
Learning the Structure of Large Networked Systems Obeying Conservation Laws*

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Abstract

Many networked systems such as electric networks, the brain, and social networks of opinion dynamics are known to obey conservation laws. Examples of this phenomenon include the Kirchoff laws in electric networks and opinion consensus in social networks. Conservation laws in networked systems are modeled as *balance equations* of the form $X = B^*Y$, where the sparsity pattern of $B^* \in \mathbb{R}^{p \times p}$ captures the connectivity of the network on p nodes, and $Y, X \in \mathbb{R}^p$ are vectors of *potentials* and *injected flows* at the nodes respectively. The node potentials Y cause flows across edges which aim to balance out the potential difference, and the flows X injected at the nodes are extraneous to the network dynamics. In several practical systems, the network structure is often unknown and needs to be estimated from data to facilitate modeling, management, and control. To this end, one has access to samples of the node potentials Y , but only the statistics of the node injections X . Motivated by this important problem, we study the estimation of the sparsity structure of the matrix B^* from n samples of Y under the assumption that the node injections X follow a Gaussian distribution with a known covariance Σ_X . We propose a new ℓ_1 -regularized maximum likelihood estimator for tackling this problem in the high-dimensional regime where the size of the network may be vastly larger than the number of samples n . We show that this optimization problem is convex in the objective and admits a unique solution. Under a new mutual incoherence condition, we establish sufficient conditions on the triple (n, p, d) for which exact sparsity recovery of B^* is possible with high probability; and d is the degree of the underlying graph. We also establish guarantees for the recovery of B^* in the element-wise maximum, Frobenius, and operator norms. Finally, we complement our theoretical results with experimental validation of the performance of the proposed estimator on synthetic and real-world data.

1 Introduction

Let $\mathcal{G} = ([p], E)$ be a directed graph on the vertex set $[p] \triangleq \{1, 2, \dots, p\}$ with a size m edge set $E \subset [p] \times [p]$. Let D denote the $p \times m$ incidence matrix that encodes the edges of \mathcal{G} as follows: each column of D is associated with an edge $(i, j) \in E$ and is a vector of zeros except at the locations i and j where it is -1 and $+1$ respectively. Let $X \in \mathbb{R}^p$ be a vector of *injected flows or signals* at the vertices and let $f \in \mathbb{R}^m$ be the vector of *flows* through the edges. Then, the basic *conservation law* between these flows may be expressed as $Df + X = 0$; that is, at each vertex, the flow (which is a linear combination of flows at the edges incident on the vertex) has to balance the injections. In physical systems, edge flows f often arise as a way to balance the differences between certain *potentials* $Y \in \mathbb{R}^p$ at the vertices. That is, the flows satisfy $f = -D^\top Y$; notice that this implies that

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the flow at the edge (i, j) is given by $Y_j - Y_i$. Thus, the above conservation law yields the following relationship, called a *balance equation*, between the node potentials and injected flows:

$$X - B^*Y = 0, \quad (1)$$

where $B^* \triangleq DD^\top \in \mathbb{R}^{p \times p}$ is the symmetric Laplacian matrix [4, 60]. In an electrical circuit (with unit resistances on edges), Y corresponds to the voltage potentials at the vertices, f corresponds to the edge currents, and X denotes the injected currents at the vertices. Indeed, this picture can be generalized by assigning weights to the edges of the network (conductances in the case of an electric network), and allowing the flows to be weighted by these weights. The model in (1) is referred to as generalized Kirchoff's law, and importantly, this models the relationship between flows and potentials in a range of systems that satisfy conservation laws such as hydraulic networks, opinion consensus in social networks, and transportation/distribution networks (see [60, 40, 55] and references therein).

It can be readily seen that the Laplacian B^* is a positive semi-definite that encodes the edges of \mathcal{G} . Specifically, $(i, j) \in E$ if and only if $B_{ij}^* \neq 0$. The Laplacian lies at the heart of spectral graph theory [15], and owing to its fascinating properties has found a range of applications in diverse areas such as image processing, manifold learning, spectral clustering, and bandits [52, 6, 62, 58, 32, 56]. In this paper, we consider a situation where the edge set E of the graph is unknown and needs to be estimated from measurements of the node potentials Y . Based on the above discussion, we will cast this as a problem of learning an unknown positive definite B^* (or the sparsity pattern thereof) from measurements of Y . Further, we suppose that we only have access to the statistics of X , namely, that it is a 0-mean Gaussian random vector with a covariance matrix Σ_X . The situations where Σ_X is unknown and B^* is non-invertible is briefly discussed in the remarks in Section 2. We list a variety of applications where this learning problem arises naturally.

1. *Topology learning in electric networks*: Consider an electric network (or circuit) with p nodes, current injections X , node voltages Y , conductances $A_{ij} \geq 0$ between nodes i and j , and shunt conductances $A_{ii} \geq 0$ connecting i -th node to the ground. The current-balance equation is given by (1), where B^* is the Laplacian with $B_{ij}^* = -A_{ij}$ and $B_{ii}^* = A_{ii} + \sum_{j=1}^n A_{ij}$ [21]. In real electric grids, current injections are unknown random variables. To ensure reliable power supply, learning B^* and its underlying graph from voltage samples is important and has been widely studied [20, 33, 13, 1]. The current-balance equation also appears in Markov chains and flow networks where Kirchoff laws apply [60, 46].
2. *Brain connectivity from graph filters*: The structural connectivity of the human brain is often studied using a network with nodes representing brain regions, and the edge weights representing the density of anatomical connections [45, 25, 61]. Recent studies showed that the weights can be inferred using *graph filters* satisfying (1) with $B^* = (\sum_{l=0}^{L-1} h_l A^l)^{-1}$, where A is the symmetric adjacency matrix; h_l is the filter coefficient; and X is the latent graph signal. For brain networks, [51, 36] showed that $L = 3$ is a reasonable choice. Additionally, for large L , the graph filter B^* can be approximated as $B^* = 1 - \alpha A$ using matrix power series expansion, thus a feasible choice for analysis. Graph filters are also used in social and protein interaction networks [50, 37].
3. *Structural equation models (SEM)*: Structural equation models are used to explain relationships among exploratory variables in several domains; for e.g., psychoanalysis [22], social sciences [16], medical research, and neuroimaging [7, 47, 38]. Using SEMs, [42] provided a causal interpretation of Linear Hawkes Processes. In SEM with no latent variables, we let $y = Ay + x$, where A is the *path matrix*. Then the SEM satisfies (1) with $B^* = I - A$.
4. *Linear dynamical (diffusion) networks*: These network dynamics are satisfied by many systems including consensus dynamics, thermal capacitance networks, power swing dynamics, [54]. Further, by lifting approach, these dynamics can be used to study periodic/cyclic behavior in atmospheric systems [53].

Before we detail our topology discovery method, we comment on a few competing approaches that only have limited utility in our setting. First, penalized (nodal) regression methods [39] are not applicable here since these require samples of both X and Y . Yet other works have considered the estimation of topology from linear measurements [2, 9, 33, 65, 11, 14, 18, 19], but these assume direct linear access to the graph (via the adjacency or appropriate covariances) or assume low-rankedness in the underlying structure. A recent line of work [20, 1], proposed estimating B^* by estimating the

inverse covariance (or precision) matrix Θ^* of Y using the graphical LASSO (GLASSO) [66, 23]. In particular, [20, 1] showed that Θ^* has non-zeros corresponding to those pairs of vertices that are connected by paths of length at most two; that is, the (i, j) -th entry of Θ^* is non zero if and only if (i, j) is an edge in \mathcal{G} or there is a $k \in [p]$ such that $i - k - j$ is a path in \mathcal{G} . The authors then estimated edges of \mathcal{G} by identifying (and eliminating) the pairs of vertices that have two-hop connections in \mathcal{G} (see Fig. 1)—for future reference, we call this estimator as GLASSO+2HR (2 hop refinement). However, this estimator requires strong structural assumptions on \mathcal{G} such as triangle-freeness. Further, the precision matrix Θ^* of Y is far more dense than the underlying graph \mathcal{G} since $\Theta^* = B^* \Sigma_X^{-1} (B^*)^\top$; this results in sub-optimal data requirements for reliable recovery (see Remark 3). Finally, if Σ_X is a diagonal matrix, we can estimate the sparsity pattern of B^* by taking the principal square root of the empirical covariance matrix of Y . Unfortunately, this method does not allow for any correlation between the node injections which is not the case in practice. Importantly, this method is numerically unstable unless one has a large number of samples (n) of Y so that the empirical covariance matrix is invertible—a requirement that is at odds with the high-dimensional regime where one typically desires n to be smaller than the number of variables p .

In light of the limitations of previous approaches, we study a natural penalized maximum likelihood estimator for B^* using the samples $\{Y_i\}_{i=1}^n$. This estimator is not only statistically efficient but also obviates the restrictive assumptions imposed by the aforementioned methods. We now summarize the main contributions of the paper:

- We propose a novel ℓ_1 regularized maximum likelihood estimator (MLE) for B^* from samples of Y . It is worth noting that the optimization program we propose is not the standard graphical LASSO program [66, 23] as it involves terms that are quadratic in the optimization variable. Our first result shows that, notwithstanding its form, the ℓ_1 regularized MLE is convex in B and it has a unique minimum even in the high-dimensional regime ($n \ll p$) under certain standard conditions.
- Under a new mutual incoherence condition, we provide a sufficient condition on the number of samples n required to recover the exact sparsity of B^* with high probability. Furthermore, under these sufficient conditions we also establish the consistency of our estimator in the element-wise maximum, Frobenius, and spectral norms. Formally, we show that if $n = \Omega(d^2 \log p)$ then with high probability $\|\hat{B} - B^*\|_\infty \in \mathcal{O}(\sqrt{\log p/n})$, where d is the degree of the underlying graph.
- Finally, we complement our theoretical results with experimental results both on the synthetic data sets and data from a benchmark power distribution system. Our experiments demonstrate the clear benefit of the proposed estimator over baseline and competing methods.

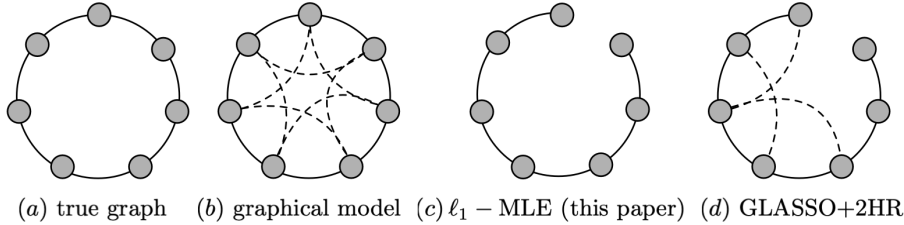


Figure 1: Stylistic visualization of ℓ_1 -MLE vs GLASSO+2HR in [20]. (a) True graph \mathcal{G} with B^* ; (b) graphical model of $\Theta^* = B^* \Sigma_X^{-1} (B^*)^\top$; (c) estimate of B^* by our proposed ℓ_1 -regularized MLE estimator; and (d) estimate of B^* by GLASSO+2HR [20]. Graph \mathcal{G} and the graphical model of Θ^* have same set of vertices; however, in the latter, there are spurious edges (dashed lines in (b)) between vertices that are two-hop neighbors in \mathcal{G} (see main text in Introduction). In Fig 1(c) and 1(d), we depict potential outputs of ℓ_1 -MLE and GLASSO+2HR respectively for some sample covariance matrix. Since the outputs are random objects, the graphs in Fig 1(c) and 1(d) can have missing and false edges that are not present in the true graph in 1(a). The GLASSO+2HR estimate advocated in [20] recovers B^* by first estimating $\Theta^* = B^* \Sigma_X^{-1} (B^*)^\top$ which is potentially dense when compared to B^* (see Fig 1(a) and Fig 1(b)). Hence, the GLASSO+2HR estimate in Fig 1(d) might have more edges than the true graph. However, since our ℓ_1 -MLE directly estimates the sparse matrix B^* , likely, the estimated and the true graph might only differ on a few edges, as shown in Fig 1 (c). Our experimental results in Section 4 validate this claim.

Organization of the paper: In Section 2, we introduce an ℓ_1 -regularized ML estimation problem for networked systems obeying conservation laws. In Section 3, we show that this optimization problem is convex in the objective and establish consistency and support recovery rates for our estimator. In Section 4, we provide simulation results. In Section 5, we summarize our paper with future directions.

Notation: For any two subsets T_1 and T_2 of $[p] \times [p]$, we denote by $A_{T_1 T_2}$ the submatrix of A with rows and columns indexed by T_1 and T_2 , respectively. When $T_1 = T_2$ we denote the submatrix by A_{T_1} . For a matrix $A = (A_{ij}) \in \mathbb{R}^{p \times p}$, we use $\|A\|_\infty \triangleq \max_{i,j} |A_{ij}|$ to denote the maximum element-wise norm, and $\|A\|_F$ and $\|A\|_2$ to denote the Frobenius norm and the operator norm. We denote the ℓ_∞ -matrix norm of A defined as $\nu_A = \|A\|_\infty \triangleq \max_{j=1,\dots,p} \sum_{i=1}^p |A_{ij}|$. We use $\|A\|_{1,\text{off}} = \sum_{i \neq j} |A_{ij}|$ to denote the off-diagonal ℓ_1 norm. We use $\text{vec}(A)$ to denote the p^2 -vector formed by stacking the columns of A and use $\Gamma(A) = (I \otimes A)$ to denote the kronecker product of A with the identity matrix I . For symmetric positive definite matrices A_1 and A_2 , we use $A_1 \succ A_2$ to denote $A_1 - A_2$ is positive definite. We define $\text{sign}(A_{ij}) = +1$ if $A_{ij} > 0$ and $\text{sign}(A_{ij}) = -1$ if $A_{ij} < 0$. For two-real valued functions $f(\cdot)$ and $g(\cdot)$, we write $f(n) = \mathcal{O}(g(n))$ if $f(n) \leq cg(n)$ and $f(n) = \Omega(g(n))$ if $f(n) \geq c'g(n)$ for constants $c, c' > 0$.

2 Problem Setup

Consider a p -dimensional random vector X following the Gaussian distribution $\mathcal{N}(0, \Sigma_X)$ with a known covariance matrix $\Sigma_X \succ 0$ (we outline a relaxation of this assumption in Remark 1). Let $Y = (B^*)^{-1}X$ with a $p \times p$ matrix $B^* \succ 0$ and note that $Y \sim \mathcal{N}(0, \Theta^{*-1})$, where $\Theta^* = B^* \Sigma_X B^*$. Define the sample covariance matrix $S = n^{-1} \sum_{i=1}^n Y_i Y_i^\top$, where $\{Y_1, \dots, Y_n\}$ are the n (possibly $n < p$) i.i.d. samples of Y . For some $\lambda_n > 0$, we consider the ℓ_1 regularized MLE for estimating B^* :

$$\arg \min_{B \succ 0; \Theta = B \Sigma_X^{-1} B^\top} [\text{Tr}(S\Theta) - \log \det(\Theta) + \lambda_n \|B\|_{1,\text{off}}], \quad (2)$$

where $\|B\|_{1,\text{off}} = \sum_{i \neq j} |B_{ij}|$ is the ℓ_1 -norm applied to the off-diagonal entries of $B \in \mathbb{R}^{p \times p}$. The loss function in (2) without the ℓ_1 penalty is the negative log-likelihood of Y , and maximizing it to estimate B^* yields an unrestricted MLE.

The optimization problem in (2) looks similar to the ℓ_1 -regularized log-determinant problem, which has a rich, long history in high-dimensional statistics, machine learning, signal processing, and network sciences (see for instance [35, 3, 66, 23]). The bulk of this literature focuses on estimating Θ^* . The resultant estimator, referred to as the graphical LASSO (or GLASSO), has many nice theoretical properties (e.g., asymptotic consistency and support recovery in the high-dimensional regime) [48, 49, 67]. However, our estimator in (2) is significantly different from GLASSO because we are estimating B^* rather than Θ^* . Other studies close to our setup estimate a sparse Cholesky factor of Θ^* [27, 17, 28]. Recall that the Cholesky decomposition is given by $\Theta^* = LL^\top$ where $L \succ 0$ is a lower triangular matrix. We differ from this line of work on multiple fronts: (i) we do not require B^* to be a lower or upper triangular Cholesky factor; (ii) our method allows for arbitrary correlations between the nodal injections resulting in an extra Σ_X^{-1} between the factors; and (iii) to the best of our knowledge, ours is the first work to provide guarantees on the sample complexity for estimating B^* in the high-dimensional regime.

Remark 1. (Unknown covariance matrix Σ_X). In problem (2), we assume that Σ_X is known. If this is not the case, we can slightly modify (2) to estimate B^*D instead of B^* , where D is the unique square root of Σ_X^{-1} satisfying $D^2 = \Sigma_X^{-1}$. This approach works best if the sparsity of B^* (approximately) equals the sparsity of B^*D , which for instance happens when Σ_X is (approximately) diagonal. \square

Remark 2. (On invertibility of B^*). The invertibility assumption of B^* ensures that B^* is identifiable from samples. This holds in several applications including the ones in (2)-(4) in Section 1. However, this might not be true if B^* is a Laplacian matrix that has k zero eigenvalues. One common work around (see e.g., [24, 20, 21]) is to work with a reduced Laplacian matrix by deleting k rows and columns of B^* (we employ this insight in our experiments; see Section 4). \square

3 A Convex Estimator and Statistical Guarantees

In this section, we first recast the objective in (2) in terms of B for a known Σ_X . We then present our main results on the performance of our estimator in (2) when X is Gaussian and non-Gaussian. We comment on extending our results to other convex loss functions and conclude with an overview of the key steps in proving our results. Full details are given in the appendix.

We begin by rewriting the problem in (2) in a form that is more suitable to our methods of analysis. Let D be the unique square root of Σ_X^{-1} satisfying $D^2 = \Sigma_X^{-1}$ (see [8]). Substituting $B = B^\top$ and

$\Theta = BD^2B^\top$ in the cost function of (2) yields the following:

$$\hat{B} = \arg \min_{B \succ 0} [\text{Tr}(DBSBD) - \log \det(B^2) + \lambda_n \|B\|_{1,\text{off}}], \quad (3)$$

where we use the fact that the trace operator is cyclic and the determinant of a matrix product equals the product of matrix determinants. We dropped constants that have no effect on the estimate. The symmetry and invertibility of B is sufficient enough to ensure that $\log(\cdot)$ is well-defined. In other words, the positive-definiteness assumption is not needed for the well-posedness of (3).

Lemma 1 below is the starting point of our analysis. It establishes two key properties of the estimator in (3) under the positive definiteness of B : (i) loss function in (3) is convex in B and (ii) \hat{B} is unique. The following result is proved in Appendix A.3.

Lemma 1. (Convexity and Uniqueness) For any $\lambda_n > 0$ and $B \succ 0$, if the diagonal elements of the sample covariance matrix $S_{ii} > 0$ for all i , then (i) the ℓ_1 -log determinant problem in (3) is convex and (ii) \hat{B} in (3) is the unique minima satisfying the sub-gradient condition $2D^2\hat{B}S - 2\hat{B}^{-1} + \lambda_n\hat{Z} = 0$. Here \hat{Z} belong to the sub-gradient $\partial\|\hat{B}\|_{1,\text{off}}$ with $\hat{Z}_{ij} = 0$, for $i = j$, and $\hat{Z}_{ij} = \text{sign}(\hat{B}_{ij})$ when $\hat{B}_{ij} \neq 0$ and $|\hat{Z}_{ij}| \leq 1$ when $\hat{B}_{ij} = 0$, for $i \neq j$.

A few comments of Lemma 1 are in order. (Convexity) First, we recall that the compositions of two convex functions is in general not convex. As an example, consider two convex functions $f(x) = x^2$ and $g(x) = -x$, however, $g(f(x)) = -x^2$ is not convex. Therefore, in light of the fact that the loss function is a composite function of B , it is not clear if (3) is convex. Nonetheless, in Lemma 1 we prove that (3) is a convex program. Key to our proof is the notion of monotone convex functions. (Uniqueness) Second, the uniqueness result is non-trivial in high-dimensions ($n < p$) because the Hessian is rank deficient, and hence, the loss function in (3) might not be strictly convex. However, in Lemma 1 (ii) we show that \hat{B} is unique. Key to our proof is the notion of coercivity and it adapts techniques in [48] to the case where the objective function is quadratic in the optimization variable.

3.1 Statement of Main result

Our first result theoretically characterizes the performance of \hat{B} in (3) when Y is Gaussian. Instead, our second result provides such a characterization for \hat{B} when Y is non-Gaussian. At a crude level, our results guarantee that when the number of samples n scales as $d^2 \log p$, our ℓ_1 -regularized MLE correctly recovers the support of B^* and is close to B^* (measured in Frobenius and operator norms) with high probability. Here, d is the degree of the graph underlying B^* .

Since we consider an ℓ_1 regularized log-determinant program for our ML estimator (3), our results might appear similar to that of [48]. However, as also pointed in Section 3.2, our main results, including the assumptions and sufficient conditions needed to derive them, are not subsumed by those in [48], or vice versa (see Remark 3 and Section 4 for more thorough discussion).

We begin with the a few assumptions that are essential to prove our theoretical statements. Similar subset of assumptions in the context of ℓ_1 regularized least squares problem appeared in [63, 57, 39, 68], and in the context of ℓ_1 regularized inverse covariance estimation problem appeared in [48, 67]. We define the edge set $\mathcal{E}(B^*) = \{(i, j) : B_{ij}^* \neq 0, \text{ for all } i \neq j\}$. Let $E := \{\mathcal{E}(B^*) \cup (1, 1) \dots \cup (p, p)\}$ be the augmented set including the diagonal elements. Let E^c be the complement of E .

[A1] Mutual incoherence condition. Let Γ^* be the Hessian of the log-determinant function in (3):

$$\Gamma^* \triangleq \nabla_B^2 \log \det(B)|_{B=B^*} = B^{*-1} \otimes B^{*-1}. \quad (4)$$

For Γ^* in (4), there exists some $\alpha \in (0, 1]$ such that, $\|\Gamma_{E^c E}^* (\Gamma_{EE}^*)^{-1}\|_\infty \leq 1 - \alpha$.

[A2] Hessian regularity condition. Let d be the maximum number of non zero entries among all the rows in B^* (i.e., the degree of the underlying graph), $\Theta^* = B^* \Sigma_X^{-1} B^*$, and $D^2 = \Sigma_X^{-1}$. Then,

$$\|\Gamma^{*-1}\|_\infty \leq \frac{1}{4d\|\Theta^{*-1}\|_\infty \|D^2\|_\infty}. \quad (5)$$

A few comments are in order. **[A1]** Our novel mutual incoherence condition on B^* regulates the influence of irrelevant variables (elements of Hessian restricted to $E^c \times E$) on relevant variables

(elements of Hessian restricted to $E \times E$). The α -incoherence assumption of the above type is standard in literature, and [48] demonstrates its validity for several graphs, including chain and grid graphs, which we will explore in experimental section. Notice that the α -incoherence in [48] is imposed on Θ^* . Instead, we require it on B^* . [A2] This condition is in parallel with bounding the maximum eigenvalue of Θ^{*-1} condition for estimating sparse Θ^* (see for e.g., [49, 30]).

Our problem set-up assumes that the injected flows $X_i, i \in [p]$, are Gaussian. Given that the vector of node potentials Y depends on X via the *balance equation* (see equation (1)), we have that Y is Gaussian. However, we work with sub-Gaussian distributions, a natural generalization to the Gaussian case, which encompasses many well known distributions that occur in practice (for e.g., bounded random variables, gaussians and mixture of gaussians). We define this distributional assumption below.

Definition 1. (*Sub-Gaussian random variable*) A zero mean random variable Z is said to be sub-Gaussian if there exists a constant $\sigma > 0$ such that for any $t \in \mathbb{R}$, $\mathbb{E}[\exp(tZ)] \leq \exp(\sigma^2 t^2/2)$.

Our first main result below provides sufficient conditions on the number of samples n needed for \hat{B} in (3) to exactly recover the sparsity structure of B^* and to achieve sign consistency, defined as $\text{sign}(\hat{B}_{ij}) = \text{sign}(B_{ij}^*)$, for all $(i, j) \in E$. We recall that $\nu_A = \|A\|_\infty \triangleq \max_{j=1, \dots, p} \sum_{i=1}^p |A_{ij}|$ and define $\Sigma^* = \Theta^{*-1}$ to be the covariance matrix of the node potential Y .

Theorem 1. (*Support Recovery: Sub-Gaussian*) Let $X = (X_1, \dots, X_p)$ be the vector of injected flows. Suppose that for all i , $X_i/\sqrt{\Sigma_X(ii)}$ is sub-Gaussian with parameter σ and assumptions [A1-A2] hold. Let the regularization parameter $\lambda_n = C_0 \sqrt{\tau(\log 4p)/n}$, where C_0 is given below. Given n independent samples from Y , if the sample size $n > C_1^2 d^2 (\tau \log p + \log 4)$, the following hold with probability at least $1 - \frac{1}{p^{\tau-2}}$, for some $\tau > 2$:

- (a) \hat{B} exactly recovers the sparsity structure of B^* ; that is, $\hat{B}_{E^c} = 0$,
- (b) \hat{B} satisfies the element-wise ℓ_∞ bound $\|\hat{B} - B^*\|_\infty \leq C_2 \sqrt{\frac{\tau \log p + \log 4}{n}}$, and
- (c) \hat{B} satisfies sign consistency if $|B_{\min}^*| \geq 2C_2 \sqrt{\frac{\tau \log p + 4}{n}}$, $B_{\min}^* \triangleq \min_{(i,j) \in \mathcal{E}(B^*)} |B_{ij}^*|$,

where $C_1 = 192\sqrt{2}[(1+4\sigma^2)\kappa(\Sigma^*) \text{Tr}(\Sigma_X)\nu_{D^2}\nu_{B^*}\nu_{B^{*-1}}] \max\{\nu_{\Gamma^{*-1}}\nu_{B^{*-1}}, 2\nu_{\Gamma^{*-1}}^2\nu_{B^{*-1}}^3, 2\alpha^{-1}d^{-1}\}$, $C_2 = [64\sqrt{2}(1+4\sigma^2)\kappa(\Sigma^*) \text{Tr}(\Sigma_X)\nu_{\Gamma^{*-1}}\nu_{D^2}\nu_{B^*}\nu_{B^{*-1}}]$, $C_0 = C_2/(4\nu_{\Gamma^{*-1}})$ and $\kappa(\cdot)$ is the condition number.

The quantities $(\nu_{\Gamma^{*-1}}, \nu_{D^2}, \nu_{B^*}, \kappa(\Sigma^*), \text{Tr}(\Sigma_X))$ capture the inherent complexity of the model and do not depend on the number of samples n . As long as the magnitude of the entries in Γ^{*-1} , D^2 , and B^* scale as $\mathcal{O}(1/d)$, the model complexity parameters do not depend on (p, d) . That is, as the size of the network grows with (p, d) the edge strengths decay with d . Suppose that the model complexity parameters are constants and that $n = \Omega(d^2 \log p)$. Then part (a) of Theorem 1 guarantees that our ML estimator does not falsely include entries (or edges in the underlying graph) that are not in the support of B^* . Part (b) establishes the element-wise ℓ_∞ norm consistency of \hat{B} ; that is, $\|\hat{B} - B^*\|_\infty = \mathcal{O}(\sqrt{(\log p)/n})$. Finally, part (c) establishes sign consistency of \hat{B} , and hence, our estimator does not falsely exclude entries that are in the support of B^* . Crucial is the requirement of $|B_{\min}^*| = \Omega(\sqrt{(\log p)/n})$, which puts a limit on the minimum (in absolute) value of the entries in B^* . This condition parallels the familiar *beta-min* condition in the LASSO literature (see [63, 59]).

We now present a corollary to Theorem 1 that gives consistency rates of convergence for \hat{B} in the Frobenius and operator norms. Let $\mathcal{E}(B^*) = \{(i, j) : B_{ij}^* \neq 0, \text{ for all } i \neq j\}$ be the edge set of B^* .

Corollary 1. Let $s = |\mathcal{E}(B^*)|$ be the cardinality of $\mathcal{E}(B^*)$. Under the same hypotheses in Theorem 1, with probability greater than $1 - \frac{1}{p^{\tau-2}}$, the estimator \hat{B} satisfies

$$\|\hat{B} - B^*\|_F \leq C_2 \sqrt{\frac{(s+p)(\tau \log p + 4)}{n}} \quad \text{and} \quad \|\hat{B} - B^*\|_2 \leq C_2 \min\{d, \sqrt{s+p}\} \sqrt{\frac{\tau \log p + 4}{n}}.$$

Proof sketch. Both the Frobenius and operator norm bounds follows by applying standard matrix norm inequalities to the ℓ_∞ consistency bound in part (b) of Theorem 1. Importantly, $s + p$ is the

bound on maximum number of non-zero entries in B^* , where s , by definition, is the total number of off-diagonal non-zeros in B^* . Complete details are provided in Appendix A.3.

Thus far we have assumed that the nodal potentials Y_i are sub-Gaussian random variables. We now explore another broad class of random variables with bounded k^{th} moments, which are known to have tails that decay according to some power law [44]. An important example of power law distributions are Pareto distributions which finds applications in a wide variety of areas [43, 41]. Motivated by such important practical considerations, we state our next result for random variables with bounded moments. We begin with the following definition.

Definition 2. (Bounded moments) A random variable Z is said to have bounded $4k^{\text{th}}$ moment if there exists a constant $M_k \in \mathbb{R}$ such that $\mathbb{E}[(Z)^{4k}] \leq M_k$.

Results below parallel Theorem 1 and Corollary 1 for random variables with bounded moments.

Theorem 2. (Support Recovery: Bounded Moments) Let $X = (X_1, \dots, X_p)$ be the vector of injected flows such that for all i , the node potential $Y_i/\sqrt{\Sigma_{ii}^*}$ has bounded moment as in Definition 2 and assumptions [A1-A2] hold. Let the regularization parameter $\lambda_n = C_0\sqrt{\tau(\log 4p)/n}$, with C_0 defined in Theorem 1. Given n independent samples from Y , if the sample size $n > C_4 d^2 p^{\tau/k}$, then with probability greater than $1 - 1/p^{\tau-2}$, for some $\tau > 2$, the following hold: (a) \hat{B} exactly recovers the sparsity structure of B^* (that is $\hat{B}_{E^c} = 0$); (b) the element-wise ℓ_∞ bound $\|\hat{B} - B^*\|_\infty \leq C_5 \sqrt{p^{\tau/k}/n}$; and (c) \hat{B} satisfies sign consistency if $|B_{\min}^*| \geq 2C_5 \sqrt{p^{\tau/k}/n}$.

The constants and their dependence on the model complexity parameters are given in Appendix A.3.

Corollary 2. Suppose the hypotheses in Theorem 2 hold. Then with probability greater than $1 - \frac{1}{p^{\tau-2}}$: $\|\hat{B} - B^*\|_F \leq C_5 \sqrt{\frac{(s+p)(p^{\tau/k})}{n}}$ and $\|\hat{B} - B^*\|_2 \leq C_5 \min\{d, \sqrt{s+p}\} \sqrt{\frac{p^{\tau/k}}{n}}$, where $s = |\mathcal{E}(B^*)|$.

Interpretations of Theorem 1 and Corollary 1 also hold for Theorem 2 and Corollary 2. However, in this setting, we have different sample size $n = \Omega(d^2 p^{\tau/k})$ and $|B_{\min}^*| = \Omega(\sqrt{p^{\tau/k}/n})$, where k is given by Definition 2. In contrast, for sub-Gaussian case we have logarithmic dependence in p (the number of vertices). Finally, albeit fundamentally different from GLASSO estimator, we were able to obtain consistency rates for \hat{B} (3) that are similar to those in [48, 12].

Remark 3. (Comparison with the GLASSO estimator). For simplicity, suppose that Σ_X is diagonal. Then, it follows that $B^* \succ 0$ is the unique square root of $\Theta^* = (B^*)^2$. Thus, a naïve way to estimate B^* is by taking the square root of the GLASSO estimate $\hat{\Theta}$. Let us call this estimator \hat{B}_{SR} and note that \hat{B}_{SR} inherits its optimal properties from $\hat{\Theta}$. We show that $\hat{\Theta}$ has sub-optimal estimation rate than \hat{B} in (3) for estimating B^* . Let B^* contains d non-zero elements in every row. Then the underlying graph of Θ^* is a two-hop network with degree d^2 . Using sample complexity results from [48], it follows that $\hat{\Theta}$ requires $n = \Omega(d^4 \log p)$ to estimate B^* . Instead, our ℓ_1 -regularized MLE requires $n = \Omega(d^2 \log p)$ samples. This reduction is more pronounced for networks with a large degree d . \square

3.2 Outline of Main Analysis

We provide an outline of our methods and main strategies to prove Theorem 1. We employ the *primal-dual witness technique*—a well-known method used derive to statistical guarantees for sparse convex estimators [63, 64, 34]. This technique involves constructing a primal-dual pair (\tilde{B}, \tilde{Z}) satisfying the zero-subgradient condition of the convex problem in (3), such that (the primal) \tilde{B} has the correct (signed) support. Suppose this construction succeeds, from the uniqueness result in Lemma 1, it follows that $\hat{B} = \tilde{B}$, and the dual \tilde{Z} is an optimal solution to the dual of (1). Thus, at the heart of our analysis is in showing that the primal-dual construction succeeds with high-probability. Similar technique is also used to prove Theorem 2 (i.e., the non-Gaussian case); see Appendix A.3.

While our proof methods are inspired from [48, 63], our analysis is more involved due to the presence of B^2 , as opposed to B , in the loss function of (3). Consequently, we require more nuanced assumption (as in [A2]) and dual feasibility condition than the ones in [48] (see below).

3.3 Primal-dual pair and supporting lemmas

We briefly introduce the primal-dual witness construction. In Lemma 2, we provide sufficient conditions under which this construction succeeds.

We construct the primal-dual pair (\tilde{B}, \tilde{Z}) as follows. The primal solution \tilde{B} is determined by solving

$$\tilde{B} \triangleq \arg \min_{B=B^T, B \succ 0, B_{E^c}=0} [\text{Tr}(DBSBD) - \log \det(B^2) + \lambda_n \|B\|_{1,\text{off}}]. \quad (6)$$

Here (6) is a restricted problem, in that, we impose $B_{E^c} = 0$. Also, we have $\tilde{B} \succ 0$ and $\tilde{B}_{E^c} = 0$. The dual $\tilde{Z} \in \partial \|\tilde{B}\|_{1,\text{off}}$ is chosen such that it satisfies the zero-subgradient condition of (6). This is obtained by setting $2\lambda_n \tilde{Z}_{ij} = [\tilde{B}^{-1}]_{ij} - [D^2 \tilde{B} S]_{ij}$, for all $(i, j) \in E^c$. It can be verified that (\tilde{B}, \tilde{Z}) satisfies the zero-subgradient condition (see the statement of Lemma 1) of the original problem in (3). Thus, it remains to establish the strict dual feasibility condition; that is $|\tilde{Z}_{ij}| < 1$, for any $(i, j) \in E^c$.

We introduce some notation. Let $W \triangleq S - \Theta^{*-1}$ be a measure of noise in the data, where S is the sample covariance and Θ^{*-1} is the true covariance of Y . Let $\Delta \triangleq \tilde{B} - B^*$ be a measure of distortion between the primal solution \tilde{B} as defined in equation (6) and the true matrix to be estimated B^* . We also need the higher order terms (denoted by $R(\Delta)$) of the Taylor expansion of the gradient $\nabla \log \det(\tilde{B})$ centered around B^* [10]:

$$\nabla \log \det(\tilde{B}) = B^{*-1} + B^{*-1} \Delta B^{*-1} + \underbrace{\tilde{B}^{-1} - B^{*-1} - B^{*-1} \Delta B^{*-1}}_{\triangleq R(\Delta)}. \quad (7)$$

Lemma 2. (Sufficient conditions for strict dual feasibility) *Let the regularization parameter $\lambda_n > 0$ and α be defined as in [A1]. Suppose the following holds*

$$\max \left\{ \|\Gamma(D^2 \Delta) + \Gamma(D^2 B^*)\|_\infty \|W\|_\infty, \|R(\Delta)\|_\infty, \|\Gamma(D^2 \Delta)\|_\infty \|\Theta^{*-1}\|_\infty \right\} \leq \frac{\lambda_n \alpha}{24}. \quad (8)$$

Then the dual vector \tilde{Z}_{E^c} satisfies $\|\tilde{Z}_{E^c}\|_\infty < 1$, and hence, $\tilde{B} = \hat{B}$.

Proof sketch: The proof essentially involves expressing the sub-gradient condition in Lemma 1 as a vectorized form using $R(\Delta)$ (in (7)) and W . By manipulating the vectorized sub-gradient condition, we obtain an expression of \tilde{Z}_{E^c} that is a function of the quantities in (8). We finish off the proof by repeated applications of triangle inequality of norms and invoking assumptions in Lemma (8).

The following results provides us with dimension and model complexity dependent bounds on the reminder term $R(\Delta)$ in (7) and the distortion Δ .

Lemma 3. (Control of reminder) *Suppose that the element-wise ℓ_∞ -bound $\|\Delta\|_\infty \leq \frac{1}{3\nu_{B^*-1} d}$ holds, where $\nu_{B^*-1} = \|B^{*-1}\|_\infty$. Then $\|R(\Delta)\|_\infty \leq \frac{3}{2} d \|\Delta\|_\infty^2 \nu_{B^*-1}^3$.*

The proof, adapted from [48], is algebraic in nature and relies on certain matrix expansions. The details are provided in Appendix. In the following result, we provide a sufficient condition under which the element-wise ℓ_∞ -bound on Δ in Lemma 3 holds.

Lemma 4. (Control of Δ) *Let $r \triangleq 4\nu_{\Gamma^*-1} [\nu_{D^2} \nu_{B^*} \|W\|_\infty + 0.5\lambda_n]$ $\leq \min \left\{ \frac{1}{3\nu_{B^*-1} d}, \frac{1}{6\nu_{\Gamma^*-1} \nu_{B^*}^3} \right\}$. Then we have the element-wise ℓ_∞ bound $\|\Delta\|_\infty = \|\tilde{B} - B^*\|_\infty \leq r$.*

Proof sketch: By construction $\tilde{B}_{E^c} = B_{E^c}^* = 0$. Hence, $\|\Delta\|_\infty = \|\Delta_E\|_\infty$, where $\Delta_E = \tilde{B}_E - B_E^*$. We construct a continous vector valued function $F : \Delta_E \rightarrow \Delta_E$ that has a unique fixed point. Invoking assumptions [A1]-[A2], we show that $F(\cdot)$ is a contractive map on the ℓ_∞ ball defined as $\mathbb{B}_r = \{A : \|A\|_\infty \leq r\}$ with r defined in the statement of the lemma. Specifically, we show that $F(\mathbb{B}_r) \subseteq \mathbb{B}_r$. Finally, we finish off the proof by an application of Brower's fixed point theorem [29] to show that the unique fixed point is inside \mathbb{B}_r . Consequently, $\|\Delta\|_\infty \leq r$.

Finally, the result in Theorem 1 follows by putting these lemmas together for an appropriate choice of λ_n and the sample size requirement, and there upon, invoking some known concentration inequalities. Refer Appendix A.3 for complete details.

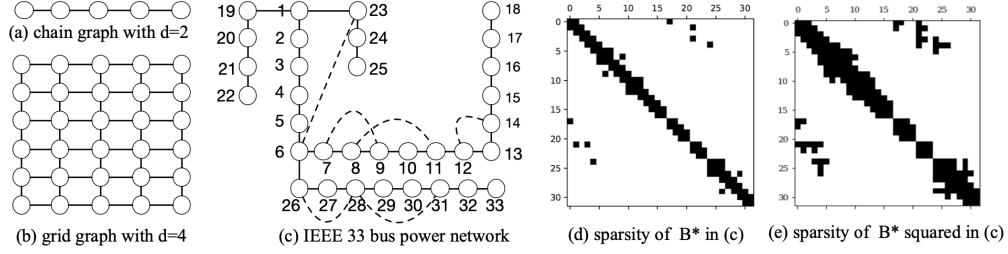


Figure 2: Graphs used in experiments. (a) Chain graph with maximum degree $d = 2$. (b) Grid graph $d = 4$. (c) IEEE 33 bus (node) distribution network with additional loops (shown in dashed lines). (d) Sparsity of B^* associated with the IEEE 33 bus network. (e) Sparsity of $(B^*)^2$. Notice that $(B^*)^2$ is denser relative to B^* . Consequently, GLASSO+HR needs more samples than ℓ_1 -MLE to recover the support (see plots in Fig. 3).

4 Experiments

We validate the support recovery performance of our ℓ_1 -regularized MLE on synthetic and a benchmark power distribution network (see Fig. 2). We choose λ_n proportional to $\sqrt{\log p/n}$. Our results are averaged over 100 trials of n independent samples of Y . We compare ℓ_1 -regularized MLE performance with (i) the square-root estimator (hereafter, GLASSO+SR) that identifies the support of B^* by determining (i, j) for which $\hat{\Theta}_{i,j}^{\frac{1}{2}} \neq 0$; and (ii) the GLASSO+2HR (Hop Refinement) estimator [20] that identifies the support of B^* by determining (i, j) for which $\hat{\Theta}_{i,j} \leq -\tau$ for $\tau = 1e - 02$. Here $\hat{\Theta}$ is the GLASSO estimate of the inverse covariance matrix of Y [23]. These estimators are described in detail in Introduction. To have a fair comparison with the GLASSO based estimators, we set Σ_X (X is the injected vector) to be diagonal. However, as discussed earlier, our ℓ_1 regularized ML estimator works for any $\Sigma_X \succ 0$. We consider p to be as large as 64 nodes. Computational examples involving large data matrices for B^* having a lower triangular matrix form has been reported in [28].

(i) *Synthetic data*: We consider two undirected graphs for B^* , the chain graph and the grid graph for $p = \{32, 64\}$ nodes. We set $B_{ij}^* = 1$ for $(i, j) \in E$ and $B_{ij}^* = 0$ for $(i, j) \in E^c$, where E can be the edge set of the chain or grid graph. We then adjust the diagonal elements of B^* to ensure $B^* \succ 0$.

(ii) *Power network*: We set B^* to be the Laplacian of the IEEE 33 bus power distribution network [69]. For this data, we note that B^* is non-invertible because of one zero eigenvalue. We obtain the reduced B^* by deleting the first row and column of B^* . We also slightly modify the network by adding three loops of cycle length three, two of cycle length four, and one loop of cycle length five (see Fig 2). We made these modifications to highlight that ℓ_1 -MLE imposes no connectivity assumptions on the graph underlying B^* , except sparsity. In contrast, GLASSO+2HR estimator [20] restricts the graph underlying B^* from having cycles of length three (i.e., triangle-free).

In Fig. 3, we show empirical support recovery probabilities for all three estimators as a function of the number of samples n . Both on synthetic and power network data, ℓ_1 -MLE achieved superior rates than the other competing estimators. In fact, ℓ_1 -MLE exactly recovers the support of B^* when the number of samples is in the order of $d^2 \log p$, which is in excellent agreement with the proposed theory. Instead, for a similar performance, GLASSO+SR needed $d^4 \log p$ samples (see Remark 3).

5 Discussions and Future Work

High dimensional networks obeying conservation laws of the form $X = B^*Y$ are often used to model and study interactions among different conserved quantities in various engineering and scientific disciplines. For such systems, with unknown structure of the network, or equivalently the sparsity pattern of B^* , we estimate the the sparsity structure of B^* using an ℓ_1 -regularized ML estimator. Our estimator relies on n samples of the node potentials Y , with some knowledge of the statistics of the node injections X . We showed that this estimator is the unique optimal solution to a variant of the convex log-determinant program. Using novel mutual incoherence conditions, we have provided (theoretical) sparsity and support recovery consistency for our estimator. Finally, using several numerical results we not only validated our theory but also showed that our ℓ_1 -regularized estimator outperforms methods that employ GLASSO based estimators.

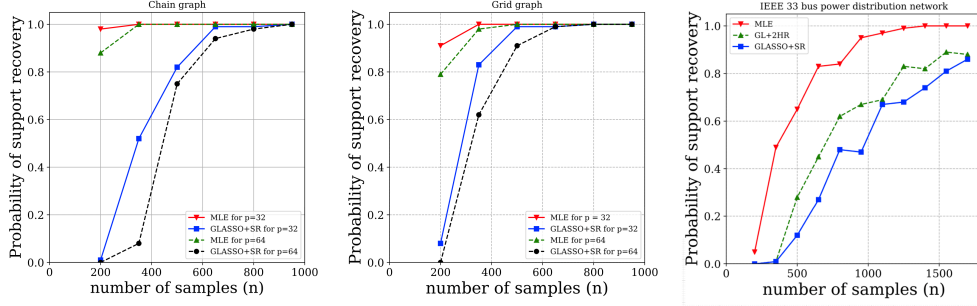


Figure 3: Empirical probability of success of various estimators versus the raw sample size n for (left) chain graph, (middle) grid graph, and (right) IEEE 33 bus network. For chain and grid graph, we compare our ℓ_1 regularized MLE performance with GLASSO+SR for $p \in \{32, 64\}$. Instead, for IEEE 33 bus network, we compare ℓ_1 regularized MLE with GLASSO+SR and GLASSO+2HR.

In our setup, we neither require actual injected flows (X), nor B^* to be a Laplacian matrix, nor $n > p$, thereby allowing our learning problem to be general enough to be applicable for a variety of domains ranging from electrical networks to social networks. Consequently, our framework and theoretical results admits many future extensions and refinements. For instance, verifying the assumptions [A1-A2] in practice is a fruitful future direction. It is worth noting that, in order to run the proposed algorithm/estimator, one need not verify the assumptions. Notice that from Lemma 1, the loss function/objective is convex and admits a unique solution if the regularization constant $\lambda_n > 0$ and B is positive definite, therefore the unknown structure of the network can be recovered by finding the unique minima. However, in order to theoretically guarantee the properties of the estimator mentioned in Theorem 1 and Theorem 2, assumptions [A1-A2] are needed. A possible extension to the current problem is to recast the objective function in (2) as the minimization of the Bregman divergence for more general loss functions. Another promising direction is to generalize (2) to include non-invertible network matrix B^* . We leave this as our future work.

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 - (b) Did you mention the license of the assets? [\[N/A\]](#)
 - (c) Did you include any new assets either in the supplemental material or as a URL? [\[N/A\]](#)
 - (d) Did you discuss whether and how consent was obtained from people whose data you’re using/curating? [\[N/A\]](#)
 - (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [\[N/A\]](#)
5. If you used crowdsourcing or conducted research with human subjects...
 - (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [\[N/A\]](#)
 - (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [\[N/A\]](#)
 - (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [\[N/A\]](#)

A Appendix

A.1 Code Availability

In Section 4, we performed numerical experiments to validate the support recovery property of the proposed ℓ_1 regularized ML estimator and compare its performance with the square root estimator (GLASSO+SR) and what we call the GLASSO+2HR (2 Hop Refinement) estimator [20]. Our benchmark examples include a synthetic data for chain and grid graphs, and a real-world electric power system network. We implement all three estimators using CVXPY 1.2 open source python package on Google Colab. All the simulation results reported in this paper can be reproduced using the code available at <https://github.com/AnirudhRayas/SLNSCL>.

A.2 Limitations

In this work we restricted B^* in the model $B^*Y - X = 0$ to be invertible and positive definite. In applications such as transportation networks, B^* might not be symmetric, and hence, not positive definite. For this setting, it is not clear if our technical analysis hold. Another limitation is that the our sample complexity result depends quadratically in the maximum degree d . This is because of the proof technique we adapted from [48]. Decreasing this dependence from d^2 to d is an open question. Finally, as is well known, verifying regularity conditions, such as the mutual incoherence,² in practice is computationally hard. Hence, there is a need to develop regularity conditions that are easily verifiable in practice.

A.3 Proofs of all technical results

Overview: We begin with a brief overview of the problem set-up and state the necessary assumptions. Then, we provide proofs for all the technical results. Recall that our observation model is $Y = B^{*-1}X$, where B^* is a $p \times p$ sparse matrix which encodes the structure of a network with the property that $B_{ij}^* = 0$ for all $(i, j) \in E^c$, $Y \in \mathbb{R}^p$ is the vector of node potentials and $X \in \mathbb{R}^p$ is the unknown random vector of injected flows with known covariance matrix Σ_X . Given n i.i.d samples of the vector Y our goal is to learn the sparsity structure of the matrix B^* . Towards this we propose an estimator \hat{B} which is the solution of the following ℓ_1 regularized log-det problem

$$\hat{B} = \arg \min_{B \succ 0} [\text{Tr}(DBSBD) - \log \det(B^2) + \lambda_n \|B\|_{1,\text{off}}]. \quad (9)$$

where $D \in \mathbb{R}^{p \times p}$ is the unique square root of Σ_X^{-1} and S is the sample covariance matrix constructed from n samples of the random vector Y . We recall the assumptions necessary to prove our results.

[A1] Mutual incoherence condition. Let Γ^* be the Hessian of the log-determinant function in (9):

$$\Gamma^* \triangleq \nabla_B^2 \log \det(B)|_{B=B^*} = B^{*-1} \otimes B^{*-1}. \quad (10)$$

For Γ^* in (10), there exists some $\alpha \in (0, 1]$ such that $\|\Gamma_{E^c E}^* (\Gamma_{EE}^*)^{-1}\|_\infty \leq 1 - \alpha$.

[A2] Hessian regularity condition. Let d be the maximum number of non zero entries among all the rows in B^* (i.e., the degree of the underlying graph), $\Theta^* = B^* \Sigma_X^{-1} B^*$, and $D^2 = \Sigma_X^{-1}$. Then,

$$\|\Gamma^{*-1}\|_\infty \leq \frac{1}{4d \|\Theta^{*-1}\|_\infty \|D^2\|_\infty}. \quad (11)$$

Our analysis is based on the Primal-Dual Witness (PDW) construction to certify the behaviour of the estimator \hat{B} . The PDW technique consists of constructing a primal-dual pair (\tilde{B}, \tilde{Z}) , where \tilde{B} is the primal solution of the restricted log-det problem defined below

$$\tilde{B} \triangleq \arg \min_{B=B^T, B \succ 0, B_{E^c} = 0} [\text{Tr}(DBSBD) - \log \det(B^2) + \lambda_n \|B\|_{1,\text{off}}]. \quad (12)$$

where \tilde{Z} is the optimal dual solution. By definition the primal solution \tilde{B} satisfies $\tilde{B}_{E^c} = B_{E^c}^* = 0$. Furthermore the pair (\tilde{B}, \tilde{Z}) are solutions to the zero gradient conditions of the restricted problem

²Interestingly, this condition is necessary and sufficient for sparse linear regression problems [63].

(12). Therefore, when the PDW construction succeeds the solution \hat{B} is equal to the primal solution \tilde{B} which guarantees the support recovery property ie. $\hat{B}_{E^c} = 0$.

We now summarize our technical results. ❶ We begin by showing that the ℓ_1 regularized log-det problem in (9) is convex and admits a unique solution \hat{B} (see Lemma 1). ❷ We then proceed to derive the sufficient conditions under which the PDW construction succeeds (see Lemma 2). ❸ We then guarantee that the remainder term $R(\Delta)$ is bounded if Δ is bounded (see Lemma 3). ❹ Furthermore, for a specific choice of radius r as a function of $\|W\|_\infty$ we show that Δ lies in a ball \mathbb{B}_r of radius r (see Lemma 4). ❺ We then derive a lemma which we call *the master lemma* which gives support recovery guarantees and element-wise ℓ_∞ norm consistency for our estimator \hat{B} under no specific distributional assumptions (see Lemma A.6). ❻ Using known concentration results on sub-gaussian and moment bounded random vectors we prove our main result for the two distributions mentioned above. Recall that our main result gives sufficient conditions on the number of samples required for our estimator \hat{B} to recover the exact sparsity structure of B^* . We also show that under these sufficient conditions \hat{B} is consistent with B^* in the element-wise ℓ_∞ norm and achieves sign consistency if $|B_{\min}^*|$ (the minimum non-zero entries of B^*) is lower bounded (see Theorem 1 and Theorem 2). ❼ Finally, we show that \hat{B} is consistent in the Frobenius and spectral norm.

Numbering convention: To make the appendix self contained we restated statements of all theorems, lemmas, and definitions with their numbers unchanged with respect to the main text. For the numbered environments that are specifically introduced in Appendix, the environment begins with the label "A" (e.g., Lemma A.1).

Lemma 1. (Convexity and uniqueness) For any $\lambda_n > 0$ and $B \succ 0$, if the diagonal elements of the sample covariance matrix $S_{ii} > 0$ for all i , then (i) the ℓ_1 -log determinant problem in (9) is convex and (ii) \hat{B} in (9) is the unique minima satisfying the sub-gradient condition $2D^2\hat{B}S - 2\hat{B}^{-1} + \lambda_n\hat{Z} = 0$. Here \hat{Z} belong to the sub-gradient $\partial\|\hat{B}\|_{1,\text{off}}$ so that $\hat{Z}_{ij} = 0$, for $i = j$, and $\hat{Z}_{ij} = \text{sign}(\hat{B}_{ij})$ when $\hat{B}_{ij} \neq 0$ and $|\hat{Z}_{ij}| \leq 1$ when $\hat{B}_{ij} = 0$, for $i \neq j$.

Proof. (i) *Convexity:* Let $S = MM^T$, for some $M \succeq 0$, and recall that $\|A\|_F^2 = \text{Tr}(AA^T)$. Then, the objective function in (9) can be expressed as

$$\|DBM\|_F^2 - \log \det(B^2) + \lambda_n \|B\|_{1,\text{off}}. \quad (13)$$

First, the square-root of the first term is convex because for any $\lambda \in (0, 1)$ and $B_1, B_2 \succ 0$, we have

$$\begin{aligned} \|D(\lambda B_1 + (1 - \lambda)B_2)M\|_F &= \|\lambda DB_1M + (1 - \lambda)DB_2M\|_F \\ &\leq \lambda \|DB_1M\|_F + (1 - \lambda) \|DB_2M\|_F. \end{aligned}$$

Now that $h_1(x) = x^2$ and $h_2(A) = \|A\|_F$ are both convex and that $h_1(x)$ is non-decreasing on the range of h_2 , that is, $[0, \infty]$, it follows that the composition $h_1 \circ h_2 = \|\cdot\|_F^2$ is convex.

Second, we show the convexity of $-\log \det(B^2)$ using the perspective function technique [10]. To this end, let $|\cdot|$ be the absolute value and note that $\log \det(B^2) = \log |\det(B^2)| = 2 \log |\det(B)|$. Let $g(t) = \log |\det(B + tV)|$ with $V \succeq 0$ be the perspective function of $\log |\det(B)|$. Since B is symmetric and invertible, there exists an orthogonal matrix Q such that $QQ^T = I$ and $B = Q\Omega Q^T$, where Ω is a diagonal matrix consisting of eigenvalues of B . Then,

$$g(t) = \log(|\det(Q\Omega Q^T + tQQ^TVQQ^T)|) \quad (14)$$

$$= \log(|\det(Q(\Omega + tQ^TVQ)Q^T)|) \quad (15)$$

$$= \log(|\det(\Omega + tQ^TVQ)|) \quad (16)$$

$$= \log(|\det(I + t\Omega^{-1}Q^TVQ)|) + \log(|\Omega|), \quad (17)$$

where we used the facts $|\det(X_1X_2)| = |\det(X_1)\det(X_2)| = |\det(X_1)||\det(X_2)|$ and Λ is full rank. Since Ω is diagonal and $Q^TVQ \succeq 0$, it follows that the eigenvalues $\{\lambda_i\}$ of $\Omega^{-1}Q^TVQ$ are real-valued (need not be positive). Thus,

$$g(t) = \log \prod_i |1 + t\lambda_i| + \log(|\Omega|).$$

Notice that $g'(t) = \sum \frac{\lambda_i(1+t\lambda_i)}{(1+t\lambda_i)^2}$ and $g''(t) = -\sum \frac{\lambda_i^2}{(1+t\lambda_i)^2} < 0$. Thus $g(t)$ is strictly concave. Hence, $2 \log |\det(B)| = \log \det(B^2)$ is strictly concave. Finally, $-\log \det(B^2)$ is strictly convex.

Third, the norm $\lambda_n \|B\|_{1,\text{off}}$ is the sum of absolute values of off-diagonal terms, and hence, convex. Because the sum of convex functions and a strictly convex function is strictly convex, we conclude that the objective function in (14) is strictly convex.

Remark: In the proof, we used the fact that B is symmetric and full rank but not the positive-definite. The proof for $B \succ 0$ is simple because we can drop the absolute values and mimic the standard log-det concavity proof [48, 23]. Finally, we required $V \succeq 0$ to not to deal with the (possible) imaginary eigenvalues of $\Omega^{-1}Q^\top VQ$. However, we conjecture that V needs to be only symmetric.

(ii) *Uniqueness:* In part (i), we showed that the objective function in (14) is strictly convex. Recall that strictly convex functions have the property that the minimum is unique if attained [10]. We show that the minimum is attained using the notion of coercivity (see Def 11.10 and Proposition 11.14 in [5]). This amounts to showing that the objective function $CO(B) \triangleq (\|DBM\|_F^2 - 2 \log \det B)$ subject to constraints (see below) tend to infinity as $\|B\|_2 \rightarrow \infty$.

By Lagrangian duality, the ℓ_1 regularized log-det problem (9) can be written as

$$\arg \min_{B \succ 0, B=B^T, \|B\|_{1,\text{off}} < \Lambda_n} \|DBM\|_F^2 - 2 \log \det B, \quad (18)$$

where Λ_n is the constraint on the off diagonal elements of B . From the constraint $\|B\|_{1,\text{off}} < \Lambda_n$, it follows that the off-diagonal elements of B lie in an ℓ_1 ball. Thus, $\|B\|_2 \rightarrow \infty$ if and only if for any sequence of diagonal elements $\| [B_{11}, \dots, B_{pp}] \|_\infty \rightarrow \infty$. On the other hand, by Hadamard's inequality for positive definite matrices [26], we have $2 \log \det B \leq \sum_k 2 \log B_{kk}$. Thus,

$$\|DBM\|_F^2 - 2 \log \det B \geq \|DBM\|_F^2 - 2 \sum_k \log B_{kk}. \quad (19)$$

We now lower bound $\|DBM\|_F^2$. Consider the following inequality:

$$\|DBM\|_F^2 = \sum_{i,j} [DBM]_{ij}^2 \quad (20)$$

$$= \sum_{i,j,k,l} (D_{ik} B_{kl} M_{lj})^2 \quad (21)$$

$$\geq \sum_{i,j,k=l} (D_{ik} B_{kk} M_{kj})^2 \quad (22)$$

$$= \sum_k (B_{kk})^2 \sum_{i,j} (D_{ik})^2 (M_{kj})^2. \quad (23)$$

The inequality in (22) follows because the off-diagonal elements of the matrix B are non-negative (i.e., $\sum_{i,j,k \neq l} (D_{ik} B_{kl} M_{lj})^2 \geq 0$). Rewriting equation (19) using the lower bound from equation (23), we have

$$\|DBM\|_F^2 - 2 \log \det B \geq \sum_k \left[(B_{kk})^2 \sum_{i,j} (D_{ik})^2 (M_{kj})^2 - 2 \log B_{kk} \right] \quad (24)$$

If the term $\sum_{i,j} (D_{ik})^2 (M_{kj})^2 > 0$, then for any sequence $\| [B_{11}, \dots, B_{pp}] \|_\infty \rightarrow \infty$, the quadratic term $(B_{kk})^2$ in the right hand side of the lower bound in equation (24) dominates the logarithmic term $\log B_{kk}$ for all k . Therefore the objective function $CO(B) = \|DBM\|_F^2 - 2 \log \det B \geq \infty$ as $\| [B_{11}, \dots, B_{pp}] \|_\infty \rightarrow \infty$. This implies that $CO(B)$ is coercive and a unique minima exists. Now it remains to show that the term $\sum_{i,j} (D_{ik})^2 (M_{kj})^2 > 0$.

One one hand, by assumption, the diagonal elements of the sample covariance matrix S are strictly positive. On the other hand, $S = M^2$, where M is the unique positive semi-definite square root. Thus,

$$S_{kk} = \sum_j (M_{kj})^2 > 0. \quad (25)$$

By the same logic, the following inequality holds for $D^2 = \Sigma_X^{-1}$, where Σ_X is positive definite matrix:

$$[\Sigma_X^{-1}]_{kk} = \sum_i (D_{ik})^2 > 0. \quad (26)$$

Multiplying terms in (25) and (26), we have that $\sum_{ij} (D_{ik})^2 (M_{kj})^2 > 0$.

□

We derive sufficient conditions under which the PDW construction (defined in Section 3.2) succeeds.

Lemma 2. (Sufficient conditions for strict dual feasibility) *Let the regularization parameter $\lambda_n > 0$ and α be defined as in [A1]. Suppose the following holds*

$$\max \left\{ \left\| \Gamma(D^2 \Delta) + \Gamma(D^2 B^*) \right\|_\infty \|W\|_\infty, \|R(\Delta)\|_\infty, \left\| \Gamma(D^2 \Delta) \right\|_\infty \|\Theta^{*-1}\|_\infty \right\} \leq \frac{\lambda_n \alpha}{24}. \quad (27)$$

Then the dual vector \tilde{Z}_{E^c} satisfies $\|\tilde{Z}_{E^c}\|_\infty < 1$, and hence, $\tilde{B} = \hat{B}$.

Proof. We begin by obtaining a suitable expression for \tilde{Z}_{E^c} using the zero-subgradient condition of the restricted ℓ_1 regularized log-det problem defined in (6):

$$\tilde{B} = \arg \min_{B=B^T, B \succ 0, B_{E^c}=0} [\text{Tr}(DBSBD) - \log \det(B^2) + \lambda_n \|B\|_{1,\text{off}}]. \quad (28)$$

The zero-subgradient of the restricted problem is given by

$$2D^2 \tilde{B} S - 2\tilde{B}^{-1} + \lambda_n \tilde{Z} = 0, \quad (29)$$

where \tilde{B} is the primal solution given by (28) and $\tilde{Z} \in \partial \|B\|_{1,\text{off}}$ is the optimal dual. Recall that $\Delta = \tilde{B} - B^*$ and $W = S - \Theta^{*-1}$ and notice the following chain of identities:

$$\begin{aligned} 2(D^2 \tilde{B} S - \tilde{B}^{-1}) + \lambda_n \tilde{Z} &= 2(D^2 \tilde{B} S - D^2 B^* S + D^2 B^* S - \tilde{B}^{-1}) + \lambda_n \tilde{Z} \\ &= 2(D^2 \Delta S + D^2 B^* S - \tilde{B}^{-1}) + \lambda_n \tilde{Z} \\ &= 2(D^2 \Delta W + D^2 B^* W + D^2 \Delta \Theta^{*-1} + D^2 B^* \Theta^{*-1} - \tilde{B}^{-1}) + \lambda_n \tilde{Z}. \end{aligned}$$

On the other hand, by definition, $\Theta^{*-1} = B^{*-1} \Sigma_X B^{*-1}$ and $(D^2)^{-1} = \Sigma_X$. Thus, $D^2 B^* \Theta^{*-1} = B^{*-1}$. Substituting these expressions in the zero-subgradient condition yields the following:

$$D^2 \Delta W + D^2 B^* W + D^2 \Delta \Theta^{*-1} + B^{*-1} - \tilde{B}^{-1} + \lambda'_n \tilde{Z} = 0, \quad (30)$$

where $\lambda'_n = 0.5\lambda_n$. By adding and subtracting $B^{*-1} \Delta B^{*-1}$ to the preceding equality and followed by some algebraic manipulations give us

$$B^{*-1} \Delta B^{*-1} + D^2 \Delta W + D^2 B^* W + D^2 \Delta \Theta^{*-1} - R(\Delta) + \lambda'_n \tilde{Z} = 0, \quad (31)$$

where $R(\Delta) = \tilde{B}^{-1} - B^{*-1} - B^{*-1} \Delta B^{*-1}$.

We now vectorize (30). We use $\text{vec}(A)$ or \bar{A} to denote the p^2 -vector formed by stacking the columns of A and use $\Gamma(A) = (I \otimes A)$ to denote the Kronecker product of A with the identity matrix I . By applying $\text{vec}()$ operator on both sides of (30) it follows that

$$\text{vec}(B^{*-1} \Delta B^{*-1} + D^2 \Delta W + D^2 B^* W + D^2 \Delta \Theta^{*-1} - R(\Delta) + \lambda'_n \tilde{Z}) = 0. \quad (32)$$

Using the standard Kronecker matrix product rules [31], we have $\text{vec}(B^{*-1} \Delta B^{*-1}) = \Gamma^* \bar{\Delta}$, where $\Gamma^* = B^{*-1} \otimes B^{*-1}$ and $\text{vec}((D^2 \Delta) W) = \Gamma(D^2 \Delta) \bar{W}$; $\Gamma(D^2 \Delta) = I \otimes D^2 \Delta$; and I is the $p \times p$ identity matrix. By substituting these observations in (32), we note that

$$\Gamma^* \bar{\Delta} + \Gamma(D^2 \Delta) \bar{W} + \Gamma(D^2 B^*) \bar{W} + \Gamma(D^2 \Delta) \bar{\Theta}^{*-1} - \bar{R}(\Delta) + \lambda'_n \tilde{\tilde{Z}} = 0. \quad (33)$$

For compactness, we suppress Δ notation in $R(\Delta)$. Recall that the E is the augmented set defined as $E := \{\mathcal{E}(B^*) \cup (1, 1) \dots \cup (p, p)\}$, where \mathcal{E} is the edge set of the network and E^c is the complement of the set E . Recall that we use the notation A_E to denote the sub-matrix of A containing all

elements A_{ij} such that $(i, j) \in E$. We partition the preceding linear equations into two separate linear equations corresponding to the sets E and E^c as

$$\Gamma_{EE}^* \bar{\Delta}_E + (\Gamma_{EE}(D^2 \Delta) + \Gamma_{EE}(D^2 B^*)) \bar{W}_E + \Gamma_{EE}(D^2 \Delta) \overline{\Theta_E^{*-1}} - \bar{R}_E + \lambda'_n \bar{\tilde{Z}}_E = 0, \quad (34)$$

$$\Gamma_{E^c E}^* \bar{\Delta}_E + (\Gamma_{E^c E}(D^2 \Delta) + \Gamma_{E^c E}(D^2 B^*)) \bar{W}_{E^c} + \Gamma_{E^c E}(D^2 \Delta) \overline{\Theta_{E^c}^{*-1}} - \bar{R}_{E^c} + \lambda'_n \bar{\tilde{Z}}_{E^c} = 0. \quad (35)$$

From (34), we can solve for $\bar{\Delta}_E$ as

$$\bar{\Delta}_E = (\Gamma_{EE}^*)^{-1} \underbrace{\left[- \left((\Gamma_{EE}(D^2 \Delta) + \Gamma_{EE}(D^2 B^*)) \bar{W}_E + \Gamma_{EE}(D^2 \Delta) \overline{\Theta_E^{*-1}} \right) + \bar{R}_E - \lambda'_n \bar{\tilde{Z}}_E \right]}_{\triangleq M}. \quad (36)$$

Substituting $\bar{\Delta}_E$ given by (36) in (35) gives us

$$\Gamma_{E^c E}^* (\Gamma_{EE}^*)^{-1} M + (\Gamma_{E^c E}(D^2 \Delta) + \Gamma_{E^c E}(D^2 B^*)) \bar{W}_{E^c} + \Gamma_{E^c E}(D^2 \Delta) \overline{\Theta_{E^c}^{*-1}} - \bar{R}_{E^c} + \lambda'_n \bar{\tilde{Z}}_{E^c} = 0. \quad (37)$$

From which we can solve for the vectorized dual $\bar{\tilde{Z}}_{E^c}$ as

$$\lambda'_n \bar{\tilde{Z}}_{E^c} = -\Gamma_{E^c E}^* (\Gamma_{EE}^*)^{-1} M - (\Gamma_{E^c E}(D^2 \Delta) + \Gamma_{E^c E}(D^2 B^*)) \bar{W}_{E^c} - \Gamma_{E^c E}(D^2 \Delta) \overline{\Theta_{E^c}^{*-1}} + \bar{R}_{E^c}. \quad (38)$$

Taking the element-wise ℓ_∞ norm on both sides of the preceding equality gives us

$$\begin{aligned} \|\bar{\tilde{Z}}_{E^c}\|_\infty &\leq \frac{1}{\lambda'_n} \|\Gamma_{E^c E}^* (\Gamma_{EE}^*)^{-1}\|_\infty \|M\|_\infty + \frac{1}{\lambda'_n} \|\Gamma_{E^c E}(D^2 \Delta)\|_\infty \|\overline{\Theta_{E^c}^{*-1}}\|_\infty \\ &\quad + \frac{1}{\lambda'_n} \|\Gamma_{E^c E}(D^2 \Delta) + \Gamma_{E^c E}(D^2 B^*)\|_\infty \|\bar{W}_{E^c}\|_\infty + \frac{1}{\lambda'_n} \|\bar{R}_{E^c}\|_\infty. \end{aligned} \quad (39)$$

We invoke the mutual incoherence condition in (10) to bound $\|\Gamma_{E^c E}^* (\Gamma_{EE}^*)^{-1}\|_\infty \leq (1 - \alpha)$ and since $\|A_{E^c}\|_\infty \leq \|A\|_\infty$ for any matrix A , we get

$$\begin{aligned} \|\bar{\tilde{Z}}_{E^c}\|_\infty &\leq \frac{1 - \alpha}{\lambda'_n} \|M\|_\infty + \frac{1}{\lambda'_n} \|\Gamma(D^2 \Delta)\|_\infty \|\Theta^{*-1}\|_\infty \\ &\quad + \frac{1}{\lambda'_n} [\|\Gamma(D^2 \Delta) + \Gamma(D^2 B^*)\|_\infty \|W\|_\infty + \|R\|_\infty]. \end{aligned} \quad (40)$$

We bound $\|M\|_\infty$ by taking the element-wise ℓ_∞ norm of M in (36) and followed by applying sub-multiplicative norm inequalities. Thus,

$$\begin{aligned} \|M\|_\infty &\leq \|\Gamma_{EE}(D^2 \Delta) + \Gamma_{EE}(D^2 B^*)\|_\infty \|W_E\|_\infty + \|\Gamma_{EE}(D^2 \Delta)\|_\infty \|\Theta_E^{*-1}\|_\infty \\ &\quad + \|R_E\|_\infty + \lambda'_n \|\bar{\tilde{Z}}_E\|_\infty. \end{aligned} \quad (41)$$

Because $\bar{\tilde{Z}}_E$ is the sub-vector of the vectorized optimal dual $\bar{\tilde{Z}}$, it follows that $\|\bar{\tilde{Z}}_E\|_\infty \leq 1$. Thus,

$$\|M\|_\infty \leq \underbrace{\left[\|\Gamma(D^2 \Delta) + \Gamma(D^2 B^*)\|_\infty \|W\|_\infty + \|\Gamma(D^2 \Delta)\|_\infty \|\Theta^{*-1}\|_\infty + \|R\|_\infty \right]}_{\triangleq H} + \lambda'_n. \quad (42)$$

On the other hand, from (42), we have $H \leq \lambda'_n \alpha / 4$. Putting together the pieces, from (42) and (40) we conclude that

$$\|\bar{\tilde{Z}}_{E^c}\|_\infty \leq (1 - \alpha) + \frac{1 - \alpha}{\lambda'_n} H + \frac{1}{\lambda'_n} H \quad (43)$$

$$= (1 - \alpha) + \frac{2 - \alpha}{\lambda'_n} H \quad (44)$$

$$\leq (1 - \alpha) + \frac{2 - \alpha}{\lambda'_n} \left(\frac{\lambda'_n \alpha}{4} \right) \quad (45)$$

$$\leq (1 - \alpha) + \frac{\alpha}{2} < 1. \quad (46)$$

Remark For comparison, consider the strict dual feasibility conditions in [48, Lemma 4]. Here, the maximum is on the noise deviation $\|W\|_\infty$ and the remainder term $\|R(\Delta)\|_\infty$. Instead, in our case, the maximum is taken over several other quantities not just $\|W\|_\infty$ and $\|R(\Delta)\|_\infty$ (see (2)). \square

The following lemma shows that the remainder term $R(\Delta)$ is bounded if Δ is bounded. The proof is adapted from [48], where a similar result is derived using matrix expansion techniques. We use this lemma in the proof of our main result (see Theorem 1 and Theorem 2) to show that with sufficient number of samples $R(\Delta) \leq \alpha\lambda_n/24$.

Lemma 3. (Control of reminder) Suppose that the element-wise ℓ_∞ -bound $\|\Delta\|_\infty \leq \frac{1}{3\nu_{B^*-1}d}$ holds, then the matrix $Q = \sum_{k=0}^{\infty} (-1)^k (B^{*-1}\Delta)^k$ satisfies the bound $\nu_{Q^T} \leq \frac{3}{2}$ and the matrix $R(\Delta) = B^{*-1}\Delta B^{*-1}\Delta Q B^{*-1}$ has the element-wise ℓ_∞ -norm bounded as

$$\|R(\Delta)\|_\infty \leq \frac{3}{2}d\|\Delta\|_\infty^2\nu_{B^*-1}^3. \quad (47)$$

We show that for a specific choice of radius r , the distortion $\Delta = \tilde{B} - B^*$ lies in a ball of radius r .

Lemma 4. (Control of Δ) Let

$$r \triangleq 4\nu_{\Gamma^*-1} [\nu_{D^2\nu_{B^*}}\|W\|_\infty + 0.5\lambda_n] \leq \min \left\{ \frac{1}{3\nu_{B^*-1}d}, \frac{1}{6\nu_{\Gamma^*-1}\nu_{B^*-1}^3d} \right\}.$$

Then we have the element-wise ℓ_∞ bound $\|\Delta\|_\infty = \|\tilde{B} - B^*\|_\infty \leq r$.

Proof. We adopt the proof technique in [48, Lemma 6]. We use the notation A_E or $[A]_E$ to denote the sub-matrix of A containing all elements A_{ij} such that $(i, j) \in E$. Let $G(\tilde{B}_E)$ be the zero sub-gradient condition of the restricted ℓ_1 log-det problem in (12):

$$G(\tilde{B}_E) = \left[D^2\tilde{B}S - \tilde{B}^{-1} + \lambda'_n\tilde{Z} \right]_E = 0. \quad (48)$$

where $\lambda'_n = 0.5\lambda_n$. Let \bar{G} denote the vectorized form of G . Recall that $\Delta = \tilde{B} - B^* = \tilde{B}_E - B_E^* \triangleq \Delta_E$. The second equality follows from PDW construction and the constraint in the restricted convex program in (6). To establish $\|\Delta\|_\infty \leq r$, we show that Δ_E lies inside the ball $\mathbb{B}_r = \{\bar{A}_E \in \mathbb{R}^{|E|} : \|\bar{A}_E\|_\infty \leq r\}$, where $\bar{A}_E = \text{vec}(A_E)$, using a contraction property of the continuous map:

$$F(\bar{\Delta}_E) \triangleq -(\Gamma_{EE}^*)^{-1} (\bar{G}(\Delta_E + B_E^*)) + \bar{\Delta}_E, \quad (49)$$

where we used the fact that $\tilde{B}_{E^c} = \tilde{B}_{E^c}^* = 0$.

Suppose that $F(\cdot)$ is a contraction on \mathbb{B}_r , i.e., $F(\mathbb{B}_r) \subseteq \mathbb{B}_r$. Then by Brower's fixed point theorem [29], it readily follows that there exists a $C \in \mathbb{B}_r$ such that $F(C) = C$. Finally, $C = \bar{\Delta}_E$ because (i) \tilde{B} that satisfies $\bar{G}(\tilde{B}) = 0$ is unique (see Lemma 1) and (ii) $F(\bar{\Delta}_E) = \bar{\Delta}_E$ if and only if $\bar{G}(\cdot) = 0$. Hence, $\bar{\Delta}_E \in \mathbb{B}_r$ is the unique fixed point of $F(\cdot)$ in (49). Consequently, $\|\Delta_E\|_\infty \leq r$.

It remains to show that $F(\cdot)$ is a contraction. Let $\Delta' \in \mathbb{R}^{p \times p}$ be a zero padded matrix on E^c such that $\bar{\Delta}'_E \in \mathbb{B}_r$. Then $F(\bar{\Delta}'_E)$ can be expanded in terms of Δ' as

$$\begin{aligned} F(\bar{\Delta}'_E) &= -(\Gamma_{EE}^*)^{-1} (\bar{G}(\Delta'_E + B_E^*)) + \bar{\Delta}'_E \\ &= -(\Gamma_{EE}^*)^{-1} \left[\text{vec}([D^2(\Delta' + B^*)S]_E - (\Delta' + B^*)^{-1} + \lambda'_n\tilde{Z}_E) + \Gamma_{EE}^*\bar{\Delta}'_E \right]. \end{aligned} \quad (50)$$

Adding and subtracting Θ^{*-1} and B_E^{*-1} to the preceding equality yields us

$$\begin{aligned} F(\bar{\Delta}'_E) &= -(\Gamma_{EE}^*)^{-1} \left[\text{vec} \left([D^2(\Delta' + B^*)W]_E + [D^2(\Delta' + B^*)\Theta^{*-1}]_E + \lambda'_n\tilde{Z}_E - B_E^{*-1} \right) \right. \\ &\quad \left. - (\Gamma_{EE}^*)^{-1} \left[-\text{vec} \left((\Delta' + B^*)^{-1} - B_E^{*-1} \right) + \Gamma_{EE}^*\bar{\Delta}'_E \right] \right]. \end{aligned} \quad (51)$$

The last $\text{vec}(\cdot)$ term can be even simplified as

$$\begin{aligned} \text{vec}\left((\Delta' + B^*)^{-1} - B^{*-1}\right) + \Gamma^* \Delta' &= \text{vec}\left((\Delta' + B^*)^{-1} - B^{*-1} + (B^{*-1} \Delta' B^{*-1})\right) \\ &= \text{vec}(R(\Delta')). \end{aligned} \quad (52)$$

Substituting this observation in (51) and rearranging the terms gives us

$$\begin{aligned} F(\bar{\Delta}'_E) &= - \underbrace{(\Gamma_{EE}^*)^{-1} \text{vec}\left[D^2 B^* W \lambda'_n \tilde{Z}\right]_E}_{\triangleq T_1} - \underbrace{(\Gamma_{EE}^*)^{-1} \text{vec}\left[D^2 \Delta' W\right]_E}_{\triangleq T_2} \\ &\quad - \underbrace{(\Gamma_{EE}^*)^{-1} [\bar{R}(\Delta')]_E}_{\triangleq T_3} - \underbrace{(\Gamma_{EE}^*)^{-1} \text{vec}\left[D^2(\Delta' + B^*)\Theta^{*-1} - B^{*-1}\right]_E}_{\triangleq T_4}. \end{aligned} \quad (53)$$

We now show that $\|F(\bar{\Delta}'_E)\|_\infty \leq r$ by bounding ℓ_∞ norms of terms (T_1) – (T_4) . Recall that $\nu_A = \|A\|_\infty \triangleq \max_{j=1,\dots,p} \sum_{i=1}^p |A_{ij}|$ and it is sub-multiplicative; that is $\|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty$. Notice that this not the case with the max norm (ℓ_∞).

(i) *upper bound on $\|T_1\|_\infty$* : Consider the following chain of inequalities.

$$\begin{aligned} \|T_1\|_\infty &\leq \left\| \Gamma^{*-1} \right\|_\infty \left\| \text{vec}(D^2 B^* W + \lambda'_n \tilde{Z}) \right\|_\infty \\ &= \left\| \Gamma^{*-1} \right\|_\infty \left\| \Gamma(D^2 B^*) \bar{W} + \lambda'_n \tilde{Z} \right\|_\infty \\ &\stackrel{(a)}{\leq} \left\| \Gamma^{*-1} \right\|_\infty \left[\left\| \Gamma(D^2 B^*) \right\|_\infty \|W\|_\infty + \lambda'_n \right] \\ &\stackrel{(b)}{\leq} \nu_{\Gamma^{*-1}} [\nu_{D^2} \nu_{B^*} \|W\|_\infty + \lambda'_n] \stackrel{(c)}{\leq} \frac{r}{4} \end{aligned} \quad (54)$$

where (a) follows because $\|\tilde{Z}\|_\infty \leq 1$ (see Lemma 1); (b) follows because $\Gamma(D^2 B^*) = (I \otimes D^2 B^*)$, and hence, $\|\Gamma(D^2 B^*)\|_\infty = \|D^2 B^*\|_\infty \leq \|D^2\|_\infty \|B^*\|_\infty = \nu_{D^2} \nu_{B^*}$; and finally, (c) follows from definition of the radius r in Lemma 4.

(ii) *upper bound on $\|T_2\|_\infty$* : For T_2 in (53), consider the following chain of inequalities.

$$\begin{aligned} \|T_2\|_\infty &\leq \nu_{\Gamma^{*-1}} \left[\left\| \Gamma(D^2 \Delta) \right\|_\infty \|W\|_\infty \right] \\ &\leq \nu_{\Gamma^{*-1}} \nu_{D^2} \|\Delta'\|_\infty \|W\|_\infty \\ &\stackrel{(a)}{\leq} \nu_{\Gamma^{*-1}} \nu_{D^2} d \|\Delta'\|_\infty \|W\|_\infty \\ &\stackrel{(b)}{\leq} \nu_{\Gamma^{*-1}} \nu_{D^2} d \|\Delta'\|_\infty \left(\frac{r}{4 \nu_{\Gamma^{*-1}} \nu_{D^2} \nu_{B^*}} \right) \\ &\stackrel{(c)}{\leq} d \left(\frac{1}{3 d \nu_{B^*-1}} \right) \left(\frac{r}{4 \nu_{B^*}} \right) \stackrel{(d)}{\leq} \frac{r}{4}, \end{aligned} \quad (55)$$

where (a) follows because by construction Δ' has at-most d non-zeros in every row and that $\|\Delta'\|_\infty \leq d \|\Delta'\|_\infty$; (b) follows from the choice of $r = 4 \nu_{\Gamma^{*-1}} (\nu_{D^2} \nu_{B^*} \|W\|_\infty + \lambda'_n)$ in Lemma 4, which is lower bounded by $4 \nu_{\Gamma^{*-1}} \nu_{D^2} \nu_{B^*} \|W\|_\infty$, for all $\lambda'_n \geq 0$. Thus, $\|W\|_\infty \leq r / (4 \nu_{\Gamma^{*-1}} \nu_{D^2} \nu_{B^*})$; (c) follows because Δ' is a zero-padded matrix of Δ . Hence $\|\Delta\|_\infty = \|\Delta'\|_\infty \leq r$, which can be upper bounded by $1/(3 d \nu_{B^*-1})$ in light of the hypothesis in Lemma 4; and finally, (d) follows because $\nu_{B^*} \nu_{B^*-1} \geq 1$.

(iii) *upper bound on $\|T_3\|_\infty$* : For T_3 in (53), consider the following chain of inequalities.

$$\|T_3\|_\infty \leq \nu_{\Gamma^{*-1}} \|R(\Delta')\|_\infty \quad (56)$$

$$\stackrel{(a)}{\leq} \frac{3}{2} d \nu_{\Gamma^{*-1}} \nu_{B^*-1}^3 \|\Delta'\|_\infty^2 \quad (57)$$

$$\stackrel{(b)}{\leq} \frac{3}{2} d \nu_{\Gamma^{*-1}} \nu_{B^*-1}^3 r(r) \stackrel{(c)}{\leq} \frac{r}{4}, \quad (58)$$

where (a) follows because Lemma 3 guarantees that $\|R(\Delta')\|_\infty \leq (3/2)d\nu_{B^{*-1}}^3 \|\Delta'\|_\infty^2$ whenever $\|\Delta'\|_\infty \leq 1/(3d\nu_{B^{*-1}})$. The latter inequality is a consequence of the hypothesis in Lemma 4; (b) is true because by construction $\Delta' \in \mathbb{B}_r$, and hence, $\|\Delta'\|_\infty \leq r$; (c) follows by invoking the hypothesis in Lemma 4, where r satisfies $r \leq 1/(6d\nu_{\Gamma^{*-1}}\nu_{B^{*-1}}^3)$.

(iv) *upper bound on $\|T_4\|_\infty$* : The expression of T_4 in (53) can be simplified as

$$\begin{aligned} T_4 &= -(\Gamma_{EE}^*)^{-1} \text{vec} \left(\left[D^2(\Delta' + B^*)\Theta^{*-1} - B^{*-1} \right]_E \right) \\ &= -(\Gamma_{EE}^*)^{-1} \text{vec} \left(\left[D^2\Delta'\Theta^{*-1} + D^2B^*\Theta^{*-1} - B^{*-1} \right]_E \right) \\ &= -(\Gamma_{EE}^*)^{-1} \text{vec} \left(\left[D^2\Delta'\Theta^{*-1} \right]_E \right). \end{aligned} \quad (59)$$

The last equality follows by observing that $D^2B^*\Theta^{*-1} = B^{*-1}$. This can be verified by plugging $\Theta^{*-1} = B^{*-1}\Sigma_X B^{*-1}$ and $\Sigma_X = (D^2)^{-1}$ in $D^2B^*\Theta^{*-1}$ and simplifying the resulting expression. By taking the ℓ_∞ bound on the both sides of (59) gives us

$$\|T_4\|_\infty \leq \nu_{\Gamma^{*-1}} \|\Gamma(D^2\Delta')\|_\infty \|\Theta^{*-1}\|_\infty \quad (60)$$

$$\leq \nu_{\Gamma^{*-1}} \nu_{D^2} d \|\Delta'\|_\infty \|\Theta^{*-1}\|_\infty \quad (61)$$

$$\leq \nu_{\Gamma^{*-1}} \nu_{D^2} r d \|\Theta^{*-1}\|_\infty \stackrel{(a)}{\leq} \frac{r}{4}, \quad (62)$$

where (a) follows by invoking the assumption in (11), and noting that $\|\Theta^{*-1}\|_\infty \leq 1/(4\nu_{\Gamma^{*-1}}\nu_{D^2}d)$.

Putting together the pieces, from the above calculations, we note that

$$\|F(\Delta'_E)\|_\infty \leq \|T_1\|_\infty + \|T_2\|_\infty + \|T_3\|_\infty + \|T_4\|_\infty \leq r. \quad (63)$$

is a contraction as claimed. This concludes the proof. \square

We borrow the following notion of tail conditions as defined in [48] to characterize the distribution. We use this characterization to prove our master lemma A.6.

Definition A.3. (Tail condition, [48]) *The random vector Y satisfies the tail condition $\mathcal{T}(f, v_*)$ if there exist a constant $v_* > 0$ and a function $f : \mathbb{N} \times (0, \infty)$ such that for any $i, j \in [p]$ and $\delta \in (0, 1/v_*)$:*

$$\mathbb{P}[|S_{ij} - \Sigma_{ij}^*| \geq \delta] \leq \frac{1}{f(n, \delta)}. \quad (64)$$

Furthermore, $f(n, \delta)$ is monotonically increasing in n (or δ) for fixed δ (or n).

Both the exponential-type tail $f(n, \delta) = \exp(cn\delta^a)$ and the polynomial-type tail $f(n, \delta) = cn^m\delta^2m$, where m is an integer and $c, a > 0$, satisfy the monotone property in Definition A.3. The following inverse functions associated with $f(n, \delta)$ are needed to prove our sample complexity result:

$$n_f(\delta, p^\tau) := \max\{n | f(n, \delta) \leq p^\tau\} \text{ and } \delta_f(n, p^\tau) := \max\{\delta | f(n, \delta) \leq p^\tau\}. \quad (65)$$

Both the functions are well-defined due to the monotonicity property of $f(n, \delta)$. Further, if $n > n_f(\delta, p^\tau)$ for some $\delta > 0$ implies that $\delta \geq \delta_f(n, p^\tau)$.

The following result presents an exponential-type tail bound for sub-Gaussian random vectors.

Lemma A.5. (Sub-Gaussian tail condition, [48]) *Consider a zero-mean random vector (Y_1, \dots, Y_p) with covariance Σ^* such that each $Y_i/\sqrt{\Sigma_{ii}^*}$ is sub-Gaussian with parameter σ . Given n i.i.d samples, the sample covariance matrix S satisfies the tail bound*

$$\mathbb{P}[|S_{ij} - \Sigma_{ij}^*| > \delta] \leq 4 \exp \left\{ - \frac{n\delta^2}{128(1 + 4\sigma^2)^2 \max_i(\Sigma_{ii}^*)^2} \right\}, \quad (66)$$

for all $\delta \in (0, 8(1 + 4\sigma^2) \max_i(\Sigma_{ii}^*))$.

Let $W_{ij} = S_{ij} - \Sigma_{ij}^*$, where $\Sigma^* = \Theta^{*-1}$. This difference quantity, which signifies the amount of noise in the data, plays a key role in bounding the error term $\|\hat{B} - B^*\|_\infty$. We later show that if W_{ij} is small, then we can guarantee that our estimator \hat{B} is close to B^* in the element-wise ℓ_∞ -norm. By taking a union bound over all entries of $|W_{ij}|$, from Lemma A.5, it follows that

$$\mathbb{P}[\|W\|_\infty \geq \delta_f(n, p^\tau)] \leq \frac{p^2}{f(n, \delta_f(n, p^\tau))} = \frac{1}{p^{\tau-2}}, \quad (67)$$

for some $\tau > 2$. The above bound gives an explicit control on the noise term.

We now state and prove our master lemma which gives support recovery guarantees and ℓ_∞ norm bounds for our estimator \hat{B} for distributions satisfying tail condition $\mathcal{T}(f, v_*)$ in Definition A.3.

Lemma A.6. (Master lemma) *Consider a distribution satisfying the incoherence assumption with parameter $\alpha \in (0, 1]$ and the tail condition $\mathcal{T}(f, v_*)$. Let \hat{B} be the unique solution of the log-determinant problem in (9) with $\lambda_n = 2\nu_{D^2}\nu_{B^*}\delta_f(n, p^\tau)$ for some $\tau > 2$. Then if the sample size is lower bounded as*

$$n > n_f(1/\max\{v_*, 24d\nu_{D^2}\nu_{B^*}\max\{\nu_{\Gamma^*-1}\nu_{B^*-1}, 2\nu_{\Gamma^*-1}^2\nu_{B^*-1}^3, 2\alpha^{-1}d^{-1}\}\}, p^\tau), \quad (68)$$

then with probability greater than $1 - \frac{1}{p^{\tau-2}}$, the estimate \hat{B} recovers the sparsity structure of B^ ie. $(\hat{B}_{E^c} = B_{E^c}^*)$. Furthermore \hat{B} satisfies the ℓ_∞ bound $\|\hat{B} - B^*\|_\infty \leq 8\nu_{\Gamma^*-1}\nu_{D^2}\nu_{B^*}\delta_f(n, p^\tau)$.*

Proof. We first show that the Primal Dual Witness (PDW) construction (see sec 3.3) succeeds with the probability stated in the lemma. This amounts to showing that the inequality in (8) holds with the required probability. To this aim, let \mathcal{A} denote the event that $\|W\|_\infty \leq \delta_f(n, p^\tau)$. We have previously shown in (67) that $\mathbb{P}[\mathcal{A}] \geq 1 - 1/p^{\tau-2}$. Conditioned on the event \mathcal{A} , we show that the inequality in (8) is satisfied.

From Lemma 4, we have

$$r = 4\nu_{\Gamma^*-1}[\nu_{D^2}\nu_{B^*}\|W\|_\infty + 0.5\lambda_n], \quad (69)$$

substituting for $\lambda_n = 2\nu_{D^2}\nu_{B^*}\delta_f(n, p^\tau)$ as given in the assumption, we get

$$r \leq 8\nu_{\Gamma^*-1}\nu_{D^2}\nu_{B^*}\delta_f(n, p^\tau). \quad (70)$$

From assumption on the sample size n in (68) and the monotonicity property (65) we have $0.5\lambda_n = \nu_{D^2}\nu_{B^*}\delta_f(n, p^\tau) \leq \alpha/48$, which implies that $\lambda_n < 1$. We also set $\delta_f(n, p^\tau) \leq \lambda_n$. Similarly from (65) and (68) we have $r \leq 8\nu_{\Gamma^*-1}\nu_{D^2}\nu_{B^*}\delta_f(n, p^\tau) \leq \min\{1/(3d\nu_{B^*-1}), 1/(6d\nu_{\Gamma^*-1}\nu_{B^*-1}^3)\}$. Therefore the assumption in Lemma 4 is satisfied resulting in

$$\|\Delta\|_\infty \leq r \leq \min\left[\frac{1}{3d\nu_{B^*-1}}, \frac{1}{6d\nu_{\Gamma^*-1}\nu_{B^*-1}^3}\right]. \quad (71)$$

Define $\delta_f \triangleq \delta_f(n, p^\tau)$. We show that the every component in the max term of (8) are bounded by $\alpha\lambda_n/24$. We begin with the first component:

$$\|\|\Gamma(D^2\Delta) + \Gamma(D^2B^*)\|\|_\infty\|W\|_\infty \leq [\|\|D^2\Delta + D^2B^*\|\|_\infty] \delta_f \quad (72)$$

$$\leq [\|\|D^2\Delta\|\|_\infty + \|\|D^2B^*\|\|_\infty] \delta_f \quad (73)$$

$$\leq [\nu_{D^2}d\|\Delta\|_\infty + \nu_{D^2}\nu_{B^*}] \frac{\alpha\lambda_n}{48\nu_{D^2}\nu_{B^*}} \quad (74)$$

$$\stackrel{(a)}{\leq} \left[1 + \frac{1}{3\nu_{B^*}\nu_{B^*-1}}\right] \frac{\alpha\lambda_n}{48} \quad (75)$$

$$\stackrel{(b)}{\leq} \frac{\alpha\lambda_n}{36} \leq \frac{\alpha\lambda_n}{24}. \quad (76)$$

where (a) follows from (71); (b) follows because $\nu_{B^*}\nu_{B^*-1} \geq 1$.

We show the second component $\|R(\Delta)\|_\infty \leq \alpha\lambda_n/24$. In fact,

$$\|R(\Delta)\|_\infty \stackrel{(a)}{\leq} \frac{3}{2}d\|\Delta\|_\infty^2\nu_{B^*-1}^3 \quad (77)$$

$$\stackrel{(b)}{\leq} \frac{3}{2}dr\nu_{B^*-1}^3 \quad (78)$$

$$\stackrel{(c)}{\leq} \frac{3}{2}d \left[\frac{1}{6d\nu_{\Gamma^*-1}\nu_{B^*-1}^3} \right] \nu_{B^*-1}^3 (8\nu_{\Gamma^*-1}\nu_{D^2}\nu_{B^*}\delta_f) \quad (79)$$

$$= 2\nu_{D^2}\nu_{B^*}\delta_f \leq \frac{\alpha\lambda_n}{24}. \quad (80)$$

where (a) holds because, as shown in (71), $\|\Delta\|_\infty$ satisfies the assumption in Lemma 3; (b) holds because $\|\Delta\|_\infty \leq r$; and (c) is a consequence of the inequality in (71).

We show that the third component $\|\Gamma(D^2\Delta)\|_\infty \|\Theta^{*-1}\|_\infty \leq \alpha\lambda_n/24$. In fact,

$$\|\Gamma(D^2\Delta)\|_\infty \|\Theta^{*-1}\|_\infty = \|D^2\Delta\|_\infty \|\Theta^{*-1}\|_\infty \quad (81)$$

$$\leq \nu_{D^2}d\|\Theta^{*-1}\|_\infty \|\Delta\|_\infty \quad (82)$$

$$\stackrel{(a)}{\leq} \nu_{D^2}d\|\Theta^{*-1}\|_\infty r \quad (83)$$

$$\stackrel{(b)}{\leq} d\nu_{D^2} \left[\frac{1}{4d\nu_{D^2}\nu_{\Gamma^*-1}} \right] [8\nu_{\Gamma^*-1}\nu_{D^2}\nu_{B^*}\delta_f] \quad (84)$$

$$\leq 2\nu_{D^2}\nu_{B^*} \left[\frac{\alpha\lambda_n}{48\nu_{D^2}\nu_{B^*}} \right] = \frac{\alpha\lambda_n}{24}. \quad (85)$$

where (a) holds because $\|\Delta\|_\infty \leq r$ and (b) follows by invoking the assumption in (11). Since the sufficient conditions for strict dual feasibility are satisfied, the PDW construction succeeds. Therefore $\hat{B} = \tilde{B}$. Since by definition $\hat{B}_E = B_{E^c}^* = 0$, the estimator \hat{B} recovers the sparsity structure of B^* . Now, since $\Delta = \hat{B} - B^*$ and $\|\Delta\|_\infty \leq 8\nu_{\Gamma^*-1}\nu_{D^2}\nu_{B^*}\delta_f(n, p^\tau)$, we have $\|\hat{B} - \tilde{B}\|_\infty \leq 8\nu_{\Gamma^*-1}\nu_{D^2}\nu_{B^*}\delta_f(n, p^\tau)$. \square

We use Lemma A.5 and Lemma A.6 to prove our main result for sub-gaussian distributions.

Theorem 1. (Support recovery: Sub-Gaussian) Let $Y = (Y_1, \dots, Y_p)$ be the node potential vector. Suppose that $Y_i/\sqrt{\Sigma_{ii}^*}$ is sub-Gaussian with parameter σ and assumptions [A1-A2] hold. Let the regularization parameter $\lambda_n = C_0\sqrt{\tau(\log 4p)/n}$, where C_0 is given below. If the sample size $n > C_1^2d^2(\tau \log p + \log 4)$, the following hold with probability at least $1 - \frac{1}{p^{\tau-2}}$, for some $\tau > 2$:

(a) \hat{B} exactly recovers the sparsity structure of B^* ; that is, $\hat{B}_{E^c} = 0$,

(b) \hat{B} satisfies the element-wise ℓ_∞ bound $\|\hat{B} - B^*\|_\infty \leq C_2\sqrt{\frac{\tau \log p + \log 4}{n}}$, and

(c) \hat{B} satisfies sign consistency if $|B_{\min}^*| \geq 2C_2\sqrt{\frac{\tau \log p + \log 4}{n}}$, $B_{\min}^* \triangleq \min_{(i,j) \in \mathcal{E}(B^*)} |B_{ij}^*|$,

where $C_1 = 192\sqrt{2} \left[(1 + 4\sigma^2) \max_i(\Sigma_{ii}^*)\nu_{D^2}\nu_{B^*} \right] \max\{\nu_{\Gamma^*-1}\nu_{B^*-1}, 2\nu_{\Gamma^*-1}^2\nu_{B^*-1}^3, 2\alpha^{-1}d^{-1}\}$, $C_2 = [64\sqrt{2}(1 + 4\sigma^2) \max_i(\Sigma_{ii}^*)\nu_{\Gamma^*-1}\nu_{D^2}\nu_{B^*}]$, and $C_0 = C_2/(4\nu_{\Gamma^*-1})$.

Proof. Part (a): From Lemma A.6, we have that if $n > n_f(\delta, p^\tau)$, then \hat{B} recovers the exact sparsity structure of B^* . We compute $n_f(\delta, p^\tau)$. Using the tail bound for sub-gaussian distributions (see Lemma A.5), the decay function $f(n, \delta) = \frac{1}{4} \exp\left\{\frac{n\delta^2}{c_*}\right\}$, where $c_* = 128(1 + 4\sigma^2)^2 \max_i(\Sigma_{ii}^*)^2$. From the definition of inverse function and monotonicity of $f(n, \delta)$ in A.3, we have $n_f(\delta, p^\tau) = \frac{c_* \log(4p^\tau)}{\delta^2}$. Substituting for δ from Lemma A.6, we get

$$n_f(\delta, p^\tau) = C_1^2d^2(\tau \log p + \log 4). \quad (86)$$

Therefore, from Lemma A.6, if $n > C_1^2 d^2 (\tau \log p + \log 4)$, the estimator \hat{B} recovers the sparsity structure of B^* .

Part(b): From Lemma A.5 we compute δ . Using the monotonicity property of $f(n, \delta)$ and setting

$$\delta \triangleq \delta_f(n, p^\tau) = \sqrt{\frac{c_* \log(4p^\tau)}{n}} = 8\sqrt{2}(1 + 4\sigma^2) \max_i(\Sigma_{ii}^*) \sqrt{\frac{\tau \log p + \log 4}{n}}. \quad (87)$$

Also we have from Lemma A.6 that $\|\hat{B} - B^*\|_\infty \leq 8\nu_{\Gamma^* - 1} \nu_{D^2} \nu_{B^*} \delta_f(n, p^\tau)$. Thus,

$$\|\hat{B} - B^*\|_\infty \leq \underbrace{64\sqrt{2}(1 + 4\sigma^2) \max_i(\Sigma_{ii}^*) \nu_{\Gamma^* - 1} \nu_{D^2} \nu_{B^*}}_{C_2} \sqrt{\frac{\tau \log p + \log 4}{n}}. \quad (88)$$

Part(c): We prove the sign consistency of \hat{B} by contradiction. Let $|B_{\min}^*| \geq 2C_2 \sqrt{\frac{\tau \log p + 4}{n}}$ be as in the theorem's hypothesis. Suppose that $\text{sign}(\hat{B}) \neq \text{sign}(B^*)$. Then, an elementary algebra shows that $\|\hat{B} - B^*\|_\infty > 2C_2 \sqrt{\frac{\tau \log p + 4}{n}}$. This contradicts the bound in part (b). Thus, $\text{sign}(\hat{B}) = \text{sign}(B^*)$. \square

We now show Frobenius and spectral norm consistency for the sub-gaussian distribution. Recall that $\mathcal{E}(B^*) = \{(i, j) : B_{ij}^* \neq 0, \text{ for all } i \neq j\}$ is the edge set of B^* . Thus, $s = |\mathcal{E}(B^*)|$ is the number of non-zero off-diagonal elements in B^* .

Corollary 1. *Let $s = |\mathcal{E}(B^*)|$ be the cardinality of $\mathcal{E}(B^*)$. Under the same hypotheses in Theorem 1, with probability greater than $1 - \frac{1}{p^{\tau-2}}$, the estimator \hat{B} satisfies*

$$\|\hat{B} - B^*\|_F \leq C_2 \sqrt{\frac{(s+p)(\tau \log p + 4)}{n}} \quad \text{and} \quad \|\hat{B} - B^*\|_2 \leq C_2 \min\{d, \sqrt{s+p}\} \sqrt{\frac{\tau \log p + 4}{n}}. \quad (89)$$

Proof. Consider the following inequality:

$$\|\hat{B} - B^*\|_F^2 = \sum_{i,j} (\hat{B}_{ij} - B_{ij}^*)^2 = \sum_i (\hat{B}_{ii} - B_{ii}^*)^2 + \sum_{i \neq j} (\hat{B}_{ij} - B_{ij}^*)^2 \quad (90)$$

$$\leq p \|\hat{B} - B^*\|_\infty^2 + s \|\hat{B} - B^*\|_\infty^2 \quad (91)$$

$$= (s+p) \|\hat{B} - B^*\|_\infty^2, \quad (92)$$

where the inequality follows because there are at most p non-zero diagonal terms and s non-zero off-diagonal terms in $\hat{B} - B^*$. The latter fact is a consequence of Theorem 1 (a), which ensures that $\hat{B}_{E^c} = B_{E^c}^*$ with high probability when $n = \Omega(d^2 \log p)$. We obtain the Frobenius norm bound in (89) by upper bounding $\|\hat{B} - B^*\|_\infty$ using the result in Theorem 1 (b).

We now spectral norm consistency. From matrix norm equivalence conditions [26], we have

$$\|\hat{B} - B^*\|_2 \leq \left\| \left\| \hat{B} - B^* \right\| \right\|_\infty \leq d \|\hat{B} - B^*\|_\infty \quad (93)$$

and that

$$\|\hat{B} - B^*\|_2 \leq \|\hat{B} - B^*\|_F \leq \sqrt{s+p} \|\hat{B} - B^*\|_\infty. \quad (94)$$

These two bounds can be unified into one single bound as

$$\|\hat{B} - B^*\|_2 \leq \min\{\sqrt{s+p}, d\} \|\hat{B} - B^*\|_\infty. \quad (95)$$

This concludes the proof. \square

Next we prove our second main result for random vectors with bounded moments. We need the following standard concentration inequality result.

Lemma A.7. (Tail bounds for random variables with bounded moments, [48]) For a random vector (Y_1, \dots, Y_p) , suppose there exists a positive integer k and scalar $M_k \in \mathbb{R}$ with

$$\mathbb{E} \left[\left| \frac{Y_i}{\sqrt{\Sigma_{ii}^*}} \right|^{4k} \right] \leq M_k. \quad (96)$$

Given n i.i.d samples, the sample covariance matrix S admits the following concentration inequality

$$\mathbb{P} [|S_{ij} - \Sigma_{ij}^*| > \delta] \leq \frac{2^{2k} (\max_i \Sigma_{ii}^*)^{2k} C_k (M_k + 1)}{n^k \delta^{2k}}. \quad (97)$$

where $C_k \geq 0$ is a constant depending only on k .

Theorem 2. (Support Recovery: Bounded Moments) Let $Y = (Y_1, \dots, Y_p)$ be the node potential vector. Suppose that $Y_i / \sqrt{\Sigma_{ii}^*}$ has bounded moment as in (96) and assumptions [A1-A2] hold. Let the regularization parameter $\lambda_n = C_0 \sqrt{\tau(\log 4p)/n}$, with C_0 defined in Theorem 1. If the sample size $n > C_4 d^2 p^{\tau/k}$, then with probability more than $1 - 1/p^{\tau-2}$, for some $\tau > 2$, the following hold:

- (a) \hat{B} exactly recovers the sparsity structure of B^* ; that is, $\hat{B}_{E^c} = 0$,
- (b) \hat{B} satisfies the element-wise ℓ_∞ bound $\|\hat{B} - B^*\|_\infty \leq C_5 \sqrt{\frac{p^{\tau/k}}{n}}$, and
- (c) \hat{B} satisfies sign consistency if $|B_{\min}^*| \geq 2C_5 \sqrt{\frac{p^{\tau/k}}{n}}$,

where $C_4 = \left[48(\max_i \Sigma_{ii}^*) (C_k(M_k + 1))^{1/2k} \nu_{D^2} \nu_{B^*} \max\{\nu_{\Gamma^*-1} \nu_{B^*-1}, 2\nu_{\Gamma^*-1}^2 \nu_{B^*-1}^3, 2\alpha^{-1} d^{-1}\} \right]^2$,
 $C_5 = 16(\max_i \Sigma_{ii}^*) (C_k(M_k + 1))^{1/2k} \nu_{\Gamma^*-1} \nu_{D^2} \nu_{B^*}$.

Proof. The proof follows along the same lines of Theorem 1. Hence, to avoid redundancy, we provide only high-level details. Part (a) We use the polynomial type tail bound in A.7 to compute $n_f(\delta, p^\tau)$, we therefore have $n_f(\delta, p^\tau) = \frac{(c_* p^\tau)^{1/k}}{\delta^2}$ and substituting for c_* and δ as given in Lemma A.7 and Lemma A.6 respectively, we get

$$n_f(\delta, p^\tau) = C_4 d^2 p^{\tau/k}. \quad (98)$$

Part(b): From Lemma A.7, we have $f(n, \delta) = \frac{n^k \delta^{2k}}{c_*}$, where $c_* = 2^{2k} (\max_i \Sigma_{ii}^*)^{2k} C_k (M_k + 1)$. Thus setting

$$\delta = \delta_f(n, p^\tau) = \left(\frac{c_* p^\tau}{n} \right)^{1/2k} = 2(\max_i \Sigma_{ii}^*) (C_k(M_k + 1))^{1/2k} \sqrt{\frac{p^{\tau/k}}{n}}. \quad (99)$$

On the other hand, from Lemma A.6, we have $\|\hat{B} - B^*\|_\infty \leq 8\nu_{\Gamma^*-1} \nu_{D^2} \nu_{B^*} \delta_f(n, p^\tau)$. Thus,

$$\|\hat{B} - B^*\|_\infty \leq 16(\max_i \Sigma_{ii}^*) (C_k(M_k + 1))^{1/2k} \nu_{\Gamma^*-1} \nu_{D^2} \nu_{B^*} \sqrt{\frac{p^{\tau/k}}{n}}. \quad (100)$$

Part (c): similar to the contradiction argument in Theorem 1. The details are omitted. \square

We present Frobenius and spectral norm consistency results for distributions with bounded moments.

Corollary 2. Suppose the hypotheses in Theorem 2 hold. Then with probability greater than $1 - \frac{1}{p^{\tau-2}}$:
 $\|\hat{B} - B^*\|_F \leq C_5 \sqrt{\frac{(s+p)(p^{\tau/k})}{n}}$ and $\|\hat{B} - B^*\|_2 \leq C_5 \min\{d, \sqrt{s+p}\} \sqrt{\frac{p^{\tau/k}}{n}}$, where $s = |\mathcal{E}(B^*)|$.

Proof. The proof follows along the same lines of Corollary 1. Hence, the details are omitted. \square