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On the Emergence of Quantum Boltzmann Fluctuation Dynamics near a Bose–Einstein Condensate

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Abstract

In this work, we study the quantum fluctuation dynamics in a Bose gas on a torus $\Lambda = (L\mathbb{T})^3$ that exhibits Bose–Einstein condensation, beyond the leading order Hartree–Fock–Bogoliubov (HFB) theory. Given a Bose–Einstein condensate (BEC) with density $N \gg 1$ surrounded by thermal fluctuations with density 1, we assume that the system dynamics is generated by a Hamiltonian with mean-field scaling. We derive a quantum Boltzmann type dynamics from a second-order Duhamel expansion upon subtracting both the BEC dynamics and the HFB dynamics, with rigorous error control. Given a quasifree initial state, we determine the time evolution of the centered correlation functions $\langle a \rangle$, $\langle aa \rangle - \langle a \rangle^2$, $\langle a^+ a \rangle - |\langle a \rangle|^2$ at mesoscopic time scales $t \sim \lambda^{-2}$, where $0 < \lambda \ll 1$ is the coupling constant determining the HFB interaction, and a, a^+ denote annihilation and creation operators. While the BEC and the HFB fluctuations both evolve at a microscopic time scale $t \sim 1$, the Boltzmann dynamics is much slower, by a factor λ^2. For large but finite N, we consider both the case of fixed system size $L \sim 1$, and the case $L \sim \lambda^{-2-}$. In the case $L \sim 1$, we show that the Boltzmann collision operator contains subleading terms that can become dominant, depending on time-dependent coefficients assuming particular values in $\mathbb{Q}$; this phenomenon is reminiscent of the Talbot effect. For the case $L \sim \lambda^{-2-}$, we prove that the collision operator is well approximated by the expression predicted in the literature. In either of those cases, we have $\lambda \sim \left(\frac{\log \log N}{\log N}\right)^{\alpha}$, for different values of $\alpha > 0$.

Keywords Nonequilibrium quantum statistical mechanics · Quantum Boltzmann equations · Boltzmann–Nordheim equations · Boltzmann–Uhlenbeck–Uehling equations · Bose–Einstein condensate · Thermal quantum fluctuations

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1 Introduction

1.1 Quantum Dynamics and Boltzmann Equations

The main question we set out to answer in this work is:

> Is there a scaling regime in an interacting quantum field theory, for which the emergence of collisional processes described by a Boltzmann equation can be rigorously established?

1.1.1 Emergence of a Quantum Boltzmann Equation

In analogy to Maxwell's and Boltzmann's theory of collisions in classical systems, Nordheim [251] in 1928 was the first to propose a Boltzmann equation for Bose and Fermi gases given by

$$\begin{aligned}\partial_t f(p) &= Q_4(f)\\ &:= \int d\mathbf{p}_4\, \delta(p_1+p_2-p_3-p_4)\delta(E(p_1)+E(p_2)-E(p_3)-E(p_4))\\ &\quad |\mathcal{M}_{22}(\mathbf{p}_4)|^2(\delta(p-p_1)+\delta(p-p_2)-\delta(p-p_3)-\delta(p-p_4))\\ &\quad \big((1\pm f(p_1))(1\pm f(p_2))f(p_3)f(p_4)-f(p_1)f(p_2)(1\pm f(p_3))(1\pm f(p_4))\big).\end{aligned} \tag{1.1}$$

Here, f denotes the particle density in the spatially homogeneous case; '$+$' refers to the bosonic, and '$-$' refers to the fermionic equation, and $\mathbf{p}_4 = (p_1, p_2, p_3, p_4)$. In addition, $E(p) = \frac{1}{2}|p|^2$ denotes the free dispersion, and $\mathcal{M}_{22}$ is the (microscopic) scattering cross section for $2 \leftrightarrow 2$ processes describing two thermal fluctuation scattering off of each other. As Nordheim already argues, the distribution of the outgoing particles needs to be taken into account, resulting in a quartic collision operator, in contrast to classical particles that are described by a quadratic collision operator. It is shown in [251] that the equilibrium is given by the Bose–Einstein, respectively the Fermi-Dirac statistics, and that an H-theorem holds true. In 1933, Uehling and Uhlenbeck [297] studied the linearization about the equilibrium, in order to determine the associated hydrodynamics, and to compute the heat conductivity and the viscosity coefficient.

Subsequently, physicists have given formal derivations of the above quantum Boltzmann equation, using diagrammatic techniques from quantum field theory, see, e.g., [1, 188, 300]. This has given rise to interesting fundamental effective theories, such as the *Kadanoff-Baym equations*, see, e.g., [181, 277]. We also mention the important contributions by Bogoliubov and collaborators [63] and Prigogine and collaborators [262].

The first mathematically rigorous works on the derivation of the classical Boltzmann equation, a billiards model for a classical gas, go back to Cercignani [79] in 1972 and Lanford [202] in 1975, where they studied the Grad-limit [161] of a hard-sphere model. These works were later revisited and completed through works by Uchiyama [296], Cercignani–Illner–Pulvirenti [81], Spohn [285], Cercignani–Gerasimenko–Petrina [80], and Gallagher–Saint–Raymond–Texier [149].

In 1983, Hugenholtz [184] considered the commutator perturbation expansion with respect to the weak coupling constant λ. Implementing the kinetic time scale $t = T\lambda^{-2}$ used by van Hove [299], it was shown that, in the translation-invariant case, terms of order $O(\lambda)$ vanish as $\lambda \to 0$, and that terms of order $O(\lambda^2)$ are proportional to T. Using a selection rule, it is conjectured in [184] that only two-point correlations of higher orders in λ survive, motivating the assumption of *quasifreeness*. Quasifreeness, as we will see below, is, in a sense, a quantum analogue to the 'Stoßzahlansatz', also known as 'molecular chaos'. Hence, Hugenholtz argues, at leading order, the Boltzmann equation should arise for the evolution of the two-point function. Ho and Landau [179] later proved that, to second order in λ, this holds true.

In 2004, Erdös, Salmhofer, and Yau [129] extended the results by Hugenholtz, and by Ho and Landau, by introducing the concept of *restricted quasi-freeness*, i.e., quasi-freeness only up to six- or eight-point correlations. Assuming propagation of restricted quasi-freeness, they showed that a (time-dependent) Boltzmann equation arises from the second-order Duhamel expansion, under certain assumptions. Around the same time, Benedetto, Castella, Esposito and Pulvirenti, began a series of works that used Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchies to derive a quadratic Boltzmann equation with a quantum collision kernel for a system obtained by iterating Duhamel's formula, ignoring the tail, and truncating the obtained hierarchy in the small-coupling limit [42, 45] respectively in the low-density regime [44]. In [43], they went on to show that the contributions to second order in the coupling recover (1.1) to leading order, evaluated at initial time with f_0 instead of f_t. Further works studying the BBGKY hierarchy in the context of the quantum Boltzmann equation include works by Gerasimenko and collaborators [152, 153, 155], see also references therein. A recent review in this direction can be found in [154]. For works using a second quantization approach, we refer to Spohn and collaborators [148, 234, 287–289], see also references therein.

In 2015, X. Chen and Y. Guo [94] showed that, if the marginals of the BBGKY hierarchy converge in the weak-coupling limit in a strong sense, and if the $W^{4,1}$-regularity per particle remains uniformly bounded for some positive time, then the limiting hierarchy is that associated to the quadratic Boltzmann equation with a quantum collision kernel, instead of the Boltzmann equation derived in [129].

In a different line of work addressing the weakly disordered Anderson model, scattering of electrons at impurities in a lattice have been investigated extensively. For works studying the derivation of the linear Boltzmann equation in this context, we refer to [82, 90, 91, 134, 178, 180, 283]. For a more recent treatment, we refer to [165].

Lukkarinen-Spohn [234, 235] showed that the nonlinear Schrödinger equation (NLS) with random initial data leads to the wave kinetic equation: They present this system as a simplified model to gain insights into the emergence of Boltzmann-type dynamics in a quantum Bose gas. The study of wave-turbulence in the context of the long-time behavior of the NLS is a very active area of research, see, e.g., [18, 75, 107, 108, 113, 114, 143, 156, 191, 270].

The derivation of Boltzmann equations from a system of classical interacting particles is an extraordinarily active research field, see [20, 21, 58–61, 80, 81, 115, 149, 193, 202, 253, 264–266, 278, 284, 296] and references therein. The methods differ vastly from the quantum field theoretic approach developed in the work at hand. For works studying well-posedness and other analytical properties of the classical Boltzmann equation, we refer to [4–8, 10, 13, 19, 25, 41, 53, 54, 81, 83, 84, 116, 118, 119, 123, 125, 158, 160, 163, 170, 172, 173, 185, 189, 238, 291–293, 298, 301–303] and references therein.

1.1.2 Well-Posedness

The first well-posedness results for (1.1) go back to Dolbeaut [124] and Lions [219] for the fermionic case. X. Lu and collaborators have made significant progress on the fermionic

Boltzmann-Uhlenbeck-Uehling (BUU) equation, see, e.g., [222, 226, 231]. For a recent work on the fermionic problem, we refer to [213].

A unified treatment of bosons and fermions can be found in [305, 306], and more recently [252]. In addition, generalized statistics such as anyons [28], and Haldane statistics [26, 30] have also been included in this line of study.

Works studying the relativistic quantum Boltzmann equation include [36, 136, 137] and references therein.

The quantum Landau equation, which can be viewed as a limit of the Boltzmann equation accounting for long-range interactions, has been studied in [11, 12, 37, 38, 205, 220]. Recently, He-Lu-Pulvirenti [175] showed that it can be obtained as a weak semi-classical limit from the quantum Boltzmann equation.

In 1924, Bose [64], and, independently in 1925, Einstein [127] predicted that, below a critical temperature, the ground state becomes gradually more populated, forming a macroscopic state called the *Bose–Einstein condensate* (BEC). In 1995, this phenomenon was independently experimentally verified by groups around Cornell and Wiemann [24] and Ketterle [111]. Both groups were awarded the 2001 Physics Nobel Prize.

In the mathematically rigorous PDE literature, the bosonic problem has been investigated first by X. Lu [221]. The long-time behavior for radial initial data was studied, obtaining global existence, local stability, conservation of energy, and estimates on moment production. Moreover, at low temperatures, it is shown in [221] that a solution concentrates at $p = 0$ for large time, and that, at high temperatures, the solution converges weakly to the Bose–Einstein distribution. Escobedo–Mischler–Valle [137] showed that bosonic entropy maximizers are given by

$$f_{eq}(p) = \frac{1}{e^{\beta(E(p)-\mu)} - 1} + m_0\delta(p)\,, \tag{1.2}$$

where $m_0 \cdot \mu = 0$, see, e.g., [141]. The BEC density m_0, the chemical potential μ and the inverse temperature β are uniquely determined by the moments $\int dp f_{eq}(p)$ and $\int dp E(p) f_{eq}(p)$, see, e.g., [223] and references therein. One has that m_0 vanishes above a critical temperature T_c, and is non-zero below T_c.

Well-posedness results have been formulated to account for solutions that form a Dirac mass at temperatures below T_c, and that stay bounded for temperatures above T_c. We refer to [26, 76, 141, 142, 223, 224, 227–229, 232] for the isotropic and space-homogeneous case, to [28, 29, 252, 272, 305, 306] for the space-dependent case, and to [73, 77, 212, 252] for the anisotropic case. Further works studying the blow-up behavior related to condensation include [40, 140, 290].

1.2 Collisions of Fluctuations About a Bose–Einstein Condensate

Pioneering works by Kirkpatrick and Dorfmann [195, 196] and Eckern [126] started analyzing the interplay of the BEC with thermal excitation cloud surrounding the condensate. They formally obtained a Boltzmann equation of the form

$$\partial_t f^{(ex)}(p) = n_c Q_3(f^{(ex)}) + Q_4(f^{(ex)})\,, \tag{1.3}$$

where Q_4 is given by (1.1) with E replaced by the Bogoliubov dispersion $\Omega = \sqrt{E(E + 2n_c\bar{\lambda})}$, and

$$Q_3(f) := \int d\mathbf{p}_3\, \delta(p_1 + p_2 - p_3)\delta(\Omega(p_1) + \Omega(p_2) - \Omega(p_3))$$

$$|\mathcal{M}_{21}(\mathbf{p}_3)|^2(\delta(p-p_1)+\delta(p-p_2)-\delta(p-p_3))$$
$$\big((1+f(p_1))(1+f(p_2))f(p_3)-f(p_1)f(p_2)(1+f(p_3))\big)\,. \tag{1.4}$$

Here n_c denotes the BEC density, $f^{(ex)}$ the density of thermal fluctuation particles, $\bar{\lambda}$ is the coupling strength of the hard-sphere pair interaction $\bar{\lambda}\delta$, $\mathcal{M}_{21}$ is the cross section for $2 \leftrightarrow 1$ processes describing collisions of 2 thermal particles, where one is either being absorbed into or emitted from the BEC, and $\mathbf{p}_3 = (p_1, p_2, p_3)$. Zaremba, Niguni and Griffin [304], see also [164], later extended their approach to include the dynamics of the condensate. In the *Hartree-Fock-Bogoliubov-Popov* approximation, and for a translation-invariant initial state, they formally argue that the condensate wave function Φ satisfies

$$\begin{aligned} i\partial_t\hat{\Phi}(p) &= (h_{Har} + 2\bar{\lambda}f^{(ex)}(p) - iQ_3(f^{(ex)}))\hat{\Phi}(p) \\ &\quad + \bar{\lambda}g^{(ex)}(p)\overline{\hat{\Phi}}(-p)\,, \end{aligned} \tag{1.5}$$

and it is linked to the density via $n_c(t,x) = |\Phi_t(x)|^2$. Here h_{Har} denotes the Hartree Hamiltonian in momentum representation, see Sect. 1.4 below, and $g^{(ex)}$ denotes the rate of pair absorption into the condensate, which they discarded as a lower-order contribution. One of the motivations to study the coupled system between the condensate and the thermal cloud is to understand the nucleation process of the BEC, see, e.g., [52, 151, 187, 201, 257, 275]. For a review, we also refer to [263, 268].

Observe that, for large values of the condensate density n_c, we expect that Q_4 is of subleading order.

For mathematical works studying the system describing the two-component gas consisting of the condensate and the excitation cloud, we refer to [14, 27, 109, 110, 139, 250, 258–260, 279, 280].

1.3 Definition of the Mathematical Model

As a starting point for our analysis, we choose a single-species Bose gas at positive temperature trapped in a periodic box $\Lambda = (L\mathbb{T})^3$ of linear length L. We assume that the gas consists of two phases:

(1) A Bose–Einstein condensate (BEC) with density N,
(2) thermal fluctuations with density ~ 1.

We note that a significant part of this paper addresses the case of a fixed volume, where we may think of N as the number of bosons when $L = 1$. On the other hand, we will also consider the limit of large volume, where $N > 0$ will denote the number of bosons per unit volume (that is, the density); for convenience, we are not changing the notation.

We assume mean-field interactions for which the kinetic energy of the condensate and the total (pair) interaction potential among particles are balanced, and analyze the interplay between the condensate dynamics and the dynamics of the fluctuation particles.

1.3.1 Definition of the Model

Let $\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n\geq 1}(L^2(\Lambda))^{\otimes_{sym} n}$ denote the bosonic Fock space, equipped with the inner product

$$\langle \Psi, \Phi \rangle_{\mathcal{F}} = \sum_{n\in\mathbb{N}_0} \langle \Psi_n, \Phi_n \rangle_{L^2(\mathbb{R}^{3n})} \tag{1.6}$$

for all $\Psi = (\Psi_n)_{n\in\mathbb{N}_0}$, $\Phi = (\Phi_n)_{n\in\mathbb{N}_0} \in \mathcal{F}$. For $n \in \mathbb{N}$, $f \in L^2(\mathbb{R}^3)$, $\Psi \in (L^2(\Lambda))^{\otimes_{sym} n}$, and $\mathbf{x}_{n-1} \in \mathbb{R}^{3(n-1)}$, we define

$$(a(f)\Psi)(\mathbf{x}_{n-1}) := \sqrt{n} \int_\Lambda dx\, \overline{f}(x)\Psi(x, \mathbf{x}_{n-1})\,. \tag{1.7}$$

For any $n \in \mathbb{N}$, let $\mathcal{S}_n$ denote the permutation group of $\{1, \ldots, n\}$. For $n \in \mathbb{N}_0$, $f \in L^2(\mathbb{R}^3)$, $\Psi \in (L^2(\Lambda))^{\otimes_{sym} n}$, and $\mathbf{x}_{n+1} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{3(n+1)}$, we define

$$(a^+(f)\Psi)(\mathbf{x}_{n+1}) := \frac{\sqrt{n+1}}{(n+1)!} \sum_{\pi\in\mathcal{S}_{n+1}} f(x_{\pi(1)})\Psi(x_{\pi(2)}, \ldots, x_{\pi(n+1)})\,. \tag{1.8}$$

Then we have that for all $f \in L^2(\Lambda)$, $\Phi, \Psi \in \mathcal{F}_{fin}$

$$\langle \Phi, a(f)\Psi\rangle_{\mathcal{F}} = \left\langle a^+(f)\Phi, \Psi\right\rangle_{\mathcal{F}}\,. \tag{1.9}$$

In addition, a and a^+ satisfy the canonical commutation relations (CCR) for any $f, g \in L^2(\Lambda)$

$$[a(f), a^+(g)] = \langle f, g\rangle \quad , \quad [a(f), a(g)] = [a^+(f), a^+(g)] = 0\,. \tag{1.10}$$

Moreover, we

introduce the operator-valued distributions a_x, a_y^+ by requiring

$$a(f) = \int_\Lambda dx\, \overline{f}(x)a_x\,, \tag{1.11}$$

$$a^+(g) = \int_\Lambda dx\, f(x)a_x^+ \tag{1.12}$$

for all $f, g \in L^2(\Lambda)$.

These satisfy the CCR

$$[a_x, a_y^+] = \delta_\Lambda(x-y) \quad , \quad [a_x, a_y] = [a_x^+, a_y^+] = 0\,. \tag{1.13}$$

We call $\Omega_0 := (1, 0, 0, \ldots) \in \mathcal{F}$ the *Fock vacuum*. Then we have that $a_x\Omega_0 = 0$ for all $x \in \Lambda$. In addition, for $\Psi \in \mathcal{F}_{fin}$, $x \in \Lambda$, $n \in \mathbb{N}$, and $\mathbf{x}_n \in \mathbb{R}^{3n}$, we have that

$$(a_x\Psi)^{(n)}(\mathbf{x}_n) = \sqrt{n+1}\Psi^{(n+1)}(x, \mathbf{x}_n)\,. \tag{1.14}$$

We introduce the number operator

$$\mathcal{N}_b := \int_\Lambda dx\, a_x^+ a_x\,, \tag{1.15}$$

which satisfies

$$(\mathcal{N}_b\Psi)^{(n)} = n\Psi^{(n)}, \tag{1.16}$$

see, e.g., [50].

The Hamiltonian of the system studied in this paper has the following form. Let v be a sufficiently regular pair potential, see Sect. 2.2, and $\lambda > 0$ a small coupling constant. Given $N > 0$, we define

$$\mathcal{H}_N := \frac{1}{2}\int_\Lambda dx\, a_x^+(-\Delta_x)a_x + \frac{\lambda}{2N}\int_{\Lambda^2} dx\, dy\, v(x-y)\, a_x^+ a_y^+ a_y a_x\,. \tag{1.17}$$

Our aim is to study the evolution associated with the Hamiltonian $\mathcal{H}_N$ given in (1.17). The condensate is accounted for by a coherent state $\mathcal{W}[\sqrt{N|\Lambda|}\phi_0]\Omega_0$, where

$$\mathcal{W}[f] := \exp(a^+(f) - a(f)) = \exp\Big(\int_\Lambda dx\,(f(x)a_x^+ - \overline{f}(x)a_x)\Big) \tag{1.18}$$

denotes the Weyl operator, and ϕ_0 is normalized, $\|\phi_0\|_{L^2(\Lambda)} = 1$. By the Baker-Campbell-Hausdorff formula,

$$\mathcal{W}[f]\Omega = e^{-\frac{1}{2}\|f\|_{L^2(\Lambda)}}\Big(\frac{f^{\otimes n}}{\sqrt{n!}}\Big)_{n\in\mathbb{N}_0}, \tag{1.19}$$

see, e.g., [50]. That is, in each fixed particle sector, the wave function is the product state determined by a single wave function f.

We assume that, at initial time $t = 0$, the system is described by a state

$$\nu_0(A) := \frac{\mathrm{Tr}\left(e^{-\mathcal{K}}A\right)}{\mathrm{Tr}(e^{-\mathcal{K}})} \tag{1.20}$$

for all observables $A \in \mathfrak{A}$, where $\mathfrak{A}$ denotes the Weyl algebra generated by $\mathcal{W}[f]$, where $f \in \mathcal{S}(\Lambda)$ is a Schwartz function, see [70], Sect. 5.2.3. In addition, the Baker-Campbell-Hausdorff formula yields

$$\mathcal{W}^*[f]a_x\mathcal{W}[f] = a_x + f(x)\,. \tag{1.21}$$

Definition 1.1 (Quasifree state) Let ν be a state and

$$\nu^{(cen)}(A) := \nu(\mathcal{W}[\nu(a)]A\mathcal{W}^*[\nu(a)]) \tag{1.22}$$

denote its centering. We say ν is quasi-free iff

$$\begin{cases} \nu^{(cen)}(a^{\#_1}a^{\#_2}\dots a^{\#_{2n}}) & = \overbrace{a^{\#_1}a^{\#_2}\dots a^{\#_{2n}}} \\ & \quad +\text{all pair contractions} \\ \nu^{(cen)}(a^{\#_1}a^{\#_2}\dots a^{\#_{2n-1}}) & = 0 \end{cases}, \tag{1.23}$$

where $\overbrace{a^{\#_1}a^{\#_2}} := \nu^{(cen)}(a^{\#_1}a^{\#_2})$. (1.23) is referred to as Wick's Theorem.

Definition 1.2 *(Number conserving state)* A state ν is called number conserving iff $\nu([A, \mathcal{N}_b]) = 0$ for every observable $A \in \mathfrak{A}$.

We assume that the initial state ν_0 is number conserving, quasifree, and translation-invariant. In particular, we assume that the translation-invariant generator $\mathcal{K}$ is given by

$$\mathcal{K} := \int_{\Lambda^*} dp\, K(p)a_p^+a_p\,, \tag{1.24}$$

where $K(p) \geq \kappa_0$ for some $\kappa_0 > 0$. Observe that, by being number conserving, ν_0 is already centered.

The state describing the two-phase Bose gas is then given by

$$\rho_0(A) := \frac{1}{\mathrm{Tr}(e^{-\mathcal{K}})}\,\mathrm{Tr}\Big(\mathcal{W}[\sqrt{N|\Lambda|}\phi_0]e^{-\mathcal{K}}\mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0]A\Big) \tag{1.25}$$

for all $A \in \mathfrak{A}$. The initial value problem (IVP) associated with the Hamiltonian $\mathcal{H}_N$ and the initial state ρ_0 is then given by the von Neumann equation

$$i\partial_t\rho_t(A) = \rho_t([A, \mathcal{H}_N]) \tag{1.26}$$

for all observables $A \in \mathfrak{A}$. Below, we impose assumptions on v ensuring that $\mathcal{H}_N$ is self-adjoint and that it induces a unitary evolution $e^{-it\mathcal{H}_N}$. Then, the solution of (1.26) is given by

$$\rho_t(A) = \rho_0(e^{it\mathcal{H}_N} A e^{-it\mathcal{H}_N}) . \tag{1.27}$$

By making specific choices for A, we will study effective equations of key correlation functions characterized by the Bose gas.

1.4 Leading Order Condensate Dynamics: Hartree Equation

We expect the leading order dynamics of (1.26) to be described by the leading order condensate dynamics, as the BEC describes the bulk component of the Bose gas. Indeed, for instance, [166–169, 269], based on Hepp's method and using coherent states, have shown, that, in a precise sense, $e^{-it\mathcal{H}_N} W[\sqrt{N|\Lambda|}\phi_0]\Omega_0$ is well approximated by $W[\sqrt{N|\Lambda|}\phi_t]\Omega_0$ for $N \gg 1$, with approximation errors $o_N(1)$, where ϕ_t satisfies the Hartree equation

$$i\partial_t\phi_t = -\frac{1}{2}\Delta\phi_t + \lambda|\Lambda|(v * |\phi_t|^2)\phi_t . \tag{1.28}$$

The volume factor in the nonlinear interaction term accounts for our assumption that the L^2-mass of the condensate is

$$\|\sqrt{N|\Lambda|}\phi_t\|^2_{L^2(\Lambda)} = N|\Lambda| . \tag{1.29}$$

In the case $v = \delta$, (1.28) yields the nonlinear Schrödinger equation (NLS). Analogous statements have been proved for different choices of v with alternative approaches involving the corresponding BBGKY hierarchy, see, e.g., [2, 85, 89, 93, 95–100, 130–133, 135, 162, 177, 182, 194, 197, 214, 286], and other approaches, see [9, 15–17, 22, 23, 39, 65, 69, 120–122, 145–147, 157, 174, 183, 186, 192, 198, 203, 204, 207, 208, 215–218, 233, 236, 237, 245, 255, 256, 271, 276, 281, 282]. For more background on the derivation of Hartree theory, we refer to [50, 159, 171, 206, 216, 249, 273].

1.4.1 Stationary, Translation-Invariant Condensate

For simplicity, we choose to consider a stationary and translation-invariant solution of (1.28). Due to the normalization constraint $\|\phi_0\|_2 = 1$, we have

$$\phi_t = \phi_0 = |\Lambda|^{-\frac{1}{2}} \in \mathbb{R}_+ . \tag{1.30}$$

Substituting this into (1.28) yields

$$0 = \lambda|\Lambda||\phi_0|^2\phi_0 \int_\Lambda dx\, v(x) = \frac{\lambda}{\sqrt{|\Lambda|}} \int_\Lambda dx\, v(x) . \tag{1.31}$$

In particular, we assume

$$\int_\Lambda dx\, v(x) = 0 , \tag{1.32}$$

with additional regularity properties introduced below. Henceforth, we assume that ϕ is stationary, translation-invariant and satisfies (1.30).

1.5 Leading Order Fluctuation Dynamics: HFB Equations

We next turn to the leading order corrections of the full dynamics past the leading order BEC Hartree dynamics. For this purpose, we consider the fluctuation dynamics described by $W^*[\sqrt{N|\Lambda|}\phi_0]e^{-it\mathcal{H}_N}W[\sqrt{N|\Lambda|}\phi_0]$. We show in Lemma B.1

$$W^*[\sqrt{N|\Lambda|}\phi_0]\mathcal{H}_N W[\sqrt{N|\Lambda|}\phi_0] = \mathcal{H}_{HFB} + \mathcal{H}_{cub} + \mathcal{H}_{quart}\,, \tag{1.33}$$

where

$$\begin{aligned}\mathcal{H}_{HFB} &:= \int_{\Lambda^2} dx\,dy\,\Big(a_x^+(-\frac{1}{2}\delta(x-y)\Delta_x + \lambda v(x-y))a_y \\ &\quad + \frac{\lambda v(x-y)}{2}(a_x^+a_y^+ + a_x a_y)\Big)\,, \end{aligned}\tag{1.34}$$

$$\mathcal{H}_{cub} := \frac{\lambda}{\sqrt{N}}\int_{\Lambda^2} dx\,dy\,v(x-y)a_x^+(a_y+a_y^+)a_y\,, \tag{1.35}$$

$$\mathcal{H}_{quart} := \frac{\lambda}{2N}\int_{\Lambda^2} dx\,dy\,v(x-y)a_x^+a_y^+a_y a_x\,, \tag{1.36}$$

noting that (1.30) has been used to obtain these expressions. In particular, (1.33) implies that the fluctuation dynamics is determined by the unitary operator

$$\widetilde{\mathcal{U}}_N(t) := W^*[\sqrt{N|\Lambda|}\phi_0]e^{-it\mathcal{H}_N}W[\sqrt{N|\Lambda|}\phi_0]\,, \tag{1.37}$$

where

$$i\partial_t\widetilde{\mathcal{U}}_N(t) = (\mathcal{H}_{HFB} + \mathcal{H}_{cub} + \mathcal{H}_{quart})\widetilde{\mathcal{U}}_N(t)\,. \tag{1.38}$$

In the unitary evolution relative to the Hartree dynamics, the dynamics of thermal bosons is determined by two types of processes:

(1) Emission and absorption of thermal bosons from and into the BEC, respectively.
(2) Collisions between thermal bosons.

In particular, the Hamiltonian $\mathcal{H}_{HFB}+\mathcal{H}_{cub}+\mathcal{H}_{quart}$ describing this relative dynamics is not number conserving, as opposed to the original Hamiltonian $\mathcal{H}_N$. Observe that conjugation by the Weyl operator $\mathcal{W}[\sqrt{N|\Lambda|}\phi_0]$ 'subtracts' the condensate dynamics, thereby revealing the relative dynamics. As a consequence, our focus will be on the IVP

$$\begin{cases} i\partial_t\rho_t^{(rel.BEC)}(A) &= \rho_t^{(rel.BEC)}([A, \mathcal{H}_{HFB}+\mathcal{H}_{cub}+\mathcal{H}_{quart}])\,, \\ \rho_0^{(rel.BEC)}(A) &= \frac{1}{\mathrm{Tr}(e^{-\mathcal{K}})}\,\mathrm{Tr}\left(e^{-\mathcal{K}}A\right). \end{cases} \tag{1.39}$$

Notice that the initial state $\rho_0^{(rel.BEC)}$ is chosen to be particle number conserving, in contrast to ρ_0.

Observe that $\mathcal{H}_{HFB} = O(1)$, while $\mathcal{H}_{cub} = O(\frac{\lambda}{\sqrt{N}})$ and $\mathcal{H}_{quart} = O(\frac{\lambda}{N})$ are of lower order as $N \gg 1$. We thus expect the leading order fluctuation dynamics to be determined by the Hartree-Fock-Bogoliubov (HFB) dynamics described by the Hamiltonian $\mathcal{H}_{HFB}$. Bogoliubov [62] observed that a Hamiltonian of the form of $\mathcal{H}_{HFB}$ can be diagonalized in terms of rotated creation and annihilation operators $b_x = a(u_x) + a^+(v_x)$. Based on this idea, and considering the more general case for a non-stationary, non-translation invariant condensate wave function ϕ_t, there are many works analyzing the emergence of the HFB dynamics, including [3, 31, 32, 55–57, 66–68, 71, 72, 78, 101–103, 105, 112, 166–168, 199, 200, 209, 239–244, 246, 247, 254] and references therein. See also [206, 249] for more details.

Let $(\mathcal{V}_{HFB}(t))_{t\in\mathbb{R}}$ denote the unitary group associated with the generator $\mathcal{H}_{HFB}$, see (3.7) below. Then the density of HFB fluctuation particles, we have that

$$\mathcal{V}_{HFB}^*(t)\mathcal{N}_b\mathcal{V}_{HFB}(t) \lesssim \mathcal{N}_b + |\Lambda|\,, \tag{1.40}$$

see Remark 4.9 for details. In particular, the HFB fluctuation density has the order of magnitude

$$\frac{1}{|\Lambda|}\rho_0^{(rel.BEC)}\big(\mathcal{V}_{HFB}^*(t)\mathcal{N}_b\mathcal{V}_{HFB}(t)\big) \lesssim 1\,, \tag{1.41}$$

compared to the BEC density N, see also Lemma 4.1.

As we verify in Lemma 3.3, the HFB evolution captures oscillations between absorption into and emission from the BEC with frequency

$$\Omega = \sqrt{E(E+2\lambda\hat{v})}\,, \tag{1.42}$$

which is the Bogoliubov dispersion relation. Here $\hat{v}$ is the Fourier transform of v, see definition (1.43), and $E(p) = |p|^2/2$ denotes the free kinetic energy. We will assume $\hat{v} \geq 0$ to be non-negative. The Bogoliubov dispersion corresponds to the propagation of acoustic excitations, see, e.g., [112, 274] for more details.

1.6 Fourier Transform

Before moving on to the next order corrections, we fix our conventions for the Fourier transform. Let the Fourier transform be given by

$$\hat{f}(p) := \int_\Lambda dx\, e^{ip\cdot x} f(x)\,, \tag{1.43}$$

for all $p \in \Lambda^* = (\frac{2\pi}{L}\mathbb{Z})^3$, where $\Lambda = (L\mathbb{T})^3$. We denote by

$$a_p := a(e^{-ip\cdot(\cdot)}) = \int_\Lambda dx\, e^{ip\cdot x} a_x\,, \tag{1.44}$$

$$a_p^+ := a^+(e^{-ip\cdot(\cdot)}) = \int_\Lambda dx\, e^{-ip\cdot x} a_x^+ \tag{1.45}$$

the Fourier transforms of the operator-valued distributions a_x, a_y^+. These satisfy the discrete CCR

$$[a_p, a_q^+] = |\Lambda|\delta_{p,q} =: \delta_{\Lambda^*}(p-q)\,, \tag{1.46}$$

$$[a_p, a_q] = [a_p^+, a_q^+] = 0\,. \tag{1.47}$$

When the context is clear, we will omit the subscript 'Λ^*' in δ_{Λ^*}.

Recalling (1.15), the number operator is given by

$$\mathcal{N}_b = \int_{\Lambda^*} dp\, a_p^+ a_p\,. \tag{1.48}$$

Then a_p, a_q^+ satisfy the bounds

$$\|a_p\Psi\|_{\mathcal{F}} \leq \|e^{-ip\cdot(\cdot)}\|_{L^2(\Lambda)}\|\mathcal{N}_b^{\frac{1}{2}}\Psi\|_{\mathcal{F}} = |\Lambda|^{\frac{1}{2}}\|\mathcal{N}_b^{\frac{1}{2}}\Psi\|_{\mathcal{F}}\,, \tag{1.49}$$

$$\|a_p^+\Psi\|_{\mathcal{F}} = \sqrt{\|a_p\Psi\|_{\mathcal{F}}^2 + |\Lambda|} \leq |\Lambda|^{\frac{1}{2}}\|(\mathcal{N}_b + 1)^{\frac{1}{2}}\Psi\|_{\mathcal{F}}, \tag{1.50}$$

see, e.g., [50].

For convenience, we will use the notation

$$\int_{\Lambda^*} dp\, f(p) \equiv \frac{1}{|\Lambda|}\sum_{p\in\Lambda^*} f(p). \tag{1.51}$$

In contrast, we will denote the Lebesgue integral by $\int_{\mathbb{R}^3} dq\, f(q)$. Moreover, we will denote an n-tuple $(q_1, q_2, \ldots, q_n)$, $n \in \mathbb{N}$, of vectors $q_j \in \mathbb{R}^d$ for some $d \in \mathbb{N}$, as

$$\mathbf{q}_n := (q_1, q_2, \ldots, q_n). \tag{1.52}$$

1.7 Lower Order Fluctuations: Emergence of Boltzmann Dynamics

In order to study corrections to the HFB dynamics, we subtract it from the dynamics (1.39) relative to the BEC Hartree dynamics by conjugation with $\mathcal{V}_{HFB}(t)$, the unitary group induced by $\mathcal{H}_{HFB}$. The resulting relative dynamics is determined by

$$\begin{cases} i\partial_t \nu_t(A) &= \nu_t([A, \mathcal{H}_{cub}(t) + \mathcal{H}_{quart}(t)]), \\ \nu_0(A) &= \frac{1}{\mathrm{Tr}(e^{-\mathcal{K}})}\,\mathrm{Tr}\left(e^{-\mathcal{K}}A\right), \end{cases} \tag{1.53}$$

where

$$\begin{aligned} \mathcal{H}_{cub}(t) &:= \mathcal{V}_{HFB}^*(t)\mathcal{H}_{cub}\mathcal{V}_{HFB}(t) \\ &= \frac{\lambda}{\sqrt{N}}\int_{(\Lambda^*)^3} d\mathbf{p}_3\,\delta(p_1 + p_2 - p_3)\hat{v}(p_2) \\ &\quad \left(e^{i(\Omega(p_1)+\Omega(p_2)-\Omega(p_3))t}a_{p_1}^+a_{p_2}^+a_{p_3} + h.c.\right) + O\left(\frac{\lambda^2}{\sqrt{N}}\right) \end{aligned} \tag{1.54}$$

$$\begin{aligned} \mathcal{H}_{quart}(t) &:= \mathcal{V}_{HFB}^*(t)\mathcal{H}_{quart}\mathcal{V}_{HFB}(t) \\ &= \frac{\lambda}{2N}\int_{(\Lambda^*)^4} d\mathbf{p}_4\,\delta(p_1 + p_2 - p_3 - p_4)\hat{v}(p_1 - p_3) \\ &\quad e^{i(\Omega(p_1)+\Omega(p_2)-\Omega(p_3)-\Omega(p_4))t}a_{p_1}^+a_{p_2}^+a_{p_3}a_{p_4} + O\left(\frac{\lambda^2}{N}\right), \end{aligned} \tag{1.55}$$

see Corollary 3.3. For a derivation of (1.53), we refer to Lemma B.1.

We are interested in the evolution of the density of fluctuation particles

$$f_t(p) := \frac{\nu_t(a_p^+a_p)}{|\Lambda|}. \tag{1.56}$$

We have that

$$f_0(p) = \frac{1}{e^{K(p)} - 1}, \tag{1.57}$$

see Remark 4.2. The case $K(p) = \beta(E(p) - \mu)$, with inverse temperature $\beta > 0$, and chemical potential $\mu < 0$, corresponds to the Bose–Einstein distribution of the ideal Bose gas.

In order to study f_t, we need to extend (1.53) to hold for a more general class of operators. We prove in Lemma 4.1 a quantitative version of

$$\nu_0(\mathcal{N}_b^k) < \infty\,, \tag{1.58}$$

and in Corollary 4.10 a quantitative version of

$$\nu_t(\mathcal{N}_b^k) < \infty \tag{1.59}$$

for all $k \in \mathbb{N}$. Thus, we may extend the IVP (1.53) to observables

$$A = a_{p_1}^{\sharp_1} a_{p_2}^{\sharp_2} \dots a_{p_k}^{\sharp_k}\,. \tag{1.60}$$

In order to expand f_t, we follow [129, 179, 184] and apply Duhamel's formula three times

$$f_t(p) - f_0(p) = -\frac{i}{|\Lambda|}\int_0^t ds\, \nu_0([a_p^+ a_p, \mathcal{H}_I(s)]) \tag{1.61}$$

$$-\frac{1}{|\Lambda|}\int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1\ge s_2}\nu_0([[a_p^+ a_p, \mathcal{H}_I(s_1)], \mathcal{H}_I(s_2)]) \tag{1.62}$$

$$+\mathrm{Rem}_t(p)\,, \tag{1.63}$$

where we abbreviated

$$\mathcal{H}_I(t) := \mathcal{H}_{cub}(t) + \mathcal{H}_{quart}(t)\,, \tag{1.64}$$

$$\mathrm{Rem}_t(p) := \frac{i}{|\Lambda|}\int_{[0,t]^3} d\mathbf{s}_3 \mathbb{1}_{s_1\ge s_2\ge s_3}\nu_{s_3}([[[a_p^+ a_p, \mathcal{H}_I(s_1)], \mathcal{H}_I(s_2)], \mathcal{H}_I(s_3)])\,, \tag{1.65}$$

and $\mathbf{s}_j = (s_1, \dots, s_j)$, $j \ge 2$.

Because $a_p^+ a_p$ commutes with $e^{-\mathcal{K}}$, the transport term (1.61) vanishes.

Due to translation-invariance and ν_0 being number conserving, the transport term (1.61) vanishes.

For (1.62), observe that $\mathcal{H}_{cub}(t) \sim \frac{\lambda}{\sqrt{N}}$ is much larger than $\mathcal{H}_{quart}(t) \sim \frac{\lambda}{N}$. We thus expect the main contribution in (1.62) to stem from the terms involving $\mathcal{H}_{cub}$.

It is key to our analysis that we exploit the fact that the HFB dynamics happens on a much *shorter* time scale than the corrections coming from $\mathcal{H}_{cub}$ and $\mathcal{H}_{quart}$. Observing that λ defines the coupling strength at the level of the HFB evolution, we consider the kinetic time scale defined by $t \sim \lambda^{-2} \gg 1$. In order to separate the corrections in (1.54) and (1.55) from the HFB oscillations, we choose $0 < \lambda \ll 1$.

Using quasifreeness of ν_0, we thus expect the main contributions in (1.62) to be given by

$$\frac{1}{N}\int_0^T dS\, Q_S^{(mol)}(f_0)(p) \tag{1.66}$$

$$+\left| -\frac{i}{|\Lambda|}\int_0^{T\lambda^{-2}} ds\, \nu_0([a_0, \mathcal{H}_{cub}(s)])\right|^2 \delta(p)\,, \tag{1.67}$$

where

$$Q_S^{(mol)}(h)(p) = \int_{(\Lambda^*)^3} d\mathbf{p}_3 \frac{\sin\left(\frac{T-S}{\lambda^2}\big(\Omega(p_1)+\Omega(p_2)-\Omega(p_3)\big)\right)}{\Omega(p_1)+\Omega(p_2)-\Omega(p_3)}\delta(p_1+p_2-p_3)$$
$$(\hat{v}(p_1)+\hat{v}(p_2))^2\big(\delta(p-p_1)+\delta(p-p_2)-\delta(p-p_3)\big)$$

$$\big((1+h(p_1))(1+h(p_2))h(p_3) \;-\; h(p_1)h(p_2)(1+h(p_3))\big) \tag{1.68}$$

is a mollification of the cubic Boltzmann operator given in (1.4). The fact that energy conservation only holds approximately up to an error of order $\lambda^2 = O(t^{-1})$ is consistent with the time-energy Heisenberg uncertainty principle, see Remark 2.5 for more details. $Q_S^{(mol)}$ describes collisions between fluctuation particles, one of which is being absorbed into or emitted from the BEC. We expect that (1.67) is of size $\frac{T^2}{N\lambda^2}$ and that it dominates the Boltzmann collision term $\frac{1}{N}\int_0^T dS\, Q_S^{(mol)}$. It turns out that the presence of (1.67) owes to the fact that, above, we only subtracted the leading-order condensate dynamics. Thus, in order to resolve Boltzmann dynamics collisions, we need to pass to centered moments according to $a_p \to a_p - \nu_t(a_p)$. Denoting

$$\begin{aligned} F_T(p) &:= \frac{\nu_{T\lambda^{-2}}\big((a_p^+ - \nu_{T\lambda^{-2}}(a_p^+))(a_p - \nu_{T\lambda^{-2}}(a_p))\big)}{|\Lambda|} \\ &= \frac{\nu_{T\lambda^{-2}}(a_p^+ a_p) - |\nu_{T\lambda^{-2}}(a_p)|^2}{|\Lambda|}\,, \end{aligned} \tag{1.69}$$

and using that ν_0 is number conserving, we expect – and will indeed prove – to have

$$F_T(p) - F_0(p) \;=\; \frac{1}{N}\int_0^T dS\, Q_S^{(mol)}(f_0)(p) + \mathrm{Rem}_{T\lambda^{-2}}(p) + l.o.t.\,, \tag{1.70}$$

where "l.o.t." abbreviates "lower order terms".

Before moving on, we would like to reflect on the validity of this identity.

1.7.1 Fixed, *N*-Independent Lattice $\Lambda^* \cong \mathbb{Z}^3$

Recall that the fluctuation particles, at leading order, propagate with the Bogoliubov dispersion Ω. These acoustic waves have the phase velocity

$$v_P(p) \;:=\; \frac{\Omega(p)}{|p|} \;=\; \sqrt{\frac{E(p)}{2} + \lambda\hat{v}(p)}\,. \tag{1.71}$$

Averaging this over all particles yields

$$\langle v_P\rangle_0 \;:=\; \int_{\Lambda^*} dp\, f_0(p) v_P(p) \sim 1\,, \tag{1.72}$$

where we assume that f_0 is sufficiently regular for this argument. During the time $t \sim \lambda^{-2}$, the corresponding acoustic waves propagate a distance $\sim \lambda^{-2}$. In particular, we have

$$\lambda^{-2} \gtrsim L \quad \Leftrightarrow \quad \text{acoustic waves interfere with themselves.}$$

Thus, when λ is small enough, lower-order terms in (1.54) and (1.55) can constructively interfere to an extent as to contribute to leading order terms of F in (1.70). As we will see below, the effect of these contributions is large, depending on whether certain time-dependent expressions, coming from HFB oscillations, have particular values in $\mathbb{Q}$. This phenomenon is slightly reminiscent of the *Talbot effect* [128, 190]. The absence of this effect has been discussed in the context of the kinetic wave equation, see, e.g., [75, 107, 108, 113, 114]. To the best of our knowledge, this phenomenon has not previously been discussed in the literature in the context of the quantum field theoretic emergence of Boltzmann equations.

1.7.2 Continuum Approximation $\Lambda^* \to \mathbb{R}^3$

In order to elucidate the link with the expression for the BUU collision operator that is widely discussed in the literature, we present a continuum limit, that is derived here. However, we emphasize that this limit applies to the kernel itself, but not to the dynamics because sharp energy conservation cannot hold for finite times, due to the Heisenberg uncertainty.

As observed above, we expect that, for $L \gg \lambda^{-2}$, the self-interactions of the HFB waves to be negligible. In fact, we show that in this limit, we can approximate sums $\int_{\Lambda^*} dp \equiv \frac{1}{|\Lambda|}\sum_p$ by Lebesgue-integrals $\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}$, with control of errors that include oscillatory contributions, see Lemma 5.5. In this case, we establish that

$$F_T(p) - F_0(p) = \frac{T}{N} Q(f_0)(p) + \mathrm{Rem}_{T\lambda^{-2}}(p) + O\left(\frac{1}{N^{1+\delta_N}}\right), \tag{1.73}$$

where

$$\begin{aligned} Q(f_0)[J] := \frac{\pi}{(2\pi)^6} \int_{\mathbb{R}^9} & dp_1\, dp_2\, dp_3\, \delta\big(E(p_1) + E(p_2) - E(p_3)\big)\delta(p_1 + p_2 - p_3) \\ & (\hat{v}(p_1) + \hat{v}(p_2))^2 \big(J(p_1) + J(p_2) - J(p_3)\big) \\ & \big((1 + f_0(p_1))(1 + f_0(p_2)) f_0(p_3) - f_0(p_1) f_0(p_2)(1 + f_0(p_3))\big) \end{aligned} \tag{1.74}$$

is the energy-conserving cubic Boltzmann operator found in the above mentioned literature.

1.7.3 Propagation of Quasifreeness

If we can show that $\mathrm{Rem}_{T\lambda^{-2}}(p)$ is, in fact, of lower order compared to $\frac{1}{N}\int_0^T dS\, Q_S^{(mol)}$, then, by rearranging (1.70), we find that

$$F_T(p) - F_0(p) = \frac{1}{N}\int_0^T dS\, Q_S^{(mol)}(F)(p) + l.o.t., \tag{1.75}$$

which would prove that the next-to-leading order correction to the HFB dynamics of the particle density is described by a cubic Boltzmann equation. Comparing with the analysis of collisions for classical systems, bounding $\mathrm{Rem}_t(p)$ in the present context includes controlling recollisions, as is necessary in the context of classical systems. For more details on the role of recollisions in the classical case, we refer, e.g., to [81].

While drawing comparisons to the classical case, we address the role of quasifreeness. With the given choice of a number conserving and translation-invariant initial state ν_0, we have that the joint distribution function of n particles satisfies

$$f_{joint}(\mathbf{p}_n) := \frac{\nu_0(a_{p_1}^+ a_{p_2}^+ \dots a_{p_n}^+ a_{p_n} \dots a_{p_2} a_{p_1})}{|\Lambda|^n} \tag{1.76}$$

$$= f_0^{\otimes n}(\mathbf{p}_n) + O(|\Lambda|^{-1}), \tag{1.77}$$

where $\mathbf{p}_n = (p_1, \dots, p_n)$. In particular, we have that asymptotically, for large $L \gg 1$, quasifreeness implies *molecular chaos* in the classical sense, which refers to factorization of the joint distribution. Propagation of the factorized form is called *propagation of chaos* in the classical context.

In the present quantum field theoretic context, it is an important task to understand in what sense and in which scaling regime propagation of quasifreeness can be observed. An important property of the HFB evolution is that it *preserves* quasifreeness. In particular, if a state $\langle\cdot\rangle_0$ is quasifree, then so is $\langle\mathcal{V}^*_{HFB}(t)(\cdot)\mathcal{V}_{HFB}(t)\rangle_0$. This is a natural consequence of the fact that the HFB dynamics arises as a quasifree reduction of the full evolution. In the recent literature, this is explained on the basis of the Dirac-Frenkel principle, see, for instance, [34] and [51] for more details.

Notice that the full evolution clearly does not preserve quasifreeness, as is expected for an interacting gas. In order to study propagation of quasifreeness, it is crucial to control the Duhamel term $\mathrm{Rem}_t(p)$ accounting for all quantum 'recollisions'. Indeed, if we expand the evolution of arbitrary expectations

$$\nu_{T\lambda^{-2}}(a^{\sharp_1}_{p_1}a^{\sharp_2}_{p_2}\dots a^{\sharp_k}_{p_k}) \;=\; \nu_0(a^{\sharp_1}_{p_1}a^{\sharp_2}_{p_2}\dots a^{\sharp_k}_{p_k}) \;+\; \mathrm{Rem}_{T\lambda^{-2}}(\mathbf{p}_k)\,, \tag{1.78}$$

controlling $\mathrm{Rem}_{T\lambda^{-2}}(\mathbf{p}_k)$ implies that $\nu_{T\lambda^{-2}}$ is quasifree to k^{th} order. Similar to [129], we do not need to propagate quasifreeness to arbitrary orders to derive a Boltzmann equation; instead, adopting their notion of *restricted quasifreeness*, it is sufficient to show that $\nu_{T\lambda^{-2}}$ is approximately restricted quasifree up to eight-point correlation functions, see (1.62). However, we choose not to explicitly prove such a statement, and, instead, calculate the evolution of f explicitly. We leave the proof of a more general result of the form (1.78) to the interested reader, which will be straightforward using the tools developed in this work.

It remains to understand how we can control $\mathrm{Rem}_t(p)$. For that, we use the fact that $\hat{v}$ is bounded to show that

$$\mathcal{H}_{cub}(t) \lesssim \frac{\lambda}{\sqrt{N}}(\mathcal{N}_b + |\Lambda|)^{\frac{3}{2}}\,, \tag{1.79}$$

$$\mathcal{H}_{quart}(t) \lesssim \frac{\lambda}{N}(\mathcal{N}_b + |\Lambda|)^{2}\,, \tag{1.80}$$

see Lemmas 4.4 and 4.5. When bounding products of non number conserving operators, one needs to take into account the growth of the particle number, see Lemma 4.3. This is consistent with the fact that fluctuation particles are being absorbed into and emitted from the BEC. It turns out that in order to control $\mathrm{Rem}_t(p)$, it suffices to only consider three, and thus a fixed number of Duhamel iterations. Hence, we apply the bounds (1.49), (1.50) on $a^+_p a_p$ with the bound (1.79) and (1.80) on $\mathcal{H}_I(t)$ to obtain

$$|\,\mathrm{Rem}_{T\lambda^{-2}}(p)| \lesssim \frac{T^3}{N^{\frac{3}{2}}\lambda^3}\sup_{t\in[0,T\lambda^{-2}]}\nu_t\left((\mathcal{N}_b+|\Lambda|)^{\frac{11}{2}}\left(1\,+\,\frac{\sqrt{\mathcal{N}_b+|\Lambda|}}{\sqrt{N}}\right)^3\right). \tag{1.81}$$

In order to bound $\nu_t(\mathcal{N}_b^k)$, we employ a result by Rodnianski and Schlein [269], see also [74], to obtain that

$$\nu_t\big((\mathcal{N}_b+|\Lambda|)^{\frac{\ell}{2}}\big) \lesssim_{|\Lambda|,\ell} e^{K_\ell\lambda|\Lambda|t}\,. \tag{1.82}$$

In particular, (1.81) then yields the constraint

$$\frac{|\Lambda|}{\lambda} \lesssim \frac{\log N}{\log\log N}\,, \tag{1.83}$$

in order to suppress $\mathrm{Rem}_{T\lambda^{-2}}(p)$ compared to the leading order Boltzmann term in the evolution of F, see (1.70). In particular, this implies the scaling

(1) $L \sim 1$ fixed: $\lambda \sim \frac{\log\log N}{\log N}$,

(2) $L \sim \lambda^{-2-}$: $\lambda \sim \left(\frac{\log\log N}{\log N}\right)^{\frac{1}{7}-}$.

1.8 Fluctuations Beyond HFB

Now that we understand that approximate restricted quasifreeness can be propagated, we would like to comment on the fact that the fluctuation dynamics does not preserve number conservation. We have that, for a non number conserving quasifree state μ, a more general version of the Wick-Theorem holds: Let $b^{\#} := a^{\#} - \mu(a^{\#})$. Then μ is quasifree if and only if it satisfies (1.23) with the operators $a^{\#}$ replaced by their centered counterparts $b^{\#}$.

Since quasifreeness implies that $n-$point correlation functions $\langle a^{\#_1} \dots a^{\#_n} \rangle$ are determined by the one- and two-point correlation functions, and since by (1.78), $\nu_{T\lambda^{-2}}$ is approximately restricted quasifree, we need to include the dynamics of

$$\Phi_t := \frac{\nu_t(a_0)}{|\Lambda|}, \tag{1.84}$$

$$g_t(p) := \frac{\nu_t(a_p a_{-p})}{|\Lambda|} \tag{1.85}$$

in our analysis. We have that Φ_t captures the corrections to the condensate dynamics. g_t is the rate of absorption into or emission from the BEC of pairs of thermal bosons.

Recall that, due to translation-invariance of the condensate, ν_t is also translation-invariant, which is why we have that $\nu_t(a_p) = \Phi_t \delta(p)$, see Lemma A.1. Due to ν_0 being number conserving, we have that $\Phi_0 = g_0 = 0$. Arguing as in the case of f above, we are also interested in the dynamics of the centered, mesoscopic counterparts

$$\Psi_T := \frac{\nu_{T\lambda^{-2}}(a_0)}{|\Lambda|}, \tag{1.86}$$

$$G_T(p) := \frac{\nu_{T\lambda^{-2}}(a_p a_{-p}) - \nu_{T\lambda^{-2}}(a_p)\nu_{T\lambda^{-2}}(a_{-p})}{|\Lambda|}. \tag{1.87}$$

We show that

$$\Psi_T = \frac{-ic_1(f_0)T}{N^{\frac{1}{2}}\lambda} + O\left(\frac{1}{N^{\frac{1}{2}+\delta_N}\lambda}\right), \tag{1.88}$$

$$G_T = (T + T^2) O\left(\frac{1}{N\lambda^2}\right), \tag{1.89}$$

where, assuming that $\hat{v}$ is real-valued, $c_1(f_0)$ is real-valued. At leading order, we show that the dynamics of G is completely determined by F. As a consequence of (1.89), and recalling (1.61) and (1.62), we have that G merely contributes lower-order corrections to the Boltzmann dynamics for F.

1.9 Scaling of λ and N

We emphasize that the parameter $N \gg 1$ accounts for the L^2 mass of the BEC per unit volume, and is unrelated to the ~ 1 density of fluctuation particles around the BEC. Hence $1/N$ yields a small perturbation parameter in the expansion of the full dynamics, in addition to the coupling constant $0 < \lambda \ll 1$ characterizing the HFB dynamics. Boßmann et al. [67] give an entire expansion of the fluctuation dynamics in powers of λ/N, in terms of effective Hamiltonians for a given order of precision. Our analysis differs in that we choose $\lambda \ll 1$ and $N \gg 1$ suitably in order to be able to extract effective equations for the moments F, G, Ψ, while keeping the error sufficiently small. In particular, our time scale is $O((\log N/\log\log N)^{\alpha})$, $\alpha > 0$, instead of $O(1)$. In the latter case, the Boltzmann dynamics cannot be observed.

In the derivation of Boltzmann equations in classical collisional systems along the lines of Lanford's approach, $O(1)$ many collisions take place during the relevant time scale (which is inversely proportional to the mean free path). In our result, we encounter a similar situation; the error of order $O(N^{-1-})$, in (1.75) dominates after $O(1)$ collisions have taken place; this can be easily seen by iterating the Duhamel formula (1.75) twice (as the k-th order terms in the Duhamel expansion, of size $O(N^{-k})$, account for k collisions). We also note that, compliant with a kinetic scaling regime, particle velocities do not scale in our problem.

Our results are limited to the parameter regime $\lambda \sim (\log\log N/\log N)^{1/7-}$ when $L \sim \lambda^{-2-}$, similar for $\lambda \sim \log\log N/\log N$ when $L \sim 1$, and $N \gg 1$. This is, in part, due to technical reasons, but we do not expect the fluctuation dynamics to remain of the form (1.75), (1.88), (1.89) for longer time scales, even if our approach is extended to the next order of magnitude. We expect the analogous to hold for the parameter regime $\lambda \sim \frac{\log\log N}{\log N}$ when $L \sim 1$.

Remark 1.3 For works studying the perturbation expansion for a fermionic gas, we refer to [33, 46–49, 51, 86, 87, 90, 91, 104, 106, 210, 211, 248, 261] and references therein.

2 Main Results

2.1 Notation

We introduce the rescaled $L^a(\Lambda^*)$ norms

$$\|f\|_{L^a(\Lambda^*)} := |\Lambda|^{-\frac{1}{a}}\|f\|_{\ell^a(\Lambda^*)} \quad \text{if } 1 \le a < \infty\,, \tag{2.1}$$

and accordingly

$$\|f\|_{a\cap\infty,d} := \|f\|_{L^a(\Lambda^*)} + \|f\|_{\ell^\infty(\Lambda^*)}\,, \tag{2.2}$$

$$\|f\|_d := \|f\|_{1\cap\infty,d}\,, \tag{2.3}$$

$$\|f\|_{m,c} := \sum_{n=0}^{m}\left\|\langle|\cdot|\rangle^n D^{m-n} f\right\|_{L^1(\mathbb{R}^3)} \quad \text{if } m \in \mathbb{N}\,. \tag{2.4}$$

Moreover, we introduce the weight

$$w(p) := 1 + \frac{1}{|p|^2}\,. \tag{2.5}$$

Whenever $wf \in L^\infty(B_R(0)\setminus\{0\})$ for some $R > 0$, we define the weighted norms

$$\|f\|_{w,d} := \|wf\|_d\,, \tag{2.6}$$

$$\|f\|_{m,w,c} := \|wf\|_{m,c}\,. \tag{2.7}$$

Recall that

$$\mathcal{H}_{cub}(t) = \frac{\lambda}{\sqrt{N}}\int_{(\Lambda^*)^3} d\mathbf{p}_3\, \hat{v}(p_2)\delta(p_1+p_2-p_3) \\ e^{it\mathcal{H}_{HFB}}\big(a^+_{p_1}a^+_{p_2}a_{p_3} + h.c.\big)e^{-it\mathcal{H}_{HFB}}\,, \tag{2.8}$$

$$\mathcal{H}_{quart}(t) = \frac{\lambda}{2N}\int_{(\Lambda^*)^3} d\mathbf{p}_3\, \hat{v}(p_1-p_3)\delta(p_1+p_2-p_3-p_4) \\ e^{it\mathcal{H}_{HFB}}a^+_{p_1}a^+_{p_2}a_{p_3}a_{p_4}e^{-it\mathcal{H}_{HFB}}\,, \tag{2.9}$$

where

$$\mathcal{H}_{HFB} = \int_{\Lambda^*} dp\, \big(E(p) + \lambda \hat{v}(p)\big) a_p^+ a_p \,+\, \frac{\lambda}{2} \int_{\Lambda^*} dp\, \hat{v}(p) \left(a_p^+ a_{-p}^+ \,+\, a_p a_{-p}\right), \tag{2.10}$$

$E(p) = \frac{|p|^2}{2}$, and $\int_{\Lambda^*} \equiv \frac{1}{|\Lambda|} \sum_{\Lambda^*}$. In this context, we also recall the Bogoliubov dispersion relation

$$\Omega(p) \,=\, \sqrt{E(p)(E(p) + 2\lambda \hat{v}(p))}. \tag{2.11}$$

We are interested in the evolution of the correlation functions

$$\Psi_T \,=\, \frac{\nu_{T\lambda^{-2}}(a_0)}{|\Lambda|}\,, \tag{2.12}$$

$$F_T(p) \,=\, \frac{\nu_{T\lambda^{-2}}(a_p^+ a_p) - |\nu_{T\lambda^{-2}}(a_p)|^2}{|\Lambda|}\,, \tag{2.13}$$

$$G_T(p) \,=\, \frac{\nu_{T\lambda^{-2}}(a_p a_{-p}) - \nu_{T\lambda^{-2}}(a_p)\nu_{T\lambda^{-2}}(a_{-p})}{|\Lambda|}\,. \tag{2.14}$$

As explained in 1.7.1, there are additional dominant terms in the Boltzmann collision terms for F, and also G in the case of $L \sim 1$ fixed. Thus, we introduce the collision operators

$$\mathcal{Q}_{d,G;T,\lambda}(h)[J] := \frac{1}{\lambda^2} \int_{[0,T]^2} \mathbb{1}_{S_1 \geq S_2} d\mathbf{S}_2 \ \mathrm{col}_d(h_{\cdot\lambda^2})[J](\mathbf{S}_2/\lambda^2)\,, \tag{2.15}$$

$$q^{(j)}_{d,F;\mathbf{S}_2,\lambda}(H_{S_2})[J] := \mathrm{bol}^{(j)}(H_{\cdot\lambda^2})[J](\mathbf{S}_2/\lambda^2)\,, \quad j \in \{1,2\}\,, \tag{2.16}$$

and the pair absorption operator

$$\begin{aligned}\mathcal{A}_{d;T,\lambda}(h)[J] := &\int_0^T dS\, \mathrm{abs}_{quart,d}(h_{\cdot\lambda^2})[J](S/\lambda^2) \\ &+ \frac{1}{\lambda^2} \int_{[0,T]^2} \mathbb{1}_{S_1 \geq S_2} d\mathbf{S}_2 \mathrm{abs}_{cub,d}(h_{\cdot\lambda^2})[J](\mathbf{S}_2/\lambda^2)\,,\end{aligned} \tag{2.17}$$

for any test function J, where $\mathbf{S}_2 = (S_1, S_2)$. The expressions for $\mathrm{bol}^{(j)}$, $j \in \{1,2\}$, col_d, $\mathrm{abs}_{quart,d}$, and $\mathrm{abs}_{cub,d}$ are lengthy, and we refer the reader to Sect. 5.1.1 for their definition. In the case $L \sim \lambda^{-2-}$, we also define the continuous counterparts $\mathcal{Q}_{c,G;T,\lambda}$, $\mathcal{A}_{c;T,\lambda}$ by replacing sums $\int_{\Lambda^*} dp$ over momenta by Lebesgue integrals $\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3}$.

The subscript 'd' in the notation refers to the fact that momentum is summed over the discrete set Λ^*; 'c' on the other hand refers to the continuum approximation. Henceforth, we refer to the case with $L \sim 1$ fixed as the 'discrete case', and $L \sim \lambda^{-2-}$ as the 'continuum approximation'.

$\mathcal{Q}_{d,G;T,\lambda}(h)$ resp.$\mathcal{Q}_{c,G;T,\lambda}(h)$ is a Boltzmann collision operator for G, and it is, after cancellations, quadratic in h, and $\mathcal{A}_{d;T,\lambda}(h)$ resp. $\mathcal{A}_{c;T,\lambda}(h)$ corresponds to the leading order of the expected rate of absorption of a pair of fluctuation bosons into the BEC, and it is linear in h.

We denote the mollified cubic Boltzmann operator in the evolution of F by

$$\begin{aligned}&Q_{d;T-S,\lambda}(F_S)[J] \\ &:= \int_{(\Lambda^*)^3} d\mathbf{p}_3 \frac{\sin\left(\frac{\Omega(p_1)+\Omega(p_2)-\Omega(p_3)}{\lambda^2}(T-S)\right)}{\Omega(p_1)+\Omega(p_2)-\Omega(p_3)} \delta(p_1+p_2-p_3)\end{aligned}$$

$$(\hat{v}(p_1)+\hat{v}(p_2))^2\big(J(p_1)+J(p_2)-J(p_3)\big)$$
$$\big((1+F_S(p_1))(1+F_S(p_2))F_S(p_3)\;-\;F_S(p_1)F_S(p_2)(1+F_S(p_3))\big)\,. \tag{2.18}$$

Again, in the continuum approximation, $Q_{c;T-S,\lambda}$ is defined by replacing the lattice sum $\int_{\Lambda^*}$ by the Lebesgue integral $\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}$.

In addition, $q^{(j)}_{d,F;\mathbf{S}_2,\lambda}$ are higher order Boltzmann type collision terms for the equations governing F_T, where j accounts for the order λ^j from which they are derived. As explained above, they are of lower order when $L\sim\lambda^{-2-}$.

2.2 Assumptions

We summarize all the assumptions described in the previous section. We also add the following restrictions required in our results.

(1)

$$\nu_0(A)=\frac{1}{\mathrm{Tr}\, e^{-\mathcal{K}}}\,\mathrm{Tr}\left(e^{-\mathcal{K}}A\right) \tag{2.19}$$

for all observables A, is a quasifree, translation-invariant state that is number conserving, with

$$\mathcal{K}=\int_{\Lambda^*}dp\,K(p)a_p^+a_p\,, \tag{2.20}$$

$$f_0(p)=\frac{\nu_0(a_p^+a_p)}{|\Lambda|}\;=\;\frac{1}{e^{K(p)}-1}\,, \tag{2.21}$$

and $K(p)\geq\kappa_0>0$.

(2) ν_t satisfies

$$\begin{cases} i\partial_t\nu_t(A) &= \nu_t([A,\mathcal{H}_{cub}(t)+\mathcal{H}_{quart}(t)])\,,\\ \nu_0(A) &= \frac{1}{\mathrm{Tr}(e^{-\mathcal{K}})}\,\mathrm{Tr}\left(e^{-\mathcal{K}}A\right)\end{cases} \tag{2.22}$$

for all observables A.

(3) The Fourier transform $\hat{v}$ of v, see Sect. 1.6, is a non-negative, radial function.

(4) If $L\sim 1$ is fixed, assume $\|\hat{v}\|_{w,d},\,\|f_0\|_d<\infty$.

(5) If $L\sim\lambda^{-2-}$, assume $\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\,\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}<\infty$.

In either case, we assume that $\hat{v}$ satisfies

$$\int_\Lambda dx\,v(x)\;=\;\hat{v}(0)\;=0\,. \tag{2.23}$$

Moreover, this implies that the leading order condensate wave function $\phi_0\equiv|\Lambda|^{-1/2}$ can be chosen to be a stationary, translation-invariant solution of the Hartree equation

$$i\partial_t\phi_t\;=\;-\Delta\phi_t+\lambda|\Lambda|v*|\phi_t|^2\phi_t\,. \tag{2.24}$$

As described above, N denotes the BEC density, and $\lambda>0$ is an (additional) coupling constant, defining the HFB coupling, see (1.34).

We are now ready to formulate our main results.

2.3 Statement of Results

Theorem 2.1 (Discrete case) *Let $T > 0$, $L \geq 1$, and let $N > 0$ denote the BEC density. Choose*

$$\lambda = \frac{\log\log N}{\log N}. \tag{2.25}$$

Then, under the assumptions stated in Sect. 2.2 and with the notations in Sect. 2.1, there exist constants

$$C_0 = C_0(\|\hat{v}\|_{w,d}, \|f_0\|_d, |\Lambda|, T), \tag{2.26}$$
$$N_0 = N_0(\|\hat{v}\|_{w,d}, |\Lambda|, T), \tag{2.27}$$

such that, for all $N \geq N_0$ we have that

$$\delta_1 := \frac{\log\frac{1}{\lambda}}{\log N} > 0, \quad \delta_2 := \frac{1}{2} - \frac{4\log\frac{1}{\lambda}}{\log N} > 0, \tag{2.28}$$

and that

$$\Big|\Psi_T + \frac{i}{N^{\frac{1}{2}}\lambda}\int_0^T dS \int_{\Lambda^*} dp\, \hat{v}(p) F_S(p)\Big| \leq \frac{C_0}{N^{\frac{1}{2}+\delta_1}\lambda}, \tag{2.29}$$

$$\Big|\int_{\Lambda^*} dp\,\big(F_T(p) - F_0(p)\big)J(p) - \frac{1}{N}\Big(\int_0^T dS\, Q_{d;T-S,\lambda}(F_S)[J] + \int_{[0,T]^2} d\boldsymbol{S}_2\, \mathbb{1}_{s_1\geq s_2} \sum_{j=1}^{2} \lambda^j q^{(j)}_{d,F;\boldsymbol{S}_2,\lambda}(F_{S_2})[J]\Big)\Big| \leq \frac{C_0 \|J\|_{\ell^\infty(\Lambda^*)}}{N^{1+\delta_1}}, \tag{2.30}$$

$$\Big|\int_{\Lambda^*} dp\, G_T(p)J(p) - \frac{1}{N}\Big(\mathcal{A}_{d;T,\lambda}(F)[J] + \mathcal{Q}_{d,G;T,\lambda}(F)[J]\Big)\Big| \leq \frac{C_0\|J\|_{2\cap\infty,d}}{N^{1+\delta_2}} \tag{2.31}$$

for all test functions J.

The error terms on the right-hand sides of (2.29)–(2.31) are subleading in N with respect to the main terms appearing on the respective left-hand sides.

Theorem 2.2 (Continuum approximation) *Let $T > 0$, $r > 6$, $\varepsilon > 0$, and let $N > 0$ denote the BEC density. Fix*

$$\lambda = \Big(\frac{\log\log N}{\log N}\Big)^{\frac{r}{(7+\varepsilon)r+6}}, \tag{2.32}$$
$$L = \lambda^{-2-\frac{2}{r}-\varepsilon}. \tag{2.33}$$

Then, under the assumptions stated in Sect. 2.2 and with the notations in Sect. 2.1, there exist constants

$$C_0 = C_0(\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}, \|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}, r, \varepsilon, T), \tag{2.34}$$

$$C_1 = C_1(r, \varepsilon)\,, \tag{2.35}$$

$$N_0 = N_0(\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, r, \varepsilon, T)\,, \tag{2.36}$$

such that, for all $N \geq N_0$, and for

$$\delta := \frac{C_1 \log \frac{1}{\lambda}}{\log N} > 0\,, \tag{2.37}$$

we have that

$$\Big|\Psi_T + \frac{i}{(2\pi)^3 N^{\frac{1}{2}} \lambda} \int_0^T dS \int_{\mathbb{R}^3} dp\, \hat{v}(p) F_S(p)\Big| \leq \frac{C_0}{N^{\frac{1}{2}+\delta}\lambda}\,, \tag{2.38}$$

$$\Big|\int_{\Lambda^*} dp\, \big(F_T(p) - F_0(p)\big) J(p) - \frac{1}{N}\int_0^T dS\, Q_{c;T-S,\lambda}(F_S)\Big| \leq \frac{C_0 \|J\|_{W^{2\lfloor \frac{r}{2} \rfloor+2,\infty}}}{N^{1+\delta}}\,, \tag{2.39}$$

$$\Big|\int_{\Lambda^*} dp\, G_T(p) J(p) - \frac{1}{N}\Big(\mathcal{A}_{c;T,\lambda}(F)[J] + \mathcal{Q}_{c,G;T,\lambda}(F)[J]\Big)\Big| \leq \frac{C_0 \|J\|_{2(\lfloor \frac{r}{2} \rfloor+1),c}}{N^{1+\delta}} \tag{2.40}$$

for all test functions J.

The error terms on the right-hand sides of (2.38)–(2.39) are subleading in N with respect to the main terms appearing on the respective left-hand sides.

The main order term in the evolution of F is given by $\frac{T}{N} Q(f_0)[J]$, where

$$\begin{aligned} Q(f_0)[J] &:= \frac{\pi}{(2\pi)^6} \int_{\mathbb{R}^9} dp_1\, dp_2\, dp_3\, \delta\big(E(p_1) + E(p_2) - E(p_3)\big)\delta(p_1 + p_2 - p_3) \\ &\quad (\hat{v}(p_1) + \hat{v}(p_2))^2 \big(J(p_1) + J(p_2) - J(p_3)\big) \\ &\quad \big((1 + f_0(p_1))(1 + f_0(p_2)) f_0(p_3) - f_0(p_1) f_0(p_2)(1 + f_0(p_3))\big) \end{aligned} \tag{2.41}$$

denotes the (energy conserving) quantum Boltzmann collision operator.

Remark 2.3 Theorem 2.2 is not the continuum limit for the dynamics. Instead, it quantifies how the collision operator can be approximated by its continuous counterpart. In particular, energy conservation cannot hold precisely for finite times, in consistence with the Heisenberg uncertainty principle, see Remark 2.5.

Remark 2.4 In this work, we are not attempting to analyze G_T in more detail beyond (2.40). We expect a more detailed analysis to yield

$$\mathcal{A}_{c;T,\lambda}(F)[J] + \mathcal{Q}_{c,G;T,\lambda}(F)[J] \sim T + T^2\,, \tag{2.42}$$

based on similar arguments as we present to control the Boltzmann collision term for F.

Remark 2.5 We point out that $Q_{d;T-S,\lambda}$ resp. $Q_{c;T-S,\lambda}$ contain the Bogoliubov dispersion Ω for sound waves propagating as fluctuations around the BEC. As stated in Theorem 2.2, the collision operator Q emerges in the limit $\lambda \searrow 0$. In the limit $\lambda \searrow 0$, we have that

$\Omega(p) \to E(p)$, whereby we retrieve the Dirac-δ on the hypersurface $\{(p_1, p_2) \in \mathbb{R}^6 \mid E(p_1) + E(p_2) = E(p_1 + p_2)\}$ of energy conservation. Let

$$\delta_\varepsilon(x) := \frac{1}{\varepsilon\pi} \frac{\sin^2(x/\varepsilon)}{(x/\varepsilon)^2} = \frac{\varepsilon}{\pi} \frac{\sin^2(x/\varepsilon)}{x^2}, \tag{2.43}$$

which defines a Dirac sequence, $\int_{\mathbb{R}} dx\, \delta_\varepsilon(x) = 1$. Observing that

$$\frac{\sin\left(\Delta_{cub}\Omega \frac{T-S}{\lambda^2}\right)}{\Delta_{cub}\Omega} = -\partial_S\Big((T-S)\delta_{\frac{2\lambda^2}{T-S}}(\Delta_{cub}\Omega)\Big), \tag{2.44}$$

where $\Delta_{cub}\Omega(\mathbf{p}_2) = \Omega(p_1) + \Omega(p_2) - \Omega(p_1 + p_2)$, this implies that the mollified quantum Boltzmann collision terms in (2.30) and (2.39) have the form

$$\begin{aligned}
&\int_0^T dS\, Q_{c/d;T-S,\lambda}(F_S)[J] \\
&= \int_0^T dS \int_{\mathcal{D}} d\mathbf{p}_3 \frac{\sin\left(\Delta_{cub}\Omega(\mathbf{p}_2)\frac{T-S}{\lambda^2}\right)}{\Delta_{cub}\Omega(\mathbf{p}_2)} H_S(\mathbf{p}_3) \\
&= \pi T \int_{\mathcal{D}} d\mathbf{p}_3\, \delta_{\frac{2\lambda^2}{T}}(\Delta_{cub}\Omega(\mathbf{p}_2)) H_0(\mathbf{p}_3; J) \\
&\quad + \pi \int_0^T dS \int_{\mathcal{D}} d\mathbf{p}_3\, \delta_{\frac{2\lambda^2}{T-S}}(\Delta_{cub}\Omega(\mathbf{p}_2))(T-S)\partial_S H_S(\mathbf{p}_3; J),
\end{aligned} \tag{2.45}$$

where $\mathbf{p}_3 = (p_1, p_2, p_3)$ and $\mathcal{D} = (\Lambda^*)^3$ in the discrete case, and $\mathcal{D} = \mathbb{R}^9$ for the continuum approximation. Here,

$$\begin{aligned}
&H_S(\mathbf{p}_3; J) \\
&:= |\hat{v}(p_1) + \hat{v}(p_2)|^2 \big(J(p_1) + J(p_2) - J(p_3)\big)\delta(p_1 + p_2 - p_3) \\
&\quad \Big((1 + F_S(p_1))(1 + F_S(p_2))F_S(p_3) - F_S(p_1)F_S(p_2)(1 + F_S(p_3))\Big),
\end{aligned} \tag{2.46}$$

where, in the continuum approximation, H contains an additional factor $1/(2\pi)^6$. In particular, this representation makes manifest that the mollification of the quantum Boltzmann collision operator corresponds to approximate energy conservation up to an error of order $O(t^{-1}) = O(\lambda^2/T)$, in compliance with the time-energy Heisenberg uncertainty principle.

Remark 2.6 Q in (2.41) can be evaluated via the Coarea Formula, yielding

$$\begin{aligned}
&Q(f_0)[J] \\
&= \frac{\pi}{(2\pi)^6} \int_{E(p_1)+E(p_2)=E(p_1+p_2)} \frac{d\mathcal{H}^5(\mathbf{p}_2)}{|\mathbf{p}_2|} (\hat{v}(p_1) + \hat{v}(p_2))^2 \\
&\big(J(p_1) + J(p_2) - J(p_1 + p_2)\big) \\
&\Big((1 + f_0(p_1))(1 + f_0(p_2))f_0(p_1 + p_2) - f_0(p_1)f_0(p_2)(1 + f_0(p_1 + p_2))\Big),
\end{aligned} \tag{2.47}$$

where $d\mathcal{H}^5$ is the induced Hausdorff measure on the hypersurface $\{(p_1, p_2) \in \mathbb{R}^6 \mid E(p_1) + E(p_2) = E(p_1 + p_2)\}$.

Remark 2.7 Theorem 2.2 implies that, for high regularity $r \gg 6$, for the maximal time scale $t_{max} \sim \lambda^{-2} \sim (\log N / \log\log N)^{\frac{2}{7}-}$ and length $L \sim \lambda^{-2-} \ll t_{max}$, we obtain that

$$\Psi_T \sim \frac{T}{N^{\frac{1}{2}-}} + O(N^{-\frac{1}{2}}), \tag{2.48}$$

$$F_T - F_0 \sim \frac{T}{N} + O(N^{-1-}) \tag{2.49}$$

for $N \gg 1$, where $F_0 \sim 1$. We point out that one of the difficulties in extracting a quantum Boltzmann dynamics for F stems from the fact that F is centered with $|\Psi|^2$, which is at least an order $O(N^{0+})$ larger than the Boltzmann collision term for F itself.

There are five characteristic length scales involved in our derivation:

(1) a BEC with large density N,
(2) thermal fluctuations with density ~ 1,
(3) the HFB coupling of size λ,
(4) the linear system size L,
(5) the time scale t.

A major difficulty that is overcome in this work is to identify a parameter regime which allows the Boltzmann dynamics to dominate over error terms.

Remark 2.8 The collision term for G is a functional only of F, due to our choice of initial data with $G_0 = 0$. Therefore, solving the Boltzmann equation for F, and substituting into

$$\mathcal{I}_G(F)[J] := \frac{1}{N}\mathcal{A}_{c;T,\lambda}(F)[J] + \frac{1}{N\lambda^2}\mathcal{Q}_{c,G;T,\lambda}(F)[J], \tag{2.50}$$

integration time, yields G_T. A key reason for which the extraction of the Boltzmann equation for F is a difficult problem, is the fact that we expect $\mathcal{I}_G(F)[J]$ to be of the same order of magnitude as the collision operator for F,

$$\mathcal{I}_F(F)[J] := \frac{1}{N}\int_0^T dS\, Q_{c;T-S,\lambda}(F_S)[J]. \tag{2.51}$$

In our case, $\mathcal{I}_G(F)[J]$ does not depend on G because the initial state ν_0 is chosen to conserve the particle number $\mathcal{N}_b$. The explicit expression for $\mathcal{I}_G(F)$ is somewhat lengthy and not sufficiently enlightening to be presented here. We obtain a closed system of equations for (Ψ, F, G) because of the approximate persistence of (restricted) quasifreeness of ν_0 in the scaling regime of this problem.

If we expand the dynamics of (Ψ, F, G) to lower orders, we expect $\mathcal{I}_F(\Psi, F, G)$ and $\mathcal{I}_G(\Psi, F, G)$ to be coupled non-trivially.

Remark 2.9 Our purpose of introducing a condensate of large density N is due to the fact that its subleading order interactions with the fluctuation field are of Boltzmann type (the leading order is determined by the HFB dynamics); the latter are not drowned out by the error term, due to the largeness of N. On the other hand, a quartic Boltzmann collision term emerges, as expected, from our analysis, but it is a lower order term that is buried in the error terms because it does not couple to the condensate.

We refer to Proposition 5.1 for more details. We do expect an analysis to the next subleading order, conjecturally involving appropriate quantum corrections to the BEC and HFB dynamics, to reveal this fourth order collision term of Boltzmann-Uhlenbeck-Uehling type, separated from the error term.

Remark 2.10 If it is assumed that the BEC is non-stationary and that both the BEC and the thermal fluctuation field are not translationally invariant, the first term in the Duhamel expansion for $\nu_t(a_p^+ a_q)$, expressed with a single time integral, does not vanish, contrary to the situation considered in the paper at hand. This additional term couples nontrivially to both the HFB and the Boltzmann dynamics, and will lead to a different system of PDEs than the one derived here. We expect the rigorous analysis of this system to be considerably more involved, and leave it for future work.

2.4 Sketch of the Proof

The strategy of our proof is to start by calculating one collision, corresponding to calculating the Duhamel expansion to second order in the coupling λ. The tail in the Duhamel expansion corresponds to recollisions. Using bounds available in the second-quantization formulation, we are able to bound this tail. For that, it is crucial to exploit the fact that the initial condensate density N provides a large perturbation parameter. Accordingly, the resulting leading order cubic Boltzmann collision operator appears with an overall factor $1/N$. This allows us to compare the density of thermal fluctuation particles for positive times with the initial density. We can observe that the superposition of HFB oscillations results in corrections to the leading order $2 \leftrightarrow 1$ processes coming from the collision of two thermal particles and one of those being absorbed into or emitted from the BEC. In addition, if we consider the continuous approximation, we show that the collision operator can be approximated by its continuous counterpart.

We start by applying Duhamel's principle to (2.22), using one recursion for Φ, and two for f and g. The main term is recovered by evaluating the terms involving ν_0. In Sect. 4, in order to control higher-order terms involving the full dynamics ν_t, we establish uniform-in-time bounds on $f[J]$, $g[J]$, $\mathcal{H}_{cub}(t)$, and $\mathcal{H}_{quart}(t)$ with respect to the number operator $\mathcal{N}_b$. We use an a-priori bound on the growth of $\nu_t(\mathcal{N}_b^k)$ established in [74, 269]. This will yield the restriction $|\Lambda|/\lambda = O(\log N/\log\log N)$. Moreover, we show closeness of ν_t to ν_0 in a suitable sense, which allows us to exploit that ν_0 is approximately quasifree.

In Sect. 5, we control the tail term in the Duhamel expansion using the previously established operator bounds and quantify propagation of moments of the number operator, using [74, 269]. The terms in the Duhamel expansion are expressed by way of $a(t) = \mathcal{V}_{HFB}^*(t) a \mathcal{V}_{HFB}(t)$ and $a^+(t)$. $\mathcal{H}_{HFB}$ is quadratic in a and a^+, thus $a(t)$ and $a^+(t)$ are linear combinations of a and a^+. That enables us to control the proximity of ν_t to ν_0, which is quasifree, in order to evaluate the main terms in the Duhamel expansion by means of Wick's Theorem.

In the discrete case, we observe interference phenomena in our scaling regime $L \leq t \sim \lambda^{-2}$. The Boltzmann collision terms are lattice sums in momentum space, and their magnitudes depend on whether terms that vary with time have particular values in $\mathbb{Q}$. This effect becomes negligible for box length $L \sim \lambda^{-2-} \gg t$ and $\lambda = \lambda(N) > 0$ chosen small enough. The latter means that we will ultimately set $\lambda = \lambda(N)$, and choose N large enough.

We prove a discretization Lemma 5.5 that allows us to improve the rate of convergence dependent on the regularity of f_0 and $\hat{v}$, beyond the trivial bound. As we noticed after completing this work, our approach here appears to be related to the numerical error estimates for the trapezoid rule in numerical mathematics via Poisson summation, see [294].

We use the Duhamel expansion and the approximate quasifreeness of ν_t to relate f_0 back to f_s. In order to control the large magnitude of condensate terms in the expansion, we need to rewrite the equations for the centered expectations $f_s(p) - |\Phi_s|^2\delta(p)$ and $g_s(p) - \Phi_s^2\delta(p)$.

In Sect. 6, we collect all results to compute the main order terms in Theorem 2.1 and 2.2, and in Sect. 7, we compute the effective equations.

Remark 2.11 We note that the Boltzmann-type equations in Theorem 2.1 and 2.2 are presented in their integral and weak forms. No smallness assumption on f is necessary for our result to hold. In this work, we will not further investigate questions regarding the well-posedness of the corresponding Cauchy problem of kinetic equations in the context of nonlinear PDE. Some works in this direction are referenced in Sect. 1.1.2, see in particular [27, 29, 140–142].

3 Preliminaries

Some of our estimates will be formulated for finite number subspaces of $\mathcal{F}$. We introduce the projectors

$$P_n \;:=\; P_{\mathcal{F}_n}\,. \tag{3.1}$$

We will consider $\mathcal{F}_n$ embedded into $\mathcal{F}$ and we identify P_n as maps $\mathcal{F} \to \mathcal{F}$. By the spectral theorem,

$$P_n \;=\; \frac{1}{2\pi i}\oint_{\partial B_{1/2}(n)} \frac{dz}{\mathcal{N}_b - z}\,. \tag{3.2}$$

Observe that $a_x \mid_{F_n}$ defines a map

$$a_x \;:\; F_n \to F_{n-1} \tag{3.3}$$

for all $n \in \mathbb{N}$, with formal adjoint $a_x^+ \mid_{F_{n-1}}: F_{n-1} \to F_n$. To study the weak formulation of the effective equations, we introduce

$$\begin{aligned} f[J] &:= \int dp\, J(p) a_p^+ a_p\,, \\ g[J] &:= \int dp\, J(p) a_{-p} a_p\,, \\ g^*[J] := g[J]^+ &= \int dp\, J(p) a_p^+ a_{-p}^+\,. \end{aligned} \tag{3.4}$$

As a convenient notation for iterating Duhamel's formula, let

$$\Delta(t, j) := \{\mathbf{s}_j \in [0, t]^j \mid s_1 \geq \ldots \geq s_j\} \tag{3.5}$$

be a j-simplex, where $t > 0$ and $j \in \mathbb{N}$. We also recall that

$$\mathcal{H}_I(t) = \mathcal{H}_{cub}(t) + \mathcal{H}_{quart}(t)\,. \tag{3.6}$$

Let $(\mathcal{V}_{HFB}(t))_{t\in\mathbb{R}}$ be the solution of

$$\begin{cases} i\partial_t \mathcal{V}_{HFB}(t) & = \mathcal{H}_{HFB}\mathcal{V}_{HFB}(t)\,, \\ \mathcal{V}_{HFB}(0) & = \mathbf{1}\,. \end{cases} \tag{3.7}$$

We denote the $d-1$-dimensional Hausdorff measure of a smooth, embedded hypersurface in $\mathbb{R}^d$ by $d\mathcal{H}^{d-1}$.

If needed, we will keep track of the dependence of constants on parameters by adding the respective parameters in a subscript. Whenever the constants have no explicit dependence, they are assumed to be universal.

The following result is a direct consequence from iterating (2.22).

Lemma 3.1 *Assume that $A \in \mathfrak{A}$ is an observable, and that ν_t is defined as in (2.22). Then, for any $k \in \mathbb{N}$, we have that*

$$\nu_t(A) = \sum_{\ell=0}^{k-1} (-i)^\ell \int_{\Delta[t,\ell]} d\boldsymbol{s}_\ell \, \nu_0([[\cdots[A, \mathcal{H}_I(s_1)], \cdots], \mathcal{H}_I(s_\ell)]) + (-i)^k \int_{\Delta[t,k]} d\boldsymbol{s}_k \, \nu_{s_k}\big([[\cdots[A, \mathcal{H}_I(s_1)], \cdots], \mathcal{H}_I(s_k)]\big) . \tag{3.8}$$

We will be particularly interested in the cases $A \in \{a_0, f[J], g[J]\}$. To study the expansion, we derive the following useful identity.

Lemma 3.2 *Let*

$$\Omega(p) := \sqrt{E(p)(E(p) + 2\lambda\widehat{v}(p))} \tag{3.9}$$

denote the dispersion function for acoustic excitations, and let

$$a_p^+(t) := \mathcal{V}_{HFB}^*(t) a_p^+ \mathcal{V}_{HFB}(t) , \tag{3.10}$$
$$a_p(t) = \mathcal{V}_{HFB}^*(t) a_p \mathcal{V}_{HFB}(t) . \tag{3.11}$$

Moreover, let

$$\mathcal{M}(p) := \begin{pmatrix} -E(p) - \lambda\widehat{v}(p) & -\lambda\widehat{v}(p) \\ \lambda\widehat{v}(p) & E(p) + \lambda\widehat{v}(p) \end{pmatrix} . \tag{3.12}$$

Then,

$$\begin{pmatrix} a_p^+(t) \\ a_{-p}(t) \end{pmatrix} = \left[\cos(t\Omega(p))\,\mathbf{1} - i \frac{\sin(t\Omega(p))}{\Omega(p)} \mathcal{M}(p) \right] \begin{pmatrix} a_p^+ \\ a_{-p} \end{pmatrix} . \tag{3.13}$$

Proof We obtain the following system of ODEs,

$$\begin{aligned} i\partial_t a_p^+(t) &= \mathcal{V}_{HFB}^*(t)\,[a_p^+, \mathcal{H}_{HFB}^{\phi_0}]\,\mathcal{V}_{HFB}(t) \\ &= -(E(p) + \lambda\widehat{v}(p)) a_p^+(t) - \lambda\widehat{v}(p) a_{-p}(t) , \end{aligned} \tag{3.14}$$

$$\begin{aligned} i\partial_t a_{-p}(t) &= \mathcal{V}_{HFB}^*(t)\,[a_{-p}, \mathcal{H}_{HFB}^{\phi_0}]\,\mathcal{V}_{HFB}(t) \\ &= (E(p) + \lambda\widehat{v}(p)) a_{-p}(t) + \lambda\widehat{v}(p) a_p^+(t) , \end{aligned} \tag{3.15}$$

where $E(p) = \frac{p^2}{2}$, so that

$$i\partial_t \begin{pmatrix} a_p^+(t) \\ a_{-p}(t) \end{pmatrix} = \mathcal{M}(p) \begin{pmatrix} a_p^+(t) \\ a_{-p}(t) \end{pmatrix} \tag{3.16}$$

with

$$\mathcal{M}(p) := \begin{pmatrix} -E(p) - \lambda\widehat{v}(p) & -\lambda\widehat{v}(p) \\ \lambda\widehat{v}(p) & E(p) + \lambda\widehat{v}(p) \end{pmatrix} . \tag{3.17}$$

We observe that $\mathcal{M}(p)$ satisfies

$$\mathcal{M}^2(p) = \Omega^2(p)\,\mathbf{1}, \tag{3.18}$$

where

$$\Omega(p) := \sqrt{E(p)(E(p) + 2\lambda\widehat{v}(p))} \tag{3.19}$$

denotes the dispersion function of acoustic excitations; see, for instance, [112, 274]. In particular, we have $\mathcal{M}^{-1}(p) = \frac{1}{\Omega^2(p)}\mathcal{M}(p)$.

Hence,

$$\begin{aligned}\begin{pmatrix} a_p^+(t) \\ a_{-p}(t) \end{pmatrix} &= \exp\left(-it\begin{pmatrix} -E(p) - \lambda\widehat{v}(p) & -\lambda\widehat{v}(p) \\ \lambda\widehat{v}(p) & E(p) + \lambda\widehat{v}(p) \end{pmatrix}\right)\begin{pmatrix} a_p^+ \\ a_{-p} \end{pmatrix} \\ &= \left(\cos(t\Omega(p))\,\mathbf{1} - i\sin(t\Omega(p))\frac{\mathcal{M}(p)}{\Omega(p)}\right)\begin{pmatrix} a_p^+ \\ a_{-p} \end{pmatrix}. \end{aligned} \tag{3.20}$$

This finishes the proof. □

Corollary 3.3 *Assume $q \in [1, \infty]$ and $n \in \{0, 1, 2, 3\}$.*

Moreover, recall the assumptions $\widehat{v} \geq 0$ with $\widehat{v}(0) = 0$ from Sect. 2.2.

V_1, V_2, defined by

$$a_p^+(t) = \left(e^{i\Omega(p)t} + i\lambda^2\sin(\Omega(p)t)V_1(p)^2\right)a_p^+ + i\lambda\sin(\Omega(p)t)V_2(p)a_{-p}, \tag{3.21}$$

are given by

$$V_1(p)^2 := \frac{\hat{v}(p)^2}{\Omega(p)(\Omega(p) + E(p) + \lambda\hat{v}(p))}, \tag{3.22}$$

$$V_2(p) := \frac{\hat{v}(p)}{\Omega(p)}. \tag{3.23}$$

They satisfy the bounds

$$\|D^n V_1^2\|_q \leq C\|\hat{v}\|^{n+2}_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}, \tag{3.24}$$

$$\|D^n V_2\|_q \leq C\|\hat{v}\|^{n+1}_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}. \tag{3.25}$$

Similarly,

$$u_\lambda(t, p) := e^{i\Omega(p)t} + i\lambda^2\sin(\Omega(p)t)V_1(p)^2 \tag{3.26}$$

$$v_\lambda(t, p) := i\lambda\sin(\Omega(p)t)V_2(p) \tag{3.27}$$

satisfy

$$\|u_\lambda(t, \cdot)\|_{\ell^\infty(\Lambda^*)} \leq C(1 + \lambda^2\|\hat{v}\|^2_{w,d}), \tag{3.28}$$

$$\|v_\lambda(t, \cdot)\|_d \leq C\lambda\|\hat{v}\|_{w,d} \tag{3.29}$$

for all $q \in [1, \infty]$, all $\lambda \in (0, 1]$, and all $t > 0$.

Proof Lemma 3.2 implies

$$a_p^+(t) = \cos(t\Omega(p))a_p^+ - i\sin(t\Omega(p))\left(\frac{-E(p) - \lambda\hat{v}(p)}{\Omega(p)}a_p^+ - \lambda\frac{\hat{v}(p)}{\Omega(p)}a_{-p}\right). \tag{3.30}$$

Define $a := E(p)$ and $b := E(p) + 2\lambda\hat{v}(p)$. Setting

$$V_1(p)^2 = \frac{1}{\lambda^2}\left(\frac{a+b}{2\sqrt{ab}} - 1\right), \tag{3.31}$$

$$V_2(p) = \frac{1}{\lambda}\frac{b-a}{2\sqrt{ab}} = \frac{\hat{v}(p)}{\Omega(p)}, \tag{3.32}$$

we have that

$$a_p^+(t) = \left(e^{i\Omega(p)t} + i\lambda^2 \sin(\Omega(p)t)V_1(p)^2\right)a_p^+ + i\lambda \sin(\Omega(p)t)V_2(p)a_{-p} \,. \tag{3.33}$$

We start by establishing pointwise bounds. We have

$$\frac{a+b}{2\sqrt{ab}} - 1 = \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2\sqrt{ab}} = \frac{(b-a)^2}{2\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)^2} \,. \tag{3.34}$$

In particular, we get

$$\begin{aligned} V_1(p)^2 &= \frac{2\hat{v}(p)^2}{\Omega(p)(\sqrt{E(p)} + \sqrt{E(p)+2\lambda\hat{v}(p)})^2} \\ &= \frac{\hat{v}(p)^2}{\Omega(p)(\Omega(p)+E(p)+\lambda\hat{v}(p))}. \end{aligned} \tag{3.35}$$

Using $\hat{v} \geq 0$, we have the pointwise bounds

$$|V_1^2| \leq 2\frac{\hat{v}^2}{E^2} \,, \tag{3.36}$$

$$|V_2| \leq \frac{\hat{v}}{E} \,. \tag{3.37}$$

We thus have that

$$\begin{aligned} \|V_1^2\|_d &\leq C\|\hat{v}\|_{w,d}^2 \,, \\ \|V_2\|_d &\leq C\|\hat{v}\|_{w,d} \,. \end{aligned} \tag{3.38}$$

Finally, set

$$h_1(x) := \frac{2x^2}{\sqrt{1+2\lambda x}\left(1+\sqrt{1+2\lambda x}\right)^2} \,, \tag{3.39}$$

$$h_2(x) := \frac{x}{\sqrt{1+2\lambda x}} \,. \tag{3.40}$$

We have that

$$V_1(p)^2 = h_1\Big(\frac{\hat{v}(p)}{E(p)}\Big) \,, \tag{3.41}$$

$$V_2(p) = h_2\Big(\frac{\hat{v}(p)}{E(p)}\Big) \,. \tag{3.42}$$

Using Lemma C.3 with $n \in \{1, 2, 3\}$, we obtain

$$\|D^n V_1^2\|_q \leq C\|h_1\|_{C^n([0,\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}])}\|\hat{v}\|^n_{2(\lfloor \frac{r}{2} \rfloor+1),w,c} \,, \tag{3.43}$$

$$\|D^n V_2\|_q \leq C\|h_2\|_{C^n([0,\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}])}\|\hat{v}\|^n_{2(\lfloor \frac{r}{2} \rfloor+1),w,c} \,. \tag{3.44}$$

A straight-forward calculation yields

$$\|h_1\|_{C^n([0,\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}])} \leq C\|\hat{v}\|^2_{2(\lfloor \frac{r}{2} \rfloor+1),w,c} \,, \tag{3.45}$$

$$\|h_2\|_{C^n([0,\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}])} \leq C\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c} \,. \tag{3.46}$$

Combining (3.43), (3.44), (3.45), and (3.46) finishes the proof. □

4 Trace Estimates

In this chapter, we omit the subscript Λ^* from $\int_{\Lambda^*}$ in our notation, as we will only consider $\int \equiv \int_{\Lambda^*}$ as sums over Λ^* in the sense of (1.51).

4.1 Computation of the Partition Function

We establish estimates on the partition function on finite particle number subspaces of the Fock space $\mathcal{F}$. The estimates derived here are needed to control the error terms coming from the expansion in Lemma 3.1.

For the next lemma, let us introduce some notation. Let us denote

$$a_p^{(1)} := a_p^+ \quad \text{and} \quad a_p^{(-1)} := a_p\,. \tag{4.1}$$

Let

$$\mathcal{P}[a, a^+] := \Big\{ \int d\mathbf{p}_k h(\mathbf{p}_k, \boldsymbol{\sigma}_k) \prod_{j=1}^{k} a_{p_j}^{(\sigma_j)} \mid k \in \mathbb{N},\ h(\cdot, \boldsymbol{\sigma}_k) \in \mathcal{S}'(\mathbb{R}^3),$$
$$\boldsymbol{\sigma}_k = (\sigma_1, \ldots, \sigma_k) \in \{\pm 1\}^k \Big\} \tag{4.2}$$

denote all monomials in a, a^+. Here and henceforth, we define ordered operator products via multiplication from the right, by

$$\prod_{j=1}^{n+1} A_j := \Big(\prod_{j=1}^{n} A_j\Big) A_{n+1}\,,\ n \in \mathbb{N}_0\,, \tag{4.3}$$

$$\prod_{j=1}^{0} A_j := \mathbf{1}\,. \tag{4.4}$$

Let us denote

$$\operatorname{sign}(A) := \sum_{j=1}^{k} \sigma_j \tag{4.5}$$

whenever $A = \int d\mathbf{p}_\mathbf{k} h(\mathbf{p}_k, \boldsymbol{\sigma}_k) \prod_{j=1}^{k} a_{p_j}^{(\sigma_j)}$. Note that we have

$$[\mathcal{N}_b, \prod_{j=1}^{k} a_{p_j}^{(\sigma_j)}] = \sum_{j=1}^{k} \sigma_j \prod_{j=1}^{k} a_{p_j}^{(\sigma_j)}\,, \tag{4.6}$$

which, in turn, gives

$$[\mathcal{N}_b, A] = \operatorname{sign}(A) A \tag{4.7}$$

whenever $A = \int d\mathbf{p}_\mathbf{k} h(\mathbf{p}_\mathbf{k}, \boldsymbol{\sigma}_\mathbf{k}) \prod_{j=1}^{k} a_{p_j}^{(\sigma_j)}$. Using the spectral decomposition

$$P_n = \frac{1}{2\pi i} \oint_{\partial B_{\frac{1}{2}}(n)} \frac{dz}{\mathcal{N}_b - z}\,, \tag{4.8}$$

with counter-clockwise contour, it follows that

$$P_n A = A P_{n-\operatorname{sign}(A)} \tag{4.9}$$

for all $A \in \mathcal{P}[a, a^+]$. We will refer to (4.9) as the *Pull-Through Formula* for projectors, see [35].

Moreover, if $\text{sign}(A) \neq 0$, (4.7) implies that

$$\nu(A) \;=\; \frac{1}{\text{sign}(A)} \nu([\mathcal{N}_b, A]) \;=\; 0\,. \tag{4.10}$$

Lemma 4.1 (Moments of the number operator) *We have for all $\ell \in \mathbb{N}_0$ that*

$$\nu_0((\mathcal{N}_b + 1)^{\frac{\ell}{2}}) \;\leq\; C_{\ell, \|f_0\|_d} |\Lambda|^{\frac{\ell}{2}}\,. \tag{4.11}$$

Proof Let $\mathcal{K}_\mu := \int_{\Lambda^*} dp\, (K(p) - \mu) a_p^+ a_p$. Then we obtain that

$$\begin{aligned} Z_0(\mu) &:= \text{Tr}(e^{-\mathcal{K}_\mu}) \\ &= \prod_{p \in \Lambda^*} \sum_{n=0}^{\infty} e^{-n(K(p)-\mu)} \\ &= \prod_{p \in \Lambda^*} \frac{1}{1 - e^{-(K(p)-\mu)}} \\ &= e^{-|\Lambda| \int_{\Lambda^*} dp \log\left(1 - e^{-(K(p)-\mu)}\right)} \end{aligned} \tag{4.12}$$

for any $\mu < \kappa_0$. Let

$$\kappa_n \;:=\; -(-\partial_\mu)^n \big|_{\mu=0} \int_{\Lambda^*} dp \log\left(1 - e^{-(K(p)-\mu)}\right) \tag{4.13}$$

denote the n^{th} cumulant for $n \in \mathbb{N}$. Let $\binom{n}{\mathbf{r}_\ell}$ denote the multinomial coefficient, and $R(\ell) := \{\mathbf{r}_\ell \in \mathbb{N}_0^\ell \mid \sum_{n=1}^{\ell} n r_n = \ell\}$. Then the Faà di Bruno formula, see [295], yields

$$\begin{aligned} \nu_0(\mathcal{N}_b^\ell) &= \frac{(-\partial_\mu)^\ell \big|_{\mu=0} Z_0(\mu)}{Z_0(0)} \\ &= \ell! \sum_{\mathbf{r}_\ell \in R(\ell)} \prod_{n=1}^{\ell} \frac{1}{r_n!} \Big(\frac{|\Lambda| \kappa_n}{n!}\Big)^{r_n}\,. \end{aligned} \tag{4.14}$$

Observe that we have

$$\kappa_1 \;=\; \int_{\Lambda^*} dp\, \frac{1}{e^{K(p)} - 1} \;=\; \int_{\Lambda^*} dp f_0(p)\,, \tag{4.15}$$

recalling f_0 from (1.57). More generally, Lemma A.2 implies

$$\kappa_n \;\leq\; C_{n, \|f_0\|_d} \tag{4.16}$$

for all $n \in \mathbb{N}$. As a consequence of (4.14) and (4.16), we find that

$$\nu_0((\mathcal{N}_b + 1)^\ell) \;\leq\; C_{\ell, \|f_0\|_d} |\Lambda|^\ell\,, \tag{4.17}$$

where we used that $|\Lambda| \geq 1$. Using

$$\nu_0((\mathcal{N}_b + 1)^{\frac{\ell}{2}}) \;\leq\; \nu_0((\mathcal{N}_b + 1)^{\ell-1})^{\frac{1}{2}} \nu_0((\mathcal{N}_b + 1)^{\ell+1})^{\frac{1}{2}} \tag{4.18}$$

yields the half-integer case. This concludes the proof. □

Remark 4.2 We use an argument from [117]. Using the fact that

$$e^{-\mathcal{K}} a_p^+ e^{\mathcal{K}} \;=\; e^{-K(p)} a_p^+ \,, \tag{4.19}$$

cyclicity of the trace implies

$$\begin{aligned} f_0(p) &= \frac{\operatorname{Tr}\left(e^{-\mathcal{K}} a_p^+ a_p\right)}{\operatorname{Tr} e^{-\mathcal{K}} |\Lambda|} \\ &= e^{-K(p)} \frac{\operatorname{Tr}\left(a_p^+ e^{-\mathcal{K}} a_p\right)}{\operatorname{Tr} e^{-\mathcal{K}} |\Lambda|} \\ &= e^{-K(p)} \frac{\operatorname{Tr}\left(e^{-\mathcal{K}} a_p a_p^+\right)}{\operatorname{Tr} e^{-\mathcal{K}} |\Lambda|} \\ &= e^{-K(p)} (f_0(p) \,+\, 1) \,, \end{aligned} \tag{4.20}$$

Thus, we have

$$f_0(p) \;=\; \frac{1}{e^{K(p)} - 1} \,. \tag{4.21}$$

Lemma 4.3 (Operator product bound) *Let $A_j \in \mathcal{P}[a, a^+]$ be monomials in a, a^+, $\gamma_j > 0$, and $k_j \in \mathbb{N}$ be such that*

$$\| P_m A_j P_{m-\operatorname{sign}(A_j)} \| \;\leq\; \gamma_j (m + |\Lambda|)^{k_j/2} \tag{4.22}$$

for all $j \in \{1, \ldots, \ell\}$ and all $m \in \mathbb{N}_0$. Then we have that

$$|\nu(\prod_{j=1}^{\ell} A_j)| \;\leq\; \Big(\prod_{j=1}^{\ell} \gamma_j\Big) \nu\Big(\big(\mathcal{N} + \sum_{m=1}^{\ell} |\operatorname{sign}(A_m)| + |\Lambda|\big)^{\sum_{j=1}^{\ell} k_j/2}\Big) \tag{4.23}$$

for any state ν.

Proof Observe that

$$\operatorname{sign}(A) \;=\; \sum_{j=1}^{\ell} \operatorname{sign}(A_j) \,. \tag{4.24}$$

In addition,

$$\sum_{n=0}^{\infty} P_n \;=\; \mathbf{1} \,, \tag{4.25}$$

$$P_n f(\mathcal{N}_b) \;=\; P_n f(n) \tag{4.26}$$

holds for any measurable function f. Hence and by applying (4.9), we have that

$$\begin{aligned} |\nu(A)| &\leq \sum_{n=0}^{\infty} |\nu(P_n A P_{n-\operatorname{sign}(A)})| \\ &= \sum_{n=0}^{\infty} \Big|\nu\Big(P_n \big(\prod_{j=1}^{\ell} P_{n-\sum_{m=1}^{j-1} \operatorname{sign}(A_m)} A_j P_{n-\sum_{m=1}^{j} \operatorname{sign}(A_m)}\big) P_{n-\operatorname{sign}(A)}\Big)\Big| \,. \end{aligned} \tag{4.27}$$

Denoting $B_n := \prod_{j=1}^{\ell} P_{n-\sum_{m=1}^{j-1} \mathrm{sign}(A_m)} A_j P_{n-\sum_{m=1}^{j} \mathrm{sign}(A_m)}$, we can apply Cauchy-Schwarz to ν to obtain the upper bound

$$
\begin{aligned}
|\nu(A)| &\le \sum_{n=0}^{\infty} \nu(P_n)^{\frac{1}{2}} \nu(P_{n-\mathrm{sign}(A)} B_n^+ B_n P_{n-\mathrm{sign}(A)})^{\frac{1}{2}} \\
&\le \sum_{n=0}^{\infty} \|B_n\| \nu(P_n)^{\frac{1}{2}} \nu(P_{n-\mathrm{sign}(A)})^{\frac{1}{2}} \\
&\le \Big(\sum_{n=0}^{\infty} \|B_n\| \nu(P_n)\Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{\infty} \|B_n\| \nu(P_{n-\mathrm{sign}(A)})\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{n=0}^{\infty} \|B_n\| \nu(P_n)\Big)^{\frac{1}{2}} \Big(\sum_{n=\mathrm{sign}(A)_-}^{\infty} \|B_{n+\mathrm{sign}(A)}\| \nu(P_n)\Big)^{\frac{1}{2}} .
\end{aligned} \tag{4.28}
$$

Here, we used Cauchy-Schwarz to $\sum_{n=0}^{\infty}$, followed by the fact that $P_{n-\mathrm{sign}(A)} \equiv 0$ for all $n < \mathrm{sign}(A)_+$, as $\mathcal{N}_b \ge 0$ in the sense of quadratic forms. Notice that we have

$$
\begin{aligned}
\|B_n\| &\le \prod_{j=1}^{\ell} \|P_{n-\sum_{m=1}^{j-1} \mathrm{sign}(A_m)} A_j P_{n-\sum_{m=1}^{j} \mathrm{sign}(A_m)}\| \\
&\le \prod_{j=1}^{\ell} \gamma_j \Big(\Big(n - \sum_{m=1}^{j-1} \mathrm{sign}(A_m)\Big)_+ + |\Lambda|\Big)^{k_j/2} \\
&\le \Big(\prod_{j=1}^{\ell} \gamma_j\Big) \Big(n + \sum_{m=1}^{\ell} |\mathrm{sign}(A_m)| + |\Lambda|\Big)^{\sum_{j=1}^{\ell} k_j/2}
\end{aligned} \tag{4.29}
$$

by assumption (4.22). Similarly, we have that

$$
\begin{aligned}
\|B_{n+\mathrm{sign}(A)}\| &\le \prod_{j=1}^{\ell} \gamma_j \Big(\Big(n + \mathrm{sign}(A) - \sum_{m=1}^{j-1} \mathrm{sign}(A_m)\Big)_+ + |\Lambda|\Big)^{k_j/2} \\
&= \prod_{j=1}^{\ell} \gamma_j \Big(\Big(n + \sum_{m=j}^{\ell} \mathrm{sign}(A_m)\Big)_+ + |\Lambda|\Big)^{k_j/2} \\
&\le \Big(\prod_{j=1}^{\ell} \gamma_j\Big) \Big(n + \sum_{m=1}^{\ell} |\mathrm{sign}(A_m)| + |\Lambda|\Big)^{\sum_{j=1}^{\ell} k_j/2}
\end{aligned} \tag{4.30}
$$

due to (4.24). Collecting (4.28)–(4.30) and applying (4.26) again, we then obtain

$$
\begin{aligned}
|\nu(A)| &\le \Big(\prod_{j=1}^{\ell} \gamma_j\Big) \sum_{n=0}^{\infty} \Big(n + \sum_{m=1}^{\ell} |\mathrm{sign}(A_m)| + \max_{1\le j\le \ell} k_j\Big)^{\sum_{j=1}^{\ell} k_j/2} \nu(P_n) \\
&= \Big(\prod_{j=1}^{\ell} \gamma_j\Big) \nu\Big(\Big(\mathcal{N} + \sum_{m=1}^{\ell} |\mathrm{sign}(A_m)| + |\Lambda|\Big)^{\sum_{j=1}^{\ell} k_j/2}\Big)
\end{aligned} \tag{4.31}
$$

□

4.2 Operator Estimates w.r.t. $\mathcal{N}_b$

In our analysis, we will bound various operators relative to powers of the particle number operator $\mathcal{N}_b$, in conjunction with previously established estimates on the partition function.

Lemma 4.4 *($\mathcal{H}_{cub}(t)$ bound) We have that*

$$\mathcal{H}_{cub}(t) = \sum_{k=1}^{16} A^{(k)}_{cub}(t) \tag{4.32}$$

with monomials $A^{(k)}_{cub}(t) \in \mathcal{P}[a, a^+]$ such that

$$\|A^{(k)}_{cub}(t) P_M\| \le C\|\hat{v}\|_{w,d}(1 + \lambda\|\hat{v}\|_{w,d})^6 \frac{\lambda(M + |\Lambda|)^{\frac{3}{2}}}{\sqrt{N}} \tag{4.33}$$

for any $M \in \mathbb{N}_0$, and $|\operatorname{sign}(A^{(k)}_{cub}(t))| \le 3$.

Proof Corollary 3.3 and evenness of $\hat{v}$ imply that

$$\begin{aligned}
&\mathcal{H}_{cub}(t)\\
&= \frac{\lambda}{\sqrt{N}} \int_{(\Lambda^*)^3} d\mathbf{p}_3\, \hat{v}(p_2)\delta(p_1 + p_2 - p_3) a^+_{p_1}(t) a^+_{p_2}(t) a_{p_3}(t) + h.c.\\
&= \frac{\lambda}{\sqrt{N}} \sum_{\boldsymbol{\sigma}_3 \in \{\pm 1\}^3} \int_{(\Lambda^*)^3} d\mathbf{p}_3\, \delta(\sum_{j=1}^3 \sigma_j p_j)\hat{v}(p_2) \prod_{j=1}^3 (g_j(p_j, \sigma_j) a^{(\sigma_j)}_{p_j}) + h.c.,
\end{aligned} \tag{4.34}$$

where $\boldsymbol{\sigma}_3 = (\sigma_1, \sigma_2, \sigma_3)$, and

$$g_j(p_j, \sigma_j) := \delta_{\sigma_j,1} u_\lambda(t, p_j) + \delta_{\sigma_j,-1} v_\lambda(t, p_j), \quad j \in \{1, 2\}, \tag{4.35}$$

$$g_3(p_3, \sigma_3) := \delta_{\sigma_3,-1} \overline{u_\lambda}(t, p_3) + \delta_{\sigma_3,1} \overline{v_\lambda}(t, p_3). \tag{4.36}$$

Abbreviating

$$A_{cub}(\boldsymbol{\sigma}_3) := \int d\mathbf{p}_3\, \delta\left(\sum_{j=1}^3 \sigma_j p_j\right) \hat{v}(p_2) \prod_{j=1}^3 \left(g_j(p_j, \sigma_j) a^{(\sigma_j)}_{p_j}\right), \tag{4.37}$$

it is sufficient to prove (4.33) for $A_{cub}(\boldsymbol{\sigma}_3)$, since the adjoint satisfies

$$\|A_{cub}(\boldsymbol{\sigma}_3)^* P_M\| = \|P_{M - \sum_{j=1}^3 \sigma_j} A_{cub}(\boldsymbol{\sigma}_3)^* P_M\| = \|A_{cub}(\boldsymbol{\sigma}_3) P_{M - \sum_{j=1}^3 \sigma_j}\| \tag{4.38}$$

due to the Pull-Through formula (4.9).

Using Wick's Theorem in Appendix A, we have that

$$\begin{aligned}
a^{(\sigma_1)}_{p_1} a^{(\sigma_2)}_{p_2} a^{(\sigma_3)}_{p_3} \delta(\sum_{j=1}^3 \sigma_j p_j) = &: a^{(\sigma_1)}_{p_1} a^{(\sigma_2)}_{p_2} a^{(\sigma_3)}_{p_3} : \delta(\sum_{j=1}^3 \sigma_j p_j)\\
&+ \delta(p_1 - p_2)\delta(p_3)\delta_{\sigma_1,-1}\delta_{\sigma_2,1} a^{(\sigma_3)}_0\\
&+ \delta(p_1 - p_3)\delta(p_2)\delta_{\sigma_1,-1}\delta_{\sigma_3,1} a^{(\sigma_2)}_0\\
&+ \delta(p_2 - p_3)\delta(p_1)\delta_{\sigma_2,-1}\delta_{\sigma_3,1} a^{(\sigma_1)}_0.
\end{aligned} \tag{4.39}$$

We want to apply Lemma A.4 to each of the terms associated with the terms in (4.39). For that, we need to ensure integrability for each of those terms. The terms involving $a_0^{(\sigma)}$ contain 2 momentum-δ where the only free momentum comes with a coefficient v_λ resp. $\overline{v_\lambda}$. Using

$$\|a_0^+ P_M\| = \|a_0 P_{M+1}\|\,, \tag{4.40}$$

Lemma A.4 then implies

$$\begin{aligned}\|a_0^{(\sigma)} P_M\| &= \delta_{\sigma,1}\|a_0^+ P_M\| + \delta_{\sigma,-1}\|a_0 P_M\| \\ &\leq \sqrt{|\Lambda|(M+1)}\,.\end{aligned} \tag{4.41}$$

For the cubic term in (4.39) and if not all $\sigma_j = 1$ or all $\sigma_j = -1$, Lemma A.4 implies

$$\begin{aligned}&\|:A_{cub}(\boldsymbol{\sigma}_3):P_M\| \\ &\leq \||\hat{v}(p_2)|^{\frac{1}{2}}\prod_{j=1}^{2}|v_\lambda(p_j)|^{\delta_{\sigma_j,-1}}|u_\lambda(p_3)|^{\delta_{\sigma_3,-1}}\delta\left(\sum_{j=1}^{3}p_j\sigma_j\right)^{\frac{1}{2}}\|_{L^\infty_{\mathbf{p}_{J_+}}L^2_{\mathbf{p}_{J_-}}} \\ &\||\hat{v}(p_2)|^{\frac{1}{2}}\prod_{j=1}^{2}|u_\lambda(p_j)|^{\delta_{\sigma_j,1}}|v_\lambda(p_3)|^{\delta_{\sigma_3,1}}\delta\left(\sum_{j=1}^{3}p_j\sigma_j\right)^{\frac{1}{2}}\|_{L^\infty_{\mathbf{p}_{J_-}}L^2_{\mathbf{p}_{J_+}}} \\ &\left(M+\sum_{j=1}^{3}\sigma_j\right)^{\frac{1}{2}}_{|J_+|}(M)^{\frac{1}{2}}_{|J_-|}\,,\end{aligned} \tag{4.42}$$

using the notation in Lemma A.4 with $n = 3$. Integrability of $\hat{v}(p_2)$ is sufficient to yield that the RHS is, indeed, finite. If $\sigma_j = 1$ for all j, we have that

$$\begin{aligned}&\|:A_{cub}(1,1,1):P_M\| \\ &\leq \big(\|\hat{v}(p_2)u_\lambda(t,p_1)u_\lambda(t,p_2)\overline{v_\lambda}(t,p_3)\|^2_{L^2_{p_2}L^\infty_{p_1,p_3}}(M-2) \\ &\quad+\|\hat{v}(p_2)u_\lambda(t,p_1)u_\lambda(t,p_2)\overline{v_\lambda}(t,p_1+p_2)\|^2_{L^2_{\mathbf{p}_2}}|\Lambda|\big)^{\frac{1}{2}}\sqrt{M(M-1)}\,.\end{aligned} \tag{4.43}$$

Similarly, we find that

$$\begin{aligned}&\|:A_{cub}(-1,-1,-1):P_M\| \\ &\leq \big(\|\hat{v}(p_2)v_\lambda(t,p_1)v_\lambda(t,p_2)\overline{u_\lambda}(t,p_3)\|^2_{L^2_{p_2}L^\infty_{p_1,p_3}}(M+1) \\ &\quad+\|\hat{v}(p_2)v_\lambda(t,p_1)v_\lambda(t,p_2)\overline{u_\lambda}(t,p_1+p_2)\|_{L^2_{p_2}L^\infty_{p_1}}|\Lambda|\big)^{\frac{1}{2}}\sqrt{(M+3)(M+2)}\end{aligned} \tag{4.44}$$

Observe that

$$\begin{aligned}\|H\|_{L^2(\Lambda^*)} &= \frac{1}{\sqrt{|\Lambda|}}\|H\|_{\ell^2(\Lambda^*)} \\ &\leq \|H\|_d\,.\end{aligned} \tag{4.45}$$

Since we are summing over a lattice, there are terms for which momentum requires the summation over $\hat{v}(2p)$ or $v_\lambda(2p)$ resp. $\overline{v_\lambda}(2p)$. In order to use the upper bound $\|\hat{v}\|_{w,d}$, we employ the fact that

$$\Big|\sum_{p\in\Lambda^*} H(2p)\Big| \leq \|H\|_{\ell^1(\Lambda^*)} . \tag{4.46}$$

Collecting (4.38), (4.41), (4.42), (4.43), (4.44), (4.45), and employing Corollary 3.3, we have shown that

$$\|A_{cub}(\boldsymbol{\sigma}_3)P_M\| \leq C\|\hat{v}\|_{w,d}(1+\lambda\|\hat{v}\|_{w,d})^6(M+|\Lambda|)^{\frac{3}{2}} , \tag{4.47}$$

where we also used the fact that $|\Lambda| \geq 1$. Together with (4.34), this finishes the proof. □

Lemma 4.5 *($\mathcal{H}_{quart}(t)$ bound) We have that*

$$\mathcal{H}_{quart}(t) = \sum_{k=1}^{16} A^{(k)}_{quart}(t) \tag{4.48}$$

with monomials $A^{(k)}_{quart}(t) \in \mathcal{P}[a, a^+]$ such that

$$\|A^{(k)}_{quart}(t)P_M\| \leq C\|\hat{v}\|_{w,d}(1+\lambda\|\hat{v}\|_{w,d})^8 \frac{\lambda(M+|\Lambda|)^2}{N} \tag{4.49}$$

for any $M \in \mathbb{N}_0$, and $|\operatorname{sign}(A^{(k)}_{quart}(t))| \leq 4$.

Proof We follow the steps of the proof of Lemma 4.4. We have that

$$\begin{aligned}
&\mathcal{H}_{quart}(t)\\
&= \frac{\lambda}{2N}\int d\mathbf{p}_4\, \hat{v}(p_1-p_3)\delta(p_1+p_2-p_3-p_4)a^+_{p_1}(t)a^+_{p_2}(t)a_{p_3}(t)a_{p_4}(t)\\
&= \frac{\lambda}{2N}\sum_{\boldsymbol{\sigma}_4\in\{\pm1\}^4}\int d\mathbf{p}_4\, \delta\left(\sum_{j=1}^4 \sigma_j p_j\right)\hat{v}(\sigma_1p_1+\sigma_3p_3)\prod_{j=1}^4\left(g_j(p_j,\sigma_j)a^{(\sigma_j)}_{p_j}\right) ,
\end{aligned} \tag{4.50}$$

where $\boldsymbol{\sigma}_4 = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, and

$$g_j(p_j,\sigma_j) := \begin{cases} u_\lambda(t,p_j)^{\delta_{\sigma_j,1}}v_\lambda(t,p_j)^{\delta_{\sigma_j,-1}} , & j\in\{1,2\} ,\\ \overline{u_\lambda}(t,p_j)^{\delta_{\sigma_j,-1}}\overline{v_\lambda}(t,p_j)^{\delta_{\sigma_j,1}} , & j\in\{3,4\} . \end{cases} \tag{4.51}$$

Analogously to above, we denote

$$A_{quart}(\boldsymbol{\sigma}_4) := \int d\mathbf{p}_4\, \delta\left(\sum_{j=1}^4\sigma_jp_j\right)\hat{v}(\sigma_1p_1+\sigma_3p_3)\prod_{j=1}^4\left(g_j(p_j,\sigma_j)a^{(\sigma_j)}_{p_j}\right) . \tag{4.52}$$

Wick-ordering using Lemma A.3 yields

$$\begin{aligned}
&\delta\left(\sum_{j=1}^4\sigma_jp_j\right)\prod_{j=1}^4 a^{(\sigma_j)}_{p_j}\\
&= \delta\left(\sum_{j=1}^4\sigma_jp_j\right) : \prod_{j=1}^4 a^{(\sigma_j)}_{p_j} : + \sum_{\substack{j_1<j_2,\\ j_3<j_4}} \delta(p_{j_1}-p_{j_2})\delta_{\sigma_{j_1},-1}\delta_{\sigma_{j_2},1}\\
&\quad\left(\delta(\sigma_{j_3}p_{j_3}+\sigma_{j_4}p_{j_4}) : a^{(\sigma_{j_3})}_{p_{j_3}}a^{(\sigma_{j_4})}_{p_{j_4}} : +|\Lambda|\delta(p_{j_3}-p_{j_4})\delta_{\sigma_{j_3},-1}\delta_{\sigma_{j_4},1}\right) ,
\end{aligned} \tag{4.53}$$

where $\{j_1, \dots, j_4\} = \{1, \dots, 4\}$.

We start with the 0-order terms. These occur whenever two pairs aa^+ exist, each with an annihilation operator to the left of a creation operator. This implies an integrable coefficient v_λ resp. $\overline{v_\lambda}$ for each of the corresponding momenta. Together with the deltas coming from the contraction, the resulting terms are integrable and are bounded by

$$\begin{aligned}
&|\Lambda| \Big\| \sum_{\substack{j_1<j_2,\\ j_3<j_4}} \int d\mathbf{p}_4\, \hat{v}(\sigma_1 p_1 + \sigma_3 p_3) \prod_{j=1}^4 g_j(p_j, \sigma_j) \\
&\delta(p_{j_1} - p_{j_2})\delta(p_{j_3} - p_{j_4})\delta_{\sigma_{j_1},-1}\delta_{\sigma_{j_2},1}\delta_{\sigma_{j_3},-1}\delta_{\sigma_{j_4},1} P_M \Big\| \\
&\leq C\lambda^2 |\Lambda| \|\hat{v}\|_{w,d}^3 (1 + \lambda^2 \|\hat{v}\|_{w,d}^2)^2 ,
\end{aligned} \tag{4.54}$$

where, as above $\{j_1, \dots, j_4\} = \{1, \dots, 4\}$.

For terms quadratic in a, a^+, there again remain two independent momenta in the integration after integrating out the $\delta' s$ coming from momentum conservation and the single contraction. Observe that one of these δ's comes from a contraction with an annihilation operator left to a creation operator. Thus, the corresponding momentum comes with an integrable factor $v_\lambda(t, p_{j_1})$ resp. $\overline{v_\lambda}(t, p_{j_1})$. The other momentum comes with a coefficient $v_\lambda(t, p_{j_3})^{\delta_{\sigma_{j_3},-1}}$ $: a_{p_{j_3}}^{(\sigma_{j_3})} a_{-\sigma_{j_3}\sigma_{j_4} p_{j_3}}^{(\sigma_{j_4})} :$, unless $j_3 = 3$. In that case, we have a coefficient $v(\sigma_1 p_1 + \sigma_3 p_3)$ $: a_{p_3}^{(\sigma_3)} a_{-\sigma_3\sigma_4 p_3}^{(\sigma_4)} :$. Then, integrating first over p_{j_2}, p_{j_4} and then p_{j_3}, Lemma A.4 implies

$$\begin{aligned}
&\Big\| \sum_{\substack{j_1<j_2,\\ j_3<j_4}} \int d\mathbf{p}_4\, \hat{v}(\sigma_1 p_1 + \sigma_3 p_3) \prod_{j=1}^4 g_j(p_j, \sigma_j) \\
&\delta(p_{j_1} - p_{j_2})\delta(\sigma_{j_3} p_{j_3} + \sigma_{j_4} p_{j_4})\delta_{\sigma_{j_1},-1}\delta_{\sigma_{j_2},1} : a_{p_{j_3}}^{(\sigma_{j_3})} a_{p_{j_4}}^{(\sigma_{j_4})} : P_M \Big\| \\
&\leq C \|\hat{v}\|_{w,d} (1 + \lambda^2 \|\hat{v}\|_{w,d})^4 (M + |\Lambda|) .
\end{aligned} \tag{4.55}$$

Finally, if not all $\sigma_j = 1$ or $\sigma_j = -1$, Lemma A.4 implies

$$\begin{aligned}
&\| : A_{quart}(\boldsymbol{\sigma}_4) : P_M \| \\
&\leq C \| |\hat{v}(\sigma_1 p_1 + \sigma_3 p_3)|^{\frac{1}{2}} \prod_{j=1}^2 |v_\lambda(t, p_j)|^{\delta_{\sigma_j,-1}} |u_\lambda(t, p_{j+2})|^{\delta_{\sigma_{j+2},-1}} \delta(\sum_{j=1}^4 \sigma_j p_j)^{\frac{1}{2}} \|_{L^\infty_{p_{J_+}} L^2_{p_{J_-}}} \\
&\| |\hat{v}(\sigma_1 p_1 + \sigma_3 p_3)|^{\frac{1}{2}} \prod_{j=1}^2 |u_\lambda(t, p_j)|^{\delta_{\sigma_j,1}} |v_\lambda(t, p_{j+2})|^{\delta_{\sigma_{j+2},1}} \delta(\sum_{j=1}^4 \sigma_j p_j)^{\frac{1}{2}} \|_{L^\infty_{p_{J_-}} L^2_{p_{J_+}}} \\
&(M + 1)^2 .
\end{aligned} \tag{4.56}$$

By flipping the roles of $(1, 2)$ and $(3, 4)$, it is sufficient to show boundedness of only one of the norms. If $1 \in J_-$, we can use integrability of $|v_\lambda(t, p_1)|$, if $2 \in J_-$, there is a factor $|v_\lambda(t, p_2)|$. Integrability w.r.t. to p_3 and p_4 is always given due to the factor $|\hat{v}(\sigma_1 p_1 + \sigma_3 p_3)|\delta(\sum_{j=1}^4 \sigma_j p_j)$. In case $J_- = \emptyset$, Lemma A.4 implies

$$\begin{aligned}
&\| : A_{quart}(1, 1, 1, 1) : P_M \| \\
&\leq (\|\hat{v}(p_1 + p_3) v_\lambda(t, p_1) v_\lambda(t, p_2) \overline{u_\lambda}(t, p_3) \overline{u_\lambda}(t, p_4)\|^2_{L^2_{\mathbf{p}_2} L^\infty_{p_3,p_4}} (M - 3)
\end{aligned}$$

$$+\|\hat{v}(p_1+p_3)v_\lambda(t,p_1)v_\lambda(t,p_2)\overline{u_\lambda}(t,p_3)\overline{u_\lambda}(t,p_1+p_2-p_3)\|^2_{L^2_{\mathbf{p}_3}}|\Lambda|)^{\frac{1}{2}}$$
$$\sqrt{M(M-1)(M-2)}\,. \tag{4.57}$$

$\|:A_{quart}(-1,-1,-1,-1):P_M\|$ can be similarly bounded. Collecting (4.54), (4.55), (4.56) and (4.57), and using $|\Lambda|\geq 1$, we have shown that

$$\|A_{quart}(\boldsymbol{\sigma}_4)P_M\| \leq C\|\hat{v}\|_{w,d}(1+\lambda\|\hat{v}\|_{w,d})^8(M+|\Lambda|)^2\,. \tag{4.58}$$

Using (4.50), this concludes the proof. □

Proposition 4.6 *Let $A\in\mathcal{P}[a,a^+]$ be a monomial in a,a^+, $\gamma>0$ and $\ell\in\mathbb{N}$ be such that*

$$\|AP_M\| \leq \gamma(M+|\Lambda|)^{\frac{\ell}{2}}\,, \tag{4.59}$$

and that $|\,\mathrm{sign}(A)|\leq\ell$. Let $\boldsymbol{s}_{j+k}\in\Delta[t,j+k]$ and $t\geq 0$. For the moment, we abbreviate by $A\mathcal{H}^j_{cub}\mathcal{H}^k_{quart}(\boldsymbol{s}_{j+k})$ all terms that contain one factor A, j factors $\mathcal{H}_{cub}$, k factors $\mathcal{H}_{quart}$, all at possibly different times s_m. Then we have that

$$\begin{aligned}&|\nu(A\mathcal{H}^j_{cub}\mathcal{H}^k_{quart}(\boldsymbol{s}_{j+k}))|\\&\quad\leq\frac{\gamma(C\lambda\|\hat{v}\|_{w,d})^{j+k}(1+\lambda\|\hat{v}\|_{w,d})^{6j+8k}}{N^{\frac{j}{2}+k}}\\&\quad\nu\Big(\big(\mathcal{N}_b+3j+4k+\ell+|\Lambda|\big)^{\frac{3j+4k+\ell}{2}}\Big)\,.\end{aligned} \tag{4.60}$$

for any state ν and all $\lambda>0$ small enough.

Proof Decomposing $\mathcal{H}_{cub}$ and $\mathcal{H}_{quart}$ into monomials as in Lemmata 4.4 and 4.5, we apply Lemmata 4.3, 4.4, 4.5 together with assumption (4.59) on A, we obtain that

$$\begin{aligned}|\nu(AA^j_{cub}A^k_{quart}(\mathbf{s}_{j+k}))|\leq&\frac{\gamma(C\lambda\|\hat{v}\|_{w,d})^{j+k}(1+\lambda\|\hat{v}\|_{w,d})^{6j+8k}}{N^{\frac{j}{2}+k}}\\&\nu\Big(\big(\mathcal{N}_b+3j+4k+\ell+|\Lambda|\big)^{\frac{3j+4k+\ell}{2}}\Big)\,.\end{aligned} \tag{4.61}$$

As a consequence, we obtain that

$$\begin{aligned}|\nu(A\mathcal{H}^j_{cub}\mathcal{H}^k_{quart}(\mathbf{s}_{j+k}))|\leq&\frac{\gamma(C\lambda\|\hat{v}\|_{w,d})^{j+k}(1+\lambda\|\hat{v}\|_{w,d})^{6j+8k}}{N^{\frac{j}{2}+k}}\\&\nu\Big(\big(\mathcal{N}_b+3j+4k+\ell+|\Lambda|\big)^{\frac{3j+4k+\ell}{2}}\Big)\,.\end{aligned} \tag{4.62}$$

This finishes the proof. □

Lemma 4.7 *Given a test function $J\in\ell^2(\Lambda^*)\cap\ell^\infty(\Lambda^*)$, we recall that*

$$f[J]=\int_{\Lambda^*}dp\,J(p)a^+_pa_p\,, \tag{4.63}$$
$$g[J]=\int_{\Lambda^*}dp\,J(p)a_{-p}a_p\,. \tag{4.64}$$

Then we have

$$\|P_mf[J]P_n\|\leq\delta_{m,n}\|J\|_{\ell^\infty(\Lambda^*)}m\,, \tag{4.65}$$
$$\|P_mg[J]P_n\|\leq\delta_{n,m+2}\|J\|_{2\cap\infty,d}(m+1+|\Lambda|)\,. \tag{4.66}$$

Proof Due to $[\mathcal{N}, f[J]] = 0$, we have that

$$P_m f[J] P_n \;=\; \delta_{m,n} \int d\mathbf{p}_2\, J(p_1)\delta(p_1 - p_2) a^+_{p_1} a_{p_2} P_n\,. \tag{4.67}$$

Then Lemma A.4 in Appendix A implies

$$\|P_m f[J] P_n\| \;\le\; \delta_{m,n} \|J\|_\infty m \tag{4.68}$$

For $g[J]$, we have that

$$P_m g[J] P_n \;=\; \delta_{n,m+2} \int d\mathbf{p}_2\, J(p_1)\delta(p_1 + p_2) a_{p_1} a_{p_2} \tag{4.69}$$

Then Lemma A.4 implies

$$\begin{aligned} \|P_m g[J] P_n\| &\le \delta_{n,m+2} \sqrt{\|J\|_\infty^2 (m+1) + |\Lambda| \|J\|_2^2} \sqrt{m+2} \\ &\le \delta_{n,m+2} (\|J\|_{L^2(\Lambda^*)} + \|J\|_\infty)(m + 1 + |\Lambda|) \end{aligned} \tag{4.70}$$

This finishes the proof. □

Lemma 4.8 (Propagation of moments in HFB evolution) *The HFB evolution $\mathcal{V}_{HFB}(t)$ satisfies*

$$\left\| (\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \mathcal{V}_{HFB}(t) (\mathcal{N}_b + |\Lambda|)^{-\frac{\ell}{2}} \right\| \;\le\; e^{K_\ell \|\hat{v}\|_{w,d} \lambda t} \tag{4.71}$$

for all $\ell \in \mathbb{N}$ and some positive constants $K_\ell > 0$.

Proof Let $\psi \in \mathcal{F}$. We have that

$$\begin{aligned} & i\partial_t \left\langle \mathcal{V}_{HFB}(t)\psi, (\mathcal{N}_b + |\Lambda|)^\ell \mathcal{V}_{HFB}(t)\psi \right\rangle \\ &= \left\langle \mathcal{V}_{HFB}(t)\psi, [(\mathcal{N}_b + |\Lambda|)^\ell, \mathcal{H}^{\phi_0}_{HFB}] \mathcal{V}_{HFB}(t)\psi \right\rangle \\ &= \sum_{n=0}^\infty (n + |\Lambda|)^\ell \left\langle \mathcal{V}_{HFB}(t)\psi, [P_n, \mathcal{H}^{\phi_0}_{cor}] \mathcal{V}_{HFB}(t)\psi \right\rangle. \end{aligned} \tag{4.72}$$

Employing (4.9) and recalling (1.34), we have that

$$[P_n, \mathcal{H}^{\phi_0}_{HFB}] = \lambda (P_n g[\hat{v}] P_{n+2} - P_{n-2} g[\hat{v}] P_n - h.c.)\,. \tag{4.73}$$

As a consequence, we have that

$$\begin{aligned} & \left| \partial_t \left\langle \mathcal{V}_{HFB}(t)\psi, (\mathcal{N}_b + |\Lambda|)^\ell \mathcal{V}_{HFB}(t)\psi \right\rangle \right| \\ &= 2\lambda \left| \sum_{n=0}^\infty (n + |\Lambda|)^\ell \mathrm{Im} \left\langle \mathcal{V}_{HFB}(t)\psi, (P_n g[\hat{v}] P_{n+2} - P_{n-2} g[\hat{v}] P_n) \mathcal{V}_{HFB}(t)\psi \right\rangle \right| \\ &= 2\lambda \left| \sum_{n=0}^\infty [(n + |\Lambda|)^\ell - (n + 2 + |\Lambda|)^\ell] \mathrm{Im} \left\langle \mathcal{V}_{HFB}(t)\psi, P_n g[\hat{v}] P_{n+2} \mathcal{V}_{HFB}(t)\psi \right\rangle \right|. \end{aligned} \tag{4.74}$$

Using the Mean-Value Theorem and Cauchy-Schwarz, the last inequality implies

$$\begin{aligned} & \left| \partial_t \left\langle \mathcal{V}_{HFB}(t)\psi, (\mathcal{N}_b + |\Lambda|)^\ell \mathcal{V}_{HFB}(t)\psi \right\rangle \right| \\ &\le K_\ell \lambda \sum_{n=0}^\infty (n + 2 + |\Lambda|)^{\ell - 1} \|P_n \mathcal{V}_{HFB}(t)\psi\| \|P_n g[\hat{v}] P_{n+2} \mathcal{V}_{HFB}(t)\psi\|\,. \end{aligned} \tag{4.75}$$

Applying Lemma 4.7 followed by Young's inequality, we find

$$\begin{aligned}
&\left|\partial_t \left\langle \mathcal{V}_{HFB}(t)\psi, (\mathcal{N}_b+|\Lambda|)^\ell \mathcal{V}_{HFB}(t)\psi\right\rangle\right| \\
&\leq K_\ell \lambda \|\hat{v}\|_{w,d} \sum_{n=0}^{\infty} (n+2+|\Lambda|)^{\ell-1}(n+1+|\Lambda|) \\
&\left(\langle \mathcal{V}_{HFB}(t)\psi, P_n \mathcal{V}_{HFB}(t)\psi\rangle + \langle \mathcal{V}_{HFB}(t)\psi, P_{n+2}\mathcal{V}_{HFB}(t)\psi\rangle\right) \\
&\leq K_\ell \lambda \|\hat{v}\|_{w,d} \left\langle \mathcal{V}_{HFB}(t)\psi, (\mathcal{N}_b+|\Lambda|)^\ell \mathcal{V}_{HFB}(t)\psi\right\rangle . \qquad (4.76)
\end{aligned}$$

Employing Gronwall's Lemma concludes the proof. □

Remark 4.9 Using Corollary 3.3, we find that

$$\begin{aligned}
\mathcal{V}_{HFB}^*(t)\mathcal{N}_b\mathcal{V}_{HFB}(t) &= \int dp \left(|u_\lambda(t,p)|^2 a_p^+ a_p + |v_\lambda(t,p)|^2 a_{-p} a_{-p}^+\right. \\
&\quad \left. + u_\lambda(t,p)\overline{v_\lambda}(t,p) a_p^+ a_{-p}^+ + \overline{u_\lambda}(t,p) v_\lambda(t,p) a_p a_{-p}\right. \\
&= \int dp \left((|u_\lambda(t,p)|^2 + |v_\lambda(t,p)|^2) a_p^+ a_p + |\Lambda||v_\lambda(t,p)|^2\right. \\
&\quad \left. + u_\lambda(t,p)\overline{v_\lambda}(t,p) a_p^+ a_{-p}^+ + \overline{u_\lambda}(t,p) v_\lambda(t,p) a_p a_{-p}\right. , \qquad (4.77)
\end{aligned}$$

where we used the CCR together with the fact that v_λ is even in p. Lemma 4.7 then implies

$$\left\| \mathcal{V}_{HFB}^*(t)(\mathcal{N}_b+|\Lambda|)^{\frac{1}{2}} \mathcal{V}_{HFB}(t)(\mathcal{N}_b+|\Lambda|)^{-\frac{1}{2}} \right\| \leq C_{\|\hat{v}\|_{w,d}} . \qquad (4.78)$$

As an immediate consequence of Lemmata 4.8 and B.2, we obtain the following statement.

Corollary 4.10 (Propagation of moments) *Let* $|\Lambda| \geq 1$. *There exist constants* $C_{\ell,\mu_0}, K_\ell > 0$ *such that for any* $\ell \in \mathbb{N}$, *we have that*

$$\nu_t\left((\mathcal{N}_b+|\Lambda|)^{\frac{\ell}{2}}\right) \leq C_{\ell,\|f_0\|_d} e^{K_\ell \|\hat{v}\|_{w,d}\lambda|\Lambda|t} |\Lambda|^{\frac{\ell}{2}} . \qquad (4.79)$$

Proof Assume that $\frac{\ell}{2} \in \mathbb{N}$. Define

$$\widetilde{\mathcal{U}}_N(t) := \mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0] e^{-i\mathcal{H}_N t} \mathcal{W}[\sqrt{N|\Lambda|}\phi_0] , \qquad (4.80)$$

and let

$$\mu_t(A) := \nu_0\left(\widetilde{\mathcal{U}}_N^*(t) A \widetilde{\mathcal{U}}_N(t)\right) . \qquad (4.81)$$

There exist $\lambda_k \geq 0$, $\sum_{k=1}^{\infty} \lambda_k = 1$, and $\Psi_k \in \mathcal{F}$ such that

$$\mu_t(A) = \sum_{k=1}^{\infty} \lambda_k \langle \Psi_k, A\Psi_k\rangle . \qquad (4.82)$$

Then we have that

$$\begin{aligned}
\nu_t\left((\mathcal{N}_b+1)^{\frac{\ell}{2}}\right) &= \mu_t\left(\mathcal{V}_{HFB}^*(t)(\mathcal{N}_b+1)^{\frac{\ell}{2}}\mathcal{V}_{HFB}(t)\right) \\
&= \sum_{k=1}^{\infty} \lambda_k \left\langle \Psi_k, \mathcal{V}_{HFB}^*(t)(\mathcal{N}_b+1)^{\frac{\ell}{2}}\mathcal{V}_{HFB}(t)\Psi_k\right\rangle \\
&\leq e^{K_\ell \|\hat{v}\|_{w,d}\lambda t} \sum_{k=1}^{\infty} \lambda_k \left\langle \Psi_k, (\mathcal{N}_b+1)^{\frac{\ell}{2}}\Psi_k\right\rangle
\end{aligned}$$

$$= e^{K_\ell \|\hat{v}\|_{w,d}\lambda t} \mu_t\big((\mathcal{N}_b+1)^{\frac{\ell}{2}}\big)\,. \tag{4.83}$$

using Lemma 4.8. Similarly, we write

$$\nu_0(A) = \sum_{k=1}^{\infty} \alpha_k \,\langle \Phi_k, A\Phi_k\rangle \tag{4.84}$$

for some $\alpha_k \geq 0$, $\sum_{k=1}^{\infty} \alpha_k = 1$, $\Phi_k \in \mathcal{F}$, and we obtain that

$$\begin{aligned}
\mu_t\big((\mathcal{N}_b+1)^{\frac{\ell}{2}}\big) &= \sum_{k=1}^{\infty} \alpha_k \left\langle \Phi_k, \widetilde{\mathcal{U}}_N^*(t)(\mathcal{N}_b+1)^{\frac{\ell}{2}}\widetilde{\mathcal{U}}_N(t)\Phi_k \right\rangle \\
&\leq C_\ell e^{K_\ell \|\hat{v}\|_{w,d}\lambda|\Lambda|t} \sum_{k=1}^{\infty} \mu_k \left\langle \Phi_k (\mathcal{N}_b+1)^{\frac{\ell}{2}} \left(1+\frac{\mathcal{N}_b}{N|\Lambda|}\right)\Phi_k \right\rangle \\
&= C_\ell e^{K_\ell \|\hat{v}\|_{w,d}\lambda|\Lambda|t} \nu_0 \left((\mathcal{N}_b+1)^{\frac{\ell}{2}}\left(1+\frac{\mathcal{N}_b}{N|\Lambda|}\right)\right) \\
&\leq C_{\ell,\|f_0\|_d} e^{K_\ell \|\hat{v}\|_{w,d}\lambda|\Lambda|t} |\Lambda|^{\frac{\ell}{2}}\,,
\end{aligned} \tag{4.85}$$

where we used Lemma B.2 followed by Lemma 4.1. Collecting (4.83) and (4.85) and using $|\Lambda| \geq 1$, we obtain that

$$\nu_t\big((\mathcal{N}_b+1)^{\frac{\ell}{2}}\big) \;\leq\; C_{\ell,\|f_0\|_d} e^{K_\ell \|\hat{v}\|_{w,d}\lambda|\Lambda|t} |\Lambda|^{\frac{\ell}{2}}\,. \tag{4.86}$$

The half-integer case follows analogously to (4.18). This concludes the proof. □

5 Control of Error Terms in the Expansion

Again, we write $\int \equiv \int_{\Lambda^*}$ for brevity, to account for lattice sums over Λ^* in the sense of (1.51).

Proposition 5.1 (Tail estimates) *Let $T > 0$ and $t \leq T\lambda^{-2}$, $\lambda \in (0,1)$, $|\Lambda| \geq 1$, and $J \in \ell^1(\Lambda^*) \cap \ell^\infty(\Lambda^*)$. Then the following holds true.*

(1)

$$\Phi_t = -i \int_0^t ds\, \frac{\nu_0\big([a_0, \mathcal{H}_{cub}(s)]\big)}{|\Lambda|} + \mathrm{Rem}_2(t;\Phi) \tag{5.1}$$

with

$$|\,\mathrm{Rem}_2(t;\Phi)| \leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d} T^2 e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T} \frac{|\Lambda|^{\frac{3}{2}}}{N\lambda^2}\left(1+\frac{|\Lambda|}{N}\right). \tag{5.2}$$

(2)

$$\begin{aligned}
\int dp\, J(p)(f_t(p)-f_0(p)) = -i \int_0^t ds\, &\frac{\nu_0\big([f[J], \mathcal{H}_{quart}(s)]\big)}{|\Lambda|} \\
&- \int_{\Delta[t,2]} ds_2 \frac{\nu_0\big([[f[J], \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)]\big)}{|\Lambda|} \\
&+ \mathrm{Rem}_2(t;\, f[J])
\end{aligned} \tag{5.3}$$

with

$$
\begin{aligned}
&|\,\mathrm{Rem}_2(t;\,f[J])| \\
&\quad \le C_{\|\hat v\|_{w,d},\|f_0\|_d}\langle T\rangle^4 \|J\|_{\ell^\infty(\Lambda^*)} \frac{e^{C\|\hat v\|_{w,d}|\Lambda|/\lambda T}|\Lambda|^6}{\lambda^4 N^2}\Big(1+\frac{|\Lambda|}{N}\Big)^2 .
\end{aligned}
\tag{5.4}
$$

(3)

$$
\begin{aligned}
\int dp\, J(p) g_t(p) = -i\int_0^t ds\, &\frac{\nu_0\big([g[J],\mathcal{H}_{quart}(s)]\big)}{|\Lambda|} \\
&-\int_{\Delta[t,2]} d\mathbf{s}_2 \frac{\nu_0\big([[g[J],\mathcal{H}_{cub}(s_1)],\mathcal{H}_{cub}(s_2)]\big)}{|\Lambda|} \\
&+\mathrm{Rem}_2(t;\,g[J])
\end{aligned}
\tag{5.5}
$$

with

$$
\begin{aligned}
&|\,\mathrm{Rem}_2(t;\,g[J])| \\
&\quad \le C_{\|\hat v\|_{w,d},\|f_0\|_d}\langle T\rangle^4 \|J\|_{2\cap\infty,d} \frac{e^{C\|\hat v\|_{w,d}|\Lambda|/\lambda T}|\Lambda|^6}{\lambda^4 N^2}\Big(1+\frac{|\Lambda|}{N}\Big)^2 .
\end{aligned}
\tag{5.6}
$$

Proof Recall that ν_0 being number conserving implies

$$
\nu_0(a_0) \;=\; \nu_0(a_p a_q) \;=\; 0\,. \tag{5.7}
$$

Let $A\in\mathcal{P}[a,a^+]$, be a monomial with data such that

$$
\|P_n A P_{n-\mathrm{sign}(A)}\| \;\le\; \gamma(n+|\Lambda|)^{\frac{\ell}{2}} \tag{5.8}
$$

with $\ell, |\,\mathrm{sign}(A)| \le 2$. Lemma 3.1 implies

$$
\begin{aligned}
\nu_t(A) &= \nu_0(A) - i\int_0^t ds\, \nu_0\big([A,\mathcal{H}_I(s)]\big) \\
&\quad -\int_{\Delta[t,2]} d\mathbf{s}_2 \nu_{s_2}\big([[A,\mathcal{H}_I(s_1)],\mathcal{H}_I(s_2)]\big) \qquad (5.9)\\
&= \nu_0(A) - i\int_0^t ds\, \nu_0\big([A,\mathcal{H}_I(s)]\big) \\
&\quad -\int_{\Delta[t,2]} d\mathbf{s}_2 \nu_0\big([[A,\mathcal{H}_I(s_1)],\mathcal{H}_I(s_2)]\big) \\
&\quad +i\int_{\Delta[t,3]} d\mathbf{s}_3 \nu_0\big([[[A,\mathcal{H}_I(s_1)],\mathcal{H}_I(s_2)],\mathcal{H}_I(s_3)]\big) \\
&\quad +\int_{\Delta[t,4]} d\mathbf{s}_4 \nu_{s_4}\big([[[[A,\mathcal{H}_I(s_1)],\mathcal{H}_I(s_2)],\mathcal{H}_I(s_3)],\mathcal{H}_I(s_4)]\big)\,, \qquad (5.10)
\end{aligned}
$$

where

$$
\mathcal{H}_I(s) \;=\; \mathcal{H}_{cub}(s)+\mathcal{H}_{quart}(s)\,, \tag{5.11}
$$

see (3.6). In order to simplify notation, we shall abbreviate by $A\mathcal{H}^j_{cub}\mathcal{H}^k_{quart}(\mathbf{s}_{j+k})$ all terms that contain one factor A, j factors $\mathcal{H}_{cub}(s_\ell)$, k factors $\mathcal{H}_{quart}(s_m)$, all at possibly different times. Let $\mathbf{s}_{j+k}\in\Delta[t,j+k]$, $t\le T\lambda^{-2}$, $\lambda\in(0,1)$. Proposition 4.6 then implies

$$
\begin{aligned}
&|\nu_{s_{j+k}}\big(A\mathcal{H}_{cub}^{j}\mathcal{H}_{quart}^{k}(\mathbf{s}_{j+k})\big)| \\
&\leq \frac{C_{j,k}\gamma\|\hat{v}\|_{w,d}^{j+k}\lambda^{j+k}(1+\lambda\|\hat{v}\|_{w,d})^{6j+8k}}{N^{\frac{j}{2}+k}}\nu_{s_{j+k}}\Big((\mathcal{N}_b+|\Lambda|)^{\frac{3j+4k+\ell}{2}}\Big)
\end{aligned}
\tag{5.12}
$$

for all $N > 0$ large enough. By Corollary 4.10, we obtain

$$
\begin{aligned}
&|\nu_{s_{j+k}}\big(A\mathcal{H}_{cub}^{j}\mathcal{H}_{quart}^{k}(\mathbf{s}_{j+k})\big)| \\
&\leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d,j,k}\gamma e^{K_{j,k}\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}\frac{\lambda^{j+k}|\Lambda|^{\frac{3j+4k+\ell}{2}}}{N^{\frac{j}{2}+k}}.
\end{aligned}
\tag{5.13}
$$

Similarly to (5.12), we have that Lemma 4.1 implies

$$
|\nu_0\big(A\mathcal{H}_{cub}^{j}\mathcal{H}_{quart}^{k}(\mathbf{s}_{j+k})\big)| \leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d,j,k}\gamma\frac{\lambda^{j+k}|\Lambda|^{\frac{3j+4k+\ell}{2}}}{N^{\frac{j}{2}+k}}.
\tag{5.14}
$$

Due to ν_0 being number conserving, see (4.10), we have that

$$
|\nu_0\big(A\mathcal{H}_{cub}^{j}\mathcal{H}_{quart}^{k}(\mathbf{s}_{j+k})\big)| = 0,
\tag{5.15}
$$

whenever the total number of creation and annihilation operators in $A\mathcal{H}_{cub}^{j}\mathcal{H}_{quart}^{k}(\mathbf{s}_{j+k})$ is odd.

In the case $A = a_0$, due to Corollary 3.3 and using $\hat{v}(0) = 0$, we have that

$$
a_0 = a_0(s)
\tag{5.16}
$$

for all $s \in \mathbb{R}$. With that, an easy computation yields

$$
\begin{aligned}
[a_0, \mathcal{H}_{cub}(s)] &= [a_0, \mathcal{H}_{cub}^{\phi_0}](s) \\
&= \frac{\lambda}{\sqrt{N}}\int dp\, \hat{v}(p)\big(a_p^+(s)a_p(s) + a_p(s)a_{-p}(s)\big).
\end{aligned}
\tag{5.17}
$$

Using Corollary 3.3, we obtain

$$
[a_0, \mathcal{H}_{cub}(s)] = f[J_1(s)] + g[J_2(s)] + g^*[J_3(s)] + J_4(s),
\tag{5.18}
$$

where, with the notation of Corollary 3.3,

$$
J_1(s,p) := \frac{\lambda\hat{v}(p)}{\sqrt{N}}\big(|u_\lambda(s,p)|^2 + |v_\lambda(s,p)|^2 + 2\overline{v_\lambda}(s,p)\overline{u_\lambda}(s,p)\big),
\tag{5.19}
$$

$$
J_2(s,p) := \frac{\lambda\hat{v}(p)}{\sqrt{N}}\big(v_\lambda(s,p) + \overline{u_\lambda}(s,p)\big)\overline{u_\lambda}(s,p),
\tag{5.20}
$$

$$
J_3(s,p) := \frac{\lambda\hat{v}(p)}{\sqrt{N}}\big(u_\lambda(s,p) + \overline{v_\lambda}(s,p)\big)\overline{v_\lambda}(s,p),
\tag{5.21}
$$

$$
J_4(s) := \frac{\lambda}{\sqrt{N}}\int dp\, \hat{v}(p)\big(|v_\lambda(s,p)|^2 + \overline{u_\lambda}(s,p)\overline{v_\lambda}(s,p)\big).
\tag{5.22}
$$

Note, that in (5.18), we used the notation

$$
g^*[J] = \int dp\, J(p)a_p^+a_{-p}^+.
\tag{5.23}
$$

Using Corollary 3.3 again, we find that

$$\|J_i(s)\|_d \leq C_{\|\hat{v}\|_{w,d}} \frac{\lambda}{\sqrt{N}} \quad \text{for } i \in \{1,2,3\}\,, \tag{5.24}$$

$$|J_4(s)| \leq C_{\|\hat{v}\|_{w,d}} \frac{\lambda}{\sqrt{N}} \tag{5.25}$$

for all $s \leq T\lambda^{-2}$.

Similarly, we compute

$$\begin{aligned}
[a_0, \mathcal{H}_{quart}(s)] &= [a_0, \mathcal{H}_{quart}](s) \\
&= \frac{\lambda}{N} \int dp\, \hat{v}(p_2)(a^+_{p_1+p_2} a_{p_2} a_{p_1})(s) \\
&= \frac{1}{\sqrt{N}} \mathcal{H}^{(2)}_{cub}(s)\,.
\end{aligned} \tag{5.26}$$

Here, we used the notation

$$\begin{aligned}
\mathcal{H}_{cub}(s) \;&=\; \frac{\lambda}{\sqrt{N}} \int dp\, \hat{v}(p_2)\big((a^+_{p_1} a^+_{p_2} a_{p_1+p_2})(s) + (a^+_{p_1+p_2} a_{p_2} a_{p_1})(s)\big) \\
&=: \mathcal{H}^{(1)}_{cub}(s) + \mathcal{H}^{(2)}_{cub}(s)\,.
\end{aligned} \tag{5.27}$$

Notice that bounds of the form (5.12)–(5.14) hold with $\mathcal{H}_{cub}$ replaced by $\mathcal{H}^{(2)}_{cub}$, due to the proof of Proposition 4.6.

Using (5.9) and (5.26), we have that

$$\Phi_t \;=\; -\frac{i}{|\Lambda|} \int_0^t ds\; \nu_0\big([a_0, \mathcal{H}_{cub}(s)]\big) \;+\; \mathrm{Rem}_2(t;\Phi)\,, \tag{5.28}$$

where

$$\begin{aligned}
\mathrm{Rem}_2(t;\Phi) :=& -\frac{i}{2\sqrt{N}} \int_0^t ds\; \frac{\nu_0(\mathcal{H}^{(2)}_{cub}(s))}{|\Lambda|} \\
& - \int_{\Delta[t,2]} d\mathbf{s}_2 \frac{\nu_{s_2}\big([f[J_1(s_1)] + g[J_2(s_1)] + g^*[J_3(s_1)], \mathcal{H}_I(s_2)]\big)}{|\Lambda|} \\
& + \frac{1}{2\sqrt{N}} \int_{\Delta[t,2]} d\mathbf{s}_2\; \frac{\nu_{s_2}\big([\mathcal{H}^{(2)}_{cub}(s_1), \mathcal{H}_I(s_2)]\big)}{|\Lambda|}\,.
\end{aligned} \tag{5.29}$$

Recall that, by definition (5.23),

$$|\nu_{s_2}([g^*[J_3(s_1)], \mathcal{H}_I(s_2)])| \;=\; |\nu_{s_2}([g[\overline{J_3}(s_1)], \mathcal{H}_I(s_2)])|\,. \tag{5.30}$$

Then Lemma 4.7 together with the bounds (5.13) and (5.24) imply

$$\begin{aligned}
& \frac{\big|\nu_{s_2}\big([f[J_1(s_1)], \mathcal{H}_I(s_2)]\big)\big|}{|\Lambda|}, \;\frac{\big|\nu_{s_2}\big([g[J_2(s_1)], \mathcal{H}_I(s_2)]\big)\big|}{|\Lambda|}, \\
& \frac{\big|\nu_{s_2}\big([g^*[J_3(s_1)], \mathcal{H}_I(s_2)]\big)\big|}{|\Lambda|} \\
& \leq \frac{C_{\|\hat{v}\|_{w,d}, \|f_0\|_d} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T} \lambda^2 |\Lambda|^{\frac{3}{2}}}{N} \left(1 + \sqrt{\frac{|\Lambda|}{N}}\right).
\end{aligned} \tag{5.31}$$

(5.15) with $A = \mathbf{1}$ implies

$$\frac{|\nu_0(\mathcal{H}^{(2)}_{cub}(s))|}{|\Lambda|\sqrt{N}} = 0\,. \tag{5.32}$$

(5.13) with $A = \mathbf{1}$ implies

$$\begin{aligned}&\frac{\left|\nu_{s_2}\big([\mathcal{H}^{(2)}_{cub}(s_1), \mathcal{H}_I(s_2)]\big)\right|}{|\Lambda|\sqrt{N}} \\ &\quad\leq \frac{C_{\|\hat{v}\|_{w,d},\|\hat{v}\|_{w,d}} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}\lambda^2|\Lambda|^2}{N^{\frac{3}{2}}}\Big(1+\sqrt{\frac{|\Lambda|}{N}}\Big)\,.\end{aligned} \tag{5.33}$$

Collecting (5.29), (5.31), (5.32), and (5.33), yields the bound

$$\begin{aligned}&|\operatorname{Rem}_2(t;\Phi)| \\ &\quad\leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}\,\frac{\lambda^2 t^2|\Lambda|^{\frac{3}{2}}}{N}\Big(1+\sqrt{\frac{|\Lambda|}{N}}\Big)^2 \\ &\quad\leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d} T^2 e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}\,\frac{|\Lambda|^{\frac{3}{2}}}{N\lambda^2}\Big(1+\frac{|\Lambda|}{N}\Big)\end{aligned} \tag{5.34}$$

for all $t \leq T\lambda^{-2}$, $\lambda \in (0,1)$.

In the cases $A = f[J]$ and $A = g[J]$, we have that

$$\begin{aligned}\frac{\nu_t(A)}{|\Lambda|} &= \frac{\nu_0(f[J])}{|\Lambda|}\mathbb{1}_{A=f[J]} - i\int_0^t ds\,\frac{\nu_0\big([A,\mathcal{H}_{quart}(s)]\big)}{|\Lambda|} \\ &\quad - \int_{\Delta[t,2]} d\mathbf{s}_2\,\frac{\nu_0\big([[A,\mathcal{H}_{cub}(s_1)],\mathcal{H}_{cub}(s_2)]\big)}{|\Lambda|} \\ &\quad + \operatorname{Rem}_2(t;A)\,,\end{aligned} \tag{5.35}$$

where, using (5.7) and (5.10),

$$\begin{aligned}\operatorname{Rem}_2(t;A) &:= -i\int_0^t ds\,\frac{\nu_0\big([A,\mathcal{H}_{cub}(s)]\big)}{|\Lambda|} \\ &\quad - \int_{\Delta[t,2]} d\mathbf{s}_2\,\frac{\nu_0\big([[A,\mathcal{H}_{cub}(s_1)],\mathcal{H}_{quart}(s_2)]\big)}{|\Lambda|} \\ &\quad - \int_{\Delta[t,2]} d\mathbf{s}_2\,\frac{\nu_0\big([[A,\mathcal{H}_{quart}(s_1)],\mathcal{H}_I(s_2)]\big)}{|\Lambda|} \\ &\quad + i\int_{\Delta[t,3]} d\mathbf{s}_3\,\frac{\nu_0\big([[[A,\mathcal{H}_I(s_1)],\mathcal{H}_I(s_2)],\mathcal{H}_I(s_3)]\big)}{|\Lambda|} \\ &\quad + \int_{\Delta[t,4]} d\mathbf{s}_4\,\frac{\nu_{s_4}\big([[[[A,\mathcal{H}_I(s_1)],\mathcal{H}_I(s_2)],\mathcal{H}_I(s_3)],\mathcal{H}_I(s_4)]\big)}{|\Lambda|}.\end{aligned} \tag{5.36}$$

Employing (5.14), Lemma 4.7 yields

$$\frac{\left|\nu_0\big(A\mathcal{H}^2_{quart}(\mathbf{s}_2)\big)\right|}{|\Lambda|} \leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d}\,\frac{\|J\|_A\lambda^2|\Lambda|^4}{N^2}\,, \tag{5.37}$$

$$\frac{\left|\nu_0\big(A\mathcal{H}^2_{cub}\mathcal{H}_{quart}(\mathbf{s}_3)\big)\right|}{|\Lambda|} \leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d}\,\frac{\|J\|_A\lambda^3|\Lambda|^5}{N^2}\,, \tag{5.38}$$

$$\frac{\left|\nu_0\left(A\mathcal{H}_{quart}^3(\mathbf{s}_3)\right)\right|}{|\Lambda|} \le C_{\|\hat{v}\|_{w,d},\|f_0\|_d} \frac{\|J\|_A \lambda^3 |\Lambda|^6}{N^3}\,, \tag{5.39}$$

where we abbreviated

$$\|J\|_A \;:=\; \|J\|_{\ell^\infty(\Lambda^*)} \mathbb{1}_{A=f[J]} \;+\; \|J\|_{2\cap\infty,d} \mathbb{1}_{A=g[J]}\,. \tag{5.40}$$

Using (5.15), we have that

$$\begin{aligned}\frac{\left|\nu_0\left([A,\mathcal{H}_{cub}(s)]\right)\right|}{|\Lambda|} &= \frac{\left|\nu_0\left(A\mathcal{H}_{cub}\mathcal{H}_{quart}(\mathbf{s}_2)\right)\right|}{|\Lambda|} \\ = \frac{\left|\nu_0\left(A\mathcal{H}_{cub}^3(\mathbf{s}_3)\right)\right|}{|\Lambda|} &= \frac{\left|\nu_0\left(A\mathcal{H}_{cub}\mathcal{H}_{quart}^2(\mathbf{s}_3)\right)\right|}{|\Lambda|} = 0\,.\end{aligned} \tag{5.41}$$

(5.12) together with Lemma 4.7 implies

$$\begin{aligned}&\frac{\left|\nu_{s_2}\left(A\mathcal{H}_I^4(\mathbf{s}_4)\right)\right|}{|\Lambda|} \\ &\le C_{\|\hat{v}\|_{w,d},\|f_0\|_d} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T} \frac{\|J\|_A \lambda^4 |\Lambda|^6}{N^2} \left(1+\sqrt{\frac{|\Lambda|}{N}}\right)^4.\end{aligned} \tag{5.42}$$

Collecting (5.37)-(5.42), we arrive at

$$\begin{aligned}&|\operatorname{Rem}_2(t;A)| \\ &\le \frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T} \|J\|_A \lambda^2 t^2 |\Lambda|^4}{N^2} \Big[1 \,+\, \lambda t|\Lambda|\Big(1+\frac{|\Lambda|}{N}\Big) \\ &\quad + \lambda^2 t^2 |\Lambda|^2 \Big(1+\sqrt{\frac{|\Lambda|}{N}}\Big)^4\Big] \\ &\le C_{\|\hat{v}\|_{w,d},\|f_0\|_d} \langle T\rangle^4 \|J\|_A \frac{e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}|\Lambda|^6}{\lambda^4 N^2} \left(\Big(1+\frac{|\Lambda|}{N}\Big)^2 + \frac{\lambda^2}{|\Lambda|^2}\right) \\ &\le C_{\|\hat{v}\|_{w,d},\|f_0\|_d} \langle T\rangle^4 \|J\|_A \frac{e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}|\Lambda|^6}{\lambda^4 N^2} \Big(1+\frac{|\Lambda|}{N}\Big)^2\end{aligned} \tag{5.43}$$

for all $t \le T\lambda^{-2}$, $\lambda \in (0,1)$, $|\Lambda| \ge 1$. This finishes the proof. □

Up until now, all calculations did not further distinguish the cases of fixed $|\Lambda|$ and $|\Lambda|$ growing with N. In the latter situation, we will approximate lattice sums over Λ^* with integrals over $\mathbb{R}^3$. It is crucial to note that oscillatory and dispersive properties differ fundamentally in these two cases.

5.1 Fixed, *N*-Independent Lattice $\Lambda^* \cong \mathbb{Z}^3$

5.1.1 Notation

For the next result, we define

$$\Delta_{cub} H(\mathbf{p_2}) := H(p_1) + H(p_2) - H(p_1+p_2)\,, \tag{5.44}$$

$$\widetilde{h} := 1 + h\,, \tag{5.45}$$

$$p_{12} := p_1 + p_2\,. \tag{5.46}$$

Recall from Corollary 3.3 that

$$V_1(p)^2 = \frac{\hat{v}(p)^2}{\Omega(p)(\Omega(p)+E(p)+\lambda\hat{v}(p))}\,, \tag{5.47}$$

$$V_2(p) = \frac{\hat{v}(p)}{\Omega(p)}\,. \tag{5.48}$$

In order to describe the dynamics of Φ, we introduce the condensate operator

$$\mathrm{Con}_d(h)(T;\lambda) := -i\int_0^T dS \int_{\Lambda^*} dp\, \hat{v}(p) h_{\frac{S}{\lambda^2}}(p)\,. \tag{5.49}$$

For the dynamics of f, we define the generalized Boltzmann operators

$$\begin{aligned}
&\mathrm{Bol}_d(h)[J](T;\lambda)\\
&\quad := \frac{1}{\lambda^2}\int_{\Delta[T,2]} d\mathbf{S}_2 \int_{(\Lambda^*)^2} d\mathbf{p}_2\, \cos\Big(\frac{\Delta_{cub}\Omega(\mathbf{p}_2)(S_1-S_2)}{\lambda^2}\Big)\\
&\quad (\hat{v}(p_1)+\hat{v}(p_2))^2\Delta_{cub}J(\mathbf{p}_2)\big(\widetilde{h}_{\frac{S_2}{\lambda^2}}(p_1)\widetilde{h}_{\frac{S_2}{\lambda^2}}(p_2)h_{\frac{S_2}{\lambda^2}}(p_1+p_2)\\
&\quad -h_{\frac{S_2}{\lambda^2}}(p_1)h_{\frac{S_2}{\lambda^2}}(p_2)\widetilde{h}_{\frac{S_2}{\lambda^2}}(p_1+p_2)\big)\,,
\end{aligned}\tag{5.50}$$

$$\begin{aligned}
&\mathrm{bol}^{(1)}(h)[J](\mathbf{S}_2/\lambda^2)\\
&\quad := \frac{2}{\lambda^2}\mathrm{Im}\int_{(\Lambda^*)^2} d\mathbf{p}_2\, (\hat{v}(p_1)+\hat{v}(p_2))(\hat{v}(p_1)+\hat{v}(p_{12}))\\
&\quad V_2(p_1)e^{-i(\Omega(p_2)-\Omega(p_{12}))(S_1-S_2)/\lambda^2}\\
&\quad \Big(\big(-J(-p_1)+J(p_2)-J(p_{12})\big)e^{i\Omega(p_1)S_2/\lambda^2}\sin(\Omega(p_1)S_1/\lambda^2)\\
&\quad \big(\widetilde{h}_{\frac{S_2}{\lambda^2}}(-p_1)h_{\frac{S_2}{\lambda^2}}(p_2)\widetilde{h}_{\frac{S_2}{\lambda^2}}(p_{12})-h_{\frac{S_2}{\lambda^2}}(-p_1)\widetilde{h}_{\frac{S_2}{\lambda^2}}(p_2)h_{\frac{S_2}{\lambda^2}}(p_{12})\big)\\
&\quad +\Delta_{cub}J(\mathbf{p}_2)e^{i\Omega(p_1)S_1/\lambda^2}\sin(\Omega(p_1)S_2/\lambda^2)\\
&\quad \big(h_{\frac{S_2}{\lambda^2}}(p_1)h_{\frac{S_2}{\lambda^2}}(p_2)\widetilde{h}_{\frac{S_2}{\lambda^2}}(p_{12})-\widetilde{h}_{\frac{S_2}{\lambda^2}}(p_1)\widetilde{h}_{\frac{S_2}{\lambda^2}}(p_2)h_{\frac{S_2}{\lambda^2}}(p_{12})\big)\Big)\,,
\end{aligned}\tag{5.51}$$

$$\begin{aligned}
&\mathrm{Bol}_d^{(1)}(h)[J](T;\lambda)\\
&\quad := \int_{\Delta[T,2]} d\mathbf{S}_2\, \mathrm{bol}^{(1)}(h)[J](\mathbf{S}_2/\lambda^2)\,,
\end{aligned}\tag{5.52}$$

$$\begin{aligned}
&\mathrm{bol}^{(2,1)}(h)[J](\mathbf{S}_2;\lambda)\\
&\quad := \frac{1}{\lambda^2}\mathrm{Im}\int_{(\Lambda^*)^2} d\mathbf{p}_2\, (\hat{v}(p_1)+\hat{v}(p_2))^2\Delta_{cub}J(\mathbf{p}_2)e^{i\Omega(\mathbf{p}_2)(S_1-S_2)/\lambda^2}\\
&\quad \Big(V_1(p_1)^2\big(\sin(\Omega(p_1)S_1/\lambda^2)e^{-i\Omega(p_1)S_1/\lambda^2}-\sin(\Omega(p_1)S_2/\lambda^2)e^{i\Omega(p_1)S_2/\lambda^2}\big)\\
&\quad +V_1(p_2)^2\big(\sin(\Omega(p_2)S_1/\lambda^2)e^{-i\Omega(p_2)S_1/\lambda^2}-\sin(\Omega(p_2)S_2/\lambda^2)e^{i\Omega(p_2)S_2/\lambda^2}\big)\\
&\quad -V_1(p_{12})^2\big(\sin(\Omega(p_{12})S_1/\lambda^2)e^{i\Omega(p_{12})S_1/\lambda^2}-\sin(\Omega(p_{12})S_2/\lambda^2)e^{i\Omega(p_{12})S_2/\lambda^2}\big)\Big)\\
&\quad \big(h_{\frac{S_2}{\lambda^2}}(p_1)h_{\frac{S_2}{\lambda^2}}(p_2)\widetilde{h}_{\frac{S_2}{\lambda^2}}(p_{12}))-\widetilde{h}_{\frac{S_2}{\lambda^2}}(p_1)\widetilde{h}_{\frac{S_2}{\lambda^2}}(p_2)h_{\frac{S_2}{\lambda^2}}(p_{12})\big)\,,
\end{aligned}\tag{5.53}$$

$$\begin{aligned}
&\mathrm{bol}^{(2,2)}(h)[J](\mathbf{S}_2/\lambda^2)\\
&\quad := \frac{2}{\lambda^2}\mathrm{Re}\int_{(\Lambda^*)^2} d\mathbf{p}_2\, (\hat{v}(p_1)+\hat{v}(p_2))
\end{aligned}$$

$$
\begin{aligned}
&\Big[\sin(\Omega(p_1)S_1/\lambda^2)\sin(\Omega(p_{12})S_1/\lambda^2) \\
&e^{i\Omega(p_2)(S_1-S_2)/\lambda^2+i(\Omega(p_1)-\Omega(p_{12}))S_2/\lambda^2} \\
&V_2(p_1)V_2(p_{12})(\hat{v}(p_2)+\hat{v}(p_{12})) \\
&(-J(-p_1)+J(p_2)+J(-p_{12})) \\
&\big(h_{\frac{s_2}{\lambda^2}}(-p_1)\widetilde{h}_{\frac{s_2}{\lambda^2}}(p_2)\widetilde{h}_{\frac{s_2}{\lambda^2}}(-p_{12}) - \widetilde{h}_{\frac{s_2}{\lambda^2}}(-p_1)h_{\frac{s_2}{\lambda^2}}(p_2)f(-p_{12})\big) \\
&+\sin(\Omega(p_1)S_1/\lambda^2)e^{i\Omega(p_2)(S_1-S_2)/\lambda^2-i\Omega(p_{12})S_1/\lambda^2}V_2(p_1) \\
&(-J(-p_1)+J(p_2)-J(p_{12})) \\
&\Big((\hat{v}(p_1)+\hat{v}(p_2))V_2(p_1)\sin(\Omega(p_1)S_2/\lambda^2)e^{i\Omega(p_{12})S_2/\lambda^2} \\
&+(\hat{v}(p_2)+\hat{v}(p_{12}))V_2(p_{12})\sin(\Omega(p_{12})S_2/\lambda^2)e^{-i\Omega(p_1)S_2/\lambda^2}\Big) \\
&\big(h_{\frac{s_2}{\lambda^2}}(-p_1)\widetilde{h}_{\frac{s_2}{\lambda^2}}(p_2)h_{\frac{s_2}{\lambda^2}}(p_{12}) - \widetilde{h}_{\frac{s_2}{\lambda^2}}(-p_1)h_{\frac{s_2}{\lambda^2}}(p_2)\widetilde{h}_{\frac{s_2}{\lambda^2}}(p_{12})\big) \\
&+\Big(\frac{1}{2}e^{-i(\Omega(p_1)+\Omega(p_2))S_2/\lambda^2}\sin(\Omega(p_{12})S_2/\lambda^2)V_2(p_{12})(\hat{v}(p_1)+\hat{v}(p_2)) \\
&+e^{-i(\Omega(p_1)+\Omega(p_{12}))s_2}\sin(\Omega(p_2)S_2/\lambda^2)V_2(p_2)(\hat{v}(p_1)+\hat{v}(p_{12}))\Big) \\
&\big(\widetilde{h}_{\frac{s_2}{\lambda^2}}(p_1)\widetilde{h}_{\frac{s_2}{\lambda^2}}(p_2)\widetilde{h}_{\frac{s_2}{\lambda^2}}(-p_{12}) - h_{\frac{s_2}{\lambda^2}}(p_1)h_{\frac{s_2}{\lambda^2}}(p_2)h_{\frac{s_2}{\lambda^2}}(-p_{12})\big)\Big], \qquad (5.54)
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{bol}^{(2)}(h)[J](\mathbf{S}_2/\lambda^2) \\
&\quad := \mathrm{bol}^{(2,1)}(h)[J](\mathbf{S}_2;\lambda) + \mathrm{bol}^{(2,2)}(h)[J](\mathbf{S}_2/\lambda^2), \qquad (5.55)
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Bol}_d^{(2)}(h)[J](T;\lambda) \\
&\quad := \int_{\Delta[T,2]} d\mathbf{S}_2\ \mathrm{bol}^{(2)}(h)[J](\mathbf{S}_2/\lambda^2). \qquad (5.56)
\end{aligned}
$$

5.1.2 Results

Proposition 5.2 *Let $T > 0$, $\lambda \in (0,1)$, $|\Lambda| \geq 1$, and $J \in L^\infty(\mathbb{R}^3;\mathbb{R})$. Then the following holds.*

1.

$$
-i\int_0^{T\lambda^{-2}} ds\, \frac{\nu_0\big([a_0, \mathcal{H}_{cub}(s)]\big)}{|\Lambda|} = \frac{1}{N^{\frac{1}{2}}\lambda}\mathrm{Con}_d(f_0)(T;\lambda) + \mathrm{err}_{1,d}^{(Bog)}\left(\frac{T}{\lambda^2};\Phi\right), \qquad (5.57)
$$

where

$$
|\,\mathrm{err}_{1,d}^{(Bog)}(\frac{T}{\lambda^2};\Phi)| \leq \frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d}T}{N^{\frac{1}{2}}}, \qquad (5.58)
$$

2.

$$
\nu_0([f[J], \mathcal{H}_{quart}(s)]) = 0, \qquad (5.59)
$$

3.

$$
-\int_{\Delta[T\lambda^{-2},2]} ds_2 \frac{\nu_0\big([[f[J], \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)]\big)}{|\Lambda|}
$$

$$= \frac{1}{N}\big(\mathrm{Bol}_d(f_0)[J](T;\lambda) + \lambda\,\mathrm{Bol}_d^{(1)}(f_0)[J](T;\lambda) + \lambda^2\,\mathrm{Bol}_d^{(2)}(f_0)[J](T;\lambda)\big)$$

$$+ J(0)|\Phi_{\frac{T}{\lambda^2}}|^2 + \mathrm{err}_{2,d}^{(Bog,\mathrm{Bol})}(\frac{T}{\lambda^2}; f[J]) + J(0)\,\mathrm{err}_{2,d}^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2}; f), \quad (5.60)$$

where

$$\left|\mathrm{err}_{2,d}^{(Bog,\mathrm{Bol})}\left(\frac{T}{\lambda^2}; f[J]\right)\right| \le C_{\|\hat{v}\|_{w,d},\|f_0\|_d}\|J\|_{\ell^\infty(\Lambda^*)}T^2\frac{\lambda}{N}, \quad (5.61)$$

$$\left|\mathrm{err}_{2,d}^{(Bog,\mathrm{Con})}(t; f)\right| \le C_{\|\hat{v}\|_{w,d},\|f_0\|_d}e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}$$

$$\frac{\langle T\rangle^4|\Lambda|^{\frac{3}{2}}}{\lambda^3 N^{\frac{3}{2}}}\Big(1+\frac{|\Lambda|}{N}\Big)^2\Big(1+\frac{|\Lambda|^{\frac{3}{2}}}{\lambda N^{\frac{1}{2}}}\Big), \quad (5.62)$$

4.

$$-i\int_0^{T\lambda^{-2}} ds\,\frac{\nu_0(g[J],\mathcal{H}_{quart}(s))}{|\Lambda|} = \frac{1}{N}\int_0^T dS\,abs_{quart,d}(f_0)[J](S/\lambda^2), \quad (5.63)$$

The absorption operator $abs_{quart,d}(f_0)[J](S/\lambda^2)$ *consists of terms of the form*

$$(-i)^\ell\lambda^j\int dp\,dk\,e^{-i(m_1\Omega(p)+m_2\Omega(k))S/\lambda^2}(1+f(p)+f(-p))J(p)$$

$$V_1(p)^{2\alpha_1}V_2(p)^{\alpha_2}\hat{v}(p-k)V_1(k)^{2\alpha_3}V_2(k)^{\alpha_4}(f_0(k)+\iota), \quad (5.64)$$

with $\ell \in \mathbb{N}_0$, $\ell \le 3$, $j \in \mathbb{N}_0$, $j \le 7$, $m_1, m_2 \in \{0, \pm 2\}$, $\alpha_j \in \{0,1,2\}$ *and* $\iota \in \{0,1\}$. *The integrand contains a factor* $f_0(k)$ *or a factor* $V_2(k)$.

5.

$$-\int_{\Delta[T\lambda^{-2},2]} d\boldsymbol{s}_2\,\frac{\nu_0\big([[g[J],\mathcal{H}_{cub}(s_1)],\mathcal{H}_{cub}(s_2)]\big)}{|\Lambda|}$$

$$= \frac{1}{N\lambda^2}\int_{\Delta[T,2]} d\boldsymbol{S}_2\,\big(\mathrm{col}_d(f_0)[J](\boldsymbol{S}_2/\lambda^2) + abs_{cub,d}(f_0)[J](\boldsymbol{S}_2/\lambda^2)\big)$$

$$+ J(0)\Big(\Phi_{T\lambda^{-2}}\Big)^2 + J(0)\,\mathrm{err}_2^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2}; g). \quad (5.65)$$

Here, the collision operator $\mathrm{col}_d(f_0)[J](\boldsymbol{s}_2)$ *consists of terms of the form*

$$(-i)^{\ell_0}\lambda^{j_0}\int_{(\Lambda^*)^3} d\boldsymbol{p}_3\,e^{-i\sum_{\ell=1}^2 s_\ell\sum_{k=1}^3\sigma_{k,\ell}\Omega(p_k)}\delta(p_1+p_2-p_3)$$

$$J(p_{j_1})\hat{v}(p_2)\hat{v}(p_{j_2})\prod_{k=1}^3 V_1(p_k)^{2\alpha_{\ell_k}}V_2(p_k)^{\beta_{\ell_k}}$$

$$\Big(\prod_{k=1}^3\big(f_0(\tau_{k,1}p_k)+\frac{1+\tau_{k,2}}{2}\big) - \prod_{k=1}^3\big(f_0(-\tau_{k,1}p_k)+\frac{1-\tau_{k,2}}{2}\big)\Big), \quad (5.66)$$

where $\ell_0 \in \mathbb{N}_0$, $\ell_0 \le 3$, $j_0 \in \mathbb{N}_0, j_0 \le 12$, $\sigma_{k,\ell}, \tau_{k,\ell} \in \{\pm 1\}$, $j_1, j_2 \in \{1,2,3\}$ *and* $\alpha_{\ell_k}, \beta_{\ell_k} \in \{0,1,2\}$. *Any term contains a product of at least two of the functions* $\hat{v}$, f_0, V_1, *and* V_2 *depending on at least two of the momenta* $\{p_1, p_2, p_3\}$, *which implies that the integrand in (5.66) exhibits the necessary regularity properties used in Propositions 5.7* *and 5.9 below. The absorption operator* $abs_{cub,d}(f_0)[J](s_2)$ *consists of terms of the form*

$$(-i)^{\ell}\lambda^{j}\int_{(\Lambda^*)^2} dp\, dk\, e^{-is_1m_1\Omega(p)-is_2m_2\Omega(k)}J(p)(1+f_0(p)+f_0(-p))$$
$$\hat{v}(p)V_1(p)^{2\alpha_1}V_2(p)^{\alpha_2}\hat{v}(k)V_1(k)^{2\alpha_3}V_2(k)^{\alpha_4}(f_0(k)+\iota)\,, \tag{5.67}$$

where $\ell\in\mathbb{N}_0$, $\ell\le 3$, $j\in\mathbb{N}_0$, $j\le 12$, $m_1,m_2\in\{0,\pm 2\}$, $\alpha_j\in\{0,1,2\}$ *and* $\iota\in\{0,1\}$. $|\operatorname{err}_2^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2};g)|$ *satisfies the same bound as* $|\operatorname{err}_2^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2};f)|$.

Proof Set $t=T\lambda^{-2}$. Since here all integrals range over Λ^* in the sense of (1.51), we will omit the domain of integration in this proof. Using that $\nu_0(Q)=0$ for $[Q,\mathcal{N}_b]\neq 0$, (5.18), and recalling J_j from (5.19) and (5.22), we start by observing that

$$-i\int_0^t ds\,\frac{\nu_0\big([a_0,\mathcal{H}_{cub}(s)]\big)}{|\Lambda|} = -i\int_0^t ds\Big(\int dp\, f_0(p)J_1(s,p)+\frac{J_4(s)}{|\Lambda|}\Big). \tag{5.68}$$

Here, we also used the fact that translation-invariance implies

$$\overline{a_p^{(\sigma)}a_q^{(-\sigma)}} = \delta(p-q)(f_0(p)+\delta_{\sigma,-1})\,. \tag{5.69}$$

Recall that, by Corollary 3.3,

$$\begin{aligned} J_1(s,p) &= \frac{\lambda\hat{v}(p)}{\sqrt{N}}\big(|u_\lambda(s,p)|^2+|v_\lambda(s,p)|^2+2\overline{v_\lambda}(s,p)\overline{u_\lambda}(s,p)\big)\\ &= \frac{\lambda\hat{v}(p)}{\sqrt{N}}\Big(1+2\lambda^2\mathrm{Re}\big(e^{-i\Omega(p)s}i\sin(\Omega(p)s)V_1(p)\big)\\ &\quad+\lambda^4\sin^2(\Omega(p)s)V_1(p)^4+|v_\lambda(s,p)|^2\\ &\quad+2\overline{v_\lambda}(s,p)\overline{u_\lambda}(s,p)\Big)\,, \end{aligned}\tag{5.70}$$

$$J_4(s) = \frac{\lambda}{\sqrt{N}}\int dp\,\hat{v}(p)\big(|v_\lambda(s,p)|^2+\overline{u_\lambda}(s,p)\overline{v_\lambda}(s,p)\big)\,. \tag{5.71}$$

Thus, again by Corollary 3.3, we obtain

$$\begin{aligned} \|J_1(s)+\frac{\lambda\hat{v}}{\sqrt{N}}\|_\infty &\le C\frac{\lambda\|\hat{v}\|_\infty}{\sqrt{N}}\Big(\lambda^2\|V_1\|_\infty^2+\lambda^4\|V_1\|_\infty^4+\|v_\lambda(s,\cdot)\|_\infty^2\\ &\quad+\|v_\lambda(s,\cdot)\|_\infty\|u_\lambda(t,\cdot)\|_\infty\Big)\\ &\le C\frac{\lambda^2\|\hat{v}\|_{w,d}^2(1+\lambda\|\hat{v}\|_{w,d})^3}{\sqrt{N}}\,, \end{aligned}\tag{5.72}$$

$$\begin{aligned} |J_4(s)| &\le C\frac{\lambda\|\hat{v}\|_1}{\sqrt{N}}\big(\|v_\lambda(s,\cdot)\|_\infty^2+\|v_\lambda(s,\cdot)\|_\infty\|u_\lambda(t,\cdot)\|_\infty\big)\\ &\le C\frac{\lambda^2\|\hat{v}\|_{w,d}^2(1+\lambda\|\hat{v}\|_{w,d})^3}{\sqrt{N}} \end{aligned}\tag{5.73}$$

for all $s\ge 0$. Then (5.68), Lemma 4.7, and the definition (5.57) of $\operatorname{err}_{1,d}^{(Bog)}(t;\Phi)$ yield

$$|\operatorname{err}_{1,d}^{(Bog)}(t;\Phi)| \le C\frac{\lambda^2 t\|\hat{v}\|_{w,d}^2(1+\lambda\|\hat{v}\|_{w,d})^3(|\Lambda|^{-1}+\|f_0\|_1)}{N^{\frac{1}{2}}}$$

$$\leq \frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d} T}{N^{\frac{1}{2}}} . \tag{5.74}$$

Collecting (5.68), (5.72), (5.73), and (5.74), we have proved (5.57).

Next, we compute the dynamics of f. We use the fact that ν_0 is quasifree to obtain that

$$\begin{aligned}\nu_0([a_p^+ a_p, \mathcal{H}_{quart}(s)]) &= [a_p^+ a_p, \mathcal{H}_{quart}(s)] + [a_p^+ a_p, \mathcal{H}_{quart}(s)] \\ &\propto f_0(p)\widetilde{f}_0(p) - \widetilde{f}_0(p) f_0(p) + 0 = 0 ,\end{aligned} \tag{5.75}$$

since, due to Corollary (3.3), $\mathcal{H}_{quart}(s)$ is a quartic polynomial.

We observe that for self-adjoint operators A, B, C and any state ν we have that

$$\nu([[A, B], C]) = \nu\big(([[A, B], C])^+\big) = \mathrm{Re}\big(\nu([[A, B], C])\big) . \tag{5.76}$$

Similarly to the previous case, we have

$$\begin{aligned}[[f[J], \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)] &= [[f[J], \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)] \\ &= [[f[J], \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)] \\ &= 0.\end{aligned} \tag{5.77}$$

This leaves us with exactly two types of possibles contractions. These are

$$(-i)^2[[f[J], \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)] \tag{5.78}$$

and, using translation invariance and employing Lemma A.1,

$$\begin{aligned}&(-i)^2[[f[J], \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)] \\ &= -\frac{2J(0)}{|\Lambda|}\mathrm{Re}\Big([a_0, \mathcal{H}_{cub}(s_1)][a_0^+, \mathcal{H}_{cub}(s_2)]\Big) .\end{aligned} \tag{5.79}$$

(5.78) corresponds to a scattering or Boltzmann term, while (5.79) describes corrections to the evolution of the condensate.

We start by analyzing the condensate term. We have that

$$\begin{aligned}&\frac{-2J(0)}{|\Lambda|^2}\mathrm{Re}\int_{\Delta[t,2]} d\mathbf{s}_2\, [a_0, \mathcal{H}_{cub}(s_1)][a_0^+, \mathcal{H}_{cub}(s_2)] \\ &= \frac{-J(0)}{|\Lambda|^2}\int_{\Delta[t,2]} d\mathbf{s}_2\, \Big([a_0, \mathcal{H}_{cub}(s_1)][a_0^+, \mathcal{H}_{cub}(s_2)] \\ &\quad + [a_0, \mathcal{H}_{cub}(s_2)][a_0^+, \mathcal{H}_{cub}(s_1)]\Big) \\ &= -\frac{J(0)}{2|\Lambda|^2}\int d\mathbf{s}_2\, \Big([a_0, \mathcal{H}_{cub}(s_1)][a_0^+, \mathcal{H}_{cub}(s_2)] \\ &\quad + [a_0, \mathcal{H}_{cub}(s_2)][a_0^+, \mathcal{H}_{cub}(s_1)]\Big) \\ &= J(0)\Big| -i\int_0^t \frac{ds}{|\Lambda|}\, [a_0, \mathcal{H}_{cub}(s)]\Big|^2 .\end{aligned} \tag{5.80}$$

Here, we used the fact that

$$\overline{[a_0, \mathcal{H}_{cub}(s)]} = -[a_0^+, \mathcal{H}_{cub}(s)] . \tag{5.81}$$

Next, we apply quasifreeness of ν_0, followed by Proposition 5.1, to get that

$$
\begin{aligned}
-i\int_0^t ds\, \frac{[a_0, \mathcal{H}_{cub}(s)]}{|\Lambda|} &= -i\int_0^t ds\, \frac{\nu_0([a_0, \mathcal{H}_{cub}(s)])}{|\Lambda|} \\
&= \Phi_t - \mathrm{Rem}_2(t;\Phi)\,.
\end{aligned} \tag{5.82}
$$

In particular, we have that the condensate term is given by

$$
J(0)|\Phi_t|^2 + J(0)\,\mathrm{err}_{2,d}^{(Bog,\mathrm{Con})}(t;f)\,, \tag{5.83}
$$

where

$$
\begin{aligned}
\mathrm{err}_{2,d}^{(Bog,\mathrm{Con})}(t;f) &= -2\mathrm{Re}\Big((-i)\int_0^t ds\, \frac{\nu_0([a_0,\mathcal{H}_{cub}(s)])}{|\Lambda|}\overline{\mathrm{Rem}_2(t;\Phi)}\Big) \\
&\quad - |\,\mathrm{Rem}_2(t;\Phi)|^2\,.
\end{aligned} \tag{5.84}
$$

Using (5.68), (5.72), and (5.73), we have that

$$
\Big|\int_0^t ds\, \frac{\nu_0([a_0,\mathcal{H}_{cub}(s)])}{|\Lambda|}\Big| \le C_{\|\hat v\|_{w,d}} \frac{\lambda t(1+\|f_0\|_d)}{\sqrt{N}} \le \frac{C_{\|\hat v\|_{w,d},\|f_0\|_d}T}{N^{\frac12}\lambda} \tag{5.85}
$$

for all $\lambda \in (0,1)$. Thus, Proposition 5.1 and (5.85) yield

$$
\begin{aligned}
&|\,\mathrm{err}_{2,d}^{(Bog,\mathrm{Con})}(t;f)| \\
&\le C_{\|\hat v\|_{w,d},\|f_0\|_d}\Big[\frac{T}{N^{\frac12}\lambda}T^2 e^{C\|\hat v\|_{w,d}|\Lambda|/\lambda T}\frac{|\Lambda|^{\frac32}}{N\lambda^2}\Big(1+\frac{|\Lambda|}{N}\Big) \\
&\quad + T^4 e^{C\|\hat v\|_{w,d}|\Lambda|/\lambda T}\frac{|\Lambda|^3}{N^2\lambda^4}\Big(1+\frac{|\Lambda|}{N}\Big)^2\Big] \\
&\le C_{\|\hat v\|_{w,d},\|f_0\|_d} e^{C\|\hat v\|_{w,d}|\Lambda|/\lambda T}\frac{\langle T\rangle^4|\Lambda|^{\frac32}}{\lambda^3 N^{\frac32}}\Big(1+\frac{|\Lambda|}{N}\Big)^2\Big(1+\frac{|\Lambda|^{\frac32}}{\lambda N^{\frac12}}\Big)\,.
\end{aligned} \tag{5.86}
$$

Next, we compute the Boltzmann term for f. Observe that with the decomposition (5.27), we have that

$$
\mathcal{H}_{cub}(t) = \mathcal{H}_{cub}^{(1)}(t) + \mathcal{H}_{cub}^{(2)}(t)\,. \tag{5.87}
$$

Notice that $\mathcal{H}_{cub}^{(1)}(t)$ and $\mathcal{H}_{cub}^{(2)}(t)$ are formal adjoints. In particular, we have that

$$
\begin{aligned}
&\nu_0\big([[f[J],\mathcal{H}_{cub}(s_1)],\mathcal{H}_{cub}(s_2)]\big) \\
&= 2\mathrm{Re}\nu_0\big([[f[J],\mathcal{H}_{cub}^{(1)}(s_1)],\mathcal{H}_{cub}^{(1)}(s_2)]\big) \\
&\quad + 2\mathrm{Re}\nu_0\big([[f[J],\mathcal{H}_{cub}^{(1)}(s_1)],\mathcal{H}_{cub}^{(2)}(s_2)]\big)
\end{aligned} \tag{5.88}
$$

With that, we sort the Boltzmann contractions of $\nu_0\big([[f[J],\mathcal{H}_{cub}(s_1)],\mathcal{H}_{cub}(s_2)]\big)$ by powers of λ, i.e.,

$$
\begin{aligned}
-\frac{1}{|\Lambda|}[[f[J],\mathcal{H}_{cub}(s_1)],\mathcal{H}_{cub}(s_2)] &= \frac{\lambda^2}{N}\big(\mathrm{bol}^{(0)}(f_0)[J](s_1,s_2) \\
&\quad + \lambda\,\mathrm{bol}^{(1)}(f_0)[J](s_1,s_2) \\
&\quad + \lambda^2\,\mathrm{bol}^{(2)}(f_0)[J](s_1,s_2) \\
&\quad + \lambda^3\,\mathrm{bol}^{(3)}(f_0)[J](s_1,s_2)\big)\,,
\end{aligned} \tag{5.89}
$$

Notice that the CCR imply

$$[f[J], a^{(\sigma_1)}_{\sigma_1 p_1} a^{(\sigma_2)}_{\sigma_2 p_2} a^{(-\sigma_3)}_{\sigma_3 p_{12}}] = \sum_{j=1}^{3} \sigma_j (-1)^{\delta_{j,3}} J(p_j) a^{(\sigma_1)}_{\sigma_1 p_1} a^{(\sigma_2)}_{\sigma_2 p_2} a^{(-\sigma_3)}_{\sigma_3 p_{12}} . \tag{5.90}$$

We have that

$$\begin{aligned} \mathrm{bol}^{(0)}(f_0)[J](\mathbf{s}_2) = & -2\mathrm{Re} \int d\mathbf{p}_2 d\mathbf{k}_2 \, e^{i\Delta_{cub}\Omega(\mathbf{p}_2)s_1 - i\Delta_{cub}\Omega(\mathbf{k}_2)s_2} \\ & \hat{v}(p_2)\hat{v}(k_2)\Delta_{cub} J(\mathbf{p}_2) \\ & \frac{1}{|\Lambda|} \Big(\big[a^+_{p_1} a^+_{p_2} a_{p_{12}}, a^+_{k_{12}} a_{k_2} a_{k_1} \big] + \big[a^+_{p_1} a^+_{p_2} a_{p_{12}}, a^+_{k_{12}} a_{k_2} a_{k_1} \big] \\ = & -2\mathrm{Re} \int d\mathbf{p}_2 e^{i\Delta_{cub}\Omega(\mathbf{p}_2)(s_1 - s_2)} \hat{v}(p_2) \\ & (\hat{v}(p_1) + \hat{v}(p_2))\Delta_{cub} J(\mathbf{p}_2) \\ & \big(f_0(p_1) f_0(p_2) \widetilde{f}_0(p_{12}) - \widetilde{f}_0(p_1)\widetilde{f}_0(p_2) f_0(p_{12}) \big) \\ = & \,\mathrm{Re} \int d\mathbf{p}_2 e^{i\Delta_{cub}\Omega(\mathbf{p}_2)(s_1 - s_2)} (\hat{v}(p_1) + \hat{v}(p_2))^2 \Delta_{cub} J(\mathbf{p}_2) \\ & \big(\widetilde{f}_0(p_1)\widetilde{f}_0(p_2) f_0(p_{12}) - f_0(p_1) f_0(p_2) \widetilde{f}_0(p_{12}) \big) . \end{aligned} \tag{5.91}$$

Here, we used (5.69) followed by symmetry $p_1 \leftrightarrow p_2$.

Recalling Corollary 3.3, $\mathrm{bol}^{(1)}(f_0)[J](s_1, s_2)$ is given by the Boltzmann contractions of

$$\begin{aligned} & -\frac{2}{|\Lambda|} \mathrm{Re}\, i \int d\mathbf{p}_2 d\mathbf{k}_2 \, \hat{v}(p_2)\hat{v}(k_2) e^{i\Delta_{cub}\Omega(\mathbf{p}_2)s_1 + i\Delta_{cub}\Omega(\mathbf{k}_2)s_2} \\ & \Big(e^{-i\Omega(p_1)s_1} \sin(\Omega(p_1)s_1) V_2(p_1) \nu_0\big([[f[J], a_{-p_1} a^+_{p_2} a_{p_{12}}], a^+_{k_1} a^+_{k_2} a_{k_{12}}] \big) \\ & + e^{-i\Omega(p_2)s_1} \sin(\Omega(p_2)s_1) V_2(p_2) \nu_0\big([[f[J], a^+_{p_1} a_{-p_2} a_{p_{12}}], a^+_{k_1} a^+_{k_2} a_{k_{12}}] \big) \\ & + e^{-i\Omega(k_1)s_2} \sin(\Omega(k_1)s_2) V_2(k_1) \nu_0\big([[f[J], a^+_{p_1} a^+_{p_2} a_{p_{12}}], a_{-k_1} a^+_{k_2} a_{k_{12}}] \big) \\ & + e^{-i\Omega(k_2)s_2} \sin(\Omega(k_2)s_2) V_2(k_2) \nu_0\big([[f[J], a^+_{p_1} a^+_{p_2} a_{p_{12}}], a^+_{k_1} a_{-k_2} a_{k_{12}}] \big) \Big) \end{aligned} \tag{5.92}$$

Using (5.90), obtain, similarly to above,

$$\begin{aligned} & 2\mathrm{Im} \int d\mathbf{p}_2 \, \hat{v}(p_2) \Big[\big(-J(-p_1) + J(p_2) - J(p_{12}) \big) \\ & (\hat{v}(p_1) + \hat{v}(p_{12})) e^{i(\Omega(p_2) - \Omega(p_{12}))(s_1 - s_2) + i\Omega(p_1)s_2} \sin(\Omega(p_1)s_1) V_2(p_1) \\ & \big(\widetilde{f}_0(-p_1) f_0(p_2) \widetilde{f}_0(p_{12}) - f_0(-p_1) \widetilde{f}_0(p_2) f_0(p_{12}) \big) \\ & + \big(J(p_1) - J(-p_2) - J(p_{12}) \big) (\hat{v}(p_2) + \hat{v}(p_{12})) \\ & e^{i(\Omega(p_1) - \Omega(p_{12}))(s_1 - s_2) + i\Omega(p_2)s_2} \sin(\Omega(p_2)s_1) V_2(p_2) \\ & \big(f_0(p_1) \widetilde{f}_0(-p_2) \widetilde{f}_0(p_{12}) - \widetilde{f}_0(p_1) f_0(-p_2) f_0(p_{12}) \big) \\ & + \Big(e^{i(\Omega(p_1) - \Omega(p_{12}))(s_1 - s_2) + i\Omega(p_2)s_1} \sin(\Omega(p_2)s_2) V_2(p_2)(\hat{v}(p_2) + \hat{v}(p_{12})) \\ & + e^{i(\Omega(p_2) - \Omega(p_{12}))(s_1 - s_2) + i\Omega(p_1)s_1} \sin(\Omega(p_1)s_2) V_2(p_1)(\hat{v}(p_1) + \hat{v}(p_{12})) \Big) \\ & \Delta_{cub} J(\mathbf{p}_2) \big(f_0(p_1) f_0(p_2) \widetilde{f}_0(p_{12}) - \widetilde{f}_0(p_1)\widetilde{f}_0(p_2) f_0(p_{12}) \big) \Big] . \end{aligned} \tag{5.93}$$

Here, we used the fact that $\hat{v}$, and thus Ω, are even functions. Using symmetry in $p_1 \leftrightarrow p_2$, we can further simplify this expression to

$$
\begin{aligned}
&\mathrm{bol}^{(1)}(f_0)[J](\mathbf{s}_2)\\
&:= 2\mathrm{Im}\int d\mathbf{p}_2\,(\hat{v}(p_1)+\hat{v}(p_2))(\hat{v}(p_1)+\hat{v}(p_{12}))\\
&V_2(p_1)e^{i(\Omega(p_2)-\Omega(p_{12}))(s_1-s_2)}\\
&\Big(\big(-J(-p_1)+J(p_2)-J(p_{12})\big)e^{i\Omega(p_1)s_2}\sin(\Omega(p_1)s_1)\\
&\big(\widetilde{f}_0(-p_1)f_0(p_2)\widetilde{f}_0(p_{12})-f_0(-p_1)\widetilde{f}_0(p_2)f_0(p_{12})\big)\\
&+\Delta_{cub}J(\mathbf{p}_2)e^{i\Omega(p_1)s_1}\sin(\Omega(p_1)s_2)\\
&\big(f_0(p_1)f_0(p_2)\widetilde{f}_0(p_{12})-\widetilde{f}_0(p_1)\widetilde{f}_0(p_2)f_0(p_{12})\big)\Big)\,.
\end{aligned}
\tag{5.94}
$$

Recalling (5.52), we have that

$$
\int_{\Delta[T\lambda^{-2},2]} d\mathbf{s}_2\ \mathrm{bol}^{(1)}(f_0)[J](\mathbf{s}_2)\ =\ \mathrm{Bol}_d^{(1)}(f_0)[J](T;\lambda)\,. \tag{5.95}
$$

Next, we compute the corrections with coefficient $\frac{\lambda^4}{N}$. These come either from a correction to the same momentum, $a_p^+ \to i\lambda^2 V_1(p)^2\sin(\Omega(p)t)a_p^+$, or from two momentum flips, $a_p^+ \to i\lambda V_2(p)\sin(\Omega(p)t)a_{-p}$. In the first case, computations analogous to (5.91) yield

$$
\begin{aligned}
&-2\mathrm{Re}\,i\int d\mathbf{p}_2\,\hat{v}(p_2)(\hat{v}(p_1)+\hat{v}(p_2))\Delta_{cub}J(\mathbf{p}_2)e^{i\Delta_{cub}\Omega(\mathbf{p}_2)(s_1-s_2)}\\
&\Big(V_1(p_1)^2\big(\sin(\Omega(p_1)s_1)e^{-i\Omega(p_1)s_1}-\sin(\Omega(p_1)s_2)e^{i\Omega(p_1)s_2}\big)\\
&+V_1(p_2)^2\big(\sin(\Omega(p_2)s_1)e^{-i\Omega(p_2)s_1}-\sin(\Omega(p_2)s_2)e^{i\Omega(p_2)s_2}\big)\\
&-V_1(p_{12})^2\big(\sin(\Omega(p_{12})s_1)e^{i\Omega(p_{12})s_1}-\sin(\Omega(p_{12})s_2)e^{i\Omega(p_{12})s_2}\big)\Big)\\
&\big(f_0(p_1)f_0(p_2)\widetilde{f}_0(p_{12})-\widetilde{f}_0(p_1)\widetilde{f}_0(p_2)f_0(p_{12})\big)\,.
\end{aligned}
\tag{5.96}
$$

Using symmetry in $p_1 \leftrightarrow p_2$ again, we obtain

$$
\begin{aligned}
&\mathrm{bol}_1^{(2)}(f_0)[J](\mathbf{s}_2)\\
&:= \mathrm{Im}\int d\mathbf{p}_2\,(\hat{v}(p_1)+\hat{v}(p_2))^2\Delta_{cub}J(\mathbf{p}_2)e^{i\Delta_{cub}\Omega(\mathbf{p}_2)(s_1-s_2)}\\
&\Big(V_1(p_1)^2\big(\sin(\Omega(p_1)s_1)e^{-i\Omega(p_1)s_1}-\sin(\Omega(p_1)s_2)e^{i\Omega(p_1)s_2}\big)\\
&+V_1(p_2)^2\big(\sin(\Omega(p_2)s_1)e^{-i\Omega(p_2)s_1}-\sin(\Omega(p_2)s_2)e^{i\Omega(p_2)s_2}\big)\\
&-V_1(p_{12})^2\big(\sin(\Omega(p_{12})s_1)e^{i\Omega(p_{12})s_1}-\sin(\Omega(p_{12})s_2)e^{i\Omega(p_{12})s_2}\big)\Big)\\
&\big(f_0(p_1)f_0(p_2)\widetilde{f}_0(p_{12})-\widetilde{f}_0(p_1)\widetilde{f}_0(p_2)f_0(p_{12})\big)\,.
\end{aligned}
\tag{5.97}
$$

In the case of two momentum flips, the correction terms are given by the Boltzmann contractions of

$$
\begin{aligned}
&-\frac{2}{|\Lambda|}\mathrm{Re}\int d\mathbf{p}_2 d\mathbf{k}_2\,\hat{v}(p_2)\hat{v}(k_2)e^{i\Delta_{cub}\Omega(\mathbf{p}_2)s_1-i\Delta_{cub}\Omega(\mathbf{k}_2)s_2}\\
&\Big(e^{-i(\Omega(p_1)-\Omega(p_{12}))s_1}i\sin(\Omega(p_1)s_1)(-i)\sin(\Omega(p_{12})s_1)V_2(p_1)V_2(p_{12})
\end{aligned}
$$

$$
\begin{aligned}
&(-J(-p_1)+J(p_2)+J(-p_{12}))\nu_0([a_{-p_1}a^+_{p_2}a^+_{-p_{12}},a^+_{k_{12}}a_{k_2}a_{k_1}])\\
&+e^{-i(\Omega(p_2)-\Omega(p_{12}))s_1}i\sin(\Omega(p_2)s_1)(-i)\sin(\Omega(p_{12})s_1)V_2(p_2)V_2(p_{12})\\
&(J(p_1)-J(-p_2)+J(-p_{12}))\nu_0([a^+_{p_1}a_{-p_2}a^+_{-p_{12}},a^+_{k_{12}}a_{k_2}a_{k_1}])\\
&+e^{-i\Omega(p_1)s_1+i\Omega(k_2)s_2}i\sin(\Omega(p_1)s_1)(-i)\sin(\Omega(k_2)s_2)V_2(p_1)V_2(k_2)\\
&(-J(-p_1)+J(p_2)-J(p_{12}))\nu_0([a_{-p_1}a^+_{p_2}a_{p_{12}},a^+_{k_{12}}a^+_{-k_2}a_{k_1}])\\
&+e^{-i\Omega(p_2)s_1+i\Omega(k_2)s_2}i\sin(\Omega(p_2)s_1)(-i)\sin(\Omega(k_2)s_2)V_2(p_2)V_2(k_2)\\
&(J(p_1)-J(-p_2)-J(p_{12}))\nu_0([a^+_{p_1}a_{-p_2}a_{p_{12}},a^+_{k_{12}}a^+_{-k_2}a_{k_1}])\\
&+e^{-i\Omega(p_1)s_1+i\Omega(k_1)s_2}i\sin(\Omega(p_1)s_1)(-i)\sin(\Omega(k_1)s_2)V_2(p_1)V_2(k_1)\\
&(-J(-p_1)+J(p_2)-J(p_{12}))\nu_0([a_{-p_1}a^+_{p_2}a_{p_{12}},a^+_{k_{12}}a_{k_2}a^+_{-k_1}])\\
&+e^{-i\Omega(p_2)s_1+i\Omega(k_1)s_2}i\sin(\Omega(p_2)s_1)(-i)\sin(\Omega(k_1)s_2)V_2(p_2)V_2(k_1)\\
&(J(p_1)-J(-p_2)-J(p_{12}))\nu_0([a^+_{p_1}a_{-p_2}a_{p_{12}},a^+_{k_{12}}a_{k_2}a^+_{-k_1}])\\
&+e^{i\Omega(p_{12})s_1-i\Omega(k_{12})s_2}(-i)\sin(\Omega(p_{12})s_1)i\sin(\Omega(k_{12})s_2)V_2(p_{12})V_2(k_{12})\\
&(J(p_1)+J(p_2)+J(-p_{12}))\nu_0([a^+_{p_1}a^+_{p_2}a^+_{-p_{12}},a_{-k_{12}}a_{k_2}a_{k_1}])\Big). \qquad (5.98)
\end{aligned}
$$

This expression equals

$$
\begin{aligned}
&-2\mathrm{Re}\int d\mathbf{p}_2\,\hat{v}(p_2)\Big[\sin(\Omega(p_1)s_1)\sin(\Omega(p_{12})s_1)e^{i\Omega(p_2)(s_1-s_2)+i(\Omega(p_1)-\Omega(p_{12}))s_2}\\
&V_2(p_1)V_2(p_{12})(\hat{v}(p_2)+\hat{v}(p_{12}))(-J(-p_1)+J(p_2)+J(-p_{12}))\\
&\big(\widetilde{f}_0(-p_1)f_0(p_2)f(-p_{12})-f_0(-p_1)\widetilde{f}_0(p_2)\widetilde{f}_0(-p_{12})\big)\\
&+\sin(\Omega(p_2)s_1)\sin(\Omega(p_{12})s_1)e^{i\Omega(p_1)(s_1-s_2)+i(\Omega(p_2)-\Omega(p_{12}))s_2}\\
&V_2(p_2)V_2(p_{12})(\hat{v}(p_1)+\hat{v}(p_{12}))(J(p_1)-J(-p_2)+J(-p_{12}))\\
&\big(f_0(p_1)\widetilde{f}_0(-p_2)f(-p_{12})-\widetilde{f}_0(p_1)f_0(-p_2)\widetilde{f}_0(-p_{12})\big)\\
&+\sin(\Omega(p_1)s_1)e^{i\Omega(p_2)(s_1-s_2)-i\Omega(p_{12})s_1}V_2(p_1)(-J(-p_1)+J(p_2)-J(p_{12}))\\
&\big(\hat{v}(p_1)V_2(p_1)\sin(\Omega(p_1)s_2)e^{i\Omega(p_{12})s_2}+\hat{v}(p_{12})V_2(p_{12})\sin(\Omega(p_{12})s_2)e^{-i\Omega(p_1)s_2}\big)\\
&\big(\widetilde{f}_0(-p_1)f_0(p_2)\widetilde{f}_0(p_{12})-f_0(-p_1)\widetilde{f}_0(p_2)f_0(p_{12})\big)\\
&+\sin(\Omega(p_2)s_1)e^{i\Omega(p_1)(s_1-s_2)-i\Omega(p_{12})s_1}V_2(p_2)(J(p_1)-J(-p_2)-J(p_{12}))\\
&\big(\hat{v}(p_2)V_2(p_2)\sin(\Omega(p_2)s_2)e^{i\Omega(p_{12})s_2}+\hat{v}(p_{12})V_2(p_{12})\sin(\Omega(p_{12})s_2)e^{-i\Omega(p_2)s_2}\big)\\
&\big(f_0(p_1)\widetilde{f}_0(-p_2)\widetilde{f}_0(p_{12})-\widetilde{f}_0(p_1)f_0(-p_2)f_0(p_{12})\big)\\
&+\sin(\Omega(p_1)s_1)e^{i\Omega(p_2)(s_1-s_2)-i\Omega(p_{12})s_1}V_2(p_1)\hat{v}(p_2)(-J(-p_1)+J(p_2)-J(p_{12}))\\
&\big(V_2(p_1)\sin(\Omega(p_1)s_2)e^{i\Omega(p_{12})s_2}+V_2(p_{12})\sin(\Omega(p_{12})s_2)e^{i\Omega(p_1)s_2}\big)\\
&\big(\widetilde{f}_0(-p_1)f_0(p_2)\widetilde{f}_0(p_{12})-f_0(-p_1)\widetilde{f}_0(p_2)f_0(p_{12})\big)\\
&+\sin(\Omega(p_2)s_1)e^{i\Omega(p_1)(s_1-s_2)-i\Omega(p_{12})s_1}V_2(p_2)\hat{v}(p_1)(J(p_1)-J(-p_2)-J(p_{12}))\\
&\big(V_2(p_2)\sin(\Omega(p_2)s_2)e^{i\Omega(p_{12})s_2}+V_2(p_{12})\sin(\Omega(p_{12})s_2)e^{i\Omega(p_2)s_2}\big)\\
&\big(f_0(p_1)\widetilde{f}_0(-p_2)\widetilde{f}_0(p_{12})-\widetilde{f}_0(p_1)f_0(-p_2)f_0(p_{12})\big)\\
&+V_2(p_2)e^{i(\Omega(p_1)+\Omega(p_2))s_1}\sin(\Omega(p_{12})s_1)\big(J(p_1)+J(p_2)+J(-p_{12})\big)\\
&+\Big(e^{-i(\Omega(p_1)+\Omega(p_2))s_2}\sin(\Omega(p_{12})s_2)V_2(p_{12})(\hat{v}(p_1)+\hat{v}(p_2))\\
&+e^{-i(\Omega(p_1)+\Omega(p_{12}))s_2}\sin(\Omega(p_2)s_2)V_2(p_2)(\hat{v}(p_1)+\hat{v}(p_{12}))
\end{aligned}
$$

$$+ \sin(\Omega(p_1)s_2)e^{-i(\Omega(p_2)+\Omega(p_{12}))s_2} V_2(p_1)(\hat{v}(p_2) + \hat{v}(p_{12}))\Big)$$
$$\big(f_0(p_1) f_0(p_2) f_0(-p_{12}) - \widetilde{f}_0(p_1)\widetilde{f}_0(p_2)\widetilde{f}_0(-p_{12})\big)\Big]. \tag{5.99}$$

Using symmetry in $p_1 \leftrightarrow p_2$, this can be reduced to

$$\begin{aligned}
&-2\mathrm{Re} \int d\mathbf{p}_2\, (\hat{v}(p_1) + \hat{v}(p_2))\\
&\Big[\sin(\Omega(p_1)s_1)\sin(\Omega(p_{12})s_1)e^{i\Omega(p_2)(s_1-s_2)+i(\Omega(p_1)-\Omega(p_{12}))s_2}\\
&V_2(p_1)V_2(p_{12})(\hat{v}(p_2) + \hat{v}(p_{12}))(-J(-p_1) + J(p_2) + J(-p_{12}))\\
&\big(\widetilde{f}_0(-p_1) f_0(p_2) f(-p_{12}) - f_0(-p_1)\widetilde{f}_0(p_2)\widetilde{f}_0(-p_{12})\big)\\
&+ \sin(\Omega(p_1)s_1)e^{i\Omega(p_2)(s_1-s_2)-i\Omega(p_{12})s_1} V_2(p_1)(-J(-p_1) + J(p_2) - J(p_{12}))\\
&\big(\hat{v}(p_1)V_2(p_1)\sin(\Omega(p_1)s_2)e^{i\Omega(p_{12})s_2} + \hat{v}(p_{12})V_2(p_{12})\sin(\Omega(p_{12})s_2)e^{-i\Omega(p_1)s_2}\big)\\
&\big(\widetilde{f}_0(-p_1) f_0(p_2)\widetilde{f}_0(p_{12}) - f_0(-p_1)\widetilde{f}_0(p_2) f_0(p_{12})\big)\\
&+ \sin(\Omega(p_1)s_1)e^{i\Omega(p_2)(s_1-s_2)-i\Omega(p_{12})s_1} V_2(p_1)\hat{v}(p_2)(-J(-p_1) + J(p_2) - J(p_{12}))\\
&\big(V_2(p_1)\sin(\Omega(p_1)s_2)e^{i\Omega(p_{12})s_2} + V_2(p_{12})\sin(\Omega(p_{12})s_2)e^{i\Omega(p_1)s_2}\big)\\
&\big(\widetilde{f}_0(-p_1) f_0(p_2)\widetilde{f}_0(p_{12}) - f_0(-p_1)\widetilde{f}_0(p_2) f_0(p_{12})\big)\\
&\Big(\frac{1}{2}e^{-i(\Omega(p_1)-\Omega(p_2))s_2}\sin(\Omega(p_{12})s_2)V_2(p_{12})(\hat{v}(p_1) + \hat{v}(p_2))\\
&+ e^{-i(\Omega(p_1)-\Omega(p_{12}))s_2}\sin(\Omega(p_2)s_2)V_2(p_2)(\hat{v}(p_1) + \hat{v}(p_{12}))\Big)\\
&\big(f_0(p_1) f_0(p_2) f_0(-p_{12}) - \widetilde{f}_0(p_1)\widetilde{f}_0(p_2)\widetilde{f}_0(-p_{12})\big)\Big].
\end{aligned} \tag{5.100}$$

This can be further be simplified to

$$\begin{aligned}
&\mathrm{bol}_2^{(2)}(f_0)[J](\mathbf{s}_2)\\
&:= 2\mathrm{Re} \int d\mathbf{p}_2\, (\hat{v}(p_1) + \hat{v}(p_2))\\
&\Big[\sin(\Omega(p_1)s_1)\sin(\Omega(p_{12})s_1)e^{i\Omega(p_2)(s_1-s_2)+i(\Omega(p_1)-\Omega(p_{12}))s_2}\\
&V_2(p_1)V_2(p_{12})(\hat{v}(p_2) + \hat{v}(p_{12}))(-J(-p_1) + J(p_2) + J(-p_{12}))\\
&\big(f_0(-p_1)\widetilde{f}_0(p_2)\widetilde{f}_0(-p_{12}) - \widetilde{f}_0(-p_1) f_0(p_2) f(-p_{12})\big)\\
&+ \sin(\Omega(p_1)s_1)e^{i\Omega(p_2)(s_1-s_2)-i\Omega(p_{12})s_1} V_2(p_1)\\
&(-J(-p_1) + J(p_2) - J(p_{12}))\\
&\Big((\hat{v}(p_1) + \hat{v}(p_2))V_2(p_1)\sin(\Omega(p_1)s_2)e^{-i\Omega(p_{12})s_2}\\
&+ (\hat{v}(p_2) + \hat{v}(p_{12}))V_2(p_{12})\sin(\Omega(p_{12})s_2)e^{-i\Omega(p_1)s_2}\Big)\\
&\big(f_0(-p_1)\widetilde{f}_0(p_2) f_0(p_{12}) - \widetilde{f}_0(-p_1) f_0(p_2)\widetilde{f}_0(p_{12})\big)\\
&\Big(\frac{1}{2}e^{-i(\Omega(p_1)+\Omega(p_2))s_2}\sin(\Omega(p_{12})s_2)V_2(p_{12})(\hat{v}(p_1) + \hat{v}(p_2))\\
&+ e^{-i(\Omega(p_1)+\Omega(p_{12}))s_2}\sin(\Omega(p_2)s_2)V_2(p_2)(\hat{v}(p_1) + \hat{v}(p_{12}))\Big)\\
&\big(\widetilde{f}_0(p_1)\widetilde{f}_0(p_2)\widetilde{f}_0(-p_{12}) - f_0(p_1) f_0(p_2) f_0(-p_{12})\big)\Big].
\end{aligned} \tag{5.101}$$

In addition, let

$$\mathrm{bol}^{(2)}(f_0)[J](\mathbf{s}_2) := \mathrm{bol}_1^{(2)}(f_0)[J](\mathbf{s}_2) + \mathrm{bol}_2^{(2)}(f_0)[J](\mathbf{s}_2) . \tag{5.102}$$

Recalling definitions (5.53), (5.54), and (5.56), we obtain that

$$\int_{\Delta[T\lambda^{-2},2]} d\mathbf{s}_2 \; \mathrm{bol}^{(2)}(f_0)[J](\mathbf{s}_2) = \mathrm{Bol}_d^{(2)}(f_0)[J](T;\lambda) . \tag{5.103}$$

In order to bound $\mathrm{bol}^{(3)}(f_0)[J](\mathbf{s}_2)$, we need to, first, look at integrability of the terms. Notice that, for Boltzmann contractions, the order of the creation/annihilation operators within a single argument of a commutator do not matter. In particular, we are interested in evaluating expressions involving the Boltzmann contractions of

$$\left[a_{\sigma_1 p_1}^{(\sigma_1)} a_{\sigma_2 p_2}^{(\sigma_2)} a_{\sigma_3 p_{12}}^{(-\sigma_3)}, a_{\tau_1 k_1}^{(\tau_1)} a_{\tau_2 k_2}^{(\tau_2)} a_{\tau_3 k_{12}}^{(-\tau_3)} \right] , \tag{5.104}$$

which contain a factor

$$\begin{aligned} &\left(f_0(\sigma_1 p_1) + \frac{1-\sigma_1}{2} \right) \left(f_0(\sigma_2 p_2) + \frac{1-\sigma_2}{2} \right) \left(f_0(\sigma_3 p_{12}) + \frac{1+\sigma_3}{2} \right) \\ &\quad - \left(f_0(\sigma_1 p_1) + \frac{1+\sigma_1}{2} \right) \left(f_0(\sigma_2 p_2) + \frac{1+\sigma_2}{2} \right) \left(f_0(\sigma_3 p_{12}) + \frac{1-\sigma_3}{2} \right) \\ &= \sigma_3 f_0(\sigma_1 p_1) f_0(\sigma_2 p_2) - \sigma_1 f_0(\sigma_2 p_2) f_0(\sigma_3 p_{12}) - \sigma_2 f_0(\sigma_1 p_1) f_0(\sigma_3 p_{12}) \\ &\quad + (\sigma_3 - \sigma_2) f_0(\sigma_1 p_1) + (\sigma_3 - \sigma_1) f_0(\sigma_2 p_2) - (\sigma_1 + \sigma_2) f_0(\sigma_3 p_{12}) . \end{aligned} \tag{5.105}$$

Observe that there is a global coefficient $\hat{v}(p_2)$. After evaluating all δ coming from contractions between p and k momenta, we want to verify integrability w.r.t. $dp_1\, dp_2$. We are thus left with verifying integrability for the terms involving $(\sigma_3 - \sigma_1) f_0(\sigma_2 p_2)$. This term occurs only if $\sigma_3 = -\sigma_1$. Another global factor then is

$$V_2(p_1)^{\frac{1-\sigma_1}{2}} V_2(p_{12})^{\frac{1-\sigma_3}{2}} = V_2(p_1)^{\frac{1-\sigma_1}{2}} V_2(p_{12})^{\frac{1+\sigma_1}{2}} , \tag{5.106}$$

where we recall from (5.88) that we only need to consider $\mathcal{H}_{cub}^{(1)}(s_1)$ in the first argument of the commutator. In particular, there is an integrable factor w.r.t. dp_1. With that, we have the estimate

$$| \mathrm{bol}^{(3)}(f_0)[J](\mathbf{s}_2) | \le C_{\|\hat{v}\|_{w,d}, \|f_0\|_d} \|J\|_{\ell^\infty(\Lambda^*)} . \tag{5.107}$$

As a consequence, we have that

$$\begin{aligned} | \mathrm{err}_{2,d}^{(Bog,\mathrm{Bol})}(t;\, f[J]) | &= \left| \frac{\lambda^5}{N} \int_{\Delta[t,2]} d\mathbf{s}_2 \; \mathrm{bol}^{(3)}(f_0)[J](\mathbf{s}_2) \right| \\ &\le C_{\|\hat{v}\|_{w,d}, \|f_0\|_d} \|J\|_{\ell^\infty(\Lambda^*)} \frac{\lambda^5 t^2}{N} \\ &\le C_{\|\hat{v}\|_{w,d}, \|f_0\|_d} \|J\|_{\ell^\infty(\Lambda^*)} T^2 \frac{\lambda}{N} . \end{aligned} \tag{5.108}$$

This concludes the proof of (5.60).

Finally, we compute the dynamics of g. We write

$$-i \int_0^t \frac{ds}{|\Lambda|} [g[J], \mathcal{H}_{quart}(s)] = \frac{\lambda^2}{N} \int_0^t ds \; \mathrm{abs}_{quart,d}(f_0)[J](s) , \tag{5.109}$$

where $\mathrm{abs}_{quart,d}(f_0)[J](s)$ consists of terms of the form

$$(-i)^\ell \lambda^j \int dp\, dk\, e^{-is(m_1\Omega(p)+m_2\Omega(k))} J(p)$$
$$[(1+f_0(p))(1+f_0(-p)) - f_0(p)f_0(-p)]$$
$$V_1(p)^{2\alpha_1} V_2(p)^{\alpha_2} \hat{v}(p \pm k) V_1(k)^{2\alpha_3} V_2(k)^{\alpha_4} (f_0(k)+\iota)\,, \tag{5.110}$$

where $\ell \in \mathbb{N}_0$, $\ell \leq 3$, $j \in \mathbb{N}_0$, $j \leq 7$, $m_1, m_2 \in \{0, \pm 2\}$, $\alpha_j \in \{0,1,2\}$ and $\iota \in \{0,1\}$. Here, we already employed the fact that $\hat{v}(0) = 0$. Using symmetry of the integrand w.r.t. $p \leftrightarrow -p$, we can further simplify the expression (5.110) to

$$(-i)^\ell \lambda^j \int dp\, dk\, e^{-is(m_1\Omega(p)+m_2\Omega(k))} (1 + f_0(p) + f_0(-p)) J(p)$$
$$V_1(p)^{2\alpha_1} V_2(p)^{\alpha_2} \hat{v}(p-k) V_1(k)^{2\alpha_3} V_2(k)^{\alpha_4} (f_0(k)+\iota)\,. \tag{5.111}$$

Observe that the only terms contributing to (5.109) are of the form $[aa, (a^+)^3 a]$ with any permutation of $(a^+)^3 a$, which is why the terms contain at least one momentum flip $a_p \to i\lambda V_2(p)\sin(\Omega(p)s) a_{-p}$. This justifies the extra factor λ on the RHS of (5.109). In the case $\iota = 1$, we need to have at least one annihilation operator left of a creation operator in the second argument of the commutator. This yields an additional factor $V_2(k)$. In particular, the integrand in (5.110) contains $\hat{v}(p \pm k) f_0(k)$ or $\hat{v}(p \pm k) V_2(k)$.

Next, we compute

$$\frac{(-i)^2}{|\Lambda|} \int_{\Delta[t,2]} d\mathbf{s}_2\, \nu_0([[g[J], \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)])$$
$$= -\frac{1}{|\Lambda|} \int_{\Delta[t,2]} d\mathbf{s}_2 \Big([[g(J), \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)] \tag{5.112}$$

$$+ [[g[J], \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)] \tag{5.113}$$

$$+ [[g[J], \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)] \tag{5.114}$$

Similarly to above, we will refer to (5.112) as condensate contraction and to (5.113) as Boltzmann contraction. In addition, we call (5.114) pair absorption contractions.

We start again with the condensate term. We obtain that

$$-\frac{1}{|\Lambda|} \int_{\Delta[t,2]} d\mathbf{s}_2\, [[g(J), \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)]$$
$$= -2J(0) \int_{\Delta[t,2]} \frac{d\mathbf{s}_2}{|\Lambda|} \Big([a_0, \mathcal{H}_{cub}(s_1)][a_0, \mathcal{H}_{cub}(s_2)] \Big)$$
$$= J(0)\Phi_t^2 + J(0)\, \mathrm{err}_{2,d}^{(Bog,\mathrm{Con})}(t; g). \tag{5.115}$$

where, analogously to (5.84),

$$\mathrm{err}_{2,d}^{(Bog,\mathrm{Con})}(t; g) := -2\Big(-i \int_0^t ds\, \frac{\nu_0([a_0, \mathcal{H}_{cub}(s)])}{|\Lambda|} \mathrm{Rem}_2(t;\Phi) \Big)$$
$$- \big(\mathrm{Rem}_2(t;\Phi)\big)^2\,. \tag{5.116}$$

Using analogous estimates as for (5.86), we find that $|\operatorname{err}_{2,d}^{(Bog,\mathrm{Con})}(t;g)|$ satisfies the same bounds as $|\operatorname{err}_{2,d}^{(Bog,\mathrm{Con})}(t;f)|$.

For the Boltzmann contraction, we write

$$\begin{aligned} &-\int_{\Delta[t,2]} \frac{d\mathbf{s}_2}{|\Lambda|} [[g[J], \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)] \\ &= \frac{\lambda^2}{N} \int_{\Delta[t,2]} d\mathbf{s}_2 \operatorname{col}_d(f_0)[J](\mathbf{s}_2)\,. \end{aligned} \tag{5.117}$$

The expressions in $\operatorname{col}_d(f_0)[J](\mathbf{s}_2)$ are of the form

$$\begin{aligned} &(-i)^{\ell_0}\lambda^{j_0} \int d\mathbf{p}_3\, e^{-i\sum_{\ell=1}^{2} s_\ell \sum_{k=1}^{3}\sigma_{k,\ell}\Omega(p_k)}\delta(p_1+p_2-p_3) \\ &J(p_{j_1})\hat{v}(p_2)\hat{v}(p_{j_2})\prod_{k=1}^{3} V_1(p_k)^{2\alpha_{\ell_k}} V_2(p_k)^{\beta_{\ell_k}} \\ &\Big(\prod_{k=1}^{3}\big(f_0(\tau_{k,1}p_k)+\frac{1+\tau_{k,2}}{2}\big) - \prod_{k=1}^{3}\big(f_0(-\tau_{k,1}p_k)+\frac{1-\tau_{k,2}}{2}\big)\Big)\,, \end{aligned} \tag{5.118}$$

where $\ell_0 \in \mathbb{N}_0^{\leq 3}$, $j_0 \in \mathbb{N}_0$, $j_0 \leq 12$, $\sigma_{k,\ell}, \tau_{k,\ell} \in \{\pm 1\}$, $j_1, j_2 \in \{1,2,3\}$ and $\alpha_{\ell_k}, \beta_{\ell_k} \in \{0,1,2\}$. We need to ensure integrability of each of these terms. More precisely, we will show that any term contains a product of at least two of the functions $\hat{v}$, f_0, V_1, and V_2 depending on at least two of the momenta p_1, p_2, or p_3. We have that

$$\int dp\, J(p)a_p a_{-p} = \int dp\, J(p)a_{-p}a_p = \int dp\, J(-p)a_p a_{-p}\,, \tag{5.119}$$

where we used the CCR followed by substitution. In particular, we may assume without loss of generality that J is even. Then the CCR imply

$$\begin{aligned} [g[J], a^{(\sigma_1)}_{\sigma_1 p_1} a^{(\sigma_2)}_{\sigma_2 p_2} a^{(-\sigma_3)}_{\sigma_3 p_3}] &= \delta_{\sigma_1,1} J(p_1) a_{-p_1} a^{(\sigma_2)}_{\sigma_2 p_2} a^{(-\sigma_3)}_{\sigma_3 p_3} \\ &\quad + \delta_{\sigma_2,1} J(p_2) a^{(\sigma_1)}_{\sigma_1 p_1} a_{-p_2} a^{(-\sigma_3)}_{\sigma_3 p_3} \\ &\quad + \delta_{\sigma_3,-1} J(p_3) a^{(\sigma_1)}_{\sigma_1 p_1} a^{(\sigma_2)}_{\sigma_2 p_2} a_{p_3}\,. \end{aligned} \tag{5.120}$$

We will discuss the expressions related to one these three terms in detail; the remaining follow with analogous computations. Consider the Boltzmann contractions of

$$\nu_0\big([a_{-p_1} a^{(\sigma_2)}_{\sigma_2 p_2} a^{(-\sigma_3)}_{\sigma_3 p_3}, a^{(\tau_1)}_{\tau_1 k_1} a^{(\tau_2)}_{\tau_2 k_2} a^{(-\tau_3)}_{\tau_3 k_3}]\big)\,, \tag{5.121}$$

which is why, again, the order of the operators a and a^+ does not matter. Observe that it is sufficient to have a factor $\hat{v}$, f_0, V_1, or V_2 with momentum p_1 or p_{12}, since, due to (5.118), we always have a coefficient $\hat{v}(p_2)$. The Boltzmann contractions in (5.121) yield a factor

$$\begin{aligned} &\big(f_0(-p_1)+1\big)\big(f_0(\sigma_2 p_2)+\frac{1-\sigma_2}{2}\big)\big(f_0(\sigma_3 p_{12})+\frac{1+\sigma_3}{2}\big) \\ &\quad - f_0(-p_1)\big(f_0(\sigma_2 p_2)+\frac{1+\sigma_2}{2}\big)\big(f_0(\sigma_3 p_{12})+\frac{1-\sigma_3}{2}\big) \\ &= \sigma_3 f_0(-p_1) f_0(\sigma_2 p_2) + f_0(\sigma_2 p_2) f_0(\sigma_3 p_{12}) - \sigma_2 f_0(-p_1) f_0(\sigma_3 p_{12}) \\ &\quad + (\sigma_3-\sigma_2) f_0(-p_1) + (1+\sigma_3) f_0(\sigma_2 p_2) + (1-\sigma_2) f_0(\sigma_3 p_{12})\,. \end{aligned} \tag{5.122}$$

The only term in (5.122) that does not already involve a factor depending on a momentum other than p_2, see also (5.118), is $(1+\sigma_3) f_0(\sigma_2 p_2)$. This term only appears if $\sigma_3 = 1$, which we now want to consider.

Due to momentum flips in p_j, the corresponding term associated with $\mathcal{H}^{(2)}_{cub}(s_1)$ in the first argument of the commutator has a coefficient $V_2(p_1)V_2(p_3)$, yielding the remaining integrability w.r.t. dp_1.

Thus, let us consider the terms associated with $\mathcal{H}^{(2)}_{cub}(s_1)$. If we contract $a^{(\tau_2)}_{\tau_2 k_2}$ in with $a^{(-\sigma_1)}_{-\sigma_1 p_1}$ or $a^{(-\sigma_3)}_{\sigma_3 p_3}$, we obtain a factor $\hat{v}(p_1)$ or $\hat{v}(p_{12})$, yielding integrability w.r.t. dp_1.

So, it remains to consider the case when $a^{(\tau_2)}_{\tau_2 k_2}$ is contracted with $a^{(\sigma_2)}_{\sigma_2 p_2}$. The remaining contractions yield either $(\tau_1, -\tau_3) = (1, 1)$ or $(-\tau_3, \tau_1) = (1, 1)$. In those cases, we obtain an additional factor $V_2(p_{12})$ or $V_2(p_1)$. This concludes the argument.

We are left with evaluating the pair absorption terms

$$
\begin{aligned}
&-\frac{1}{|\Lambda|} \int_{\Delta[t,2]} d\mathbf{s}_2 \, [[g[J], \mathcal{H}_{cub}(s_1)], \mathcal{H}_{cub}(s_2)] \\
&= \frac{\lambda^2}{N} \int_{\Delta[t,2]} d\mathbf{s}_2 \, \mathrm{abs}_{cub,d}(f_0)[J](\mathbf{s}_2) \, .
\end{aligned} \tag{5.123}
$$

$\mathrm{abs}_{cub,d}(f_0)[J](\mathbf{s}_2)$ consists of terms of the form

$$
\begin{aligned}
&(-i)^{\ell} \lambda^{j} \int dp \, dk \, e^{-is_1 m_1 \Omega(p) - is_2 m_2 \Omega(k)} J(p)(1 + f_0(p) + f_0(-p))\hat{v}(p) \\
&V_1(p)^{2\alpha_1} V_2(p)^{\alpha_2} \hat{v}(k) V_1(k)^{2\alpha_3} V_2(k)^{\alpha_4} (f_0(k) + \iota) \, ,
\end{aligned} \tag{5.124}
$$

where $\ell \in \mathbb{N}_0$, $\ell \leq 3$, $j \in \mathbb{N}_0$, $j \leq 12$, $m_1, m_2 \in \{0, \pm 2\}$, $\alpha_j \in \{0, 1, 2\}$ and $\iota \in \{0, 1\}$. Here again, we take into account that terms involving $\hat{v}(0) = 0$ vanish. This concludes the proof.
□

Lemma 5.3 *We have the following expansions.*

1. *Identifying the RHS with its continuous extension, we have that*

$$
\begin{aligned}
\int_{\Delta[t,2]} d\mathbf{s}_2 \, e^{i(\omega_1 s_1 + \omega_2 s_2)} &= \frac{2}{\omega_2} \Big(\frac{\sin^2\big((\omega_1+\omega_2)t/2\big)}{\omega_1+\omega_2} - \frac{\sin^2\big(\omega_1 t/2\big)}{\omega_1} \Big) \\
&\quad - \frac{i}{\omega_2} \Big(\frac{\sin\big((\omega_1+\omega_2)t\big)}{\omega_1+\omega_2} - \frac{\sin(\omega_1 t)}{\omega_1} \Big)
\end{aligned} \tag{5.125}
$$

for all $\omega_1, \omega_2 \in \mathbb{R}$.

2. *The Bogoliubov dispersion Ω in Lemma 3.2 satisfies*

$$
\begin{aligned}
\Omega &= E + \lambda \hat{v}_{Bog} \\
&= E + \lambda \hat{v} - \lambda^2 \frac{\hat{v}^2}{2E} + \lambda^3 \mathrm{err}_{Bog} \, ,
\end{aligned} \tag{5.126}
$$

where $\hat{v}_{Bog} := \frac{2\hat{v}}{1+\sqrt{1+2\lambda\frac{\hat{v}}{E}}}$ and err_{Bog} satisfies

$$
\|\mathrm{err}_{Bog}\|_\infty \leq C \|\hat{v}\|^3_{w,d} \tag{5.127}
$$

for all $\lambda > 0$.

Proof For the first part, let $\omega_1, \omega_2, \omega_1 + \omega_2 \neq 0$. Then

$$\int_{\Delta[t,2]} d\mathbf{s}_2\, e^{i(\omega_1 s_1 + \omega_2 s_2)} = \int_0^t ds\, \frac{e^{i\omega_1 s}}{i\omega_2}(e^{i\omega_2 s} - 1)$$
$$= -\frac{1}{\omega_2}\Big(\frac{e^{i(\omega_1+\omega_2)t} - 1}{\omega_1 + \omega_2} - \frac{e^{i\omega_1 t} - 1}{\omega_1}\Big). \tag{5.128}$$

Using $1 - \cos(x) = 2\sin(x/2)$, we have shown the first statement.

For the second part, we expand Ω using

$$\Omega = E + \frac{\Omega^2 - E^2}{E + \Omega}$$
$$= E + \frac{2\lambda\hat{v}}{1 + \sqrt{1 + 2\lambda\frac{\hat{v}}{E}}}. \tag{5.129}$$

We Taylor expand

$$\frac{1}{1 + \sqrt{1 + x}} = \frac{1}{2} - \frac{x}{8} + R(x) \tag{5.130}$$

for some $R(x)$. An easy computation yields

$$\frac{|R(x)|}{x^2} \leq C \tag{5.131}$$

for all $x > 0$, and in the limit $x \searrow 0$. This concludes the proof. □

Remark 5.4 (Talbot effect) Lemma 5.3 shows that the leading term in $\mathrm{Bol}_d(f_0)[J]$ is a sum of the form

$$\frac{T}{\lambda^2}\int_{(\Lambda^*)^2} d\mathbf{p}_2\, \frac{\sin^2\Big(\frac{T(\Delta_{cub}E + \lambda\Delta_{cub}\hat{v}_{Bog})}{2\lambda^2}\Big)}{\Big(\frac{T(\Delta_{cub}E + \lambda\Delta_{cub}\hat{v}_{Bog})}{2\lambda^2}\Big)^2}\Delta_{cub}J(\mathbf{p}_2)H(\mathbf{p}_2) \tag{5.132}$$

for some $H \sim 1$ with $\hat{v}_{Bog}$ as defined in Lemma 5.3. Thus, as a function of T, the modulus of this sum oscillates between 0 and $O(\lambda^{-2})$. The size of these oscillations is dependent on the the interaction profile $\hat{v}$, the lattice Λ, and the chosen sequence $\lambda = \lambda(N)$. A similar phenomenon occurs for $\mathrm{Bol}_d^{(j)}$, which thereby also oscillate in modulus between 0 and $O(\lambda^{j-2})$, depending on T, and we also observe it for col_d, $\mathrm{abs}_{quart,d}$, and $\mathrm{abs}_{cub,d}$. Therefore, $\mathrm{Bol}_d^{(j)}$ can dominate Bol_d, depending on whether $\frac{T}{4\pi\lambda^2}\Delta_{cub}\Omega(\mathbf{p}_2)$ lies in a small vicinity of $\frac{1}{4} + \mathbb{Z}$ for some $\mathbf{p}_2 \in (\Lambda^*)^2$. This is reminiscent of the Talbot effect, see [128, 190]. As heuristically explained above, $\mathrm{Bol}_d^{(j)}$ are negligible in the case of large system length $L \sim \lambda^{-2-}$. Next, we will present a proof of this fact.

5.2 Continuum Approximation $\Lambda^* \to \mathbb{R}^3$

In this section, we will write out the summation over the lattice Λ^* explicitly. In contrast, all integrals will be understood as integrals over $\mathbb{R}^d$ in the respective dimension d.

Lemma 5.5 *Let $F_1(\boldsymbol{p}_2), F_2(\boldsymbol{p}_2) \in \{\pm\Omega(p_1) \pm \Omega(p_2) \pm \Omega(p_1+p_2), 0\}$, $H \in C_c^\infty(\mathbb{R}^6)$, $\chi \in C_c^\infty(\mathbb{R}^3)$, and $\tau_1, \tau_2 \in \mathbb{R}$. Let $\langle x\rangle := (1+x^2)^{\frac{1}{2}}$. Abbreviate*

$$\begin{aligned}
\mathrm{err}^{(1)}_{disc} &:= \lambda^2 \int_0^{T\lambda^{-2}} ds \Big(\frac{1}{|\Lambda|^2} \sum_{\boldsymbol{p}_2 \in (\Lambda^*)^2} e^{i(\tau_1\Omega(p_1)+\tau_2\Omega(p_2))s} H(\boldsymbol{p}_2) \\
&\quad - \frac{1}{(2\pi)^6} \int d\boldsymbol{p}_2\, e^{i(\tau_1\Omega(p_1)+\tau_2\Omega(p_2))s} H(\boldsymbol{p}_2) \Big), \\
\mathrm{err}^{(2)}_{disc} &:= \lambda^2 \int_{\Delta[T\lambda^{-2},2]} d\boldsymbol{s}_2 \Big(\frac{1}{|\Lambda|^2} \sum_{\boldsymbol{p}_2 \in (\Lambda^*)^2} e^{iF_1(\boldsymbol{p}_2)s_1 + iF_2(\boldsymbol{p}_2)s_2} H(\boldsymbol{p}_2) \\
&\quad - \frac{1}{(2\pi)^6} \int d\boldsymbol{p}_2\, e^{iF_1(\boldsymbol{p}_2)s_1 + iF_2(\boldsymbol{p}_2)s_2} H(\boldsymbol{p}_2) \Big). \qquad (5.133)
\end{aligned}$$

Then we have for any $r > 6$ and $\lambda > 0$ the following.

1.

$$\Big| \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} \chi(p) - \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dp\, \chi(p) \Big| \le C_r \frac{\| |\nabla|^{\frac{r}{2}} \chi \|_1}{|\Lambda|^{\frac{r}{6}}}, \qquad (5.134)$$

2.

$$|\mathrm{err}^{(1)}_{disc}| \le C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, r} \frac{\langle T\rangle^{r+1}}{\lambda^{2r} |\Lambda|^{\frac{r}{3}}} \sum_{n=0}^{2(\lfloor \frac{r}{2} \rfloor+1)} \Big\| \langle |\boldsymbol{p}_2| \rangle^n D^{2(\lfloor \frac{r}{2} \rfloor+1)-n} H \Big\|_1, \qquad (5.135)$$

3. *if $F_1 = F_2 \equiv 0$,*

$$|\mathrm{err}^{(2)}_{disc}| \le C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, r} \frac{T^2}{\lambda^2 |\Lambda|^{\frac{r}{3}}} \| |\nabla|^r H \|_1, \qquad (5.136)$$

4. *if $(F_1, F_2) \not\equiv (0,0)$,*

$$|\mathrm{err}^{(2)}_{disc}| \le C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, r} \frac{\langle T\rangle^{r+2}}{\lambda^{2r+2} |\Lambda|^{\frac{r}{3}}} \sum_{n=0}^{2(\lfloor \frac{r}{2} \rfloor+1)} \Big\| \langle |\boldsymbol{p}_2| \rangle^n D^{2(\lfloor \frac{r}{2} \rfloor+1)-n} H \Big\|_1. \qquad (5.137)$$

Proof Let

$$\tau := \frac{2\pi}{|\Lambda|^{\frac{1}{3}}}, \qquad (5.138)$$

so that

$$\int_{\Lambda^*} dp\, \tilde{H}(p) = \frac{\tau^3}{(2\pi)^3} \sum_{p \in \Lambda^*} \tilde{H}(p). \qquad (5.139)$$

Denote

$$G_\lambda(\mathbf{p}_2) := \lambda^2 \int_{\Delta[T\lambda^{-2},2]} d\mathbf{s}_2\, e^{iF_1(\mathbf{p}_2)s_1 + iF_2(\mathbf{p}_2)s_2}. \qquad (5.140)$$

Poisson summation implies

$$\begin{aligned}\tau^6 \sum_{\mathbf{p}_2 \in (\Lambda^*)^2} G_\lambda(\mathbf{p}_2) H(\mathbf{p}_2) &= \tau^6 \sum_{\mathbf{P}_2 \in \mathbb{Z}^6} G_\lambda(\tau \mathbf{P}_2) H(\tau \mathbf{P}_2) \\ &= \sum_{\mathbf{X}_2 \in \mathbb{Z}^6} \tau^6 \int_{\mathbb{R}^6} d\mathbf{P}_2 \, e^{2\pi i \mathbf{P}_2 \cdot \mathbf{X}_2} G_\lambda(\tau \mathbf{P}_2) H(\tau \mathbf{P}_2) \\ &= \sum_{\mathbf{X}_2 \in \mathbb{Z}^6} \int_{\mathbb{R}^6} d\mathbf{p}_2 \, e^{2\pi i \mathbf{p}_2 \cdot \mathbf{X}_2/\tau} G_\lambda(\mathbf{p}_2) H(\mathbf{p}_2) \, . \end{aligned} \tag{5.141}$$

As a consequence of (5.141), we obtain that

$$\mathrm{err}^{(2)}_{disc} = \frac{1}{(2\pi)^6} \sum_{\mathbf{X}_2 \in \mathbb{Z}^6 \setminus \{0\}} \int_{\mathbb{R}^6} d\mathbf{p}_2 \, e^{2\pi i \mathbf{p}_2 \cdot \mathbf{X}_2/\tau} G_\lambda(\mathbf{p}_2) H(\mathbf{p}_2) \, . \tag{5.142}$$

Next, we have that

$$\begin{aligned} &|\mathbf{X}_2|^r \int_{\mathbb{R}^6} d\mathbf{p}_2 \, e^{2\pi i \mathbf{p}_2 \cdot \mathbf{X}_2/\tau} G_\lambda(\mathbf{p}_2) H(\mathbf{p}_2) \\ &= \frac{\tau^r}{(2\pi)^r} \int_{\mathbb{R}^6} d\mathbf{p}_2 \, e^{2\pi i \mathbf{p}_2 \cdot \mathbf{X}_2/\tau} |\nabla_{\mathbf{p}_2}|^r \big(G_\lambda(\mathbf{p}_2) H(\mathbf{p}_2) \big) \, . \end{aligned} \tag{5.143}$$

In particular, we have that

$$\begin{aligned} |\mathrm{err}^{(2)}_{disc}| &\leq C_r \tau^r \sum_{\mathbf{X}_2 \in \mathbb{Z}^6 \setminus \{0\}} \frac{1}{|\mathbf{X}_2|^r} \| |\nabla|^r \big(G_\lambda H \big) \|_1 \\ &\leq \frac{C_r}{|\Lambda|^{\frac{r}{3}}} \| |\nabla|^r \big(G_\lambda H \big) \|_1 \end{aligned} \tag{5.144}$$

due to $r > 6$. With analogous steps, we obtain

$$\begin{aligned} \Big| \tau^3 \sum_{p \in \Lambda^*} \chi(p) - \int_{\mathbb{R}^3} dp \, \chi(p) \Big| &= \frac{\tau^{\frac{r}{2}}}{(2\pi)^{\frac{r}{2}}} \Big| \sum_{X \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{|X|^{\frac{r}{2}}} \int_{\mathbb{R}^3} dp \, e^{2\pi i p \cdot X/\tau} |\nabla|^{\frac{r}{2}} \chi(p) \Big| \\ &\leq \frac{C_r}{|\Lambda|^{\frac{r}{6}}} \| |\nabla|^{\frac{r}{2}} \chi \|_1 \, . \end{aligned} \tag{5.145}$$

Interpolation implies

$$\| |\nabla|^r \big(G_\lambda H \big) \|_1 \leq C \| (-\Delta)^{\lfloor \frac{r}{2} \rfloor + 1} \big(G_\lambda H \big) \|_1^{\frac{\frac{r}{2}}{\lfloor \frac{r}{2} \rfloor + 1}} \| G_\lambda H \|_1^{\frac{1 + \lfloor \frac{r}{2} \rfloor - \frac{r}{2}}{\lfloor \frac{r}{2} \rfloor + 1}}) \, , \tag{5.146}$$

where $t_0 > 0$ will be fixed below. The Leibniz rule implies for $k \in \mathbb{N}_0$ that

$$|(-\Delta)^k G_\lambda H| \leq C_k \sum_{n=0}^{2k} |D^n G_\lambda| |D^{2k-n} H| \, . \tag{5.147}$$

We start with the case $F_1 = F_2 \equiv 0$. Lemma 5.3 yields $G_\lambda = C \frac{T^2}{2} \lambda^{-2}$, which, employing (5.144), implies

$$|\mathrm{err}^{(2)}_{disc}| \leq C_r T^2 \lambda^{-2} \tau^r \| |\nabla|^r H \|_1 \leq C_r \frac{T^2}{\lambda^2 |\Lambda|^{\frac{r}{3}}} \| |\nabla|^r H \|_1 \, . \tag{5.148}$$

Next, let $F_1 \equiv 0 \wedge F_2 \not\equiv 0$, $F_1 \not\equiv 0 \wedge F_2 \equiv 0$, or $F_1 = -F_2 \not\equiv 0$. Let $F_j \not\equiv 0$ for some $j \in \{1, 2, 3\}$. Then, by Lemma 5.3, we obtain that

$$G_\lambda(\mathbf{p}_2) \;=\; \frac{T^2}{\lambda^2} G(F_j(\mathbf{p}_2)T/\lambda^2) \tag{5.149}$$

for some continuous function $G \in C_b^\infty(\mathbb{R})$. Moreover, the Faà di Bruno formula, see [295], implies

$$|D^n G(F_j T/\lambda^2)| \;\leq\; C_n \sum_{\substack{\mathbf{r}_n \in \\ R(n)}} |G^{(S(\mathbf{r}_n))}|(F_j T/\lambda^2) \prod_{\ell=1}^{n} \Big[\frac{T}{\lambda^2} |D^\ell F_j|\Big]^{r_\ell}\,, \tag{5.150}$$

where

$$R(n) := \{\mathbf{r}_n \in \mathbb{N}_0^n \mid \sum_{\ell=1}^{n} \ell r_\ell = n\}\,, \tag{5.151}$$

$$S(\mathbf{r}_n) := \sum_{\ell=1}^{n} r_\ell\,. \tag{5.152}$$

Observe that due to Lemma 5.3, we have that

$$|\nabla F_j| \leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}} \langle |\mathbf{p}_2| \rangle\,, \tag{5.153}$$

$$|D^\ell F_j| \leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}} \tag{5.154}$$

for all $\lambda \in (0, 1)$, and all $\ell \geq 2$. Then (5.150) together with (5.153), (5.154), and the fact that $1 \leq S(\mathbf{r}_n) \leq n$ imply

$$\begin{aligned} |D^n G(F_j T/\lambda^2)| &\leq C_n \frac{T\langle T\rangle^{n-1} \langle |\nabla F_j| \rangle^n}{\lambda^{2n}} \Big(\sum_{\ell=1}^{n} |G^{(\ell)}|\Big)(F_j T/\lambda^2) \\ &\leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c},n} \frac{T\langle T\rangle^{n-1} \langle |\mathbf{p}_2| \rangle^n}{\lambda^{2n}} \Big(\sum_{\ell=1}^{n} |G^{(\ell)}|\Big)(F_j T/\lambda^2) \end{aligned} \tag{5.155}$$

for all $\lambda \in (0, 1)$. Collecting (5.147), 5.155, we have that

$$\begin{aligned} &\|(-\Delta)^k \big(G_\lambda H\big)\|_1 \\ &\leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c},r} \sum_{n=0}^{2k} \frac{T^3\langle T\rangle^{n-1}}{\lambda^{2n+2}} \Big\| \Big(\sum_{\ell=1}^{n} |G^{(\ell)}|\Big)(F_j T/\lambda^2) \langle |\mathbf{p}_2| \rangle^n D^{2k-n} H \Big\|_1 \\ &\leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c},r} \frac{\langle T\rangle^{2k+2}}{\lambda^{4k+2}} \Big\| \Big(\sum_{\ell=0}^{2k} |G^{(\ell)}|\Big) \Big\|_\infty \sum_{n=0}^{2k} \Big\| \langle |\mathbf{p}_2| \rangle^n D^{2k-n} H \Big\|_1 \end{aligned} \tag{5.156}$$

for all $k \in \{0, 1, 2, \ldots, \lfloor \frac{r}{2} \rfloor + 1\}$. Then (5.146) and (5.156) yield

$$\begin{aligned} &\||\nabla|^r \big(G_\lambda H\big)\|_1 \\ &\leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c},r} \frac{\langle T\rangle^{r+2}}{\lambda^{2r+2}} \sum_{n=0}^{2(\lfloor \frac{r}{2} \rfloor+1)} \Big\| \langle |\mathbf{p}_2| \rangle^n D^{2(\lfloor \frac{r}{2} \rfloor+1)-n} H \Big\|_1\,, \end{aligned} \tag{5.157}$$

where we also used the fact that $\|H\|_1 \leq \|\langle \mathbf{p}_2\rangle^{2(\lfloor \frac{r}{2}\rfloor+1)} H\|_1$. (5.144) and (5.157) yield

$$|\mathrm{err}_{disc}^{(2)}| \;\leq\; C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2}\rfloor+1),w,c},r} \frac{\langle T\rangle^{r+2}}{\lambda^{2r+2}|\Lambda|^{\frac{r}{3}}} \sum_{n=0}^{2(\lfloor \frac{r}{2}\rfloor+1)} \Big\| \langle|\mathbf{p}_2|\rangle^n D^{2(\lfloor \frac{r}{2}\rfloor+1)-n} H \Big\|_1 . \tag{5.158}$$

Now, let $F_j \not\equiv 0$ for all $j \in \{1, 2, 3\}$. In this case, Lemma 5.3 implies

$$G_\lambda(\mathbf{p}_2) \;=\; \frac{T^2}{\lambda^2} G\Big(\frac{T}{\lambda^2}(F_1(\mathbf{p}_2), F_2(\mathbf{p}_2))\Big) \tag{5.159}$$

for some smooth function $G \in C_b^\infty(\mathbb{R}^2)$. The Faà di Bruno formula implies

$$\begin{aligned}
\Big|D^n G\Big(\frac{T}{\lambda^2}(F_1(\mathbf{p}_2), F_2(\mathbf{p}_2))\Big)\Big| &\leq C_n \sum_{\substack{\mathbf{r}_n \in \\ R(n)}} |D^{S(\mathbf{r}_n)} G|\Big(\frac{T}{\lambda^2}(F_1(\mathbf{p}_2), F_2(\mathbf{p}_2))\Big) \\
&\qquad \prod_{\ell=1}^{n} \Big[\frac{T}{\lambda^2}\sum_{j=1}^{2} |D^\ell F_j|\Big]^{r_\ell} \\
&\leq C_n \frac{T\langle T\rangle^{n-1}\langle \sum_{j=1}^2 |\nabla F_j|\rangle^n}{\lambda^{2n}} \\
&\qquad \Big(\sum_{\ell=1}^{n} |D^\ell G|\Big)\Big(\frac{T}{\lambda^2}(F_1(\mathbf{p}_2), F_2(\mathbf{p}_2))\Big) .
\end{aligned} \tag{5.160}$$

Collecting (5.147), (5.153), (5.154), and 5.160, we have that

$$\begin{aligned}
&\|(-\Delta)^k \big(G_\lambda H\big)\|_1 \\
&\leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2}\rfloor+1),w,c},r} \sum_{n=0}^{2k} \frac{T^3\langle T\rangle^{n-1}}{\lambda^{2n+2}} \\
&\quad \Big\| \Big(\sum_{\ell=1}^{n} |D^\ell G|\Big)\Big(\frac{T}{\lambda^2}(F_1, F_2)\Big)\langle \sum_{j=1}^{2} |\nabla F_j|\rangle^n D^{2k-n} H \Big\|_1 \\
&\leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2}\rfloor+1),w,c},r} \frac{\langle T\rangle^{2k+2}}{\lambda^{4k+2}} \Big\| \Big(\sum_{\ell=0}^{2k} |D^\ell G|\Big)\Big\|_\infty \sum_{n=0}^{2k} \Big\| \langle|\mathbf{p}_2|\rangle^n D^{2k-n} H \Big\|_1
\end{aligned} \tag{5.161}$$

for all $k \in \{0, 1, 2, \ldots, \lfloor \frac{r}{2}\rfloor + 1\}$. As a consequence of (5.146), we thus obtain

$$\||\nabla|^r \big(G_\lambda H\big)\|_1 \leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2}\rfloor+1),w,c},r} \frac{\langle T\rangle^{r+2}}{\lambda^{2r+2}} \sum_{n=0}^{2(\lfloor \frac{r}{2}\rfloor+1)} \Big\| \langle|\mathbf{p}_2|\rangle^n D^{2(\lfloor \frac{r}{2}\rfloor+1)-n} H \Big\|_1 . \tag{5.162}$$

(5.144) and (5.162) imply

$$|\mathrm{err}_{disc}^{(2)}| \;\leq\; C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2}\rfloor+1),w,c},r} \frac{\langle T\rangle^{r+2}}{\lambda^{2r+2}|\Lambda|^{\frac{r}{3}}} \sum_{n=0}^{2(\lfloor \frac{r}{2}\rfloor+1)} \Big\| \langle|\mathbf{p}_2|\rangle^n D^{2(\lfloor \frac{r}{2}\rfloor+1)-n} H \Big\|_1 . \tag{5.163}$$

Finally, let

$$\mathring{G}_\lambda(\mathbf{p}_2) \;:=\; \lambda^2 \int_0^{T\lambda^{-2}} ds\, e^{i(\tau_1\Omega(p_1)+\tau_2\Omega(p_2))s} . \tag{5.164}$$

A simple computation shows

$$\mathring{G}_\lambda(\mathbf{p}_2) \;=\; T\mathring{G}\big((\tau_1\Omega(p_1)+\tau_2\Omega(p_2))\frac{T}{\lambda^2}\big) \tag{5.165}$$

for some smooth $\mathring{G} \in C_b^\infty(\mathbb{R})$. If $\tau_1 = \tau_2 = 0$, we have $\mathring{G} \equiv C$. If $(\tau_1, \tau_2) \neq (0,0)$, a computation analogous to (5.155) yields

$$\begin{aligned} &|D^n\mathring{G}\big((\tau_1\Omega(p_1)+\tau_2\Omega(p_2))T/\lambda^2\big)| \\ &\le C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},n}\frac{T\langle T\rangle^{n-1}\langle|\mathbf{p}_2|\rangle^n}{\lambda^{2n}}\Big(\sum_{\ell=1}^{n}|\mathring{G}^{(\ell)}|\Big)((\tau_1\Omega(p_1)+\tau_2\Omega(p_2))T/\lambda^2) \end{aligned} \tag{5.166}$$

for all $n \in \{0,1,2,\dots,2(\lfloor\frac{r}{2}\rfloor+1)\}$. In particular, with analogous steps that led to (5.158), we obtain

$$|\mathrm{err}^{(1)}_{disc}| \;\le\; C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},r}\frac{\langle T\rangle^{r+1}}{\lambda^{2r}|\Lambda|^{\frac{r}{3}}}\sum_{n=0}^{2(\lfloor\frac{r}{2}\rfloor+1)}\Big\|\langle|\mathbf{p}_2|\rangle^n D^{2(\lfloor\frac{r}{2}\rfloor+1)-n}H\Big\|_1. \tag{5.167}$$

This concludes the proof. □

Remark 5.6 Observe that Lemma 5.5 implies that

$$\begin{aligned} \int_{\Lambda^*} dp\, f_0(p) &\le \|f_0\|_1 + C_r\frac{\||\nabla|^{\frac{r}{2}}f_0\|_1}{|\Lambda|^{\frac{r}{6}}} \\ &\le C_r\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c} \end{aligned} \tag{5.168}$$

for any $r > 6$ as in Theorem 2.2, where C_r is independent of $|\Lambda| \ge 1$. In particular, we have that

$$\|f_0\|_d \;\le\; C_r\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}\,. \tag{5.169}$$

This is allows us to use the previous estimates to prove Theorem 2.2 as well. Analogously, we have that

$$\|\hat{v}\|_{w,d} \;\le\; C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\,. \tag{5.170}$$

in the assumption of Theorem 2.1, where, again C_r is independent of $|\Lambda| \ge 1$.

Likewise, we have that

$$\|J\|_{2\cap\infty,d} \;\le\; C_r\|J\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}\,. \tag{5.171}$$

For the next statement, define the continuous analogues Con_c, Bol_c, etc. of Con_d, Bol_d, etc. in (5.49)–(5.56), and (5.63), (5.65) by replacing the lattice sums $\int_{\Lambda^*}$ over Λ^* in the sense of (1.51) by Lebesgue integrals $\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}$ over $\mathbb{R}^3$.

Proposition 5.7 *Let $T > 0$, $J \in L^\infty(\mathbb{R}^3;\mathbb{R})$ and $r > 6$. Then the following holds for all $\lambda > 0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}$.*

1.

$$-i\int_0^{T\lambda^{-2}} ds\,\frac{\nu_0\big([a_0,\mathcal{H}_{cub}(s)]\big)}{|\Lambda|} = \frac{1}{N^{\frac{1}{2}}\lambda}\mathrm{Con}_c(f_0)(T;\lambda) \,+\, \mathrm{err}^{(Bog)}_{1,c}\left(\frac{T}{\lambda^2};\Phi\right), \tag{5.172}$$

where

$$|\operatorname{err}_{1,c}^{(Bog)}(\frac{T}{\lambda^2};\Phi)| \le \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}T}{N^{\frac{1}{2}}\lambda}\Big(\lambda + \frac{1}{|\Lambda|^{\frac{r}{6}}}\Big), \tag{5.173}$$

2.

$$\begin{aligned}&-\int_{\Delta[t,2]} d\boldsymbol{s}_2 \frac{\nu_0\big([[f[J],\mathcal{H}_{cub}(s_1)],\mathcal{H}_{cub}(s_2)]\big)}{|\Lambda|}\\&= \frac{1}{N}\operatorname{Bol}_c(f_0)[J](T;\lambda) + J(0)|\Phi_{\frac{T}{\lambda^2}}|^2\\&\quad + \operatorname{err}_{2,c}^{(Bog,\mathrm{Bol})}(\frac{T}{\lambda^2};f[J]) + J(0)\operatorname{err}_{2,c}^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2};f),\end{aligned} \tag{5.174}$$

where

$$\begin{aligned}|\operatorname{err}_{2,c}^{(Bog,\mathrm{Bol})}(\frac{T}{\lambda^2};f[J])| &\le \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}\langle T\rangle^{r+2}\|J\|_{W^{2\lfloor\frac{r}{2}\rfloor+2,\infty}}}{N}\\&\quad\Big(\frac{1}{\lambda^{2(r+1)}|\Lambda|^{\frac{r}{3}}} + \lambda|\log(\lambda)|\Big),\\|\operatorname{err}_{2,c}^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2};f[J])| &\le C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}e^{C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}|\Lambda|/\lambda T}\\&\quad\frac{\langle T\rangle^4|\Lambda|^{\frac{3}{2}}}{\lambda^3N^{\frac{3}{2}}}\Big(1+\frac{|\Lambda|}{N}\Big)^2\Big(1+\frac{|\Lambda|^{\frac{3}{2}}}{\lambda N^{\frac{1}{2}}}\Big),\end{aligned} \tag{5.175}$$

3.

$$\begin{aligned}&-i\int_0^{T\lambda^{-2}} ds\,\frac{\nu_0(g[J],\mathcal{H}_{quart}(s))}{|\Lambda|}\\&= \frac{1}{N}\int_0^T dS\,abs_{quart,c}(f_0)[J](S/\lambda^2) + \operatorname{err}_{1,c}^{(dis)}(\frac{T}{\lambda^2};g[J])\end{aligned} \tag{5.176}$$

with

$$|\operatorname{err}_{1,c}^{(dis)}(\frac{T}{\lambda^2};g[J])| \le \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}\langle T\rangle^{r+1}\|J\|_{W^{2\lfloor\frac{r}{2}\rfloor+2,\infty}}}{N\lambda^{2r}|\Lambda|^{\frac{r}{3}}}, \tag{5.177}$$

4.

$$\begin{aligned}&-\int_{\Delta[T\lambda^{-2},2]} d\boldsymbol{s}_2 \frac{\nu_0\big([[g[J],\mathcal{H}_{cub}(s_1)],\mathcal{H}_{cub}(s_2)]\big)}{|\Lambda|}\\&= \frac{1}{N\lambda^2}\int_{\Delta[T,2]} d\boldsymbol{S}_2\,\big(\operatorname{col}_c(f_0)[J](\boldsymbol{S}_2/\lambda^2) + abs_{cub,c}(f_0)[J](\boldsymbol{S}_2/\lambda^2)\big)\\&\quad + \operatorname{err}_{2,c}^{(dis)}(\frac{T}{\lambda^2};g[J]) + J(0)\Big(\Phi_{T\lambda^{-2}}\Big)^2 + J(0)\operatorname{err}_{2,c}^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2};g).\end{aligned} \tag{5.178}$$

where

$$|\operatorname{err}_{2,c}^{(dis)}(\frac{T}{\lambda^2};g[J])| \le \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}\langle T\rangle^{r+2}\|J\|_{W^{2\lfloor\frac{r}{2}\rfloor+2,\infty}}}{N\lambda^{2(r+1)}|\Lambda|^{\frac{r}{3}}}$$

(5.179)

and $\mathrm{err}_{2,c}^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2};g)$ *satisfies the same bound as* $\mathrm{err}_{2,c}^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2};f)$.

Proof Recall from Proposition 5.2 that

$$\begin{aligned} &-i\int_0^{T\lambda^{-2}} ds\,\frac{\nu_0\big([a_0,\mathcal{H}_{cub}(s)]\big)}{|\Lambda|} \\ &\quad= \frac{1}{N^{\frac12}\lambda}\,\mathrm{Con}_d(f_0)(T;\lambda)\;+\;\mathrm{err}_{1,d}^{(Bog)}(\frac{T}{\lambda^2};\Phi) \end{aligned} \tag{5.180}$$

with

$$\begin{aligned} \left|\mathrm{err}_{1,d}^{(Bog)}\left(\frac{T}{\lambda^2};\Phi\right)\right| &\le \frac{C_{\|\hat v\|_{w,d},\|f_0\|_d}T}{N^{\frac12}} \\ &\le \frac{C_{\|\hat v\|_{2(\lfloor\frac r2\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac r2\rfloor+1),c},r}T}{N^{\frac12}} \end{aligned} \tag{5.181}$$

due to Remark 5.6. Let

$$\mathrm{err}_{1,c}^{(dis)}(\frac{T}{\lambda^2};\Phi)\;:=\;\frac{1}{N^{\frac12}\lambda}\Big(\mathrm{Con}_d(f_0)(T;\lambda)-\mathrm{Con}_c(f_0)(T;\lambda)\Big)\,. \tag{5.182}$$

Then Lemma 5.5 implies

$$\begin{aligned} \left|\mathrm{err}_{1,c}^{(dis)}\left(\frac{T}{\lambda^2};\Phi\right)\right| &\le \frac{T}{N^{\frac12}\lambda}\left|\frac{(2\pi)^3}{|\Lambda|}\sum_{p\in\Lambda^*}\hat v(p)f_0(p)\;-\;\int_{\mathbb{R}^3}dp\,\hat v(p)f_0(p)\right| \\ &\le C_r\frac{T}{N^{\frac12}\lambda|\Lambda|^{\frac r6}}\,\||\nabla|^{\frac r2}\hat v f_0\|_1 \\ &\le \frac{C_{\|\hat v\|_{2(\lfloor\frac r2\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac r2\rfloor+1),c},r}T}{N^{\frac12}\lambda|\Lambda|^{\frac r6}}\,. \end{aligned} \tag{5.183}$$

Here, we used interpolation as in (5.146), together with the Leibniz rule. Applying (5.181), (5.183) implies

$$\begin{aligned} \left|\mathrm{err}_{1,c}^{(Bog)}\left(\frac{T}{\lambda^2};\Phi\right)\right| &\le |\,\mathrm{err}_{1,d}^{(Bog)}(\frac{T}{\lambda^2};\Phi)|\;+\;\left|\mathrm{err}_{1,c}^{(dis)}\left(\frac{T}{\lambda^2};\Phi\right)\right| \\ &\le \frac{C_{\|\hat v\|_{2(\lfloor\frac r2\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac r2\rfloor+1),c},r}T}{N^{\frac12}\lambda}\left(\lambda\;+\;\frac{1}{|\Lambda|^{\frac r6}}\right). \end{aligned} \tag{5.184}$$

Next, Proposition 5.2 together with Remark 5.6 yield that

$$\begin{aligned} &-\int_{\Delta[T\lambda^{-2},2]} d\mathbf{s}_2\,\frac{\nu_0\big([[f[J],\mathcal{H}_{cub}(s_1)],\mathcal{H}_{cub}(s_2)]\big)}{|\Lambda|} \\ &= \frac1N\Big(\mathrm{Bol}_d(f_0)[J](T;\lambda)\;+\;\lambda\,\mathrm{Bol}_d^{(1)}(f_0)[J](T;\lambda)\;+\;\lambda^2\,\mathrm{Bol}_d^{(2)}(f_0)[J](T;\lambda)\Big) \\ &\quad+J(0)|\Phi_{\frac{T}{\lambda^2}}|^2\;+\;\mathrm{err}_{2,d}^{(Bog,\mathrm{Bol})}(\frac{T}{\lambda^2};f[J])\;+\;J(0)\,\mathrm{err}_{2,d}^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2};f)\,, \end{aligned} \tag{5.185}$$

where

$$|\,\mathrm{err}_{2,d}^{(Bog,\mathrm{Bol})}(\frac{T}{\lambda^2};f[J])|\le C_{\|\hat v\|_{2(\lfloor\frac r2\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac r2\rfloor+1),c},r}T^2\|J\|_{\ell^\infty(\Lambda^*)}\frac{\lambda}{N}\,, \tag{5.186}$$

$$|\operatorname{err}_{2,d}^{(Bog,\mathrm{Con})}(t;f)| \leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r} e^{C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}|\Lambda|/\lambda T} \frac{\langle T\rangle^4|\Lambda|^{\frac{3}{2}}}{\lambda^3 N^{\frac{3}{2}}}\Big(1+\frac{|\Lambda|}{N}\Big)^2\Big(1+\frac{|\Lambda|^{\frac{3}{2}}}{\lambda N^{\frac{1}{2}}}\Big). \tag{5.187}$$

Let

$$\operatorname{err}_{2,0}^{(dis)}(\frac{T}{\lambda^2};f[J]) := \frac{1}{N}\Big(\mathrm{Bol}_d(f_0)[J](T;\lambda) - \mathrm{Bol}_c(f_0)[J](T;\lambda)\Big), \tag{5.188}$$

$$\operatorname{err}_{2,1}^{(dis)}(\frac{T}{\lambda^2};f[J]) := \frac{\lambda}{N}\Big(\mathrm{Bol}_d^{(1)}(f_0)[J](T;\lambda) - \mathrm{Bol}_c^{(1)}(f_0)[J](T;\lambda)\Big), \tag{5.189}$$

$$\operatorname{err}_{2,2}^{(dis)}(\frac{T}{\lambda^2};f[J]) := \frac{\lambda^2}{N}\Big(\mathrm{Bol}_d^{(2)}(f_0)[J](T;\lambda) - \mathrm{Bol}_c^{(2)}(f_0)[J](T;\lambda)\Big). \tag{5.190}$$

Recalling (5.50), (5.52), (5.56), (5.53), and (5.54), Lemma 5.5 then implies

$$\begin{aligned}\operatorname{err}_{2,j}^{(dis)}(\frac{T}{\lambda^2};f[J]) &\leq \frac{\lambda^j}{N\lambda^{2r+2}|\Lambda|^{\frac{r}{3}}}\\ &\leq \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}\langle T\rangle^{r+2}\|J\|_{W^{2\lfloor\frac{r}{2}\rfloor+2,\infty}}}{N\lambda^{2r+2}|\Lambda|^{\frac{r}{3}}}\end{aligned} \tag{5.191}$$

for $j\in\{0,1,2\}$. Lemma C.5 yields

$$\begin{aligned}&\frac{\lambda^j}{N}|\mathrm{Bol}_c^{(j)}(f_0)[J](T;\lambda)|\\ &\leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r} T(1+\log(T))\|J\|_{C^1}\frac{\lambda|\log(\lambda)|}{N}\end{aligned} \tag{5.192}$$

for $j\in\{1,2\}$, and all $\lambda>0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}$ Let

$$\begin{aligned}&\frac{1}{N}\mathrm{Bol}_c(f_0)[J](T;\lambda) + \operatorname{err}_{2,c}^{(Bog,\mathrm{Bol})}(\frac{T}{\lambda^2};f[J])\\ &:= \frac{1}{N}\big(\mathrm{Bol}_d(f_0)[J](T;\lambda) + \lambda\,\mathrm{Bol}_d^{(1)}(f_0)[J](T;\lambda)\\ &\quad+\lambda^2\,\mathrm{Bol}_d^{(2)}(f_0)[J](T;\lambda)\big) + \operatorname{err}_{2,d}^{(Bog,\mathrm{Bol})}(\frac{T}{\lambda^2};f[J]).\end{aligned} \tag{5.193}$$

Collecting (5.186), (5.191), and (5.192), we thus proved that

$$\begin{aligned}&|\operatorname{err}_{2,c}^{(Bog,\mathrm{Bol})}(\frac{T}{\lambda^2};f[J])|\\ &\leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}\Big(\frac{T^2\|J\|_{\ell^\infty(\Lambda^*)}\lambda}{N}\\ &\quad+\frac{\langle T\rangle^{r+2}\|J\|_{W^{2\lfloor\frac{r}{2}\rfloor+2,\infty}}}{N\lambda^{2(r+1)}|\Lambda|^{\frac{r}{3}}}\\ &\quad+\frac{\lambda|\log(\lambda)|T(1+\log(T))\|J\|_{W^{1,\infty}}}{N}\Big)\\ &\leq \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}\langle T\rangle^{r+2}\|J\|_{W^{2\lfloor\frac{r}{2}\rfloor+2,\infty}}}{N}\\ &\quad\Big(\frac{1}{\lambda^{2(r+1)}|\Lambda|^{\frac{r}{3}}}+\lambda|\log(\lambda)|\Big).\end{aligned} \tag{5.194}$$

Finally, we compute the discretization error in the dynamics of g. Let

$$\begin{aligned}&\mathrm{err}_{1,c}^{(dis)}\left(\frac{T}{\lambda^2};g[J]\right)\\&:=\frac{1}{N}\int_0^T dS\left(\mathrm{abs}_{quart,d}(f_0)[J](S/\lambda^2)\;-\;\mathrm{abs}_{quart,c}(f_0)[J](S/\lambda^2)\right).\end{aligned} \tag{5.195}$$

Lemma 5.5 together with Proposition 5.2 implies

$$\left|\mathrm{err}_{1,c}^{(dis)}\left(\frac{T}{\lambda^2};g[J]\right)\right|\le\frac{C_{\|\hat v\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}\langle T\rangle^{r+1}\|J\|_{W^{2\lfloor\frac{r}{2}\rfloor+2,\infty}}}{N\lambda^{2r}|\Lambda|^{\frac{r}{3}}}. \tag{5.196}$$

Moreover, let

$$\begin{aligned}&\mathrm{err}_{2,c}^{(dis)}\left(\frac{T}{\lambda^2};g[J]\right)\\&:=\frac{1}{N\lambda^2}\int_{\Delta[T,2]}d\mathbf{S}_2\left(\mathrm{col}_d(f_0)[J](\mathbf{S}_2/\lambda^2))\;-\;\mathrm{col}_c(f_0)[J](\mathbf{S}_2/\lambda^2))\right.\\&\quad\left.+\,\mathrm{abs}_{cub,d}(f_0)[J](\mathbf{S}_2/\lambda^2))\;-\;\mathrm{abs}_{cub,c}(f_0)[J](\mathbf{S}_2/\lambda^2))\right)\end{aligned} \tag{5.197}$$

Again, applying Lemma 5.5 together with Proposition 5.2, yields

$$\left|\mathrm{err}_{2,c}^{(dis)}\left(\frac{T}{\lambda^2};g[J]\right)\right|\le\frac{C_{\|\hat v\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}\langle T\rangle^{r+2}\|J\|_{W^{2\lfloor\frac{r}{2}\rfloor+2,\infty}}}{N\lambda^{2r+2}|\Lambda|^{\frac{r}{3}}} \tag{5.198}$$

for all $\lambda>0$ small enough, dependent on $\|\hat v\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}$. This concludes the proof. □

For the next result, observe that

$$\Delta_{cub}E\;=\;E(p_1)+E(p_2)-E(p_1+p_2)\;=\;-p_1\cdot p_2\,. \tag{5.199}$$

Let

$$\begin{aligned}&\mathrm{Bol}_{fec}(f_0)[J](T)\\&:=\frac{\pi T}{(2\pi)^6}\int_{p_1\perp p_2}d\mathcal{H}^5(\mathbf{p}_2)\,\Delta_{cub}J(\mathbf{p}_2)\frac{(\hat v(p_1)+\hat v(p_2))^2}{|\mathbf{p}_2|}\\&\left(\widetilde{f}_0(p_1)\widetilde{f}_0(p_2)f_0(p_1+p_2)\;-\;f_0(p_1)f_0(p_2)\widetilde{f}_0(p_1+p_2)\right)\end{aligned} \tag{5.200}$$

be the Boltzmann operator with energy conserving collision kernel for the free energy dispersion $E(p)=\frac{|p|^2}{2}$.

Proposition 5.8 *We have that*

$$\begin{aligned}&\mathrm{Bol}_c(f_0)[J](T;\lambda)\\&=\frac{\pi T}{(2\pi)^6}\int d\boldsymbol{p}_2\,\delta_{\frac{2\lambda^2}{T}}(\Delta_{cub}\Omega(\boldsymbol{p}_2))\Delta_{cub}J(\boldsymbol{p}_2)(\hat v(p_1)+\hat v(p_2))^2\\&\left(\widetilde{f}_0(p_1)\widetilde{f}_0(p_2)f_0(p_1+p_2)\;-\;f_0(p_1)f_0(p_2)\widetilde{f}_0(p_1+p_2)\right),\end{aligned} \tag{5.201}$$

where $\delta_\varepsilon(x):=\frac{\varepsilon\sin^2(x/\varepsilon)}{\pi x^2}$. *Moreover, approximating* $\mathrm{Bol}(f_0)[J](T\lambda^{-2})$ *with its continuum counterpart, i.e., replacing* $\frac{1}{|\Lambda|}\sum_{\Lambda^*}$ *by the (Lebesgue-) integral* $\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}$*, we find that*

$$\frac{1}{N}\mathrm{Bol}_c(f_0)[J](T;\lambda)=\frac{1}{N}\mathrm{Bol}_{fec}(f_0)[J](T)\;+\;\mathrm{err}_{2,c}^{(fec)}\left(\frac{T}{\lambda^2};f[J]\right), \tag{5.202}$$

where

$$|\,\mathrm{err}^{(fec)}_{2,c}(\frac{T}{\lambda^2}; f[J])| \leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}, \|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}, r} \langle T\rangle \|J\|_{W^{2,\infty}} \frac{\lambda}{N} \tag{5.203}$$

for all $\lambda > 0$ *small enough, dependent on* $\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}$.

Proof Observe that $\Delta_{cub}E = 0$ is equivalent to $p_1 \perp p_2$, see (5.199), and that

$$|\nabla\Delta_{cub}E(\mathbf{p}_2)| = \left|\begin{pmatrix} -p_2 \\ -p_1 \end{pmatrix}\right| = |\mathbf{p}_2|\,. \tag{5.204}$$

Moreover, notice that for any $\omega \in \mathbb{R}\setminus\{0\}$, we have

$$\begin{aligned} \mathrm{Re}\int_{\Delta[t,2]} d\mathbf{s}_2\, e^{-i\omega(s_1-s_2)} &= \frac{1-\cos(\omega t)}{\omega^2} \\ &= 2\frac{\sin^2(\frac{\omega t}{2})}{\omega^2}\,. \end{aligned} \tag{5.205}$$

Observe that $\delta_a(x) = \frac{a\sin^2(x/a)}{\pi x^2}$ defines an approximate identity in the sense that we have

$$\delta_a(x) \geq 0 \quad \forall a > 0\,, \tag{5.206}$$

$$\int dx\, \delta_a(x) = 1 \quad \forall a > 0\,, \tag{5.207}$$

$$\int_{|x|>\rho} dx\, \delta_a(x) \leq \frac{Ca}{\rho} \quad \forall a, \rho > 0\,. \tag{5.208}$$

With that, we obtain

$$\begin{aligned} &\mathrm{Bol}(f_0)[J](T;\lambda) \\ &= \frac{\pi T}{(2\pi)^6}\int d\mathbf{p}_2\, \delta_{\frac{2\lambda^2}{T}}(\Delta_{cub}\Omega(\mathbf{p}_2))\Delta_{cub}J(\mathbf{p}_2)(\hat{v}(p_1)+\hat{v}(p_2))^2 \\ &\quad \big(f_0(p_1+p_2) + f_0(p_1)f_0(p_1+p_2) + f_0(p_1+p_2)f_0(p_2) \\ &\quad - f_0(p_1)f_0(p_2)\big) \end{aligned} \tag{5.209}$$

This proves the first part of the statement.

For the second part, assume $\mathbf{p}_2$ ranges over $\mathbb{R}^6$. Emphasizing the dependence of $\mathcal{E}(\lambda, \mathbf{p}_2) := \Delta_{cub}\Omega_\lambda(\mathbf{p}_2)$ on $\lambda > 0$, we abbreviate (5.209) as

$$\lambda^2\,\mathrm{Bol}(f_0)[J](T\lambda^{-2}) =: \int d\mathbf{p}_2\, \delta_{\frac{2\lambda^2}{T}}(\mathcal{E}(\lambda,\mathbf{p}_2))H(\mathbf{p}_2)\,, \tag{5.210}$$

where we emphasize. Observe that $\mathcal{E}(0, \mathbf{p}_2) = \Delta_{cub}E(\mathbf{p}_2)$. Using the Fundamental Theorem of Calculus by the Coarea Formula, we obtain that

$$\begin{aligned} N\,\mathrm{err}^{(fd)}_2(\frac{T}{\lambda^2}; f[J]) &:= \int d\mathbf{p}_2 \Big[\delta_{\frac{2\lambda^2}{T}}(\mathcal{E}(\lambda,\mathbf{p}_2)) - \delta_{\frac{2\lambda^2}{T}}(\mathcal{E}(0,\mathbf{p}_2))\Big]H(\mathbf{p}_2) \\ &= \int_0^\lambda d\tau\, \delta'_{\frac{2\lambda^2}{T}}(\mathcal{E}(\tau,\mathbf{p}_2))\partial_\tau\mathcal{E}(\tau,\mathbf{p}_2)H(\mathbf{p}_2) \\ &= \int_0^\lambda d\tau \int d\omega\, \delta'_{\frac{2\lambda^2}{T}}(\omega) \\ &\quad \int_{\mathcal{E}(\tau,\mathbf{p}_2)=\omega} d\mathcal{H}^5\, \frac{\partial_\tau\mathcal{E}(\tau,\mathbf{p}_2)H(\mathbf{p}_2)}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|}\,. \end{aligned} \tag{5.211}$$

We prove in Lemma C.6 that

$$| \operatorname{err}_2^{(fd)}(t; f[J])| \leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, \|f_0\|_{2(\lfloor \frac{r}{2} \rfloor+1),c}, r} T \|J\|_{W^{2,\infty}} \frac{\lambda}{N} \tag{5.212}$$

for all $\lambda > 0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}$.

Next, using (5.207) together with the Coarea Formula, we have that

$$\begin{aligned} N \operatorname{err}_2^{(ec)}(\frac{T}{\lambda^2}; f[J]) := & \int d\mathbf{p}_2\, \delta_{\frac{2\lambda^2}{T}}(\Delta_{cub} E(\mathbf{p}_2)) H(\mathbf{p}_2) \\ & - \int_{\Delta_{cub} E=0} d\mathcal{H}^5 \frac{H}{|\nabla \Delta_{cub} E|} \\ = & \int d\omega\, \delta_{\frac{2\lambda^2}{T}}(\omega) \Big(\int_{\Delta_{cub} E=\omega} d\mathcal{H}^5 \frac{H}{|\nabla \Delta_{cub} E|} \\ & + \int_{\Delta_{cub} E=0} d\mathcal{H}^5 \frac{H}{|\nabla \Delta_{cub} E|} \Big). \end{aligned} \tag{5.213}$$

We prove in Lemma C.7 that

$$| \operatorname{err}_2^{(ec)}(\frac{T}{\lambda^2}; f[J])| \leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, \|f_0\|_{2(\lfloor \frac{r}{2} \rfloor+1),c}, r} \sqrt{T} \|J\|_{W^{2,\infty}} \frac{\lambda}{N} \tag{5.214}$$

for all $\lambda > 0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}$. In particular, we have that

$$\operatorname{err}_{2,c}^{(fec)}\left(\frac{T}{\lambda^2}; f[J]\right) = \operatorname{err}_2^{(fd)}(\frac{T}{\lambda^2}; f[J]) + \operatorname{err}_2^{(ec)}(\frac{T}{\lambda^2}; f[J]) \tag{5.215}$$

satisfies

$$\left|\operatorname{err}_{2,c}^{(fec)}\left(\frac{T}{\lambda^2}; f[J]\right)\right| \leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, \|f_0\|_{2(\lfloor \frac{r}{2} \rfloor+1),c}, r} \langle T \rangle \|J\|_{W^{2,\infty}} \frac{\lambda}{N}. \tag{5.216}$$

This concludes the proof. □

5.3 Centered Expectations

In order to resolve the fluctuations around the HFB dynamics, we have to consider the dynamics relative to the condensate term, i.e.,

$$f_t^{(\Phi)}(p) := f_t(p) - |\Phi_t|^2 \delta(p). \tag{5.217}$$

Proposition 5.9 *Let $T > 0$, $\lambda \in (0, 1)$ and $|\Lambda| \geq 1$. Let $J \in L^\infty(\mathbb{R}^3; \mathbb{R})$. Then the following holds.*

(1)

$$\frac{1}{N^{\frac{1}{2}}\lambda} \operatorname{Con}_d(f_0)(T; \lambda) = \frac{1}{N^{\frac{1}{2}}\lambda} \operatorname{Con}_d(f^{(\Phi)})(T; \lambda) + \operatorname{err}_{1,d}^{(cen)}(\frac{T}{\lambda^2}; \Phi), \tag{5.218}$$

where

$$\begin{aligned} \left|\operatorname{err}_{1,d}^{(cen)}\left(\frac{T}{\lambda^2}; \Phi\right)\right| \leq & C_{\|\hat{v}\|_{w,d}, \|f_0\|_d} \langle T \rangle^3 \frac{|\Lambda|^{\frac{3}{2}} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N\lambda^2} \\ & \left(1 + \frac{|\Lambda|}{N}\right)\left(1 + \frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\right), \end{aligned} \tag{5.219}$$

(2) (a)

$$
\begin{aligned}
&\frac{1}{N}\big(\mathrm{Bol}_d(f_0)[J](T;\lambda) \,+\, \lambda\,\mathrm{Bol}_d^{(1)}(f_0)[J](T;\lambda) \,+\, \lambda^2\,\mathrm{Bol}_d^{(2)}(f_0)[J](T;\lambda)\big)\\
&\quad= \frac{1}{N}\big(\mathrm{Bol}_d(f^{(\Phi)})[J](T;\lambda) \,+\, \lambda\,\mathrm{Bol}_d^{(1)}(f^{(\Phi)})[J](T;\lambda)\\
&\qquad+\lambda^2\,\mathrm{Bol}_d^{(2)}(f^{(\Phi)})[J](T;\lambda)\big) \,+\, \mathrm{err}_{2,d}^{(cen)}\left(\frac{T}{\lambda^2}; f[J]\right), \qquad (5.220)
\end{aligned}
$$

with

$$
\begin{aligned}
&\left|\mathrm{err}_{2,d}^{(cen)}\left(\frac{T}{\lambda^2}; f[J]\right)\right|\\
&\quad\le C_{\|\hat{v}\|_{w,d},\|f_0\|_d}\,\|J\|_{\ell^\infty(\Lambda^*)}\,\langle T\rangle^4\,\frac{|\Lambda|^{\frac{3}{2}}\,e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{3}{2}}\lambda^3}\\
&\quad\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1\,+\,\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\\
&\quad\Big[1\,+\,\langle T\rangle^2\,\frac{|\Lambda|^{\frac{3}{2}}\,e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda}\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1\,+\,\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\Big], \qquad (5.221)
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\frac{1}{N}\,\mathrm{Bol}_c(f_0)[J](T;\lambda)\\
&\quad= \frac{1}{N}\,\mathrm{Bol}_c(f^{(\Phi)})[J](T;\lambda) \,+\, \mathrm{err}_{2,c}^{(cen)}\Big(\frac{T}{\lambda^2}; f[J]\Big), \qquad (5.222)
\end{aligned}
$$

where $\mathrm{err}_{2,c}^{(cen)}(\frac{T}{\lambda^2}; f[J])$ *satisfies the same bound as* $\mathrm{err}_{2,d}^{(cen)}(\frac{T}{\lambda^2}; f[J])$*,*

(3) (a)

$$
\begin{aligned}
&\frac{1}{N}\int_0^T dS\,abs_{quart,d}(f_0)[J](S/\lambda^2)\\
&\quad= \frac{1}{N}\int_0^T dS\,abs_{quart,d}(f^{(\Phi)})[J](S/\lambda^2) \,+\, \mathrm{err}_{1,d}^{(fd)}\left(\frac{T}{\lambda^2}; g[J]\right) \qquad (5.223)
\end{aligned}
$$

with

$$
\begin{aligned}
&\left|\mathrm{err}_{1,d}^{(cen)}\left(\frac{T}{\lambda^2}; g[J]\right)\right|\\
&\quad\le C_{\|\hat{v}\|_{w,d},\|f_0\|_d}\,\|J\|_{\ell^\infty(\Lambda^*)}\,\langle T\rangle^3\,\frac{|\Lambda|^{\frac{3}{2}}\,e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{3}{2}}\lambda}\\
&\quad\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1\,+\,\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\\
&\quad\Big[1\,+\,\langle T\rangle^2\,\frac{|\Lambda|^{\frac{3}{2}}\,e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda}\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1\,+\,\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\Big], \qquad (5.224)
\end{aligned}
$$

(b)

$$
\frac{1}{N\lambda^2}\int_{\Delta[T,2]} d\boldsymbol{S}_2\,\big(\mathrm{col}_d(f_0)[J](\boldsymbol{S}_2/\lambda^2) + abs_{cub,d}(f_0)[J](\boldsymbol{S}_2/\lambda^2)\big)
$$

$$= \frac{1}{N\lambda^2}\int_{\Delta[T,2]} d\boldsymbol{S}_2 \Big(\mathrm{col}_d(f^{(\Phi)})[J](\boldsymbol{S}_2/\lambda^2) + abs_{cub,d}(f^{(\Phi)})[J](\boldsymbol{S}_2/\lambda^2)\Big) + \mathrm{err}^{(cen)}_{2,d}(\frac{T}{\lambda^2}; g[J]) \tag{5.225}$$

with

$$\Big|\mathrm{err}^{(cen)}_{2,d}(\frac{T}{\lambda^2}; g[J])\Big| \leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d}\|J\|_{\ell^\infty(\Lambda^*)}\langle T\rangle^3 \frac{|\Lambda|^{\frac{3}{2}}e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{3}{2}}\lambda^3} \Big(1+\frac{|\Lambda|}{N}\Big)\Big(1 + \frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big) \Big[1 + \langle T\rangle^2\frac{|\Lambda|^{\frac{3}{2}}e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda}\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1 + \frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\Big]. \tag{5.226}$$

The analogous statements hold true for the continuum approximation if one replaces $(\|\hat{v}\|_{w,d}, \|f_0\|_d)$ *on the RHS of the inequalities by* $(C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}, C_r\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c})$.

Proof In the notation of the proof, we will focus on the discrete case. The bounds for the continuum approximation follow by replacing $(\|\hat{v}\|_{w,d}, \|f_0\|_d)$ by

$$(C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}, C_r\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}), \tag{5.227}$$

see Remark 5.6.

Let $t = T\lambda^{-2}$. Next, Lemma 3.1 and the definition (5.217) of $f^{(\Phi)}$ imply that

$$\int dp\, J(p)\big(f_0(p) - f_t^{(\Phi)}(p)\big) = \int dp\, J(p)\big(f_0(p) - f_t(p)\big) + J(0)|\Phi_t|^2 = -i\int_0^t ds\,\frac{\nu_s([f[J],\mathcal{H}_I(s)])}{|\Lambda|} + J(0)|\Phi_t|^2. \tag{5.228}$$

Using (5.13), we have the estimate

$$\Big|\frac{\nu_s([f[J],\mathcal{H}_I(s)])}{|\Lambda|}\Big| \leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d}e^{C\|f_0\|_d|\Lambda|/\lambda T}\|J\|_{\ell^\infty(\Lambda^*)}\frac{\lambda|\Lambda|^{\frac{3}{2}}}{N^{\frac{1}{2}}}\Big(1 + \frac{|\Lambda|^{\frac{1}{2}}}{N^{\frac{1}{2}}}\Big). \tag{5.229}$$

We have that

$$\Phi_t = -i\int_0^t ds\,\frac{\nu_s\big([a_0,\mathcal{H}_I(s)]\big)}{|\Lambda|}. \tag{5.230}$$

Recall from (5.18) and (5.26) that

$$[a_0,\mathcal{H}_{cub}(s)] = f[J_1(s)] + g[J_2(s)] + g^*[J_3(s)] + J_4(s), \tag{5.231}$$

$$[a_0,\mathcal{H}_{quart}(s)] = \frac{1}{N^{\frac{1}{2}}}\mathcal{H}^{(1)}_{cub}(s). \tag{5.232}$$

Collecting (5.24), (5.25), and using Lemma 4.4, (5.13) yields

$$|\Phi_t| \leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d} T e^{C\|f_0\|_d|\Lambda|/\lambda T} \frac{1}{N^{\frac{1}{2}}\lambda}\Big(1 + \frac{|\Lambda|^{\frac{1}{2}}}{N^{\frac{1}{2}}}\Big). \tag{5.233}$$

Then (5.228), (5.229), and (5.233) yield that

$$\begin{aligned}
&\Big|\int dp\, J(p)\big(f_0(p) - f_t^{(\Phi)}(p)\big)\Big| \\
&\leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d} \langle T\rangle^2 \|J\|_{\ell^\infty(\Lambda^*)} \frac{|\Lambda|^{\frac{3}{2}} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda} \\
&\quad \Big(1 + \frac{|\Lambda|}{N}\Big)\Big(1 + \frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big).
\end{aligned} \tag{5.234}$$

As a consequence of (5.234) and recalling definition (5.49) of $\mathrm{Con}(f)$, we find that

$$\begin{aligned}
&\Big|\mathrm{err}_1^{(cen)}\Big(\frac{T}{\lambda^2}; \Phi\Big)\Big| \\
&\leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d} \langle T\rangle^3 \frac{|\Lambda|^{\frac{3}{2}} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N\lambda^2} \\
&\quad \Big(1 + \frac{|\Lambda|}{N}\Big)\Big(1 + \frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big).
\end{aligned} \tag{5.235}$$

After substitution and using the notation in the proof of Proposition 5.1, the terms in $\mathrm{bol}^{(j)}(f_0)[J](t) - \mathrm{bol}^{(j)}(f^{(\Phi)})[J](t)$ and $\mathrm{col}^{(k)}(f_0)[J](t) - \mathrm{col}^{(k)}(f^{(\Phi)})[J](t)$ are of the form

$$\int d\mathbf{p}_2\, H_1(p_1)H_2(p_2)H_3(p_1 \pm p_2) f_{s_2}^{(\Phi)}(p_1)(f_0(p_2) - f_{s_2}^{(\Phi)}(p_2)), \tag{5.236}$$

$$\int d\mathbf{p}_2\, H_1(p_1)H_2(p_2)H_3(p_1 \pm p_2) f_0(p_1)(f_0(p_2) - f_{s_2}^{(\Phi)}(p_2)), \tag{5.237}$$

$$\int d\mathbf{p}_2\, H_1(p_1)H_2(p_2)H_3(p_{12})(f_0(p_{12}) - f_{s_2}^{(\Phi)}(p_{12})), \tag{5.238}$$

where $H_j \in L^\infty_{\mathbf{s}_2}(\mathbb{R}^2; L^\infty_p(\mathbb{R}^3))$ can differ in every line and in the last line, we require $H_1 \in L^\infty_{\mathbf{s}_2}(\mathbb{R}^2; L^1_p(\mathbb{R}^3))$. We have that $s_2 \leq T\lambda^{-2}$ with $\lambda \in (0, 1)$.

Observe that

$$\begin{aligned}
\int dp\, |H_1(p)| f_0(p) &= \frac{\nu_0(f(|H_1|))}{|\Lambda|} \\
&\leq \frac{\|H_1\|_\infty \nu_0(\mathcal{N}_b)}{|\Lambda|} \\
&\leq \|f_0\|_d \|H_1\|_\infty
\end{aligned} \tag{5.239}$$

due to Lemma 4.7 and

$$\frac{\nu_0(\mathcal{N}_b)}{|\Lambda|} = \int dp\, f_0(p) \leq \|f_0\|_d\,. \tag{5.240}$$

(5.234) and (5.239) imply that the terms of the form (5.237) satisfy

$$\Big|\int d\mathbf{p}_2\, H_1(p_1)H_2(p_2)H_3(p_1 \pm p_2) f_0(p_1)(f_0(p_2) - f_{s_2}^{(\Phi)}(p_2))\Big|$$

$$
\begin{aligned}
&\leq \int dp_1 \left| H_1(p_1) | f_0(p_1) \right| \int dp_2\, H_2(p_2) H_3(p_1 \pm p_2)(f_0(p_2) - f_{s_2}^{(\Phi)}(p_2)) \Big| \\
&\leq C_{\|\hat{v}\|_{w,d}, \|f_0\|_d} \|H_1\|_\infty \|H_2\|_\infty \|H_3\|_\infty \langle T \rangle^2 \frac{|\Lambda|^{\frac{3}{2}} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda} \\
&\quad \Big(1 + \frac{|\Lambda|}{N}\Big)\Big(1 + \frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big).
\end{aligned}
\tag{5.241}
$$

For (5.236), (5.234) and (5.241) yield that

$$
\begin{aligned}
&\left| \int d\mathbf{p}_2\, H_1(p_1) H_2(p_2) H_3(p_1 \pm p_2) f_{s_2}^{(\Phi)}(p_1)(f_0(p_2) - f_{s_2}^{(\Phi)}(p_2)) \right| \\
&\leq \left| \int d\mathbf{p}_2\, H_1(p_1) H_2(p_2) H_3(p_1 \pm p_2) f_0(p_1)(f_0(p_2) - f_{s_2}^{(\Phi)}(p_2)) \right| \\
&\quad + \left| \int d\mathbf{p}_2\, H_1(p_1) H_2(p_2) H_3(p_1 \pm p_2)\big(f_{s_2}^{(\Phi)}(p_1) - f_0(p_1)\big)(f_0(p_2) - f_{s_2}^{(\Phi)}(p_2)) \right| \\
&\leq C_{\|\hat{v}\|_{w,d}, \|f_0\|_d} \|H_1\|_\infty \langle T \rangle^2 \frac{|\Lambda|^{\frac{3}{2}} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda} \\
&\quad \Big(1 + \frac{|\Lambda|}{N}\Big)\Big(1 + \frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big) \\
&\quad \Big(\|H_2\|_\infty \|H_3\|_\infty + \sup_{p_1} \Big| \int dp_2\, H_2(p_2) H_3(p_1 \pm p_2)(f_0(p_2) - f_{s_2}^{(\Phi)}(p_2)) \Big| \Big) \\
&\leq C_{\|\hat{v}\|_{w,d}, \|f_0\|_d} \|H_1\|_\infty \|H_2\|_\infty \|H_3\|_\infty \langle T \rangle^2 \frac{|\Lambda|^{\frac{3}{2}} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda} \\
&\quad \Big(1 + \frac{|\Lambda|}{N}\Big)\Big(1 + \frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big) \\
&\quad \Big[1 + \langle T \rangle^2 \frac{|\Lambda|^{\frac{3}{2}} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda} \Big(1 + \frac{|\Lambda|}{N}\Big)\Big(1 + \frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\Big].
\end{aligned}
\tag{5.242}
$$

Employing (5.234) again, we estimate (5.238) by

$$
\begin{aligned}
&\left| \int d\mathbf{p}_2\, H_1(p_1) H_2(p_2) H_3(p_1 + p_2)(f_0(p_1 + p_2) - f_{s_2}^{(\Phi)}(p_1 + p_2)) \right| \\
&\leq C_{\|\hat{v}\|_{w,d}, \|f_0\|_d} \|H_1\|_1 \|H_2\|_\infty \|H_3\|_\infty \langle T \rangle^2 \frac{|\Lambda|^{\frac{3}{2}} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda} \\
&\quad \Big(1 + \frac{|\Lambda|}{N}\Big)\Big(1 + \frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big).
\end{aligned}
\tag{5.243}
$$

Collecting (5.241)–(5.243), recalling definitions (5.50)–(5.56), and denoting $\mathrm{Bol}^{(0)} := \mathrm{Bol}$, we find the upper bound

$$
\begin{aligned}
&\frac{\lambda^j}{N} |\,\mathrm{Bol}^{(j)}(f_0)[J](T;\lambda) - \mathrm{Bol}^{(j)}(f^{(\Phi)})[J](T;\lambda)| \\
&\quad \leq C_{\|\hat{v}\|_{w,d}, \|f_0\|_d} \frac{\lambda^{2+j} t^2 \|J\|_{\ell^\infty(\Lambda^*)}}{N} \langle T \rangle^2 \frac{|\Lambda|^{\frac{3}{2}} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda}
\end{aligned}
$$

$$\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1\,+\,\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)$$
$$\Big[1\,+\,\langle T\rangle^2\frac{|\Lambda|^{\frac{3}{2}}e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda}\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1\,+\,\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\Big]. \tag{5.244}$$

In particular, we have that

$$\begin{aligned}
&|\,\mathrm{err}_{2,d}^{(cen)}(t;\,f[J])|\,,\;|\,\mathrm{err}_{2,c}^{(cen)}(t;\,f[J])|\\
&\le C_{\|\hat{v}\|_{w,d},\|f_0\|_d}\|J\|_{\ell^\infty(\Lambda^*)}\langle T\rangle^4\frac{|\Lambda|^{\frac{3}{2}}e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{3}{2}}\lambda^3}\\
&\quad\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1\,+\,\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\\
&\quad\Big[1\,+\,\langle T\rangle^2\frac{|\Lambda|^{\frac{3}{2}}e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda}\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1\,+\,\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\Big].
\end{aligned} \tag{5.245}$$

Finally, it remains to estimate $\mathrm{err}_j^{(cen)}(\frac{T}{\lambda^2};\,g[J])$, $j\in\{1,2\}$. Following analogous steps that lead to (5.245), Proposition 5.2 implies

$$\begin{aligned}
&\left|\mathrm{err}_1^{(cen)}(\frac{T}{\lambda^2};\,g[J])\right|\\
&\le C_{\|\hat{v}\|_{w,d},\|f_0\|_d}\|J\|_{\ell^\infty(\Lambda^*)}\langle T\rangle^3\frac{|\Lambda|^{\frac{3}{2}}e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{3}{2}}\lambda}\\
&\quad\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1\,+\,\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\\
&\quad\Big[1\,+\,\langle T\rangle^2\frac{|\Lambda|^{\frac{3}{2}}e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda}\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1\,+\,\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\Big],
\end{aligned} \tag{5.246}$$

$$\begin{aligned}
&\left|\mathrm{err}_2^{(cen)}\left(\frac{T}{\lambda^2};\,g[J]\right)\right|\\
&\le C_{\|\hat{v}\|_{w,d},\|f_0\|_d}\|J\|_{\ell^\infty(\Lambda^*)}\langle T\rangle^3\frac{|\Lambda|^{\frac{3}{2}}e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{3}{2}}\lambda^3}\\
&\quad\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1\,+\,\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\\
&\quad\Big[1\,+\,\langle T\rangle^2\frac{|\Lambda|^{\frac{3}{2}}e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda}\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1\,+\,\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\Big].
\end{aligned} \tag{5.247}$$

This concludes the proof. □

6 Main Order Terms in the Evolution of (Φ, f, g)

In this section, we will prove Theorem 2.2. We will always assume that $\lambda\in(0,1)$ is small enough to comply with the estimates proven in all preceding steps.

6.1 Discrete Case

Fix $L \geq 1$. Recall that we refer to the case of fixed Λ as the 'discrete case'.

6.1.1 Main Order Term of Φ

Using Propositions 5.1, and 5.2, we have that

$$\Phi_{\frac{T}{\lambda^2}} = \frac{1}{N^{\frac{1}{2}}\lambda} \mathrm{Con}_d(f_0)(T;\lambda) + \mathrm{Rem}_2(\frac{T}{\lambda^2};\Phi) + \mathrm{err}_{1,d}^{(Bog)}(\frac{T}{\lambda^2};\Phi) \tag{6.1}$$

with

$$|\,\mathrm{Rem}_2(\frac{T}{\lambda^2};\Phi)| \leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d} T^2 e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T} \frac{|\Lambda|^{\frac{3}{2}}}{N\lambda^2}, \tag{6.2}$$

$$|\,\mathrm{err}_{1,d}^{(Bog)}(\frac{T}{\lambda^2};\Phi)| \leq \frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d} T}{N^{\frac{1}{2}}}. \tag{6.3}$$

In particular, the main term is given by

$$\frac{1}{N^{\frac{1}{2}}\lambda} \mathrm{Con}_d(f_0)(T;\lambda) = -\frac{iT}{N^{\frac{1}{2}}\lambda} \int_{\Lambda^*} dp\, \hat{v}(p) f_0(p), \tag{6.4}$$

see (5.49), and thus it is of size $N^{-\frac{1}{2}}\lambda^{-1}$. In order to suppress $\mathrm{Rem}_2(\frac{T}{\lambda^2};\Phi)$, we choose $\lambda = \log\log N/\log N$.

6.1.2 Main Order Term of f

For f, we apply Propositions 5.1, and 5.2 to obtain that

$$\begin{aligned}
&\int_{\Lambda^*} dp\, \big(f^{(\Phi)}_{\frac{T}{\lambda^2}} - f^{(\Phi)}_0\big) J(p) \\
&= \frac{1}{N}\Big(\mathrm{Bol}_d(f_0)[J](T;\lambda) + \lambda\,\mathrm{Bol}_d^{(1)}(f_0)[J](T;\lambda) + \lambda^2\,\mathrm{Bol}_d^{(2)}(f_0)[J](T;\lambda)\Big) \\
&+ \mathrm{Rem}_2(\frac{T}{\lambda^2}; f[J]) + \mathrm{err}_{2,d}^{(Bog,\mathrm{Bol})}(\frac{T}{\lambda^2}; f[J]) + J(0)\,\mathrm{err}_2^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2}; f),
\end{aligned} \tag{6.5}$$

where

$$\left|\mathrm{Rem}_2\left(\frac{T}{\lambda^2}; f[J]\right)\right| \leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d} \langle T\rangle^4 \|J\|_{\ell^\infty(\Lambda^*)} \frac{e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}|\Lambda|^6}{\lambda^4 N^2}, \tag{6.6}$$

$$\left|\mathrm{err}_{2,d}^{(Bog,\mathrm{Bol})}\left(\frac{T}{\lambda^2}; f[J]\right)\right| \leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d} \|J\|_{\ell^\infty(\Lambda^*)} T^2 \frac{\lambda}{N}, \tag{6.7}$$

$$\left|\mathrm{err}_{2,d}^{(Bog,\mathrm{Con})}(t; f)\right| \leq C_{\|\hat{v}\|_{w,d},\|f_0\|_d} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T} \frac{\langle T\rangle^4|\Lambda|^{\frac{3}{2}}}{\lambda^3 N^{\frac{3}{2}}}\left(1 + \frac{|\Lambda|^{\frac{3}{2}}}{\lambda N^{\frac{1}{2}}}\right). \tag{6.8}$$

Due to Lemma 5.3, we can expand the oscillatory factors in $\mathrm{Bol}_d^{(j)}$ in powers of λ. Then the main order term depends on $\hat{v}$, $\lambda(N)$, and T. We are interested in terms up to $O(N^{-1})$ which, as we will see below, is the size of the main order term in the continuum approximation. For the same reasons as above, we choose $\lambda = \log\log N/\log N$.

6.1.3 Main Order Term of g

Similar to the case of f, we need to subtract the dynamics of the condensate in order to resolve the fluctuations defined by g. For that, we introduce

$$g_t^{(\Phi)}(p) \;:=\; g_t(p) \;-\; \Phi_t^2 \delta(p)\,. \tag{6.9}$$

Propositions 5.1 and 5.2 then yield

$$\begin{aligned}
&\int_{\Lambda^*} dp\, g^{(\Phi)}_{\frac{T}{\lambda^2}}(p) J(p) \\
&\quad = \frac{1}{N}\int_0^T dS\, \mathrm{abs}_{quart,d}(f_0)[J](S/\lambda^2) \\
&\qquad + \frac{1}{N\lambda^2}\int_{\Delta[T,2]} d\mathbf{S}_2 \left(\mathrm{col}_d(f_0)[J](\mathbf{S}_2/\lambda^2) + \mathrm{abs}_{cub,d}(f_0)[J](\mathbf{S}_2/\lambda^2)\right) \\
&\qquad + \mathrm{Rem}_2(\frac{T}{\lambda^2}; g[J]) \;+\; J(0)\,\mathrm{err}_2^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2}; g)
\end{aligned} \tag{6.10}$$

with

$$|\,\mathrm{Rem}_2(\frac{T}{\lambda^2}; g[J])| \le C_{\|\hat{v}\|_{w,d},\|f_0\|_d} \langle T\rangle^4 \|J\|_{2\cap\infty,d} \frac{e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}|\Lambda|^6}{\lambda^4 N^2}\,, \tag{6.11}$$

$$|\,\mathrm{err}_{2,d}^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2}; g)| \le C_{\|\hat{v}\|_{w,d},\|f_0\|_d} e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T} \frac{\langle T\rangle^4|\Lambda|^{\frac{3}{2}}}{\lambda^3 N^{\frac{3}{2}}}\left(1+\frac{|\Lambda|^{\frac{3}{2}}}{\lambda N^{\frac{1}{2}}}\right). \tag{6.12}$$

As in the case of f, we again observe a phenomenon similar to the Talbot effect on $\mathrm{abs}_{quart,d}$, col_d, and $\mathrm{abs}_{cub,d}$. We choose N^{-1} as reference order.

6.1.4 Conclusion for $L \sim 1$

As described above, we choose $\lambda = \log\log(N)/\log(N)$. Let

$$\delta_{\Psi,d} = \min\left\{\frac{\log\frac{1}{\lambda}}{\log N}, \frac{1}{2} - \frac{\log\frac{1}{\lambda}}{\log N} - \frac{C\|\hat{v}\|_{w,d}|\Lambda|T}{\log\log N}\right\}, \tag{6.13}$$

$$\delta_{F,d} = \min\left\{\frac{\log\frac{1}{\lambda}}{\log N}, \frac{1}{2} - \frac{3\log\frac{1}{\lambda}}{\log N} - \frac{C\|\hat{v}\|_{w,d}|\Lambda|T}{\log\log N}\right\}, \tag{6.14}$$

$$\delta_{G,d} = \frac{1}{2} - \frac{3\log\frac{1}{\lambda}}{\log N} - \frac{C\|\hat{v}\|_{w,d}|\Lambda|T}{\log\log N}\,. \tag{6.15}$$

Then we obtain

$$\begin{aligned}
&\left|\Phi_{\frac{T}{\lambda^2}} \;-\; \frac{1}{N^{\frac{1}{2}}\lambda}\mathrm{Con}_d(f_0)(T;\lambda)\right| \\
&\quad \le \frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d,|\Lambda|,T}}{N^{\frac{1}{2}+\delta_{\Psi,d}}\lambda}\,,
\end{aligned} \tag{6.16}$$

$$\begin{aligned}
&\Big|\int_{\Lambda^*} dp\left(f^{(\Phi)}_{\frac{T}{\lambda^2}}(p) - f_0^{(\Phi)}(p)\right)J(p) \\
&\quad - \frac{1}{N}\Big(\mathrm{Bol}_d(f_0)[J](T;\lambda) \;+\; \sum_{j=1}^{2}\lambda^j\,\mathrm{Bol}_d^{(j)}(f_0)[J](T;\lambda)\Big)\Big|
\end{aligned}$$

$$\leq \frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d,|\Lambda|,T}\|J\|_{\ell^\infty(\Lambda^*)}}{N^{1+\delta_{F,d}}}, \tag{6.17}$$

$$\begin{aligned}&\Big|\int_{\Lambda^*} dp\, g^{(\Phi)}_{\frac{T}{\lambda^2}}(p)J(p) - \frac{1}{N}\int_0^T dS\, \mathrm{abs}_{quart,d}(f_0)[J](S/\lambda^2)\\ &- \frac{1}{N\lambda^2}\int_{\Delta[T,2]} d\mathbf{S}_2 \left(\mathrm{col}_d(f_0)[J](\mathbf{S}_2/\lambda^2) + \mathrm{abs}_{cub,d}(f_0)[J](\mathbf{S}_2/\lambda^2)\right)\Big|\\ &\leq \frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d,|\Lambda|,T}\|J\|_{2\cap\infty,d}}{N^{1+\delta_{G,d}}}\end{aligned} \tag{6.18}$$

for all N larger than a universal constant. Notice that for all N large enough, we have that $\delta_{j,d} = \delta_{j,d}(N, |\Lambda|, T) > 0$ for $j \in \{\Psi, F, G\}$.

6.2 Continuum Approximation

We have that errors coming from the tail in the Duhamel expansion grow like $\exp(C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}T|\Lambda|/\lambda)$. As a consequence, we require

$$\frac{|\Lambda|}{\lambda} = O(\frac{\log(N)}{\log\log(N)}) \tag{6.19}$$

as $N \to \infty$. In order to use bounds established for general values of $|\Lambda|$ to the limit $|\Lambda| \to \infty$, for specific expressions of interest, we employ Remark 5.6.

6.2.1 Main Order Term of Φ

Using Propositions 5.1 and 5.7, we have that

$$\Phi_{\frac{T}{\lambda^2}} = \frac{1}{N^{\frac{1}{2}}\lambda}\mathrm{Con}_c(f_0)(T;\lambda) + \mathrm{Rem}_2(\frac{T}{\lambda^2};\Phi) + \mathrm{err}^{(Bog)}_{1,c}(\frac{T}{\lambda^2};\Phi) \tag{6.20}$$

with

$$\begin{aligned}\left|\mathrm{Rem}_2\left(\frac{T}{\lambda^2};\Phi\right)\right| &\leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}T^2 e^{C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}|\Lambda|/\lambda T}\frac{|\Lambda|^{\frac{3}{2}}}{N\lambda^2}\\ &\quad\left(1+\frac{|\Lambda|}{N}\right),\\ \left|\mathrm{err}^{(Bog)}_{1,c}(t;\Phi)\right| &\leq \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}T}{N^{\frac{1}{2}}\lambda}\left(\lambda + \frac{1}{|\Lambda|^{\frac{r}{6}}}\right)\end{aligned} \tag{6.21}$$

for some $r > 6$. The main order term is given by

$$\frac{1}{N^{\frac{1}{2}}\lambda}\mathrm{Con}_c(f_0)(T;\lambda) = -\frac{iT}{(2\pi)^3N^{\frac{1}{2}}\lambda}\int_{\mathbb{R}^3} dp\,\hat{v}(p)f_0(p), \tag{6.22}$$

and it is of size $N^{\frac{1}{2}}\lambda^{-1}$.

6.2.2 Main Order Term of *f*

For f, we apply Propositions 5.1, 5.7, and 5.8 to obtain that

$$\int_{\Lambda^*} dp\left(f^{(\Phi)}_{\frac{T}{\lambda^2}}(p) - f^{(\Phi)}_0(p)\right)J(p)$$

$$
\begin{aligned}
&= \frac{1}{N} \mathrm{Bol}_{fec}(f_0)[J](T) \\
&+ \mathrm{Rem}_2(\frac{T}{\lambda^2}; f[J]) + \mathrm{err}_{2,c}^{(Bog,\mathrm{Bol})}(\frac{T}{\lambda^2}; f[J]) \\
&+ J(0)\, \mathrm{err}_{2,c}^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2}; f) + \mathrm{err}_{2,c}^{(fec)}(T\lambda^{-2}; f[J]) ,
\end{aligned}
\tag{6.23}
$$

where

$$
\left|\mathrm{Rem}_2\left(\frac{T}{\lambda^2}; f[J]\right)\right| \leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, \|f_0\|_{2(\lfloor \frac{r}{2} \rfloor+1),c}, r} \langle T\rangle^4 \|J\|_\infty \frac{e^{C_r \|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c} |\Lambda|/\lambda T} |\Lambda|^6}{\lambda^4 N^2} \Big(1 + \frac{|\Lambda|}{N}\Big)^2 , \tag{6.24}
$$

$$
\left|\mathrm{err}_{2,c}^{(Bog,\mathrm{Bol})}\left(\frac{T}{\lambda^2}; f[J]\right)\right| \leq \frac{C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, \|f_0\|_{2(\lfloor \frac{r}{2} \rfloor+1),c}, r} \langle T\rangle^{r+2} \|J\|_{W^{2\lfloor \frac{r}{2} \rfloor+2,\infty}}}{N} \Big(\frac{1}{\lambda^{2(r+1)} |\Lambda|^{\frac{r}{3}}} + \lambda |\log(\lambda)|\Big) , \tag{6.25}
$$

$$
\left|\mathrm{err}_{2,c}^{(Bog,\mathrm{Con})}\left(\frac{T}{\lambda^2}; f[J]\right)\right| \leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, \|f_0\|_{2(\lfloor \frac{r}{2} \rfloor+1),c}, r} e^{C_r \|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c} |\Lambda|/\lambda T} \frac{\langle T\rangle^4 |\Lambda|^{\frac{3}{2}}}{\lambda^3 N^{\frac{3}{2}}} \Big(1 + \frac{|\Lambda|}{N}\Big)^2 \Big(1 + \frac{|\Lambda|^{\frac{3}{2}}}{\lambda N^{\frac{1}{2}}}\Big) , \tag{6.26}
$$

$$
\left|\mathrm{err}_{2,c}^{(fec)}\left(\frac{T}{\lambda^2}; f[J]\right)\right| \leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, \|f_0\|_{2(\lfloor \frac{r}{2} \rfloor+1),c}, r} \langle T\rangle \|J\|_{W^{2,\infty}} \frac{\lambda}{N} \tag{6.27}
$$

for all $\lambda > 0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}$. Recalling definition (5.200) of Bol_{fec}, the main term is given by

$$
\frac{\pi T}{(2\pi)^6 N} \int_{p_1 \perp p_2} d\mathcal{H}^5(\mathbf{p}_2)\, \Delta_{cub} J(\mathbf{p}_2) \frac{(\hat{v}(p_1) + \hat{v}(p_2))^2}{|\mathbf{p}_2|} \big(\widetilde{f}_0(p_1)\widetilde{f}_0(p_2) f_0(p_1+p_2) - f_0(p_1) f_0(p_2) \widetilde{f}_0(p_1+p_2)\big) , \tag{6.28}
$$

and it is of size N^{-1}. In order to have that the discretization error is negligible, we need to impose $\frac{1}{\lambda} = o(|\Lambda|^{\frac{r}{6(r+1)}})$ as $|\Lambda| \to \infty$. In particular, we may choose

$$
L = \lambda^{-2-\frac{2}{r}-\varepsilon} \tag{6.29}
$$

for any arbitrary $\varepsilon > 0$. Recall that we required

$$
\frac{|\Lambda|}{\lambda} = O\left(\frac{\log N}{\log\log N}\right) \tag{6.30}
$$

as $N \to \infty$ to suppress the tail in the Duhamel expansion. We may thus choose

$$
\lambda = \Big(\frac{\log\log N}{\log N}\Big)^{\frac{r}{(7+\varepsilon)r+6}} , \tag{6.31}
$$

$$
L = \Big(\frac{\log N}{\log\log N}\Big)^{\frac{(2+\varepsilon)r+2}{(7+\varepsilon)r+6}} . \tag{6.32}
$$

6.2.3 Main Order Term of *g*

Propositions 5.1 and 5.7 imply

$$
\begin{aligned}
&\int_{\Lambda^*} dp\, g^{(\Phi)}_{\frac{T}{\lambda^2}}(p) J(p) \\
&= \frac{1}{N}\int_0^T dS\, \mathrm{abs}_{quart,c}(f_0)[J](S/\lambda^2) \\
&\quad + \frac{1}{N\lambda^2}\int_{\Delta[T,2]} d\mathbf{S}_2 \left(\mathrm{col}_c(f_0)[J](\mathbf{S}_2/\lambda^2) + \mathrm{abs}_{cub,c}(f_0)[J](\mathbf{S}_2/\lambda^2)\right) \\
&\quad + \mathrm{Rem}_2(\frac{T}{\lambda^2}; g[J]) \;+\; \mathrm{err}^{(dis)}_{1,c}(\frac{T}{\lambda^2}; g[J]) \\
&\quad + \mathrm{err}^{(dis)}_{2,c}(\frac{T}{\lambda^2}; g[J]) \;+\; J(0)\,\mathrm{err}^{(Bog,\mathrm{Con})}_{2,c}(\frac{T}{\lambda^2}; g)
\end{aligned} \tag{6.33}
$$

with

$$
\begin{aligned}
\left|\mathrm{Rem}_2\left(\frac{T}{\lambda^2}; g[J]\right)\right| &\le C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r} \langle T\rangle^4 \|J\|_{2(\lfloor\frac{r}{2}\rfloor+1),c} \\
&\qquad \frac{e^{C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}|\Lambda|/\lambda T}|\Lambda|^6}{\lambda^4 N^2}\left(1+\frac{|\Lambda|}{N}\right)^2,
\end{aligned} \tag{6.34}
$$

$$
\left|\mathrm{err}^{(dis)}_{1,c}\left(\frac{T}{\lambda^2}; g[J]\right)\right| \le \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}\langle T\rangle^{r+1}\|J\|_{W^{2\lfloor\frac{r}{2}\rfloor+2,\infty}}}{N\lambda^{2r}|\Lambda|^{\frac{r}{3}}}, \tag{6.35}
$$

$$
\left|\mathrm{err}^{(dis)}_{2,c}\left(\frac{T}{\lambda^2}; g[J]\right)\right| \le \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}\langle T\rangle^{r+2}\|J\|_{W^{2\lfloor\frac{r}{2}\rfloor+2,\infty}}}{N\lambda^{2(r+1)}|\Lambda|^{\frac{r}{3}}}, \tag{6.36}
$$

$$
\begin{aligned}
\left|\mathrm{err}^{(Bog,\mathrm{Con})}_{2,c}\left(\frac{T}{\lambda^2}; g[J]\right)\right| &\le C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}\, e^{C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}|\Lambda|/\lambda T} \\
&\qquad \frac{\langle T\rangle^4|\Lambda|^{\frac{3}{2}}}{\lambda^3 N^{\frac{3}{2}}}\left(1+\frac{|\Lambda|}{N}\right)^2\left(1+\frac{|\Lambda|^{\frac{3}{2}}}{\lambda N^{\frac{1}{2}}}\right)
\end{aligned} \tag{6.37}
$$

for all $\lambda > 0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}$. All the errors are suppressed for the choices of $|\Lambda|$ and λ as in the case of f above.

6.2.4 Conclusion for $L \sim \lambda^{-2-}$

Recall that we impose

$$
\lambda = \left(\frac{\log\log N}{\log N}\right)^{\frac{r}{(7+\varepsilon)r+6}}, \tag{6.38}
$$

$$
L = \lambda^{-2-\frac{2}{r}-\varepsilon} = \left(\frac{\log N}{\log\log N}\right)^{\frac{(2+\varepsilon)r+2}{(7+\varepsilon)r+6}}. \tag{6.39}
$$

Let

$$
\begin{aligned}
\delta_{\Psi,c} &:= \min\left\{\frac{1}{2} - C_{r,\varepsilon}\frac{\log\frac{1}{\lambda}}{\log N} - \frac{C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}T}{\log\log N}, \frac{\log\frac{1}{\lambda}}{\log N}\right\},
\end{aligned} \tag{6.40}
$$

$$
\delta_{F,c} := \min\left\{\frac{1}{2} - C_{r,\varepsilon}\frac{\log\frac{1}{\lambda}}{\log N} - \frac{C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}T}{\log\log N}, \frac{\varepsilon r\log\frac{1}{\lambda}}{\log N},\right.
$$

$$\frac{\log\frac{1}{\lambda}-\log\log\frac{1}{\lambda}}{\log N}\Big\}, \tag{6.41}$$

$$\delta_{G,c} := \min\left\{\frac{1}{2}-C_{r,\varepsilon}\frac{\log\frac{1}{\lambda}}{\log N}-\frac{C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}T}{\log\log N}, \frac{\varepsilon r\log\frac{1}{\lambda}}{\log N}\right\}. \tag{6.42}$$

Observe that we have

$$\|J\|_{W^{2,\infty}(\mathbb{R}^3)} \leq \|J\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}\,. \tag{6.43}$$

Then we have proved that, for some $N_0 = N_0(\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c})$,

$$\begin{aligned}
&\left|\Phi_{\frac{T}{\lambda^2}} - \frac{1}{N^{\frac{1}{2}}\lambda}\mathrm{Con}_c(f_0)(T;\lambda)\right| \\
&\qquad\leq \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r,\varepsilon,T}}{N^{\frac{1}{2}+\delta_{\Psi,c}}\lambda}, \qquad (6.44)\\
&\left|\int_{\Lambda^*}dp\,\big(f^{(\Phi)}_{\frac{T}{\lambda^2}}(p)-f^{(\Phi)}_0(p)\big)J(p) - \frac{1}{N}\mathrm{Bol}_{fec}(f_0)[J](T)\right| \\
&\qquad\leq \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r,\varepsilon,T}\|J\|_{W^{2\lfloor\frac{r}{2}\rfloor+2,\infty}}}{N^{1+\delta_{F,c}}}, \qquad (6.45)\\
&\Big|\int_{\Lambda^*}dp\,g^{(\Phi)}_{\frac{T}{\lambda^2}}(p)J(p) - \frac{1}{N}\int_0^T dS\,\mathrm{abs}_{quart,c}(f_0)[J](S/\lambda^2) \\
&-\frac{1}{N\lambda^2}\int_{\Delta[T,2]}d\mathbf{S}_2\,\big(\mathrm{col}_c(f_0)[J](\mathbf{S}_2/\lambda^2) \\
&+\mathrm{abs}_{cub,c}(f_0)[J](\mathbf{S}_2/\lambda^2)\big)\Big| \\
&\qquad\leq \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r,\varepsilon,T}\|J\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}}{N^{1+\delta_{G,c}}} \qquad (6.46)
\end{aligned}$$

for all $N \geq N_0(\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c})$. Observe that for all N large enough, we have that $\delta_{j,c} = \delta_{j,c}(N,T) > 0$ for $j \in \{\Psi, F, G\}$.

7 Effective Equations

After establishing the sizes of the leading order terms, we can derive the effective equations. This will prove Theorem 2.1.

7.1 Discrete Case

Let $L \geq 1$ be fixed. Choose $\lambda = \log\log(N)/\log(N)$ as explained in section 6.1.4.

7.1.1 Evolution of Φ

We have that

$$\Phi_{\frac{T}{\lambda^2}} = \frac{1}{N^{\frac{1}{2}}\lambda}\mathrm{Con}_d(f^{(\Phi)})(T;\lambda)$$

$$+\mathrm{Rem}_2\left(\frac{T}{\lambda^2};\Phi\right)+\mathrm{err}_{1,d}^{(Bog)}\left(\frac{T}{\lambda^2};\Phi\right)+\mathrm{err}_{1,d}^{(cen)}\left(\frac{T}{\lambda^2};\Phi\right), \tag{7.1}$$

see (6.1). Proposition 5.9 implies

$$\begin{aligned}\left|\mathrm{err}_{1,d}^{(cen)}\left(\frac{T}{\lambda^2};\Phi\right)\right| &\le C_{\|\hat{v}\|_{w,d},\|f_0\|_d}\langle T\rangle^3\frac{|\Lambda|^{\frac{3}{2}}e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N\lambda^2}\\ &\quad\left(1+\frac{|\Lambda|}{N}\right)\left(1+\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\right)\\ &\le\frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},|\Lambda|,T}}{N^{\frac{1}{2}+\delta_{\Psi,d}}\lambda}.\end{aligned} \tag{7.2}$$

for all N large enough. Observe that

$$\frac{1}{N^{\frac{1}{2}}\lambda}\mathrm{Con}_d(f^{(\Phi)})(T;\lambda)=-\frac{i}{N^{\frac{1}{2}}\lambda}\int_0^T dS\int_{\Lambda^*}dp\,\hat{v}(p)f_{\frac{S}{\lambda^2}}^{(\Phi)}(p). \tag{7.3}$$

Then, analogously to (6.16) and employing (7.2), we obtain that

$$\left|\Phi_{\frac{T}{\lambda^2}}-\frac{1}{N^{\frac{1}{2}}\lambda}\mathrm{Con}_d(f^{(\Phi)})(T;\lambda)\right|\le\frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d,|\Lambda|,T}}{N^{\frac{1}{2}+\delta_{\Psi,d}}\lambda}. \tag{7.4}$$

7.1.2 Evolution of *f*

(6.5) and Proposition 5.9 imply

$$\begin{aligned}&\int_{\Lambda^*}dp\,\big(f_{\frac{T}{\lambda^2}}^{(\Phi)}(p)-f_0^{(\Phi)}(p)\big)J(p)\\ &=\frac{1}{N}\Big(\mathrm{Bol}_d(f^{(\Phi)})[J](T;\lambda)+\sum_{j=1}^{2}\lambda^j\,\mathrm{Bol}_d^{(1)}(f^{(\Phi)})[J](T;\lambda)\Big)\\ &+\mathrm{Rem}_2(\frac{T}{\lambda^2};f[J])+\mathrm{err}_{2,d}^{(Bog,\mathrm{Bol})}(\frac{T}{\lambda^2};f[J])\\ &+J(0)\,\mathrm{err}_{2,d}^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2};f)+\mathrm{err}_{2,d}^{(cen)}(\frac{T}{\lambda^2};f[J]).\end{aligned} \tag{7.5}$$

Recall that, due to Proposition 5.9,

$$\begin{aligned}|\,\mathrm{err}_{2,d}^{(cen)}(\frac{T}{\lambda^2};f[J])|&\le C_{\|\hat{v}\|_{w,d},\|f_0\|_d}\|J\|_{\ell^\infty(\Lambda^*)}\langle T\rangle^4\frac{|\Lambda|^{\frac{3}{2}}e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{3}{2}}\lambda^3}\\ &\quad\left(1+\frac{|\Lambda|}{N}\right)\left(1+\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\right)\\ &\quad\left[1+\langle T\rangle^2\frac{|\Lambda|^{\frac{3}{2}}e^{C\|\hat{v}\|_{w,d}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda}\left(1+\frac{|\Lambda|}{N}\right)\left(1+\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\right)\right]\\ &\le\frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d,|\Lambda|,T}\|J\|_{\ell^\infty(\Lambda^*)}}{N^{1+\delta_{F,d}(N)}},\end{aligned} \tag{7.6}$$

where we possibly enlarge the constant in the definition of $\delta_{F,d}$. In particular, similar to (6.17) we obtain that

$$\Big|\int_{\Lambda^*}dp\,\big(f_{\frac{T}{\lambda^2}}^{(\Phi)}(p)-f_0^{(\Phi)}(p)\big)J(p)-\frac{1}{N}\Big(\mathrm{Bol}_d(f^{(\Phi)})[J](T;\lambda)$$

$$
\begin{aligned}
&+ \sum_{j=1}^{2} \lambda^j \, \mathrm{Bol}_d^{(j)}(f^{(\Phi)})[J](T;\lambda)\Big)\Big| \\
&\leq \frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d,|\Lambda|,T} \|J\|_{\ell^\infty(\Lambda^*)}}{N^{1+\delta_{F,d}}} \,.
\end{aligned} \tag{7.7}
$$

7.1.3 Evolution of g

(6.10) yields

$$
\begin{aligned}
&\int_{\Lambda^*} dp \, g^{(\Phi)}_{\frac{T}{\lambda^2}}(p) J(p) \\
&\quad = \frac{1}{N} \int_0^T dS \, \mathrm{abs}_{quart,d}(f^{(\Phi)})[J](S/\lambda^2) \\
&\qquad + \frac{1}{N\lambda^2} \int_{\Delta[T,2]} d\mathbf{S}_2 \left(\mathrm{col}_d(f^{(\Phi)})[J](\mathbf{S}_2/\lambda^2) + \mathrm{abs}_{cub,d}(f^{(\Phi)})[J](\mathbf{S}_2/\lambda^2) \right) \\
&\qquad + \mathrm{Rem}_2(\frac{T}{\lambda^2}; g[J]) \, + \, J(0) \, \mathrm{err}_{2,d}^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2}; g) \\
&\qquad + \mathrm{err}_1^{(cen)}(\frac{T}{\lambda^2}; g[J]) \, + \, \mathrm{err}_2^{(cen)}(\frac{T}{\lambda^2}; g[J]) \,,
\end{aligned} \tag{7.8}
$$

where, due to Proposition 5.9 and analogously to (7.6),

$$
\begin{aligned}
&|\,\mathrm{err}_1^{(cen)}(\frac{T}{\lambda^2}; g[J])| \,, \; |\,\mathrm{err}_2^{(cen)}(\frac{T}{\lambda^2}; g[J])| \\
&\quad \leq \frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d,|\Lambda|,T} \|J\|_{\ell^\infty(\Lambda^*)}}{N^{1+\delta_{G,d}}} \,,
\end{aligned} \tag{7.9}
$$

where we possibly enlarge the constant in the definition of $\delta_{G,d}$. As a consequence of (6.18), we thus obtain that

$$
\begin{aligned}
&\Big| \int_{\Lambda^*} dp \, g^{(\Phi)}_{\frac{T}{\lambda^2}}(p) J(p) \, - \, \frac{1}{N} \int_0^T dS \, \mathrm{abs}_{quart,d}(f^{(\Phi)})[J](S/\lambda^2) \\
&\quad - \frac{1}{N\lambda^2} \int_{\Delta[T,2]} d\mathbf{S}_2 \left(\mathrm{col}_d(f^{(\Phi)})[J](\mathbf{S}_2/\lambda^2) + \mathrm{abs}_{cub,d}(f^{(\Phi)})[J](\mathbf{S}_2/\lambda^2) \right) \Big| \\
&\quad \leq \frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d,|\Lambda|,T} \|J\|_{2\cap\infty,d}}{N^{1+\delta_{G,d}(N)}} \,.
\end{aligned} \tag{7.10}
$$

7.1.4 Conclusion for $L \sim 1$

Recall that $\lambda = \log\log(N)/\log(N)$. We consider the effective quantities

$$
\Psi_T = \Phi_{\frac{T}{\lambda^2}} \,, \tag{7.11}
$$

$$
F_T = f^{(\Phi)}_{\frac{T}{\lambda^2}} \,, \tag{7.12}
$$

$$
G_T = g^{(\Phi)}_{\frac{T}{\lambda^2}} \,. \tag{7.13}
$$

We have that

$$
\lambda^2 \mathrm{Re} \int_{[\Delta[T\lambda^{-2},2]} d\mathbf{s}_2 \, e^{-i\omega(s_1-s_2)} h(s_2) = \lambda^2 \int_0^{\frac{T}{\lambda^2}} ds \, \frac{\sin(\omega(\frac{T}{\lambda^2}-s))}{\omega} h(s)
$$

$$= \int_0^T dS \, \frac{\sin\left(\frac{\omega}{\lambda^2}(T-S)\right)}{\omega} h(\frac{S}{\lambda^2}) \,. \qquad (7.14)$$

For a function $H_t(p)$ and $j \in \{1, 2\}$, denote

$$\begin{aligned}
Q_{d;T-S,\lambda}(H_S)[J] &= \int_{(\Lambda^*)^2} d\mathbf{p}_2 \frac{\sin\left(\frac{\Omega(p_1)+\Omega(p_2)-\Omega(p_1+p_2)}{\lambda^2}(T-S)\right)}{\Omega(p_1)+\Omega(p_2)-\Omega(p_1+p_2)} \\
&\quad (\hat{v}(p_1)+\hat{v}(p_2))^2 \big(J(p_1)+J(p_2)-J(p_1+p_2)\big) \\
&\quad \big(\widetilde{H}_S(p_1)\widetilde{H}_S(p_2)H_S(p_1+p_2) \\
&\quad - H_S(p_1)H_S(p_2)\widetilde{H}_S(p_1+p_2)\big) \,, \qquad (7.15) \\
q^{(j)}_{d,F;\mathbf{S}_2,\lambda}(H_{S_2})[J] &= \mathrm{bol}^{(j)}(H_{\cdot\lambda^2})[J](\mathbf{S}_2/\lambda^2) \,, \qquad (7.16) \\
\mathcal{Q}_{d,G;T,\lambda}(H)[J] &= \frac{1}{\lambda^2}\int_{\Delta[T,2]} d\mathbf{S}_2 \; \mathrm{col}_d(H_{\cdot\lambda^2})[J](\mathbf{S}_2/\lambda^2) \,, \qquad (7.17) \\
\mathcal{A}_{d;T,\lambda}(H)[J] &= \int_0^T dS \, \mathrm{abs}_{quart,d}(H_{\cdot\lambda^2})[J](S/\lambda^2) \\
&\quad + \frac{1}{\lambda^2}\int_{\Delta[T,2]} d\mathbf{S}_2 \mathrm{abs}_{cub,d}(H_{\cdot\lambda^2})[J](\mathbf{S}_2/\lambda^2) \,, \qquad (7.18)
\end{aligned}$$

where we used Proposition 5.2. $\mathcal{Q}_{d,G;T,\lambda}$ denotes a generalized collision operator and $\mathcal{A}_{d;T,\lambda}$ a generalized absorption operator. We have proved that

$$\begin{aligned}
&\left|\Psi_T + \frac{i}{N^{\frac{1}{2}}\lambda}\int_0^T dS \int_{\Lambda^*} dp \, \hat{v}(p)F_S(p)\right| \\
&\quad \leq \frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d,|\Lambda|,T}}{N^{\frac{1}{2}+\delta_{\Psi,d}}\lambda} \,, \qquad (7.19) \\
&\Big|\int_{\Lambda^*} dp \, \big(F_T(p)-F_0(p)\big)J(p) \;-\; \frac{1}{N}\Big(\int_0^T dS \, Q_{d;T-S,\lambda}(F_S)[J] \\
&\quad + \int_{\Delta[T,2]} d\mathbf{S}_2 \sum_{j=1}^{2} \lambda^j q^{(j)}_{d,F;\mathbf{S}_2,\lambda}(F_{S_2})[J]\Big)\Big| \\
&\quad \leq \frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d,|\Lambda|,T}\|J\|_{\ell^\infty(\Lambda^*)}}{N^{1+\delta_{F,d}}} \,, \qquad (7.20) \\
&\Big|\int_{\Lambda^*} dp \, G_T(p)J(p) \;-\; \frac{1}{N}\Big(\mathcal{A}_{d;T,\lambda}(F)[J] \;+\; \mathcal{Q}_{d,G;T,\lambda}(F)[J]\Big)\Big| \\
&\qquad\qquad \leq \frac{C_{\|\hat{v}\|_{w,d},\|f_0\|_d,|\Lambda|,T}\|J\|_{2\cap\infty,d}}{N^{1+\delta_{G,d}}} \qquad (7.21)
\end{aligned}$$

for all $N > 0$ larger than a universal constant.

7.2 Continuum Approximation

Recall that

$$\lambda = \Big(\frac{\log\log N}{\log N}\Big)^{\frac{r}{(7+\varepsilon)r+6}}, \tag{7.22}$$

$$L = \lambda^{-2-\frac{2}{r}-\varepsilon} = \Big(\frac{\log N}{\log\log N}\Big)^{\frac{(2+\varepsilon)r+2}{(7+\varepsilon)r+6}}. \tag{7.23}$$

7.2.1 Evolution of Φ

(6.20) and Proposition 5.9 imply that

$$\begin{aligned}\Phi_{\frac{T}{\lambda^2}} &= \frac{1}{N^{\frac{1}{2}}\lambda}\,\mathrm{Con}_c(f^{(\Phi)})(T;\lambda)\\ &\quad + \mathrm{Rem}_2(\frac{T}{\lambda^2};\Phi) + \mathrm{err}_{1,c}^{(Bog)}(\frac{T}{\lambda^2};\Phi) + \mathrm{err}_1^{(cen)}(t;\Phi)\,.\end{aligned} \tag{7.24}$$

We have that

$$\begin{aligned}|\,\mathrm{err}_1^{(cen)}(t;\Phi)| &\le C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}\langle T\rangle^3\,\frac{|\Lambda|^{\frac{3}{2}}e^{C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}|\Lambda|/\lambda T}}{N\lambda^2}\\ &\qquad\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1+\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\\ &\le \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r,T}}{N^{\frac{1}{2}+\delta_{\Psi,c}}\lambda}\end{aligned} \tag{7.25}$$

by possibly enlarging the constant C in the definition (6.40) of $\delta_{\Psi,c}$. Combining this inequality with (6.44), we find that,

$$\Big|\Phi_{\frac{T}{\lambda^2}} - \frac{1}{N^{\frac{1}{2}}\lambda}\mathrm{Con}_c(f^{(\Phi)})(T;\lambda)\Big| \le \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r,\varepsilon,T}}{N^{\frac{1}{2}+\delta_{\Psi,c}}\lambda}\,. \tag{7.26}$$

7.2.2 Evolution of f

As a consequence of (6.23), and Proposition 5.8, 5.9, we have that

$$\begin{aligned}&\int_{\Lambda^*} dp\,\big(f^{(\Phi)}_{\frac{T}{\lambda^2}}(p) - f^{(\Phi)}_0(p)\big)J(p)\\ &= \frac{1}{N}\,\mathrm{Bol}_c(f^{(\Phi)})[J](T;\lambda)\\ &\quad + \mathrm{Rem}_2(\frac{T}{\lambda^2};f[J]) + \mathrm{err}_{2,c}^{(Bog,\mathrm{Bol})}(\frac{T}{\lambda^2};f[J])\\ &\quad + J(0)\,\mathrm{err}_2^{(Bog,\mathrm{Con})}(\frac{T}{\lambda^2};f) + \mathrm{err}_{2,c}^{(cen)}(\frac{T}{\lambda^2};f[J])\,.\end{aligned} \tag{7.27}$$

Observe that

$$\begin{aligned}|\,\mathrm{err}_{2,c}^{(cen)}(\frac{T}{\lambda^2};f[J])| &\le C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r}\|J\|_\infty\langle T\rangle^4\,\frac{|\Lambda|^{\frac{3}{2}}e^{C_r\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}|\Lambda|/\lambda T}}{N^{\frac{3}{2}}\lambda^3}\\ &\qquad\Big(1+\frac{|\Lambda|}{N}\Big)\Big(1+\frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\end{aligned}$$

$$\left[1 + \langle T\rangle^2 \frac{|\Lambda|^{\frac{3}{2}} e^{C_r \|\hat{v}\|_{2(\lfloor \frac{r}{2}\rfloor+1),w,c}|\Lambda|/\lambda T}}{N^{\frac{1}{2}}\lambda}\Big(1 + \frac{|\Lambda|}{N}\Big)\Big(1 + \frac{1}{N^{\frac{1}{2}}\lambda|\Lambda|^{\frac{3}{2}}}\Big)\right]$$
$$\leq \frac{C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor \frac{r}{2}\rfloor+1),c},r,T}\|J\|_\infty}{N^{1+\delta_{F,c}}} \tag{7.28}$$

by possibly enlarging the constants in the definition (6.41) of $\delta_{F,c}(N,\lambda)$. Then, (6.45) yields

$$\Big|\int_{\Lambda^*} dp\,\big(f^{(\Phi)}_{\frac{T}{\lambda^2}}(p) - f^{(\Phi)}_0(p)\big)J(p) - \frac{1}{N}\,\mathrm{Bol}_c(f^{(\Phi)})[J](T;\lambda)\Big|$$
$$\leq \frac{C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor \frac{r}{2}\rfloor+1),c},r,\varepsilon,T}\|J\|_{W^{2\lfloor \frac{r}{2}\rfloor+2,\infty}}}{N^{1+\delta_{F,c}}}. \tag{7.29}$$

7.2.3 Evolution of *g*

(6.33) and Proposition 5.9 imply

$$\begin{aligned}
&\int_{\Lambda^*} dp\, g^{(\Phi)}_{\frac{T}{\lambda^2}}(p)J(p)\\
&= \frac{1}{N}\int_0^T dS\,\mathrm{abs}_{quart,c}(f^{(\Phi)})[J](S/\lambda^2)\\
&\quad + \frac{1}{N\lambda^2}\int_{\Delta[T,2]} d\mathbf{S}_2\,\big(\mathrm{col}_c(f^{(\Phi)})[J](\mathbf{S}_2/\lambda^2) + \mathrm{abs}_{cub,c}(f^{(\Phi)})[J](\mathbf{S}_2/\lambda^2)\big)\\
&\quad + \mathrm{Rem}_2(\frac{T}{\lambda^2}; g[J]) + \mathrm{err}^{(dis)}_1(\frac{T}{\lambda^2}; g[J]) + \mathrm{err}^{(dis)}_2(\frac{T}{\lambda^2}; g[J])\\
&\quad + J(0)\,\mathrm{err}^{(Bog,\mathrm{Con})}_2(\frac{T}{\lambda^2}; g) + \mathrm{err}^{(cen)}_1(\frac{T}{\lambda^2}; g[J]) + \mathrm{err}^{(cen)}_2(\frac{T}{\lambda^2}; g[J]).
\end{aligned} \tag{7.30}$$

Analogously to (7.28), we have that

$$|\,\mathrm{err}^{(cen)}_1(\frac{T}{\lambda^2}; g[J])|\,,\ |\,\mathrm{err}^{(cen)}_2(\frac{T}{\lambda^2}; g[J])|$$
$$\leq \frac{C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor \frac{r}{2}\rfloor+1),c},r,\varepsilon,T}\|J\|_\infty}{N^{1+\delta_{G,c}}}, \tag{7.31}$$

by again possibly enlarging the constants in the definition (6.42) of $\delta_{G,c}$. Applying (6.46), we obtain that

$$\begin{aligned}
&\Big|\int_{\Lambda^*} dp\, g^{(\Phi)}_{\frac{T}{\lambda^2}}(p)J(p) - \frac{1}{N}\int_0^T dS\,\mathrm{abs}_{quart,c}(f^{(\Phi)})[J](S/\lambda^2)\\
&\quad - \frac{1}{N\lambda^2}\int_{\Delta[T,2]} d\mathbf{S}_2\,\big(\mathrm{col}_c(f^{(\Phi)})[J](\mathbf{S}_2/\lambda^2)\\
&\quad + \mathrm{abs}_{cub,c}(f^{(\Phi)})[J](\mathbf{S}_2/\lambda^2)\big)\Big|\\
&\leq \frac{C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor \frac{r}{2}\rfloor+1),c},r,\varepsilon,T}\|J\|_{2(\lfloor \frac{r}{2}\rfloor+1),c}}{N^{1+\delta_{G,c}}}.
\end{aligned} \tag{7.32}$$

7.2.4 Conclusion for $L \sim \lambda^{-2-}$

Recall that we impose

$$\lambda = \left(\frac{\log\log N}{\log N}\right)^{\frac{r}{(7+\varepsilon)r+6}}, \tag{7.33}$$

$$L = \lambda^{-2-\frac{2}{r}-\varepsilon} = \left(\frac{\log N}{\log\log N}\right)^{\frac{(2+\varepsilon)r+2}{(7+\varepsilon)r+6}}. \tag{7.34}$$

Again, we consider the effective quantities

$$\Psi_T = \Phi_{\frac{T}{\lambda^2}}, \tag{7.35}$$

$$F_T = f^{(\Phi)}_{\frac{T}{\lambda^2}}, \tag{7.36}$$

$$G_T = g^{(\Phi)}_{\frac{T}{\lambda^2}}. \tag{7.37}$$

Let $Q_{c;T-S,\lambda}$, $\mathcal{Q}_{c,G;T,\lambda}$ and $\mathcal{A}_{c;T,\lambda}$ be defined analogously to (2.18), (2.15) respectively (2.17) with sums over Λ^* replaced by integrals $\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}$ over $\mathbb{R}^3$. Collecting (7.26), (7.29), and (7.32), we have proved that, for some possibly larger constant $N_0 = N_0(\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c})$,

$$\begin{aligned}
&\left|\Psi_T + \frac{i}{(2\pi)^3 N^{\frac{1}{2}}\lambda}\int_0^T dS \int_{\mathbb{R}^3} dp\, \hat{v}(p) F_S(p)\right| \\
&\quad\leq \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r,\varepsilon,T}}{N^{\frac{1}{2}+\delta_{\Psi,c}}\lambda}, && (7.38)\\
&\left|\int_{\Lambda^*} dp\,\big(F_T(p)-F_0(p)\big)J(p) - \frac{1}{N}\int_0^T dS\, Q_{c;T-S,\lambda}(F_S)\right| \\
&\quad\leq \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r,\varepsilon,T}\|J\|_{W^{2\lfloor\frac{r}{2}\rfloor+2,\infty}}}{N^{1+\delta_{F,c}}}, && (7.39)\\
&\left|\int_{\Lambda^*} dp\, G_T(p)J(p) - \frac{1}{N}\Big(\mathcal{A}_{c;T,\lambda}(F)[J] + \mathcal{Q}_{c,G;T,\lambda}(F)[J]\Big)\right| \\
&\quad\leq \frac{C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c},r,\varepsilon,T}\|J\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}}{N^{1+\delta_{G,c}}} && (7.40)
\end{aligned}$$

for all $N \geq N_0$.

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Declarations

Conflict of interest No conflicts of interests are connected to this article.

Appendix A. Calculus for Creation and Annihilation Operators

Lemma A.1 *Let ν be a translation invariant state, i.e.,*

$$\nu(A) \;=\; \nu(e^{ix\cdot\mathcal{P}} A e^{-ix\cdot\mathcal{P}}) \tag{A.1}$$

for all $x \in \mathbb{R}^3$ and all observables A. Then we have

$$\nu\left(\prod_{i=1}^{m} a_{p_i}^{(\sigma_i)}\right) \;=\; \frac{\delta(\sum_{i=1}^{m}\sigma_i p_i)}{|\Lambda|}\,\nu\left(\prod_{i=1}^{m} a_{p_i}^{(\sigma_i)}\right). \tag{A.2}$$

Proof By translation invariance, we have that

$$\begin{aligned}\nu\left(\prod_{i=1}^{m} a_{p_i}^{(\sigma_i)}\right) &= \nu\left(e^{ix\cdot\mathcal{P}}\prod_{i=1}^{m} a_{p_i}^{(\sigma_i)} e^{-ix\cdot\mathcal{P}}\right)\\ &= e^{ix\cdot(\sum_{i=1}^{m}\sigma_i p_i)}\,\nu\left(\prod_{i=1}^{m} a_{p_i}^{(\sigma_i)}\right)\end{aligned} \tag{A.3}$$

for all $x \in \mathbb{R}^3$. Integrating both sides $\int_\Lambda dx$, we obtain

$$|\Lambda|\nu\left(\prod_{i=1}^{m} a_{p_i}^{(\sigma_i)}\right) \;=\; \delta\left(\sum_{i=1}^{m}\sigma_i p_i\right)\nu\left(\prod_{i=1}^{m} a_{p_i}^{(\sigma_i)}\right), \tag{A.4}$$

which yields the statement. □

Lemma A.2 *(Cumulant Formula) Recall from (4.13) that*

$$\kappa_n \;=\; (-\partial_\mu)^n\big|_{\mu=0}\int_{\Lambda^*} dp\,\log\left(1 - e^{-(K(p)-\mu)}\right). \tag{A.5}$$

Then there are constants $a_{n,k} \in \mathbb{R}$ such that

$$\kappa_n \;=\; \sum_{k=1}^{n-1} a_{n,k}\int_{\Lambda^*} dp\; f_0(p)^k(1+f_0(p))^{n-k} \;+\; \delta_{n,1}\int_{\Lambda^*} dp\; f_0(p)\,, \tag{A.6}$$

where $f_0(p) = (e^{K(p)} - 1)^{-1}$.

Proof We will show, in more generality that

$$\kappa_n(\mu) := (-\partial_\mu)^n\int_{\Lambda^*} dp\,\log\left(1 - e^{-(K(p)-\mu)}\right) \tag{A.7}$$

$$= \sum_{k=1}^{n-1} a_{n,k}\int_{\Lambda^*} dp\; f_\mu(p)^k(1+f_\mu(p))^{n-k} \;+\; \delta_{n,1}\int_{\Lambda^*} dp\; f_\mu(p) \tag{A.8}$$

for all $\mu \le 0$, where $f_\mu(p) = (e^{K(p)-\mu} - 1)^{-1}$. A straightforward calculation yields

$$\kappa_1(\mu) = \int_{\Lambda^*} dp\; f_\mu(p)\,, \tag{A.9}$$

$$\kappa_2(\mu) = \int_{\Lambda^*} dp\, f_\mu(p)(1 + f_\mu(p))\,. \tag{A.10}$$

Observe that

$$-\,\partial_\mu f_\mu \;=\; f_\mu(1 + f_\mu)\,. \tag{A.11}$$

Now assume that (A.8) for some fixed $n \in \mathbb{N}$, $n \geq 2$. By definition, we have that

$$\begin{aligned}
\kappa_{n+1}(\mu) &= (-\partial_\mu)\kappa_n(\mu) \\
&= \int_{\Lambda^*} dp \sum_{k=1}^{n-1} a_{n,k}\big(k f_\mu(p)^k (1 + f_\mu(p))^{n+1-k} \\
&\quad + (n-k) f_\mu(p)^{k+1}(1 + f_\mu(p))^{n-k}\big)\,.
\end{aligned} \tag{A.12}$$

After an index shift, we can further simplify this to

$$\begin{aligned}
\kappa_{n+1}(\mu) &= a_{n,1} \int_{\Lambda^*} dp\, f_\mu(p)(1 + f_\mu(p))^n \\
&\quad + a_{n,n-1} \int_{\Lambda^*} dp\, f_\mu(p)^n (1 + f_\mu(p)) \\
&\quad + \sum_{k=2}^{n-1} (k a_{n,k} + (n+1-k) a_{n,k-1}) \\
&\quad \int_{\Lambda^*} dp\, f_\mu(p)^k (1 + f_\mu(p))^{n+1-k} \\
&= \sum_{k=1}^{n} a_{n+1,k} \int_{\Lambda^*} dp\, f_\mu(p)^k (1 + f_\mu(p))^{n+1-k}\,,
\end{aligned} \tag{A.13}$$

for some $a_{n+1,k} \in \mathbb{R}$. This finishes the proof. □

For the following standard result, we need to introduce some notation. For a proof of the statement, we refer, e.g., to [34, 35]. Given a finite ordered subset $J = \{j_1 < j_2 < \ldots < j_r\} \subset \mathbb{N}$ and $\sigma_{j_k} \in \{\pm 1\}$, we define the ordered product

$$\prod_{j \in J} a_{p_j}^{(\sigma_j)} \;:=\; a_{p_{j_1}}^{(\sigma_{j_1})} \ldots a_{p_{j_r}}^{(\sigma_{j_r})}\,. \tag{A.14}$$

In addition, we abbreviate

$$\mathbf{p}_J \;:=\; (p_{j_k})_{k=1}^r\,, \tag{A.15}$$

as well as

$$a^{(\sigma)}(\mathbf{p}_J) \;:=\; \prod_{j \in J} a_{p_j}^{(\sigma)}\,. \tag{A.16}$$

Furthermore, we define the sets

$$J_\pm \;:=\; \{j \in J \mid \sigma_j = \pm 1\} \tag{A.17}$$

and the Wick-ordered product

$$:\prod_{j \in J} a_{p_j}^{(\sigma_j)} : \;:=\; a^+(\mathbf{p}_{J_+}) a(\mathbf{p}_{J_-}) \tag{A.18}$$

with all creation operators to the left, and all annihilation operators to the right.

Finally, in order to keep track of the correct scaling, it is useful to work with the rescaled $\ell^2(\Lambda^*)$-norm

$$\|H\|_{L^2(\Lambda^*)} = \frac{1}{\sqrt{|\Lambda|}}\|H\|_{\ell^2(\Lambda^*)}, \tag{A.19}$$

see (2.1). More generally, we also define

$$\|H\|_{L^\infty_{\mathbf{p}_m}L^2_{\mathbf{k}_n}((\Lambda^*)^{m+n})} := \sup_{\mathbf{p}_m\in(\Lambda^*)^m}\left(\int_{(\Lambda^*)^n} d\mathbf{k}_n |H(\mathbf{p}_m,\mathbf{k}_n)|^2\right)^{\frac{1}{2}}, \tag{A.20}$$

$$\|H\|_{L^2_{\mathbf{k}_n}L^\infty_{\mathbf{p}_m}((\Lambda^*)^{m+n})} := \left(\int_{(\Lambda^*)^n} d\mathbf{k}_n \sup_{\mathbf{p}_m\in(\Lambda^*)^m}|H(\mathbf{p}_m,\mathbf{k}_n)|^2\right)^{\frac{1}{2}}, \tag{A.21}$$

where in the case $n=0$, this norm reduces to $\|H\|_{L^\infty_{\mathbf{p}_m}((\Lambda^*)^m)}$, and in the case $m=0$, to $\|H\|_{L^2_{\mathbf{k}_n}((\Lambda^*)^n)}$.

Lemma A.3 *(Wick's Theorem) Let $\sigma_j\in\{\pm1\}$, $p_j\in\mathbb{R}^3$ for all $j\in\{1,\ldots,n\}$, $n\in\mathbb{N}$. Then we have that*

$$\prod_{j=1}^n a_{p_j}^{(\sigma_j)} = \sum_{\substack{J\subseteq\\ \{1,\ldots,n\}}}\left\langle\Omega_0,\prod_{\substack{j\in\\ \{1,\ldots,n\}\setminus J}} a_{p_j}^{(\sigma_j)}\Omega_0\right\rangle : \prod_{j\in J} a_{p_j}^{(\sigma_j)} : . \tag{A.22}$$

Lemma A.4 *(Wick-ordered operator bound) Let $M\in\mathbb{N}_0$, $n\in\mathbb{N}$, $J:=\{1,\ldots,n\}$, $\sigma_j\in\{\pm1\}$ for all $j\in J$. Let $H:(\Lambda^*)^n\to\mathbb{C}$, and $g_j:\Lambda^*\to\mathbb{C}$ be given functions. Then the following holds true*

(1) If $J_\pm\neq\emptyset$, we have that

$$\begin{aligned}&\left\|\int_{(\Lambda^*)^n} d\boldsymbol{p}_n H(\boldsymbol{p}_n)\delta\left(\sum_{j=1}^n p_j\sigma_j\right)\prod_{j=1}^n g_j(p_j) : \prod_{j=1}^n a_{p_j}^{(\sigma_j)} : P_M\right\| \\ &\leq \left\||H|^{\frac{1}{2}}\prod_{j\in J_-} g_j(p_j)\delta\left(\sum_{j=1}^n\sigma_j p_j\right)^{\frac{1}{2}}\right\|_{L^\infty_{p_{J_+}}L^2_{p_{J_-}}} \\ &\quad \left\||H|^{\frac{1}{2}}\prod_{j\in J_+} g_j(p_j)\delta\left(\sum_{j=1}^n\sigma_j p_j\right)^{\frac{1}{2}}\right\|_{L^\infty_{p_{J_-}}L^2_{p_{J_+}}}\left(M+\sum_{j=1}^n\sigma_j\right)_{|J_+|}^{\frac{1}{2}}(M)_{|J_-|}^{\frac{1}{2}}\mathbb{1}_{M\geq|J_-|},\end{aligned}$$

where

$$(x)_m := \prod_{k=0}^{m-1}(x-k) \tag{A.23}$$

denotes the falling factorial.

(2) If $J_+=\emptyset$ and $n\geq 2$, we find that

$$\left\|\int_{(\Lambda^*)^n} d\boldsymbol{p}_n H(\boldsymbol{p}_n)\delta(\sum_{j=1}^n p_j)\prod_{j=1}^n a_{p_j} P_M\right\|$$

$$\leq \big(\|H\|^2_{L^2_{p_{n-2}}L^\infty_{p_{n-1},p_n}}(M-n+1) + |\Lambda| \|\delta(\sum_{j=1}^n p_j)^{\frac{1}{2}} H\|^2_{L^2_{p_n}}\big)^{\frac{1}{2}} (M)^{\frac{1}{2}}_{n-1} \,. \tag{A.24}$$

Similarly, in the case $J_- = \emptyset$, we have that

$$\Big\| \int_{(\Lambda^*)^n} d\boldsymbol{p}_n \, H(\boldsymbol{p}_n) \delta\left(\sum_{j=1}^n p_j\right) a^+(\boldsymbol{p}_n) P_M \Big\| \tag{A.25}$$

$$\leq \big(\|H\|^2_{L^2_{p_{n-2}}L^\infty_{p_{n-1},p_n}}(M+1) + |\Lambda| \|\delta\left(\sum_{j=1}^n p_j\right)^{\frac{1}{2}} H\|^2_{L^2_{p_n}}\big)^{\frac{1}{2}} (M+n)^{\frac{1}{2}}_{n-1} \,. \tag{A.26}$$

(3) If $n=1$, we obtain

$$\|a_0 P_M\| = \sqrt{M|\Lambda|} \,. \tag{A.27}$$

Proof Let $\Phi_+ \in \mathcal{F}_{M+\sum_{j=1}^n \sigma_j}$ and $\Phi_- \in \mathcal{F}_M$ be two normalized test functions.

In the case $J_\pm \neq \emptyset$, we have that

$$\Big|\Big\langle \Phi_+, \int_{(\Lambda^*)^n} d\mathbf{p}_n H(\mathbf{p}_n) \delta\left(\sum_{j=1}^n p_j \sigma_j\right) \prod_{j=1}^n g_j(p_j) : \prod_{j=1}^n a^{(\sigma_j)}_{p_j} : \Phi_- \Big\rangle\Big|$$

$$\leq \int_{(\Lambda^*)^n} d\mathbf{p}_n |H(\mathbf{p}_n)| \delta\left(\sum_{j=1}^n p_j \sigma_j\right) \prod_{j=1}^n |g_j(p_j)| \|a(\mathbf{p}_{J_+})\Phi_+\| \|a(\mathbf{p}_{J_-})\Phi_-\| \,, \tag{A.28}$$

where we applied Cauchy-Schwarz w.r.t. the inner product on $\mathcal{F}$. Let $\alpha \in [0,1]$ be arbitrary. Then Cauchy-Schwarz w.r.t. $d\mathbf{p}_n$ implies that we can estimate the last expression by

$$\Big(\int_{(\Lambda^*)^n} d\mathbf{p}_n \delta\left(\sum_{j=1}^n p_j \sigma_j\right) |H(\mathbf{p}_n)| \prod_{j\in J_-} |g_j(p_j)|^2 \|a(\mathbf{p}_{J_+})\Phi_+\|^2\Big)^{\frac{1}{2}}$$

$$\Big(\int_{(\Lambda^*)^n} d\mathbf{p}_n \delta\left(\sum_{j=1}^n p_j \sigma_j\right) |H(\mathbf{p}_n)| \prod_{j\in J_+} |g_j(p_j)|^2 \|a(\mathbf{p}_{J_-})\Phi_-\|^2\Big)^{\frac{1}{2}} \,. \tag{A.29}$$

Observe that the Pull-Through Formula implies that

$$\begin{aligned}\int d\mathbf{p}_m a^+(\mathbf{p}_m) a(\mathbf{p}_m) &= \int d\mathbf{p}_{m-1} a^+(\mathbf{p}_{m-1}) a(\mathbf{p}_{m-1})(\mathcal{N}_b - m + 1)_+ \\ &= (\mathcal{N}_b)_m \,,\end{aligned} \tag{A.30}$$

where we define $(\mathcal{N}_b)_m\big|_{\mathcal{F}_j} := 0$ for all $j \in \{0, 1, \ldots, m-1\}$, in accordance with (3.3). In particular, we have that $(\mathcal{N}_b)_m \geq 0$ as a quadratic form on $\mathcal{F}$.

Then, integrating the first factor first w.r.t. the momenta p_{J_-} and then p_{J_+}, and opposite for the second factor, (A.29) has the upper bound

$$\||H|^{\frac{1}{2}} \prod_{j\in J_-} g_j(p_j) \delta\left(\sum_{j=1}^n \sigma_j p_j\right)^{\frac{1}{2}} \Big\|_{L^\infty_{\mathbf{p}_{J_+}} L^2_{\mathbf{p}_{J_-}}} \|(\mathcal{N}_b)^{\frac{1}{2}}_{|J_+|} \Phi_+\Big\|$$

$$\| |H|^{\frac{1}{2}} \prod_{j \in J_+} g_j(p_j) \delta \left(\sum_{j=1}^{n} \sigma_j p_j \right)^{\frac{1}{2}} \|_{L^\infty_{\mathbf{p}_{J_-}} L^2_{\mathbf{p}_{J_+}}} \left\| (\mathcal{N}_b)^{\frac{1}{2}}_{|J_-|} \Phi_- \right\| . \tag{A.31}$$

Using the fact that $\Phi_+ \in \mathcal{F}_{M+\sum_{j=1}^n \sigma_j}$, $\Phi_- \in \mathcal{F}_M$ and that both are normalized with norm 1, we have that

$$\|(\mathcal{N}_b)^{\frac{1}{2}}_{|J_+|} \Phi_+\| \leq (M + \sum_{j=1}^{n} \sigma_j)^{\frac{1}{2}}_{|J_+|} \mathbb{1}_{M+\sum_{j=1}^n \sigma_j \geq |J_+|} , \tag{A.32}$$

$$\|(\mathcal{N}_b)^{\frac{1}{2}}_{|J_-|} \Phi_-\| \leq (M)^{\frac{1}{2}}_{|J_-|} \mathbb{1}_{M \geq |J_-|} . \tag{A.33}$$

Observe that $\sum_{j=1}^n \sigma_j = |J_+| - |J_-|$. Collecting (A.28)–(A.33), we have proved that

$$\begin{aligned}
&\left|\left\langle \Phi_+, \int_{(\Lambda^*)^n} d\mathbf{p}_n H(\mathbf{p}_n) \delta \left(\sum_{j=1}^{n} p_j \sigma_j \right) \prod_{j=1}^{n} g_j(p_j) : \prod_{j=1}^{n} a_{p_j}^{(\sigma_j)} : \Phi_- \right\rangle\right| \\
&\leq \| |H|^{\frac{1}{2}} \prod_{j \in J_-} g_j(p_j) \delta \left(\sum_{j=1}^{n} \sigma_j p_j \right)^{\frac{1}{2}} \|_{L^\infty_{\mathbf{p}_{J_+}} L^2_{\mathbf{p}_{J_-}}} \| |H|^{\frac{1}{2}} \prod_{j \in J_+} g_j(p_j) \delta \left(\sum_{j=1}^{n} \sigma_j p_j \right)^{\frac{1}{2}} \|_{L^\infty_{\mathbf{p}_{J_-}} L^2_{\mathbf{p}_{J_+}}} \\
&\quad \left(M + \sum_{j=1}^{n} \sigma_j \right)^{\frac{1}{2}}_{|J_+|} (M)^{\frac{1}{2}}_{|J_-|} \mathbb{1}_{M \geq |J_-|} .
\end{aligned} \tag{A.34}$$

In the case $J_- = \emptyset$, we have that

$$\left\| \int_{(\Lambda^*)^n} d\mathbf{p}_n \, H(\mathbf{p}_n) \delta \left(\sum_{j=1}^{n} p_j \right) a^+(\mathbf{p}_n) P_M \right\| \tag{A.35}$$

$$= \left\| \int_{(\Lambda^*)^n} d\mathbf{p}_n \, \overline{H}(\mathbf{p}_n) \delta (\sum_{j=1}^{n} p_j) a(\mathbf{p}_n) P_{M+n} \right\| , \tag{A.36}$$

which reduces to the case $J_+ = \emptyset$.

In the case $J_+ = \emptyset$ and $n \geq 2$, we find that

$$\begin{aligned}
&\left\| \int_{(\Lambda^*)^n} d\mathbf{p}_n \, H(\mathbf{p}_n) \delta \left(\sum_{j=1}^{n} p_j \right) a(\mathbf{p}_n) \Phi_- \right\|^2 \\
&= \int_{(\Lambda^*)^{2n}} d\mathbf{p}_n d\mathbf{q}_n \, \overline{H}(\mathbf{p}_n) H(\mathbf{q}_n) \delta \left(\sum_{j=1}^{n} p_j \right) \delta \left(\sum_{j=1}^{n} q_j \right) \left\langle \Phi_-, a^+(\mathbf{p}_n) a(\mathbf{q}_n) \Phi_- \right\rangle \\
&= \int_{(\Lambda^*)^{2n}} d\mathbf{p}_n d\mathbf{q}_n \, \overline{H}(\mathbf{p}_n) H(\mathbf{q}_n) \delta \left(\sum_{j=1}^{n} p_j \right) \delta \left(\sum_{j=1}^{n} q_j \right) \\
&\quad \left\langle \Phi_-, a^+(\mathbf{p}_{n-1}) a_{q_n} a^+_{p_n} a(\mathbf{q}_{n-1}) \Phi_- \right\rangle \\
&\quad - \int_{(\Lambda^*)} dp_n \, \| A_{p_n} \Phi_- \|^2 ,
\end{aligned} \tag{A.37}$$

where

$$A_{p_n} := \int_{(\Lambda^*)^{n-1}} d\mathbf{p}_{n-1}\, H(\mathbf{p}_n)\delta\left(\sum_{j=1}^{n} p_j\right) a(\mathbf{p}_{n-1})\,. \tag{A.38}$$

Using Cauchy-Schwarz first on the inner product on $\mathcal{F}$, and then w.r.t. $d\mathbf{p}_n d\mathbf{q}_n$, we find the upper bound

$$\begin{aligned}
&\Big\| \int_{(\Lambda^*)^n} d\mathbf{p}_n\, H(\mathbf{p}_n)\delta(\sum_{j=1}^{n} p_j) a(\mathbf{p}_n)\Phi_- \Big\|^2 \\
&\leq \int_{(\Lambda^*)^{2n}} d\mathbf{p}_n d\mathbf{q}_n\, |H(\mathbf{p}_n)||H(\mathbf{q}_n)|\delta\left(\sum_{j=1}^{n} p_j\right)\delta\left(\sum_{j=1}^{n} q_j\right) \\
&\quad \|a^+_{q_n} a(\mathbf{p}_{n-1})\Phi_-\|\,\|a^+_{p_n} a(\mathbf{q}_{n-1})\Phi_-\| \\
&\leq \Big(\int_{(\Lambda^*)^{2n}} d\mathbf{p}_n d\mathbf{q}_n\, \delta\left(\sum_{j=1}^{n} p_j\right)\delta\left(\sum_{j=1}^{n} q_j\right) |H(\mathbf{q}_n)|^2 \|a^+_{q_n} a(\mathbf{p}_{n-1})\Phi_-\|^2\Big)^{\frac{1}{2}} \\
&\quad \Big(\int_{(\Lambda^*)^{2n}} d\mathbf{p}_n d\mathbf{q}_n\, \delta\left(\sum_{j=1}^{n} p_j\right)\delta\left(\sum_{j=1}^{n} q_j\right) |H(\mathbf{p}_n)|^2 \|a^+_{p_n} a(\mathbf{q}_{n-1})\Phi_-\|^2\Big)^{\frac{1}{2}} \\
&= \int_{(\Lambda^*)^{2n-1}} d\mathbf{p}_{n-1} d\mathbf{q}_n\, \delta\left(\sum_{j=1}^{n} q_j\right) |H(\mathbf{q}_n)|^2 \|a_{q_n} a(\mathbf{p}_{n-1})\Phi_-\|^2 \\
&\quad + |\Lambda| \int_{(\Lambda^*)^n} d\mathbf{q}_n\, \delta\left(\sum_{j=1}^{n} q_j\right) |H(\mathbf{q}_n)|^2 \|(\mathcal{N}_b)^{\frac{1}{2}}_{n-1}\Phi_-\|^2\,,
\end{aligned} \tag{A.39}$$

where we used that $[a_{q_n}, a^+_{q_n}] = |\Lambda|$ together with (A.30). Using (A.30) again, we conclude

$$\begin{aligned}
&\Big\| \int_{(\Lambda^*)^n} d\mathbf{p}_n\, H(\mathbf{p}_n)\delta\left(\sum_{j=1}^{n} p_j\right) a(\mathbf{p}_n)\Phi_- \Big\|^2 \\
&\leq \|H\|^2_{L^2_{\mathbf{p}_{n-2}} L^\infty_{p_{n-1},p_n}} \|(\mathcal{N}_b)^{\frac{1}{2}}_n \Phi_-\|^2 + |\Lambda| \|\delta\left(\sum_{j=1}^{n} p_j\right)^{\frac{1}{2}} H\|^2_{L^2_{\mathbf{p}_n}} \|(\mathcal{N}_b)^{\frac{1}{2}}_{n-1}\Phi_-\|^2 \\
&\leq \big(\|H\|^2_{L^2_{\mathbf{p}_{n-2}} L^\infty_{p_{n-1},p_n}} (M-n+1) + |\Lambda| \|\delta\left(\sum_{j=1}^{n} p_j\right)^{\frac{1}{2}} H\|^2_{L^2_{\mathbf{p}_n}}\big)(M)_{n-1}\,.
\end{aligned} \tag{A.40}$$

Finally, we have

$$\begin{aligned}
\|a_0 P_M\|^2 &= \sup_{\|\Phi\|=1} \left\langle P_M\Phi, a_0^+ a_0 P_M\Phi\right\rangle \\
&= \left\langle P_M\Phi, a_0 a_0^+ P_M\Phi\right\rangle - |\Lambda| \|P_M\| \\
&= \|a_0^+ P_M\|^2 - |\Lambda|\,,
\end{aligned} \tag{A.41}$$

where we used that $\|P_M\| = 1$. In addition, we have that

$$\|a_0^+ P_M\| \;=\; \|P_{M+1} a_0^+\| \;=\; \|a_0 P_{M+1}\|\,. \tag{A.42}$$

(A.41) and (A.42) imply

$$\|a_0 P_M\|^2 \;=\; M|\Lambda|\,. \tag{A.43}$$

This finishes the proof. □

Appendix B. Propagation of Approximate Restricted Quasifreeness

Lemma B.1 *Let*

$$\mathcal{U}_N(t) \;:=\; \mathcal{V}^*_{HFB}(t)\mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0]e^{-it\mathcal{H}_N}\mathcal{W}[\sqrt{N|\Lambda|}\phi_0]\,, \tag{B.1}$$

where for the definition of $\mathcal{V}_{HFB}$, we refer to (3.7). Then fluctuation dynamics $\mathcal{U}_N$ obeys

$$\begin{cases} i\partial_t\mathcal{U}_N(t) & = (\mathcal{H}_{cub}(t) + \mathcal{H}_{quart}(t))\mathcal{U}_N(t), \\ \mathcal{U}_N(0) & = \mathbf{1}\,, \end{cases} \tag{B.2}$$

where $\mathcal{H}_{cub}(t)$ is defined in (2.8), $\mathcal{H}_{quart}(t)$ in (2.9).

Proof We start by defining the auxiliary dynamics

$$\widetilde{\mathcal{U}}_N(t) := \mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0]e^{-i\mathcal{H}_N t}\mathcal{W}[\sqrt{N|\Lambda|}\phi_0]. \tag{B.3}$$

We have that

$$i\partial_t\widetilde{\mathcal{U}}_N(t) \;=\; \mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0]\mathcal{H}_N\mathcal{W}[\sqrt{N|\Lambda|}\phi_0]\widetilde{\mathcal{U}}_N(t)\,. \tag{B.4}$$

Using $\phi_0 = |\Lambda|^{-1/2}$, the explicit expressions for the terms on the right hand side are given by

$$\begin{aligned} &\mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0]\,\mathcal{H}_N\,\mathcal{W}[\sqrt{N|\Lambda|}\phi_0] \\ &= \frac{N|\Lambda|\lambda}{2}\int_\Lambda dx\, v(x) \;+\; \frac{1}{2}\int_\Lambda dx\, a_x^+(-\Delta_x)a_x \\ &\quad +\lambda\sqrt{N}\int_{\Lambda^2} dx\,dy\, v(x-y)(a_y^+ + a_y) \;+\; \lambda\int_{\Lambda^2} dx\,dy\, v(x-y)a_y^+a_y \\ &\quad +\lambda\int_{\Lambda^2} dx\,dy\, v(x-y)\left(a_x^+a_y \;+\; \frac{1}{2}(a_x^+a_y^+ \;+\; a_ya_x)\right) \\ &\quad +\frac{\lambda}{\sqrt{N}}\int_{\Lambda^2} dx\,dy\, v(x-y)a_x^+(a_y + a_y^+)a_y \\ &\quad +\frac{\lambda}{2N}\int_{\Lambda^2} dx\,dy\, v(x-y)a_x^+a_y^+a_ya_x\,. \end{aligned} \tag{B.5}$$

Using the fact that $\hat{v}(0) = 0$, and recalling definitions (1.34)–(1.35), we thus obtain

$$\begin{cases} i\partial_t\widetilde{\mathcal{U}}_N(t) & = (\mathcal{H}_{HFB} + \mathcal{H}_{cub} + \mathcal{H}_{quart})\widetilde{\mathcal{U}}_N(t), \\ \widetilde{\mathcal{U}}_N(0) & = \mathbf{1}\,. \end{cases} \tag{B.6}$$

In particular, using

$$\mathcal{U}_N(t) \;=\; \mathcal{V}^*_{HFB}(t)\widetilde{\mathcal{U}}_N(t)\,, \tag{B.7}$$

a straight-forward calculation yields that $\mathcal{U}_N$ satisfies (B.2). This concludes the proof. □

Next, we adjust Proposition 3.1 in [74] to our present context.

Lemma B.2 *Assume* $|\Lambda| \geq 1$. *Let* $\widetilde{\mathcal{U}}_N$ *be defined as in (B.6). Then there are* $C_\ell, K_\ell > 0$ *such that*

$$\left\| (\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \widetilde{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|)^{-\frac{\ell}{2}} \left(1 + \frac{\mathcal{N}_b}{N|\Lambda|}\right)^{-\frac{1}{2}} \right\| \leq C_\ell e^{K_\ell \|\hat{v}\|_{w,d} \lambda |\Lambda| t} \tag{B.8}$$

for all $\ell \in \mathbb{N}$.

Proof We follow the steps of the proof in [74] and point out the differences. We show the statement by induction. Let $\psi \in \mathcal{F}$ be arbitrary.
Step 1: (B.6) implies that

$$\begin{aligned}\partial_t \left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)\widetilde{\mathcal{U}}_N(t)\psi \right\rangle &= -i \left\langle \widetilde{\mathcal{U}}_N(t)\psi, [\mathcal{N}_b, \mathcal{H}_{HFB} + \mathcal{H}_{cub} + \mathcal{H}_{quart}]\widetilde{\mathcal{U}}_N(t)\psi \right\rangle \\ &= -i \left\langle \widetilde{\mathcal{U}}_N(t)\psi, [\mathcal{N}_b, \mathcal{H}_{HFB} + \mathcal{H}_{cub}]\widetilde{\mathcal{U}}_N(t)\psi \right\rangle . \end{aligned} \tag{B.9}$$

Recalling (1.34) and (1.35), we have that

$$[\mathcal{N}_b, \mathcal{H}_{HFB}] = \lambda \int d\mathbf{p}_2\, \hat{v}(p_1)\delta(p_1 - p_2)(a^+_{p_1} a^+_{-p_2} - a_{p_1} a_{-p_2}) , \tag{B.10}$$

$$[\mathcal{N}_b, \mathcal{H}_{cub}] = \frac{\lambda}{\sqrt{N}} \int d\mathbf{p}_3\, \hat{v}(p_2)\delta(p_1 + p_2 - p_3)(a^+_{p_1} a^+_{p_2} a_{p_3} - a^+_{p_3} a_{p_2} a_{p_1}) . \tag{B.11}$$

Employing Lemma A.4, (B.9) thus yields

$$\begin{aligned} &|\partial_t \left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)\widetilde{\mathcal{U}}_N(t)\psi \right\rangle | \\ &\leq C\|\hat{v}\|_{w,d}\lambda \Big(\left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)\widetilde{\mathcal{U}}_N(t)\psi \right\rangle \\ &\quad + \frac{1}{\sqrt{N}} \left\langle \widetilde{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^{\frac{3}{2}} \widetilde{\mathcal{U}}_N(t)\psi \right\rangle \Big) \\ &\leq C\|\hat{v}\|_{w,d}\lambda \Big(\left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)\widetilde{\mathcal{U}}_N(t)\psi \right\rangle \\ &\quad + \frac{1}{N} \left\langle \widetilde{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^2 \widetilde{\mathcal{U}}_N(t)\psi \right\rangle \Big) . \end{aligned} \tag{B.12}$$

Using

$$\mathcal{W}[\sqrt{N|\Lambda|}\phi_0] a_p \mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0] = a_p - \sqrt{N}\delta(p) , \tag{B.13}$$

we derive that

$$\begin{aligned} [\mathcal{N}_b, \mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0]] &= -\sqrt{N}\mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0](a_0 + a_0^+) + N|\Lambda|\mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0] \\ &= -\big(\sqrt{N}(a_0 + a_0^+) + N|\Lambda|\big)\mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0] , \end{aligned} \tag{B.14}$$

$$[\mathcal{N}_b, \mathcal{W}[\sqrt{N|\Lambda|}\phi_0]] = \mathcal{W}[\sqrt{N|\Lambda|}\phi_0]\big(\sqrt{N}(a_0 + a_0^+) + N|\Lambda|\big) . \tag{B.15}$$

From these identities and using $[\mathcal{N}_b, \mathcal{H}_N] = 0$, we obtain that

$$\begin{aligned} [\mathcal{N}_b, \widetilde{\mathcal{U}}_N(t)] &= \Big[\mathcal{N}_b, \mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0]\Big] e^{-it\mathcal{H}_N} \mathcal{W}[\sqrt{N|\Lambda|}\phi_0] \\ &\quad + \mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0] e^{-it\mathcal{H}_N} [\mathcal{N}_b, \mathcal{W}[\sqrt{N|\Lambda|}\phi_0]] \\ &= -\sqrt{N}(a_0 + a_0^+)\widetilde{\mathcal{U}}_N(t) + \sqrt{N}\widetilde{\mathcal{U}}_N(t)(a_0 + a_0^+) . \end{aligned} \tag{B.16}$$

As a consequence, we have that

$$\begin{aligned}&\langle \widetilde{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^2 \widetilde{\mathcal{U}}_N(t)\psi\rangle \\ &\quad = \langle \mathcal{N}_b \widetilde{\mathcal{U}}_N(t)\psi, \widetilde{\mathcal{U}}_N(t)\mathcal{N}_b\psi\rangle - \sqrt{N}\langle \mathcal{N}_b \widetilde{\mathcal{U}}_N(t)\psi, (a_0 + a_0^+)\widetilde{\mathcal{U}}_N(t)\psi\rangle \\ &\qquad + \sqrt{N}\langle \mathcal{N}_b \widetilde{\mathcal{U}}_N(t)\psi, \widetilde{\mathcal{U}}_N(t)(a_0 + a_0^+)\psi\rangle . \end{aligned} \tag{B.17}$$

Using Cauchy-Schwarz, we thus obtain

$$\begin{aligned}&\langle \widetilde{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^2 \widetilde{\mathcal{U}}_N(t)\psi\rangle \\ &\quad \leq \|\mathcal{N}_b \widetilde{\mathcal{U}}_N(t)\psi\|\Big(\|\widetilde{\mathcal{U}}_N(t)\mathcal{N}_b\psi\| + \sqrt{N}\big(\|(a_0 + a_0^+)\widetilde{\mathcal{U}}_N(t)\psi\| + \|\widetilde{\mathcal{U}}_N(t)(a_0 + a_0^+)\psi\|\big)\Big) \\ &\quad \leq \|\mathcal{N}_b \widetilde{\mathcal{U}}_N(t)\psi\|\Big(\|\mathcal{N}_b\psi\| + \sqrt{N}\big(\|a_0\widetilde{\mathcal{U}}_N(t)\psi\| + \|a_0^+\widetilde{\mathcal{U}}_N(t)\psi\| \\ &\qquad + \|a_0\psi\| + \|a_0^+\psi\|\big)\Big), \end{aligned} \tag{B.18}$$

where we also applied that $\widetilde{\mathcal{U}}_N(t) : \mathcal{F} \to \mathcal{F}$ is a unitary transformation. Lemma A.4 implies

$$\begin{aligned}\|a_0\psi\|^2 &= \sum_{M=0} \|P_M a_0 \psi\|^2 \\ &= \sum_{M=0} \|a_0 P_{M-1}\psi\|^2 \\ &\leq \sum_{M=0} (M-1)|\Lambda| \|P_{M-1}\psi\|^2 \\ &= |\Lambda| \|\sqrt{\mathcal{N}_b}\psi\|^2 . \end{aligned} \tag{B.19}$$

Similarly, we have that

$$\|a_0^+\psi\| \leq \sqrt{|\Lambda|}\|\sqrt{\mathcal{N}_b + 1}\psi\| \tag{B.20}$$

Employing (B.19) and (B.20), (B.18) implies

$$\begin{aligned}&\langle \widetilde{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^2 \widetilde{\mathcal{U}}_N(t)\psi\rangle \\ &\quad \leq \|\mathcal{N}_b \widetilde{\mathcal{U}}_N(t)\psi\|\big(\|\mathcal{N}_b\psi\| + 2\sqrt{N|\Lambda|}(\|\sqrt{\mathcal{N}_b + 1}\widetilde{\mathcal{U}}_N(t)\psi\| + \|\sqrt{\mathcal{N}_b + 1}\psi\|)\big) . \end{aligned} \tag{B.21}$$

Using Young's inequality implies

$$\begin{aligned}&\langle \widetilde{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^2 \widetilde{\mathcal{U}}_N(t)\psi\rangle \\ &\quad \leq \frac{1}{2}\langle \widetilde{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^2 \widetilde{\mathcal{U}}_N(t)\psi\rangle + C\Big(\langle \psi, \big(\mathcal{N}_b^2 + N|\Lambda|(\mathcal{N}_b + 1)\big)\psi\rangle \\ &\qquad N|\Lambda|\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + 1)\widetilde{\mathcal{U}}_N(t)\psi\rangle\Big) \end{aligned} \tag{B.22}$$

As a consequence, we find that

$$\begin{aligned}&\frac{1}{N}\langle \widetilde{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^2 \widetilde{\mathcal{U}}_N(t)\psi\rangle \\ &\quad \leq C|\Lambda|\Big(\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + 1)\widetilde{\mathcal{U}}_N(t)\psi\rangle + \Big\langle \psi, \big(\mathcal{N}_b + 1 + \frac{\mathcal{N}_b^2}{N|\Lambda|}\big)\psi\Big\rangle\Big). \end{aligned} \tag{B.23}$$

Plugging this into (B.12) and using $|\Lambda| \geq 1$, we obtain that

$$
\begin{aligned}
&|\partial_t \langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)\widetilde{\mathcal{U}}_N(t)\psi\rangle| \\
&\leq C\|\hat{v}\|_{w,d}\lambda|\Lambda|\Big(\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)\widetilde{\mathcal{U}}_N(t)\psi\rangle + \Big\langle \psi, \big(\mathcal{N}_b+1+\frac{\mathcal{N}_b^2}{N|\Lambda|}\big)\psi\Big\rangle\Big).
\end{aligned} \tag{B.24}
$$

Gronwall's Lemma then implies

$$
\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)\widetilde{\mathcal{U}}_N(t)\psi\rangle \leq e^{C\|\hat{v}\|_{w,d}\lambda|\Lambda|t}\Big\langle \psi, \big(\mathcal{N}_b+|\Lambda|+\frac{\mathcal{N}_b^2}{N|\Lambda|}\big)\psi\Big\rangle. \tag{B.25}
$$

Step 2: Assume that

$$
\begin{aligned}
&\Big\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^j\widetilde{\mathcal{U}}_N(t)\psi\Big\rangle \\
&\leq C_j e^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|t}\Big\langle \psi, (\mathcal{N}_b+|\Lambda|)^j\big(1+\frac{\mathcal{N}_b}{N|\Lambda|}\big)\psi\Big\rangle
\end{aligned} \tag{B.26}
$$

for all $1\leq j\leq \ell$ and some constants C_j, K_j, and any $\psi\in\mathcal{F}$. We compute

$$
\begin{aligned}
&i\partial_t \Big\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\widetilde{\mathcal{U}}_N(t)\psi\Big\rangle \\
&= \Big\langle \widetilde{\mathcal{U}}_N(t)\psi, \big[(\mathcal{N}_b+|\Lambda|)^{\ell+1}, \mathcal{H}_{cor}^{\phi_0}+\mathcal{H}_{cub}^{\phi_0}\big]\widetilde{\mathcal{U}}_N(t)\psi\Big\rangle \\
&= \sum_{j=1}^{\ell+1}\Big\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{j-1}\big[\mathcal{N}_b, \mathcal{H}_{cor}^{\phi_0}+\mathcal{H}_{cub}^{\phi_0}\big](\mathcal{N}_b+|\Lambda|)^{\ell+1-j}\widetilde{\mathcal{U}}_N(t)\psi\Big\rangle.
\end{aligned} \tag{B.27}
$$

Let

$$
A_{cub}[\hat{v}] := \int d\mathbf{p}_3\, \hat{v}(p_2)\delta(p_1+p_2-p_3)a_{p_1}^+a_{p_2}^+a_{p_3}. \tag{B.28}
$$

Applying (B.10) and (B.11), (B.27) yields

$$
\begin{aligned}
&i\partial_t \Big\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\widetilde{\mathcal{U}}_N(t)\psi\Big\rangle \\
&= 2\sum_{j=1}^{\ell+1}\mathrm{Im}\Big\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{j-1}(g[\hat{v}]-A_{cub}[\hat{v}])(\mathcal{N}_b+|\Lambda|)^{\ell+1-j}\widetilde{\mathcal{U}}_N(t)\psi\Big\rangle \\
&= 2\sum_{j=1}^{\ell+1}\mathrm{Im}\sum_{m,n=0}^{\infty}(m+|\Lambda|)^{j-1}(n+|\Lambda|)^{\ell+1-j} \\
&\Big\langle \widetilde{\mathcal{U}}_N(t)\psi, P_m(\lambda g[\hat{v}]-\frac{\lambda}{\sqrt{N}}A_{cub}[\hat{v}])P_n\widetilde{\mathcal{U}}_N(t)\psi\Big\rangle.
\end{aligned} \tag{B.29}
$$

Observe that we have

$$
P_m g[\hat{v}]P_n = P_m g[\hat{v}]P_{m+2}\delta_{n,m+2}, \tag{B.30}
$$

$$
P_m A_{cub}[\hat{v}]P_n = P_m A_{cub}[\hat{v}]P_{m-1}\delta_{n,m-1}. \tag{B.31}
$$

Lemma A.4 and (B.29) then imply

$$
|\partial_t \Big\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\widetilde{\mathcal{U}}_N(t)\psi\Big\rangle|
$$

$$
\begin{aligned}
&\leq C\lambda \sum_{j=1}^{\ell+1} \sum_{m,n=0}^{\infty} (m+|\Lambda|)^{j-1}(n+|\Lambda|)^{\ell+1-j} \|P_m \widetilde{\mathcal{U}}_N(t)\psi\| \|P_n \widetilde{\mathcal{U}}_N(t)\psi\| \\
&\quad \big(\|g[\hat{v}]P_{m+2}\|\delta_{n,m+2} + \frac{\|A_{cub}[\hat{v}]P_{m-1}\|}{\sqrt{N}}\delta_{n,m-1} \big) \\
&\leq C\|\hat{v}\|_{w,d}\lambda \sum_{j=1}^{\ell+1} \sum_{m,n=0}^{\infty} (m+|\Lambda|)^{j-1}(n+|\Lambda|)^{\ell+1-j} \\
&\quad \big(\|P_m \widetilde{\mathcal{U}}_N(t)\psi\|^2 + \|P_n \widetilde{\mathcal{U}}_N(t)\psi\|^2 \big)\big((m+2+|\Lambda|)\delta_{n,m+2} + \frac{m^{\frac{3}{2}}}{\sqrt{N}}\delta_{n,m-1} \big) \\
&\leq C\|\hat{v}\|_{w,d}\lambda \sum_{j=1}^{\ell+1} \sum_{m=0}^{\infty} (m+|\Lambda|)^{j-1}(m+2+|\Lambda|)^{\ell+1-j} \|P_m \widetilde{\mathcal{U}}_N(t)\psi\|^2 \\
&\quad \big(m+2+|\Lambda| + \frac{(m+1)^{\frac{3}{2}}}{\sqrt{N}} \big) .
\end{aligned} \tag{B.32}
$$

We can further estimate this by

$$
\begin{aligned}
&|\partial_t \left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\widetilde{\mathcal{U}}_N(t)\psi \right\rangle | \\
&\quad \leq C(\ell+1)\|\hat{v}\|_{w,d}\lambda \Big(\left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|+2)^{\ell+1}\widetilde{\mathcal{U}}_N(t)\psi \right\rangle \\
&\qquad + \left\langle \widetilde{\mathcal{U}}_N(t)\psi, \frac{(\mathcal{N}_b+|\Lambda|+2)^{\ell+\frac{3}{2}}}{\sqrt{N}}\widetilde{\mathcal{U}}_N(t)\psi \right\rangle \Big) \\
&\quad \leq C_\ell \|\hat{v}\|_{w,d}\lambda \Big(\left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\widetilde{\mathcal{U}}_N(t)\psi \right\rangle \\
&\qquad + \frac{1}{N} \left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+2}\widetilde{\mathcal{U}}_N(t)\psi \right\rangle
\end{aligned} \tag{B.33}
$$

We claim that

$$
\begin{aligned}
&\frac{1}{N|\Lambda|} \left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi \right\rangle \\
&\quad \leq C_j \Big(e^{K_j \|\hat{v}\|_{w,d}\lambda|\Lambda|t} \left\langle \psi, (\mathcal{N}_b+|\Lambda|)^j \big(1+\frac{\mathcal{N}_b}{N|\Lambda|}\big)\psi \right\rangle \\
&\qquad + \left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^j \widetilde{\mathcal{U}}_N(t)\psi \right\rangle \Big)
\end{aligned} \tag{B.34}
$$

for all $1 \leq j \leq \ell+1$ and all $\psi \in \mathcal{F}$. (B.34) for $j = \ell+1$ together with (B.33) implies

$$
\begin{aligned}
&|\partial_t \left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\widetilde{\mathcal{U}}_N(t)\psi \right\rangle | \\
&\quad \leq C_\ell \|\hat{v}\|_{w,d}\lambda|\Lambda| \Big(\left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\widetilde{\mathcal{U}}_N(t)\psi \right\rangle \\
&\qquad + e^{K_\ell \|\hat{v}\|_{w,d}\lambda|\Lambda|t} \left\langle \psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\big(1+\frac{\mathcal{N}_b}{N|\Lambda|}\big)\psi \right\rangle \Big) .
\end{aligned} \tag{B.35}
$$

Gronwall's Lemma then implies

$$
\begin{aligned}
&\left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\widetilde{\mathcal{U}}_N(t)\psi \right\rangle | \\
&\quad \leq e^{K_\ell \|\hat{v}\|_{w,d}\lambda|\Lambda|t} \left\langle \psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\big)\psi \right\rangle
\end{aligned}
$$

$$+ C_\ell \|\hat{v}\|_{w,d}\lambda|\Lambda| t e^{K_\ell \|\hat{v}\|_{w,d}\lambda|\Lambda| t} \left\langle \psi, (\mathcal{N}_b + |\Lambda|)^{\ell+1}\big(1 + \frac{\mathcal{N}_b}{N|\Lambda|}\big)\psi \right\rangle$$
$$\leq C_\ell e^{K_\ell \|\hat{v}\|_{w,d}\lambda|\Lambda| t} \left\langle \psi, (\mathcal{N}_b + |\Lambda|)^{\ell+1}\big(1 + \frac{\mathcal{N}_b}{N|\Lambda|}\big)\psi \right\rangle . \tag{B.36}$$

Thus, proving (B.34) for $j = \ell + 1$ concludes the proof. We have proved (B.34) for $j = 1$ in Step 1, (B.23). We have that (B.34) also holds for $j = 0$, observing that

$$\begin{aligned} &\mathcal{W}[\sqrt{N|\Lambda|}\phi_0]\mathcal{N}_b\mathcal{W}^*[\sqrt{N|\Lambda|}\phi_0] \\ &\quad = \mathcal{N}_b - \sqrt{N}(a_0 + a_0^+) + N|\Lambda| \\ &\quad \leq 2(\mathcal{N}_b + 1 + N|\Lambda|)\,, \end{aligned} \tag{B.37}$$

which then commutes with $e^{-it\mathcal{H}_N}$.

Suppose (B.34) holds up to some $1 \leq j \leq \ell - 1$. Applying (B.16), we have that

$$\begin{aligned} &\frac{1}{N|\Lambda|}\left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{j+2}\widetilde{\mathcal{U}}_N(t)\psi \right\rangle \\ &\quad = \frac{1}{N|\Lambda|}\left\langle (\mathcal{N}_b + |\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi, \widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b + |\Lambda|)\psi \right\rangle \\ &\qquad - \frac{1}{\sqrt{N}|\Lambda|}\left\langle (\mathcal{N}_b + |\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi, (a_0 + a_0^+)\widetilde{\mathcal{U}}_N(t)\psi \right\rangle \\ &\qquad + \frac{1}{\sqrt{N}|\Lambda|}\left\langle (\mathcal{N}_b + |\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi, \widetilde{\mathcal{U}}_N(t)(a_0 + a_0^+)\psi \right\rangle . \end{aligned} \tag{B.38}$$

We can bound the second term by

$$\begin{aligned} &\frac{1}{\sqrt{N}|\Lambda|}\left|\left\langle (\mathcal{N}_b + |\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi, (a_0 + a_0^+)\widetilde{\mathcal{U}}_N(t)\psi \right\rangle\right| \\ &\quad \leq \alpha \left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi \right\rangle \\ &\qquad + \frac{1}{\alpha N|\Lambda|^2}\left\langle \widetilde{\mathcal{U}}_N(t)\psi, (a_0 + a_0^+)(\mathcal{N}_b + |\Lambda|)^{j+1}(a_0 + a_0^+)\widetilde{\mathcal{U}}_N(t)\psi \right\rangle . \end{aligned} \tag{B.39}$$

Employing (B.19) and (B.20), we find that, for any $\tilde{\psi} \in \mathcal{F}$,

$$\begin{aligned} &\left\langle \tilde{\psi}, (a_0 + a_0)^+(\mathcal{N}_b + |\Lambda|)^k(a_0 + a_0^+)\tilde{\psi} \right\rangle \\ &\quad \leq (\|(\mathcal{N}_b + |\Lambda|)^{\frac{k}{2}}a_0\tilde{\psi}\| + \|(\mathcal{N}_b + |\Lambda|)^{\frac{k}{2}}a_0^+\tilde{\psi}\|)^2 \\ &\quad = (\|a_0(\mathcal{N}_b + |\Lambda| - 1)^{\frac{k}{2}}\tilde{\psi}\| + \|a_0^+(\mathcal{N}_b + |\Lambda| + 1)^{\frac{k}{2}}\tilde{\psi}\|)^2 \\ &\quad \leq C_k|\Lambda|\left\langle \tilde{\psi}, (\mathcal{N}_b + |\Lambda|)^{k+1}\tilde{\psi} \right\rangle , \end{aligned} \tag{B.40}$$

i.e.,

$$(a_0 + a_0)^+(\mathcal{N}_b + |\Lambda|)^k(a_0 + a_0^+) \;\leq\; C_k|\Lambda|(\mathcal{N}_b + |\Lambda|)^{k+1} . \tag{B.41}$$

Employing (B.41) and choosing $\alpha > 0$ sufficiently large, (B.39) implies

$$\begin{aligned} &\frac{1}{\sqrt{N}|\Lambda|}\left|\left\langle (\mathcal{N}_b + |\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi, (a_0 + a_0^+)\widetilde{\mathcal{U}}_N(t)\psi \right\rangle\right| \\ &\quad \leq C_j\left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi \right\rangle \end{aligned}$$

$$+\frac{1}{4N|\Lambda|}\left\langle\widetilde{\mathcal{U}}_N(t)\psi,\,(\mathcal{N}_b+|\Lambda|)^{j+2}\widetilde{\mathcal{U}}_N(t)\psi\right\rangle. \tag{B.42}$$

We bound the third term in (B.38) by

$$\begin{aligned}
&\frac{1}{\sqrt{N}|\Lambda|}\left|\left\langle(\mathcal{N}_b+|\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi,\widetilde{\mathcal{U}}_N(t)(a_0+a_0^+)\psi\right\rangle\right|\\
&\quad\le\frac{4}{|\Lambda|}\left\langle\widetilde{\mathcal{U}}_N(t)(a_0+a_0^+)\psi,\,(\mathcal{N}_b+|\Lambda|)^{j}\widetilde{\mathcal{U}}_N(t)(a_0+a_0^+)\psi\right\rangle\\
&\quad\quad+\frac{1}{4N|\Lambda|}\left\langle\widetilde{\mathcal{U}}_N(t)\psi,\,(\mathcal{N}_b+|\Lambda|)^{j+2}\widetilde{\mathcal{U}}_N(t)\psi\right\rangle.
\end{aligned} \tag{B.43}$$

The induction hypothesis (B.26) and (B.41) hence imply

$$\begin{aligned}
&\frac{1}{\sqrt{N}|\Lambda|}\left|\left\langle(\mathcal{N}_b+|\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi,\widetilde{\mathcal{U}}_N(t)(a_0+a_0^+)\psi\right\rangle\right|\\
&\quad\le\frac{C_j}{|\Lambda|}e^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|t}\left\langle(a_0+a_0^+)\psi,\,(\mathcal{N}_b+|\Lambda|)^{j}\Big(1+\frac{\mathcal{N}_b}{N|\Lambda|}\Big)(a_0+a_0^+)\psi\right\rangle\\
&\quad\quad+\frac{1}{4N|\Lambda|}\left\langle\widetilde{\mathcal{U}}_N(t)\psi,\,(\mathcal{N}_b+|\Lambda|)^{j+2}\widetilde{\mathcal{U}}_N(t)\psi\right\rangle\\
&\quad\le C_je^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|t}\left\langle\psi,\,(\mathcal{N}_b+|\Lambda|)^{j+1}\Big(1+\frac{\mathcal{N}_b}{N|\Lambda|}\Big)\psi\right\rangle\\
&\quad\quad+\frac{1}{4N|\Lambda|}\left\langle\widetilde{\mathcal{U}}_N(t)\psi,\,(\mathcal{N}_b+|\Lambda|)^{j+2}\widetilde{\mathcal{U}}_N(t)\psi\right\rangle
\end{aligned} \tag{B.44}$$

For the first term in (B.38), we apply (B.16) to the left and obtain

$$\begin{aligned}
&\frac{1}{N|\Lambda|}\left\langle(\mathcal{N}_b+|\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi,\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\right\rangle\\
&\quad=\frac{1}{N|\Lambda|}\left\langle\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi,\,(\mathcal{N}_b+|\Lambda|)^{j}\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\right\rangle\\
&\quad\quad-\frac{1}{\sqrt{N}|\Lambda|}\left\langle(\mathcal{N}_b+|\Lambda|)^{j}(a_0+a_0^+)\widetilde{\mathcal{U}}_N(t)\psi,\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\right\rangle\\
&\quad\quad+\frac{1}{\sqrt{N}|\Lambda|}\left\langle(\mathcal{N}_b+|\Lambda|)^{j}\widetilde{\mathcal{U}}_N(t)(a_0+a_0^+)\psi,\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\right\rangle.
\end{aligned} \tag{B.45}$$

For the first term in (B.45), we use the induction hypothesis (B.34), and obtain

$$\begin{aligned}
&\frac{1}{N|\Lambda|}\left\langle\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi,\,(\mathcal{N}_b+|\Lambda|)^{j}\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\right\rangle\\
&\quad\le C_j\Big(e^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|t}\left\langle\psi,\,(\mathcal{N}_b+|\Lambda|)^{j+1}\Big(1+\frac{\mathcal{N}_b}{N|\Lambda|}\Big)\psi\right\rangle\\
&\quad\quad+\left\langle\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi,\,(\mathcal{N}_b+|\Lambda|)^{j-1}\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\right\rangle\Big)\\
&\quad\le C_je^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|t}\left\langle\psi,\,(\mathcal{N}_b+|\Lambda|)^{j+1}\Big(1+\frac{\mathcal{N}_b}{N|\Lambda|}\Big)\psi\right\rangle,
\end{aligned} \tag{B.46}$$

where in the last step we employed (B.26). Using Cauchy-Schwarz, followed by Young's inequality, the second term in (B.45) can be bounded by

$$\begin{aligned}
&\frac{1}{\sqrt{N}|\Lambda|}\Big|\Big\langle(\mathcal{N}_b+|\Lambda|)^j(a_0+a_0^+)\widetilde{\mathcal{U}}_N(t)\psi,\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\Big\rangle\Big| \\
&\leq \frac{1}{\sqrt{N}|\Lambda|}\|(\mathcal{N}_b+|\Lambda|)^{\frac{j}{2}}(a_0+a_0^+)\widetilde{\mathcal{U}}_N(t)\psi\|\|(\mathcal{N}_b+|\Lambda|)^{\frac{j}{2}}\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\| \\
&\leq \frac{1}{|\Lambda|}\Big(\Big\langle\widetilde{\mathcal{U}}_N(t)\psi,(a_0+a_0^+)(\mathcal{N}_b+|\Lambda|)^j(a_0+a_0^+)\widetilde{\mathcal{U}}_N(t)\psi\Big\rangle \\
&\quad+\frac{1}{N}\Big\langle\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi,(\mathcal{N}_b+|\Lambda|)^j\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\Big\rangle\Big).
\end{aligned} \tag{B.47}$$

Employing (B.41) and then (B.46), we thus obtain

$$\begin{aligned}
&\frac{1}{\sqrt{N}|\Lambda|}\Big|\Big\langle(\mathcal{N}_b+|\Lambda|)^j(a_0+a_0^+)\widetilde{\mathcal{U}}_N(t)\psi,\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\Big\rangle\Big| \\
&\leq C_j\Big\langle\widetilde{\mathcal{U}}_N(t)\psi,(\mathcal{N}_b+|\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi\Big\rangle \\
&\quad+\frac{1}{N|\Lambda|}\Big\langle\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi,(\mathcal{N}_b+|\Lambda|)^j\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\Big\rangle \\
&\leq C_j\Big\langle\widetilde{\mathcal{U}}_N(t)\psi,(\mathcal{N}_b+|\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi\Big\rangle \\
&\quad+C_je^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|t}\Big\langle\psi,(\mathcal{N}_b+|\Lambda|)^{j+1}\big(1+\frac{\mathcal{N}_b}{N|\Lambda|}\big)\psi\Big\rangle.
\end{aligned} \tag{B.48}$$

Similarly, the third term in (B.45) can be estimated using

$$\begin{aligned}
&\frac{1}{\sqrt{N}|\Lambda|}\Big|\Big\langle(\mathcal{N}_b+|\Lambda|)^j\widetilde{\mathcal{U}}_N(t)(a_0+a_0^+)\psi,\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\Big\rangle\Big| \\
&\leq \frac{1}{|\Lambda|}\Big\langle\widetilde{\mathcal{U}}_N(t)(a_0+a_0^+)\psi,(\mathcal{N}_b+|\Lambda|)^j\widetilde{\mathcal{U}}_N(t)(a_0+a_0^+)\psi\Big\rangle \\
&\quad+\frac{1}{N|\Lambda|}\Big\langle\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi,(\mathcal{N}_b+|\Lambda|)^j\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\Big\rangle \\
&\leq \frac{C_j}{|\Lambda|}e^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|t}\Big\langle\psi,(a_0+a_0^+)(\mathcal{N}_b+|\Lambda|)^j\big(1+\frac{\mathcal{N}_b}{N|\Lambda|}\big)(a_0+a_0^+)\psi\Big\rangle \\
&\quad+C_je^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|t}\Big\langle\psi,(\mathcal{N}_b+|\Lambda|)^{j+1}\big(1+\frac{\mathcal{N}_b}{N|\Lambda|}\big)\psi\Big\rangle \\
&\leq C_je^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|t}\Big\langle\psi,(\mathcal{N}_b+|\Lambda|)^{j+1}\big(1+\frac{\mathcal{N}_b}{N|\Lambda|}\big)\psi\Big\rangle,
\end{aligned} \tag{B.49}$$

where we used (B.26) and (B.46), followed by (B.41). Inserting (B.46), (B.48), and (B.49) into (B.45), we obtain that

$$\begin{aligned}
&\frac{1}{N|\Lambda|}\Big\langle(\mathcal{N}_b+|\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi,\widetilde{\mathcal{U}}_N(t)(\mathcal{N}_b+|\Lambda|)\psi\Big\rangle \\
&\leq C_je^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|t}\Big\langle\psi,(\mathcal{N}_b+|\Lambda|)^{j+1}\big(1+\frac{\mathcal{N}_b}{N|\Lambda|}\big)\psi\Big\rangle \\
&\quad+C_j\Big\langle\widetilde{\mathcal{U}}_N(t)\psi,(\mathcal{N}_b+|\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi\Big\rangle.
\end{aligned} \tag{B.50}$$

Plugging (B.42), (B.44), and (B.50) into (B.38), we find that

$$
\begin{aligned}
&\frac{1}{N|\Lambda|}\left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{j+2}\widetilde{\mathcal{U}}_N(t)\psi\right\rangle \\
&\leq C_j e^{K_j \|\hat{v}\|_{w,d}\lambda|\Lambda|t}\left\langle \psi, (\mathcal{N}_b+|\Lambda|)^{j+1}\big(1+\frac{\mathcal{N}_b}{N|\Lambda|}\big)\psi\right\rangle \\
&\quad + C_j\left\langle \widetilde{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{j+1}\widetilde{\mathcal{U}}_N(t)\psi\right\rangle . \qquad \text{(B.51)}
\end{aligned}
$$

This concludes the proof. □

Appendix C. Proof of Convergence to Mean Field Equations

Lemma C.1 *Let $H, F \in C^1(\mathbb{R}^n)$ be such that*

$$
\left\| \nabla\cdot\Big(\frac{H\nabla F}{|\nabla F|}\Big)\right\|_1 < \infty . \qquad \text{(C.1)}
$$

Then the following holds true for all $\omega\in\mathbb{R}$ and $g\in C_0^1(\mathbb{R})$:

(1) $\int_{F=\omega}\frac{d\mathcal{H}^{n-1}}{|\nabla F|}H = \int_{F<\omega} dp\, \nabla\cdot\Big(\frac{H\nabla F}{|\nabla F|^2}\Big)$,

(2) $\int d\omega\, g(\omega)\partial_\omega \int_{F=\omega}\frac{d\mathcal{H}^{n-1}}{|\nabla F|}H = \int d\omega\, g(\omega)\int_{F=\omega}\frac{d\mathcal{H}^{n-1}}{|\nabla F|}\nabla\cdot\Big(\frac{H\nabla F}{|\nabla F|^2}\Big)$.

Proof The first statement is a direct consequence of the Divergence Theorem together with the fact that $\nabla F/|\nabla F|$ is the outer normal for $\{F<\omega\}$. For the second statement, we have that the divergence theorem implies that

$$
\int d\omega\, g(\omega)\partial_\omega \int_{F=\omega}\frac{d\mathcal{H}^{n-1}}{|\nabla F|}H = -\int d\omega\, g'(\omega)\int_{F=\omega}\frac{d\mathcal{H}^{n-1}}{|\nabla F|}H , \qquad \text{(C.2)}
$$

where we used the assumptions $g\in C_0^1(\mathbb{R})$, (C.1), and the first statement. Employing the Coarea Formula, we have that

$$
\int d\omega\, g(\omega)\partial_\omega \int_{F=\omega}\frac{d\mathcal{H}^{n-1}}{|\nabla F|}H = -\int dp\, g'(F)H . \qquad \text{(C.3)}
$$

Expanding with the factor $\frac{\nabla F\cdot\nabla F}{|\nabla F|^2}$, the Divergence Theorem, together with $g\in C_0^1(\mathbb{R})$, (C.1), then implies

$$
\int d\omega\, g(\omega)\partial_\omega \int_{F=\omega}\frac{d\mathcal{H}^{n-1}}{|\nabla F|}H = \int dp\, g(F)\nabla\cdot\Big(\frac{H\nabla F}{|\nabla F|^2}\Big) . \qquad \text{(C.4)}
$$

Finally, the Coarea Formula implies

$$
\int d\omega\, g(\omega)\partial_\omega \int_{F=\omega}\frac{d\mathcal{H}^{n-1}}{|\nabla F|}H = \int d\omega\, g(\omega)\int_{F=\omega}\frac{d\mathcal{H}^{n-1}}{|\nabla F|}\nabla\cdot\Big(\frac{H\nabla F}{|\nabla F|^2}\Big) . \qquad \text{(C.5)}
$$

This finishes the proof. □

Lemma C.2 *Let $H, F_1, F_2 \in C^2(\mathbb{R}^6)$ be s.t.*

$$
H\, ,\ \nabla\cdot\frac{\nabla F_2 |H|}{|\nabla F_2|^2} \in L^1(\mathbb{R}^6) . \qquad \text{(C.6)}
$$

Then, for all $t \geq 0$, we have that

$$\Big| \int d\boldsymbol{p}_2 \int_{\Delta[t,2]} d\boldsymbol{s}_2 \, e^{-i(F_1(\boldsymbol{p}_2)s_1 + F_2(\boldsymbol{p}_2)s_2)} H(\boldsymbol{p}_2) \Big| \\ \leq Ct \Big(\log(t+1) \Big\| \nabla \cdot \frac{\nabla F_2 |H|}{|\nabla F_2|^2} \Big\|_1 + \|H\|_1 \Big). \tag{C.7}$$

Proof Observe that

$$\begin{aligned} \Big| \int_{\Delta[t,2]} d\mathbf{s}_2 \, e^{-i(\omega_1 s_1 + \omega_2 s_2)} \Big| &\leq \int_0^t ds_1 \Big| \int_0^{s_1} ds_2 \, e^{-i\omega_2 s_2} \Big| \\ &\leq \int_0^t ds_1 \frac{2}{|\omega_2| + \frac{1}{s_1}} \\ &\leq \frac{2t}{|\omega_2| + \frac{1}{t}} \end{aligned} \tag{C.8}$$

for all $\omega_1, \omega_2 \in \mathbb{R}$, $t \geq 0$, due to the fact that

$$\Big| \int_{x_1}^{x_2} dy \, e^{-iay} \Big| \leq \min\{x_2 - x_1; \frac{2}{|a|}\} \leq \frac{2}{|a| + \frac{1}{x_2 - x_1}} \tag{C.9}$$

for all $x_1, x_2, a \in \mathbb{R}$, $x_2 \geq x_1$.

Then (C.8) followed by the Coarea Formula implies

$$\begin{aligned} &\Big| \int d\mathbf{p}_2 \int_{\Delta[t,2]} d\mathbf{s}_2 \, e^{-i(F_1(\mathbf{p}_2)s_1 + F_2(\mathbf{p}_2)s_2)} H(\mathbf{p}_2) \Big| \\ &\leq Ct \int d\mathbf{p}_2 \frac{|H(\mathbf{p}_2)|}{|F_2(\mathbf{p}_2)| + \frac{1}{t}} \\ &\leq Ct \int d\omega \frac{1}{|\omega| + \frac{1}{t}} \int_{F_2 = \omega} d\mathcal{H}^5(\mathbf{p}_2) \frac{|H(\mathbf{p}_2)|}{|\nabla F_2(\mathbf{p}_2)|}. \end{aligned} \tag{C.10}$$

Lemma C.1 yields

$$\begin{aligned} \Big| \int_{F_2 = \omega} d\mathcal{H}^5(\mathbf{p}_2) \frac{|H(\mathbf{p}_2)|}{|\nabla F_2(\mathbf{p}_2)|} \Big| &\leq \int_{F_2 > \omega} d\mathbf{p}_2 \Big| \nabla \cdot \frac{\nabla F_2(\mathbf{p}_2) |H(p_2)|}{|\nabla F_2(\mathbf{p}_2)|^2} \Big| \\ &\leq \Big\| \nabla \cdot \frac{\nabla F_2 |H|}{|\nabla F_2|^2} \Big\|_1 . \end{aligned} \tag{C.11}$$

Then, after splitting the domain of integration w.r.t. $d\omega$ into $(-1, 1) \cup (-1, 1)^c$, (C.10) can be bounded by

$$\begin{aligned} &\int_{-1}^{1} d\omega \frac{1}{|\omega| + \frac{1}{t}} \sup_\omega \Big| \int_{F_2 = \omega} d\mathcal{H}^5(\mathbf{p}_2) \frac{|H(\mathbf{p}_2)|}{|\nabla F_2(\mathbf{p}_2)|} \Big| \\ &\quad + \int d\mathbf{p}_2 \, \mathbb{1}_{|F_2(\mathbf{p}_2)| \geq 1} |H(\mathbf{p}_2)| \\ &\leq Ct \Big(\log(t+1) \Big\| \nabla \cdot \frac{\nabla F_2 |H|}{|\nabla F_2|^2} \Big\|_1 + \|H\|_1 \Big), \end{aligned} \tag{C.12}$$

where, again, we used the Coarea Formula. □

Lemma C.3 *Let $h \in C_b^n([0, \|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}])$, $n \in \mathbb{N}$, $n \leq 3$. Then the following holds for all $\lambda \in (0,1]$. We have that*

$$\left\| D^n\left[h\left(\frac{\lambda\hat{v}}{E}\right)\right]\right\|_\infty \leq C_n\lambda\|h\|_{C^n[0,\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\lambda]}\|\hat{v}\|^n_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\,. \tag{C.13}$$

Moreover, Ω satisfies

$$\nabla\Omega(p) = (1+\lambda m_\Omega(p))p \tag{C.14}$$

$$\|m_\Omega\|_\infty \leq C\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\,, \tag{C.15}$$

$$\|D^2\Omega - I\|_\infty \leq C\lambda\|\hat{v}\|^2_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\,,$$

$$\|D^3\Omega\|_\infty \leq C\lambda\|\hat{v}\|^3_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\,. \tag{C.16}$$

Proof By the Faà di Bruno formula, we have that

$$|D^n\left[h\left(\frac{\lambda\hat{v}}{E}\right)\right]| \leq C_n \sum_{\mathbf{r}_n\in R(n)} \left|h^{(S(\mathbf{r}_n))}\left(\frac{\lambda\hat{v}}{E}\right)\right|\lambda^{S(\mathbf{r}_n)}\prod_{j=1}^n\left[\left|D^j\left(\frac{\hat{v}}{E}\right)\right|\right]^{r_j}\,, \tag{C.17}$$

where, $R(n)$ is defined as in (5.151) and

$$S(\mathbf{r}_n) := \sum_{k=1}^n r_k\,, \tag{C.18}$$

see (5.152). Notice that $S(\mathbf{r}_n)$ satisfies

$$1 \leq S(\mathbf{r}_n) \leq n \tag{C.19}$$

due to the summation condition

$$\sum_{j=1}^n jr_j = n\,. \tag{C.20}$$

(C.19) allows us to extract a factor λ in (C.17) since it appears within the sum with power $S(\mathbf{r}_n)$. Then (C.17) implies

$$\left\| D^n\left[h\left(\frac{\lambda\hat{v}}{E}\right)\right]\right\|_\infty \leq C_n\lambda\|h\|_{C^n[0,\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\lambda]}\|\hat{v}\|^n_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\,. \tag{C.21}$$

With analogous steps, we have that

$$\left\| \partial^n_{|p|}\left[h\left(\frac{\lambda\hat{v}}{E}\right)\right]\right\|_\infty \leq C_n\lambda\|h\|_{C^n[0,\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\lambda]}\|\hat{v}\|^n_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\,. \tag{C.22}$$

We have that $\Omega = E\sqrt{1+\frac{2\lambda\hat{v}}{E}}$, see (3.19), is radial since $\hat{v}$ is radial. Thus we have that

$$\nabla\Omega(p) = p\left[\sqrt{1+\frac{2\lambda\hat{v}(p)}{E(p)}} + \frac{|p|}{2}\left(\sqrt{1+\frac{2\lambda\hat{v}}{E}}\right)'(p)\right]. \tag{C.23}$$

Observe that

$$\left|\sqrt{1+\frac{2\lambda\hat{v}}{E}} - 1\right| \leq C\frac{\lambda\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}}{\sqrt{1+\frac{2\lambda\hat{v}}{E}}+1}$$

$$\leq C\lambda \|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c} \,. \tag{C.24}$$

Then (C.22), and (C.24) imply that

$$\nabla \Omega(p) = (1+\lambda m_\Omega(p))p \tag{C.25}$$

$$\|m_\Omega\|_\infty \leq C\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c} \,. \tag{C.26}$$

Next, using the multivariate Leibniz rule together with (C.24), we then find that

$$\begin{aligned} D^2\Omega &= \sqrt{1+\frac{2\lambda\hat{v}}{E}}I + 2p \vee D\sqrt{1+\frac{2\lambda\hat{v}}{E}} + ED^2\sqrt{1+\frac{2\lambda\hat{v}}{E}} \\ &= I + \lambda A \end{aligned} \tag{C.27}$$

for some bounded matrix A. Using (C.21) with $h(x) = \sqrt{1+2x}$, and (C.24), we have that

$$\|A\|_\infty \leq C\|\hat{v}\|^2_{2(\lfloor \frac{r}{2} \rfloor+1),w,c} \tag{C.28}$$

for all $\lambda \in (0, 1]$. Here, we used the fact that $\|h\|_{C^2([0,\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}])} \leq C$.

Finally, the multivariate Leibniz rule implies

$$\begin{aligned} \|D^3\Omega\|_\infty &\leq C\sum_{k=0}^{3}\Big\|D^k(|p|^2)D^{3-k}\sqrt{1+\frac{2\lambda\hat{v}}{E}}\Big\|_\infty \\ &\leq C\sum_{k=0}^{2}\Big\||p|^{2-k}D^{3-k}\sqrt{1+\frac{2\lambda\hat{v}}{E}}\Big\|_\infty \,. \end{aligned} \tag{C.29}$$

Using (C.17) and the fact that

$$\||p|^{2-k}D^j\big(\frac{\hat{v}}{E}\big)\|_\infty \leq C\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c} \tag{C.30}$$

for $j \in \{1, 2, 3\}$, we find that

$$\|D^3\Omega\|_\infty \leq C\lambda\|\hat{v}\|^3_{2(\lfloor \frac{r}{2} \rfloor+1),w,c} \,. \tag{C.31}$$

This concludes the proof. □

Lemma C.4 *Let*

$$F_2(\boldsymbol{p}_2) := \sigma_1\Omega(p_1) + \sigma_2\Omega(p_2) + \sigma_{12}\Omega(p_{12}) \tag{C.32}$$

and

$$T := \begin{pmatrix} (\sigma_1+\sigma_{12})I & \sigma_{12}I \\ \sigma_{12}I & (\sigma_2+\sigma_{12})I \end{pmatrix} . \tag{C.33}$$

F_2 satisfies

$$\nabla F_2(0) = 0 \,, \tag{C.34}$$

$$\|\det(D^2F_2) - \sigma_{12}(\sigma_{12}\sigma_1\sigma_2 + \sigma_1 + \sigma_2)^3\|_\infty \leq C\lambda\|\hat{v}\|^2_{2(\lfloor \frac{r}{2} \rfloor+1),w,c} \,, \tag{C.35}$$

$$\|D^2F_2\|_\infty \leq C \tag{C.36}$$

For all $\boldsymbol{p}_2, \boldsymbol{k}_2 \in \mathbb{R}^6$, and all $\lambda > 0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}$, we have that

$$|D^2F_2(\boldsymbol{k}_2)\boldsymbol{p}_2| \in \Big[\frac{|T\boldsymbol{p}_2|}{2}, \frac{3|T\boldsymbol{p}_2|}{2}\Big] . \tag{C.37}$$

In particular, in this case, we have

$$|D^2 F_2(\boldsymbol{k}_2)\boldsymbol{p}_2| \;\geq\; \begin{cases} |\boldsymbol{p}_2|/2\,, & \textit{if}\ \sigma_1 = \sigma_2 = \pm\sigma_{12}\,, \\ |p_1|/2\,, & \textit{if}\ \sigma_1 = -\sigma_2 = \sigma_{12}\,, \\ |p_2|/2\,, & \textit{if}\ -\sigma_1 = \sigma_2 = \sigma_{12}\,. \end{cases}$$

Proof Thanks to Lemma C.3, we have that

$$\nabla\Omega(p) \;=\; p(1 + \lambda m_\Omega(p)) \tag{C.38}$$

with

$$\|m_\Omega\|_\infty \;\leq\; C\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\,. \tag{C.39}$$

This immediately implies (C.34).

Next, again due to Lemma C.3, we have that

$$D^2\Omega \;=\; I + \lambda A \tag{C.40}$$

for some matrix $A \in \mathbb{R}^{3\times 3}$ with

$$\|A\|_\infty \;\leq\; C\|\hat{v}\|^2_{2(\lfloor\frac{r}{2}\rfloor+1),w,c} \tag{C.41}$$

for all $\lambda \in (0,1)$. Denoting $A_j := A(p_j)$, we obtain that

$$\begin{aligned} D^2 F_2 \;&=\; \begin{pmatrix} \sigma_1(I+\lambda A_1) + \sigma_{12}(I+\lambda A_{12}) & \sigma_{12}(I+\lambda A_{12}) \\ \sigma_{12}(I+\lambda A_{12}) & \sigma_2(I+\lambda A_2) + \sigma_{12}(I+\lambda A_{12}) \end{pmatrix} \\ &=\; \begin{pmatrix} (\sigma_1+\sigma_{12})I & \sigma_{12}I \\ \sigma_{12}I & (\sigma_2+\sigma_{12})I \end{pmatrix} + \lambda \begin{pmatrix} \sigma_1 A_1 + \sigma_{12}A_{12} & \sigma_{12}A_{12} \\ \sigma_{12}A_{12} & \sigma_2 A_2 + \sigma_{12}A_{12} \end{pmatrix} \\ &=: T + \lambda\tilde{B}\,. \end{aligned} \tag{C.42}$$

This identity together with (C.41) immediately implies (C.36). Notice that due to (C.41), we have that

$$\|\tilde{B}\|_\infty \;\leq\; C\|\hat{v}\|^2_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\,. \tag{C.43}$$

Now, observe that

$$\begin{aligned} \det(T) &= \det\begin{pmatrix} 0 & -(\sigma_{12}\sigma_1\sigma_2 + \sigma_1 + \sigma_2)I \\ \sigma_{12}I & (\sigma_2+\sigma_{12})I \end{pmatrix} \\ &= -\det\begin{pmatrix} \sigma_{12}I & (\sigma_2+\sigma_{12})I \\ 0 & -(\sigma_{12}\sigma_1\sigma_2 + \sigma_1 + \sigma_2)I \end{pmatrix} \\ &= \sigma_{12}(\sigma_{12}\sigma_1\sigma_2 + \sigma_1 + \sigma_2)^3 \;\neq\; 0\,. \end{aligned} \tag{C.44}$$

Thus we may rewrite $T + \lambda\tilde{B} =: (I + \lambda B)T$ with

$$\|B\|_\infty \;\leq\; C\|\hat{v}\|^2_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\,. \tag{C.45}$$

due to (C.43) and (C.44). Then, we have that

$$\|\det(I+\lambda B) - 1\|_\infty \;\leq\; C\lambda\|B\|_\infty \tag{C.46}$$

for all $\lambda \in (0,1)$.

Finally, Gershgorin's Circle Theorem implies that

$$\sigma(|I+\lambda B|^2) \subseteq [1 - 2\lambda\|B\|_\infty - \lambda^2\|B\|^2_\infty,\, 1 + 2\lambda\|B\|_\infty + \lambda^2\|B\|^2_\infty]$$

$$\subseteq \left[\frac{1}{4}, \frac{9}{4}\right] \tag{C.47}$$

for all $\lambda \in (0,1)$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}$. Then we have that

$$|D^2 F_2(\mathbf{k}_2)\mathbf{p}_2|^2 \;=\; (T\mathbf{p}_2)^T |I + \lambda B(\mathbf{k}_2)|^2 T\mathbf{p}_2\,, \tag{C.48}$$

which together with (C.47) implies that

$$|D^2 F_2(\mathbf{k}_2)\mathbf{p}_2| \in \left[\frac{|T\mathbf{p}_2|}{2}, \frac{3|T\mathbf{p}_2|}{2}\right]. \tag{C.49}$$

This finishes the proof. □

C.1. Error Bounds Due to HFB Evolution

Lemma C.5 *We have that*

$$\frac{\lambda^j}{N} |\operatorname{Bol}_c^{(j)}(f_0)[J](T;\lambda)| \\ \leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c},\|f_0\|_{2(\lfloor \frac{r}{2} \rfloor+1),c},r}\, T(1+\log(T))\|J\|_{W^{1,\infty}} \frac{\lambda|\log(\lambda)|}{N} \tag{C.50}$$

for $j \in \{1,2\}$, and all $\lambda > 0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}$.

Proof $\frac{\lambda^j}{N} |\operatorname{Bol}_c^{(j)}(f_0)[J](T;\lambda)|$ are of the form

$$\frac{\lambda^{j-2}}{N} \int_{\mathbb{R}^6} d\mathbf{p}_2 \int_{\Delta[T,2]} d\mathbf{S}_2\, e^{-i(F_1(\mathbf{p}_2)S_1+F_2(\mathbf{p}_2)S_2)/\lambda^2} H(\mathbf{p}_2) \tag{C.51}$$

with

$$F_2(\mathbf{p}_2) \;=\; \sigma_1 \Omega(p_1) + \sigma_2 \Omega(p_2) + \sigma_{12}\Omega(p_{12}) \tag{C.52}$$

for some $\sigma_j \in \{\pm 1\}$ and $H : \mathbb{R}^6 \to \mathbb{R}$. Employing the bound on $D^2 F_2$ given in Lemma C.4 and the diamagnetic inequality $|\nabla|h|| \leq |\nabla h|$, we find the estimate

$$\left\|\nabla \cdot \frac{\nabla F_2 |H|}{|\nabla F_2|^2}\right\|_1 \leq \left\|\frac{\Delta F_2 H}{|\nabla F_2|^2}\right\|_1 + \left\|\frac{\nabla F_2^T D^2 F_2 \nabla F_2 H}{|\nabla F_2|^4}\right\|_1 + \left\|\frac{\nabla |H|}{|\nabla F_2|}\right\|_1 \\ \leq C\Big(\left\|\frac{H}{|\nabla F_2|^2}\right\|_1 + \left\|\frac{\nabla H}{|\nabla F_2|}\right\|_1\Big). \tag{C.53}$$

Let $B_{F_2} := \{|\nabla F_2| \leq 1\}$ denote the unit ball induced by ∇F_2, and let $J_{\nabla F_2} := |\det(D^2 F_2)|$. Notice that for any $m \in [0,4)$ and test function h, we have that

$$\left\|\frac{h}{|\nabla F_2|^m}\right\|_1 \leq \left\|\frac{J_{\nabla F_2}^{\frac{12-m}{24-3m}}}{|\nabla F_2|^m}\right\|_{L^{\frac{24-3m}{12-m}}(B_{F_2})} \left\|\frac{h}{J_{\nabla F_2}^{\frac{12-m}{24-3m}}}\right\|_{L^{\frac{24-3m}{12-2m}}(B_{F_2})} \\ + \left\|\frac{h}{|\nabla F_2|^m}\right\|_{L^1(B_{F_2}^c)} \\ \leq C\big(\|h\|_{\frac{24-3m}{12-2m}} + \|h\|_1\big) \tag{C.54}$$

for all $\lambda > 0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}$. Here, we used Lemma C.4 to bound $J_{\nabla F_2} \geq C$ for $\lambda \in (0,1)$ small enough, substitution, and the fact that

$$\int_{B_1} \frac{d\mathbf{x}_6}{|\mathbf{x}_6|^{m\frac{24-3m}{12-m}}} \leq C \int_0^1 \frac{d|\mathbf{x}_6|}{|\mathbf{x}_6|^{m\frac{24-3m}{12-m}-5}} < \infty \tag{C.55}$$

since, due to $m < 4$,

$$m\frac{24-3m}{12-m} - 5 = -3\frac{(m-5)^2-1}{12-m} + 1 < 1. \tag{C.56}$$

In particular, (C.53) and (C.54) imply

$$\begin{aligned}\Big\| \nabla \cdot \frac{\nabla F_2 |H|}{|\nabla F_2|^2} \Big\|_1 &\leq C\big(\|H\|_{W^{1,1}} + \|H\|_{\frac{9}{4}} + \|\nabla H\|_{\frac{21}{10}}\big) \\ &\leq C\big(\|H\|_{W^{1,1}} + \|H\|_{W^{1,\frac{9}{4}}}\big),\end{aligned} \tag{C.57}$$

where we employed the fact that $m \mapsto \frac{24-3m}{12-2m}$ is an increasing function, and interpolation. With that, we apply Lemma C.2 to obtain

$$\begin{aligned}&\Big| \int d\mathbf{p}_2 \int_{\Delta[t,2]} d\mathbf{s}_2\, e^{-i(F_1(\mathbf{p}_2)s_1 + F_2(\mathbf{p}_2)s_2)} H(\mathbf{p}_2) \Big| \\ &\leq Ct\Big(\log(1+t)\Big\| \nabla \cdot \frac{\nabla F_2 |H|}{|\nabla F_2|^2}\Big\|_1 + \|H\|_1\Big) \\ &\leq Ct(1+\log(1+t))\big(\|H\|_{W^{1,1}} + \|H\|_{W^{1,\frac{9}{4}}}\big).\end{aligned} \tag{C.58}$$

As a consequence of (C.58) and using the chain rule, we obtain that

$$\begin{aligned}&\frac{\lambda^j}{N} |\operatorname{Bol}_c^{(j)}(f_0)[J](T;\lambda)| \\ &\leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, \|f_0\|_{2(\lfloor \frac{r}{2} \rfloor+1),c}, r} T(1+\log(T))\|J\|_{W^{1,\infty}} \frac{\lambda|\log(\lambda)|}{N}\end{aligned} \tag{C.59}$$

for $j \in \{1,2\}$, and all $\lambda > 0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}$. This finishes the proof.

□

C.2. Mollified Energy Conservation

Lemma C.6 *Let* $\operatorname{err}_2^{(fd)}(t; f[J])$ *be defined as in (5.211). Then*

$$|\operatorname{err}_2^{(fd)}(t; f[J])| \leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, \|f_0\|_{2(\lfloor \frac{r}{2} \rfloor+1),c}, r} T\|J\|_{W^{2,\infty}} \frac{\lambda}{N} \tag{C.60}$$

for all $\lambda > 0$ *small enough, dependent on* $\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}$.

Proof We want to apply integration by parts to

$$\begin{aligned}\operatorname{err}_2^{(fd)}(t; f[J]) = \frac{1}{N}\int_0^\lambda d\tau \int d\omega\, \delta'_{\frac{2\lambda^2}{T}}(\omega) \\ \int_{\mathcal{E}(\tau,\mathbf{p}_2)=\omega} d\mathcal{H}^5\, \frac{\partial_\tau \mathcal{E}(\tau,\mathbf{p}_2) H(\mathbf{p}_2)}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|}.\end{aligned} \tag{C.61}$$

For that, we need to establish a uniform bound on

$$\int_{\mathcal{E}(\tau,\mathbf{p}_2)=\omega} d\mathcal{H}^5 \, \frac{\partial_\tau \mathcal{E}(\tau,\mathbf{p}_2) H(\mathbf{p}_2)}{|\nabla_{\mathbf{p}_2} \mathcal{E}(\tau,\mathbf{p}_2)|} \,. \tag{C.62}$$

The divergence theorem implies that

$$\Big| \int_{\mathcal{E}(\tau,\mathbf{p}_2)=\omega} d\mathcal{H}^5 \, \frac{\partial_\tau \mathcal{E}(\tau,\mathbf{p}_2) H(\mathbf{p}_2)}{|\nabla_{\mathbf{p}_2} \mathcal{E}(\tau,\mathbf{p}_2)|} \Big| \leq \Big\| \nabla_{\mathbf{p}_2} \cdot \Big(\frac{\nabla_{\mathbf{p}_2} \mathcal{E}(\tau) \partial_\tau \mathcal{E}(\tau) H}{|\nabla_{\mathbf{p}_2} \mathcal{E}(\tau)|^2} \Big) \Big\|_1 \,. \tag{C.63}$$

We compute

$$\partial_\tau \Omega_\tau \; = \; \frac{2\hat{v}}{\sqrt{1 + \frac{2\tau\hat{v}}{E}}} \,. \tag{C.64}$$

Then $\partial_\tau \Omega_\tau$ satisfies

$$\|\partial_\tau \Omega_\tau\|_\infty \leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor + 1), w, c}} \,, \tag{C.65}$$

$$\begin{aligned} \|\nabla_p \partial_\tau \Omega_\tau\|_\infty &\leq C\Big(\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor + 1), w, c} \; + \; \|\hat{v}\|_\infty \Big\| D\Big(1 + \frac{2\tau\hat{v}}{E}\Big)^{-\frac{1}{2}} \Big\|_\infty \Big) \\ &\leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor + 1), w, c}} \,, \end{aligned} \tag{C.66}$$

$$\begin{aligned} \|D_p^2 \partial_\tau \Omega_\tau\|_\infty &\leq C\Big(\|D^2 \hat{v}\|_\infty \; + \; \|D\hat{v}\|_\infty \Big\| D\Big(1 + \frac{2\tau\hat{v}}{E}\Big)^{-\frac{1}{2}} \Big\|_\infty \\ &\quad + \|\hat{v}\|_\infty \Big\| D^2\Big(1 + \frac{2\tau\hat{v}}{E}\Big)^{-\frac{1}{2}} \Big\|_\infty \Big) \leq C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor + 1), w, c}} \end{aligned} \tag{C.67}$$

for all $\tau \in [0, \lambda]$, $\lambda \in (0, 1]$. Here we applied the Leibniz Formula, Lemma C.3. Then (C.65)-(C.67) imply

$$\|\partial_\tau \mathcal{E}(\tau,\mathbf{p}_2)\|_{W^{2,\infty}} \; \leq \; C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor + 1), w, c}} \tag{C.68}$$

for all $\tau \in [0, \lambda]$, $\lambda \in (0, 1)$. Following the steps of (C.53) in Lemma C.5, we find that

$$\begin{aligned} \Big\| \nabla_{\mathbf{p}_2} \cdot \Big(\frac{\nabla_{\mathbf{p}_2} \mathcal{E}(\tau) \partial_\tau \mathcal{E}(\tau) H}{|\nabla_{\mathbf{p}_2} \mathcal{E}(\tau)|^2} \Big) \Big\|_1 &\leq C\Big(\Big\| \frac{\partial_\tau \mathcal{E}(\tau) H}{|\nabla_{\mathbf{p}_2} \mathcal{E}(\tau)|^2} \Big\|_1 \\ &\quad + \Big\| \frac{\nabla_{\mathbf{p}_2} \big(\partial_\tau \mathcal{E}(\tau) H \big)}{|\nabla_{\mathbf{p}_2} \mathcal{E}(\tau)|} \Big\|_1 \Big) \end{aligned} \tag{C.69}$$

due to Lemma C.4. Using the Mean-Value Theorem together with $\nabla_{\tau,\mathbf{p}_2} \mathcal{E}(\tau, 0) = 0$, see Lemma C.4, we find that, for some $\zeta_{\tau,\mathbf{p}_2} \in [0, \mathbf{p}_2]$,

$$\nabla_{\mathbf{p}_2} \mathcal{E}(\tau,\mathbf{p}_2) \; = \; D^2_{\mathbf{p}_2} \mathcal{E}(\tau, \zeta_{\tau,\mathbf{p}_2}) \mathbf{p}_2 \,. \tag{C.70}$$

Using C.4 again, we conclude that

$$|\nabla_{\mathbf{p}_2} \mathcal{E}(\tau,\mathbf{p}_2)| \; \geq \; \frac{1}{2} \Big| \begin{pmatrix} p_2 \\ p_1 \end{pmatrix} \Big| \; = \; \frac{|\mathbf{p}_2|}{2} \tag{C.71}$$

for all $\lambda > 0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor + 1), w, c}$. Thus, we obtain that

$$
\begin{aligned}
&\Big| \frac{D^\ell\big[(\hat{v}(p_1)+\hat{v}(p_2))^2 f_0(p_{12})\big]}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|^m}\Big| \\
&\quad \leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}}\Big(\frac{1}{|\mathbf{p}_2|^m}\mathbb{1}_{B_1}(\mathbf{p}_2) \\
&\qquad + D^\ell\big[(\hat{v}(p_1)+\hat{v}(p_2))^2 f_0(p_{12})\big]\mathbb{1}_{B_1^c}(\mathbf{p}_2)\Big)
\end{aligned} \tag{C.72}
$$

for all $\ell, m \in \mathbb{N}_0$, $\ell \leq 2$. Hence,

$$
\Big\| \frac{D^\ell\big[(\hat{v}(p_1)+\hat{v}(p_2))^2 f_0(p_{12})\big]}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau)|^m}\Big\|_1 \leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}} \tag{C.73}
$$

for all $\ell, m \in \mathbb{N}_0$, $\ell \leq 2$, $m \leq 5$, and all $\lambda > 0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}$. With analogous arguments, we may replace $f_0(p_{12})$ in (C.73) by $f_0(p_{12})f_0(p_1)$, $f_0(p_{12})f_0(p_2)$, or $f_0(p_1)f_0(p_2)$ and obtain an analogous inequality. Then, (C.63), (C.68), (C.69), (C.73), and the definition (5.210) of H imply

$$
\Big| \int_{\mathcal{E}(\tau,\mathbf{p}_2)=\omega} d\mathcal{H}^5\, \frac{\partial_\tau\mathcal{E}(\tau,\mathbf{p}_2)H(\mathbf{p}_2)}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|}\Big| \leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}} T\|J\|_{W^{1,\infty}} \tag{C.74}
$$

for all $\omega \in \mathbb{R}$, $\tau \in [0,\lambda]$, λ small enough, dependent on $\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}$.

Next, by integration by parts w.r.t. $d\omega$, $\delta'_{\frac{2\lambda^2}{T}}(\omega) \to 0$ as $|\omega| \to \infty$, (C.74), and employing Lemma C.1, (C.61) implies that

$$
\begin{aligned}
\mathrm{err}_2^{(fd)}(t; f[J]) = -\frac{1}{N}\int_0^\lambda d\tau \int d\omega\, \delta_{\frac{2\lambda^2}{T}}(\omega) \int_{\mathcal{E}(\tau,\mathbf{p}_2)=\omega} \frac{d\mathcal{H}^5}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|} \\
\nabla_{\mathbf{p}_2}\cdot\Big(\frac{\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)\partial_\tau\mathcal{E}(\tau,\mathbf{p}_2)H(\mathbf{p}_2)}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|^2}\Big)
\end{aligned} \tag{C.75}
$$

Lemma C.1 implies

$$
\begin{aligned}
&\Big| \int_{\mathcal{E}(\tau,\mathbf{p}_2)=\omega} \frac{d\mathcal{H}^5}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|}\nabla_{\mathbf{p}_2}\cdot\Big(\frac{\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)\partial_\tau\mathcal{E}(\tau,\mathbf{p}_2)H(\mathbf{p}_2)}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|^2}\Big)\Big| \\
&\leq \Big\|\nabla_{\mathbf{p}_2}\cdot\Big[\frac{\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|^2}\nabla_{\mathbf{p}_2}\cdot\Big(\frac{\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)\partial_\tau\mathcal{E}(\tau,\mathbf{p}_2)H(\mathbf{p}_2)}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|^2}\Big)\Big]\Big\|_1 \\
&\leq C\Big(\|D^2_{\mathbf{p}_2}\mathcal{E}(\tau)\|^2_\infty\Big\|\frac{\partial_\tau\mathcal{E}(\tau)H}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|^4}\Big\|_1 + \|D^3_{\mathbf{p}_2}\mathcal{E}(\tau)\|_\infty\Big\|\frac{\partial_\tau\mathcal{E}(\tau)H}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|^3}\Big\|_1 \\
&\quad + \|D^2_{\mathbf{p}_2}\mathcal{E}(\tau)\|_\infty\Big\|\frac{D_{\mathbf{p}_2}\big(\partial_\tau\mathcal{E}(\tau)H\big)}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|^3}\Big\|_1 + \Big\|\frac{D^2_{\mathbf{p}_2}\big(\partial_\tau\mathcal{E}(\tau)H\big)}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|^2}\Big\|_1\Big).
\end{aligned} \tag{C.76}
$$

Lemma C.3 implies

$$
\|D^2_{\mathbf{p}_2}\mathcal{E}(\tau)\|_\infty \leq 1 + C\|\hat{v}\|^2_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\tau\,, \tag{C.77}
$$

$$
\|D^3_{\mathbf{p}_2}\mathcal{E}(\tau)\|_\infty \leq C\|\hat{v}\|^3_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}\tau\,. \tag{C.78}
$$

Collecting (C.68), (C.73), (C.76), (C.77), and (C.78), we find that

$$
\Big| \int_{\mathcal{E}(\tau,\mathbf{p}_2)=\omega} \frac{d\mathcal{H}^5}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|}\nabla_{\mathbf{p}_2}\cdot\Big(\frac{\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)\partial_\tau\mathcal{E}(\tau,\mathbf{p}_2)H(\mathbf{p}_2)}{|\nabla_{\mathbf{p}_2}\mathcal{E}(\tau,\mathbf{p}_2)|^2}\Big)\Big|
$$

$$\leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}} T\|J\|_{W^{2,\infty}} . \tag{C.79}$$

Employing (C.73), (C.75), and (C.79), we have proved that

$$\begin{aligned} |\operatorname{err}_2^{(fd)}(t; f[J])| &\leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}} \frac{T\|J\|_{W^{2,\infty}}}{N} \int_0^\lambda d\tau \int d\omega\, \delta_{\frac{2\lambda^2}{T}}(\omega) \\ &\leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}} \frac{T\lambda\|J\|_{W^{2,\infty}}}{N} , \end{aligned} \tag{C.80}$$

where in the last step, we applied the normalization (5.207) of $\delta_{\frac{2\lambda^2}{T}}$. This finishes the proof. □

Lemma C.7 *Recall that, due to (5.213),*

$$\begin{aligned} \operatorname{err}_2^{(ec)}(t; f[J]) = \frac{1}{N} \int d\omega\, \delta_{\frac{2\lambda^2}{T}}(\omega) \Big(\int_{\Delta_{cub}E=\omega} d\mathcal{H}^5 \frac{H}{|\nabla\Delta_{cub}E|} \\ + \int_{\Delta_{cub}E=0} d\mathcal{H}^5 \frac{H}{|\nabla\Delta_{cub}E|} \Big) \end{aligned} \tag{C.81}$$

with H as defined in (5.210). Then we have that

$$|\operatorname{err}_2^{(ec)}(t; f[J])| \leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}} \sqrt{T}\|J\|_{W^{2,\infty}} \frac{\lambda}{N} \tag{C.82}$$

for all $\lambda > 0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}$.

Proof We start by writing

$$\operatorname{err}_2^{(ec)}(t; f[J]) =: \frac{1}{N} \int d\omega\, \delta_{\frac{2\lambda^2}{T}}(\omega) \mathcal{I}(\omega) . \tag{C.83}$$

Analogously to (C.74), we have that

$$|\mathcal{I}(\omega)| \leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}} T\|J\|_{W^{1,\infty}} \frac{1}{N} . \tag{C.84}$$

Notice that, due to Lemma C.1, we have that

$$\mathcal{I}(\omega) = \int_0^\omega d\tau \int_{\Delta_{cub}E=\tau} \frac{d\mathcal{H}^5}{|\nabla\Delta_{cub}E|} \nabla\cdot\Big(\frac{\nabla\Delta_{cub}E\,H}{|\nabla\Delta_{cub}E|^2}\Big) , \tag{C.85}$$

where the integral respects the orientation of $[0, \omega]$ resp. $[\omega, 0]$. Using Lemma C.1, we thus obtain

$$\begin{aligned} |\mathcal{I}(\omega)| &\leq |\omega| \Big\| \nabla\cdot\Big[\frac{\nabla\Delta_{cub}E}{|\nabla\Delta_{cub}E|^2}\nabla\cdot\Big(\frac{\nabla\Delta_{cub}E\,H}{|\nabla\Delta_{cub}E|^2}\Big)\Big]\Big\|_1 \\ &\leq C|\omega| \Big(\|D^2\Delta_{cub}E\|_\infty^2 \Big\|\frac{H}{|\nabla\Delta_{cub}E|^4}\Big\|_1 + \Big\|\frac{|D^3\Delta_{cub}E|H}{|\nabla\Delta_{cub}E|^3}\Big\|_1 \\ &\quad + \|D^2\Delta_{cub}E\|_\infty \Big\|\frac{DH}{|\nabla\Delta_{cub}E|^3}\Big\|_1 + \Big\|\frac{D^2H}{|\nabla\Delta_{cub}E|^2}\Big\|_1 \Big) , \end{aligned} \tag{C.86}$$

for all $\lambda > 0$ small enough, dependent on $\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c}$, analogously to (C.76). Notice that $D^3\Delta_{cub}E = 0$ and $\|D^2\Delta_{cub}E\|_\infty \leq C$. Analogously to (C.80), we thus obtain the upper bound

$$|\mathcal{I}(\omega)| \leq C_{\|\hat{v}\|_{2(\lfloor\frac{r}{2}\rfloor+1),w,c},\|f_0\|_{2(\lfloor\frac{r}{2}\rfloor+1),c}} |\omega| T\|J\|_{W^{2,\infty}} . \tag{C.87}$$

We split the integral in (C.83) into the regions $(-\omega_0, \omega_0) \cup (-\omega_0, \omega_0)^c$ with ω_0 to be determined below. Then (C.84) and (C.87) yield

$$
\begin{aligned}
&|\operatorname{err}_2^{(ec)}(t; f[J])| \\
&\leq \frac{1}{N} \int_{(-\omega_0,\omega_0)} d\omega\, \delta_{\frac{2\lambda^2}{T}}(\omega) |\mathcal{I}(\omega)| + \int_{(-\omega_0,\omega_0)^c} d\omega\, \delta_{\frac{2\lambda^2}{T}}(\omega) |\mathcal{I}(\omega)| \\
&\leq \frac{C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, \|f_0\|_{2(\lfloor \frac{r}{2} \rfloor+1),c}} T}{N} \Big(\|J\|_{W^{2,\infty}} \int_{(-\omega_0,\omega_0)} d\omega\, \delta_{\frac{2\lambda^2}{T}}(\omega) |\omega| \\
&\quad + \|J\|_{W^{1,\infty}} \int_{(-\omega_0,\omega_0)^c} d\omega\, \delta_{\frac{2\lambda^2}{T}}(\omega) \Big).
\end{aligned} \tag{C.88}
$$

Employing the normalization condition (5.207) and the decay condition (5.208) on $\delta_{\frac{2\lambda^2}{T}}$, we thus obtain

$$
|\operatorname{err}_2^{(ec)}(t; f[J])| \leq \frac{C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, \|f_0\|_{2(\lfloor \frac{r}{2} \rfloor+1),c}} T \|J\|_{W^{2,\infty}}}{N} \Big(\omega_0 + \frac{\lambda^2}{\omega_0 T} \Big). \tag{C.89}
$$

By now choosing $\omega_0 = \frac{\lambda}{\sqrt{T}}$, we have hence proved that

$$
|\operatorname{err}_2^{(ec)}(t; f[J])| \leq \frac{C_{\|\hat{v}\|_{2(\lfloor \frac{r}{2} \rfloor+1),w,c}, \|f_0\|_{2(\lfloor \frac{r}{2} \rfloor+1),c}} \sqrt{T} \|J\|_{W^{2,\infty}} \lambda}{N}. \tag{C.90}
$$

□

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