**Zeitschrift für angewandte
Mathematik und Physik ZAMP**

On well-posedness of generalized Hall-magneto-hydrodynamics

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Abstract. We obtain local well-posedness result for the generalized Hall-magneto-hydrodynamics system in Besov spaces $\dot{B}_{\infty,\infty}^{-(2\alpha_1-\gamma)} \times \dot{B}_{\infty,\infty}^{-(2\alpha_2-\beta)}(\mathbb{R}^3)$ with suitable indexes $\alpha_1, \alpha_2, \beta$ and γ. As a corollary, the hyperdissipative electron magneto-hydrodynamics system is globally well-posed in $\dot{B}_{\infty,\infty}^{-(2\alpha_2-2)}(\mathbb{R}^3)$ for small initial data.

Mathematics Subject Classification. 35Q35, 35Q60, 35A05.

Keywords. Local well-posedness, Global well-posedness, Hall-MHD, Electron-MHD, Besov spaces.

1. Introduction

In this paper, we study the well-posedness problem of the following generalized Hall-magneto-hydrodynamics (Hall-MHD) system

$$\begin{cases} u_t + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla p = -\nu(-\Delta)^{\alpha_1}u, \\ b_t + (u \cdot \nabla)b - (b \cdot \nabla)u + \eta\nabla \times ((\nabla \times b) \times b) = -\mu(-\Delta)^{\alpha_2}b, \\ \nabla \cdot u = 0, \ \nabla \cdot b = 0, \\ u(0,x) = u_0, \ b(0,x) = b_0, \ t \in \mathbb{R}^+, x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

with the parameters $\alpha_1, \alpha_2 > 0$ and the constants $\nu, \mu > 0, \eta \geq 0$.

In particular, the fourth term on the left-hand side of the second equation is called the Hall term. When $\alpha_1 = \alpha_2 = 1, \eta > 0$, system (1.1) becomes the standard Hall-MHD system, whereas the case $\eta = 0$ corresponds to the generalized magneto-hydrodynamics (MHD) system.

Derived in [1] as the incompressible limit of a two-fluid isothermal Euler–Maxwell system for electrons and ions, the Hall-MHD system describes the evolution of a system consisting of charged particles that can be approximated as a conducting fluid, in the presence of a magnetic field b, with u denoting the fluid velocity, p the pressure, ν the viscosity, μ the magnetic resistivity and η a constant determined by the ion inertial length. The MHD and Hall-MHD systems have a wide range of applications in plasma physics and astrophysics, including modeling solar wind turbulence, designing tokamaks as well as studying the origin and dynamics of the terrestrial magnetosphere. Notably, the Hall-MHD system serves a vital role in interpreting the magnetic reconnection phenomenon, frequently observed in space plasmas. For more physical backgrounds, we refer readers to [4, 22–24, 38, 46].

Over the past decade, various mathematical results concerning the Hall-MHD system have been obtained. A mathematically rigorous derivation of the system is due to Acheritogaray et al. [1]. Concerning the solvability of the system, Chae et al. [5] obtained global-in-time existence of weak solutions and local-in-time existence of classical solutions. In [6], Chae and Lee established a blow-up criterion and a

Birkhäuser

small data global existence result. In addition, local well-posedness results can be found in the works by Dai [11,12], and global existence results for small data were also proved by Wan and Zhou [49] as well as by Kwak and Lkhagvasuren [33]. By treating the current density $j = \nabla \times b$ as an additional unknown function, Danchin and Tan [15,16] proved well-posedness results in Besov and Sobolev spaces, which have been extended by Liu and Tan in [39]. For various regularity criteria, readers are referred to [10,18,19,28,51,60–62,69]. Regarding the properties of the solutions, the temporary decay of weak solutions was studied by Chae and Schonbek [7], while the stability of global strong solutions is due to Benvenutti and Ferreira [2]. On the other hand, in the irresistive setting, there are striking ill-posedness results due to Chae and Weng [9] as well as Jeong and Oh [30]. Recently, Dai [13] proved the non-uniqueness of the Leray–Hopf weak solution via a convex integration scheme.

As generalized Laplacians may appear in realistic circumstances in which the viscosity (or resistance) is enhanced or attenuated (c.f. [52–54]), the MHD system with generalized Laplacians has been intensively studied and there is a sizable literature. Without attempting to write an exhaustive list, we recall a few key results on the generalized MHD system. Wu [52] showed that there exists a global weak solution to the n-dimensional generalized MHD system, i.e., system (1.1) with $\nu > 0$, $\mu > 0$ and $\eta = 0$, for any $\alpha_1 > 0$ and $\alpha_2 > 0$, and that the weak solutions associated with $\alpha_1 \geq \frac{1}{2} + \frac{n}{4}$ and $\alpha_2 \geq \frac{1}{2} + \frac{n}{4}$ are classical for sufficiently smooth initial data; the constraints on the parameters were further relaxed to $\alpha_1 \geq \frac{1}{2} + \frac{n}{4}$, $\alpha_2 > 0$ and $\alpha_1 + \alpha_2 \geq 1 + \frac{n}{2}$ in a subsequent work of the same author [54]. Global existence of mild solutions for small initial data in various function spaces such as the Fourier–Besov–Morrey spaces, Fourier–Herz spaces, pseudomeasure spaces, anisotropic Sobolev/Besov spaces and Lei-Lin type spaces can be found in [17,35–37,40,56,57,63]. In 2D, the generalized MHD system is well-posed even if α_1 falls in the the hypodissipative range, e.g., $0 < \alpha_1 \leq 1$ and $\alpha_2 = 1$, or even in the non-resistive case, i.e., $\alpha_2 = 0$, as shown in [20,29,47]. Results in the form of blow-up criterion, e.g., [53,70], are abundant. For asymptotic behaviors of the generalized MHD system, we refer readers to [59,65] and the references therein.

Our study of the generalized system (1.1) is motivated by the results on the generalized MHD system. Mathematically, it is easier to exploit dissipation to overcome the Hall term, which tends to be an obstacle to well-posedness. Chae et al. [8] proved local well-posedness in the case $\alpha_1 = 0$, $\alpha_2 > \frac{1}{2}$, while local well-posedness result for $0 < \alpha_1 \leq 2$, $1 < \alpha_2 \leq 2$ and global well-posedness result for $\alpha_1 \geq \frac{5}{4}$, $\alpha_2 \geq \frac{7}{4}$ were obtained, respectively, by Wan and Zhou [50] and Wan [48]. Small data global solutions were established in [44,55,58]. Results concerning the asymptotic behavior of solutions to the generalized Hall-MHD can be found in [66–68]. In addition, decay result of global smooth solutions in the cases where either α_1 or $\alpha_2 = 0$ is due to Dai and Liu [14]. We refer readers to [21,27,31,45] for a number of regularity criteria.

In this paper, we shall prove that system (1.1) is locally well-posed in the Besov space $\dot{B}_{\infty,\infty}^{-(2\alpha_1-\gamma)} \times \dot{B}_{\infty,\infty}^{-(2\alpha_2-\beta)}(\mathbb{R}^3)$ for suitable choices of $\alpha_1, \alpha_2, \beta$ and γ. We are curious about how the parameters α_1 and α_2 affect the well-posedness of the system in large Besov spaces. For generalized MHD system, local and global well-posedness results in Besov spaces were proved in [64] via the same mechanism as in this paper, in spite of a major difference between the MHD and Hall-MHD systems in terms of scaling properties. In brief, the generalized MHD system scales as

$$u_\lambda(t,x) = \lambda^{2\alpha_1-1}u(\lambda^{2\alpha_1}t, \lambda x), b_\lambda(t,x) = \lambda^{2\alpha_2-1}b(\lambda^{2\alpha_2}t, \lambda x),$$

while the electron-MHD (EMHD) equations, i.e., the fluid-free version of system (1.1), scale as $b_\lambda(t,x) = \lambda^{2\alpha_2-2}b(\lambda^{2\alpha_2}t, \lambda x)$, resulting in the absence of a genuine scaling along with a lack of the notion of criticality in the Hall-MHD system, which seems to render the global well-posedness for the full system (1.1) rather elusive. For system (1.1), we can only establish local well-posedness, in contrast to the generalized MHD system, which possesses global-in-time solutions in the largest critical space $\dot{B}_{\infty,\infty}^{-(2\alpha_1-1)} \times \dot{B}_{\infty,\infty}^{-(2\alpha_2-1)}(\mathbb{R}^3)$, with $\alpha_1 = \alpha_2$, $\frac{1}{2} < \alpha_1, \alpha_2 < 1$ for small initial data, as proven in [64]. The fact that the well-posedness result for the Hall-MHD system deviates from that for the MHD system is an evidence that the new scale and nonlinear interactions introduced by the Hall term $\eta\nabla \times ((\nabla \times b) \times b)$ play a significant role.

Our main result states as follows.

Theorem 1.1. (Local well-posedness) *For* $(u_0, b_0) \in \dot{B}_{\infty,\infty}^{-(2\alpha_1-\gamma)} \times \dot{B}_{\infty,\infty}^{-(2\alpha_2-\beta)}(\mathbb{R}^3)$, *there exists a unique local-in-time solution* (u, b) *to system* (1.1) *such that*

$$(u, b) \in L^\infty\left(0, T; \dot{B}_{\infty,\infty}^{-(2\alpha_1-\gamma)} \times \dot{B}_{\infty,\infty}^{-(2\alpha_2-\beta)}(\mathbb{R}^3)\right)$$

with $T = T\left(\nu, \mu, \eta, \|u_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_1-\gamma)}}, \|b_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_2-\beta)}}\right)$, *provided that the parameters* $\alpha_1, \alpha_2, \beta$ *and* γ *satisfy the following constraints*

$$\begin{cases} \gamma \geq \max\{1, \frac{\alpha_1}{\alpha_2}\}, \\ \beta \geq \max\{2, \frac{(\gamma+1)\alpha_2}{2\alpha_1}\}, \\ \frac{\gamma}{2} < \alpha_1 < \gamma, \\ \frac{\beta}{2} < \alpha_2 < \beta. \end{cases} \tag{1.2}$$

An interesting byproduct of the above result is small data global well-posedness for the EMHD equations.

Theorem 1.2. (Global existence for small data) *Let* $1 < \alpha_2 < 2$. *There exists some* $\varepsilon = \varepsilon(\mu) > 0$ *such that if* $\|b_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_2-2)}(\mathbb{R}^3)} \leq \varepsilon$, *then there exists a solution* b *to the EMHD system, i.e., system* (1.1) *with* $u \equiv 0$, *satisfying*

$$b \in L^\infty\left(0, +\infty; \dot{B}_{\infty,\infty}^{-(2\alpha_2-2)}(\mathbb{R}^3)\right) \ \text{and} \ \sup_{t>0} t^{\frac{\alpha_2-1}{\alpha_2}} \|b\|_{L^\infty(\mathbb{R}^3)} < \infty.$$

2. Preliminaries

2.1. Notation

Throughout the paper, we will use C to denote different constants. The notation $A \lesssim B$ means that $A \leq CB$ for some constant C. For simplicity, we denote the caloric extensions $e^{-\nu t(-\Delta)^{\alpha_1}} u_0$ and $e^{-\mu t(-\Delta)^{\alpha_2}} b_0$ by $\tilde{u}_0$ and $\tilde{b}_0$, respectively. In addition, we use $\mathbb{P}$ to denote the Helmholtz–Leray projection onto solenoidal vector fields, which acts on a vector field ϕ as

$$\mathbb{P}\phi = \phi + \nabla \cdot (-\Delta)^{-1} \text{div}\phi.$$

2.2. Besov spaces via Littlewood–Paley theory

We shall briefly recall the homogeneous Littlewood–Paley decomposition, through which we shall define the homogeneous Besov space. For a complete description of Littlewood–Paley theory and its applications, we refer readers to [3,26].

We introduce the radial function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \chi \leq 1$ and

$$\chi(\xi) = \begin{cases} 1, & \text{for } |\xi| \leq \frac{3}{4} \\ 0, & \text{for } |\xi| \geq 1. \end{cases}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be such that $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$. We construct a family of smooth functions $\{\varphi_q\}_{q\in\mathbb{Z}}$ supported on dyadic annuli in the frequency space, defined as

$$\varphi_q(\xi) = \varphi(2^{-q}\xi), \ q \in \mathbb{Z}.$$

We can see that $\{\varphi_q\}_{q\in\mathbb{Z}}$ is a partition of unity in $\mathbb{R}^n$.

Denoting the Fourier transform and its inverse by $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively, we introduce $h := \mathcal{F}^{-1}\varphi$. For $u \in \mathcal{S}'$, the homogeneous Littlewood–Paley projections are defined as

$$\dot{\Delta}_q u := \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}u) = 2^{nq} \int_{\mathbb{R}^n} h(2^q y)u(x - y)\mathrm{d}y, \ q \in \mathbb{Z}.$$

In view of the above definitions, we note that the following identity holds in the sense of distributions
-

$$u = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u.$$

With each $\dot{\Delta}_q u$ supported in some annular domain in the Fourier space, Littlewood–Paley projections provide us with a way to decompose a function into pieces with localized frequencies.

For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, we define the homogeneous Besov space $\dot{B}_{p,q}^s$ as

$$\dot{B}_{p,q}^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} < \infty \right\},$$

with the norm given by

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} = \begin{cases} \left(\sum_{j \in \mathbb{Z}} (2^{sj}\|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^n)})^q \right)^{\frac{1}{q}}, & \text{if } 1 \le q < \infty, \\ \sup_{j \in \mathbb{Z}} \left(2^{sj}\|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^n)} \right), & \text{if } q = \infty. \end{cases}$$

In this paper, we are primarily interested in the L^∞, ℓ^∞-based Besov spaces $\dot{B}_{\infty,\infty}^s$.

2.3. Besov spaces and the heat kernel

It turns out that negative order Besov spaces can also be characterized via the action of the heat kernel. In particular, we have the following lemma, for whose proof we refer readers to [34].

Lemma 2.1. *Let* $f \in \dot{B}_{\infty,\infty}^s$ *for some* $s < 0$. *The following norm equivalence holds.*

$$\|f\|_{\dot{B}_{\infty,\infty}^s} = \sup_{t>0} t^{-\frac{s}{2\alpha}}\|\mathrm{e}^{-t(-\Delta)^\alpha} f\|_{L^\infty}, \ \text{where } \alpha > 0. \tag{2.1}$$

More generally, the following lemma concerning the action of the heat semigroup in Besov spaces holds true and shall be extensively used in this paper.

Lemma 2.2. (i) *For* $\alpha > 0$, *the following inequalities hold.*

$$\|\mathrm{e}^{-t(-\Delta)^\alpha} f\|_{L^\infty} \le C\|f\|_{L^\infty},$$
$$\|\nabla \mathrm{e}^{-t(-\Delta)^\alpha} f\|_{L^\infty} \le Ct^{-\frac{1}{2\alpha}}\|f\|_{L^\infty},$$
$$\|\nabla \mathbb{P} \mathrm{e}^{-t(-\Delta)^\alpha} f\|_{L^\infty} \le Ct^{-\frac{1}{2\alpha}}\|f\|_{L^\infty}.$$

(ii) *For* $\alpha > 0$ *and* $s_0 \le s_1$, *the following inequalities hold.*

$$\|\mathrm{e}^{-t(-\Delta)^\alpha} f\|_{\dot{B}_{\infty,\infty}^{s_1}} \le Ct^{-\frac{1}{2\alpha}(s_1-s_0)}\|f\|_{\dot{B}_{\infty,\infty}^{s_0}},$$
$$\|\nabla^k \mathrm{e}^{-t(-\Delta)^\alpha} f\|_{\dot{B}_{\infty,\infty}^{s_1}} \le Ct^{-\frac{1}{2\alpha}(s_1-s_0+k)}\|f\|_{\dot{B}_{\infty,\infty}^{s_0}}.$$

Proofs of Lemma 2.2 can be found in [32,43].

2.4. Mild solutions

A mild solution to system (1.1) is the fix point of the map

$$S(u,b) := \begin{pmatrix} S_1(u,b) \\ S_2(u,b) \end{pmatrix}, \tag{2.2}$$

where $S_1(u,b)$ and $S_2(u,b)$ are given by the following Duhamel's formulae-

$$S_1(u,b) := u(t,x) = e^{-\nu t(-\Delta)^{\alpha_1}} u_0(x) - \int_0^t e^{-\nu(t-s)(-\Delta)^{\alpha_1}} \mathbb{P}\nabla \cdot (u \otimes u)(s)\mathrm{d}s$$

$$+ \int_0^t e^{-\nu(t-s)(-\Delta)^{\alpha_1}} \mathbb{P}\nabla \cdot (b \otimes b)(s)\mathrm{d}s, \tag{2.3}$$

$$S_2(u,b) := b(t,x) = e^{-\mu t(-\Delta)^{\alpha_2}} b_0(x) - \int_0^t e^{-\mu(t-s)(-\Delta)^{\alpha_2}} \mathbb{P}\nabla \cdot (u \otimes b)(s)\mathrm{d}s$$

$$+ \int_0^t e^{-\mu(t-s)(-\Delta)^{\alpha_2}} \mathbb{P}\nabla \cdot (b \otimes u)(s)\mathrm{d}s \tag{2.4}$$

$$- \eta \int_0^t e^{-\mu(t-s)(-\Delta)^{\alpha_2}} \nabla \times (\nabla \cdot (b \otimes b))(s)\mathrm{d}s.$$

In (2.4), we have applied the vector identity $\nabla \times (\nabla \cdot (b \otimes b)) = \nabla \times ((\nabla \times b) \times b)$ to the Hall term. To further simplify notations, we view the integrals in expressions (2.3) and (2.4) as bilinear forms.

Definition 2.3. (*Bilinear forms*) Let $f,g \in \mathcal{S}'$. The bilinear forms $\mathcal{B}_{\alpha_1}(\cdot,\cdot)$, $\mathcal{B}_{\alpha_2}(\cdot,\cdot)$ and $\mathfrak{B}_{\alpha_2}(\cdot,\cdot)$ are defined as follows.

$$\mathcal{B}_{\alpha_1}(f,g) = \int_0^t e^{-\nu(t-s)(-\Delta)^{\alpha_1}} \mathbb{P}\nabla \cdot (f \otimes g)(s)\mathrm{d}s;$$

$$\mathcal{B}_{\alpha_2}(f,g) = \int_0^t e^{-\mu(t-s)(-\Delta)^{\alpha_2}} \mathbb{P}\nabla \cdot (f \otimes g)(s)\mathrm{d}s;$$

$$\mathfrak{B}_{\alpha_2}(f,g) = \eta \int_0^t e^{-\mu(t-s)(-\Delta)^{\alpha_2}} \nabla \times (\nabla \cdot (b \otimes b))(s)\mathrm{d}s.$$

In view of the above, we can write the formulae (2.2), (2.3) and (2.4) as

$$\begin{aligned} S_1(u,b) &= \tilde{u}_0(x) - \mathcal{B}_{\alpha_1}(u,u) + \mathcal{B}_{\alpha_1}(b,b), \\ S_2(u,b) &= \tilde{b}_0(x) - \mathcal{B}_{\alpha_2}(u,b) + \mathcal{B}_{\alpha_2}(b,u) - \mathfrak{B}_{\alpha_2}(b,b). \end{aligned} \tag{2.5}$$

2.5. The contraction principle

Given the mild solution formulation (2.2), a traditional approach is to find a fixed point by iterating the map $(u,b) \mapsto S(u,b)$. In order to do so, it is essential to find a space $\mathcal{E}$ such that the bilinear forms $\mathcal{B}_\alpha(\cdot,\cdot)$

and $\mathfrak{B}_\alpha(\cdot, \cdot)$ are bounded from $\mathcal{E} \times \mathcal{E}$ to $\mathcal{E}$. In this paper, we shall use the following lemma, proven in [34] and [41] as a simple consequence of Banach fixed point theorem.

Lemma 2.4. *Let $\mathcal{E}$ be a Banach space. Given a bilinear form $\mathbb{B} : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ such that $\|\mathbb{B}(u, v)\|_\mathcal{E} \leq C_0 \|u\|_\mathcal{E} \|v\|_\mathcal{E}, \forall u, v \in \mathcal{E}$, for some constant $C_0 > 0$, we have the following assertions for the equation*

$$u = y + \mathbb{B}(u, u). \tag{2.6}$$

(i) *Suppose that $y \in B_\varepsilon(0) := \{f \in \mathcal{E} : \|f\|_\mathcal{E} < \varepsilon\}$ for some $\varepsilon \in \left(0, \frac{1}{4C_0}\right)$, then the Eq. (2.6) has a solution $u \in B_{2\varepsilon}(0) := \{f \in \mathcal{E} : \|f\|_\mathcal{E} < 2\varepsilon\}$, which is, in fact, the unique solution in the ball $\overline{B_{2\varepsilon}(0)}$.*

(ii) *On top of* (i)*, suppose that $\bar{y} \in B_\varepsilon(0), \bar{u} \in B_{2\varepsilon}(0)$ and $\bar{u} = \bar{y} + \mathbb{B}(\bar{u}, \bar{u})$, then the following continuous dependence is true.*

$$\|u - \bar{u}\|_\mathcal{E} \leq \frac{1}{1 - 4\varepsilon C_0} \|y - \bar{y}\|_\mathcal{E}. \tag{2.7}$$

It can be seen from inequality (2.7) that to ensure local well-posedness, it suffices that $C_0 = CT^a$ for some $a > 0$, while global well-posedness would require C_0 to be bounded above by a time-independent constant.

3. Proofs of Theorems

This section is devoted to the proofs of Theorems 1.1 and 1.2. We work within a framework based on the concepts of the "admissible path space" and "adapted value space," as formulated in [34]. The idea is to first identify an "admissible path space" $\mathcal{E}_T$ in which we may apply the contraction principle, then characterize the "adapted value space" E_T associated with $\mathcal{E}_T$. In our case, we consider the space

$$E_T = \{f : f \in \mathcal{S}', \ e^{-t(-\Delta)^{\alpha_i}} f \in \mathcal{E}_T, \ 0 < t < T\}, \ i = 1 \text{ or } 2.$$

To start, we define the Banach spaces X_T and Y_T and the admissible path space $\mathcal{E}_T := X_T \times Y_T$.

$$X_T = \left\{ f : \mathbb{R}^+ \to L^\infty(\mathbb{R}^3) : \nabla \cdot f = 0 \text{ and } \sup_{0<t<T} t^{\frac{2\alpha_1 - \gamma}{2\alpha_1}} \|f(t)\|_{L^\infty(\mathbb{R}^3)} < \infty \right\} \tag{3.1}$$

$$Y_T = \left\{ f : \mathbb{R}^+ \to L^\infty(\mathbb{R}^3) : \nabla \cdot f = 0 \text{ and } \sup_{0<t<T} t^{\frac{2\alpha_2 - \beta}{2\alpha_2}} \|f(t)\|_{L^\infty(\mathbb{R}^3)} < \infty \right\} \tag{3.2}$$

By formulae (2.3) and (2.4) along with the characterization of homogeneous Besov spaces in terms of the heat flow (2.2), we have the following inequalities -

$$\|u\|_{X_T} \leq \sup_{t>0} t^{\frac{2\alpha_1 - \gamma}{2\alpha_1}} \|\tilde{u}_0\|_\infty + \|\mathcal{B}_{\alpha_1}(u, u)\|_{X_T} + \|\mathcal{B}_{\alpha_1}(b, b)\|_{X_T}$$

$$\leq C_\nu \|u_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_1 - \gamma)}} + \|\mathcal{B}_{\alpha_1}(u, u)\|_{X_T} + \|\mathcal{B}_{\alpha_1}(b, b)\|_{X_T},$$

$$\|b\|_Y \leq \sup_{t>0} t^{\frac{2\alpha_2 - \beta}{2\alpha_2}} \|\tilde{b}_0\|_\infty + \|\mathcal{B}_{\alpha_2}(u, b)\|_{Y_T} + \|\mathcal{B}_{\alpha_2}(b, u)\|_{Y_T} + \|\mathfrak{B}_{\alpha_2}(b, b)\|_{Y_T}$$

$$\leq C_\mu \|b_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_2 - \beta)}} + \|\mathcal{B}_{\alpha_2}(u, b)\|_{Y_T} + \|\mathcal{B}_{\alpha_2}(b, u)\|_{Y_T} + \|\mathfrak{B}_{\alpha_2}(b, b)\|_{Y_T}.$$

Clearly, $\dot{B}_{\infty,\infty}^{-(2\alpha_1 - \gamma)} \times \dot{B}_{\infty,\infty}^{-(2\alpha_2 - \beta)}(\mathbb{R}^3)$ is an adapted value space corresponding to the admissible path space $\mathcal{E}_T$ given by Definitions 3.1 and 3.2.

We proceed to prove the following proposition.

Proposition 3.1. *Suppose that the parameters $\alpha_1, \alpha_2, \beta$ and γ satisfy*

$$\begin{cases} \gamma \geq \max\{1, \frac{\alpha_1}{\alpha_2}\}, \\ \beta \geq \max\{2, \frac{(\gamma+1)\alpha_2}{2\alpha_1}\}, \\ \frac{\gamma}{2} < \alpha_1 < \gamma, \\ \frac{\beta}{2} < \alpha_2 < \beta. \end{cases} \tag{3.3}$$

If $(u,b) \in \mathcal{E}_T$ for some $0 < T < \infty$, then $\|S(u,b) - (\tilde{u}_0, \tilde{b}_0)\| \in \mathcal{E}_T$. In particular,

$$\|S(u,b) - (\tilde{u}_0, \tilde{b}_0)\|_{\mathcal{E}_T} \leq CT^a \|(u,b)\|_{\mathcal{E}_T}^2 \tag{3.4}$$

for some $a > 0$ and $C = C(\nu, \mu, \eta) > 0$.

Proof. First, we remark that the constraints on the parameters indeed yield a non-empty set, since the combination $\gamma = 1, \beta = 2, \alpha_1 = 1 - \delta$ and $\alpha_2 = 2 - 2\delta$ with $\frac{1}{4} < \delta < \frac{1}{2}$ clearly satisfies (3.3).

To prove (3.4), it suffices to show that the bilinear forms are bounded from $\mathcal{E}_T \times \mathcal{E}_T$ to $\mathcal{E}_T$, with bounds dependent on ν, μ, η and T. To this end, we invoke the property of the Beta function. More specifically, for $\alpha > 1$ and $0 < \theta < \alpha$, we have

$$\int_0^t (t - \tau)^{-\frac{1}{\alpha}} \tau^{-\frac{\theta}{\alpha}} \, d\tau = t^{1 - \frac{1}{\alpha} - \frac{\theta}{\alpha}} B\left(1 - \frac{\theta}{\alpha}, 1 - \frac{1}{\alpha}\right) \leq C t^{1 - \frac{1}{\alpha} - \frac{\theta}{\alpha}}. \tag{3.5}$$

Let $\gamma \geq 1$ and $\frac{\gamma}{2} < \alpha_1 < \gamma$. Via integration by parts, Hölder's inequality, identity (3.5) and Definition 3.1, we have the following inequalities.

$$\|\mathcal{B}_{\alpha_1}(u,u)\|_{X_T} \leq C_\nu \sup_{0 < t < T} t^{\frac{2\alpha_1 - \gamma}{2\alpha_1}} \int_0^t (t - s)^{-\frac{1}{2\alpha_1}} \|u(s)\|_\infty \|u(s)\|_\infty d$$

$$\leq C_\nu \|u\|_{X_T}^2 \sup_{0 < t < T} t^{\frac{2\alpha_1 - \gamma}{2\alpha_1}} \int_0^t (t - s)^{-\frac{1}{2\alpha_1}} s^{-2 + \frac{\gamma}{\alpha_1}} \, ds$$

$$\leq C_\nu T^{\frac{\gamma - 1}{2\alpha_1}} \|u\|_{X_T}^2.$$

Similarly, the following estimates are true provided that $\gamma \geq 1$, $\frac{\gamma}{2} < \alpha_1 < \gamma$, $\frac{\beta}{2} < \alpha_2 < \beta$ and $\beta \geq \frac{(\gamma+1)\alpha_2}{2\alpha_1}$.

$$\|\mathcal{B}_{\alpha_1}(b,b)\|_{X_T} \leq C_\nu \sup_{0 < t < T} t^{\frac{2\alpha_1 - \gamma}{2\alpha_1}} \int_0^t (t - s)^{-\frac{1}{2\alpha_1}} \|b(s)\|_\infty \|b(s)\|_\infty ds$$

$$\leq C_\nu \|b\|_{X_T}^2 \sup_{0 < t < T} t^{\frac{2\alpha_1 - \gamma}{2\alpha_1}} \int_0^t (t - s)^{-\frac{1}{2\alpha_1}} s^{-2 + \frac{\beta}{\alpha_2}} \, ds$$

$$\leq C_\nu T^{\frac{\beta}{\alpha_2} - \frac{\gamma + 1}{2\alpha_1}} \|b\|_{X_T}^2.$$

To bound the term $\|\mathcal{B}_{\alpha_2}(b,u)\|_Y$, we further require that $\alpha_2 > \frac{1}{2}$ and $\gamma \geq \frac{\alpha_1}{\alpha_2}$.

$$\|\mathcal{B}_{\alpha_2}(b,u)\|_{Y_T} \leq C_\mu \sup_{0<t<T} t^{\frac{2\alpha_2-\beta}{2\alpha_2}} \int_0^t (t-s)^{-\frac{1}{2\alpha_2}} \|u(s)\|_\infty \|b(s)\|_\infty \mathrm{d}s$$

$$\leq C_\mu \|u\|_X \|b\|_Y \sup_{0<t<T} t^{\frac{2\alpha_2-\beta}{2\alpha_2}} \int_0^t (t-s)^{-\frac{1}{2\alpha_2}} s^{-2+\frac{\gamma}{2\alpha_1}+\frac{\beta}{2\alpha_2}} \mathrm{d}$$

$$\leq C_\mu T^{\frac{\gamma}{2\alpha_1}-\frac{1}{2\alpha_2}} \|u\|_{X_T} \|b\|_{Y_T}.$$

We note that the term $\|\mathcal{B}_{\alpha_2}(u,b)\|_Y$ can be estimated in an identical manner.

Finally, we integrate by parts twice to estimate the Hall term. We end up with the condition $\alpha_2 > 1$ along with all the constraints from previous estimates.

$$\|\mathfrak{B}_{\alpha_2}(b,b)\|_{Y_T} \leq C_{\mu,\eta} \sup_{0<t<T} t^{\frac{2\alpha_2-\beta}{2\alpha_2}} \int_0^t (t-s)^{-\frac{1}{\alpha_2}} \|b(s)\|_\infty \|b(s)\|_\infty \mathrm{d}s$$

$$\leq C_{\mu,\eta} \|b\|_{Y_T}^2 \sup_{0<t<T} t^{\frac{2\alpha_2-\beta}{2\alpha_2}} \int_0^t (t-s)^{-\frac{1}{\alpha_2}} s^{-2+\frac{\beta}{\alpha_2}} \mathrm{d}s$$

$$\leq C_{\mu,\eta} T^{\frac{\beta-2}{2\alpha_2}} \|b\|_{Y_T}^2.$$

<div align="right">□</div>

Proof of Theorem 1.1. By inequality (3.4), Lemmas 2.2 and 2.4, there exists a solution $(u,b) \in \mathcal{E}_T$ provided that the initial data (u_0,b_0) and the time T satisfy

$$4CT^a \left(C_\nu \|u_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_1-\gamma)}} + C_{\mu,\eta} \|b_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_2-\beta)}} \right) < 1.$$

It remains to be shown that $(u,b) \in L^\infty\left(0,T; \dot{B}_{\infty,\infty}^{-(2\alpha_1-\gamma)} \times \dot{B}_{\infty,\infty}^{-(2\alpha_2-\beta)}(\mathbb{R}^3)\right)$. By (2.3) and Lemma 2.2, it holds that

$$\|S_1 u(t)\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_1-\gamma)}} = \sup_{0<\tau<T} \tau^{\frac{2\alpha_1-\gamma}{2\alpha_1}} \|\mathrm{e}^{-\nu\tau(-\Delta)^{\alpha_1}} S_1 u(t)\|_{L^\infty}$$

$$\lesssim \sup_{0<\tau<T} \tau^{\frac{2\alpha_1-\gamma}{2\alpha_1}} \|\mathrm{e}^{-\nu(\tau+t)(-\Delta)^{\alpha_1}} u_0\|_{L^\infty}$$

$$+ \sup_{0<\tau<T} \tau^{\frac{2\alpha_1-\gamma}{2\alpha_1}} \|u\|_{X_T}^2 \int_0^{\tau+t} (\tau+t-s)^{-\frac{1}{2\alpha_1}} s^{-2+\frac{\gamma}{\alpha_1}} \mathrm{d}s$$

$$+ \sup_{0<\tau<T} \tau^{\frac{2\alpha_1-\gamma}{2\alpha_1}} \|b\|_{Y_T}^2 \int_0^{\tau+t} (\tau+t-s)^{-\frac{1}{2\alpha_1}} s^{-2+\frac{\beta}{\alpha_2}} \mathrm{d}s.$$

Estimating with the help of (3.5), we have

$$\|S_1 u(t)\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_1-\gamma)}} \lesssim \sup_{0<\tau<T} \tau^{\frac{2\alpha_1-\gamma}{2\alpha_1}} \left(\|\mathrm{e}^{-\nu\tau(-\Delta)^{\alpha_1}} u_0\|_{L^\infty} + (\tau+t)^{-1+\frac{2\gamma-1}{2\alpha_1}} \|u\|_{X_T}^2 \right.$$

$$\left. + (\tau+t)^{-1-\frac{1}{2\alpha_1}+\frac{2\beta}{2\alpha_2}} \|b\|_{Y_T}^2 \right)$$

$$\lesssim \|u_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_1-\gamma)}} + T^a \|(u,b)\|_{\mathcal{E}_T}^2.$$

In a similar fashion, the following inequalities follows from (2.4) and Lemma 2.2.

$$\|S_2 b(t)\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_2-\beta)}} = \sup_{0<\tau<T} \tau^{\frac{2\alpha_2-\beta}{2\alpha_2}} \|\mathrm{e}^{-\mu\tau(-\Delta)^{\alpha_2}} S_2 b(t)\|_{L^\infty}$$

$$\lesssim \sup_{0<\tau<T} \tau^{\frac{2\alpha_2-\beta}{2\alpha_2}} \left(\|\mathrm{e}^{-\mu(\tau+t)(-\Delta)^{\alpha_2}} b_0\|_{L^\infty} \right.$$

$$+ 2\|u\|_{X_T} \|b\|_{Y_T} \int_0^{\tau+t} (\tau+t-s)^{-\frac{1}{2\alpha_2}} s^{-2+\frac{\gamma}{2\alpha_1}+\frac{\beta}{2\alpha_2}} \,\mathrm{d}s$$

$$\left. + \|b\|_{Y_T}^2 \int_0^{\tau+t} (\tau+t-s)^{-\frac{1}{\alpha_2}} s^{-2+\frac{\beta}{\alpha_2}} \,\mathrm{d}s \right).$$

The integrals can be evaluated thanks to (3.5), which yields the bound on $S_2 b$.

$$\|S_2 b(t)\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_2-\beta)}} \lesssim \sup_{0<\tau<T} \tau^{\frac{2\alpha_2-\beta}{2\alpha_2}} \left(\|\mathrm{e}^{-\nu\tau(-\Delta)^{\alpha_1}} b_0\|_{L^\infty} \right.$$

$$\left. + (\tau+t)^{-1+\frac{\gamma}{2\alpha_1}+\frac{\beta-1}{2\alpha_2}} \|u\|_{X_T} \|b\|_{Y_T} + (\tau+t)^{-1+\frac{\beta-1}{\alpha_2}} \|b\|_{Y_T}^2 \right)$$

$$\lesssim \|b_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_2-\beta)}} + T^a \|(u,b)\|_{\mathcal{E}_T^2}.$$

The inequalities above imply that

$$(u,b) \in L^\infty\big(0,T; \dot{B}_{\infty,\infty}^{-(2\alpha_1-\gamma)} \times \dot{B}_{\infty,\infty}^{-(2\alpha_2-\beta)}(\mathbb{R}^3)\big).$$

$\square$

However, well-posedness result for the standard Hall-MHD system, i.e., the case $\alpha_1 = \alpha_2 = 1$, is unattainable as the above method breaks down in this case.

We now turn to the hyper-resistive EMHD equations, written as

$$\begin{cases} b_t + \eta \nabla \times ((\nabla \times b) \times b) = -\mu(-\Delta)^{\alpha_2} b, \\ \nabla \cdot b = 0, \\ b(0,x) = b_0, \ t \in \mathbb{R}^+, x \in \mathbb{R}^3, \end{cases} \tag{3.6}$$

where $1 < \alpha_2 < 2$.

The above system is the small-scale limit of the Hall-MHD system, corresponding to the scenario in which the ions are practically static, simply forming a neutralizing background for the moving electrons. It is named electron MHD as the system is solely determined by the electrons. In astrophysics, system (3.6) makes frequent appearances in the study of the magnetosphere and the solar wind, whose dynamics can be puzzling due to high frequency magnetic fluctuations. Readers may consult [22,25,42] for relevant physics backgrounds.

Unlike the complete system (1.1), system (3.6) possesses the property of scaling invariance. More specifically, if $b(t,x)$ solves system (3.6) with initial data b_0, then $b_\lambda(t,x) = \lambda^{2\alpha_2-2} b(\lambda^{2\alpha_2} t, \lambda x)$ is a solution subject to the initial data $\lambda^{2\alpha_2-2} b_0(\lambda x)$. One can see that the space $L^\infty\big(0,\infty; \dot{B}_{\infty,\infty}^{-(2\alpha_2-2)}(\mathbb{R}^3)\big)$ is the largest critical space according to the scaling property. Unfortunately, our pathway to small data global well-posedness fails just when $\alpha_2 = 1$, leaving the question of the standard EMHD equations' solvability in the largest critical space $\dot{B}_{\infty,\infty}^0(\mathbb{R}^3)$ unanswered.

We proceed to prove Theorem 1.2 by finding a ball $B \subset Y_T$ where the solution map S_2 is a contraction mapping. We have the following two propositions.

Proposition 3.2. *Let $\alpha_2 \in (1,2)$ and $\beta = 2$. For $0 < T \le \infty$, the map S_2 satisfies*

$$\|S_2 b - \tilde{b}_0\|_{Y_T} \le C \|b\|_{Y_T}^2. \tag{3.7}$$

Therefore, there exists some $\varepsilon_1 > 0$, such that S_2 is a self-mapping on the ball

$$B_{\varepsilon_1}(\tilde{b}_0) =: \{f \in Y_T : \|f - \tilde{b}_0\|_{Y_T} < \varepsilon_1\},$$

provided that $\|b_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_2-2)}(\mathbb{R}^3)} < \varepsilon_1$.

Proof. The inequality (3.7) follows from the following estimate.

$$\|\mathfrak{B}_{\alpha_2}(b,b)\|_{Y_T} \leq \sup_{t>0} t^{\frac{2\alpha_2-2}{2\alpha}} \int_0^t (t-s)^{-\frac{1}{\alpha_2}} \|b(s)\|_\infty \|b(s)\|_\infty \mathrm{d}s$$

$$\leq \|b\|_{Y_T}^2 \sup_{t>0} t^{\frac{2\alpha_2-2}{2\alpha_2}} \int_0^t (t-s)^{-\frac{1}{\alpha_2}} s^{-2+\frac{2}{\alpha_2}} \mathrm{d}s$$

$$\leq C_{\mu,\eta} \|b\|_{Y_T}^2.$$

Since it is assumed that $b \in B_{\varepsilon_1}(\tilde{b}_0)$ and $\|b_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_2-2)}(\mathbb{R}^3)} < \varepsilon_1$, it follows from inequality (3.7) and lemma (2.2) that

$$\|S_2 b - \tilde{b}_0\|_{Y_T} \leq C\|b\|_{Y_T}^2 \leq C(\|b - \tilde{b}_0\|_{Y_T}^2 + \|\tilde{b}_0\|_{Y_T}^2) \leq C\varepsilon_1^2.$$

$\square$

Proposition 3.3. *Let $1 < a_2 < 2$ and $\beta = 2$. For any $T \in (0, \infty]$, there exists some $\varepsilon_2 \in (0, \varepsilon_1)$ such that if $\|b_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_2-2)}(\mathbb{R}^3)} < \varepsilon_2$, then the solution map S_2 is a contraction mapping on the ball*

$$B_{\varepsilon_2}(\tilde{b}_0) =: \{f \in Y_T : \|f - \tilde{b}_0\|_{Y_T} < \varepsilon_2\}.$$

Proof. Let $b, \bar{b} \in B_{\varepsilon_2}(\tilde{b}_0)$. Clearly, the following inequalities hold.

$$\begin{aligned}
\|S_2 b - S_2 \bar{b}\|_{Y_T} &= \|\mathfrak{B}_{\alpha_2}(b,b) - \mathfrak{B}_{\alpha_2}(\bar{b},\bar{b})\|_{Y_T} \\
&\leq \|\mathfrak{B}_{\alpha_2}(b,b) - \mathfrak{B}_{\alpha_2}(b,\bar{b})\|_{Y_T} + \|\mathfrak{B}_{\alpha_2}(b,\bar{b}) - \mathfrak{B}_{\alpha_2}(\bar{b},\bar{b})\|_{Y_T} \\
&\leq C_{\mu,\eta} \max\{\|b\|_{Y_T}, \|\bar{b}\|_{Y_T}\}\|b - \bar{b}\|_{Y_T} \\
&\leq C_{\mu,\eta} \varepsilon_2 \|b - \bar{b}\|_{Y_T}.
\end{aligned}$$

We can ensure that S_2 is a contraction mapping by choosing $\varepsilon_2 < 1/2C_{\mu,\eta}$.

$\square$

Proof of Theorem 1.2. As a result of Proposition 3.3, we know that for some $\varepsilon_2 > 0$, S_2 has a fixed point, which is a mild solution to system (3.6), in

$$B_{\varepsilon_2}(\tilde{b}_0) =: \{f \in Y_T : \|f - \tilde{b}_0\|_{Y_T} < \varepsilon_2, \ T = +\infty\},$$

provided that $\|b_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_2-2)}(\mathbb{R}^3)} < \varepsilon_2$.

To see that the solution b is in $L^\infty(0, \infty; \dot{B}_{\infty,\infty}^{-(2\alpha_2-2)}(\mathbb{R}^3))$, we just calculate

$$\begin{aligned}
\|S_2 b(t)\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_2-2)}} &\lesssim \sup_{\tau>0} \tau^{\frac{2\alpha_2-2}{2\alpha_2}} \bigg(\|\mathrm{e}^{-\mu(\tau+t)(-\Delta)^{\alpha_2}} b_0\|_{L^\infty} \\
&\qquad + \|b\|_{Y_T}^2 \int_0^{\tau+t} (\tau+t-s)^{-\frac{1}{\alpha_2}} s^{-2+\frac{2}{\alpha_2}} \mathrm{d}s \bigg) \\
&\lesssim \|b_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha_2-2)}} + \|b\|_{Y_T}^2.
\end{aligned}$$

$\square$

Acknowledgements

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

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