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Uniqueness and Non-Uniqueness Results for Forced Dyadic MHD Models

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Abstract

We construct non-unique Leray–Hopf solutions for some forced dyadic models for magnetohydrodynamics (MHD) when the intermittency dimension δ is less than 1. Conventionally, the interaction of the velocity and magnetic fields is a major challenge in the context of MHD. However, in the dyadic MHD model scenario, we exploit to our benefit certain symmetries in the interactions of the fields to obtain a non-uniqueness result. In contrast, uniqueness of the Leray–Hopf solution to the dyadic MHD models is established in the case of $\delta \geq 1$. Analogous results on uniqueness and non-uniqueness of Leray–Hopf solution are also obtained for dyadic models of MHD with fractional diffusion.

Keywords Magnetohydrodynamics · Intermittency · Dyadic model · Uniqueness and non-uniqueness

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1 Introduction

1.1 Magnetohydrodynamics

In geophysics and astrophysics, incompressible magnetohydrodynamics (MHD) governed by the equations

$$u_t + (u \cdot \nabla)u - (B \cdot \nabla)B + \nabla P = \nu \Delta u + f, \tag{1.1a}$$

$$B_t + (u \cdot \nabla)B - (B \cdot \nabla)u = \mu \Delta B, \tag{1.1b}$$

$$\nabla \cdot u = 0, \quad \nabla \cdot B = 0, \tag{1.1c}$$

is a fundamental model in the investigation of electrically conducting fluids. The system is posed on the spatial domain $\mathbb{R}^3$ or $\mathbb{T}^3$. The vector fields u and B represent the fluid velocity and magnetic field, respectively; the scalar function P denotes the pressure; the parameters ν and μ denote, respectively, the viscosity and the magnetic resistivity; and f stands for an external force acting on the fluid. When $B = 0$, system (1.1a)–(1.1c) becomes the Navier–Stokes equation (NSE) (1.4) which will be discussed later.

It is evident that the MHD system inherits challenges from the NSE, but also exhibits its own complexity which is mainly caused by the nonlinear interactions between the fluid velocity field and the magnetic field. The unsolved problems for the NSE usually also hang in the air for the MHD system. In particular, it is not clear whether either the NSE or MHD has a classical solution for all the time, given arbitrary initial data. The concept of Leray–Hopf solution for the NSE was introduced by Leray (1934) and Hopf (1951). A Leray–Hopf solution is a weak solution in the standard distributional sense, which satisfies the basic energy inequality. Such a concept is naturally adapted to other partial differential equations. Since the pioneering work of Leray, the well-posedness problem for the Leray–Hopf solutions to the NSE in three-dimensional (3D) space is not completely understood yet. In particular, the uniqueness of a Leray–Hopf solution to the 3D NSE without external forcing remains unsolved. However, in the recent remarkable work (Albritton et al. 2022) of Albritton, Brué and Colombo, the authors constructed non-unique Leray–Hopf solutions for the forced 3D NSE, building on the seminal works of Vishik (2018a, b) for the 2D Euler equation. On the other hand, the well-posedness problem of Leray–Hopf solutions for the 3D MHD (1.1a)–(1.1c) is largely open. Nonetheless, wild weak solutions have been constructed for the ideal MHD, i.e. $\nu = \mu = 0$ and $f = 0$ in (1.1a)–(1.1c) by Beekie et al. (2020). The weak solutions constructed in Beekie et al. (2020) have finite total energy, but do not conserve the magnetic helicity which is an invariant quantity for smooth solutions. Interestingly, Faraco et al. (2021) constructed infinitely many bounded solutions which violate conservation of the total energy and cross helicity but preserve magnetic helicity. In a more recent work (Faraco et al. 2021), the same authors further showed the sharpness of the L^3 integrability condition for the conservation of the magnetic helicity. For the 3D hypoviscous incompressible elastodynamics which is similar to the MHD, Chen and Liu (2021) constructed weak solutions with finite kinetic energy.

The main objective of this paper is to investigate the problem of uniqueness/non-uniqueness of Leray–Hopf solution for the forced diffusive dyadic models of the MHD system (1.1a)–(1.1c). The following dyadic model for the MHD system was proposed in Dai (2021):

$$\frac{\mathrm{d}}{\mathrm{d}t}a_j + \nu\lambda_j^2 a_j + \kappa_1\left(\lambda_j^\theta a_j a_{j+1} - \lambda_{j-1}^\theta a_{j-1}^2\right) - \kappa_2\left(\lambda_j^\theta b_j b_{j+1} - \lambda_{j-1}^\theta b_{j-1}^2\right) = f_j, \tag{1.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}b_j + \mu\lambda_j^2 b_j + \kappa_2\left(\lambda_j^\theta a_j b_{j+1} - \lambda_j^\theta b_j a_{j+1}\right) = 0 \tag{1.2b}$$

for $j \geq 0$, $\lambda_j = \lambda^j$ with a fixed constant $\lambda > 1$, and $a_{-1} = b_{-1} = 0$. The variables (a_j, b_j) are quantities related to the energy in the sense that $\frac{1}{2}a_j^2$ and $\frac{1}{2}b_j^2$ are the kinetic energy and magnetic energy in the j-th shell, respectively. The parameter θ is defined as $\theta = \frac{5-\delta}{2}$, where $\delta \in [0, 3]$ is the intermittency dimension for the 3D turbulent vector field (cf. Cheskidov and Dai 2019). Naturally, $\theta \in [1, \frac{5}{2}]$. Notice that smaller δ corresponds to larger θ, and hence stronger nonlinearity. The parameters κ_1 and κ_2 placed in front of the nonlinear terms represent the energy transfer direction and strength among shells. Similar dyadic models have been presented by physicists for the MHD system, for instance, see Gloaguen et al. (1985), Plunian et al. (2013).

Denote the total energy by

$$E(t) = \frac{1}{2}\sum_{j=0}^{\infty}\left(a_j^2 + b_j^2\right)$$

and the flux through the j-th shell by

$$\Pi_j = \lambda_j^\theta(\kappa_1 a_j^2 - \kappa_2 b_j^2)a_{j+1}, \quad j \geq 0.$$

The energy balance for the j-th shell of the system (1.2a)–(1.2b) is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(a_j^2 + b_j^2\right) = -\nu\lambda_j^2 a_j^2 - \mu\lambda_j^2 b_j^2 + \Pi_{j-1} - \Pi_j + f_j a_j.$$

Thus, system (1.2a)–(1.2b) obeys the formal energy law

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\nu\sum_{j=0}^{\infty}\lambda_j^2 a_j^2 - \mu\sum_{j=0}^{\infty}\lambda_j^2 b_j^2 + \sum_{j=0}^{\infty} f_j a_j.$$

It is clear to see that the energy is invariant for (1.2a)–(1.2b) if $\nu = \mu = 0$ and $f_j = 0$ for $j \geq 0$. We will consider the four particular cases of the general model (1.2a)–(1.2b) with $\kappa_1 = \pm 1$ and $\kappa_2 = \pm 1$.

We will provide a definition of Leray–Hopf solution for the dyadic models (1.2a)–(1.2b) in analogy with the Leray–Hopf solution for the original MHD equations (1.1a)–(1.1c). The main goal is to: (i) establish global in time existence of Leray–Hopf solutions for the dyadic models; (ii) show the uniqueness of Leray–Hopf solution when $1 \le \theta \le 2$; (iii) construct non-unique Leray–Hopf solutions in the case of $2 < \theta \le \frac{5}{2}$. Philosophically, the process of constructing non-uniqueness resembles the convex integration method in the sense that it takes advantage of the forcing term in the construction. Technically, it is much simpler than convex integration since no iteration or approximation is involved.

1.2 Main Results for Dyadic MHD Models

In this part, we lay out the results regarding Leray–Hopf solutions for the dyadic MHD model (1.2a)–(1.2b). First, for any initial data with finite total energy, we show the existence of global Leray–Hopf solutions.

Theorem 1.1 *Let* $\theta \in [1, \frac{5}{2}]$, $a^0 = \{a_j^0\}_{j\ge 0} \in l^2$ *and* $b^0 = \{b_j^0\}_{j\ge 0} \in l^2$. *For any* $T > 0$, *assume*

$$\sum_{j=0}^{\infty} \lambda_j^{-2} \int_0^T f_j^2(t)\,\mathrm{d}t < \infty,$$

i.e. $f \in L^2(0, T; H^{-1})$. *Then, there exists a Leray–Hopf solution to system* (1.2a)–(1.2b) *accompanied with the initial data* (a^0, b^0) *on* $[0, T]$.

The next result concerns the weak-strong type of uniqueness.

Theorem 1.2 *Let* $\theta \in [1, \frac{5}{2}]$. *Let* $(a(t), b(t))$ *and* $(u(t), v(t))$ *be Leray–Hopf solutions to* (1.2a)–(1.2b) *with the same initial data* $(a^0, b^0) \in l^2 \times l^2$. *Assume in addition that there is a number* J *such that*

$$|a_j(t)| \le C_0 \lambda_j^{2-\theta}, \quad |b_j(t)| \le C_0 \lambda_j^{2-\theta} \quad \text{for} \quad j \ge J, \quad t \in [0, T] \tag{1.3}$$

with a constant C_0 *depending on* λ *and* θ. *Then,*

$$a_j \equiv u_j, \quad b_j \equiv v_j, \quad \text{on} \quad [0, T] \text{ for all } \ j \ge 0.$$

As a consequence of Theorem 1.2, the uniqueness of the Leray–Hopf solution in the case of $\theta \le 2$ follows immediately.

Theorem 1.3 *Let* $1 \le \theta \le 2$. *Let* $a^0 = \{a_j^0\}_{j\ge 0} \in l^2$, $b^0 = \{b_j^0\}_{j\ge 0} \in l^2$ *and* $f \in L^2(0, T; H^{-1})$. *Then, the Leray–Hopf solution to* (1.2a)–(1.2b) *is unique.*

When $\theta > 2$, we adapt the construction strategy of Filonov and Khodunov (2021) for the forced dyadic NSE model and show that the forced dyadic MHD models have more than one Leray–Hopf solutions. Specifically, we prove:

Theorem 1.4 *Let $\theta \in (2, \frac{5}{2}]$. Let $a^0 = 0$ and $b^0 = 0$, i.e. $a_j^0 = b_j^0 = 0$ for all $j \geq 0$. There exists $T > 0$ and functions $\{f_j(t)\}$ satisfying $f = \{f_j\}_{j\geq 0} \in L^2(0, T; H^{-1})$ such that system (1.2a)–(1.2b) with initial data (a^0, b^0) has at least two Leray–Hopf solutions $(a(t), b(t))$, one of which has non-vanishing $a(t)$ and $b(t)$ on $[0, T]$.*

Remark 1.5 The solutions constructed in Theorem 1.4 satisfy the energy identity.

Remark 1.6 We see that the threshold value of θ that separates the uniqueness and non-uniqueness results is $\theta = 2$. Notice that since $\theta = \frac{5-\delta}{2}$, $\theta = 2$ corresponds to the intermittency dimension $\delta = 1$. In fact, there is evidence that $\delta = 1$ is critical for 3D turbulent flows, see Cheskidov and Shvydkoy (2014).

1.3 Weak Solutions for Dyadic NSE

The incompressible Navier–Stokes equation

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla P &= \nu \Delta u + f, \\ \nabla \cdot u &= 0, \end{aligned} \tag{1.4}$$

is a central topic in the study of fluids. Although there has been much progress in the past century concerning fundamental properties of the NSE, many significant questions remain open. Partly for this reason various so-called dyadic models have been proposed. One such model for oceanographic turbulence was presented by Desnyanskiy and Novikov (1974) and later with motivation from harmonic analysis by Katz and Pavlović (2005). This model takes the form:

$$\frac{\mathrm{d}}{\mathrm{d}t} a_j + \nu \lambda_j^2 a_j + \lambda_j^\theta a_j a_{j+1} - \lambda_{j-1}^\theta a_{j-1}^2 = f_j, \tag{1.5}$$

for $j \geq 1$ and $a_0 = 0$. A crucial property of this particular model is the persistence of positivity, namely that with nonnegative forcing a solution starting from positive initial data remains positive for all time. This attribute of the system (1.5) was essential for the proof of many interesting results, for example, see Barbato et al. (2011), Barbato et al. (2011), Cheskidov (2008), Cheskidov and Friedlander (2009), Cheskidov et al. (2007), Cheskidov et al. (2010). However, as was recently observed by Filonov and Khodunov (2021), the energy cascade in a turbulent fluid is a random process with no physical reason for the conservation of positivity. Hence, there is an intrinsic desirability for techniques that do not depend on positivity. In Filonov and Khodunov (2021), the authors introduced a novel approach that does not depend on positivity. Filonov (2017) proved for (1.5) existence and uniqueness of Leray–Hopf solution with $\theta \leq 2$. In Filonov and Khodunov (2021), they proved that there exist more than one Leray–Hopf solutions with $\theta > 2$. Specifically, they proved the following theorem:

Theorem 1.7 (Filonov and Khodunov 2021) *Let $\theta \in (2, \frac{5}{2}]$ and $a^0 = 0$. There exists $T > 0$ and functions $f_j(t)$ satisfying $f = \{f_j\}_{j\geq 0} \in L^2(0, T; H^{-1})$ such that the dyadic NSE model (1.5) with initial data a^0 has at least two Leray–Hopf solutions.*

It is interesting to point out that this non-uniqueness result for the forced dyadic NSE shares some superficial similarity with that of Albritton et al. (2022) for the original NSE. Both constructions start from zero initial data, and the forcing term is critical in both cases.

Returning to the dyadic MHD model (1.2a)–(1.2b), the delicate interactions between the velocity and the magnetic fields preclude the possibility of making a sign choice of the parameters that ensures the persistence of positivity. The techniques that we use to prove the results stated in Theorems 1.2–1.4 are motivated by the approach used for the NSE in Filonov and Khodunov (2021) which does not depend on positivity. We observe that in the MHD model context the complexity and symmetry of the nonlinear coupling of the two fields are actually benefits that give us additional freedom in constructing the scheme used to prove non-uniqueness.

1.4 Dyadic Models with Fractional Laplacian Scaling

We note that the dyadic MHD equations (1.2a)–(1.2b) can also be rescaled to

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}a_j + \nu\lambda_j^{2\alpha}a_j + \kappa_1(\lambda_j a_j a_{j+1} - \lambda_{j-1}a_{j-1}^2) \\ &\quad - \kappa_2(\lambda_j b_j b_{j+1} - \lambda_{j-1}b_{j-1}^2) = f_j, \\ &\frac{\mathrm{d}}{\mathrm{d}t}b_j + \mu\lambda_j^{2\alpha}b_j + \kappa_2(\lambda_j a_j b_{j+1} - \lambda_j b_j a_{j+1}) = 0, \end{aligned} \tag{1.6}$$

for $j \geq 1$, $a_0 = b_0 = 0$ and $\alpha = \frac{1}{\theta}$.

The analogous dyadic model for the fractional MHD with diffusion terms $(-\Delta)^{\alpha}u$ and $(-\Delta)^{\beta}B$ is

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}a_j + \nu\lambda_j^{2\alpha}a_j + \kappa_1(\lambda_j a_j a_{j+1} - \lambda_{j-1}a_{j-1}^2) \\ &\quad - \kappa_2(\lambda_j b_j b_{j+1} - \lambda_{j-1}b_{j-1}^2) = f_j, \\ &\frac{\mathrm{d}}{\mathrm{d}t}b_j + \mu\lambda_j^{2\beta}b_j + \kappa_2(\lambda_j a_j b_{j+1} - \lambda_j b_j a_{j+1}) = 0, \end{aligned} \tag{1.7}$$

with $j \geq 1$, $a_0 = b_0 = 0$, and $\alpha > 0$, $\beta > 0$. Obviously (1.6) is a special case of (1.7) with $\alpha = \beta$. We mention that fractional dissipation in the context of dyadic models may be physically relevant for the hydrodynamics and MHD, see the discussion in Mailybaev's work (Mailybaev 2015).

With slight modifications of the proof for Theorem 1.1, we can prove that:

Theorem 1.8 *Let $\alpha > 0$ and $\beta > 0$. Let $a^0 = \{a_j^0\}_{j\geq 0} \in l^2$ and $b^0 = \{b_j^0\}_{j\geq 0} \in l^2$. Assume $f \in L^2(0, T; H^{-\alpha})$ for any $T > 0$. Then, there exists a Leray–Hopf solution to system (1.7) accompanied with the initial data (a^0, b^0) on $[0, T]$.*

In analogy with Theorem 1.2, we can prove the following weak-strong type of uniqueness for a Leray–Hopf solution to (1.7).

Theorem 1.9 *Let $\alpha > 0$ and $\beta > 0$. Let $(a(t), b(t))$ and $(u(t), v(t))$ be Leray–Hopf solutions to (1.7) with the same initial data $(a^0, b^0) \in l^2 \times l^2$. Assume in addition that there is a number J such that*

$$|a_j(t)| \le C_0 \left(\lambda_j^{2\alpha-1} + \lambda_j^{2\beta-1}\right), \quad |b_j(t)| \le C_0 \lambda_j^{\alpha+\beta-1} \tag{1.8}$$

for all $j \ge J$ and $t \in [0, T]$, with a constant C_0 depending on λ and θ. Then,

$$a_j \equiv u_j, \quad b_j \equiv v_j, \quad \text{on } [0, T] \text{ for all } j \ge 0.$$

The following uniqueness of a Leray–Hopf solution to (1.7) with $\alpha \ge \frac{1}{2}$ and $\beta \ge \frac{1}{2}$ is an immediate consequence of Theorem 1.9.

Theorem 1.10 *Let $\alpha \ge \frac{1}{2}$ and $\beta \ge \frac{1}{2}$. Let $a^0 = \{a_j^0\}_{j \ge 0} \in l^2$, $b^0 = \{b_j^0\}_{j \ge 0} \in l^2$ and $f \in L^2(0, T; H^{-\alpha})$. Then, the Leray–Hopf solution to (1.7) is unique.*

We also construct non-unique Leray–Hopf solutions to (1.7) for appropriate values of α and β. Namely, we will show:

Theorem 1.11 *Let $0 < \alpha \le \beta < \frac{1}{2}$ and $3\beta - \alpha < 1$. Let $a^0 = 0$ and $b^0 = 0$. There exists $T > 0$ and functions $\{f_j(t)\}$ satisfying*

$$\sum_{j=0}^{\infty} \lambda_j^{-2\alpha} \int_0^T f_j^2(t)\, \mathrm{d}t < \infty,$$

such that system (1.7) with initial data (a^0, b^0) has at least two Leray–Hopf solutions.

When $0 < \alpha = \beta < \frac{1}{2}$, we automatically have the same result for system (1.6).

Remark 1.12 In view of Theorem 1.10 and Theorem 1.11, the value of $\frac{1}{2}$ for α and β is a sharp threshold to separate uniqueness from non-uniqueness result for system (1.6). On the other hand, for (1.7) the additional conditions of $\alpha \le \beta$ and $3\beta - \alpha < 1$ leave some gap where neither uniqueness nor non-uniqueness is known to hold.

1.5 Organization of the Paper

We provide an outline of the rest of the paper.

- Section 2 introduces notations and definitions of solutions for dyadic systems.
- Section 3 is devoted to a proof of Theorem 1.1 on the existence of Leray–Hopf solutions.
- Section 4 addresses the weak-strong uniqueness and uniqueness of Leray–Hopf solution for system (1.2a)–(1.2b) with $1 \le \theta \le 2$.
- In Sect. 5, we construct non-unique Leray–Hopf solutions for (1.2a)–(1.2b) with $\theta \in (2, \frac{5}{2}]$.
- Section 6 outlines constructions to establish firstly conditions for uniqueness and secondly conditions for non-uniqueness of Leray–Hopf solutions to the dyadic model for MHD with fractional diffusion.

2 Notations and Notion of Solutions

2.1 Notations

The space l^2 is endowed with the standard scalar product and norm,

$$(u, v) := \sum_{n=1}^{\infty} u_n v_n, \quad |u| := \sqrt{(u, u)}.$$

It is regarded as the energy space in this paper. We use H^s to represent the space of sequences equipped with the scalar product

$$(u, v)_s := \sum_{n=1}^{\infty} \lambda_n^{2s} u_n v_n$$

and norm

$$\|u\|_s := \sqrt{(u, u)_s}.$$

Notice that $H^0 = l^2$.

2.2 Notion of Solutions

In the following, we introduce the concept of solutions for dyadic systems.

Definition 2.1 A pair of l^2-valued functions $(a(t), b(t))$ defined on $[0, \infty)$ is said to be a weak solution of (1.2a)–(1.2b) if a_j and b_j satisfy (1.2a)–(1.2b) and $a_j, b_j \in C^1([0, \infty))$ for all $j \geq 0$.

Definition 2.2 A Leray–Hopf solution $(a(t), b(t))$ of (1.2a)–(1.2b) on $[0, T)$ is a weak solution satisfying

$$a_j, b_j \in L^\infty([0, T); l^2) \cap L^2([0, T); H^1), \quad \forall \ j \geq 0,$$

and

$$\|a(t)\|_{l^2}^2 + \|b(t)\|_{l^2}^2 + 2\nu \int_0^t \|a(\tau)\|_{H^1}^2 \, d\tau + 2\mu \int_0^t \|b(\tau)\|_{H^1}^2 \, d\tau \leq \|a(0)\|_{l^2}^2 + \|b(0)\|_{l^2}^2$$

for all $0 \leq t < T$.

Definition 2.3 A Leray–Hopf solution $(a(t), b(t))$ of (1.6) on $[0, T)$ is a weak solution satisfying

$$a_j, b_j \in L^\infty([0, T); l^2) \cap L^2([0, T); H^\alpha), \quad \forall \ j \geq 0,$$

and

$$\|a(t)\|_{l^2}^2 + \|b(t)\|_{l^2}^2 + 2\nu \int_0^t \|a(\tau)\|_{H^\alpha}^2 \,\mathrm{d}\tau + 2\mu \int_0^t \|b(\tau)\|_{H^\alpha}^2 \,\mathrm{d}\tau \\ \le \|a(0)\|_{l^2}^2 + \|b(0)\|_{l^2}^2$$

for all $0 \le t < T$.

Weak solution and Leray–Hopf solution of other dyadic systems in the paper can be defined analogously.

To reduce the number of parameters, we take $\nu = \mu = 1$ in the rest of the paper since they do not affect the estimates or constructions.

3 Existence of Leray–Hopf Solutions

In this section, we apply the Galerkin approximating approach to show the existence of Leray–Hopf solutions to (1.2a)–(1.2b). Since the value of κ_1 and κ_2 does not play a role in the proof, without loss of generality, we set $\kappa_1 = -\kappa_2 = 1$. Fix any integer $N \ge 1$. Denote the sequences

$$a^N(t) = \{a_j^N(t)\}_{j\ge 0}, \quad b^N(t) = \{b_j^N(t)\}_{j\ge 0}, \quad \text{with } a_j^N = b_j^N \equiv 0, \quad \forall \ j \ge N+1.$$

That is,

$$a^N(t) = \left(a_0^N(t), a_1^N(t), a_2^N(t), \ldots, a_N^N(t), 0, 0, 0, \ldots\right), \\ b^N(t) = \left(b_0^N(t), b_1^N(t), b_2^N(t), \ldots, b_N^N(t), 0, 0, 0, \ldots\right).$$

Consider the truncated system for $(a^N(t), b^N(t))$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} a_j^N &= -\lambda_j^2 a_j^N - \lambda_j^\theta a_j^N a_{j+1}^N + \lambda_{j-1}^\theta (a_{j-1}^N)^2 - \lambda_j^\theta b_j^N b_{j+1}^N \\ &\quad + \lambda_{j-1}^\theta (b_{j-1}^N)^2 + f_j, \quad 0 \le j \le N \\ \frac{\mathrm{d}}{\mathrm{d}t} b_j^N &= -\lambda_j^2 b_j^N + \lambda_j^\theta a_j^N b_{j+1}^N - \lambda_j^\theta b_j^N a_{j+1}^N, \quad 0 \le j \le N \\ a_j^N(0) &= a_j^0, \quad b_j^N(0) = b_j^0, \quad 0 \le j \le N. \end{aligned} \tag{3.1}$$

By convention, $a_{-1}^N = b_{-1}^N = 0$.

In the following, we proceed with the standard Galerkin approximating framework: (i) for any $N \ge 1$, there is a solution $(a^N(t), b^N(t))$ to (3.1) with $a^N(t)$ and $b^N(t)$ in the space $L^\infty(0, T; l^2) \cap L^2(0, T; H^1)$ and satisfying the corresponding energy inequality; (ii) we pass the sequence $\left\{(a^N(t), b^N(t))\right\}_{N\ge 1}$ (or a subsequence of it) to a limit $(a(t), b(t))$; (iii) the limit $(a(t), b(t))$ is shown to be a Leray–Hopf solution of (1.2a)–(1.2b).

The integral form of (3.1) is

$$\begin{aligned} a_j^N(t) = \ & a_j^0 + \int_0^t \Big(-\lambda_j^2 a_j^N(\tau) - \lambda_j^\theta a_j^N(\tau) a_{j+1}^N(\tau) + \lambda_{j-1}^\theta (a_{j-1}^N(\tau))^2 \\ & \qquad -\lambda_j^\theta b_j^N(\tau) b_{j+1}^N(\tau) + \lambda_{j-1}^\theta (b_{j-1}^N(\tau))^2 + f_j(\tau) \Big) \, \mathrm{d}\tau, \\ b_j^N(t) = \ & b_j^0 + \int_0^t \Big(-\lambda_j^2 b_j^N(\tau) + \lambda_j^\theta a_j^N(\tau) b_{j+1}^N(\tau) - \lambda_j^\theta b_j^N(\tau) a_{j+1}^N(\tau) \Big) \, \mathrm{d}\tau, \end{aligned} \tag{3.2}$$

for $0 \le j \le N$. Denote

$$\begin{aligned} F^N(a^N, b^N, t) &= \Big(F_0^N(a^N, b^N, t), F_1^N(a^N, b^N, t), \dots, F_N^N(a^N, b^N, t) \Big), \\ G^N(a^N, b^N) &= \Big(G_0^N(a^N, b^N), G_1^N(a^N, b^N), \dots, G_N^N(a^N, b^N) \Big), \end{aligned}$$

with

$$\begin{aligned} F_j^N(a^N, b^N, t) &= -\lambda_j^2 a_j^N(t) - \lambda_j^\theta a_j^N(t) a_{j+1}^N(t) + \lambda_{j-1}^\theta (a_{j-1}^N(t))^2 \\ &\quad - \lambda_j^\theta b_j^N(t) b_{j+1}^N(t) + \lambda_{j-1}^\theta (b_{j-1}^N(t))^2 + f_j(t), \\ G_j^N(a^N, b^N) &= -\lambda_j^2 b_j^N(t) + \lambda_j^\theta a_j^N(t) b_{j+1}^N(t) - \lambda_j^\theta b_j^N(t) a_{j+1}^N(t), \end{aligned}$$

for $0 \le j \le N$. Denote $a^{0,N} = (a_0^0, a_1^0, \dots, a_N^0)$, $b^{0,N} = (b_0^0, b_1^0, \dots, b_N^0)$ and $f^N = (f_0, f_1, \dots, f_N)$. Thus, system (3.2) can be written as:

$$\begin{aligned} a^N(t) &= a^{0,N} + \int_0^t F^N(a^N(\tau), b^N(\tau), \tau) \, \mathrm{d}\tau, \\ b^N(t) &= b^{0,N} + \int_0^t G^N(a^N(\tau), b^N(\tau)) \, \mathrm{d}\tau. \end{aligned} \tag{3.3}$$

Denote the map

$$M_N(a^N, b^N)(t) = \begin{pmatrix} a^{0,N} + \int_0^t F^N(a^N(\tau), b^N(\tau), \tau) \, \mathrm{d}\tau \\ b^{0,N} + \int_0^t G^N(a^N(\tau), b^N(\tau)) \, \mathrm{d}\tau \end{pmatrix}.$$

Notice that there exists a constant C_N depending on N such that

$$\left| F^N(a^N, b^N, t) \right| \le C_N \left(|a^N| + |a^N|^2 + |b^N|^2 \right) + |f^N|, \tag{3.4}$$

$$\left| G^N(a^N, b^N, t) \right| \le C_N \left(|b^N| + |a^N|^2 + |b^N|^2 \right), \tag{3.5}$$

and moreover

$$\left| F^N(a^N, b^N, t) - F^N(\tilde{a}^N, \tilde{b}^N, t) \right|$$

$$\leq C_N \left(1 + |a^N| + |\tilde{a}^N| + |b^N| + |\tilde{b}^N|\right)\left(|a^N - \tilde{a}^N| + |b^N - \tilde{b}^N|\right), \quad (3.6)$$

$$\left|G^N(a^N, b^N) - G^N(\tilde{a}^N, \tilde{b}^N)\right|$$
$$\leq C_N \left(1 + |a^N| + |b^N| + |\tilde{a}^N| + |\tilde{b}^N|\right)\left(|a^N - \tilde{a}^N| + |b^N - \tilde{b}^N|\right). \quad (3.7)$$

Choose

$$R_N = 2|a^{0,N}| + 2|b^{0,N}| + 2\int_0^T \left|f^N(t)\right| \mathrm{d}t, \quad (3.8)$$

and

$$t_{N,1} = \frac{1}{2C_N(2R_N + 1)}. \quad (3.9)$$

Consider the map $M_N(a^N, b^N)$ on the following closed subset of the space of continuous functions $C([0, t_{N,1}]; \mathbb{R}^N)$

$$B_N = \left\{(u, v) \in C([0, t_{N,1}]; \mathbb{R}^N) \times C([0, t_{N,1}]; \mathbb{R}^N) : \|u\|_C \leq R_N, \|v\|_C \leq R_N.\right\}$$

We claim that M_N is a contraction mapping on B_N. Indeed, for any $(a^N, b^N) \in B_N$, it follows from (3.4), (3.8) and (3.9) that for $0 < t \leq t_{N,1} \leq T$

$$\begin{aligned}
&\left|a^{0,N} + \int_0^t F^N(a^N(\tau), b^N(\tau), \tau)\,\mathrm{d}\tau\right| \\
&\leq \left|a^{0,N}\right| + \int_0^t \left|F^N(a^N(\tau), b^N(\tau), \tau)\right| \mathrm{d}\tau \\
&\leq \left|a^{0,N}\right| + tC_N\left(R_N + 2R_N^2\right) + \int_0^T |f^N(t)|\,\mathrm{d}t \\
&\leq \frac{1}{2}R_N + \frac{1}{2C_N(2R_N + 1)}C_N\left(R_N + 2R_N^2\right) \\
&= R_N;
\end{aligned}$$

and similarly, by (3.5), (3.8) and (3.9)

$$\begin{aligned}
&\left|b^{0,N} + \int_0^t G^N(a^N(\tau), b^N(\tau))\,\mathrm{d}\tau\right| \\
&\leq \left|b^{0,N}\right| + \int_0^t \left|G^N(a^N(\tau), b^N(\tau))\right| \mathrm{d}\tau \\
&\leq \left|b^{0,N}\right| + tC_N\left(R_N + 2R_N^2\right) \\
&\leq \frac{1}{2}R_N + \frac{1}{2C_N(2R_N + 1)}C_N\left(R_N + 2R_N^2\right) \\
&= R_N.
\end{aligned}$$

Thus, M_N maps B_N to itself. On the other hand, the property of contraction follows from (3.6), (3.7) and the choice of time $t_{N,1}$ in (3.9). Therefore, system (3.3) has a solution $(a^N(t), b^N(t))$ on $[0, t_{N,1}]$, and so does system (3.1). Next, we show that the solution satisfies the energy inequality. Multiplying the first equation of (3.1) by a_j^N and the second one by b_j^N, taking the sum for $0 \le j \le N$ and integrating over $[0, t]$ for $0 < t \le t_{N,1}$, we obtain

$$\begin{aligned}&\sum_{j=0}^{N}\left(a_j^N(t)^2+b_j^N(t)^2\right)+2\sum_{j=0}^{N}\int_0^t \lambda_j^2\left(a_j^N(\tau)^2+b_j^N(\tau)^2\right)\,\mathrm{d}\tau\\&\quad=\sum_{j=0}^{N}\left(\left(a_j^{0,N}\right)^2+\left(b_j^{0,N}\right)^2\right)+2\sum_{j=0}^{N}\int_0^t f_j(\tau)a_j^N(\tau)\,\mathrm{d}\tau.\end{aligned}\tag{3.10}$$

Applying the Cauchy–Schwarz inequality, we have

$$2\sum_{j=0}^{N}\int_0^t f_j(\tau)a_j^N(\tau)\,\mathrm{d}\tau \le \sum_{j=0}^{N}\int_0^t \lambda_j^{-2}f_j^2(\tau)\,\mathrm{d}\tau+\sum_{j=0}^{N}\int_0^t \lambda_j^2 a_j^N(\tau)^2\,\mathrm{d}\tau.$$

Hence, it follows from (3.10)

$$\begin{aligned}&\sum_{j=0}^{N}\left(a_j^N(t)^2+b_j^N(t)^2\right)+\sum_{j=0}^{N}\int_0^t \lambda_j^2\left(a_j^N(\tau)^2+b_j^N(\tau)^2\right)\,\mathrm{d}\tau\\&\quad\le\sum_{j=0}^{N}\left(\left(a_j^{0,N}\right)^2+\left(b_j^{0,N}\right)^2\right)+\sum_{j=0}^{N}\int_0^t \lambda_j^{-2}f_j^2(\tau)\,\mathrm{d}\tau.\end{aligned}\tag{3.11}$$

We can iterate the process above to construct the solution on time intervals $[t_{N,1}, t_{N,2}]$, $[t_{N,2}, t_{N,3}]$,..., $[t_{N,k}, t_{N,k+1}]$,..., and finally reach the time T. Indeed, repeating the contraction argument above starting from $t_{N,1}$, we obtain a solution on a time interval $[t_{N,1}, t_{N,2}]$ with $t_{N,2}$ satisfying

$$t_{N,2}-t_{N,1}=\frac{1}{2C_N\left(4|a^N(t_{N,1})|+4|b^N(t_{N,1})|+4\int_0^T\left|f^N(t)\right|\,\mathrm{d}t+1\right)}.\tag{3.12}$$

Note that (3.12) is obtained in analogy with (3.8) and (3.9). Iteratively, the time $t_{N,k+1}$ for any $k \ge 1$ is given by

$$t_{N,k+1}-t_{N,k}=\frac{1}{2C_N\left(4|a^N(t_{N,k})|+4|b^N(t_{N,k})|+4\int_0^T\left|f^N(t)\right|\,\mathrm{d}t+1\right)}.\tag{3.13}$$

On the other hand, we observe the energy inequality (3.11) holds on $[t_{N,k-1}, t_{N,k}]$ for any $k \geq 1$, i.e. for $t \in [t_{N,k-1}, t_{N,k}]$

$$\begin{aligned}&\sum_{j=0}^{N}\left(a_j^N(t_{N,k})^2 + b_j^N(t_{N,k})^2\right) + \sum_{j=0}^{N}\int_{t_{N,k-1}}^{t}\lambda_j^2\left(a_j^N(\tau)^2 + b_j^N(\tau)^2\right)\mathrm{d}\tau \\ &\quad\leq \sum_{j=0}^{N}\left(\left(a_j^N(t_{N,k-1})\right)^2 + \left(b_j^N(t_{N,k-1})\right)^2\right) + \sum_{j=0}^{N}\int_{t_{N,k-1}}^{t}\lambda_j^{-2}f_j^2(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{3.14}$$

Taking the sum of (3.14) over $k \geq 1$ yields

$$\begin{aligned}&\sum_{j=0}^{N}\left(a_j^N(t_{N,k})^2 + b_j^N(t_{N,k})^2\right) \\ &\quad\leq \sum_{j=0}^{N}\left(\left(a_j^{0,N}\right)^2 + \left(b_j^{0,N}\right)^2\right) + k\sum_{j=0}^{N}\int_0^T\lambda_j^{-2}f_j^2(\tau)\,\mathrm{d}\tau\end{aligned}$$

which implies

$$\left|a^N(t_{N,k})\right| + \left|b^N(t_{N,k})\right| \leq |a^{0,N}| + |b^{0,N}| + \sqrt{k}\left(\sum_{j=0}^{N}\int_0^T\lambda_j^{-2}f_j^2(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{2}}. \tag{3.15}$$

Combining (3.13) and (3.15) gives

$$\begin{aligned}&t_{N,k+1} - t_{N,k} \\ &\geq \left[C_N\left(8|a^{0,N}| + 8|b^{0,N}| + 8\int_0^T|f^N(t)|\,\mathrm{d}t + 2 + 8\sqrt{k}\left(\sum_{j=0}^{N}\int_0^T\lambda_j^{-2}f_j^2(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{2}}\right)\right]^{-1} \\ &\gtrsim \frac{1}{\sqrt{k}}.\end{aligned}$$

Therefore, the sum $\sum_k(t_{N,k+1} - t_{N,k})$ diverges and will reach T after a certain number of iterations. In conclusion, we obtain a solution $(a^N(t), b^N(t))$ of (3.1) on the interval $[0, T]$, which satisfies the energy inequality (3.11) for all $t \in [0, T]$.

The next step is to extract a limit from the sequence $\{(a^N(t), b^N(t))\}_{N\geq 1}$. In view of the energy inequality (3.11), we know $a^N, b^N \in L^\infty(0, T; l^2) \cap L^2(0, T; H^1)$ for any $N \geq 1$. Moreover, by a standard analysis (see Cheskidov 2008), we can show that $\{(a^N(t), b^N(t))\}_{N\geq 1}$ is equicontinuous on $C([0, T]; l_w^2) \times C([0, T]; l_w^2)$ where l_w^2 denotes the space l^2 equipped with a certain weak topology. As a consequence, there exists a subsequence $\{(a^{N_k}(t), b^{N_k}(t))\}_{k\geq 1}$ which converges to $(a(t), b(t))$ in $C[0, T]$ such that (by employing a diagonal process)

$$a_j^{N_k} \to a_j, \ \ b_j^{N_k} \to b_j, \ \text{ in } \ C(0,T) \ \text{ as } \ k \to \infty, \ \ \forall \ j \geq 0.$$

The last step is to show that the limit $(a(t), b(t))$ is a Leray–Hopf solution of (1.2a)–(1.2b). Replacing N by N_k in (3.2) and taking the limit $k \to \infty$, we see that $(a(t), b(t))$ satisfies the integral system

$$\begin{aligned} a_j(t) = \ & a_j^0 + \int_0^t \Big(-\lambda_j^2 a_j(\tau) - \lambda_j^\theta a_j(\tau) a_{j+1}(\tau) + \lambda_{j-1}^\theta (a_{j-1}(\tau))^2 \\ & -\lambda_j^\theta b_j(\tau) b_{j+1}(\tau) + \lambda_{j-1}^\theta (b_{j-1}(\tau))^2 + f_j(\tau) \Big) \, \mathrm{d}\tau, \\ b_j(t) = \ & b_j^0 + \int_0^t \Big(-\lambda_j^2 b_j(\tau) + \lambda_j^\theta a_j(\tau) b_{j+1}(\tau) - \lambda_j^\theta b_j(\tau) a_{j+1}(\tau) \Big) \, \mathrm{d}\tau, \end{aligned}$$

for all $j \geq 0$. Hence, $(a(t), b(t))$ satisfies system (1.2a)–(1.2b). Moreover, $a_j, b_j \in C^1[0,T]$ for all $j \geq 0$. In addition, taking the limit in the energy inequality (3.11) yields

$$a, b \in L^\infty(0,T; l^2) \cap L^2(0,T; H^1).$$

Notice that $a^{N_k} \in L^\infty(0,T;l^2) \cap L^2(0,T;H^1)$ for all k and N_k. Thus, the sequence $\{a_j^{N_k}\}_{k \geq 1}$ converges weakly in $L^2(0,T)$ for any fixed $j \geq 0$, and the limit coincides with a_j. Consequently, we have

$$\sum_{j=0}^\infty \int_0^t f_j(\tau) a_j^{N_k}(\tau) \, \mathrm{d}\tau \to \sum_{j=0}^\infty \int_0^t f_j(\tau) a_j(\tau) \, \mathrm{d}\tau, \ \ \text{as} \ \ k \to \infty, \ \ \forall \ \ t \in [0,T]. \tag{3.16}$$

Passing the limit in (3.10) and applying (3.16), it leads to the energy inequality satisfied by the limit $(a(t), b(t))$

$$\begin{aligned} & \sum_{j=0}^\infty \left(a_j(t)^2 + b_j(t)^2 \right) + 2 \sum_{j=0}^\infty \int_0^t \lambda_j^2 \left(a_j(\tau)^2 + b_j(\tau)^2 \right) \mathrm{d}\tau \\ & \quad \leq \sum_{j=0}^\infty \left(\left(a_j^0\right)^2 + \left(b_j^0\right)^2 \right) + 2 \sum_{j=0}^\infty \int_0^t f_j(\tau) a_j(\tau) \, \mathrm{d}\tau. \end{aligned}$$

It completes the proof of Theorem 1.1.

4 Weak–Strong Uniqueness

In order to show the weak–strong uniqueness, a standard argument involving Grönwall's inequality will be applied to the difference $(a(t) - u(t), b(t) - v(t))$ of the two solutions $(a(t), b(t))$ and $(u(t), v(t))$.

Proof of Theorem 1.2 As in the previous section, we set $\kappa_1 = -\kappa_2 = 1$. We start with the energy balance through the j-th shell

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big((a_j-u_j)^2+(b_j-v_j)^2\Big)\\
&\quad=\Big((2a_ja_j'+2b_jb_j')+(2u_ju_j'+2v_jv_j')\Big)\\
&\qquad-\Big((2a_ju_j'+2a_j'u_j)+(2b_jv_j'+2b_j'v_j)\Big)
\end{aligned}\tag{4.1}$$

and continue to estimate the four groups on the right-hand side. In view of equations (1.2a)–(1.2b) satisfied by (a_j, b_j) and (u_j, v_j), respectively, we have

$$a_ja_j'+b_jb_j' = -\lambda_j^2(a_j^2+b_j^2)-\lambda_j^\theta(a_j^2+b_j^2)a_{j+1}+\lambda_{j-1}^\theta(a_{j-1}^2+b_{j-1}^2)a_j+f_ja_j, \tag{4.2}$$

$$u_ju_j'+v_jv_j' = -\lambda_j^2(u_j^2+v_j^2)-\lambda_j^\theta(u_j^2+v_j^2)u_{j+1}+\lambda_{j-1}^\theta(u_{j-1}^2+v_{j-1}^2)u_j+f_ju_j, \tag{4.3}$$

$$\begin{aligned}
(a_ju_j)' &= -2\lambda_j^2a_ju_j-\lambda_j^\theta a_ju_ja_{j+1}-\lambda_j^\theta a_ju_ju_{j+1}\\
&\quad-\lambda_j^\theta b_ju_jb_{j+1}-\lambda_j^\theta a_jv_jv_{j+1}+\lambda_{j-1}^\theta a_{j-1}^2u_j+\lambda_{j-1}^\theta b_{j-1}^2u_j\\
&\quad+\lambda_{j-1}^\theta u_{j-1}^2a_j+\lambda_{j-1}^\theta v_{j-1}^2a_j+f_j(a_j+u_j),
\end{aligned}\tag{4.4}$$

$$(b_jv_j)' = -2\lambda_j^2b_jv_j+\lambda_j^\theta a_jv_jb_{j+1}+\lambda_j^\theta b_ju_jv_{j+1}-\lambda_j^\theta b_jv_ja_{j+1}-\lambda_j^\theta b_jv_ju_{j+1}. \tag{4.5}$$

Combining (4.1)–(4.5) and grouping the terms appropriately gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big((a_j-u_j)^2+(b_j-v_j)^2\Big)+2\lambda_j^2(a_j-u_j)^2+2\lambda_j^2(b_j-v_j)^2\\
&\quad=-\lambda_j^\theta(a_j^2+b_j^2)a_{j+1}+\lambda_{j-1}^\theta(a_{j-1}^2+b_{j-1}^2)a_j\\
&\qquad-\lambda_j^\theta(u_j^2+v_j^2)u_{j+1}+\lambda_{j-1}^\theta(u_{j-1}^2+v_{j-1}^2)u_j\\
&\qquad+\Big(-\lambda_j^\theta a_ju_ja_{j+1}-\lambda_j^\theta a_ju_ju_{j+1}+\lambda_{j-1}^\theta a_{j-1}^2u_j+\lambda_{j-1}^\theta u_{j-1}^2a_j\Big)\\
&\qquad+\Big(-\lambda_j^\theta b_ju_jb_{j+1}-\lambda_j^\theta a_jv_jv_{j+1}+\lambda_j^\theta a_jv_jb_{j+1}+\lambda_j^\theta b_ju_jv_{j+1}\Big)\\
&\qquad+\Big(\lambda_{j-1}^\theta b_{j-1}^2u_j+\lambda_{j-1}^\theta v_{j-1}^2a_j-\lambda_j^\theta b_jv_ja_{j+1}-\lambda_j^\theta b_jv_ju_{j+1}\Big).
\end{aligned}\tag{4.6}$$

We further rearrange the terms in the last three parentheses of (4.6) to create terms in differences, for instance, $a_j - u_j$ and $b_j - v_j$. Shifting the sub-index j to $j+1$ in the last two terms of

$$\Big(-\lambda_j^\theta a_ju_ja_{j+1}-\lambda_j^\theta a_ju_ju_{j+1}+\lambda_{j-1}^\theta a_{j-1}^2u_j+\lambda_{j-1}^\theta u_{j-1}^2a_j\Big),$$

we have

$$\begin{aligned}
&-\lambda_j^\theta a_ju_ja_{j+1}-\lambda_j^\theta a_ju_ju_{j+1}+\lambda_{j-1}^\theta a_{j-1}^2u_j+\lambda_{j-1}^\theta u_{j-1}^2a_j\\
&\quad=-\lambda_j^\theta a_ju_ja_{j+1}-\lambda_j^\theta a_ju_ju_{j+1}+\lambda_j^\theta a_j^2u_{j+1}+\lambda_j^\theta u_j^2a_{j+1}\\
&\quad=\ \lambda_j^\theta\left(a_ju_{j+1}(a_j-u_j)-u_ja_{j+1}(a_j-u_j)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda_j^\theta (a_j - u_j)\left(a_j u_{j+1} - a_j a_{j+1} + a_j a_{j+1} - u_j a_{j+1}\right)\\
&= -\lambda_j^\theta a_j (a_j - u_j)\left(a_{j+1} - u_{j+1}\right) + \lambda_j^\theta \left(a_j - u_j\right)^2 a_{j+1}.
\end{aligned} \tag{4.7}
$$

Similarly, with a shift of sub-index in the first two terms of

$$
\left(\lambda_{j-1}^\theta b_{j-1}^2 u_j + \lambda_{j-1}^\theta v_{j-1}^2 a_j - \lambda_j^\theta b_j v_j a_{j+1} - \lambda_j^\theta b_j v_j u_{j+1}\right),
$$

we obtain

$$
\begin{aligned}
&\lambda_j^\theta b_j^2 u_{j+1} + \lambda_j^\theta v_j^2 a_{j+1} - \lambda_j^\theta b_j v_j a_{j+1} - \lambda_j^\theta b_j v_j u_{j+1}\\
&\quad = \lambda_j^\theta b_j u_{j+1}(b_j - v_j) - \lambda_j^\theta v_j a_{j+1}(b_j - v_j)\\
&\quad = \lambda_j^\theta (b_j - v_j)(b_j u_{j+1} - b_j a_{j+1} + b_j a_{j+1} - v_j a_{j+1})\\
&\quad = -\lambda_j^\theta b_j (b_j - v_j)(a_{j+1} - u_{j+1}) + \lambda_j^\theta (b_j - v_j)^2 a_{j+1}.
\end{aligned} \tag{4.8}
$$

We rearrange the terms of

$$
\left(-\lambda_j^\theta b_j u_j b_{j+1} - \lambda_j^\theta a_j v_j v_{j+1} + \lambda_j^\theta a_j v_j b_{j+1} + \lambda_j^\theta b_j u_j v_{j+1}\right)
$$

as

$$
\begin{aligned}
&-\lambda_j^\theta b_j u_j b_{j+1} - \lambda_j^\theta a_j v_j v_{j+1} + \lambda_j^\theta a_j v_j b_{j+1} + \lambda_j^\theta b_j u_j v_{j+1}\\
&\quad = \lambda_j^\theta a_j v_j (b_{j+1} - v_{j+1}) - \lambda_j^\theta b_j u_j (b_{j+1} - v_{j+1})\\
&\quad = \lambda_j^\theta (a_j v_j - a_j b_j + a_j b_j - b_j u_j)(b_{j+1} - v_{j+1})\\
&\quad = -\lambda_j^\theta a_j (b_j - v_j)(b_{j+1} - v_{j+1}) + \lambda_j^\theta b_j (a_j - u_j)(b_{j+1} - v_{j+1}).
\end{aligned} \tag{4.9}
$$

Since $(a(t), b(t))$ and $(u(t), v(t))$ are Leray–Hopf solutions, we have that the following two series with telescope sums vanish,

$$
\begin{aligned}
&\sum_{j=0}^\infty \int_0^t \left(-\lambda_j^\theta (a_j^2 + b_j^2) a_{j+1} + \lambda_{j-1}^\theta (a_{j-1}^2 + b_{j-1}^2) a_j\right) \mathrm{d}\tau = 0,\\
&\sum_{j=0}^\infty \int_0^t \left(-\lambda_j^\theta (u_j^2 + v_j^2) u_{j+1} + \lambda_{j-1}^\theta (u_{j-1}^2 + v_{j-1}^2) u_j\right) \mathrm{d}\tau = 0.
\end{aligned} \tag{4.10}
$$

Integrating (4.6) over $[0, t]$, taking the sum for $j \geq 0$, using the fact $a_{-1} = b_{-1} = u_{-1} = v_{-1} = 0$, shifting the sub-index in the terms with sub-index $j-1$, and applying

(4.7)–(4.10), we deduce

$$
\begin{aligned}
&\sum_{j=0}^{\infty}\Big((a_j(t)-u_j(t))^2+(b_j(t)-v_j(t))^2\Big)\\
&\quad+2\sum_{j=0}^{\infty}\lambda_j^2\int_0^t (a_j(\tau)-u_j(\tau))^2+(b_j(\tau)-v_j(\tau))^2\mathrm{d}\tau\\
&=-\sum_{j=0}^{\infty}\int_0^t \lambda_j^{\theta}a_j(\tau)(a_j(\tau)-u_j(\tau))\left(a_{j+1}(\tau)-u_{j+1}(\tau)\right)\mathrm{d}\tau\\
&\quad-\sum_{j=0}^{\infty}\int_0^t \lambda_j^{\theta}b_j(\tau)(b_j(\tau)-v_j(\tau))\left(a_{j+1}(\tau)-u_{j+1}(\tau)\right)\mathrm{d}\tau\\
&\quad+\sum_{j=0}^{\infty}\int_0^t \lambda_j^{\theta}(a_j(\tau)-u_j(\tau))^2a_{j+1}(\tau)\mathrm{d}\tau\\
&\quad+\sum_{j=0}^{\infty}\int_0^t \lambda_j^{\theta}(b_j(\tau)-v_j(\tau))^2a_{j+1}(\tau)\mathrm{d}\tau\\
&\quad-\sum_{j=0}^{\infty}\int_0^t \lambda_j^{\theta}a_j(\tau)(b_j(\tau)-v_j(\tau))\left(b_{j+1}(\tau)-v_{j+1}(\tau)\right)\mathrm{d}\tau\\
&\quad+\sum_{j=0}^{\infty}\int_0^t \lambda_j^{\theta}b_j(\tau)(a_j(\tau)-u_j(\tau))\left(b_{j+1}(\tau)-v_{j+1}(\tau)\right)\mathrm{d}\tau.
\end{aligned}
\tag{4.11}
$$

We claim that the series on the right-hand side of (4.11) are well defined. Indeed, since $(a(t), b(t))$ and $(u(t), v(t))$ are Leray–Hopf solutions, it is clear that

$$
\sum_{j=0}^{\infty}\lambda_j^2\int_0^t a_j^2(\tau)+b_j^2(\tau)\mathrm{d}\tau<\infty,\quad \sum_{j=0}^{\infty}\lambda_j^2\int_0^t u_j^2(\tau)+v_j^2(\tau)\mathrm{d}\tau<\infty.
$$

As a consequence, applying the assumption (1.3), we infer

$$
\begin{aligned}
&\sum_{j=J}^{\infty}\int_0^t \left|\lambda_j^{\theta}a_j(\tau)(a_j(\tau)-u_j(\tau))\left(a_{j+1}(\tau)-u_{j+1}(\tau)\right)\right|\mathrm{d}\tau\\
&\quad\le C\sum_{j=J}^{\infty}\int_0^t \left|\lambda_j^{2}(a_j(\tau)-u_j(\tau))\left(a_{j+1}(\tau)-u_{j+1}(\tau)\right)\right|\mathrm{d}\tau\\
&\quad\le 4C\sum_{j=J}^{\infty}\lambda_j^2\int_0^t a_j^2(\tau)+b_j^2(\tau)+u_j^2(\tau)+v_j^2(\tau)\mathrm{d}\tau\\
&\quad<\infty
\end{aligned}
$$

for a constant C. Other series can be shown to converge analogously. Next, we estimate these series starting from the J-th shell. We only need to show details for one of them, for instance, thanks to the assumption (1.3)

$$\begin{aligned}
&\sum_{j=J}^{\infty}\int_0^t \left|\lambda_j^{\theta} b_j(\tau)(b_j(\tau)-v_j(\tau))\left(a_{j+1}(\tau)-u_{j+1}(\tau)\right)\right| \mathrm{d}\tau \\
&\quad\le C_0\sum_{j=J}^{\infty}\int_0^t \left|\lambda_j^{2}(b_j(\tau)-v_j(\tau))\left(a_{j+1}(\tau)-u_{j+1}(\tau)\right)\right| \mathrm{d}\tau \\
&\quad\le \frac{C_0}{2}\sum_{j=J}^{\infty}\int_0^t \lambda_j^{2}(b_j(\tau)-v_j(\tau))^2 \mathrm{d}\tau + \frac{C_0}{2\lambda^2}\sum_{j=J}^{\infty}\int_0^t \lambda_{j+1}^{2}(a_{j+1}(\tau)-u_{j+1}(\tau))^2 \mathrm{d}\tau.
\end{aligned}$$

Similarly, the other series have the estimates

$$\begin{aligned}
&\sum_{j=J}^{\infty}\int_0^t \left|\lambda_j^{\theta} a_j(\tau)(a_j(\tau)-u_j(\tau))\left(a_{j+1}(\tau)-u_{j+1}(\tau)\right)\right| \mathrm{d}\tau \\
&\quad\le \frac{C_0}{2}\sum_{j=J}^{\infty}\int_0^t \lambda_j^{2}(a_j(\tau)-u_j(\tau))^2 \mathrm{d}\tau + \frac{C_0}{2\lambda^2}\sum_{j=J}^{\infty}\int_0^t \lambda_{j+1}^{2}(a_{j+1}(\tau)-u_{j+1}(\tau))^2 \mathrm{d}\tau, \\
&\sum_{j=J}^{\infty}\int_0^t \left|\lambda_j^{\theta} a_j(\tau)(b_j(\tau)-v_j(\tau))\left(b_{j+1}(\tau)-v_{j+1}(\tau)\right)\right| \mathrm{d}\tau \\
&\quad\le \frac{C_0}{2}\sum_{j=J}^{\infty}\int_0^t \lambda_j^{2}(b_j(\tau)-v_j(\tau))^2 \mathrm{d}\tau + \frac{C_0}{2\lambda^2}\sum_{j=J}^{\infty}\int_0^t \lambda_{j+1}^{2}(b_{j+1}(\tau)-v_{j+1}(\tau))^2 \mathrm{d}\tau, \\
&\sum_{j=J}^{\infty}\int_0^t \left|\lambda_j^{\theta} b_j(\tau)(a_j(\tau)-u_j(\tau))\left(b_{j+1}(\tau)-v_{j+1}(\tau)\right)\right| \mathrm{d}\tau \\
&\quad\le \frac{C_0}{2}\sum_{j=J}^{\infty}\int_0^t \lambda_j^{2}(a_j(\tau)-u_j(\tau))^2 \mathrm{d}\tau + \frac{C_0}{2\lambda^2}\sum_{j=J}^{\infty}\int_0^t \lambda_{j+1}^{2}(b_{j+1}(\tau)-v_{j+1}(\tau))^2 \mathrm{d}\tau, \\
&\sum_{j=J}^{\infty}\int_0^t \left|\lambda_j^{\theta}(a_j(\tau)-u_j(\tau))^2 a_{j+1}(\tau)\right| \mathrm{d}\tau \le C_0\lambda^{2-\theta}\sum_{j=J}^{\infty}\int_0^t \lambda_j^{2}(a_j(\tau)-u_j(\tau))^2 \mathrm{d}\tau, \\
&\sum_{j=J}^{\infty}\int_0^t \left|\lambda_j^{\theta}(b_j(\tau)-v_j(\tau))^2 a_{j+1}(\tau)\right| \mathrm{d}\tau \le C_0\lambda^{2-\theta}\sum_{j=J}^{\infty}\int_0^t \lambda_j^{2}(b_j(\tau)-v_j(\tau))^2 \mathrm{d}\tau.
\end{aligned}$$

Combining the estimates above and (4.11), we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty}\left((a_j(t)-u_j(t))^2+(b_j(t)-v_j(t))^2\right) \\
&+2\sum_{j=0}^{\infty}\lambda_j^2\int_0^t (a_j(\tau)-u_j(\tau))^2+(b_j(\tau)-v_j(\tau))^2 \mathrm{d}\tau \\
&\le C_1\sum_{j=0}^{J}\int_0^t (a_j(\tau)-u_j(\tau))^2+(b_j(\tau)-v_j(\tau))^2 \mathrm{d}\tau
\end{aligned}$$

$$+C_0\left(1+\lambda^{2-\theta}+\lambda^{-2}\right)\sum_{j=J}^{\infty}\lambda_j^2\int_0^t (a_j(\tau)-u_j(\tau))^2+(b_j(\tau)-v_j(\tau))^2 \mathrm{d}\tau \tag{4.12}$$

where the constant C_1 is given by

$$C_1 = 32\lambda_J^{\theta}\sup_{0\le j\le J+1}\left(\|a_j\|_C+\|b_j\|_C\right)\le 32\lambda_J^{\theta}\left(\|a^0\|_{l^2}+\|b^0\|_{l^2}\right).$$

We take C_0 such that $C_0\left(1+\lambda^{2-\theta}+\lambda^{-2}\right)\le 2$. Hence, it follows from (4.12) that

$$\sum_{j=0}^{\infty}\left((a_j(t)-u_j(t))^2+(b_j(t)-v_j(t))^2\right)$$
$$\le C_1\sum_{j=0}^{\infty}\int_0^t (a_j(\tau)-u_j(\tau))^2+(b_j(\tau)-v_j(\tau))^2\mathrm{d}\tau.$$

Therefore, Grönwall's inequality implies that

$$a_j\equiv u_j,\quad b_j\equiv v_j,\quad \forall\ j\ge 0.$$

□

Proof of Theorem 1.3 Since $1\le\theta\le 2$, for any Leray–Hopf solution $(a(t),b(t))$, there exists $J>0$ such that

$$|a_j(t)|\le C_0\lambda_j^{2-\theta},\quad |b_j(t)|\le C_0\lambda_j^{2-\theta},\quad \forall\ j\ge J.$$

That is, assumption (1.3) is satisfied and hence uniqueness follows. □

5 Non-uniqueness of Leray–Hopf Solutions for $\theta\in(2,\frac{5}{2}]$

We prove Theorem 1.4 in this section. We adapt the construction scheme for the dyadic NSE in Filonov and Khodunov (2021) in order to construct a solution $(a(t),b(t))$ of (1.2a)–(1.2b) with zero initial data such that both $a(t)$ and $b(t)$ are non-vanishing. We first present the proof for the special case $\kappa_1=-\kappa_2=1$ and then point out modifications to prove other cases when changing the signs of κ_1 and κ_2.

Fix $T=\frac{1}{\lambda^2-1}$. Define

$$t_j=\lambda_j^{-2}T,\quad j\ge 0.$$

We note

$$t_{j-1}-t_j=\lambda_j^{-2},\quad j\ge 1,$$

$$(0, T) = \cup_{j=1}^{\infty}[t_j, t_{j-1}).$$

For $p, q \in C_c^{\infty}(0, 1)$ (class of smooth functions compactly supported on [0, 1]) and constant $\rho > \lambda^{\theta}$, we construct a_j and b_j as follows:

$$a_j(t) = \begin{cases} 0, & t < t_{j+1}, \\ \lambda_{j+1}^{2-\theta} p\left(\lambda_{j+1}^2 (t - t_{j+1})\right), & t_{j+1} < t < t_j, \\ -\lambda_j^{2-\theta} q\left(\lambda_j^2 (t - t_j)\right), & t_j < t < t_{j-1}, \\ 0, & t > t_{j-1}. \end{cases} \tag{5.1}$$

$$b_j(t) = \begin{cases} 0, & t < t_{j+1}, \\ \rho^{-j-1} h_1\left(\lambda_{j+1}^2 (t - t_{j+1})\right), & t_{j+1} < t < t_j, \\ \rho^{-j} h_2\left(\lambda_j^2 (t - t_j)\right), & t_j < t < t_{j-1}, \\ \rho^{-j+1} h_3\left(\lambda_{j-1}^2 (t - t_{j-1})\right), & t_{j-1} < t < t_{j-2}, \\ \rho^{-j+1} h_3(1) e^{-\lambda_j^2 (t - t_{j-2})}, & t > t_{j-2}, \end{cases} \tag{5.2}$$

such that h_1, h_2 and h_3 satisfy the ODE system on [0, 1]

$$\frac{\mathrm{d}}{\mathrm{d}t} h_1 + \left(\lambda^{-2} - \lambda^{-\theta} q\right) h_1 - \lambda^{-\theta} p h_2 = 0, \tag{5.3a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} h_2 + h_2 + q h_3 = 0, \tag{5.3b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} h_3 + \lambda^2 h_3 = 0, \tag{5.3c}$$

$$h_1(0) = 0, \quad h_2(0) = c_0, \quad h_3(0) = d_0. \tag{5.3d}$$

In addition, we assume

$$h_1(1) = \rho c_0, \quad h_2(1) = \rho d_0. \tag{5.4}$$

With (a_j, b_j) constructed in (5.1)–(5.2), we define the forcing by

$$f_j = \frac{\mathrm{d}}{\mathrm{d}t} a_j + \lambda_j^2 a_j + \lambda_j^{\theta} a_j a_{j+1} + \lambda_j^{\theta} b_j b_{j+1} - \lambda_{j-1}^{\theta} a_{j-1}^2 - \lambda_{j-1}^{\theta} b_{j-1}^2 \tag{5.5}$$

for all $j \geq 0$.

Lemma 5.1 *Let a_j and b_j be constructed as in (5.1)–(5.2). Then, the following properties hold:*

(i) $a_j \in C_c^{\infty}(0, T)$ *for all* $j \geq 0$;
(ii) b_j *are piecewise smooth and* $b_j \in H^1(0, T)$ *for all* $j \geq 0$;
(iii)

$$a_j(0) = b_j(0) = 0, \quad \forall \ j \geq 0;$$

(iv)

$$a_j(t) = O(\lambda_j^{2-\theta}), \;\; a_j'(t) = O(\lambda_j^{4-\theta}), \;\; b_j(t) = O(\rho^{-j}), \quad j \to \infty.$$

Proof Since $p, q \in C_c^\infty(0, 1)$, we only need to verify the values of the functions at t_{j+1}, t_j, t_{j-1} and t_{j-2}. The functions b_j are piecewise smooth and continuous at these times; hence, $b_j \in H^1(0, T)$. It is obvious to see (iii) and (iv) from (5.1)–(5.2). □

Lemma 5.2 *The functions a_j and b_j defined in (5.1)–(5.2) satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}t} b_j + \lambda_j^2 b_j - \lambda_j^\theta a_j b_{j+1} + \lambda_j^\theta b_j a_{j+1} = 0$$

for all $j \geq 0$.

Proof Denote

$$A_j(t) = \frac{\mathrm{d}}{\mathrm{d}t} b_j(t) + \lambda_j^2 b_j(t) - \lambda_j^\theta a_j(t) b_{j+1}(t) + \lambda_j^\theta b_j(t) a_{j+1}(t).$$

For $t < t_{j+1}$, we see $a_j(t) = b_j(t) = 0$ from (5.1) and (5.2), and hence $A_j(t) = 0$.

For $t_{j+1} < t < t_j$, we denote $\tau = \lambda_{j+1}^2 (t - t_{j+1}) \in (0, 1)$. It follows from (5.1)–(5.2) that

$$\begin{aligned} A_j(t) = &\ \rho^{-j-1} \lambda_{j+1}^2 \frac{\mathrm{d}}{\mathrm{d}\tau} h_1(\tau) + \rho^{-j-1} \lambda_j^2 h_1(\tau) \\ &- \rho^{-j-1} \lambda_{j+1}^2 \lambda^{-\theta} p(\tau) h_2(\tau) - \rho^{-j-1} \lambda_{j+1}^2 \lambda^{-\theta} q(\tau) h_1(\tau) \\ = &\ \rho^{-j-1} \lambda_{j+1}^2 \left(\frac{\mathrm{d}}{\mathrm{d}\tau} h_1(\tau) + \lambda^{-2} h_1(\tau) - \lambda^{-\theta} q(\tau) h_1(\tau) - \lambda^{-\theta} p(\tau) h_2(\tau) \right) \\ = &\ 0 \end{aligned}$$

thanks to (5.3a).

For $t_j < t < t_{j-1}$, we denote $\tau = \lambda_j^2 (t - t_j) \in (0, 1)$. We note $a_{j+1}(t) = 0$ by (5.1). Moreover, we have

$$\begin{aligned} A_j(t) = &\ \rho^{-j} \lambda_j^2 \frac{\mathrm{d}}{\mathrm{d}\tau} h_2(\tau) + \rho^{-j} \lambda_j^2 h_2(\tau) + \rho^{-j} \lambda_j^2 q(\tau) h_3(\tau) \\ = &\ \rho^{-j} \lambda_j^2 \left(\frac{\mathrm{d}}{\mathrm{d}\tau} h_2(\tau) + h_2(\tau) + q(\tau) h_3(\tau) \right) \\ = &\ 0 \end{aligned}$$

where we applied (5.3b).

For $t_{j-1} < t < t_{j-2}$, we denote $\tau = \lambda_{j-1}^2 (t - t_{j-1}) \in (0, 1)$. On this interval, we have $a_j(t) = a_{j+1}(t) = 0$, and by (5.2) and (5.3c)

$$A_j(t) = \rho^{-j+1} \lambda_{j-1}^2 \frac{\mathrm{d}}{\mathrm{d}\tau} h_3(\tau) + \rho^{-j+1} \lambda_j^2 h_3(\tau) = 0.$$

For $t > t_{j-2}$, we note $a_j(t) = a_{j+1}(t) = 0$, and

$$A_j(t) = -\rho^{-j+1}\lambda_j^2 h_3(1) e^{-\lambda_j^2(t-t_{j-2})} + \rho^{-j+1}\lambda_j^2 h_3(1) e^{-\lambda_j^2(t-t_{j-2})} = 0.$$

□

Lemma 5.3 *The forcing $f = \{f_j(t)\}_{j\ge 0}$ constructed in (5.5) satisfies*

$$\sum_{j=0}^{\infty} \lambda_j^{-2} \int_0^T f_j^2(t)\,\mathrm{d}t < \infty.$$

Proof It follows from (5.1)–(5.2), (5.5) and straightforward computations that

$$f_j(t) = \begin{cases} 0, & t < t_{j+1}, \\ O(\lambda_j^{4-\theta}), & t_{j+1} < t < t_{j-2}, \\ O(\lambda_j^{-\theta}), & t > t_{j-2}. \end{cases}$$

Hence,

$$\begin{aligned} \lambda_j^{-2} \int_0^T f_j^2(t)\,\mathrm{d}t &= \lambda_j^{-2} \int_{t_{j+1}}^{t_{j-2}} O(\lambda_j^{8-2\theta})\,\mathrm{d}t + \lambda_j^{-2} \int_{t_{j-2}}^T O(\lambda_j^{-2\theta})\,\mathrm{d}t \\ &= O(\lambda_j^{4-2\theta}) + O(\lambda_j^{-2-2\theta}). \end{aligned}$$

Since $4 - 2\theta < 0$ for $2 < \theta \le \frac{5}{2}$, it is clear that

$$\sum_{j=0}^{\infty} \lambda_j^{-2} \int_0^T f_j^2(t)\,\mathrm{d}t \le \sum_{j=0}^{\infty} \Big(O(\lambda_j^{4-2\theta}) + O(\lambda_j^{-2-2\theta}) \Big) < \infty.$$

□

Lemma 5.4 *There exist functions $p, q \in C_c^\infty(0,1)$ and constants c_0 and d_0 with $c_0^2 + d_0^2 \neq 0$ such that there exists a unique solution $h = (h_1, h_2, h_3)$ of system (5.3a)–(5.3d) satisfying (5.4) and $h \in C^\infty([0,1]; \mathbb{R}^3)$.*

Proof It is obvious from (5.3c) and the initial data $h_3(0) = d_0$ that

$$h_3(t) = d_0 e^{-\lambda^2 t}.$$

It then follows from (5.3b) and $h_2(0) = c_0$ that

$$h_2(t) = c_0 e^{-t} - \int_0^t e^{s-t} q(s) h_3(s)\,\mathrm{d}s = c_0 e^{-t} - d_0 e^{-t} \int_0^t e^{(1-\lambda^2)s} q(s)\,\mathrm{d}s.$$

Since $h_2(1) = \rho d_0$, we have the constraint

$$c_0 - d_0 \int_0^1 e^{(1-\lambda^2)s} q(s)\,\mathrm{d}s = e\rho d_0. \tag{5.6}$$

In the end, we solve (5.3a) with $h_1(0) = 0$ as:

$$h_1(t) = \int_0^t e^{-\int_s^t (\lambda^{-2} - \lambda^{-\theta} q(\tau))\,\mathrm{d}\tau} \lambda^{-\theta} p(s) h_2(s)\,\mathrm{d}s.$$

The assumption $h_1(1) = \rho c_0$ gives another constraint,

$$\int_0^1 e^{-\int_s^1 (\lambda^{-2} - \lambda^{-\theta} q(\tau))\,\mathrm{d}\tau} \lambda^{-\theta} p(s) h_2(s)\,\mathrm{d}s = \rho c_0. \tag{5.7}$$

We note that in the case of constant p and q, equations (5.6)–(5.7) have a unique solution (c_0, d_0). Thus, by a continuity argument, we know that there exist functions $p, q \in C_c^\infty(0, 1)$ such that (5.6) and (5.7) are satisfied for some constants c_0, d_0 with $c_0^2 + d_0^2 \neq 0$. Since $p, q \in C_c^\infty(0, 1)$, it is clear that $h_1, h_2, h_3 \in C^\infty(0, 1)$. □

Proof of Theorem 1.4 Let $a = (a_j)_{j\geq 0}$ and $b = (b_j)_{j\geq 0}$ be constructed as in (5.1)–(5.2). According to Lemma 5.2, we have shown (a, b) satisfies the model (1.2a)–(1.2b) with $\kappa_1 = -\kappa_2 = 1$ and with forcing f_j defined in (5.5). It is shown in Lemma 5.1 that $b \in H^1(0, T)$. We are left to show that $a \in l^2 \cap H^1$ and (a, b) satisfies the energy estimate.

Since $2 < \theta \leq \frac{5}{2}$,

$$a_j(t) = \begin{cases} 0, & t < t_{j+1}, \\ O(\lambda_j^{2-\theta}), & t_{j+1} < t < t_{j-1}, \\ 0, & t > t_{j-1}, \end{cases}$$

which implies

$$\sup_{t\in[0,T]} \sum_{j\geq 0}^{\infty} a_j^2(t) < \infty.$$

Hence, we have $a \in l^2$.

Notice

$$t_{j-1} - t_j = \lambda_j^{-2}, \quad t_j - t_{j+1} = \lambda_{j+1}^{-2}, \quad T - t_{j-1} = \frac{1}{\lambda^2 - 1}\left(1 - \frac{1}{\lambda_{j-1}^2}\right) < \frac{1}{\lambda^2 - 1}.$$

As a consequence, we have

$$\int_0^T a_j^2(t)\,\mathrm{d}t = \int_{t_{j+1}}^{t_{j-1}} a_j^2(t)\,\mathrm{d}t = \int_{t_{j+1}}^{t_{j-1}} O(\lambda_j^{4-2\theta})\,\mathrm{d}t = O(\lambda_j^{2-2\theta}).$$

Hence,

$$\sum_{j=0}^{\infty}\int_0^T \lambda_j^2 a_j^2(t)\,\mathrm{d}t = \sum_{j=0}^{\infty} O(\lambda_j^{4-2\theta}) < \infty$$

provided $\theta \in (2, \frac{5}{2}]$. That is, $a \in H^1(0,T)$.

Next, we show that (a, b) satisfies the energy identity. Since $(a(t), b(t))$ is a solution of (1.2a)–(1.2b), it follows

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(a_j^2(t)+b_j^2(t)\right) &= -\lambda_j^2 a_j^2 - \lambda_j^2 b_j^2 - \lambda_j^\theta a_j^2 a_{j+1} + \lambda_{j-1}^\theta a_{j-1}^2 a_j \\ &\quad - \lambda_j^\theta b_j^2 a_{j+1} + \lambda_{j-1}^\theta b_{j-1}^2 a_j + f_j a_j;\end{aligned}$$

and hence

$$\begin{aligned}&\frac{1}{2}\left(a_j^2(t)+b_j^2(t)\right) - \frac{1}{2}\left(a_j^2(0)+b_j^2(0)\right) \\ &= -\int_0^t \lambda_j^2 a_j^2\,\mathrm{d}t - \int_0^t \lambda_j^2 b_j^2\,\mathrm{d}t - \int_0^t \lambda_j^\theta a_j^2 a_{j+1}\,\mathrm{d}t + \int_0^t \lambda_{j-1}^\theta a_{j-1}^2 a_j\,\mathrm{d}t \\ &\quad - \int_0^t \lambda_j^\theta b_j^2 a_{j+1}\,\mathrm{d}t + \int_0^t \lambda_{j-1}^\theta b_{j-1}^2 a_j\,\mathrm{d}t + \int_0^t f_j a_j\,\mathrm{d}t.\end{aligned} \tag{5.8}$$

Again, notice

$$a_j(t) = \begin{cases} 0, & t < t_{j+1}, \\ O(\lambda_j^{2-\theta}), & t_{j+1} < t < t_{j-1}, \\ 0, & t > t_{j-1}, \end{cases}$$

and

$$a_{j+1}(t) = \begin{cases} 0, & t < t_{j+2}, \\ O(\lambda_{j+1}^{2-\theta}), & t_{j+2} < t < t_j, \\ 0, & t > t_j. \end{cases}$$

Thus, we have

$$\int_0^t \lambda_j^\theta \left|a_j^2 a_{j+1}\right|\,\mathrm{d}t = \int_{t_{j+2}}^{t_{j-1}} \lambda_j^\theta O(\lambda_j^{6-3\theta})\,\mathrm{d}t = O(\lambda_j^{\theta-2+6-3\theta}) < \infty$$

since $\theta \in (2, \frac{5}{2}]$. Obviously, we also have

$$\int_0^t \lambda_{j-1}^\theta \left|a_{j-1}^2 a_j\right|\,\mathrm{d}t < \infty.$$

On the other hand, we note

$$b_j(t) = \begin{cases} 0, & t < t_{j+1}, \\ O(\rho^{-j-1}), & t_{j+1} < t < t_j, \\ O(\rho^{-j}), & t_j < t < t_{j-1}, \\ O(\rho^{-j+1}), & t > t_{j-1}. \end{cases}$$

Thus, we have

$$\begin{aligned} \int_0^t \lambda_j^\theta \left| b_j^2 a_{j+1} \right| \, \mathrm{d}t &= \int_{t_{j+1}}^{t_j} \lambda_j^\theta O(\rho^{-2j-2} \lambda_{j+1}^{2-\theta}) \, \mathrm{d}t + \int_{t_j}^{t_{j-1}} \lambda_j^\theta O(\rho^{-2j} \lambda_{j+1}^{-\theta}) \, \mathrm{d}t \\ &\quad + \int_{t_{j-1}}^{T} \lambda_j^\theta O(\rho^{-2j+2} \lambda_{j+1}^{-\theta}) \, \mathrm{d}t \\ &= O(\rho^{-2j-2}) + O(\rho^{-2j} \lambda_j^{-2}) + O(\rho^{-2j+2}) \\ &< \infty. \end{aligned}$$

Similarly,

$$\int_0^t \lambda_{j-1}^\theta \left| b_{j-1}^2 a_j \right| \, \mathrm{d}t < \infty.$$

Therefore, we can take the sum of (5.8) over $j \geq 0$ and obtain

$$\begin{aligned} &\frac{1}{2} \sum_{j=0}^\infty \left(a_j^2(t) + b_j^2(t) \right) - \frac{1}{2} \sum_{j=0}^\infty \left(a_j^2(0) + b_j^2(0) \right) \\ &\quad = - \sum_{j=0}^\infty \int_0^t \lambda_j^2 a_j^2 \, \mathrm{d}t - \sum_{j=0}^\infty \int_0^t \lambda_j^2 b_j^2 \, \mathrm{d}t + \sum_{j=0}^\infty \int_0^t f_j a_j \, \mathrm{d}t. \end{aligned}$$

Thus, we conclude (a, b) is a Leray–Hopf solution of (1.2a)–(1.2b) with zero initial data; however, $a \neq 0$ and $b \neq 0$. Non-uniqueness then follows. Indeed, for such forcing $f(t)$ as in (5.5), considering $b(t) \equiv 0$ in (1.2a)–(1.2b), the forced dyadic model of the NSE has a solution $\tilde{a}(t)$. Hence, $(\tilde{a}(t), 0)$ is a trivial solution for the dyadic MHD model (1.2a)–(1.2b). □

Remark 5.5 When κ_1 and κ_2 take different signs, we can choose the same constructions for $(a_j(t), b_j(t))$ as in (5.1) and (5.2). The difference comes in the ODE system (5.3a)–(5.3d) for the profile functions h_1, h_2 and h_3. For instance, if $\kappa_1 = \kappa_2 = 1$, the functions h_1, h_2 and h_3 satisfy the following system:

$$\frac{\mathrm{d}}{\mathrm{d}t} h_1 + \left(\lambda^{-2} + \lambda^{-\theta} q \right) h_1 + \lambda^{-\theta} p h_2 = 0, \tag{5.9a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} h_2 + h_2 - q h_3 = 0, \tag{5.9b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}h_3 + \lambda^2 h_3 = 0, \tag{5.9c}$$

accompanied with

$$h_1(0) = 0, \quad h_2(0) = c_0, \quad h_3(0) = d_0, \tag{5.9d}$$

$$h_1(1) = \rho c_0, \quad h_2(1) = \rho d_0. \tag{5.10}$$

We note that the structure of system (5.9a)–(5.9d) remains similar to that of system (5.3a)–(5.3d). Thus, in analogy with Lemma 5.4, it is not hard to show the existence of a solution (h_1, h_2, h_3) to system (5.9a)–(5.9d) satisfying (5.10). The rest analysis of Sect. 5 also holds for system (1.2a)–(1.2b) with $\kappa_1 = \kappa_2 = 1$.

Remark 5.6 We expect the framework of the non-uniqueness construction presented here may also be appropriate for other shell models, such as L'vov et al. (1998), GOY (Gledzer 1973; Ohkitani and Yamada 1989) and Hall MHD dyadic models (Dai 2021). The framework is flexible as it does not require unidirectional energy cascade mechanism. The analysis would lead to ODE systems analogous to the system (5.3a)–(5.3d). The examination of such ODE systems will be left for future work.

6 Uniqueness and Non-uniqueness Results for the Dyadic MHD Model with Fractional Laplacian

6.1 Uniqueness

The weak-strong uniqueness stated in Theorem 1.9 under assumption (1.8) can be proved by following the steps described in Sect. 4. We briefly present the main steps and emphasize why assumption (1.8) is required for the uniqueness.

Let $(a(t), b(t))$ and $(u(t), v(t))$ be two Leray–Hopf solutions of (1.7) with $(a(t), b(t))$ satisfying (1.8). The difference of the two solutions satisfies the energy estimate

$$\begin{aligned}
&\sum_{j=0}^{\infty}\left((a_j(t)-u_j(t))^2+(b_j(t)-v_j(t))^2\right)\\
&\quad+2\sum_{j=0}^{\infty}\int_0^t \lambda_j^{2\alpha}(a_j(\tau)-u_j(\tau))^2+\lambda_j^{2\beta}(b_j(\tau)-v_j(\tau))^2\mathrm{d}\tau\\
&=-\sum_{j=0}^{\infty}\int_0^t \lambda_j a_j(\tau)(a_j(\tau)-u_j(\tau))\left(a_{j+1}(\tau)-u_{j+1}(\tau)\right)\mathrm{d}\tau\\
&\quad-\sum_{j=0}^{\infty}\int_0^t \lambda_j b_j(\tau)(b_j(\tau)-v_j(\tau))\left(a_{j+1}(\tau)-u_{j+1}(\tau)\right)\mathrm{d}\tau\\
&\quad+\sum_{j=0}^{\infty}\int_0^t \lambda_j(a_j(\tau)-u_j(\tau))^2 a_{j+1}(\tau)\mathrm{d}\tau+\sum_{j=0}^{\infty}\int_0^t \lambda_j(b_j(\tau)-v_j(\tau))^2 a_{j+1}(\tau)\mathrm{d}\tau\\
&\quad-\sum_{j=0}^{\infty}\int_0^t \lambda_j a_j(\tau)(b_j(\tau)-v_j(\tau))\left(b_{j+1}(\tau)-v_{j+1}(\tau)\right)\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{\infty}\int_0^t \lambda_j b_j(\tau)(a_j(\tau)-u_j(\tau))\left(b_{j+1}(\tau)-v_{j+1}(\tau)\right)\mathrm{d}\tau\\
&\equiv\ I_1+I_2+I_3+I_4+I_5+I_6. \qquad (6.1)
\end{aligned}
$$

With $(a(t), b(t))$ and $(u(t), v(t))$ being Leray–Hopf solutions of (1.7), it holds

$$
\sum_{j=0}^{\infty}\int_0^t \left(\lambda_j^{2\alpha}a_j^2(\tau)+\lambda_j^{2\beta}b_j^2(\tau)\right)\mathrm{d}\tau\ <\infty,\quad \sum_{j=0}^{\infty}\int_0^t \left(\lambda_j^{2\alpha}u_j^2(\tau)+\lambda_j^{2\beta}v_j^2(\tau)\right)\mathrm{d}\tau\ <\infty.
$$

Thus, combining assumption (1.8) we know that all of the series on the right-hand side of (6.1) are well defined. Moreover, these series can be estimated in the following way by using (1.8). For example, we estimate I_2 thanks to the condition $|b_j|\le C_0\lambda_j^{\alpha+\beta-1}$ of (1.8),

$$
\begin{aligned}
&\sum_{j=J}^{\infty}\int_0^t \lambda_j\left|b_j(\tau)(b_j(\tau)-v_j(\tau))\left(a_{j+1}(\tau)-u_{j+1}(\tau)\right)\right|\mathrm{d}\tau\\
&\le\ C_0\sum_{j=J}^{\infty}\int_0^t\left|\lambda_j^{\alpha+\beta}(b_j(\tau)-v_j(\tau))\left(a_{j+1}(\tau)-u_{j+1}(\tau)\right)\right|\mathrm{d}\tau\\
&\le\ \frac{C_0}{2}\sum_{j=J}^{\infty}\int_0^t\lambda_j^{2\beta}(b_j(\tau)-v_j(\tau))^2\mathrm{d}\tau+\frac{C_0}{2\lambda^{2\alpha}}\sum_{j=J}^{\infty}\int_0^t\lambda_{j+1}^{2\alpha}(a_{j+1}(\tau)-u_{j+1}(\tau))^2\mathrm{d}\tau.
\end{aligned}
$$

The term I_6 can be handled similarly. The condition $|a_j|\le C_0\lambda_j^{2\alpha-1}$ is posed to estimate I_1 and I_3, and $|a_j|\le C_0\lambda_j^{2\beta-1}$ is for I_4 and I_5. With the estimates, it follows from (6.1) that

$$
\begin{aligned}
&\sum_{j=0}^{\infty}\left((a_j(t)-u_j(t))^2+(b_j(t)-v_j(t))^2\right)\\
&\qquad+2\sum_{j=0}^{\infty}\lambda_j^2\int_0^t(a_j(\tau)-u_j(\tau))^2+(b_j(\tau)-v_j(\tau))^2\mathrm{d}\tau\\
&\quad\le C_1\sum_{j=0}^{J}\int_0^t(a_j(\tau)-u_j(\tau))^2+(b_j(\tau)-v_j(\tau))^2\mathrm{d}\tau \qquad (6.2)\\
&\qquad+C_0\left(1+\lambda^{2\alpha-1}+\lambda^{-2\alpha}\right)\sum_{j=J}^{\infty}\int_0^t\lambda_j^{2\alpha}(a_j(\tau)-u_j(\tau))^2\mathrm{d}\tau\\
&\qquad+C_0\left(1+\lambda^{2\beta-1}+\lambda^{-2\beta}\right)\sum_{j=J}^{\infty}\int_0^t\lambda_j^{2\alpha}(b_j(\tau)-v_j(\tau))^2\mathrm{d}\tau
\end{aligned}
$$

with a constant $C_1 > 0$. The constant C_0 can be chosen small enough such that $C_0\left(1+\lambda^{2\alpha-1}+\lambda^{-2\alpha}\right) \leq 2$ and $C_0\left(1+\lambda^{2\beta-1}+\lambda^{-2\beta}\right) \leq 2$. Consequently, we have from (6.2) that

$$\sum_{j=0}^{\infty}\left((a_j(t)-u_j(t))^2+(b_j(t)-v_j(t))^2\right)$$
$$\leq C_1\sum_{j=0}^{\infty}\int_0^t (a_j(\tau)-u_j(\tau))^2+(b_j(\tau)-v_j(\tau))^2\mathrm{d}\tau.$$

Grönwall's inequality immediately implies that $a_j \equiv u_j$ and $b_j \equiv v_j$ for all $j \geq 0$.

6.2 Non-uniqueness

The construction scheme to prove Theorem 1.11 is similar to that presented in Sect. 5. The main effort is to determine the scaling in constructing a_j and b_j. To be complete, we specify the constructions as follows. For $T = \frac{1}{\lambda^{2\beta}-1}$, we take the partition

$$t_j = \lambda_j^{-2\beta}T, \quad j \geq 0,$$

with

$$t_{j-1}-t_j = \lambda_j^{-2\beta}, \quad j \geq 1; \quad (0,T) = \cup_{j=1}^{\infty}[t_j, t_{j-1}).$$

For $p, q \in C_c^{\infty}(0,1)$ and constant $\rho > \lambda$, we choose a_j and b_j as:

$$a_j(t) = \begin{cases} 0, & t < t_{j+1}, \\ \lambda_{j+1}^{2\beta-1}p\left(\lambda_{j+1}^{2\beta}(t-t_{j+1})\right), & t_{j+1} < t < t_j, \\ -\lambda_j^{2\beta-1}q\left(\lambda_j^{2\beta}(t-t_j)\right), & t_j < t < t_{j-1}, \\ 0, & t > t_{j-1}. \end{cases} \tag{6.3}$$

$$b_j(t) = \begin{cases} 0, & t < t_{j+1}, \\ \rho^{-j-1}h_1\left(\lambda_{j+1}^{2\beta}(t-t_{j+1})\right), & t_{j+1} < t < t_j, \\ \rho^{-j}h_2\left(\lambda_j^{2\beta}(t-t_j)\right), & t_j < t < t_{j-1}, \\ \rho^{-j+1}h_3\left(\lambda_{j-1}^{2\beta}(t-t_{j-1})\right), & t_{j-1} < t < t_{j-2}, \\ \rho^{-j+1}h_3(1)e^{-\lambda_j^{2\beta}(t-t_{j-2})}, & t > t_{j-2}, \end{cases} \tag{6.4}$$

where h_1, h_2 and h_3 are functions satisfying the following ODE system on $[0, 1]$

$$\frac{\mathrm{d}}{\mathrm{d}t}h_1+\left(\lambda^{-2\beta}-\lambda^{-1}q\right)h_1-\lambda^{-1}ph_2 = 0, \tag{6.5a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}h_2 + h_2 + qh_3 = 0, \tag{6.5b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}h_3 + \lambda^{2\beta}h_3 = 0, \tag{6.5c}$$

$$h_1(0) = 0, \quad h_2(0) = c_0, \quad h_3(0) = d_0, \tag{6.5d}$$

$$h_1(1) = \rho c_0, \quad h_2(1) = \rho d_0. \tag{6.5e}$$

We take the forcing f_j as:

$$f_j = \frac{\mathrm{d}}{\mathrm{d}t}a_j + \lambda_j^{2\alpha}a_j + \lambda_j a_j a_{j+1} + \lambda_j b_j b_{j+1} - \lambda_{j-1}a_{j-1}^2 - \lambda_{j-1}b_{j-1}^2 \tag{6.6}$$

for all $j \geq 0$.

For f_j defined by (6.6), we can show that $(a(t), b(t))$ with components constructed as in (6.3)–(6.4) is a Leray–Hopf solution of system (1.7) with non-vanishing $b(t)$. We state the main ingredients to prove Theorem 1.11 in the following lemmas, the proof of which are omitted.

Lemma 6.1 *Let $\rho > \lambda > 1$ and $0 < \alpha \leq \beta < \frac{1}{2}$. The following properties hold for a_j and b_j as constructed in (6.3)–(6.4):*

(i) $a_j \in C_c^\infty(0, T)$ and $a_j \in H^\alpha(0, T)$ for all $j \geq 0$;
(ii) b_j are piecewise smooth and $b_j \in H^\beta(0, T)$ for all $j \geq 0$;
(iii) $a_j(0) = b_j(0) = 0, \quad \forall\ j \geq 0$;
(iv) $a_j(t) = O(\lambda_j^{2\beta-1}), \ \ a'_j(t) = O(\lambda_j^{4\beta-1}), \ \ b_j(t) = O(\rho^{-j}), \quad j \to \infty$.

Lemma 6.2 *The functions a_j and b_j defined in (6.3)–(6.4) satisfy system (1.7) with forcing f_j defined by (6.6).*

Lemma 6.3 *Let $\rho > \lambda > 1$. Assume $0 < \alpha \leq \beta < \frac{1}{2}$ and $3\beta - \alpha < 1$. The forcing f_j defined by (6.6) satisfies*

$$\sum_{j=0}^{\infty} \lambda_j^{-2\alpha} \int_0^T f_j^2(t)\,\mathrm{d}t < \infty.$$

Lemma 6.4 *Let $\rho > \lambda > 1$ and $0 < \alpha \leq \beta < \frac{1}{2}$. There exist functions $p, q \in C_c^\infty(0, 1)$ and constants c_0, d_0 satisfying $c_0^2 + d_0^2 \neq 0$ such that there exists a unique solution $h = (h_1, h_2, h_3) \in C^\infty([0, 1]; \mathbb{R}^3)$ of system (6.5a)–(6.5e).*

Lemma 6.5 *Let $\rho > \lambda > 1$ and $0 < \alpha \leq \beta < \frac{1}{2}$. Then, $(a(t), b(t))$ satisfies the energy identity*

$$\frac{1}{2}\sum_{j=0}^{\infty}\left(a_j^2(t) + b_j^2(t)\right) - \frac{1}{2}\sum_{j=0}^{\infty}\left(a_j^2(0) + b_j^2(0)\right)$$
$$= -\sum_{j=0}^{\infty}\int_0^t \lambda_j^{2\alpha}a_j^2\,\mathrm{d}t - \sum_{j=0}^{\infty}\int_0^t \lambda_j^{2\beta}b_j^2\,\mathrm{d}t + \sum_{j=0}^{\infty}\int_0^t f_j a_j\,\mathrm{d}t$$

with f_j defined by (6.6).

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