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SPACE-TIME VARIABLE DENSITY SAMPLINGS FOR SPARSE BANDLIMITED GRAPH SIGNALS DRIVEN BY DIFFUSION OPERATORS

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ABSTRACT

We consider the space-time sampling and reconstruction of sparse bandlimited graph signals driven by a heat diffusion process. In this paper, we develop a sampling framework consisting of selecting a small subset of space-time nodes at random according to some probability distribution, generalizing the classical variable density sampling to the heat diffusion field. We show that the number of space-time samples required to ensure stable recovery depends on an incoherence parameter determined by the interplay between graph topology, temporal dynamics, and sampling probability distributions. In optimal scenarios, as few as $\mathcal{O}(s\log k)$ space-time samples are sufficient to ensure accurate recovery of all k-bandlimited graph signals that are additionally s-sparse. Our proposed sampling method requires much fewer spatial samples than the static case by leveraging temporal information. Finally, we test our sampling techniques on a wide variety of graphs. The numerical results on synthetic and real climate data sets support our theoretical findings and demonstrate the practical applicability.

Index Terms— Graph signal recovery, sampling theorem, sparse signals, random space-time sampling, compressive sensing.

1. INTRODUCTION

In the era of big data, graph signals are ubiquitous. Graph signal processing has become a very active research topic [1, 2, 3, 4]. Sampling theory is a fundamental component of graph signal processing and has attracted considerable research attention. It concerns when a graph signal can be recovered from its discrete sampled values. In classical sampling theory, the signals of interest are defined on regular domains and are smooth, where smoothness is built upon concepts of frequency analysis, such as bandlimitness and shift-invariance. It becomes more challenging in the graph setting, as espoused in review paper [5]. One of the difficulties lies in how we can develop and connect notions of frequencies for graphs that model the actual properties of signals of interest. Several graph sampling approaches [6, 7, 8] have recently been developed based on different notions of graph frequency, bandlimitedness, and shift-invariance, and have found various applications in modeling real graph data sets.

In modern applications, there are many situations where the graph signals to be sampled and recovered are evolving in time. Examples include propagation of rumors over social networks [9], neural activities transfer in different regions of the brain [10], and spatial temperature profiles measured by a wireless sensor network [11]. Due to application specific constraints, one may not be able to get enough samples to reconstruct the graph signal at any single time snapshot. However, it is sometimes convenient to obtain sampled values of graph signals at multiple time instances. Reconstruction of time-varying graph signals from space-time samples has received much attention in recent years [12, 13, 14, 15, 16, 17]. Different models for time series over graphs have been considered. The examples include bandlimited graph processes [12], dynamical processes with smooth temporal difference [13], and signals defined on time-varying graphs [17]. Various reconstruction algorithms [12, 13, 14] are proposed. In most of these works, the space-time samples are often chosen uniformly at random or according to sampling theory for static graph signals [14]. Despite the superior empirical performance, the sampling and recovery methods are short of theoretical guarantees. Sampling theory for dynamical processes over graphs is relatively scarce in the literature. For the smooth temporal difference model, [13] provides a characterization for feasible deterministic sampling sets for their reconstruction algorithm. For bandlimited graph processes, [15] considered both deterministic and Bernoulli space-time sampling. Necessary conditions on probabilities are derived to ensure the exact reconstruction of signals and a convex optimization approach was proposed to choose the optimal sampling design.

In this paper, we proposed a random space-time sampling approach for s-sparse bandlimited graph signals with bandwidth k driven by a heat diffusion operator. We select space-

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time nodes randomly according to a predetermined probability distribution and reconstruct the signal via solving an ℓ_1-minimization problem. We provide a theoretical sampling bound on the sampling complexity to guarantee the exact recovery of the signal. There are several merits of our sampling method: (i) Our sampling method is computationally cheap. Once the probability distribution is determined, the node selection can be realized quickly; (ii) It allows us to perform physics-informed space-time sampling. We develop a notion of optimal sampling distribution that depends on the graph topology, the signal bandwidth, and the temporal dynamics [18]. We show in optimal scenarios, $\mathcal{O}(s \log k)$ space-time samples are enough to guarantee robust recovery. Further, numerical examples on synthetic and real data sets demonstrate the effectiveness of the proposed approach.

2. NOTATION, PRELIMINARIES AND SET-UP

We provide some formal definitions related to graph operators and signals. For an undirected weighted graph $\mathcal{G} = (V, E, W)$, where $V = \{v_1, \cdots, v_n\}$ is a set of vertices, $E \subseteq V \times V$ is a set of edges, and W is the weighted adjacency matrix. Specifically, if we denote (v_i, v_j) as the edge between v_i and v_j if they are connected with positive weight, $w_{ij} > 0$, W is defined as

$$W(i,j) = \begin{cases} w_{ij}, & (v_i, v_j) \in E \\ 0, & \text{otherwise} \end{cases}.$$

In particular, W is symmetric because $\mathcal{G}$ is undirected. The degree of a vertex v_i is defined by $\deg(v_i) = \sum_{j=1}^{n} W(i,j)$, and the diagonal degree matrix of $\mathcal{G}$ is denoted by $D = \operatorname{diag}(\deg(v_i))$.

With these definitions in place, we introduce the Laplacian operator of $\mathcal{G}$ as $L = D - W$. L is a positive-semidefinite operator, hence it admits an eigendecomposition as $L = U\Sigma U^\top$ where the columns of U are orthonormal, and Σ is a diagonal matrix containing the eigenvalues $\sigma_1, \cdots, \sigma_n \geq 0$. According to spectral graph theory, the multiplicity of 0 equals the number of connected components of the graph. In this paper, we consider connected undirected graphs and assume the multiplicity of each eigenvalue is 1. So we shall have $\sigma_1 = 0 < \sigma_2 < \cdots < \sigma_n$.

A graph signal is a vector $\mathbf{x} \in \mathbb{R}^n$ defined on the vertices V of $\mathcal{G}$, i.e. $\mathbf{x}(i)$ is the signal value associated with the node v_i. For any graph signal $\mathbf{x}$ on $\mathcal{G}$, the graph Fourier transform of $\mathbf{x}$ is defined as $\hat{\mathbf{x}} = U^\top \mathbf{x}$, where $\hat{\mathbf{x}}$ contains the Fourier coefficients of $\mathbf{x}$ ordered in increasing frequencies. The inverse Fourier transform is defined naturally as $\mathbf{x} = U\hat{\mathbf{x}}$.

A graph signal can be considered smooth if neighboring nodes have similar signal values. As such, we may define the smoothness of a graph signal $\mathbf{x}$ to be the weighted sum of squared differences between every pair of neighboring nodes. This quantity can be represented via the Laplacian as

$$\mathbf{x}^\top L \mathbf{x} = \sum_{(v_i, v_j) \in E} w_{ij} (\mathbf{x}(i) - \mathbf{x}(j))^2.$$

For a smooth signal $\mathbf{x}$, $\mathbf{x}^\top L \mathbf{x}$ is small, and since $\mathbf{x}^\top L \mathbf{x} = \sum_{i=1}^{n} \sigma_i \hat{\mathbf{x}}(i)^2$, we expect $\hat{\mathbf{x}}(i)$ to be near zero for sufficiently large indices of i. This motivates the definition of k-bandlimited signals and its subset of s-sparse signals.

Definition 1. A graph signal $\mathbf{x}$ on $\mathcal{G}$ is k-bandlimited for some $k \in \mathbb{Z}_+$ if $\mathbf{x} \in \operatorname{span}(U_k)$, where U_k denotes the first k columns of U. Equivalently, $\mathbf{x}$ is bandlimited if the only nonzero entries of $\hat{\mathbf{x}}$ are in the first k positions. Furthermore, A k-bandlimited graph signal $\mathbf{x}$ on $\mathcal{G}$ is s-sparse for $s \in \mathbb{Z}_+$ with $s \ll k$ if

$$|\text{supp}(\hat{\mathbf{x}})| = |\{i : \hat{\mathbf{x}}(i) \neq 0\}| \leq s.$$

This generalized the sparse bandlimited signals from classical setting [19] to finite graphs. In practice, the bandwidth is typically unknown, so one can choose a large upper bound k. In many cases, signals of interest are s-sparse in the spectral domain. In this paper, we will consider s-sparse k-bandlimited graph signals that diffuse according to the heat equation, given by

$$\frac{\partial}{\partial t}\mathbf{x}_t = -L\mathbf{x}_t, \quad t \geq 0.$$

This diffusive process is determined completely by the initial condition, since $\mathbf{x}_t = e^{-tL}\mathbf{x}_0$. Restricting to only discrete uniform observations, we label the time-steps by $\{0, \Delta t, \cdots, (T-1)\Delta t\}$ for some $\Delta t > 0$ and $T \in \mathbb{Z}_+$. Define $A = e^{-\Delta t L}$ as the signal evolution operator, then the diffusive signals are given by $\mathbf{x}, A\mathbf{x}, \cdots, A^{T-1}\mathbf{x}$. Note that $A = U\Lambda U^\top$, where $\Lambda = e^{-\Delta t \Sigma}$. If we denote $\lambda_1, \cdots, \lambda_n$ as the diagonal entries of Λ, then $1 = \lambda_1 > \cdots > \lambda_n$.

The heat diffusion process over graphs is a simple yet effective model of real-world dynamics. It finds applications in modeling diffusion phenomena such as temperature variations, pollution dispersion, and functional connectivity of neurons in different regions of the brain. We assume that A is known and consider how we can leverage the underlying physics to perform efficient space-time sampling.

2.1. Space-time Variable Density Sampling

The space-time sampling and reconstruction problem can be formally stated as follows. Let Ω_t denote the subset of nodes at which the signal $\mathbf{x}_t$ is observed, then we wish to determine sufficient conditions on $\{\Omega_t\}_{t=0}^{T-1}$ such that $\mathbf{x}_0$ can be stably recovered from its space-time samples. When $T = 1$, then this problem reduces to the sampling of s-sparse and k-bandlimited static graph signals.

We will consider the following random sampling procedure on the space-time nodes with respect to a probability distribution on all space-time locations.

Let $\mathbf{p}$ be a probability distribution on $\mathfrak{I} = \{1, 2, \cdots, Tn\}$, we construct the sampling set $\Omega = \{\omega_1, \cdots, \omega_m\}$ by drawing m indices with replacement from $\mathfrak{I}$ according to $\mathbf{p}$, i.e.

$$P(\omega_j = i) = \mathbf{p}(i) \quad \forall j = 1, \cdots, m; i \in \mathfrak{I}.$$

The corresponding sampling matrix $S \in \mathbb{R}^{m \times Tn}$ is defined as

$$S(i,j) = \begin{cases} 1, & j = \omega_i \\ 0, & \text{otherwise} \end{cases}.$$

We collect the probabilities of selecting each element in Ω into a diagonal matrix $P_\Omega \in \mathbb{R}^{m \times m}$ given by

$$P_\Omega = S\text{diag}(\mathbf{p})S^\top = \text{diag}([\mathbf{p}(\omega_i)]_{i=1}^m).$$

We may apply the sampling matrix to the extended space-time signal in $\mathbb{R}^{nT}$ formed as

$$\pi_{A,T}(\mathbf{x}) = \left[\mathbf{x}^\top, (A\mathbf{x})^\top, \cdots, \left(A^{T-1}\mathbf{x}\right)^\top\right]^\top.$$

Our goal is to recover $\mathbf{x}$ from its space-time samples $\mathbf{y} = S\pi_{A,T}(\mathbf{x})$.

2.2. Sparse Representation of $\pi_{A,T}(x)$

To establish the main sampling and reconstruction theorem, our key idea is to produce a sparse representation of the diffusion trajectory $\pi_{A,T}(\mathbf{x})$ generated by $\mathbf{x}$. We recall first that $\mathbf{x}$ is sparse relative to U_k. This sparsity is preserved through the diffusion process, that is, $A^t\mathbf{x}$ is also sparse relative to U_k. As such, we can introduce a natural extension $\tilde{U}_{k,T}$ of U_k such that $\pi_{A,T}(\mathbf{x})$ is sparse relative to a space-time orthonormal dictionary $\tilde{U}_{k,T}$ defined as:

$$\tilde{U}_{k,T} = \begin{bmatrix} \frac{1}{f_T(\lambda_1)}\mathbf{u}_1 & \frac{1}{f_T(\lambda_2)}\mathbf{u}_2 & \cdots & \frac{1}{f_T(\lambda_k)}\mathbf{u}_k \\ \frac{\lambda_1}{f_T(\lambda_1)}\mathbf{u}_1 & \frac{\lambda_2}{f_T(\lambda_2)}\mathbf{u}_2 & \cdots & \frac{\lambda_k}{f_T(\lambda_k)}\mathbf{u}_k \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\lambda_1^{T-1}}{f_T(\lambda_1)}\mathbf{u}_1 & \frac{\lambda_2^{T-1}}{f_T(\lambda_2)}\mathbf{u}_2 & \cdots & \frac{\lambda_k^{T-1}}{f_T(\lambda_k)}\mathbf{u}_k \end{bmatrix} \in \mathbb{R}^{Tn \times k},$$

where $f_T(\lambda_i) = \sqrt{\sum_{j=0}^{T-1} \lambda_i^{2j}}$ is introduced to normalize the columns. Note that in the static case (i.e. $T = 1$), $\tilde{U}_{k,T} = U_k$.

Now we verify the claim that if $\mathbf{x}$ is s-sparse and k-bandlimited, then $\tilde{U}_{k,T}^\top \pi_{A,T}(\mathbf{x})$ is s-sparse. If $\mathbf{x}$ is bandlimited of the form $\mathbf{x} = \sum_{i=1}^k \hat{c}_i \mathbf{u}_i$, denote $\hat{\mathbf{c}} = [\hat{c}_1, \cdots, \hat{c}_k]^\top$, then $\pi_{A,T}(\mathbf{x}) = \tilde{U}_{k,T}\text{diag}\left([f_T(\lambda_i)]_{i=1}^k\right)\hat{\mathbf{c}}$ with $\text{diag}([f_T(\lambda_i)]_{i=1}^k)\hat{\mathbf{c}}$ being s-sparse.

3. RECOVERY ℓ_1-MINIMIZATION PROBLEM

With a sparse representation of $\pi_{A,T}(\mathbf{x})$ in place, we propose the following ℓ_1-minimization problem for signal recovery. Consider

$$\min_{\mathbf{c} \in \mathbb{R}^k} \|\mathbf{c}\|_1 \text{ subject to } P_\Omega^{-\frac{1}{2}} S\tilde{U}_{k,T}\mathbf{c} = P_\Omega^{-\frac{1}{2}}\mathbf{y}, \tag{3.1}$$

where a weighted sampling matrix $P_\Omega^{-\frac{1}{2}}$ is applied to ensure numerical stability. As the true solution $\text{diag}([f_T(\lambda_i)]_{i=1}^k)\hat{\mathbf{c}}$ is s-sparse, if it is the unique solution to the ℓ_1-minimization problem (3.1), then we can recover $\mathbf{x} = U_k\hat{\mathbf{c}}$ uniquely by solving the ℓ_1-minimization problem. The following theorem, derived from theorem 12.20 in [20], provides a sufficient condition on the number of space-time samples required for consistent recovery.

Theorem 1. *Let $\mathbf{x}$ be a s-sparse, k-bandlimited signal. Assume that $\Omega = \{\omega_1, \cdots, \omega_m\}$ are selected according to $\mathbf{p}$ as in subsection 2.1. Denote $\tilde{U}_{k,T}^\top\delta_j$ as the jth row of $\tilde{U}_{k,T}$, and suppose the incoherence parameter $K(\mathbf{p})$ is finite, where*

$$K(\mathbf{p}) = \max_{1 \le j \le Tn} \frac{\left\|\tilde{U}_{k,T}^\top \delta_j\right\|_\infty}{\sqrt{\mathbf{p}(j)}}. \tag{3.2}$$

Then as long as $m \geq CK^2(\mathbf{p})s\log(k)\log\left(\epsilon^{-1}\right)$, for some constant $C > 0$, with probability at least $1 - \epsilon$, the vector $\hat{\mathbf{c}}$ is the unique solution to the ℓ_1-minimization problem in 3.1.

3.1. Optimal Sampling Distribution

A natural follow-up investigation to Theorem 1 is how we can minimize the number of samples necessary for recovery. It is clear that the relevant factor is $K^2(\mathbf{p})$. The following proposition gives us an 'optimal' probability distribution $\mathbf{p}_{\text{opt}}$ that minimizes $K^2(\mathbf{p})$.

Proposition 1. *Define $K(\mathbf{p})$ as in 3.2. $K^2(\mathbf{p})$ is minimized when $\mathbf{p} = \mathbf{p}_{\text{opt}}$, where*

$$\mathbf{p}_{\text{opt}}(i) = \frac{\left\|\tilde{U}_{k,T}^\top \delta_i\right\|_\infty^2}{\sum_{j=1}^{Tn}\left\|\tilde{U}_{k,T}^\top \delta_j\right\|_\infty^2}.$$

In the optimal case, we have

$$K^2(\mathbf{p}_{\text{opt}}) = \sum_{j=1}^{Tn} \|\tilde{U}_{k,T}^\top \delta_j\|_\infty^2.$$

Notice that the optimal sampling distribution $\mathbf{p}_{\text{opt}}$ is not dependent on signal sparsity, so a one-time calculation is sufficient for the optimal recovery of all space-time signals in $\text{span}\left(\tilde{U}_{k,T}\right)$. From now on, we only consider the optimal case, and denote $\mathbf{p} = \mathbf{p}_{\text{opt}}$ and $K = K(\mathbf{p}_{\text{opt}})$.

As $\tilde{U}_{k,T}$ contains k normalized columns, we obtain an upper-bound for K^2 as

$$K^2 = \sum_{d=1}^{Tn}\left\|\tilde{U}_{k,T}^\top \delta_d\right\|_\infty^2 \le \sum_{i=1}^{Tn}\sum_{j=1}^{k}\tilde{U}_{k,T}(i,j)^2 = k.$$

Our numerical experiments show that for highly irregular graphs, K^2 could be of the order $\mathcal{O}(k)$. For regular graphs, K^2 are often of order $\mathcal{O}(1)$, such as in the case of the unweighted ring graph, which has sinusoidal eigenvectors and therefore the entries of $\tilde{U}_{k,T}$ have approximately equal norms.

4. EXPERIMENTS

In this section, we will empirically investigate three aspects of the recovery problem: (i) verify the sampling complexity of Theorem 1; (ii) study behavior of K^2 for certain graphs; (iii) test our methods on real data. We use the Lasso algorithm [21] as the ℓ_1-minimization solver.

4.1. Number of Samples for Accurate Recovery

In this subsection, our main goal is to investigate how the required number of space-time samples m varies with respect to the sparsity s of the graph signal. Specifically, we want to find the smallest such m such that at least 95 out of 100 randomly generated sparse and bandlimited signals can be recovered via Theorem 1 with less than 5% relative error.

We perform this test on the Minnesota graph ($n = 2642, k = 1000$) and the unweighted ring graph ($n = 1000, k = 400$) from the GSP Toolbox [22], comparing optimal dynamic sampling and uniform dynamic sampling methods. The dynamics are determined by the heat diffusion process in Section 2 with $\Delta t = 4$ and $T = 10$.

The results are summarized in the left panel of Figure 1. We note that the number of samples required for consistent recovery is roughly proportional to s, as shown by the linear best-fit dashed lines, supporting the claims of Theorem 1.

4.2. Incoherence Parameter K^2

We now discuss the relationship between K^2 and k, T for the Minnesota graph and the unweighted ring graph, shown in the right panel of Figure 1.

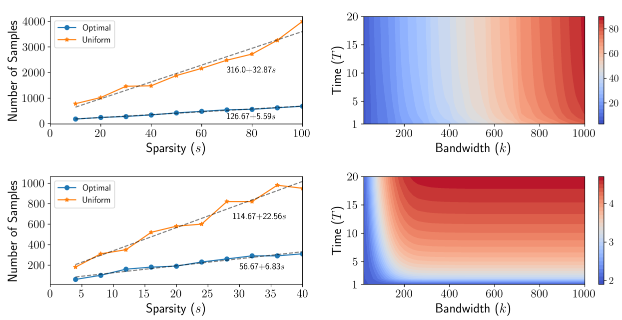

Fig. 1: Left: The number of space-time samples v.s. the sparsity for Minnesota graph (top) and ring graph (bottom) with linear best-fit dashed line. Blue: optimal sampling distribution. Yellow: uniform sampling distribution. Right: K^2 v.s. bandwidth and time for Minnesota graph (top) and ring graph (bottom).

We see that for both graphs, K^2 has a very weak positive correlation with T. However, K^2 is highly dependent on k. For the irregular Minnesota graph, K^2 is roughly linearly related to k with coefficient 0.08. As for the unweighted ring graph, K^2 is bounded by a constant 5 as a function of k. As such, we may treat K^2 for the ring graph as a constant. In fact, the class of regular graphs with reasonable weights all have near-constant bounds on K^2 as a function of k.

4.3. Real Data: Sea Level Measurements

We now apply our recovery methods to the sea level pressure data set from [23]. The signals ($n = 500$) are naturally bandlimited over a KNN graph (see [18]). We choose a time series consisting of $T = 50$ snapshots and bandwidth $k = 200$. Such signals are approximately to be sparse with $s = 30$. We remark that the KNN graph is approximately regular, and as such the optimal sampling distribution is close to the uniform sampling distribution.

We first numerically demonstrate the advantage of dynamic optimal sampling ($T = 50$) versus static optimal sampling ($T = 1$). For each fixed number of samples m, we conduct 100 trials and record the recovery results with an error plot, representing the average relative error and the standard deviation. The results are shown in the left panel of Figure 2. With more than 20 samples, the dynamic sampling method

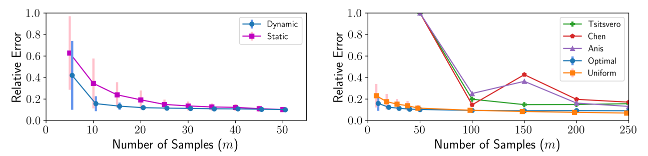

Fig. 2: Recovery results for real sea level pressure data. Left: space-time recovery versus static recovery. Right: comparison of space-time variable density sampling with static optimal sampling approaches.

can reconstruct signals with around 10% relative error with high probability, as indicated by the tight error bar.

We also compare the average relative error of the proposed optimal and uniform sampling method with existing sampling algorithms from Tsitsvero et al. [6], Chen et al. [7], and Anis et al. [8] for bandlimited signal recovery. For each of these three sampling methods, we optimally choose several spatial locations at $t = 1$ and fix it until $t = 50$. For each sampling method, we perform 100 trials at each sampling rate with additive Gaussian noise of mean 0 and variance 0.1^2, and perform recovery via the Lasso algorithm. As our proposed optimal and uniform sampling methods are probabilistic, we included error bars that represent the standard deviation of recovery errors. The results are shown in the right panel of Figure 2. We see that optimal sampling has better performance compared to existing methods when the observational samples are scarce.

5. CONCLUSION AND FUTURE WORK

We propose a space-time variable density sampling algorithm for the recovery of sparse bandlimited graph signal in a heat diffusion process. We provide theoretical guarantees in the noiseless case and present empirical success with noise and model error on synthetic and real data. Future work includes theoretical analysis with noise, extension to general linear time-invariant system, and validating this method on more real data sets.

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