle$ denotes its dual. We often use the trace pairing to identify J^* with J. The Lie bracket relations are described in [26].

Using the above description of $\mathfrak{h}$, we may now describe the dual pair $G_2 \times (PU_3(K) \rtimes \mathbf{Z}/2\mathbf{Z})$. We let $\mathfrak{g}^0$ denote the centralizer of e in $\mathfrak{l}$:

$$\mathfrak{g}^0 = C_{\mathfrak{l}}(e).$$

We then set

$$\mathfrak{g} = C_{\mathfrak{h}}(\mathfrak{g}^0) = \mathfrak{sl}_3 \oplus W \otimes e \oplus W^* \otimes e.$$

We let G and G^0 denote the closed subgroups of H which correspond to $\mathfrak{g}$ and $\mathfrak{g}^0$, respectively. Then $G \times G^0$ is a dual pair inside H. Furthermore, G is a split group of type G_2, and G^0 is isomorphic to $PU_3(K) \rtimes \mathbf{Z}/2\mathbf{Z}$. Indeed, we may describe G^0 directly as the subgroup of L_J which fixes e.

The proof of Howe duality will require us to consider another dual pair inside H, which we now describe. Let E be an etale cubic F-algebra. We consider the set of E-isomorphism classes of embeddings $E \hookrightarrow J$. This set is in bijection with the set of (E-isomorphism) classes of twisted composition algebras C such that $J = E \oplus C$; see Theorem 1.1 in [8] for additional details. Fixing such a C, we let $i_C : E \to J$ denote an embedding in the corresponding isomorphism class; note that this also gives us embedding of E into J. Let $G^0_{E,C}$ denote the subgroup of G^0 fixing i_C. The centralizer of $G^0_{E,C}$ in $\mathfrak{h}$ contains

$$\mathfrak{sl}_3 \oplus \mathfrak{t}_E \oplus W \otimes E \oplus W^* \otimes E,$$

where $\mathfrak{t}_E$ is the Lie algebra of trace 0 elements in E, and E is embedded into J using i_C. The above Lie algebra is isomorphic to $\mathrm{Lie}(G_E)$, where G_E is the simply-connected quasi-split group Spin_8^E. We thus get the dual pair $G_E \times G^0_{E,C}$ inside H.

1.5 Minimal representations and theta correspondence

We will be interested in studying the minimal representation Π of H. We recall one possible definition here. Let π be an irreducible representation of H. A result of Harish-Chandra then says that the character distribution of π can be expressed as

$$\Theta_\pi = \sum_{\mathcal{O}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}$$

where the sum is taken over all the nilpotent H-orbits in h, and ^O is the Fourier transform of a (suitably normalized) H-invariant measure on O. There exists a minimal non-trivial orbit O_{min} in h(F). Assuming O_{min} \ h consists of a single H-orbit O_{min}, we have the following

Denition 1.3. We say that is minimal if

$$= c_0 + \hat{}_{O_{min}}$$

For a detailed exposition of minimal representations and a construction of for exceptional group, we refer the reader to [7].

Our goal is to study the restriction of to the dual pair G G^0 introduced above, and the exceptional theta correspondence which arises in this way. Fixing an irreducible representation of G, the maximal -isotypic quotient of is of the form

$$();$$

for an admissible representation () of G^0 [21, Lemme 2.III.4]. This is the so-called big theta lift of . Of course, one may start from 2 Irr(G^0) to obtain the big theta lift () in the same way.

2 Parabolic subgroups

In this section we describe the three maximal parabolic subgroups of H we consider in this paper.

2.1 Three-step parabolic

The rst parabolic subgroup we consider is the maximal parabolic $P_1 = M_1N_1$ which corresponds to the vertex $_4$ of the Dynkin diagram. On the level of Lie algebras, it can be constructed as follows: let

$$s = @^1 \quad 1 \quad A \; 2 \; sl_3:$$
0 1 2

Now set h(i) = fx 2 h : [s; x] = ixg. Then the Lie algebra of P_1 is $p_1 = m_1 + n_1$, where

$$m_1 = h(0) = @ \quad A \quad l$$
0 1

and $n_1 = h(1) + h(2) + h(3)$ with

$$h(1) = he_1; e_2i$$
$$J; \; h(2) = {}_3hei$$
$$J; \quad 0 \quad 1$$
$$h(3) = @ \quad A \quad sl_3:$$

We also have

$$\mathfrak{h}(1) = \langle e_1; e_2\rangle \otimes J; \quad \mathfrak{h}(2) = \langle e_3\rangle \otimes J:$$

Since N_1 is a 3-step nilpotent group, we call P_1 the 3-step parabolic. We let Ω_i denote the minimal non-trivial M_1-orbit on $\mathfrak{h}(i)$, $i = 1; 2$.

Setting $\mathfrak{u}_1(i) = \mathfrak{h}(i) \cap \mathfrak{g}$ for $i = 1; 2$, we see that

$$\mathfrak{u}_1(1) = \langle e_1; e_2\rangle \otimes e; \quad \mathfrak{u}_1(2) = \langle e_3\rangle \otimes e; \quad \mathfrak{u}_1(3) = \mathfrak{h}(3):$$

Looking at the intersection of P_1 with $G \times G^0$, we get

$$(G \times G^0) \cap P_1 = Q_1 \times G^0:$$

Here $Q_1 = L_1U_1$ is the maximal parabolic subgroup of $G = G_2$; we identify the Levi factor L_1 with GL_2 so that the action on $\mathfrak{u}_1(1)$ is the standard representation.

Now let V_i be the orthogonal complement of $\mathfrak{u}_1(i)$ in $\mathfrak{h}(i)$ (for $i = 1; 2$) with respect to the Killing form. Then

$$V_1 = \langle e_1; e_2\rangle \otimes J_0 \quad V_2 = \langle e_3\rangle \otimes J_0;$$

where J_0 denotes the set of all elements X in J such that $tr(Xe) = 0$, and J_0 is identied with J_0 using the trace pairing. We need to describe $\Omega_i \cap V_i$, for $i = 1; 2$. Letting $r(X)$ denote the rank (over K) of $X \in J$, we have (cf. [6], Lemma 4.1)

Lemma 2.1. *The group $L_1 \times G^0$ acts transitively on $\Omega_i \cap V_i$, $i = 1; 2$. Furthermore,*

$$\Omega_1 \cap V_1 = \{w \otimes X : w \in \langle e_1; e_2\rangle; X \in J_0; r(X) = 1\}$$
$$\Omega_2 \cap V_2 = \{e_3 \otimes X : X \in J_0; r(X) = 1\}:$$

2.2 Heisenberg parabolic

Here we consider the maximal parabolic $P_2 = M_2N_2$ which corresponds to the vertex α_2 of the Dynkin diagram. On the level of Lie algebras, it can be constructed as follows: let

$$s = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \in \mathfrak{sl}_3:$$

Again, set $\mathfrak{h}(i) = \{x \in \mathfrak{h} : [s; x] = ix\}$. We intentionally abuse notation by reusing s and $\mathfrak{h}(i)$, not only to reduce the number of unnecessary symbols, but also to emphasize the analogy in our constructions related to dierent parabolics. Since we never use these symbols for dierent parabolics at the same time, there is no fear of confusion.

The Lie algebra of P_2 is $p_2 = m_2 + n_2$, where

$$m_2 = h(0) = @ \quad A \quad l \quad he_2 i \quad J \quad he_2 i \quad J;$$

(with bracket markers 0 and 1 printed above)

and $n_2 = h(1) + h(2)$ with

$$h(1) = @ \quad A \quad he_1 i \quad J \quad he_3 i \quad J;$$

$$h(2) = @ \quad 1 \quad A \quad sl_3:$$

(with bracket markers 0 and 1 printed above)

We also note that

$$h(\;\; 1) = @ \quad A \quad he_3 i \quad J \quad he_1 i \quad J:$$

(with bracket markers 0 and 1 printed above)

We often refer to P_2 as the Heisenberg parabolic, because its unipotent radical N_2 is a Heisenberg group with center $Z = h(2)$, attached to the symplectic space $N_2 = Z = h(1)$. We let denote the minimal non-trivial M_2-orbit on $h(\;\; 1)$. It is the orbit of a highest weight vector.

We have

$$u_2(1) = g \setminus h(1) = F \quad he_1 \quad ei \quad he \quad ei \quad F; \qquad 3$$

we identify the intersection $h(1) \setminus sl_3$ with $F \;\; F$. Looking at the intersection of P_2 with $G \;\; G^0$, we get

$$(G \;\; G^0) \setminus P_2 = Q_2 \;\; G^0:$$

Here $Q_2 = L_2 U_2$ is the maximal parabolic subgroup of $G = G_2$ whose Levi factor L_2 we identify with GL_2 so that its action on $u_2(1)$ is the symmetric cube representation twisted by $jdetj^{\;\;1}$ (see Section 3 in [3]). Once again, U_2 is a Heisenberg group (with center Z) attached to the space $u_2(1)$.

Now let V be the orthogonal complement of $u_2(1)$ in $h(\;\; 1)$ with respect to the Killing form. Then

$$V = he_3 i \quad J_0 \quad hei \quad J \quad J_0 \quad J_0: \qquad 1 \quad 0 =$$

Once again, we need to describe $\setminus V$. Following the proof of Proposition 7.4 in [19], one shows the following

Lemma 2.2. We have

$$\setminus V = f(X; Y) \; 2 \; J_0 \;\; J_0 : \qquad r(X); r(Y) \;\; 1; \qquad \dim hX; Y i = 1g:$$

Furthermore, $L_2 \;\; G^0$ acts transitively on this set.

2.3 B_3 parabolic

Finally, we consider the maximal parabolic $P' = M'N'$ which corresponds to the vertex $_1$ of the relative Dynkin diagram. Set

$$s = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \in sl_3(K)$$

and $\mathfrak{h}(i) = \{x \in \mathfrak{h} : [s, x] = ix\}$. Then the Lie algebra of P' is $\mathfrak{p}' = \mathfrak{m}' + \mathfrak{n}'$. Here

$$\mathfrak{m}' = \mathfrak{h}(0) = sl_3 \oplus l(0) \otimes W \ \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \ \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} W$$

and $\mathfrak{n}_1 = \mathfrak{h}(1) + \mathfrak{h}(2)$, with

$$\mathfrak{h}(1) = l(1) \otimes W \otimes J_{12} \oplus W \otimes J_{23}, \quad \mathfrak{h}(2) = l(2) \otimes W \otimes J_{11} \oplus W \otimes J_{33}.$$

We also have

$$\mathfrak{h}(-1) = l(-1) \otimes W \otimes J_{23} \oplus W \otimes J_{12}; \quad \mathfrak{h}(-2) = l(-2) \otimes W \otimes J_{33} \oplus W \otimes J_{11};$$

where

$$l(-1) = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \in sl_3(K) \quad \text{and} \quad l(-2) = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \in sl_3(K):$$

We let Ω'_i denote the minimal non-trivial M_1-orbit on $\mathfrak{h}(-i)$, for $i = 1, 2$. The intersection of P' with $G \times G'$ is

$$(G \times G') \cap P' = G \times B'.$$

Here B' denotes the semidirect product of $\mathbf{Z}/2\mathbf{Z}$ with the Borel subgroup consisting of all upper-triangular matrices in $PU_3(K)$. There is a Levi decomposition $B' = T'U'$ with $T' = K^\times \rtimes \mathbf{Z}/2\mathbf{Z}$; we identify the diagonal torus in $PU_3(K)$ with $K^\times$ using the isomorphism

$$\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \mapsto \frac{a}{b}$$

(a, b, c satisfy $a\bar{c} = 1$ and $b\bar{b} = 1$).

Just like before, we let $u^0(i) = h(i) \cap g^0$ for $i = 1; 2$. We get

$$u^0(1) = \{ \begin{pmatrix} 0 & & 1 \\ & n & \end{pmatrix} \bar{n} \in sl_3(K)\} = K$$

$$u^0(2) = \{ \begin{pmatrix} 0 & & 1 \\ & y & \end{pmatrix} \in sl_3(K) : Tr(y) = 0\}:$$

We let V_i^0 be the orthogonal complement of $u^0(i)$ in $h(\ i)$ (for $i = 1; 2$) with respect to the Killing form. Direct computation shows that we have

$$V_1^{\complement} = \{ \begin{pmatrix} 0 & & 1 \\ x & & \end{pmatrix} : x \in K\} \quad W$$

J_{23} W

$$J_{12}; \begin{pmatrix} 0 & & 1 \\ & x & \end{pmatrix}$$

$$V_2^{\complement} = \{ \begin{pmatrix} \\ \end{pmatrix} : y \in F\} \quad W$$

J_{33} W

$J_{11} : y$

It is convenient to use the following identications (cf. [6, p.137]):

$$V_1^{\complement} = \mathbf{O}_0 \quad K; \qquad V_2^{\complement} = \mathbf{O}_0 \quad F;$$

here we use $\mathbf{O}_0$ to denote the space of traceless octonions. Then $G = G_2$ acts naturally on $\mathbf{O}_0$, whereas $z \in K \quad T^0$ acts by $\frac{1}{z}$ and $1=N_{K=F}(z)$ on K and F, respectively. As before, we want to describe the GT^0 orbits on $_i^0 \cap V_i^0$ for $i = 1; 2$. However, a direct computation now shows that $_1 \cap V_1 = ;$. On the other hand, we have (cf. [6], Lemma 2.4.11) Lemma 2.3. The group $G \quad T^0$ acts transitively on $_2^0 \cap V_2^0$.

We close this section with the following

Remark 2.4. One can work in a more general setting, starting with a Jordan algebra J of Hermitian symmetric $3 \quad 3$ matrices with coecients in a composition algebra C. In particular, we have an exceptional Lie algebra

$$h = (sl_3(F) \quad l) \quad W \quad J \quad (W \quad J):$$

Observe that $sl_3(F) \quad l$, where $x \in sl_3(F)$ acts on $y \in J$ by $xy + yx$ where y is the transpose of y. This works even when C is the non-associative algebra of octonions.

In particular, one can dene the parabolic subgroup B^0 in $G^0 = Aut(J)$ starting with the same choice of s as above (note that $s \in sl_3(F) \quad l$). Let U^0 be the unipotent radical of B^0, and $U^0(2)$ the center of U^0. Then $U^0(2) = C_0$, trace 0 elements in C, and $U^0=U^0(2) = C$. We will need this in x4 where we prove Howe duality.

3 Jacquet modules

Recall that is the minimal representation of H. Our goal in this section is to compute $_{U_1}$, $_{U_2}$, and $_{U^0}$.

3.1 Three-step parabolic

To compute $_{U_1}$, we begin by looking at $_{N_1}$. Recall that N_1 is a three-step nilpotent group: we have

$$f1g = N_1(4) \ N_1(3) \ N_1(2) \ N_1(1) = N_1$$

with $N_1(i)=N_1(i+1) = h(i)$. This gives us a ltration of ,

(3.1) $$f0g = {}_4 \ {}_3 \ {}_2 \ {}_1 \ {}_0 = ;$$

where $_i = _{i+1} = ({}_i)_{N_1(i+1)}$. We have

$$0 \,!\ {}_3 \,!\ \,!\ ={}_3 \,!\ 0$$
$$0 \,!\ {}_2={}_3 \,!\ ={}_3 \,!\ ={}_2 \,!\ 0 \quad 0 \,!\ {}_1={}_2 \,!$$
$$={}_2 \,!\ ={}_1 \,!\ 0:$$

We need to compute $=_1$, $_1=_2$, and $_2=_3$. The rst quotient is simply $_{N_1}$; the remaining two can be computed using the work of Mglin and Waldspurger [22]. We provide only a rough outline here; see e.g. [6] or [19] for additional details.

Recall that $_1$ is the minimal M_1-orbit in h(1). Let f_1 be an arbitrary element of $_1$, and denote by M_{f_1} its stabilizer in M_1. Then f_1 denes a character $_{f^1}$ on $N_1=N_1(2)$. Then [22] shows that the space $_{N_1;}{}^{f_1}$ (i.e. the maximal quotient of on which N_1 acts by $_{f_1}$) is 1-dimensional; M_{f_1} acts on it by a character $_1$. In short, we get

$$_1=_2 = \text{c-ind}_{M_{f_1}N_1}^{P_1}(_1 \ _{f_1}) = C_c(\ _1):$$

We compute $_2=_3$ similarly. We choose an arbitrary element f_2 2 $_2$ and we let M_{f_2} denote its stabilizer in M_1. Again, the results of [22] (see also [6, x5]) show that f_2 denes a certain Heisenberg representation, which we denote by W_{f_2}. We get

$$_2=_3 = \text{c-ind}_{M_{f_2}N_1}^{P_1}(W_{f_2}) = C_c(\ _2; W_{f_2}):$$

Having computed the N_1-coinvariants, we proceed to investigate the U_1-coinvariants. The unipotent radical U_1 of Q_1 inherits the ltration from N_1:

(3.2) $$f0g = U_1(4) \ U_1(3) \ U_1(2) \ U_1(1) = U_1;$$

where $U_1(i) = U_1 \setminus N_1(i)$. In particular, $U_1(3) = N_1(3)$. We apply the U_1-coinvariants functor to the exact sequences above. From the rst one, we see that $_{U_1} = (=_3)_{U_1}$. The remaining two sequences become

$$0 \,!\ ({}_2={}_3)_{U_1} \,!\ {}_{U_1} \,!\ (={}_2)_{U_1} \,!\ 0$$
$$0 \,!\ ({}_1={}_2)_{U_1} \,!\ (={}_2)_{U_1} \,!\ (={}_1)_{U_1} \,!\ 0:$$

Thus, to determine $_{U_1}$, we need to describe $(=_1)_{U_1}$, $(_1=_2)_{U_1}$, and $(_2=_3)_{U_1}$.

First, notice that $(=_1)_{U_1} = (_{N_1})_{U_1} = _{N_1}$. The N_1-coinvariants can be computed following [27, x4], and the exponents of have been determined in Proposition 8.4 of [7]. As an L_1 G^0-module,

$$_{N_1} = \text{jdetj}^{\frac{7}{2}} \mathbf{1}_{A_2} \ !_{K=F} \ \text{jdetj}^{\frac{7}{2}} \ _{A_1}:$$

Recall that the Levi factor M_1 consists of two parts (which correspond to the two parts of the F_4 diagram one obtains by removing the vertex ${}_4$): A_2 and A_1. Here $\mathbf{1}_{A_2}$ is the trivial representation of A_2 (in this case, $SL_3(K)$), whereas ${}_{A_1}$ is a principal series representation of A_1 (i.e. $SL_2(F)$). Furthermore, jdetj denotes the standard determinant of $L_1 = GL_2$.

Next, we consider $({}_1={}_2)_{U_1}$. Just like in [19, Lemma 2.2], we obtain

$$({}_1={}_2)_{U_1} = C_c({}_1 \setminus V_1):$$

Recall that ${}_1 \setminus V_1^{1\ 2\ 0} = \{w\ X : w \in \langle e; ei; X \in J; r(X) = 1\}$. The stabilizer of a line in ${}_1 \setminus V_1$ (excluding 0) is a product of Borel subgroups. For the sake of concreteness, we consider the line through e_1 xx, where $x = (1;0;0)$ (once more, we identify J with J_0). The stabilizer of e_1 in $L_1 = GL_2$ is the subgroup $B = TU$ consisting of all lower-triangular matrices in GL_2 (recall that we are considering the action of GL_2 on W, the dual of the standard representation); here T denotes the diagonal torus. The stabilizer of xx is the Borel subgroup $B^0 = T^0U^0$ of G^0 introduced in x2.3. The group T T^0 acts transitively on the above line, which we identify with F: The action of

$$a\quad b\ ;z \in T\ K$$

on $C_c(F)$ is translation by $a^{1}\ N_{K=F}(z)$ (and $\mathbf{Z}=2\mathbf{Z}$ acts trivially).

We deduce that

$$C_c({}_1 \setminus V_1) = i^{L_1 G^0}_{\overline{B} B}({}_1^{\ c}\ C_c(F))$$

(normalized induction) where ${}_1$ is a character of the diagonal torus in L_1 GL_2 which is yet to be determined.

Finally, we determine $({}_2={}_3)_{U_1}$. Recall that ${}_2 \setminus V_2 = \{e_3\ X : X \in J_0; r(X) = 1\}$ is a single L_1 G^0-orbit. We simplify the notation by identifying $\langle e_3\rangle$ J with J, keeping in mind that $L_1 = GL_2$ acts on ${}_2 \setminus V_2$ by det. We start by observing that

$$C_c({}_2; W_{f_2})_{U_1(2)} = C_c({}_2 \setminus V_2; W_{f_2}):$$

Notice that $(L_1\ G^0) \setminus M_{f_2} = RU$, where

$$R = \{(g;(z;)) \in L_1\ (K \rtimes \mathbf{Z}=2\mathbf{Z}) : \det(g) = N_{K=F}(z)\}$$

and U is the unipotent radical. Therefore $C_c({}_2 \setminus V_2; W_{f_2}) = \text{c-ind}^{L_1G^0}_{RU} W_{f_2}$ and thus

$$({}_2={}_3)_{U_1} = \text{c-ind}^{L_1G^0}_{RU}((W_{f_2})_{U_1}):$$

It remains to determine $(W_{f_2})_{U_1}$; to do that, we need an explicit model for W_{f_2}. With this in mind, we choose $f_2 = xx \in J$, with $x = (1;0;0)$. Following [22] (see also [6]), we consider the alternating form on $h(1) = \langle e_1; e_2\rangle$ J given by

$$(v\ X; w\ Y) = (v;w)\ (f_2;X;Y):$$

Here $(v;w)$ is the standard symplectic form on $\langle e_1; e_2\rangle$, and $(f_2; X; Y)$ is the natural trilinear form on J. With our choice of f_2, the kernel of the bilinear form $(f_2; X; Y)$ is

$$= \{\begin{pmatrix} a & x & y \\ & & x \\ & & \end{pmatrix} \in J\} : y$$

We let $^{?}$ denote the orthogonal complement of in J:

$$^{?} = \{\begin{pmatrix} 0 & 0 & 0 \\ 0 & b & z \\ 0 & \bar{z} & c \end{pmatrix} \in J\} : 0$$

The corresponding quadratic form is given by $2 \begin{vmatrix} a & z \\ \bar{z} & b \end{vmatrix}$. We x the maximal isotropic subspace consisting of elements of the form

$$e_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & & 0 \\ 0 & 0 & \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ & & z \\ 0 & 0 & \end{pmatrix} + e_1, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_1 & 0 & 0 \end{pmatrix} + b_2 e_2 : 0 \quad z \quad 0 \quad 0 \quad 0$$

With this choice of polarization, the action of $U_1=U_1(2) = \langle e_1; e_2\rangle$ hei is given by

$$(u)f(z; b_1; b_2) = (u_1 b_2 \quad u_2 b_1) f(z; b_1; b_2);$$

where $u = (u_1 e_1 + u_2 e_2)$ e. We see that $U_1=U_1(2)$ acts trivially on functions supported on the subspace

$$\{e_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & & z \\ 0 & z & 0 \end{pmatrix} : z \in K\} : 0$$

It follows that

$$(W_{f_2})_{U_1=U_1(2)} = W;$$

where W is the Heisenberg representation associated with the symplectic space $\langle e_1; e_2\rangle$ K; here we identify $z \in K$ with

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & \bar{z} & 0 \end{pmatrix}:$$

Unraveling the denitions, we see that $w \in K \quad T^0$ acts on $z \in K$ by

$$(w; z) \mapsto \frac{z}{w};$$

and $\mathbf{Z}=2\mathbf{Z}$ acts by Galois conjugation. Recall that we also have a GL_2 action on the orbit $_2 \setminus V_2$: an element $g \in L_1 = GL_2$ acts by $\det(g)^{-1}$. In summary, we have

$$(_2=_3)_{U_1} = \text{c-ind}^{L_1 G^0}_{RU} W:$$

We now recall the work of Roberts [24]: there is a Weil representation ! of R which induces to a representation of $\tilde{R} = L_1 \quad T^0$. The correspondence which arises from can be thought of as a similitude version of the usual SL_2 $SO(K)$ corrsependence. As noted before, we have $M_{f_2} \setminus \tilde{R} = R$, so we are precisely in the situation studied in [24].

An application of Schur's Lemma now shows that the action of R on W is a twist (by a character of R) of the action of R on !. However, as one checks directly, every character of R is a restriction of a character of $\tilde{R}$. We thus get

$$(_{2=3})_{U_1} = i_{L_1}{}^{B}\!(^{1G}{}^{c}\ ^{\dagger});$$

where is a character yet to be determined. In fact, in x3.4, we show that is the trivial character. Thus, we have

Proposition 3.1. As a representation of L_1 G^0, $r_{U_1}()$ has a ltration with successive (top to bottom) subquotients

(T1) $\ _{Q_1}^{1=2}\ _{N_1}$ = jdetj $\mathbf{1}_{A_2}$ $!_{K=F}$ jdetj $_{A_1}$ (M1)

$i^{L_1 G^0}(_1$ $\overline{B}B$ ^{c}

$C_c(F))$

(B1) $i^{L_1 G^0}_{L_1 B}(^{c}\ ^{\dagger})$:

Recall that $\mathbf{1}_{A_2}$ is the trivial representation of A_2, whereas $_{A_1}$ is a principal series representation of A_1. Furthermore, jdetj is the standard determinant of $L_1 = GL_2$ and $_{Q_1}$ = jdetj is the modular character of Q_1. The center of $L_1 = GL_2$ acts trivially on $_{A_2}$ and $_{A_1}$. In x3.4, we show that $_1 = 1$ $!_{K=F}$.

3.1.1 The Fourier{Jacobi period

We digress slightly to describe the Fourier{Jacobi period of the minimal representation . Although it is not required for the main results of the present paper (i.e. for the proof of Howe duality), the Fourier{Jacobi period becomes useful in various similar settings. Since the computation is similar to the one we just performed to obtain $_{U_1}$, we take a moment to describe it here.

Recall the ltration (3.2): U_1 is a three-step nilpotent group, and the quotient U_1=$U_1(3)$ is a three-dimensional Heisenberg group. Let be a character of its center $U_1(2)$=$U_1(3)$. Our goal is to describe the space of -twisted coinvariants

$$_{U_1(2);}$$

i.e. the maximal quotient of on which $U_1(2)$=$U_1(3)$ acts by . Notice that this is (in addition to being a G^0 module) a module for the Jacobi group, F J = Q_1^{der}=$U_1(3)$. Here Q_1^{der} denotes the derived group of Q_1.

Using the notation from (3.1), we have $_{U_1(2);}$ = $(_{2=3})_{U_1(2);}$. Recall that $_{2=3}$ = $C_c(_2; W_{f_2})$. To nd the $U_1(2)$ coinvariants we looked at $C_c(_2 \setminus V_2; W_{f_2})$, where V_2 was the orthogonal complement of $u_1(2)$ in h(2); this was identied with the space J_0 of traceless elements in J. However, we are now looking for the twisted coinvariants. We get

$$C_c(_2; W_{f_2})_{U_1(2);} = C_c(_2 \setminus J_1; W_{f_2}):$$

Here J_1 is the set of trace 1 elements in J. One checks that G^0 has two orbits on $_2 \setminus J_1$ (the set of rank 1, trace 1 elements in J). Thus, as a representation of G^0 = $PU_3(K) \rtimes \mathbf{Z}/2\mathbf{Z}$,

the space $C_c($
$_2 \backslash J_1; W_{f_2})$ splits into a direct sum of two induced representations. Indeed, one choice of representatives for these orbits is

$$i = @ \quad 1 \quad A; \qquad j = @ \quad " \quad 1{=}2 \quad A \quad 1{=}2 \quad 1{=}(4")$$

where " is an element of F which is not in the image of the norm map N : K ! F. (Indeed, j cannot be written as xx for some x 2 K^3, which shows that it is not in the same orbit as i). The stabilizer of i (resp. j) in $PU_3(K)$ is a unitary group in two variables, which we denote by $U(2)_i$ (resp. $U(2)_j$). We thus have

$$_{U_1(2);} \;=\; i^{G^0}_{U(2)_i o \mathbf{z}=2\mathbf{z}}(W_i) \quad i^{G^0}_{U(2)_i o \mathbf{z}=2\mathbf{z}}(W_j):$$

Here W_i (resp. W_j) denotes the corresponding Weil representation, i.e. the ber at i (resp. j).

For example, the stabilizer of i in $PU_3(K)$ consists of all elements of the form

$$@ \quad 1 \quad A \; 2 \; U_3(K):$$

Thus we may identify the group $U(2)_i$ with the unitary group

$$2 \; GL_2(K) : \quad g \quad {}_1{}^1 \quad g = \quad {}_1{}^1 \quad \text{fg g}$$

in the obvious way. We obtain an explicit model of W_i the same way we found W_{f_2} above. Here it is convenient to x f_2 = i; then

$$= f@x - \begin{matrix} x \\ b \\ \overline{z} \end{matrix} \; zA \; 2 \; Jg \quad \text{and} \quad ^? = f@ \begin{matrix} a & y \\ y & c \end{matrix} {}^{-1}A \; 2 \; Jg:$$

Here $^?$ can be identied with the space I = f $\begin{matrix} a & \overline{y} \\ y & c \end{matrix}$ g of 2 2 Hermitian matrices. Thus the representation W_i can be realized on the space $C_c(I)$, where the action of $U(2)_i$ on A 2 I is given by (g; A) ! gAg.

3.2 Heisenberg parabolic

We compute $_{U_2}$ using the same general approach. Since N_2 is a Heisenberg group, there are only two subquotients we need to consider: $=_1$ and $_1=_2$. Just like in the case of the three-step parabolic, we have $(=_1)_{U_2}$ = $(_{N_2})_{U_2}$ = $_{N_2}$. Again, we may compute the N_2-coinvariants following [27, x4]. We get

$$_{N_2} = \text{jdetj}^{\frac{3}{2}}\,{}_{C_3} \quad !_{K=F}\,\text{jdetj}^2:$$

Here $_{C_3}$ denotes the minimal representation of M_2 (corresponding to the F_4 diagram with the $_2$ vertex removed).

The description of $(_1=_2)_{U_2}$ is similar to the one we had in the three-step case: we have

$$(_1=_2)_{U_2} = C_c(\cap V):$$

Recall that $\cap V$ is a single orbit for $L_2 \times G^0$, and that

$$\cap V = f(X; Y) 2 J_0 \times J_0 : \quad r(X); r(Y) \; 1; \quad \dim hX; Y i = 1g:$$

For the sake of concreteness, we consider the line through $e_3 \; xx \; 2 \; \cap V$, where $x = (1; 0; 0)$. The stabilizer of this line is again the product $B \times B^0$ of Borel subgroups (here we are abusing notation by using $B = TU$ to denote the group of all lower-triangular matrices, but this time in L_2). The group $T \times T^0$ acts transitively on the above line, which we identify with F: The action of a b $;z \; 2 \; T \times K$ on $C_c(F)$ is translation by $a \; N_{K=F}(z)$. We deduce that

$$C_c(\cap V) = i^{L_1 \times G^0}_{B \times B^0}(_2 \otimes C_c(F))$$

where $_2$ is a character yet to be determined. To summarize, we have

Proposition 3.2. As a representation of $L_2 \times G^0$, $r_{U_2}()$ has a ltration with successive (top to bottom) subquotients

(T2) $^{1=2}_{Q_2} N_2 = {}_{C_3} \otimes !_{K=F} \; |\det|^{\frac{1}{2}}$: (B2) $i^{L_2 \times G^0}_{B \times B^0}(_2 \otimes C_c(F))$:

Here $_{Q_2} = |\det|^3$ denotes the modular character of Q_2. The center of $L_2 = GL_2$ acts trivially on $_{C_3}$. In x3.4 we show that $_2 = 1 \otimes !_{K=F}$

3.3 B_3 parabolic

This case is entirely analogous to the previous two, so we just briey sketch the results.

First, the top part in the ltration of $_{U^0}$ is simply $_{N^0}$. Secondly, recall that in this case $_1$ does not intersect V_1, so the middle part of the ltration vanishes. The computation of the bottom (subrepresentation) part is equivalent to the one we described in detail in x3.1; in fact, the bottom part is induced from the same representation as the bottom part in Proposition 3.1. We omit the details and simply state the results:

Proposition 3.3. As a representation of $G \times T^0$, $r_{U^0}()$ has a ltration with successive (top to bottom) subquotients

(T3) $^{1=2}_{B} N^0 = {}_{B_3} \otimes |N_{K=F}(z)|$ (B3) $i^{G \times T}_{L_1 \times T^0}(\dagger)$:

Here $_{B_3}$ denotes the minimal representation of the Levi factor M^0, and $_{B^0} = |N_{K=F}(z)|^2$ is the modular character of B.

We take a moment to describe the restriction of the representation $_B$ to G_3 T^0. Recall that M^0 is the Levi factor of the parabolic P^0 which corresponds to the D_4 part of the Dynkin diagram (i.e. B_3 in the relative diagram). The derived group of M^0 is D_4^E, the simply connected quasi-split form of $Spin_8$ attached to the etale cubic algebra $E = K + F$; these groups are described in detail in Section 2 of [8]. By looking at the exponents, one veries that the restriction of the representation (T3) to $Spin_8$ is the minimal representation. Note that K T^0 acts trivially on $_{B_3}$. However, the action of the Galois group $\mathbf{Z}=2\mathbf{Z}$ is non-trivial; in fact, this is precisely the situation studied in [15] and [5]. There is a dual pair G_2 S_E inside D_4^E (here S_E denotes the twisted form of S_3 attached to E). In [5], the authors use the correspondence arising in this way to construct the so-called cubic unipotent A-packets of G_2. In our case, $S_E = \mathbf{Z}=2\mathbf{Z}$, so the corresponding A-packet contains two elements. One of them is a supercuspidal representation; the other is the Langlands quotient of $i^{G_2}_{Q_1}(jdetj$), with equal to the tempered representation $1 !_{K=F}$ of GL_2. See Proposition 6.2 of [5] for a detailed description of local A-packets arising in this way.

3.4 Filling in the details

In this section, we determine the characters $_1$, $_2$, and that appear in the ltrations discussed above (recall that is introduced in the discussion preceding Proposition 3.1).

Both the bottom piece of $r^0_U()$ (B3) and the bottom piece of $r_{U_1}()$ (B1) are in-duced from the Weil representation . The similitude correspondence between $GL_2(F)$ and $GO(K)$ = K o $\mathbf{Z}=2\mathbf{Z}$ established by amounts to the usual base change K ! $GL_2(F)$. Let be a character of K. By Lemma 1.1, there are two possibilities:

(i) = . In this case, extends to two characters of K o $\mathbf{Z}=2\mathbf{Z}$, only one of which appears in the correspondence. We label that character $^+$, and we let be the other one. Then $^+$ lifts to the principal series $_1$ $_2$ (see Lemma 1.1).

(ii) = . In this case, lifts to a cuspidal representation of $GL_2(F)$.

See Section 7 of [25] for a brief account of this correspondence.

We now prove that the character introduced in the discussion preceding Proposition 3.1 is trivial. First, note that is the restriction of a character $j j^s$ ~ of R, where s 2 $\mathbf{R}$ and is a unitary character of T^0. The above description of the T^0 \$ $GL_2(F)$ correspondence shows that

$$j\ j^s{}_1\ j\ j^s{}_2 \quad\quad {}_B^c \quad i^G{}_0(^+)$$

appears as ${}_{L_1B}^c$ a quotient of $i^{L_1G}{}_0($). (To simplify notation, we write instead of $^+$ in the rest of this section.) For a generic choice of we may assume that $i^G_{Q_1}(j\ j^s{}_1\ j\ j^s{}_2)$ and $i_{B^0}()$ are irreducible, so we get

$$i^G_{Q_1}(j\ j^s{}_1\ j\ j^s{}_2) \quad {}_B^c \quad i^G{}_0():$$

But now notice that $i^G_{B^0}{}^0() = i^G{}_0({}_B{}^1\ {}^1) = i^G{}_0(\ ({}_B{}^2\ {}^1))$ and we can apply the same reasoning to the character $^0 = {}^2\ {}^1$. This shows that

$$j\ j^s\ {}^2{}_1{}^1$$
$$j\ j^s\ {}^2{}_2{}^1$$

needs to appear in the Jacquet module of $i^G_{Q_1}(j j^s{}_1 j j^s{}_2)$. Computing the said Jacquet module shows this to be possible only if s = 0 and ^{2} = 1.

It remains to prove that is trivial. Looking at the quotient (T2) in $r_{U_2}()$ and applying Frobenius reciprocity, we see that

$$i^G_{Q_2}(!_{K=F}\ jdetj_2)^1 \quad \mathbf{1};$$

where **1** denotes the trivial representation of $PU_3(K)$. The representation $i^G_{Q_2}(!_{K=F}\ jdetj_2)^1$ is of length 2 (cf. Proposition 4.1 in [23]); from the above map we get

$$\mathbf{1};$$

where is the unique (Langlands) quotient of $i^G_{Q_1}$ (jdetj $(1 !_{K=F}$)). Applying the Jacquet functor r_{U_1} to the above map, and comparing wtih the subrepresentation (B1) in the ltra-tion $r_{U_1}()$, we conclude that = 1.

Once we have established that is the trivial character, it is not hard to determine $_1$, the character that appears in the middle piece of the ltration (M1). Recall that $_1$ is a character of the diagonal torus in $L_1 = GL_2$. As explained above, the bottom piece of the ltration (B1) shows that we have

$$() \quad i_{Q_1}({}_1{}^G{}_2) \quad {}^G{}^C \quad i_{B^0}():$$

Here we are still assuming the choice of is such that both representations appearing on the right-hand side are irreducible. Applying the Jacquet module with respect to U_1, we see that, in addition to ^C_B $({}_1 \quad {}_2)$ $i^G_{Q_1}()$ (which appears in the bottom piece of the ltration), contains

$${}_1{}^1 !_{K=F} \quad {}^0_B \quad i^G{}_0() \quad \text{and} \quad {}_2{}^1 !_{K=F} \quad {}^0_B \quad i^G{}_0():$$

These quotients come from the middle part of the ltration; in other words, we have

$$i^{L_1 G^0}_{\overline{B} B}({}_1 \ C^c_c(F)) \quad {}^1 !_{K=F} \quad {}_i \quad {}^c_B \quad i^G{}_0();$$

for i = 1; 2. Using the Bernstein form of Frobenius reciprocity, and computing the appropriate Jacquet modules, we see that this is possible if and only if

$${}_1 = 1 \quad !_{K=F}:$$

A similar argument can be used to determine $_2$: we apply the Jacquet module r_{U_2} to (), observing that $i^G_{Q_1}({}_1{}_2) = i^G_{Q_2}({}_2 !_{K=F}$). Then certain quotients of $r_{U_2}(i^G_{Q_2}({}_2 \ !_{K=F}$)) come from the bottom of the ltration (B2), and one veries that $_2$ = 1 $!_{K=F}$.

Remark 3.4. The fact that $i^{L_1 G^0}_{\overline{B} B}(C_c(F))$ is responsible for the two quotients appearing above (even though we have a single orbit) is explained by the action of $\overline{B}\,B^0$ on $C_c(F)$. Recall that

$$\begin{matrix} a & \\ & b \end{matrix} \; ;(z;) \quad 2 \ T \ T^0$$

acts by a $^1 N_{K=F}(z)$, so we view this as an action of F $N_{K=F}(K)$. Since a character of $N_{K=F}$ (K) (or equivalently, a Galois-invariant character of K) extends to a character of F in two ways, both

$${}^1 \; !^1_{K=F} \; !_{K=F} \quad \text{and} \quad {}^2_1$$

appear as quotients of $C_c(F)$.

4 Howe duality

Our main result is the following theorem (Howe duality).

Theorem 4.1. (i) Let be an irreducible representation of G_2. If () = 0, then it is a representation of nite length, with a unique irreducible quotient ().

(ii) For $_1$; $_2$ 2 Irr(G_2),

$$0 = (_1) = (_2) =) \quad _1 = _2:$$

(iii) If () = 0, then () is tempered if and only if is tempered.

(iv) Let 2 Irr(G^0). Then () is either 0 or a representation of nite length.

The proof will take up the rest of this section; we provide an outline:

1) First, we consider the non-tempered correspondence in x4.1. In particular, we prove (i) and (ii) for non-tempered ; (iii) is then a consequence of the proof. These results will follow from our computations of Jacquet modules in x3.

Next, we study the lifts of tempered representations. If is tempered, we decompose () into its cuspidal and non-cuspidal part: () = $()_c$ $()_{nc}$. We have the analogous decomposition () = $()_c$ $()_{nc}$ for tempered 2 Irr(G^0).

2) The niteness of $()_{nc}$ and $()_{nc}$ is proved in Proposition 4.5 using the Jacquet module computations from x3.

To analyze the cuspidal part we employ the strategy from [10]. The main idea is the \period ping-pong" introduced there | see Lemma 4.7 and 4.12.

3) We show that $()_c$ is either irreducible or zero in Proposition 4.9 (for generic) and Proposition 4.14 (for non-generic). The uniqueness of the irreducible quotient in (i) is then deduced easily as a consequence of the period ping-pong; see Proposition 4.16. This proves (i).

4) Part (ii) is also shown to be a consequence of the period ping-pong; see Prop. 4.17.

5) Finally, the niteness of $()_c$ in (iv) follows from Propositions 4.8 and 4.15.

4.1 Non-tempered correspondence

Using the results of the Section 3, we now compute the lifts of non-tempered representations. We begin by recalling the Langlands classication for G = G_2. Any non-tempered 2 Irr(G) is isomorphic to exactly one of the following representations:

a) Unique irreducible quotient of $i^G_{Q_1}()$ for = $j\det j^s$ $_0$, where $_0$ is a tempered irreducible representation and $s > 0$.

b) Unique irreducible quotient of $i^G_{Q_2}()$ for = $j\det j^s$ $_0$, where $_0$ is a tempered irreducible representation and $s > 0$.

c) Unique irreducible quotient of $i^G_{Q_1}()$, where is the unique (Langlands) quotient of $_1$ $_2$; here $j_1 j = j\ j^{s_1}$ and $j_2 j = j\ j^{s_2}$ with $s_1 > s_2 > 0$.

In case of $PU_3(K)$, the situation is even simpler. Any character of K can be written as a product $jN_{K=F}\ j^s$ where is unitary and $s \in \mathbf{R}$. We let $I(; s)$ denote the principal series representation of $PU_3(K)$ obtained by inducing this character of K. If $s > 0$ this is a standard module and has a unique irreducible quotient. Before doing computations, we need to address the question of distinguishing extensions to $G^0 = PU_3(K) \rtimes Gal(K=F)$ of $Gal(K=F)$-invariant representations of $PU_3(K)$. Fortunately, for principal series the extension can be done at the level of inducing data. Given a $Gal(K=F)$-invariant character of K, only one extension to $K \rtimes Gal(K=F)$, denoted by $^+$, appears in the quadratic base change (cf.3.4). Let denote the other extension. Thus, for Galois-invariant, $I(; s)$ extends to G^0 in two ways: $I(^+; s)$ and $I(\ ; s)$. When is not invariant, only one extension exists; in the following proposition, we denote it by $I(^+; s)$ to enable uniform statements.

Proposition 4.2. (i) Let $\twoheadrightarrow i^G_{Q_1}(-)$ with as in (a) above. If comes from a char-acter $jN_{K=F}\ j^s$ of K via base change $K \to GL_2(F)$ (with $s > 0$ and unitary), then $()$ is a non-zero quotient of $I(\ ; s)$; in particular, it has finite length. If does not come from a character of K via base change, then does not appear in the theta correspondence.

Conversely, let $^+$ denote the unique irreducible quotient of $I(^+; s)$. Then $(^+) = 0$.

(ii) Let $\twoheadrightarrow i^G_{Q_2}(-)$ with as in (b) or (c) above. Then does not appear in the theta correspondence.

Proof. We use the fact that $() = Hom_G(;)$ (non-smooth linear dual). Thus, from $\twoheadrightarrow i^G_{Q_i}(-)$ we get $() = Hom_G(;) \supseteq Hom_G(; i^G_{Q_i}(-)) = Hom_{L_i}(r_{U_i}(); -)$ using Frobenius reciprocity. We now analyze the space $Hom_{L_i}(r_{U_i}(); \ )$ using Propositions 3.1 and 3.2.

(i) Let $S_1; S_2$ and S_3 denote the subquotients of $r_{U_1}()$ appearing in (T1), (M1) and (B1), respectively. Comparing the central characters, one sees that $Ext_{L_1}(S_1; -) = 0$ (recall that $s > 0$, so the central character is a negative power of $j\ j$). We thus get the following exact sequence:

$$0 \to Hom_{L_1}(S_2; -) \to Hom_{L_1}(r_{U_1}(); -) \to Hom_{L_1}(S_3; -) \to Ext_{L_1}(S_2; -):$$

Recall that $S_2 = i^{L_1 G^0}_{B\ B}(1 \otimes C_c(F))$. Using the Bernstein form of Frobenius reci-procity we see that

$$Ext_{L_1}(S_2; -) = Ext_{GL_1 GL_1}(1 \otimes C_c(F); r_B(-)):$$

Recall that the second GL_1 factors acts on $_1 \otimes C_c(F)$ by $\omega_{K=F}$; by our assumption on, this is different from the corresponding action on $r_B(-)$. Therefore $Ext^i_{L}(S_2; -) = 0$ for all i, and the above long exact sequence becomes

$$Hom_{L_1}(r_{U_1}(); -) = Hom_{L_1}(S_3; -):$$

Now (B1) shows that $S_3 = {}^{L}i_{B}^{G}(\cdot)$. By Lemma 9.4 of [4], the maximal –-isotypic quotient of i^{G}_{B} is – $i^{G}_{0}((-))$, where $(-)$ is the big theta lift of – with respect to the similitude correspondence described in x3.4. Hence, if () is non-zero, then – must come from a character of K via base change. Note that the character corresponding to $_1$ $_2$ (or) is in fact ^{1} (and not); this is accounted for by the fact that w 2 K acts on K by 1=w (see x3.1). Thus, if = $_1$j j^s $_2$j j^s or = $_{jj^s}$ for a unitary character , we get () = $^+$j j^s. This proves () Hom_{L_1} (S_3; –) = I($^+$; s). Taking the smooth vectors (and the contragredient), we see that () is a quotient of I($^+$; s), as claimed. Furthermore, notice that the above proof shows that I($^+$; s) is a quotient of , so () = 0.

(ii) This is proved by comparing the central character, the same way we did in (i). We omit the details.

□

Not surprisingly, we get analogous results for lifts from $G^0 = PU_3(K) \rtimes \mathbf{Z}/2\mathbf{Z}$. The following proposition is proved just like Proposition 4.2, by analyzing $r_{U^0}()$:

Proposition 4.3. As before, let $^+$ (resp.) denote the unique irreducible quotient of I($^+$; s) (resp. I(; s)), where is a unitary character of K and s > 0. Then ( ) = 0, and ($^+$) is a non-zero quotient of $i^{G}_{Q_1}$ (), where is the representation of GL_2 obtained from j j^s by base change K ! $GL_2(F)$. In particular, () has nite length. Further-more, () = 0, where is the unique irreducible quotient of $i^{G}_{Q_1}$().

Remark 4.4. Notice that Propositions 4.2 and 4.3 combine to give us the following: Assume that and are irreducible representations of G and G^0, respectively, such that is a quotient of . Then

$$\text{is tempered} \iff \text{is tempered:}$$

4.2 Finiteness of theta lifts

Our rst task is to prove that the big theta lift () has nite length. To do this, we recall that () can be decomposed as

$$() = ()_{nc} \oplus ()_c;$$

the sum of its non-cuspidal and cuspidal part. We rst prove the following

Proposition 4.5. (i) Let 2 Irr(G^0) be tempered. Then $()_{nc}$ has nite length.

(ii) Let 2 Irr(G) be tempered. Then $()_{nc}$ has nite length.

Proof. (i) Recall that we have two maximal parabolic subgroups in G, $Q_i = L_i U_i$ for i = 1; 2. It suces to show that the Jacquet module $r_{U_i}(())$ is a nite-length representation of L_i, i.e. that the -isotypic quotient of $r_{U_i}()$ has nite length. To do that, we use the Jacquet module ltrations computed in Section 3.

Consider $r_{U_1}()$ rst. Again, we let $S_1; S_2$ and S_3 denote the subquotients appearing in (T1), (M1) and (B1), respectively. We need to show that the multiplicity space of the -isotypic quotient of S_i has nite length, for i = 1; 2; 3.

At the bottom, we have

$$S_3() := \mathrm{Hom}_{G^0}(S_3;) \quad \mathrm{Hom}_{G^0}(i_{L_1}^{B_0}(\Upsilon)^L_1;)^G \mathrm{Hom}_{T^0}(\Upsilon; r_{\overline{B}^0}()):$$

Now $r_{\overline{B}^0}$ is a nite-length representation of T^0 K. Taking any irreducible subquotient (which is in fact a character of K), we have $\mathrm{Hom}_{T^0}(\Upsilon;)$ $_\Upsilon()$ where $_\Upsilon()$ is the theta lift of with respect to the Weil representation Υ. The fact that $_\Upsilon()$ has nite length follows from the Howe duality theorem for classical (similitude) correspondences, cf. x3.4. This in turn shows (after taking the smooth vectors) that $S_3()$ itself is of nite length as an L_1-module.

In the middle, we have

$$S_2() := \mathrm{Hom}_{G^0}(S_2;) = \mathrm{Hom}_{G^0}(i^{L_1 G^0}(1_{B\,B}\; C_c(F));) \quad \mathrm{Hom}_{T^0}(i_B(1 \; C_c(F)); r_{\overline{B}^0}()):$$

Again, $r_B()$ is a nite-length representation of T. Taking an irreducible subquotient of $r_B()$, we see that

$$\mathrm{Hom}_{T^0}(i_B^G(1 \; C_c(F));) = i_B^G(1^1 \; !_{K=F}) \quad i_B^G(2^1 \; !_{K=F})$$

(see Remark 3.4); in particular, we get a representation of nite length. Taking the smooth vectors, we see that $S_2()$ has nite length.

Finally, we need to check S_1, the top part of the ltration. However, B^0 acts trivially on S_1, so the -isotypic quotient is zero.

The Jacquet module with respect to U_2 is analyzed in the same way. Let S_1 and S_2 be the subquotients of $r_{U_2}()$ appearing in (T2) and (B2), respectively. To show that the -isotypic quotient of S_2 has nite length we proceed just like in the U_1 case; we omit the details.

As for S_1, recall that $P_2 = M_2N_2$ is the Heisenberg parabolic in H. The Levi factor M_2 (which corresponds to the C_3 part of the relative diagram) has been described in [9, x7.2]|one can think of it roughly as a unitary group $U_6(K)$|and $_{C_3}$ is its minimal representation. We thus need to analyze the -isotypic quotient of $_C$ as a representation of $L_2 = GL_2(F)$. On the other hand, we have the central isogeny $GL_2(F)$ K $!^3$ $GU_2(K)$. Thus the correspondence that arises from $_{C_3}$ can roughly be viewed as (the similitude version) of the classical correspondence

$$U_2(K) \quad ! \quad U_3(K)$$

for unitary groups. It follows that the theta lift of with respect to $_{C_3}$ is a nite length representation of $GL_2(F)$.

Part (ii) can be proved in the same way, by analyzing $r_{U^0}()$ as a T^0-module. We let S_1 and S_3 denote the subquotients appearing in (T3) and (B3), respectively. To prove that the -isotypic quotient of S_3 has nite length we repeat the arguments from the U_1 case; we leave the details to the reader.

In order to analyze S_3 we need to consider the -isotypic quotient of the representation $_{B_3}$. Recall that $_{B_3}$ is the minimal representation of the Levi factor M^0. This Levi factor is a quasi-split group D_4^E where E = K F, and we are thus looking at the correspondence for the dual pair $G_2 \times Aut(E)$ inside the group D_4^E. The finiteness now follows from the results of [15], where this correspondence has been studied in detail.

□

Having established the finiteness of $()_{nc}$, we turn to $()_c$. Here our approach is based on the "ping-pong" of periods utilized in [10]. We will need to consider generic and non-generic representation separately. We begin by recalling the relevant periods.

4.3 Shalika periods

First, we recall the parabolic subgroup B^0 of G^0 discussed in Remark 2.4. The unipotent radical U^0 of B^0 has a filtration $\{0\} \subseteq U^0(2) \subseteq U^0$ with $U^0(2) \cong C_0$ and $U^0/U^0(2) \cong C$. We let ψ_{U^0} be the character of U^0 which, via the identification $U^0/U^0(2) = C$ is given by $Tr_{C/F}$. Finally, let S be the semi-direct product of U^0 and the stabilizer of of ψ_{U^0} in the Levi of B_C. We note that the stabilizer is isomorphic to $Aut(C)$. We denote by ψ_S the (Shalika) character of S equal to ψ_{U^0} on U^0 and trivial on $Aut(C)$.

Let V be the unipotent radical of the Borel subgroup $Q_1 \cap Q_2$ in G, and let $\psi_V : V \to \mathbf{C}$ be a Whittaker character for $G = G_2$. Just like in [28, Lemma 4.5], one shows that

$$(4.1) \qquad \Pi_{V,\psi_V} = \text{c-ind}_S^{G^0} \psi_S$$

holds for general C. Here Π_{V,ψ_V} denotes the maximal quotient of Π on which V acts by ψ_V. This immediately implies

Corollary 4.6. Let π be an irreducible representation of G^0 Then $\Theta(\pi)$ is (non-zero) generic if and only if π has a non-trivial Shalika period.

For $C = K$ we have $Aut(C) = \mathbf{Z}/2\mathbf{Z}$, and the Shalika functional is simply the Whittaker functional extended trivially to $\mathbf{Z}/2\mathbf{Z}$.

Conversely, recall that Q_1 is the three step maximal parabolic in G. Let Q_1^{der} be its derived group. In particular $Q_1^{der}/U_1 \cong SL_2(F)$, and $U_1/U_1(3)$ is a three-dimensional Heisenberg group. Then

$$\Pi_{S,\psi_S} \cong \text{c-ind}^G_{Q_1^{der}}(\rho \otimes \Theta(1))$$

where ρ is the unique irreducible representation of $U_1/U_1(3)$, extended to $\widetilde{SL_2}(F)$, and $\Theta(1)$ is the big theta lift of the trivial representation of $Aut(C)$ to $\widetilde{SL_2}(F)$, via the correspondence arising from $\widetilde{SL_2}(F) \times Aut(C)$ acting on the Weil representation on $C_c(C_0)$ given by ω_ψ. If C is the algebra of 2×2 matrices, this is given by Proposition 11.6 in [10]. Of course, the proof generalizes. If $C = K$ then $\Theta(1)$ is an irreducible even Weil representation. The even Weil representation is a quotient of a principal series representation $I(\chi)$ (notation of [10, Section 11]) for a character χ such that $|\chi| = |\cdot|^{1/2}$. Thus, by [10, Corollary 11.4], we have

$$(4.2) \qquad \dim \text{Hom}_G(\Pi_{S,\psi_S}, \pi) \leq 1$$

for any Whittaker generic and tempered irreducible representation π of G.

4.4 Howe duality for tempered representations

We now have all the ingredients required for the ping-pong game:

Lemma 4.7. Let be the minimal representation of H. Let 2 Irr(G) be tempered, and let 2 Irr(G^0) be tempered such that

$$\mathrm{Hom}_{GG^0}(;\ ) = 0:$$

Then we have the following (natural) inclusions

$$\mathrm{Hom}_V(;\ _V) \overset{(1)}{} \mathrm{Hom}_V(();\ _V) \overset{(2)}{} \mathrm{Hom}_S(-;\ _S)$$

$$\overset{(3)}{} \mathrm{Hom}_S((-);\ _S) \overset{(4)}{} \mathrm{Hom}_G(_{S;}\ _S\ ;-):$$ If is

generic, than all the above spaces are one-dimensional.

Proof. This is analogous to Lemma 12.1 in [10]. First, (1) follows from () . The isomorphism (2) follows from

$$\mathrm{Hom}_V(();\ _V) = \mathrm{Hom}_{VG^0}(;\ _V) = \mathrm{Hom}_{G^0}(\ _V;)$$

combined with (4.1). Next, (3) follows from the fact that () is the complex conjugate of (). Since = – and = –, we have (–) –. Finally, (4) is

$$\mathrm{Hom}_S((-);\ _S) = \mathrm{Hom}_{GS}(;-\ _S) = \mathrm{Hom}_G(_{S;}\ _S\ ;-):$$

If the representation is generic, then $\mathrm{Hom}_V(;\ _V)$ is one-dimensional. However, (4.2) shows that $\mathrm{Hom}_G(_{S;}\ _S-;-)$ is one-dimensional as well. The lemma follows. □

We now have two immediate consequences of the above lemma (cf. Propoisition 12.2 and 12.3 of [10]):

Proposition 4.8. Let 2 Irr(G^0) be tempered. Then () cannot have two irreducible tempered and generic quotients.

Proof. Assume that $_1$ and $_2$ are tempered and generic such that () $_1$ $_2$. Then dim $()_{V;}$ 2. However, Lemma 4.7 asserts that dim $()_{V;}$ $_V$ = 1, so we have a contradiction. □

Proposition 4.9. Let 2 Irr(G) be tempered and generic. Then () cannot have two tempered irreducible quotients. In particular, its cuspidal part $()_c$ is either irreducible or zero.

Proof. Let $_1;_2$ be irreducible and tempered such that () $_1$ $_2$. Lemma 4.7 (applied to $-;-_1$, and again to $-;-_2$) implies

$$1 = \dim \mathrm{Hom}_S(_1;\ _S) = \dim \mathrm{Hom}_S(();\ _S) = \dim \mathrm{Hom}_S(_2;\ _S):$$

But $_1$ $_2$ is a quotient of (), so we have

$$1 = \dim \mathrm{Hom}_S(();\ _S)\ \dim \mathrm{Hom}_S(_1;\ _S) + \dim \mathrm{Hom}_S(_2;\ _S) = 2;$$

a contradiction. □

Thus, we have proved that $()_C$ has nite length in case is generic. To prove the same result for non-generic we need another version of period ping-pong, which we now describe.

Recall the groups $G_E = Spin_8^E$ and $G^0_{E;C} = Aut(i_C : E ,! J)$ introduced in x1.4. Together with G and G^0, they constitute a see-saw dual pair

$$\begin{array}{ccc} G_E = Spin_8^E & & G^0 \\ | & \times & | \\ G_2 & & G^0_{E;C} = Aut(i_C : E ,! J): \end{array}$$

This gives us the standard see-saw identity

$$Hom_{G^0_{E;C}}((); 1) = Hom_{G_2}(R_C(E););$$

where $R_C(E) = (1)$ denotes the big theta lift of the trivial representation of $G^0_{E;C}$ to G_E.

To better understand the representations $R_C(E)$ (for various C), we need to relate them to a certain degenerate principal series of G_E. Here we use the results of [10, x5]. Let $P_E = M_E N_E$ be the Heisenberg parabolic subgroup of $G_E = Spin_8{}^E$. We consider the degenerate principal series

$$I_{E;!_{K=F}}(s) = Ind_{P_E}^{G_E}(!_{K=F}\, j\det j^s):$$

We then have

Proposition 4.10. Let 2 $Irr(G_2)$ be tempered. Then

(i) $I_{E;!_{K=F}}(1=2) \quad \bigoplus_C R_C(E)$, where the sum is taken over all C such that E C = J.

(ii) $Hom_{G_2}(I_{E;!_{K=F}}(1=2);) = Hom_{N_2}(-; \quad _E)$.

Proof. These results are taken, mutatis mutandis, from propositions 5.2 and 5.5 in [10]. □

The nal ingredient we need is a description of the twisted N_2-coinvariants of :

Lemma 4.11. We have
$$_{N_2;\ _E} = \bigoplus_C c\text{-}ind^{G^0}_{G^0_{E;C}}(1);$$

where the sum is taken over all twisted composition algebras C such that E C = J.

Proof. This is essentially Lemma 2.9 in [13]; the only dierence is that here we have more than one isomorphism class of embeddings E ,! J. □

We are now ready for the second game of period ping-pong.

Lemma 4.12. Let 2 $Irr(G_2)$ and 2 $Irr(G^0)$ be tempered representations such that

$$Hom_{G_2 G^0}(; \ ) = 0:$$

Then we have the following sequence of natural inclusions:

$$\mathrm{Hom}_{N_2}(\;;\ _E)\overset{(1)}{\ }\mathrm{Hom}_{N_2}((\,);\ _E)\overset{(2)}{\ }\bigoplus_C \mathrm{Hom}_{G_{E;C}}(-;1)$$

$$\overset{(3)}{\ }\bigoplus_C \mathrm{Hom}_{G_{E;C}}((-);1)\overset{(4)}{=}\mathrm{Hom}_{G_2}\Big(\bigoplus_C R_C(E);-\Big):$$

Here the sum is taken over all C such that E C = J . If any one of these spaces is nite-dimensional, then inclusions (1) and (3) are in fact isomorphisms.

Proof. This is analogous to Lemma 6.4 in [10]. First, (1) follows from () . Next, (2) follows from

$$\mathrm{Hom}_{N_2}((\,);\ _E) = \mathrm{Hom}_{G^0}(_{N_2;\ _E};\,)$$

combined with Lemma 4.11 and Frobenius reciprocity. The inclusion (3) is a consequence of (−) −; this follows from the fact that () is the complex conjugate of (), combined with $\overline{\ }=$ and −. Finally, (4) is the see-saw identity.

Now if any one of the above spaces is nite-dimensional, it follows that $\mathrm{Hom}_{N_2}(\;;\ _E)$ is nite-dimensional as well. By Proposition 4.10, we then have

$$\dim \mathrm{Hom}_{G_2}\Big(\bigoplus_C R_C(E);-\Big)\ \dim \mathrm{Hom}_{G_2}(I_E(1/2;\,!_{K/F});-) = \dim \mathrm{Hom}_{N_2}(\;;\ _E):$$

The result follows. □

The following result will allow us to use the above lemma when analyzing non-generic representations:

Lemma 4.13. Let be an irreducible non-generic innite-dimensional representation of G_2. Then there exists an etale cubic algebra E such that $_{N_2;\ _E}$ is non-zero. Moreover, $_{N_2;\ _E}$ is nite-dimensional for any E .

Proof. This is Lemma 3.4 in [10] □

The two games of period ping-pong allow us to conclude the proof of Theorem 4.1. First, we prove the niteness of $()_c$ (cf. Proposition 6.7 in [10]).

Proposition 4.14. Let 2 Irr(G) be tempered and non-generic. Then () cannot have two tempered irreducible quotients. In particular, $()_c$ is irreducible or 0.

Proof. Let $_1;_2$ 2 Irr(G^0) be irreducible and tempered; assume that () $_1$ $_2$. Since is non-generic, there is an etale cubic algebra E such that d := dim Hom_N (−; $_E$) is nite-dimensional and non-zero. Lemma 4.12 (applied rst to $-;-_1$, and then to $-$, $-_2$) now shows

$$d = \dim \bigoplus_C \mathrm{Hom}_{G^0_{E;C}}(_1;1) = \dim \bigoplus_C \mathrm{Hom}_{G^0_{E;C}}((\,);1) = \dim \bigoplus_C \mathrm{Hom}_{G^0_{E;C}}(_2;1):$$

However, this is impossible, since it would imply

$$d = \dim \bigoplus \mathrm{Hom}_{G^0_{E;C}}((\,);1)\ \dim \bigoplus_C \mathrm{Hom}_{G^0_{E;C}}(_1;1) + \dim \bigoplus_C \mathrm{Hom}_{G^0_{E;C}}(_2;1) = 2d:$$

Therefore, we have arrived at a contradiction, and the Proposition is proved. □

We will also need the following result:

Proposition 4.15. Let 2 Irr(G^0) be tempered. Let 2 Irr(G) be a tempered, non-generic quotient of (). Then is the unique irreducible tempered quotient of ().

Proof. Since is non-generic, Lemma 4.13 shows that there is a cubic algebra E such that d := dim Hom_{N_2} (; E) is nite-dimensional and non-zero. Now Lemma 4.12 shows that dim Hom_{N_2} ((); E) = d. If ^{0} is another tempered irreducible quotient of (), then Lemma 4.12 shows that dim Hom_{N_2}(^{0}; E) = d. But this implies

$$d = \dim \mathrm{Hom}_{N_2}((); \ _E) \ \dim \mathrm{Hom}_{N_2}(; \ _E) + \dim \mathrm{Hom}_{N_2}(^0; \ _E) = 2d;$$

which is a contradiction. The Proposition is proved. □

Next, we prove that (), if non-zero, has a unique irreducible quotient.

Proposition 4.16. Let 2 Irr(G) be tempered such that () = 0. Then () has a unique irreducible quotient.

Proof. Remark 4.4 shows that () can only have tempered quotients. The result now follows from Proposition 4.9 (when is generic) and Proposition 4.14 (when is non-generic). □

Proposition 4.17. Let $_1; _2$ 2 Irr(G) be tempered. Then 0 = ($_1$) = ($_2$) implies $_1$ = $_2$.

Proof. If $_1$ and $_2$ are both generic, this follows from Proposition 4.8 applied to = ($_1$) = ($_2$). If either is non-generic, then the result follows from Proposition 4.15. □

5 Explicit theta correspondences

In this section we discuss lifts of (non-cuspidal) tempered representations of G^0.

5.1 Representations of the unitary group

Recall that I(; s) denotes the principal series representation of $PU_3(K)$ obtained by inducing $jN_{K=F}j^s$ (with unitary) from T^0 = K . We shall denote I(; 0) simply by I(). The unique non-trivial element of the Weyl group conjugates ^{1} to , where is the non-trivial element of Gal(K=F). It is easy to argue that the principal series representations I(; s) have reducibility points only if ^{1} = , that is, is conjugate-dual. Then there is a dichotomy at play. If I() is reducible, then I(; s) is irreducible for all s = 0. If I() is irreducible then there exists s_0 > 0 such that I(; s) are irreducible for s = s_0. The representation I(; s_0) is a standard module of length two, with a unique irreducible quotient and a discrete series representation as a unique submodule. By [12] I() is irreducible if and only if

$$L(; s)L(As \ (); 2s)$$

has a (simple) pole at s = 0. More precisely, if L(; s) has a pole at 0, then = 1 and s_0 = 1; if L(As (); 2s) has a pole at 0, then is conjugate-symplectic and s_0 = 1=2. Now the following summarizes our discussion:

Proposition 5.1. Let be a conjugate-dual character of K. Then the principal series I(; s), for s 0, reduces as follows:

1. If = 1, the trivial representation is the quotient, and the Steinberg representation St is a submodule at s = 1.

2. If = 1 is conjugate-orthogonal then reduction occurs at s = 0,

$$I() = I()_{gen} \; I()_{deg}$$

   where $I()_{gen}$ is Whittaker generic.

3. If is conjugate-symplectic then I(; s) reduces at s = 1=2. The Whittaker generic submodule is a discrete series () whose Langlands parameter [2, Section 10] is a 3-dimensional conjugate-orthogonal representation

$$\overset{2}{V_2}$$

   of K SL$_2$, a quotient of the Weil{Deligne group of K by the commutator of W_K, where V_2 is the irreducible two-dimensional representation of SL_2.

Remark 5.2. The Langlands parameter of () should perhaps be expressed using ^{1} = instead of . However, the theta lift of both () and () is the same representation of G_2, so this imprecision is harmless.

We may now describe the theta correspondence for (limits of) discrete series of $PU_3(K)$ discussed above, and for extensions of those representations to $PU_3(K)$ o Gal(K=F), when Gal(K=F)-invariant. These lifts will be computed using Jacquet functors and the following additional inputs that are, roughly speaking,

The correspondence is one-to-one.

It preserves tempered representations.

It preserves generic representations.

Recall that there are two ways to extend I() to $PU_3(K)$ o Gal(K=F) when is Galois-invariant: $I(^+)$ and I( ). Here $^+$ is the extension of which appears in the quadratic base change (see x3.4). Another way to characterize $I(^+)$ is via Whittaker functionals: Gal(K=F) acts trivially on the one-dimensional space of Whittaker functionals. If is a constituent of I(), let $^+$ (resp.) be the extension of contained in $I(^+)$ (resp. I( )). Using Jacquet functors like in Proposition 4.2, it follows that theta lifts of constituents of I( ) are trivial unless the constituent is St .

In order to state the result, let be a conjugate-dual character of K and let () be the irreducible representation of $GL_2(F)$ corresponding to the two-dimensional representation () of the Weil group W_F. Let $I^G_{Q_1}((); s)$ denote the principal series where we induce () twisted by jdetj^s. If is conjugate-orthogonal, = 1, then the central character of () is $!_{K=F}$ and $I^G_{Q_1}(())$ is reducible,

$$I^G_{Q_1}(()) = I^G_{Q_1}(())_{gen} \; I^G_{Q_1}(())_{deg}:$$

If is conjugate-symplectic, then the central character of () is trivial and $I^G_{Q_1}$ ((); 1=2) has a Whittaker generic tempered submodule. If is not Gal(K=F)-invariant, then () is a cuspidal representation and the tempered submodule is a discrete series representation. If is Gal(K=F)-invariant, then () = $_1\,_2$ for a pair of mutually inverse characters of F, and the tempered submodule is

$$I^G_{Q_2}(st_1) = I^G_{Q_2}(st_2);$$

where st_i denotes a twist of the Steinberg representation of $GL_2(F)$ by the character $_i$(det).

Finally, we recall the A-packet discussed in Section 3.3. The packet contains two representations: a supercuspidal, and the Langlands quotient of $i^{G_2}_{Q_1}$(jdetj), with equal to the tempered representation 1 $!_{K=F}$. In the following proposition, we consider the corre-sponding discrete series L-packet (again attached to the subregular unipotent orbit and the cubic etale algebra F + K). Its elements are obtained by applying the Aubert involutions to the elements of the A-packet; in particular, we have a supercuspidal representation, and a generic discrete series representation contained in $i^{G_2}_{Q_1}$(jdetj) as a submodule.

Proposition 5.3. We have:

1. f(St^+); (St)g is the discrete series L-packet of G attached to the subregular unipotent orbit and the cubic etale algebra F + K. (St) is supercuspidal.

2. If = 1 is conjugate-orthogonal then

   $$(I()_{gen}) = I^G_{Q_1}(())_{gen} \text{ and } (I()_{deg}) = I^G_{Q_1}(())_{deg}:$$

   If is Gal(K=F)-invariant, then the statements involve constituents of I($^+$).

3. If is conjugate-symplectic then (()) is a Whittaker generic, tempered submodule of $I^G_{Q_1}$ ((); 1=2). If is Gal(K=F)-invariant, the lift is of ()$^+$.

Proof. 1. Let f; g be the L-packet, with supercuspidal. We use the description of r_{U^0}() from Proposition 3.3. First, note that () (a representation of G^0) is non-trivial, because appears as a quotient of (T3). Thus () appears as a quotient of the minimal representation . It follows that r_{U^0}(()) is a quotient of r_{U^0}(). Since is supercuspidal, r_{U^0}(()) cannot appear in (B3); therefore, it is a quotient of (T3). Thus r_{U^0}(()) is one-dimensional, a quotient of (T3); notice that r_{U^0}(()) is precisely the exponent of St . Moreover, the supercuspidal part of () is necessarily 0 because of the one-to-one property (Theorem 4.1). This shows () = St .

It remains to determine (St^+). We know that (St^+) = 0 (because St^+ is generic; see Corollary 4.6) and tempered. This implies, using Proposition 3.3 again, that (St^+) appears as a quotient in B3. From here it follows that (St^+) is a subquotient of $i^{G_2}_{Q_1}$(jdetj^1 (1 $!_{K=F}$)). Since (St^+) is tempered, the claim follows.

2. Again, we look at the description of the Jacquet module in Proposition 3.3. The two constituents of I() are not distinguished by Jacquet modules: we have $r_{U^0}(I()_{gen})$ = $r_{U^0}(I()_{deg})$ = . Note that appears as a quotient in (B3). As explained in 3.4,

the similitude correspondence arising from the representation Ϯ in (B3) now shows that ^{G} () is a quotient of Ϯ. Inducing, we get that I_Q^{2} (()) is a quotient of r^0 (); in other words (using Frobenius reciprocity), I^{G_2} (()) is a quotient of . In particular, both $I^{G_2}_{Q_1}(())_{gen}$ and $I^{G_2}_{Q_1}(())_{gen}$ have non-zero theta lifts which are constituents of I(). Since lifts of generic representations remain generic (Corollary 4.6), we must have $(I()_{gen})$ = $I_Q^{2}(())_{gen}$; by the one-to-one property it now follows that $(I()_{deg})$ = $I_Q^{2}(())_{deg}$.

3. The proof here is the same as for (St^+) in case (i). □

6 Mini theta

It is possible that a cuspidal representations of G^0 lift to non-cuspidal representations of G. From Jacquet functors, it is clear that such representations of G^0 are lifts from the Levi $L_2 = GL_2(F)$ via the minimal representation of M_2. We shall describe this mini-theta correspondence by relating it to a classical theta correspondence for unitary groups.

6.1 The similitude theta correspondence for $GU(2)^{det}$ GU(3)

Let GU(n) denote the group of similitudes of an n-dimensional Hermitian space, and let $GU(n)^{det}$ denote the index two subgroup of elements such that the similitude takes value in $N_{K=F}(K)$. The forms of GU(2) can be described using quaternion algebras. Let B be a quaternion algebra and x an embedding of K into B. The right multiplication by K turns B into a 2-dimensional symmetric Hermitian space, the Hermitian form given by the quaternion norm. We have

$$GU(2) = (B \; K)=F$$

where B acts on B from the left, and K from the right, by inverse. The center of this group is (F K)=F = K. The subgroup $GU(2)^{det}$ consists of pairs (g; z) such that the norm of g is in $N_{K=F}(K)$.

The mini-theta correspondence is related to the similitude theta correspondence for $GU(2)^{det}$ GU(3), which ts into the seesaw

$$\begin{array}{ccc} GU(3) & & GU(6)^{det} \\ & @ & \\ & @ & \\ & @@ & \\ GU(1) = K & & GU(2)^{det} \end{array}$$

One uses the splitting character 1 on GU(3) and GU(1) on the left hand side, on $GU(6)^{det}$ and ^{3} on $GU(2)^{det}$ [1, Section 7]; here is a character of K which restricts to $!_{K=F}$ on F. We start by considering this, without referring to the mini-theta yet.

In this see-saw, one starts with the trivial representation of GU(1) on the bottom left, and one takes an irreducible representation of $GU(2)^{det}$ on the bottom right. Then the see-saw identity is

$$\mathrm{Hom}_{GU(2)^{det}}((1);) = \mathrm{Hom}_{GU(1)}((); 1):$$

Hence, () is a representation of GU(3)=GU(1) i.e. a representation of GU(3) with trivial central character. Moreover, with the choice of splitting characters as above, the theta correspondence carries representations of GU(3) with trivial central character to representations of $GU(2)^{det}$ with central character ^{3}. One can describe this theta correspondence, as a lifting from $GU(2)^{det}$ to U(3)=U(1) ⊴GU(3)=GU(1) as follows.

An irreducible representation of GU(2) ⊴(GL$_2$(F) K)=F is of the form for an irreducible representation of GL$_2$(F) and a character of K, so that the central character of is j$_F$. Since we are interested only in those irreducible representations of GU(2) with the central character ^{3}, we must take = ^{3}. In other words, we are looking at irreducible representations of GL$_2$(F) whose central character is $!_{K=F}$ (since j_F = $!_{K=F}$).

Note that:

- the contragredient of such a is – = $!_{K=F}$, so is dihedral with respect to K=F if and only if is self-dual.
- hence, the restriction of to GL$_2$(F)^{det} is irreducible if and only if is not self-dual; if is self-dual, the restriction breaks into 2 pieces.

With ^{3} given, it gives an L-packet of

$$U(2) = f(g; z) : det(g) = N(z)g=F \quad GU(2):$$

If denotes the L-parameter of (as a GL$_2$(F)-representation), then the L-parameter of this U(2) L-packet is the conjugate-symplectic representation of the Weil group W_K

$$j_{W_K} \, ^3:$$

Assume now that is a discrete series representation of GL$_2$(F). Then j_{W_K} is irre-ducible if and only if is non-dihedral with respect to K=F, i.e. is not self-dual. In any case, the representations of U(3) we get by theta lifting from $GU(2)^{det}$ to U(3)=U(1) have the L-parameter

$$= j_{W_K} + 1;$$

see Theorem A in [11]. Observe that this is a conjugate-orthogonal representation of W_K of determinant one: j_{W_K} has trivial determinant, since det() = $!_{K=F}$.

More precisely:

- if is not self-dual, i.e. not dihedral, we are starting with a singleton L-packet on $GU(2)^{det}$ and its lift is the unique generic representation in the L-packet of U(3) with the L-parameter . That L-packet has another element which is lifted from the non-quasi-split form of $GU(2)^{det}$.
- if is self-dual, then the L-packet we started with on $GU(2)^{det}$ has 2 elements, distin-guished by their Whittaker support, and their theta lifts are two representation $_{gen}$ and $_{deg}$ in the L-packet with the L-parameter ; there are two other representations lifted from he non-quasi-split form of $GU(2)^{det}$.

6.2 The lifts coming from mini-theta

With the above understanding of the similitude theta lifting in hand, we return to the problem of mini-theta.

Let M_2 be the connected component of M_2. Take a generator of the group of algebraic characters of M_2, and let M_2^{det} be the index two subgroup of M_2 of elements such that the character takes values in $N_{K=F}(K\ )$. Furthermore, the restriction of the character to $L_2 = GL_2(F)$ is the determinant (or its inverse) and we can dene $L^{det} = GL_2(F)^{det}$ analogously. Thus we have a group isomorphic to $PU(3)\ GL_2(F)^{det}$ contained in M_2^{det}. We can describe M_2^{det} and this embedding explicitly within the framework of x6.1.

Assume that the quaternion algebra B from x6.1 is split, so that $B = GL_2(F)$. Let U(6) be the unitary group corresponding to the Hermitian space B B B. Then by Section 7.2 in [9]

$$M_2^{det} = f(z; g)\ 2\ (K\ \ U(6))=U(1)\ j\ z=z\!\!= \ \det g g$$

where U(1) is embedded into K U(6) by z ! $(z^3; z)$ and zdenotes the action of the non-trivial element in Gal(K=F). Let U(3) U(2) be a dual pair in U(6) so that U(2) acts diagonally on the three copies of B. We map U(3) into M_2^{det} by

$$g\ !\ \ (\det(g); g):$$

It is easy to check that this map is well dened and trivial on the center U(1). Now let g 2 $GL_2(F)^{det}$. Let z 2 K such that $N_{K=F}(z) = \det(g)$. Then (g; z) denes an element in U(2), using the above description of GU(2), and

$$g\ !\ \ (z\ \ ^3; (g; z))$$

is a well dened map from GL_2^{det} into M_2^{det} which does not depend on the choice of z.

In this way, we may view $PU(3)\ GL_2(F)^{det}$ as (a subgroup of) the dual pair discussed in x6.1. The representation ^{1} (1) of K U(6) descends to a representation of M_2^{det} which is now independent of ; see Section 8.4 in [9]. Furthermore,

$$\mathrm{Ind}_{M_2^{det}}^{M_2}\ \ ^1\ (1)$$

is the minimal representation of M_2. From the formulas for the embedding of PU(3) and $GL_2(F)^{det}$ into M_2^{det}, it is easy to check that these two groups act on ^{1} (1) in the same way as they act in the classical see-saw pair above. Combining this with the two bullet points at the end of x6.1, we get:

Proposition 6.1. Let be a discrete series representation of $GL_2(F)$ with the central character $!_{K=F}$. Then $= j_{W_K} + 1$ is a PU(3)-parameter and:

- If is not self-dual, i.e. not dihedral w.r.t. K=F, the L-packet of has one represen-tation we are considering. Under the mini-theta, it lifts to + –, and under the theta lift to G_2, it lifts to $\mathrm{Ind}_{Q_2}^{G_2} = \mathrm{Ind}_{Q_2}^{G_2}(-)$
- If is self-dual, i.e. dihedral w.r.t. K=F, then the L-packet of has two representa-tions $_{gen}$ and $_{deg}$ we are considering. Under the mini-theta, they lift to . Under the theta lift to G_2, these two representations of PU(3) lift to the two constituents of $\mathrm{Ind}_{Q_2}^{G_2}()$.

We still need to go from PU(3) to PU(3) o Z=2Z. The action of Z=2Z on Irr(PU(3)) is sending to – (see [21]). The representations in the L-packet of are all self-dual, and hence each has two extensions to PU(3) o Z=2Z. Of course, by the one-to-one result, only one of these extensions can lift to a summand of Ind^{G_2}. The$_P$other extension should not lift to G_2, however, we cannot exclude that it lifts to a cuspidal representation of G_2.

Finally, let us discuss what happens on the level of L-parameters. The L-parameter is a 3-dimensional rep : W_K ! $SL_3(\mathbf{C})$ of the form = j_{W_K} + 1, where : W_F ! $GL_2(\mathbf{C})$ has det() = $!_{K=F}$. Also, is the restriction to W_K of an L-parameter

$$^0: W_F \; ! \; SL(3) \text{ o } \mathbf{Z}=2\mathbf{Z}$$

where the latter is the L-group of PU(3). Using the further inclusion SL(3) o Z=2Z ! G_2 the 7-dimensional representation of $G_2(\mathbf{C})$, as a W_F-module, decomposes as

$$(\mathrm{Ind}_{W_K}^{W_F}) + !_{K=F} = (!_{K=F} + + 1 + !_{K=F}) + !_{K=F} :$$

Recalling that $!_{K=F}$ = –, and regrouping (i.e. conjugating), we rewrite this as: (+

$$!_{K=F}) + (- + !_{K=F}) + 1:$$

This parameter factors through the Levi $GL_2(\mathbf{C})$ in $SL_3(\mathbf{C})$ $G_2(\mathbf{C})$, therefore it is a parameter of the induced representation $\mathrm{Ind}_P^{G_2}$ = Ind_P ^{G_2}(–).

7 Statements and declarations

On behalf of all authors, the corresponding author states that there is no conict of interest. Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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