The Computational Complexity of Multi-player Concave Games and Kakutani Fixed Points

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Kakutani's Fixed Point theorem is a fundamental theorem in topology with numerous applications in game theory and economics. Formally, Kakutani's theorem states that for any set-valued function mapping F, also known as correspondence, from a compact, convex set to itself in a locally convex topological vector space, if the function is upper hemicontinuous, has a closed graph, and its output at any given point is a non-empty and convex set, then there exists a fixed point x, namely a point in the domain which is mapped to itself by the function $x \in F(x)$. Interestingly, computational formulations of Kakutani exist only in special cases and are too restrictive to be useful in reductions.

In this paper, we provide a general computational formulation of finding an approximate Kakutani's Fixed Point Theorem using polynomial sized circuits that computes weak separation oracles for set-valued map F and we prove that it is PPAD-complete. As an application of our theorem we are able to characterize the computational complexity of the following fundamental problems:

(1) **Concave Games.** Introduced by the celebrated works of Debreu and Rosen in the 1950s and 60s, concave n-person games have found many important applications in Economics and Game Theory. We characterize the computational complexity of finding an equilibrium in such games. We show that a general formulation of this problem belongs to PPAD, and that finding an equilibrium is PPAD-hard even for a rather restricted games of this kind: strongly-concave utilities that can be expressed as multivariate polynomials of a constant degree with axis aligned box constraints.
(2) **Walrasian Equilibrium.** Using Kakutani's fixed point Arrow and Debreu we resolve an open problem related to Walras's theorem on the existence of price equilibria in general economies. There are many results about the PPAD-hardness of Walrasian equilibria, but the inclusion in PPAD is only known for piecewise linear utilities. We show that the problem with general convex utilities is in PPAD.

A core part of our proof involves demonstrating how to tolerate potential inaccuracies in intricate computational processes, such as projection and optimization subroutines in the application of Sperner's Lemma. Along the way, we introduce a novel Lipschitz continuous version of Berge's maximum theorem, which may be of independent interest.

A full version of this paper can be found at https://arxiv.org/abs/2207.07557.

CCS Concepts: • **Theory of computation → Exact and approximate computation of equilibria**.

Additional Key Words and Phrases: Kakutani' Fixed Point Theorem, Nash equilibria, Concave Games, Walrasian Equilibria, PPAD (Polynomial Parity Arguments on Directed graphs)

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