# 1D MODEL FOR THE 3D MAGNETOHYDRODYNAMICS

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Abstract. We propose a one-dimensional (1D) model for the three-dimensional (3D) incompressible ideal magnetohydrodynamics. For this 1D model, local well-posedness is established, and a regularity criterion of the Beale-Kato-Majda type is obtained. Without the stretching effect, the model with only transport effect is shown to have global in time strong solution. Some numerical simulations suggest that solutions of the model with certain smooth periodic initial data are not likely to develop singularities in finite time, while solutions starting from other initial data have the tendency to form singularities.

KEY WORDS: magnetohydrodynamics; regularity; Hilbert transform; 1D simplified model; singularity formation.
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## 1. Introduction

The ideal incompressible magnetohydrodynamics (MHD) governed by the set of partial differential equations

$$\begin{aligned} u_t + (u\cdot\nabla)u - (B\cdot\nabla)B + \nabla\Pi &= 0,\\ B_t + (u\cdot\nabla)B - (B\cdot\nabla)u &= 0,\\ \nabla\cdot u = 0, \quad \nabla\cdot B &= 0, \end{aligned} \tag{1.1}$$

is an important model in geophysics and astrophysics. In the system, the vector fields $u$ and $B$ denote the fluid velocity and magnetic field respectively; the scalar function $\Pi$ is the pressure. We notice that (1.1) reduces to the incompressible Euler equation if $B \equiv 0$,

$$\begin{aligned} u_t + (u\cdot\nabla)u + \nabla\Pi &= 0,\\ \nabla\cdot u &= 0. \end{aligned} \tag{1.2}$$

The mathematical question of whether or not a solution of the 3D Euler (1.2) develops singularity at finite time remains open. So does it for the 3D MHD (1.1).

Denote the vorticity by $\omega = \nabla\times u$. Taking a curl on (1.2) gives

$$\omega_t + (u\cdot\nabla)\omega + (\omega\cdot\nabla)u = 0, \tag{1.3a}$$

$$u = \nabla\times(-\Delta)^{-1}\omega. \tag{1.3b}$$

We note that $u$ can be recovered from $\omega$ through the Biot-Savart law (1.3b) which involves a nonlocal operator. In (1.3a), the quadratic term $(u\cdot\nabla)\omega$ is regarded as the transport term, while $(\omega\cdot\nabla)u$ represents the stretching effect. The general

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belief is that the stretching effect is responsible for dramatic wild behaviours of solutions, for instance, the appearance of finite-time singularity.

1.1. **1D models for Euler equation and related equations.** To gain insights towards understanding the properties of solutions to the Euler equation (1.2), approximating models and toy models have been proposed and studied in the literature. One type of 1D models for the vorticity form of Euler equation has attracted a great deal of attention, which can be traced back to the work of Constantin, Lax and Majda [6]. The authors of [6] proposed the following 1D model for system (1.3a)-(1.3b),

$$\omega_t = \omega H\omega, \tag{1.4a}$$

$$u_x = H\omega, \tag{1.4b}$$

with $\omega = \omega(t, x)$ and $u = u(t, x)$ for $t \geq 0$ and $x \in \mathbb{R}$. In the system, $H$ denotes the Hilbert transform defined by

$$Hf = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x - y}\, dy. \tag{1.5}$$

We note that equation (1.4b) is a 1D analogue of the Biot-Savart law (1.3b). With only stretching effect in equation (1.4a), the authors solved system (1.4a)-(1.4b) exactly and showed the formation of finite-time singularities for a class of initial data. Since then, various generalisations of (1.4a)-(1.4b) have been studied both analytically and numerically. The De Gregorio model [9, 10]

$$\omega_t + u\omega_x - \omega H\omega = 0, \tag{1.6a}$$

$$u_x = H\omega, \tag{1.6b}$$

includes both transport and stretching effects. Numerical results of [9, 10] provide evidence that finite-time blow-up may not occur for system (1.6a)-(1.6b) with some smooth periodic initial data. It indicates that the convection (transport) term has a regularization effect. Later on, in order to understand the competing effects of convection and stretching terms, Okamoto, Sakajo and Wunsch [21] suggested to study the following family of models

$$\omega_t + au\omega_x - \omega H\omega = 0, \tag{1.7a}$$

$$u_x = H\omega, \tag{1.7b}$$

with a parameter $a \in \mathbb{R}$. The authors also conjectured global in time existence of solutions to (1.7a)-(1.7b) with $a = 1$ which is the De Gregorio model (1.6a)-(1.6b). Indeed, Jia, Stewart and Šverák [16] proved that solutions of (1.6a)-(1.6b) with initial data near a steady state are global and converge to this steady state. Lei, Liu and Ren [18] showed that the De Gregorio model with non-negative vorticity initial data is globally well-posed. In contrast, Elgindi and Jeong [12] showed singularity formation for (1.6a)-(1.6b) in classes of Hölder continuous solutions. Moreover, the authors of [12] established that, there exists smooth initial data such that solution of the Okamoto-Sakajo-Wunsch model (1.7a)-(1.7b) with small $|a|$ develops self-similar type of blow-up at finite time. Later on, Elgindi, Ghoul and Masmoudi [11] further showed that such self-similar blow-up is stable. In [5], Chen, Hou and Huang provided a novel method of analysis and established self-similar blowup for the De Gregorio model with certain initial data on both $\mathbb{R}$ and $\mathbb{S}^1$. For (1.7a)-(1.7b), Chen [3] showed finite time singularity from some smooth initial data when

$a < 1$ and close to 1, and global well-posedness with the same initial data when $a > 1$. When $a = 1$, Chen [2] proved finite time blowup for (1.7a)-(1.7b) on $\mathbb{S}^1$ with $C^\alpha$ data for any $\alpha < 1$. The model (1.7a)-(1.7b) with a viscosity term was also studied in [4] for $a \in \mathbb{R}$. Recently, Lushnikov, Silantyev and Siegel [19] performed an extensive numerical and analytical study of (1.7a)-(1.7b) on the topic of global existence versus finite time singularity formation for different values of $a$. They identified a new critical value $a_c = 0.6890665337007457...$ below which self-similar type of singularity develops in finite time. Moreover, for $a = 0$ and $a = \frac{1}{2}$, the authors constructed exact analytical solutions with pole singularity.

In the literature, many other 1D simplified models for fluid equations have been studied. A notable model is the nonlocal transport equation

$$\theta_t - (H\theta)\theta_x = 0 \tag{1.8}$$

which has a connection with the integrodifferential Birkhoff-Rott equation modeling vortex sheets, see [1, 20]. It has an analogy with (1.7a)-(1.7b) as well. Indeed, taking derivative $\partial_x$ on (1.8), the resulted equation is equivalent to (1.7a)-(1.7b) with $a = -1$. Moreover, it serves as a 1D simplified model for the surface quasi-geostrophic equation. Córdoba, Córdoba and Fontelos [7, 8] showed finite-time singularity formation for (1.8) with a general class of initial data. For axisymmetric 3D incompressible Navier-Stokes equation with swirl, Hou, Li, Shi, Wang and Yu [14] proposed a 1D nonlocal model for a simplified 3D nonlocal system [15]. For this 1D model, the authors proved finite-time singularity formation rigorously and showed numerical evidences.

1.2. **1D models for MHD.** Inspired by the works discussed above, we will propose a family of nonlocal nonlinear models for the MHD system (1.1) as an attempt to understand the intricate structures involved in this system. In the context of MHD, besides the convection and stretching effects, the coupling and interaction between the fluid velocity and magnetic field also play crucial roles, which naturally introduce additional challenges.

Denote the Elsässer variables by

$$p = u + B, \quad m = u - B.$$

Equivalent to (1.1), $(p, m)$ satisfies the system

$$\begin{aligned} p_t + (m \cdot \nabla)p + \nabla\Pi &= 0, \\ m_t + (p \cdot \nabla)m + \nabla\Pi &= 0, \\ \nabla \cdot p = 0, \quad \nabla \cdot m &= 0. \end{aligned} \tag{1.9}$$

The structure of system (1.9) indicates that $p$ and $m$ are transported by each other. We also note that (1.9) appears in a rather symmetric form. Denote the vorticity of $p$ and $m$ by

$$\Omega = \nabla \times p, \quad \omega = \nabla \times m.$$

It follows from the Biot-Savart law that

$$p = \nabla \times (-\Delta)^{-1}\Omega, \quad m = \nabla \times (-\Delta)^{-1}\omega.$$

Taking the curl $\nabla\times$ on the equations of (1.9) gives

$$\begin{aligned} \Omega_t + (m\cdot\nabla)\Omega - (\Omega\cdot\nabla)m + \nabla\times(m\nabla p) &= 0, \\ \omega_t + (p\cdot\nabla)\omega - (\omega\cdot\nabla)p + \nabla\times(p\nabla m) &= 0, \\ p &= \nabla\times(-\Delta)^{-1}\Omega, \\ m &= \nabla\times(-\Delta)^{-1}\omega, \end{aligned} \tag{1.10}$$

where $(m\nabla p)_j = m_i\partial_j p_i$ and $(p\nabla m)_j = p_i\partial_j m_i$ for $1\le j\le 3$. Note

$$\nabla\times(m\nabla p) = (\partial_2 m_i\partial_3 p_i - \partial_3 m_i\partial_2 p_i, \partial_3 m_i\partial_1 p_i - \partial_1 m_i\partial_3 p_i, \partial_1 m_i\partial_2 p_i - \partial_2 m_i\partial_1 p_i),$$
$$\nabla\times(p\nabla m) = (\partial_2 p_i\partial_3 m_i - \partial_3 p_i\partial_2 m_i, \partial_3 p_i\partial_1 m_i - \partial_1 p_i\partial_3 m_i, \partial_1 p_i\partial_2 m_i - \partial_2 p_i\partial_1 m_i)$$

and $\nabla\times(m\nabla p) = -\nabla\times(p\nabla m)$. To reveal the anti-symmetry feature, we can rewrite

$$\begin{aligned} \nabla\times(m\nabla p) &= \frac{1}{2}\nabla\times(m\nabla p) - \frac{1}{2}\nabla\times(p\nabla m), \\ \nabla\times(p\nabla m) &= \frac{1}{2}\nabla\times(p\nabla m) - \frac{1}{2}\nabla\times(m\nabla p). \end{aligned}$$

Superficially we view $\nabla\times(m\nabla p)$ and $\nabla\times(p\nabla m)$ in analogy with $(\nabla\times m)\nabla p$ and $(\nabla\times p)\nabla m$, respectively. Thus we propose the following 1D model to mimic system (1.10),

$$\begin{aligned} \Omega_t + m\Omega_x - \Omega m_x + \frac{1}{2}\omega p_x - \frac{1}{2}\Omega m_x &= 0, \\ \omega_t + p\omega_x - \omega p_x + \frac{1}{2}\Omega m_x - \frac{1}{2}\omega p_x &= 0, \\ p_x = H\Omega, \quad m_x &= H\omega. \end{aligned} \tag{1.11}$$

In this paper, we will work with a simplified version of (1.11) by dropping the stretching effects $\Omega m_x$ from the first equation and $\omega p_x$ from the second equation, and focusing on the transport effects and the nonlocal coupling, namely

$$\begin{aligned} \Omega_t + \tilde{a}m\Omega_x + \omega p_x &= 0, \\ \omega_t + \tilde{a}p\omega_x + \Omega m_x &= 0, \\ p_x = H\Omega, \quad m_x &= H\omega, \end{aligned}$$

with a parameter $\tilde{a}\in\mathbb{R}$. Applying the transform $(\Omega,\omega)\to(-\Omega,-\omega)$, the system above is equivalent to the form

$$\begin{aligned} \Omega_t + am\Omega_x - \omega p_x &= 0, \\ \omega_t + ap\omega_x - \Omega m_x &= 0, \\ p_x = H\Omega, \quad m_x &= H\omega, \end{aligned} \tag{1.12}$$

with $a = -\tilde{a}\in\mathbb{R}$. We will investigate (1.12) on the periodic interval $\mathbb{S}^1 = [-\pi,\pi]$. Correspondingly, the Hilbert transform for periodic functions on $\mathbb{S}^1$ can be defined as

$$Hf(x) = \frac{1}{2\pi}P.V.\int_{-\pi}^{\pi} f(y)\cot\left(\frac{x-y}{2}\right)dy. \tag{1.13}$$

Indeed, the Cauchy kernel $\frac{1}{x}$ in definition (1.5) can be made periodic using the following identity

$$\frac{1}{2}\cot\left(\frac{x-y}{2}\right) = \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{x+2n\pi} + \frac{1}{x-2n\pi}\right).$$

To uniquely determine $p$ from $\Omega$ and $m$ from $\omega$, we make the choice of Gauge by taking zero-mean value

$$\int_{-\pi}^{\pi} p(t,x)\,dx = \int_{-\pi}^{\pi} m(t,x)\,dx = 0. \tag{1.14}$$

We note that the mean value of $\Omega$ and $\omega$ is invariant for system (1.12) with $a = 1$. Indeed, we have for a smooth solution $(\Omega, \omega)$ that

$$\begin{aligned}\frac{d}{dt}\int_{-\pi}^{\pi} \Omega(t,x)\,dx =& \int_{-\pi}^{\pi} \left(-am\Omega_x + \omega p_x\right)\,dx \\ =& \int_{-\pi}^{\pi} \left(am_x\Omega + \omega p_x\right)\,dx \\ =& \int_{-\pi}^{\pi} \left(a\Omega H\omega + \omega H\Omega\right)\,dx \\ =&(1-a)\int_{-\pi}^{\pi} \omega H\Omega\,dx\end{aligned}$$

where we have used integration by parts and the skew symmetry property of the Hilbert transform. Similarly, we have

$$\frac{d}{dt}\int_{-\pi}^{\pi} \omega(t,x)\,dx = \int_{-\pi}^{\pi} \left(-ap\omega_x + \Omega m_x\right)\,dx = (1-a)\int_{-\pi}^{\pi} \Omega H\omega\,dx.$$

Obviously when $a = 1$, it follows

$$\frac{d}{dt}\int_{-\pi}^{\pi} \Omega(t,x)\,dx = \frac{d}{dt}\int_{-\pi}^{\pi} \omega(t,x)\,dx = 0,$$

and this is not true in general for $a \neq 1$. However, we observe that for odd initial data $(\Omega_0, \omega_0)$, the solution $(\Omega, \omega)$ of (1.12) remains odd. While for odd functions $\Omega$ and $\omega$, the Hilbert transform $H\Omega$ and $H\omega$ are even, and hence

$$\int_{-\pi}^{\pi} \omega H\Omega\,dx = \int_{-\pi}^{\pi} \Omega H\omega\,dx = 0.$$

Therefore it is appropriate to consider solutions of (1.12) in spaces of functions with zero mean for general value of $a$, since at least odd solutions of (1.12) automatically have zero mean.

Regarding the parameter $a$, for the Euler model (1.7a)-(1.7b), it is believed that $a = 1$ is the most relevant case. In contrast, it is not clear for which value $a$, model (1.12) is more relevant for the original MHD system. A speculation is that $a = -2$ may be more relevant by comparing (1.11) and (1.12). This indicates the difference of the ideal MHD from the Euler equation due to the interaction of the velocity and magnetic field. It is an interesting question to be further investigated in future work.

Consider the rescaled variables

$$\tilde{\Omega} = a\Omega, \quad \tilde{\omega} = a\omega$$

with corresponding $\tilde{p}$ and $\tilde{m}$ such that

$$\tilde{p}_x = H\tilde{\Omega}, \quad \tilde{m}_x = H\tilde{\omega}.$$

We can verify that $\tilde{p} = ap$ and $\tilde{m} = am$. In view of (1.12), $(\tilde{\Omega}, \tilde{\omega})$ satisfies the system

$$\begin{aligned} \tilde{\Omega}_t + \tilde{m}\tilde{\Omega}_x - a^{-1}\tilde{\omega}\tilde{p}_x &= 0, \\ \tilde{\omega}_t + \tilde{p}\tilde{\omega}_x - a^{-1}\tilde{\Omega}\tilde{m}_x &= 0. \end{aligned} \tag{1.15}$$

Formally, taking $a \to \infty$, (1.15) turns to the system with only convection effect (with the tilde sign suppressed),

$$\begin{aligned} \Omega_t + m\Omega_x &= 0, \\ \omega_t + p\omega_x &= 0, \\ p_x = H\Omega, \quad m_x &= H\omega. \end{aligned} \tag{1.16}$$

We will investigate both systems (1.12) and (1.16) in the paper. We point out that formulating the problem in Elsässer variables does not give us essential advantage; rather it has the benefit of dealing with less nonlinear terms.

1.3. **Main results.** For general $a \in \mathbb{R}$, we show the existence of local in time solutions to (1.12) in the space $\mathcal{H}^1(\mathbb{S}^1)$.

**Theorem 1.1.** *Let $a \in \mathbb{R}$ and $\Omega_0, \omega_0 \in \mathcal{H}^1(\mathbb{S}^1)$. There exists a time $T > 0$ which depends on $\|\Omega_{0,x}\|_{L^2}$ and $\|\omega_{0,x}\|_{L^2}$ such that there exists a unique solution $(\Omega(t,x), \omega(t,x))$ to (1.12) with initial data $\Omega(0,x) = \Omega_0$ and $\omega(0,x) = \omega_0$ on $[0,T)$, which satisfies*

$$\Omega, \omega \in C^0\left([0,T); \mathcal{H}^1(\mathbb{S}^1)\right) \cap C^1\left([0,T); L^2(\mathbb{S}^1)\right).$$

The following theorem provides a Beale-Kato-Majda type of regularity criterion.

**Theorem 1.2.** *Let $(\Omega(t,x), \omega(t,x))$ be the solution of (1.12) on $[0,T)$ obtained in Theorem 1.1. If*

$$\int_0^T \left(\|H\Omega(t)\|_{L^\infty} + \|H\omega(t)\|_{L^\infty}\right) dt < \infty, \tag{1.17}$$

*the solution can be extended beyond $T$ in the space $\mathcal{H}^1(\mathbb{S}^1) \times \mathcal{H}^1(\mathbb{S}^1)$.*

Furthermore, if the initial data is in a space with higher regularity, the solution obtained in Theorem 1.1 also has higher regularity. Specifically, we will show:

**Theorem 1.3.** *Assume $\Omega_0, \omega_0 \in \mathcal{H}^n(\mathbb{S}^1)$ with $n \geq 2$. Let $(\Omega, \omega)$ be a solution of (1.12) with initial data $(\Omega_0, \omega_0)$ on $[0,T)$, satisfying $\Omega, \omega \in C([0,T); \mathcal{H}^1)$. Then, we have*

$$\sup_{0 \leq t < T} \left(\|\Omega(t)\|_{\mathcal{H}^n} + \|\omega(t)\|_{\mathcal{H}^n}\right) < \infty.$$

With only transport effect, the solution of (1.16) can be shown to exist in the space $H^1(\mathbb{S}^1)$ for all the time. Namely, we have

**Theorem 1.4.** *Assume $\Omega_0, \omega_0 \in \mathcal{H}^1(\mathbb{S}^1)$. Then there exists a unique solution $(\Omega(t), \omega(t))$ of (1.16) with initial data $(\Omega_0, \omega_0)$ on $[0, \infty)$.*

**Remark 1.5.** If $p = m$ and $\Omega = \omega$, system (1.12) reduces to the 1D Euler model (1.7a)-(1.7b). Therefore, in this special situation, the aforementioned solutions with finite time singularity for the Euler model with various values of the parameter $a$ are also (trivial) solutions of (1.12).

**Remark 1.6.** For the original ideal MHD (1.1) in 2D or 3D, it is known that the Beale-Kato-Majda type regularity criterion with condition imposed only on the velocity is valid. The main reason is that the magnetic field equation in (1.1) is linear in $B$. A common interpretation of the BKM criterion with only velocity dependence is that the velocity field plays a more dominant role for incompressible MHD. In the 1D situation, the criterion of Theorem 1.2 relies on both the velocity and magnetic field. The additional dependence of magnetic field is essentially due to the loss of divergence free condition in 1D, that is, $\nabla \cdot u = \nabla \cdot B = 0$ is not valid any more. In general, the loss of divergence free is an artefact for 1D simplified models, which causes deviation for the simplified models from the original PDE systems in some aspects. For instance, the 1D model (1.8) does not conserve the $L^2$ norm of smooth solution $\theta$, while the 2D SQG does conserve the $L^2$ norm. Back to the 1D model of MHD, such artefact brings forth more influence of the magnetic field on the entire system.

Numerical study is presented in Section 6. The numerical results suggest that starting from some smooth periodic initial data, solutions of the model (1.12) with some values of $a$ are unlikely to develop singularities in finite time; while solutions of the model with some initial data and certain $a$ have the tendency to form singularities. In particular, one of the observations agrees with the numerical results done by De Gregorio [9, 10] and Okamoto, Sakajo and Wunsch [21] for the De Gregorio model (1.6a)-(1.6b). Another interesting observation is that the solution of (1.12) with an initial data and $a = -1$ seems regular. It is worth to point out that the numerical indication of no singularity for (1.12) with $a = -1$ does not contradict the finite time singularity formation result of [7]. Our numerical simulation is performed for some particular initial data on the periodic domain $\mathbb{S}^1$, while the singular solution of [7] is constructed on $\mathbb{R}$ for a specific class of initial data. In addition, we note finite time blowup for (1.12) with $a = -1$ was also established on $\mathbb{S}^1$ for a class of initial data by Chen, Hou and Huang [5]. Therefore it seems that the choice of initial data plays an important role for the phenomena of finite time singularity formation when $a = -1$.

To conclude, we mention that the analytical results established in Theorems 1.1, 1.2, 1.3 and 1.4 hold on the space $\mathbb{R}$ as well, with slight modifications of the proofs. We present the results on the periodic domain $\mathbb{S}^1$ such that, as a consistent followup in Section 6, numerical study in periodic settings is performed. We would like to add that, we learned an interesting approach to transform 1D models between periodic domain and $\mathbb{R}$ from [19] after the completion of the first version of our manuscript. With the transform given by arctan function, one can study these 1D models numerically on $\mathbb{R}$ as well.

## 2. Notations and preliminaries

### 2.1. **Functional setting.** Denote

$$
\begin{aligned}
&L^2(\mathbb{S}^1) = \left\{ f | f \in L^2(-\pi,\pi), f \text{ is periodic on} [-\pi,\pi] \right\},\\
&\mathcal{H}^k(\mathbb{S}^1) = \left\{ f | f^{(s)} \in L^2(-\pi,\pi), f^{(s)} \text{ is periodic on} [-\pi,\pi], \quad \text{for all } \ 0 \le s \le k \right\}.
\end{aligned}
$$

In particular, we consider the triplet of spaces

$$\mathcal{V} = \left\{ f \middle| f \in \mathcal{H}^2(\mathbb{S}^1), \ \int_{-\pi}^{\pi} f(x)\, dx = 0 \right\}$$

$$\mathcal{W} = \mathcal{H}^1(\mathbb{S}^1), \quad \mathcal{X} = L^2(\mathbb{S}^1),$$

with the obvious embedding $\mathcal{V} \subset \mathcal{W} \subset \mathcal{X}$.

We denote $(,)$ by

$$(f, g) = \int_{-\pi}^{\pi} fg\, dx.$$

The space $\mathcal{H}^k(\mathbb{S}^1)$ is a Hilbert space endowed with the natural inner product

$$(f, g)_{\mathcal{H}^k} = \sum_{s=0}^{k} \left( f^{(s)}, g^{(s)} \right) \quad \text{for functions} \quad f, g \in \mathcal{H}^k(\mathbb{S}^1),$$

and norm $(f, f)_{\mathcal{H}^k}^{\frac{1}{2}}$.

A bilinear form $\langle , \rangle : \mathcal{V} \times \mathcal{X} \to \mathbb{R}$ is defined as

$$\langle f, g \rangle = -\int_{-\pi}^{\pi} f_{xx} g\, dx.$$

Applying the integration by parts, we have for all $f \in \mathcal{V}$ and $g \in \mathcal{W}$

$$\langle f, g \rangle = (f_x, g_x).$$

For a space $\mathcal{Z}$, we denote $\mathcal{Z}^2 = \mathcal{Z} \times \mathcal{Z}$ by convention. In the context of a coupled system, for instance (1.12), it is convenient to introduce the triplet $\{\mathcal{V}^2, \mathcal{W}^2, \mathcal{X}^2\}$. Naturally, the Hilbert space $\mathcal{W}^2$ is endowed with the inner product

$$(f, g)_{\mathcal{W}^2} = (f_1, g_1)_{\mathcal{W}} + (f_2, g_2)_{\mathcal{W}} \quad \forall \ \ f = (f_1, f_2) \in \mathcal{W}^2, \quad g = (g_1, g_2) \in \mathcal{W}^2.$$

In an analogous way, inner product can be defined for $\mathcal{V}^2$ and $\mathcal{X}^2$. A bilinear form $\langle , \rangle : \mathcal{V}^2 \times \mathcal{X}^2 \to \mathbb{R}$ is defined as

$$\langle f, g \rangle = -\int_{-\pi}^{\pi} f_{1,xx} g_1\, dx - \int_{-\pi}^{\pi} f_{2,xx} g_2\, dx. \tag{2.1}$$

For all $f = (f_1, f_2) \in \mathcal{V}^2$ and $g = (g_1, g_2) \in \mathcal{W}^2$, we also have

$$\langle f, g \rangle = (f_{1,x}, g_{1,x}) + (f_{2,x}, g_{2,x}).$$

**Definition 2.1.** A family $\{\mathcal{Z}, \mathcal{H}, \mathcal{Y}\}$ of three real separable Banach spaces is called an admissible triplet if the following conditions hold:
(i) The inclusions $\mathcal{Z} \subset \mathcal{H} \subset \mathcal{Y}$ are continuous and dense.
(ii) $\mathcal{H}$ is a Hilbert space endowed with inner product $(,)_{\mathcal{H}}$ and norm $\|\|_{\mathcal{H}} = (,)_{\mathcal{H}}^{\frac{1}{2}}$.
(iii) There is a continuous non-degenerate bilinear form on $\mathcal{Z} \times \mathcal{Y}$, denoted by $\langle , \rangle$, such that

$$\langle v, u \rangle = (v, u)_{\mathcal{H}}, \quad \text{for} \quad v \in \mathcal{Z} \quad \text{and} \quad u \in \mathcal{H}. \tag{2.2}$$

Denote $C_w$ by the space of functions with weak continuity and $C_w^1$ the space of functions with weak differentiability.

An abstract theorem of existence of Kato-Lai [17] is stated as follows.

**Theorem 2.2.** *Let $\{\mathcal{Z}, \mathcal{H}, \mathcal{Y}\}$ be an admissible triplet. Let $A : \mathcal{H} \to \mathcal{Y}$ be a weakly continuous map such that*

$$\langle v, A(v)\rangle \geq -\beta(\|v\|_{\mathcal{H}}^2), \quad \forall \quad v \in \mathcal{Z} \tag{2.3}$$

*where $\beta(r) \geq 0$ is a monotone increasing function of $r \geq 0$. Then for any $u_0 \in \mathcal{H}$, there exists a time $T > 0$ such that the Cauchy problem*

$$u_t + A(u) = 0, \quad u(0, x) = u_0$$

*has a solution $u(t, x)$ on $[0, T]$ satisfying*

$$u \in C_w([0, T]; \mathcal{H}) \cap C_w^1([0, T]; \mathcal{Y}).$$

*Moreover,* $\sup_{0<t<T} \|u(t)\|_H$ *depends only on $T$, $\beta$ and $\|u_0\|_{\mathcal{H}}$.*

In order to prove the existence part of Theorem 1.1, the Kato-Lai theorem will be applied to system (1.12) with the admissible triplet $\{\mathcal{V}^2, \mathcal{W}^2, \mathcal{X}^2\}$.

2.2. **Properties of Hilbert transform.** The Hilbert transform has the following simple properties

$$H(cf) = cHf, \quad \text{for a constant} \quad c,$$
$$H \sin(kx) = -\cos(kx), \quad H \cos(kx) = \sin(kx).$$

And more generally, we have

$$H \sin(kx + \theta) = -\cos(kx + \theta), \quad H \cos(kx + \theta) = \sin(kx + \theta).$$

For any periodic function $f$, the mean value of its Hilbert transform is zero, that is

$$\int_{-\pi}^{\pi} Hf \, dx = 0. \tag{2.4}$$

**Lemma 2.3.** [22] *The Hilbert transform $H$ is a bounded linear operator from space $L^p$ to $L^p$ with $1 < p < \infty$ and*

$$\|Hf\|_{L^p} \leq C_p \|f\|_{L^p} \tag{2.5}$$

*for a constant $C_p > 0$ depending on $p$.*

## 3. Local existence

This section is devoted to a proof of Theorem 1.1. The proof includes three steps: (i) establishing the local existence of a solution by employing Theorem 2.2; (ii) showing the uniqueness of solution by a rather standard argument; (iii) justifying the strong continuity which is a consequence of the uniqueness and the time-reversible property of system (1.12).

**Proof of Theorem 1.1:** Denote $u = (\Omega, \omega)$, $q = (p, m)$, and naturally $q_x = Hu = (H\Omega, H\omega)$. Denote $A(u) = (A_1(u), A_2(u))$ with

$$A_1(u) = am\Omega_x - \omega p_x, \quad A_2(u) = ap\omega_x - \Omega m_x.$$

Thus, system (1.12) can be written as

$$u_t + A(u) = 0.$$

It is obvious that the family $\{\mathcal{V}^2, \mathcal{W}^2, \mathcal{X}^2\}$ is an admissible triplet associated with the bilinear form $\langle , \rangle$ defined in (2.1). To apply Theorem 2.2, we will need to show that the operator $A$ maps $\mathcal{W}^2$ into $\mathcal{X}^2$ continuously and it satisfies (2.3). Indeed, for any $u = (\Omega, \omega) \in \mathcal{W}^2$ with $q = (p, m) \in \mathcal{V}^2$, we have

$$\begin{aligned}
\|A(u)\|_{\mathcal{X}^2} &= \left(\|am\Omega_x - \omega p_x\|_{L^2}^2 + \|ap\omega_x - \Omega m_x\|_{L^2}^2\right)^{\frac{1}{2}} \\
&\le \|am\Omega_x - \omega p_x\|_{L^2} + \|ap\omega_x - \Omega m_x\|_{L^2} \\
&\le |a|\|m\|_{L^\infty}\|\Omega_x\|_{L^2} + \|\omega\|_{L^\infty}\|p_x\|_{L^2} \\
&\quad + |a|\|p\|_{L^\infty}\|\omega_x\|_{L^2} + \|\Omega\|_{L^\infty}\|m_x\|_{L^2} \\
&\le c_0\left(|a|\|m_x\|_{L^2}\|\Omega_x\|_{L^2} + \|\omega\|_{\mathcal{H}^1}\|p_x\|_{L^2}\right. \\
&\quad \left. + |a|\|p_x\|_{L^2}\|\omega_x\|_{L^2} + \|\Omega\|_{\mathcal{H}^1}\|m_x\|_{L^2}\right) \\
&\le c_0\left(|a|+1\right)\left(\|H\omega\|_{L^2}\|\Omega\|_{\mathcal{H}^1} + \|\omega\|_{\mathcal{H}^1}\|H\Omega\|_{L^2}\right) \\
&\le c_0\left(|a|+1\right)\left(\|\omega\|_{L^2}\|\Omega\|_{\mathcal{H}^1} + \|\omega\|_{\mathcal{H}^1}\|\Omega\|_{L^2}\right)
\end{aligned}$$

where we have used the Hölder inequality, Sobolev inequality, the fact that $p$ and $m$ have zero mean, and the property (2.5). It follows that $A$ maps $\mathcal{W}^2$ into $\mathcal{X}^2$. On the other hand, for any $u_1 = (\Omega_1, \omega_1) \in \mathcal{W}^2$ with $q_1 = (p_1, m_1) \in \mathcal{V}^2$ and $u_2 = (\Omega_2, \omega_2) \in \mathcal{W}^2$ with $q_2 = (p_2, m_2) \in \mathcal{V}^2$, we deduce

$$\begin{aligned}
\|A(u_1) - A(u_2)\|_{\mathcal{X}^2} &= \Big(\|(am_1\Omega_{1,x} - \omega_1 p_{1,x}) - (am_2\Omega_{2,x} - \omega_2 p_{2,x})\|_{L^2}^2 \\
&\qquad + \|(ap_1\omega_{1,x} - \Omega_1 m_{1,x}) - (ap_2\omega_{2,x} - \Omega_2 m_{2,x})\|_{L^2}^2\Big)^{\frac{1}{2}} \\
&\le \|(am_1\Omega_{1,x} - \omega_1 p_{1,x}) - (am_2\Omega_{2,x} - \omega_2 p_{2,x})\|_{L^2} \\
&\quad + \|(ap_1\omega_{1,x} - \Omega_1 m_{1,x}) - (ap_2\omega_{2,x} - \Omega_2 m_{2,x})\|_{L^2}.
\end{aligned} \tag{3.1}$$

Applying the Hölder inequality, Sobolev inequality, and (2.5) leads to

$$\begin{aligned}
&\|(am_1\Omega_{1,x} - \omega_1 p_{1,x}) - (am_2\Omega_{2,x} - \omega_2 p_{2,x})\|_{L^2} \\
&\le |a|\|\Omega_{1,x}\|_{L^2}\|m_1 - m_2\|_{L^\infty} + |a|\|\Omega_{1,x} - \Omega_{2,x}\|_{L^2}\|m_2\|_{L^\infty} \\
&\quad + \|\omega_2\|_{L^\infty}\|p_{2,x} - p_{1,x}\|_{L^2} + |a|\|\omega_2 - \omega_1\|_{L^\infty}\|p_{1,x}\|_{L^2} \\
&\le c_0|a|\|\Omega_{1,x}\|_{L^2}\|m_{1,x} - m_{2,x}\|_{L^2} + c_0|a|\|\Omega_{1,x} - \Omega_{2,x}\|_{L^2}\|m_{2,x}\|_{L^2} \\
&\quad + c_0\|\omega_2\|_{\mathcal{H}^1}\|p_{2,x} - p_{1,x}\|_{L^2} + c_0|a|\|\omega_2 - \omega_1\|_{\mathcal{H}^1}\|p_{1,x}\|_{L^2} \\
&\le c_0(|a|+1)\left(\|\Omega_1\|_{\mathcal{H}^1} + \|\omega_2\|_{\mathcal{H}^1}\right)\left(\|\Omega_1 - \Omega_2\|_{\mathcal{H}^1} + \|\omega_1 - \omega_2\|_{\mathcal{H}^1}\right),
\end{aligned} \tag{3.2}$$

and similarly

$$\begin{aligned}
&\|(ap_1\omega_{1,x} - \Omega_1 m_{1,x}) - (ap_2\omega_{2,x} - \Omega_2 m_{2,x})\|_{L^2} \\
&\le c_0(|a|+1)\left(\|\Omega_1\|_{\mathcal{H}^1} + \|\omega_2\|_{\mathcal{H}^1}\right)\left(\|\Omega_1 - \Omega_2\|_{\mathcal{H}^1} + \|\omega_1 - \omega_2\|_{\mathcal{H}^1}\right).
\end{aligned} \tag{3.3}$$

The estimates (3.1)-(3.3) together indicate that $A : \mathcal{W}^2 \to \mathcal{X}^2$ is strongly continuous.

By the definition of the bilinear form in (2.1), we have for any $u = (\Omega, \omega) \in \mathcal{V}^2$

$$\begin{aligned}
\langle u, A(u)\rangle &= -\left(u_{xx}, A(u)\right) = \left(u_x, (A(u))_x\right) \\
&= \left(\Omega_x, (am\Omega_x - \omega p_x)_x\right) + \left(\omega_x, (ap\omega_x - \Omega m_x)_x\right) \\
&= \int_{-\pi}^{\pi} \Omega_x (am_x\Omega_x + am\Omega_{xx} - \omega_x p_x - \omega p_{xx})\, dx \\
&\quad + \int_{-\pi}^{\pi} \omega_x (ap_x\omega_x + ap\omega_{xx} - \Omega_x m_x - \Omega m_{xx})\, dx.
\end{aligned} \tag{3.4}$$

Note that $A(u) \in \mathcal{X}^2$ and (3.4) can be made rigorous through a standard approximating procedure. Applying integration by parts to the right hand side of (3.4), it has

$$a\int_{-\pi}^{\pi} m\Omega_x\Omega_{xx}\,dx = -a\int_{-\pi}^{\pi} m_x\Omega_x\Omega_x\,dx - a\int_{-\pi}^{\pi} m\Omega_{xx}\Omega_x\,dx.$$

Hence we conclude

$$a\int_{-\pi}^{\pi} m\Omega_x\Omega_{xx}\,dx = -\frac{a}{2}\int_{-\pi}^{\pi} m_x\Omega_x^2\,dx, \tag{3.5}$$

and similarly

$$a\int_{-\pi}^{\pi} p\omega_x\omega_{xx}\,dx = -\frac{a}{2}\int_{-\pi}^{\pi} p_x\omega_x^2\,dx. \tag{3.6}$$

Since $p_x = H\Omega$ and $m_x = H\omega$, combining (3.4)-(3.6) gives

$$\begin{aligned} \langle u, A(u)\rangle &= (u_x, (A(u))_x) \\ &= \frac{a}{2}\int_{-\pi}^{\pi}(H\omega)\Omega_x^2\,dx + \frac{a}{2}\int_{-\pi}^{\pi}(H\Omega)\omega_x^2\,dx \\ &\quad - \int_{-\pi}^{\pi}\Omega_x\omega_x H\Omega\,dx - \int_{-\pi}^{\pi}\omega\Omega_x H\Omega_x\,dx \\ &\quad - \int_{-\pi}^{\pi}\Omega_x\omega_x H\omega\,dx - \int_{-\pi}^{\pi}\Omega\omega_x H\omega_x\,dx. \end{aligned} \tag{3.7}$$

Applying Hölder's inequality, Sobolev's inequality, (2.4) and (2.5), we have

$$\begin{aligned} \left|\int_{-\pi}^{\pi} H\omega\Omega_x^2\,dx\right| &\le \|H\omega\|_{L^\infty}\|\Omega_x\|_{L^2}^2 \\ &\le c_0\|H\omega_x\|_{L^2}\|\Omega_x\|_{L^2}^2 \\ &\le c_0\|\omega_x\|_{L^2}\|\Omega_x\|_{L^2}^2, \end{aligned} \tag{3.8}$$

and similarly

$$\begin{aligned} \left|\int_{-\pi}^{\pi}\Omega_x\omega_x H\Omega\,dx\right| + \left|\int_{-\pi}^{\pi}\omega\Omega_x H\Omega_x\,dx\right| &\le c_0\|\omega\|_{\mathcal{H}^1}\|\Omega_x\|_{L^2}^2, \\ \left|\int_{-\pi}^{\pi}(H\Omega)\omega_x^2\,dx\right| &\le c_0\|\omega_x\|_{L^2}^2\|\Omega_x\|_{L^2}, \\ \left|\int_{-\pi}^{\pi}\omega_x\Omega_x H\omega\,dx\right| + \left|\int_{-\pi}^{\pi}\omega_x\Omega H\omega_x\,dx\right| &\le c_0\|\omega_x\|_{L^2}^2\|\Omega\|_{\mathcal{H}^1}. \end{aligned} \tag{3.9}$$

Therefore, putting together (3.7)-(3.9), we deduce

$$\begin{aligned} |\langle u, A(u)\rangle| &\le c_0(|a|+1)\left(\|\omega\|_{\mathcal{H}^1}\|\Omega_x\|_{L^2}^2 + \|\omega_x\|_{L^2}^2\|\Omega\|_{\mathcal{H}^1}\right) \\ &\le c_0(|a|+1)\left(\|\omega\|_{\mathcal{H}^1} + \|\Omega\|_{\mathcal{H}^1}\right)^3. \end{aligned} \tag{3.10}$$

Hence, the operator $A$ satisfies (2.3) with $\beta(r) = c_0(|a|+1)r^{\frac{3}{2}}$. As a consequence, applying Theorem 2.2, we conclude that there exists a time $T > 0$ such that system (1.12) has a solution $(\Omega(t,x), \omega(t,x))$ on $[0,T]$ satisfying

$$\Omega, \omega \in C_w([0,T];\mathcal{W}) \cap C_w^1([0,T];\mathcal{X}).$$

Next we show the uniqueness of solution to (1.12). Let $u_1 = (\Omega_1, \omega_1)$ be a solution to (1.12) with initial data $u_0 = (\Omega_0, \omega_0)$. Let $q_1 = (p_1, m_1)$ such that $p_{1,x} = H\Omega_1$ and $m_{1,x} = H\omega_1$. Let $u_2 = (\Omega_2, \omega_2)$ be another solution to (1.12) with the same initial data $(\Omega_0, \omega_0)$ and associated with $q_2 = (p_2, m_2)$. Since both $(\Omega_1, \omega_1)$ and $(\Omega_2, \omega_2)$ satisfy (1.12), we are able to show that (details omitted)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Omega_1(t)-\Omega_2(t)\|_{L^2}^2+\|\omega_1(t)-\omega_2(t)\|_{L^2}^2\right)\\
\le & c_0(|a|+1)\max_{0\le t\le T}\left(\|\Omega_1\|_{\mathcal{H}^1}+\|\Omega_2\|_{\mathcal{H}^1}+\|\omega_1\|_{\mathcal{H}^1}+\|\omega_2\|_{\mathcal{H}^1}\right)\\
&\cdot\left(\|\Omega_1(t)-\Omega_2(t)\|_{L^2}^2+\|\omega_1(t)-\omega_2(t)\|_{L^2}^2\right).
\end{aligned}\tag{3.11}$$

Thus, uniqueness follows from (3.11) and Grönwall's inequality.

Strong continuity in time follows from the uniqueness and the fact that system (1.12) is time-reversible. Indeed, it follows from (3.10) that

$$\|\Omega_x(t)\|_{L^2}+\|\omega_x(t)\|_{L^2}\to\|\Omega_{0,x}\|_{L^2}+\|\omega_{0,x}\|_{L^2} \quad \text{as} \quad t\to 0.$$

Hence, we know

$$\Omega(t)\to\Omega_0, \quad \omega(t)\to\omega_0 \quad \text{strongly in} \quad \mathcal{H}^1 \quad \text{as} \quad t\to 0.$$

As a consequence of uniqueness, $\Omega$ and $\omega$ are strongly right-continuous. In addition, the property of time-reversibility implies that $\Omega$ and $\omega$ are strongly left-continuous as well.

□

## 4. Regularity criterion

In this section, we prove Theorem 1.2 and the higher regularity result in Theorem 1.3.

**Proof of Theorem 1.2:** In view of the local existence theorem, we just need to show that the $\mathcal{H}^1$ norm of $\Omega(t)$ and $\omega(t)$ remains bounded as $t\to T$ under condition (1.17).

Assume $(\Omega, \omega)$ is a solution of (1.12) on $[0, T)$. We note that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Omega_x\|_{L^2}^2+\|\omega_x\|_{L^2}^2\right)\\
=&(\Omega_x,\Omega_{tx})+(\omega_x,\omega_{tx})\\
=&(\Omega_x,-(am\Omega_x-\omega p_x)_x)+(\omega_x,-(ap\omega_x-\Omega m_x)_x)\\
=&-\frac{a}{2}\int_{-\pi}^{\pi}(H\omega)\Omega_x^2\,dx-\frac{a}{2}\int_{-\pi}^{\pi}(H\Omega)\omega_x^2\,dx\\
&+\int_{-\pi}^{\pi}\Omega_x\omega_x H\Omega\,dx+\int_{-\pi}^{\pi}\omega\Omega_x H\Omega_x\,dx\\
&+\int_{-\pi}^{\pi}\Omega_x\omega_x H\omega\,dx+\int_{-\pi}^{\pi}\Omega\omega_x H\omega_x\,dx
\end{aligned}\tag{4.1}$$

where we used (3.7) in the last step. Applying the identities

$$(v,u)=(Hv,Hu), \quad H(vHv)=\frac{1}{2}\left((Hv)^2-v^2\right),$$

we infer

$$\begin{aligned}
\int_{-\pi}^{\pi}\omega\Omega_x H\Omega_x\,dx&=\int_{-\pi}^{\pi}(H\omega)H(\Omega_x H\Omega_x)\,dx\\
&=\frac{1}{2}\int_{-\pi}^{\pi}(H\omega)\left((H\Omega_x)^2-(\Omega_x)^2\right)\,dx,
\end{aligned}\tag{4.2}$$

$$\int_{-\pi}^{\pi} \Omega\omega_x H\omega_x \, dx = \int_{-\pi}^{\pi} (H\Omega) H(\omega_x H\omega_x) \, dx \\ = \frac{1}{2} \int_{-\pi}^{\pi} (H\Omega) \left( (H\omega_x)^2 - (\omega_x)^2 \right) \, dx. \tag{4.3}$$

Combining (4.1)-(4.3), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \|\Omega_x\|_{L^2}^2 + \|\omega_x\|_{L^2}^2 \right) \\ = & - \frac{a+1}{2} \int_{-\pi}^{\pi} (H\omega)\Omega_x^2 \, dx - \frac{a+1}{2} \int_{-\pi}^{\pi} (H\Omega)\omega_x^2 \, dx \\ & + \frac{1}{2} \int_{-\pi}^{\pi} (H\omega)(H\Omega_x)^2 \, dx + \frac{1}{2} \int_{-\pi}^{\pi} (H\Omega)(H\omega_x)^2 \, dx \\ & + \int_{-\pi}^{\pi} \Omega_x \omega_x \left( H\Omega + H\omega \right) \, dx \\ \leq & \frac{|a+1|}{2} \|H\omega\|_{L^\infty} \|\Omega_x\|_{L^2}^2 + \frac{|a+1|}{2} \|H\Omega\|_{L^\infty} \|\omega_x\|_{L^2}^2 \\ & + \frac{1}{2} \|H\omega\|_{L^\infty} \|\Omega_x\|_{L^2}^2 + \frac{1}{2} \|H\Omega\|_{L^\infty} \|\omega_x\|_{L^2}^2 \\ & + \|H\Omega + H\omega\|_{L^\infty} \|\Omega_x\|_{L^2} \|\omega_x\|_{L^2} \\ \leq & \ c_0(|a|+1) \left( \|H\Omega\|_{L^\infty} + \|H\omega\|_{L^\infty} \right) \left( \|\Omega_x\|_{L^2}^2 + \|\omega_x\|_{L^2}^2 \right) \end{aligned} \tag{4.4}$$

for a constant $c_0 > 0$. It follows from Grönwall's inequality that

$$\begin{aligned} & \left( \|\Omega_x(t)\|_{L^2}^2 + \|\omega_x(t)\|_{L^2}^2 \right) \\ \leq & \left( \|\Omega_x(0)\|_{L^2}^2 + \|\omega_x(0)\|_{L^2}^2 \right) \exp \left\{ 2c_0(|a|+1) \int_0^t \left( \|H\Omega(\tau)\|_{L^\infty} + \|H\omega(\tau)\|_{L^\infty} \right) \, d\tau \right\}. \end{aligned}$$

Thus, the statement of the theorem is justified.

□

**Proof of Theorem 1.3:** The statement can be established through standard energy method. We only deal with the case of $n = 2$ and obtain the a priori estimate for $\|\Omega(t)\|_{\mathcal{H}^2}$ and $\|\omega(t)\|_{\mathcal{H}^2}$. Formally, differentiating the equations of (1.12) twice in space yields

$$\begin{aligned} \Omega_{txx} = & - 2am_x\Omega_{xx} + 2\omega_x p_{xx} - am_{xx}\Omega_x \\ & + \omega_{xx} p_x - am\Omega_{xxx} + \omega p_{xxx}, \\ \omega_{txx} = & - 2ap_x\omega_{xx} + 2\Omega_x m_{xx} - ap_{xx}\omega_x \\ & + \Omega_{xx} m_x - ap\omega_{xxx} + \Omega m_{xxx}. \end{aligned} \tag{4.5}$$

Taking the inner product of the first equation with $\Omega_{xx}$ and the second one with $\omega_{xx}$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \|\Omega_{xx}\|_{L^2}^2 + \|\omega_{xx}\|_{L^2}^2 \right) \\ = & - 2a(\Omega_{xx}, m_x\Omega_{xx}) + 2(\Omega_{xx}, \omega_x p_{xx}) - a(\Omega_{xx}, m_{xx}\Omega_x) \\ & + (\Omega_{xx}, \omega_{xx} p_x) - a(\Omega_{xx}, m\Omega_{xxx}) + (\Omega_{xx}, \omega p_{xxx}) \\ & - 2a(\omega_{xx}, p_x\omega_{xx}) + (\omega_{xx}, \Omega_x m_{xx}) - a(\omega_{xx}, p_{xx}\omega_x) \\ & + (\omega_{xx}, \Omega_{xx} m_x) - a(\omega_{xx}, p\omega_{xxx}) + (\omega_{xx}, \Omega m_{xxx}). \end{aligned} \tag{4.6}$$

Notice that, by integration by parts,

$$-a(\Omega_{xx}, m\Omega_{xxx}) = a(\Omega_{xxx}, m\Omega_{xx}) + a(\Omega_{xx}, m_x\Omega_{xx})$$

which implies

$$-a(\Omega_{xx}, m\Omega_{xxx}) = \frac{a}{2}(\Omega_{xx}, m_x\Omega_{xx}).$$

Similarly, we have

$$-a(\omega_{xx}, p\omega_{xxx}) = \frac{a}{2}(\omega_{xx}, p_x\omega_{xx}).$$

Applying Hölder's inequality, the Hilbert transform boundedness on $L^p$, it follows

$$\begin{aligned}|(\Omega_{xx}, m_x\Omega_{xx})| &= |(\Omega_{xx}, (H\omega)\Omega_{xx})| \\ &\le c_0\|H\omega\|_{L^\infty}\|\Omega_{xx}\|_{L^2}^2 \\ &\le c_0\|\omega_x\|_{L^2}\|\Omega_{xx}\|_{L^2}^2,\end{aligned}$$

and similarly

$$|(\omega_{xx}, p_x\omega_{xx})| = |(\Omega_{xx}, (H\omega)\Omega_{xx})| \le c_0\|\Omega_x\|_{L^2}\|\omega_{xx}\|_{L^2}^2.$$

We estimate $(\Omega_{xx}, \omega_x p_{xx})$ as

$$\begin{aligned}|(\Omega_{xx}, \omega_x p_{xx})| &= |(\Omega_{xx}, \omega_x H\Omega_x)| \\ &\le c_0\|\Omega_{xx}\|_{L^2}\|\omega_x\|_{L^4}\|H\Omega_x\|_{L^4} \\ &\le c_0\|\Omega_{xx}\|_{L^2}\|\omega_x\|_{L^2}^{\frac{3}{4}}\|\omega_{xx}\|_{L^2}^{\frac{1}{4}}\|H\Omega_x\|_{L^2}^{\frac{3}{4}}\|H\Omega_{xx}\|_{L^2}^{\frac{1}{4}} \\ &\le c_0\|\Omega_{xx}\|_{L^2}^{\frac{5}{4}}\|H\Omega_x\|_{L^2}^{\frac{3}{4}}\|\omega_x\|_{L^2}^{\frac{3}{4}}\|\omega_{xx}\|_{L^2}^{\frac{1}{4}} \\ &\le c_0\|\Omega_{xx}\|_{L^2}^{\frac{15}{8}}\|H\Omega_x\|_{L^2}^{\frac{9}{8}} + c_0\|\omega_x\|_{L^2}^{\frac{9}{4}}\|\omega_{xx}\|_{L^2}^{\frac{3}{4}} \\ &\le c_0\|\Omega_x\|_{L^2}\|\Omega_{xx}\|_{L^2}^2 + c_0\|\omega_x\|_{L^2}\|\omega_{xx}\|_{L^2}^2,\end{aligned}$$

where we used the inequalities of Hölder, Galiardo-Nirenberg and Young, and the facts that $\|\omega_x\|_{L^2} \le \|\omega_{xx}\|_{L^2}$ and $\|\Omega_x\|_{L^2} \le \|\Omega_{xx}\|_{L^2}$. Other terms on the right hand side of (4.6) can be handled similarly as above. We conclude

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\Omega_{xx}\|_{L^2}^2 + \|\omega_{xx}\|_{L^2}^2\right) \\ &\le c_0(1+|a|)\left(\|\Omega_x\|_{L^2} + \|\omega_x\|_{L^2}\right)\left(\|\Omega_{xx}\|_{L^2}^2 + \|\omega_{xx}\|_{L^2}^2\right),\end{aligned}$$

which immediately gives, by Grönwall's inequality

$$\begin{aligned}&\left(\|\Omega_{xx}(t)\|_{L^2}^2 + \|\omega_{xx}(t)\|_{L^2}^2\right) \\ &\le \left(\|\Omega_{xx}(0)\|_{L^2}^2 + \|\omega_{xx}(0)\|_{L^2}^2\right) e^{\int_0^t 2c_0(1+|a|)\left(\|\Omega_x(\tau)\|_{L^2} + \|\omega_x(\tau)\|_{L^2}\right) d\tau}.\end{aligned} \tag{4.7}$$

Combining (4.7) with the assumption that $\Omega, \omega \in C([0,T]; H^1)$, it follows that

$$\sup_{0\le t\le T}\left(\|\Omega(t)\|_{H^2} + \|\omega(t)\|_{H^2}\right) < \infty.$$

$\square$

## 5. Pure transport case

In this section we prove Theorem 1.4. According to Theorem 1.1, there exists a unique solution $(\Omega(t), \omega(t))$ of (1.16) on $[0,T]$ for some $T>0$. In view of Theorem 1.2, in order to show the global existence, it is sufficient to prove

$$\int_0^T \left(\|H\Omega(t)\|_{L^\infty} + \|H\omega(t)\|_{L^\infty}\right) dt < \infty \quad \text{for all} \quad T > 0.$$

On the other hand, due to the boundedness of Hilbert transform, we have

$$\|H\Omega(t)\|_{L^\infty} \le c_0 \|H\Omega(t)\|_{C^\beta} \le c_0 \|\Omega(t)\|_{C^\beta},$$
$$\|H\omega(t)\|_{L^\infty} \le c_0 \|H\omega(t)\|_{C^\beta} \le c_0 \|\omega(t)\|_{C^\beta}$$

for $\beta \in (0,1)$. As a consequence, we only need to prove:

**Proposition 5.1.** *Assume $\Omega_0, \omega_0 \in H^1(S^1)$. Let $(\Omega(t), \omega(t))$ be the solution of (1.16) with initial data $(\Omega_0, \omega_0)$ on $[0,T]$. Then there exists $\beta_1, \beta_2 \in (0,1)$ such that*

$$\sup_{0\le t\le T} \left(\|\omega(t)\|_{C^{\beta_1}} + \|\Omega(t)\|_{C^{\beta_2}}\right) < \infty. \tag{5.1}$$

**Proof:** Recall the equations satisfied by $(\Omega, \omega)$,

$$\Omega_t + m\Omega_x = 0,$$
$$\omega_t + p\omega_x = 0.$$

Consider the characteristics $X_t(x)$ and $Y_t(x)$ satisfying

$$\frac{d}{dt} X_t = p(t, X_t(\xi)), \quad X_0(\xi) = \xi, \tag{5.2}$$

$$\frac{d}{dt} Y_t = m(t, Y_t(\xi)), \quad Y_0(\xi) = \xi, \tag{5.3}$$

such that

$$\Omega(t, Y_t(x)) = \Omega_0(x), \quad \omega(t, X_t(x)) = \omega_0(x). \tag{5.4}$$

We notice that there exists a unique solution $X_t(x)$ to the Cauchy problem (5.2) and a unique solution $Y_t(x)$ to (5.3). Indeed, since $\Omega(t), \omega(t) \in H^1(S^1) \subset C^{\frac{1}{2}}(S^1)$ and the Hilbert transform is bounded on $C^\beta$, we have

$$\|p(t)\|_{C^{1,\frac{1}{2}}} \le c_0 \|\Omega(t)\|_{C^{1,\frac{1}{2}}} < \infty,$$
$$\|m(t)\|_{C^{1,\frac{1}{2}}} \le c_0 \|\omega(t)\|_{C^{1,\frac{1}{2}}} < \infty.$$

Hence, $p$ and $m$ are Lipschitz in time. Thus, the standard ordinary differential equation theory implies existence and uniqueness of solution to (5.2) and (5.3).

Denote the inverse (backward) trajectory of $X_t(x)$ and $Y_t(x)$ by $q_1(t,x) = X_t^{-1}(x)$ and $q_2(t,x) = Y_t^{-1}(x)$, respectively. Note that $q_1(t,x)$ and $q_2(t,x)$ satisfy respectively,

$$\partial_t q_1 = -p(t, q_1(t,x)), \quad q_1(0,x) = x, \tag{5.5}$$

$$\partial_t q_2 = -m(t, q_2(t,x)), \quad q_2(0,x) = x. \tag{5.6}$$

We claim that $p$ and $m$ satisfy the estimate

$$\begin{aligned} |p(t,x) - p(t,y)| &\le F(|x-y|), \quad x,y \in [-\pi,\pi], \\ |m(t,x) - m(t,y)| &\le G(|x-y|), \quad x,y \in [-\pi,\pi], \end{aligned} \tag{5.7}$$

with

$$F(s) = \begin{cases} c_0\|\Omega_0\|_{L^\infty} s(1-\log s), & 0 \le s \le 1, \\ c_0\|\Omega_0\|_{L^\infty}, & s > 1, \end{cases} \tag{5.8}$$

and

$$G(s) = \begin{cases} c_0\|\omega_0\|_{L^\infty} s(1-\log s), & 0 \le s \le 1, \\ c_0\|\omega_0\|_{L^\infty}, & s > 1, \end{cases} \tag{5.9}$$

for a universal constant $c_0 > 0$. We only need to show one of them, for instance, the estimate for $p$. Recall that, by (1.13)

$$p_x(t,x) = H\Omega = \frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} \Omega(t,y) \cot\left(\frac{x-y}{2}\right) dy.$$

Hence, we have

$$p(t,x) = \frac{1}{\pi} P.V. \int_{-\pi}^{\pi} \Omega(t,y) \log\left|\sin\left(\frac{x-y}{2}\right)\right| dy.$$

Without loss of generality, we take $x, y \in (-\pi, \pi)$ such that $-\pi < x < y < \pi$ and $\delta = y - x$. We split the interval $[-\pi, \pi]$ into subintervals

$$I_1 = [-\pi, x - \frac{\delta}{2}), \quad I_2 = [x - \frac{\delta}{2}, x + \frac{\delta}{2}), \quad I_3 = [x + \frac{\delta}{2}, y + \frac{\delta}{2}), \quad I_4 = [y + \frac{\delta}{2}, \pi].$$

In the case of $x - \frac{\delta}{2} \le -\pi$ or $y + \frac{\delta}{2} > \pi$, we treat $I_1$ or $I_4$ as an empty set. In order to prove the estimate on $p$ in (5.7), we proceed as

$$\begin{aligned} |p(t,x) - p(t,y)| &= \left| \frac{1}{\pi} P.V. \int_{-\pi}^{\pi} \Omega(t,z) \left( \log\left|\sin\left(\frac{x-z}{2}\right)\right| - \log\left|\sin\left(\frac{y-z}{2}\right)\right| \right) dz \right| \\ &\le \left| \frac{1}{\pi} P.V. \int_{I_1} \Omega(t,z) \left( \log\left|\sin\left(\frac{x-z}{2}\right)\right| - \log\left|\sin\left(\frac{y-z}{2}\right)\right| \right) dz \right| \\ &\quad + \left| \frac{1}{\pi} P.V. \int_{I_2} \cdots dz \right| + \left| \frac{1}{\pi} P.V. \int_{I_3} \cdots dz \right| + \left| \frac{1}{\pi} P.V. \int_{I_4} \cdots dz \right|. \end{aligned}$$

The second term on the right hand side can be estimated as

$$\begin{aligned} &\left| \frac{1}{\pi} P.V. \int_{I_2} \Omega(t,z) \left( \log\left|\sin\left(\frac{x-z}{2}\right)\right| - \log\left|\sin\left(\frac{y-z}{2}\right)\right| \right) dz \right| \\ \le & c_0 \|\Omega(t)\|_{L^\infty} P.V. \int_{x-\frac{\delta}{2}}^{x-\frac{\delta}{2}} |\log|x-z|| + |\log|y-z|| \, dz \\ \le & c_0 \|\Omega(t)\|_{L^\infty} \delta \left(1 + |\log \delta|\right) \\ \le & \begin{cases} c_0\|\Omega_0\|_{L^\infty} \delta\left(1 - \log\delta\right), & 0 < \delta < 1 \\ c_0\|\Omega_0\|_{L^\infty}, & \delta \ge 1. \end{cases} \end{aligned}$$

The integrals on $I_1$, $I_3$ and $I_4$ can be estimated similarly. The estimate for $m$ in (5.7) can be established in an analogous way.

In view of (5.2)-(5.3) and (5.7), we have

$$\begin{aligned} \partial_t |q_1(t,x) - q_1(t,y)| &\le F(|q_1(t,x) - q_1(t,y)|), \\ \partial_t |q_2(t,x) - q_2(t,y)| &\le G(|q_2(t,x) - q_2(t,y)|). \end{aligned} \tag{5.10}$$

Denote $\beta_1(t) = e^{-c_0\|\Omega_0\|_{L^\infty} t}$ and $\beta_2(t) = e^{-c_0\|\omega_0\|_{L^\infty} t}$. For fixed $x$ and $y$ with $|x-y|<1$, define

$$z_1(t) = \begin{cases} |x-y|^{\beta_1(t)} e^{1-\beta_1(t)}, & 0 \le t < t_0, \\ 1 + c_0\|\Omega_0\|_{L^\infty}(t-t_0), & t \ge t_0, \end{cases}$$

where $t_0$ is such that $|x-y|^{\beta_1(t_0)} e^{1-\beta_1(t_0)} = 1$. Note that $\beta_1(0) = 1$ and $z_1(0) = |x-y| < 1$. Hence, $z_1(t)$ is well-defined on $[0,\infty)$. One can verify that $z_1(t)$ is the solution of the differential equation

$$\partial_t z = F(z), \quad z(0) = |x-y|.$$

Combining with the first inequality of (5.10), we conclude

$$|q_1(t,x) - q_1(t,y)| \le z_1(t). \tag{5.11}$$

Similarly, we define

$$z_2(t) = \begin{cases} |x-y|^{\beta_2(t)} e^{1-\beta_2(t)}, & 0 \le t < t_0, \\ 1 + c_0\|\omega_0\|_{L^\infty}(t-t_0), & t \ge t_0, \end{cases}$$

with $t_0$ such that $|x-y|^{\beta_2(t_0)} e^{1-\beta_2(t_0)} = 1$. Analogously, using the second inequality of (5.10), we infer

$$|q_2(t,x) - q_2(t,y)| \le z_2(t). \tag{5.12}$$

We are ready to show (5.1). Noticing that $\omega(t,x) = \omega(0, X_t^{-1}(x))$, we deduce

$$\begin{aligned} |\omega(t,x) - \omega(t,y)| &= \left|\omega(0, X_t^{-1}(x)) - \omega(0, X_t^{-1}(y))\right| \\ &= \left| \int_{X_t^{-1}(y)}^{X_t^{-1}(x)} \omega_{0,x}(\zeta)\, d\zeta \right| \\ &\le c_0 \|\omega_{0,x}\|_{L^2} \left| X_t^{-1}(x) - X_t^{-1}(y) \right|^{\frac{1}{2}} \\ &\le c_0 \|\omega_{0,x}\|_{L^2} \left| q_1(t,x) - q_1(t,y) \right|^{\frac{1}{2}} \end{aligned}$$

where mean value theorem and Hölder's inequality were applied. As a consequence, we conclude

$$\sup_{0\le t\le T} \|\omega(t)\|_{C^{\beta_1}} < \infty.$$

thanks to (5.11). Analogously, we can show

$$\sup_{0\le t\le T} \|\Omega(t)\|_{C^{\beta_2}} < \infty.$$

It completes the proof of the proposition.

□

## 6. Numerical simulations

In this section, we perform some numerical study for the 1D model (1.12) of MHD. For convenience, we recall (1.12) here,

$$\begin{aligned} \Omega_t + am\Omega_x - \omega p_x &= 0, \\ \omega_t + ap\omega_x - \Omega m_x &= 0, \qquad x \in [-\pi,\pi] \\ p_x = H\Omega, \quad m_x &= H\omega, \end{aligned} \tag{6.1}$$

and the Hilbert transform for a periodic function

$$Hf = \frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} f(y) \cot\left(\frac{x-y}{2}\right) dy.$$

As mentioned earlier, in order for $p$ and $m$ to be uniquely defined, we can choose the gauge and set them to have either zero mean over the interval $[-\pi, \pi]$ or zero point value at a fixed point, e.g., $p(x_0, t) = m(x_0, t) = 0$ for some $x_0$, see [16].

We use a Fourier-collocation spectral method for the spatial approximation and a five stage fourth order low storage Runge-Kutta method for time discretization. An exponential type filter is used for stabilization of the spectral method, see [13]. For a periodic function $f(x)$, its Hilbert transform can be approximated in spectral method via the following formula:

$$\widehat{Hf}(k) = -i\text{sgn}(k)\hat{f}(k),$$

where $\hat{f}(k)$ are coefficients in Fourier series of $f(x)$, see [14, 21]. Similarly, for periodic functions $p(x)$ and $\Omega(x)$, the equation $p_x = H\Omega$ can be approximated in spectral method through the relation

$$ik\hat{p}(k) = -i\text{sgn}(k)\hat{\Omega}(k).$$

6.1. **Numerical results for the 1D model of MHD.** One can check that, for arbitrary constants $A_1$, $A_2$, $\theta_1$, $\theta_2$ and $k$

$$\Omega(x) = A_1 \sin(kx + \theta_1), \quad \omega(x) = A_2 \sin(kx + \theta_2)$$

and

$$\Omega(x) = A_1 \cos(kx + \theta_1), \quad \omega(x) = A_2 \cos(kx + \theta_2)$$

are steady states of system (6.1). Thus, we choose to consider the following initial conditions composed of steady states with possible perturbations

$$\Omega_0 = \sin(x) + \cos(4x) + 5, \quad \omega_0 = \sin(2x) + 2, \tag{6.2}$$

$$\Omega_0 = \sin(x) + \sin(4x) + 0.05, \quad \omega_0 = \sin(2x) + 0.02, \tag{6.3}$$

$$\Omega_0 = \omega_0 = -\frac{4}{3}\left(\sin x + \frac{1}{2}\sin(2x)\right). \tag{6.4}$$

We conduct simulations for (6.1) with initial data (6.2)-(6.4) and various values of $a$: $a = 1$, $a = \frac{1}{2}$, $a = 0$, $a = -1$ and $a = -2$. In the computation, we take $N = 12800$ points in the Fourier-collocation spectral method. The outcome indicates that for some data and value of $a$, solutions are likely regular, while for some data and $a$ we observe the tendency of singularity formation. In particular, (i) the numerical solutions of (6.1) with data (6.2) and the values of $a = 1$, $a = 0$, $a = -1$ and $a = -2$ look regular; (ii) solutions of (6.1) with data (6.3) and $a = 1$, $a = 0$ tend to develop singularities; with the same data and $a = -1$, $a = -2$, solutions seem regular; (iii) solutions of (6.1) with data (6.4) and $a = 0.5$, $a = 0$ are likely to develop singularities. Rigorous analysis on the possible singularity formation scenarios is forthcoming in a follow-up work. More details on the numerical study are provided below.

6.1.1. *Solutions of (6.1) with the initial data (6.2).* Figure 1 shows the numerical results for the solution to (6.1) with data (6.2) and $a = 1$. The time evolution of $\Omega(t, x)$ and $\omega(t, x)$ are plotted in Figure 1(a) and Figure 1(b), respectively. One can see that $\Omega(t, x)$ and $\omega(t, x)$ are rather smooth. The first order derivative $\Omega_x$ shown in Figure 1(c) seems smooth as well, while $\omega_x$ illustrated in Figure 1 (d) develops some mild spines at time $t = 4$. However, we observe spines for the second derivatives $\Omega_{xx}$ and $\omega_{xx}$ at larger time in Figure 1(e) and (f). In particular, there is a notable spine near $x = 0$ at $t = 4$. Notice that

$$u = \frac{1}{2}(p + m), \quad B = \frac{1}{2}(p - m),$$

and hence

$$\begin{aligned} u_x = \frac{1}{2}(p_x + m_x) =& \frac{1}{2}H(\Omega + \omega), B_x = \frac{1}{2}(p_x - m_x) = \frac{1}{2}H(\Omega - \omega), \\ H\Omega = u_x &+ B_x, \quad H\omega = u_x - B_x. \end{aligned} \tag{6.5}$$

Figure 1(g) shows the time evolution of $\|H\Omega\|_{L^\infty} + \|H\omega\|_{L^\infty}$, while Figure 1(h) shows $\|u_x(t)\|_{L^\infty}$ and $\|B_x(t)\|_{L^\infty}$. We observe oscillations in these graphs and the amplitudes grow slowly in a linear manner. Combined with the regularity criterion (1.17), it seems that the solution starting with data (6.2) may not develop singularities in finite time.

The evolution of numerical solution to (6.1) with data (6.2) and $a = 0$ is illustrated in Figure 2. It is easy to notice that the behavior of the solution is similar to that in Figure 1. The solution of (6.1) with data (6.2) and $a = -1$ is plotted in Figure 3. One can see from Figure 3(a) and (b) that the solution is less regular compared to the solutions in Figure 1(a) and (b) and Figure 2(a) and (b). This suggests that the convection term with a negative sign causes the solution to behave more singularly. Nevertheless, 3(c) and 3(d) show that the amplitudes of $\|H\Omega\|_{L^\infty} + \|H\omega\|_{L^\infty}$, $\|u_x(t)\|_{L^\infty}$ and $\|B_x(t)\|_{L^\infty}$ grow faster than that in Figure 1(c) and (d), but remain in a linear growth. Thus one may speculate that the solution of system (6.1) with $a = -1$ starting from the initial data (6.2) do not develop singularity in finite time. We also note that the solution of (6.1) with data (6.2) and $a = -2$ shown in Figure 4 behaves similarly as the solution in Figure 3.

6.1.2. *Solutions of (6.1) with the initial data (6.3).* The behavior of the numerical solution of (6.1) with the initial data (6.3) and $a = 1$ is shown in Figure 5. We observe dramatic oscillations of $\Omega$ and $\omega$ near $x = 0$ in (a) and (b), and spines of derivatives near $x = 0$ in (c), (d), (e) and (f) with large amplitudes. Moreover, the norms $\|H\Omega(t)\|_\infty + \|H\omega(t)\|_\infty$ and $\|H(\Omega - \omega)(t)\|_\infty$ tend to grow fast as seen in (g) and (h). In the situation of $a = 0$ with the same initial data, the solution is more singular, see Figure 6. Spines with large amplitudes appear for $\Omega$ and $\omega$ shown in (a) and (b), and for their derivatives shown in (c), (d), (e) and (f). We also notice that the amplitudes are of much higher orders compared to (c), (d), (e) and (f) in Figure 5. In the end, the exponential like growth of the norms $\|H\Omega(t)\|_\infty + \|H\omega(t)\|_\infty$ and $\|H(\Omega + \omega)(t)\|_\infty$ as shown in Figure 6(g) and (h) indicates the formation of singularity. The singularity seems to develop after the time $t = 1.8$ and near $t = 2$. Indeed, the evolution of the solution before time $t = 1.8$ is shown in Figure 7. Comparing the amplitudes of $\Omega$, $\omega$ and their derivatives between Figure 6 and Figure 7, it seems that the dramatic behavior of the solution occurs after time $t = 1.8$.

In contrast, no evidence of singularity is observed for the solutions of (6.1) with data (6.3) in the cases of $a = -1, -2$, see Figure 8 and Figure 9. Although high concentrations and spines are noted for $\Omega$ and $\omega$ and their derivatives near $x = -\pi$, $x = 0$ and $x = \pi$, the norms of $\|H\Omega(t)\|_\infty + \|H\omega(t)\|_\infty$ and $\|H(\Omega \pm \omega)(t)\|_\infty$ shown in Figure 8(g) and (h), Figure 9(g) and (h), grow mildly in the beginning and then become stabilized. Thus according to the Beale-Kato-Majda type of regularity criterion established in Theorem 1.2, we speculate that no singularity is to occur in the situations of $a = -1, -2$.

6.1.3. *Solutions of (6.1) with the initial data (6.4).* Data (6.4) was used in [19] to produce solutions with potential singularities for the generalized Constantin-Lax-Majda model. Recall that if $\Omega = \omega$, system (6.1) reduces to the generalized Constantin-Lax-Majda model. Hence we investigate the solutions of (6.1) with the initial data (6.4). When $a = 0.5$ and $a = 0$, the solutions appear regular in the early time, see Figure 12. However, rapid growth of the solutions $\Omega$ and $\omega$ and their derivatives are observed near $x = 0$ after certain time, as shown in Figure 10 and Figure 11. It looks like that the fast growth starts after the time $t = 1.4$ in the case of $a = 0.5$ from Figure 10, and the fast growth starts after $t = 0.8$ when $a = 0$ from Figure 11. In particular, the exponential like growth of $\|H\Omega(t)\|_\infty + \|H\omega(t)\|_\infty$ and $\|H(\Omega + \omega)(t)\|_\infty$ seen in Figure 10(g), (h) and Figure 11(g), (h) suggests that singularities are likely to develop in finite time. In fact the data (6.4) falls in the class of the initial data used in [6]; hence the numerical result here reproduces the numeric evidence of blowup discussed in [6].

6.2. **Numerical results for the De Gregorio model revisited.** Numerical simulations for the De Gregorio model (1.6a)-(1.6b) have been performed in [9, 10, 19, 21] among others. One outcome is that singularity formation for this model with certain smooth initial data is unlikely to happen in the periodic case.

We apply our numerical scheme to (1.6a)-(1.6b) with the initial data

$$\omega_0(x) = \sin x + 0.1 \sin(2x)$$

by taking $N = 12800$ points in the Fourier-collocation spectral method. The obtained simulations are shown in Figure 13, which recover the numerical results done by Okamoto, Sakajo, and Wunsch [21].

We note that $u_x = H\omega$ for the De Gregorio model (1.6a)-(1.6b) and $u_x = \frac{1}{2}H(\Omega + \omega)$ for our 1D MHD model (6.1), see (6.5). Comparing Figure 1(h) and Figure 13(e), we observe oscillations of $\|u_x\|_{L^\infty}$ for the 1D MHD model and absence of such oscillations for the pure fluid model. It is reasonable to infer that the interactions between fluid velocity and magnetic field cause such oscillations and more complicated dynamics.

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## Data availability statement

Data is available upon reasonable request.

## Conflict of interest statement

The authors state that there is no conflict of interest for this manuscript.

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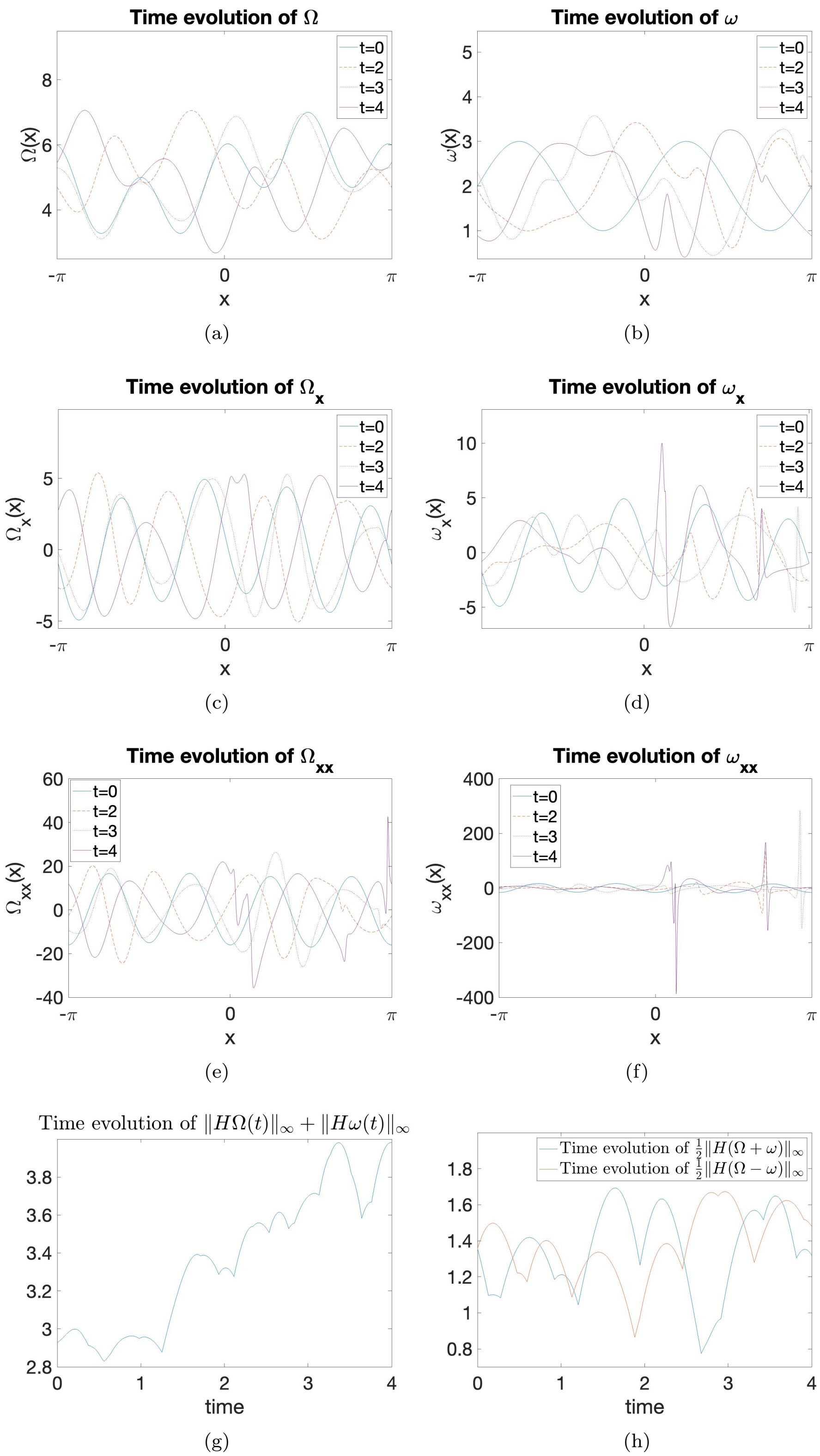



FIGURE 1. $a = 1$ in (6.1) with initial data (6.2).

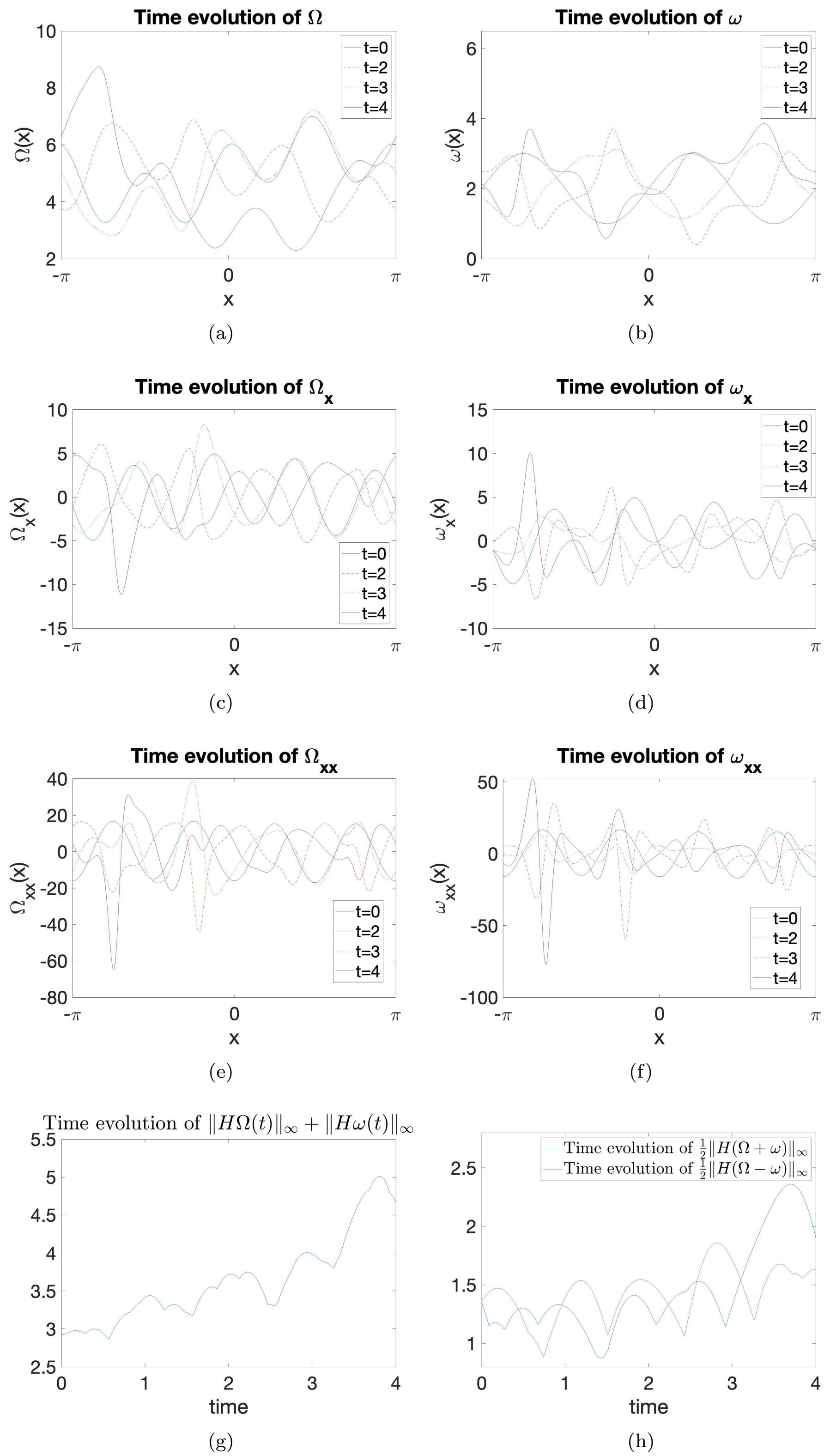



Figure 2. $a = 0$ in (6.1) with initial data (6.2).

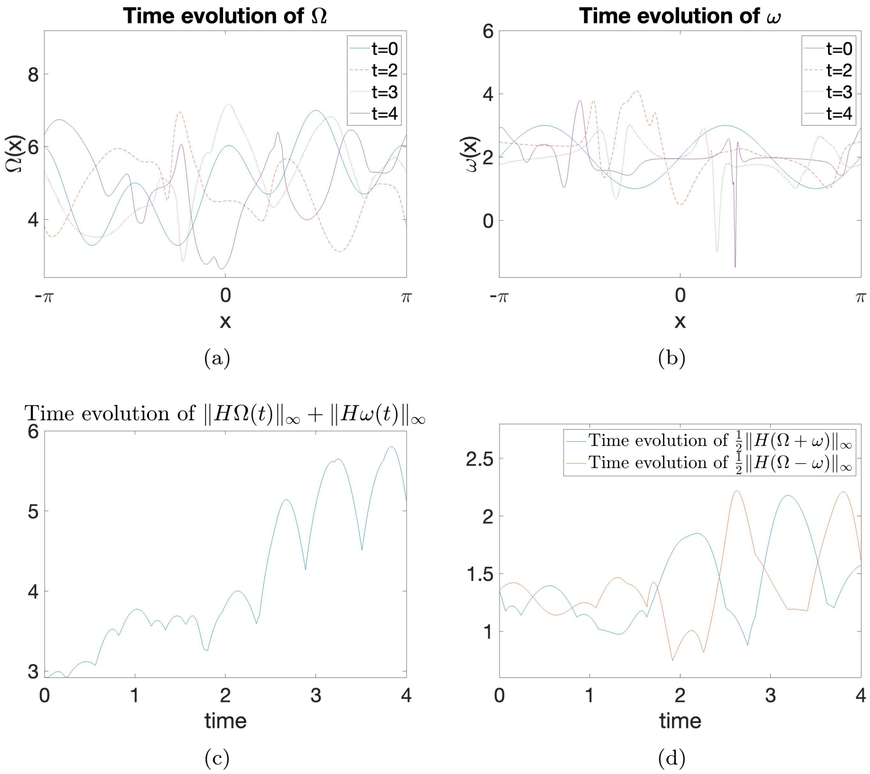



FIGURE 3. $a = -1$ in (6.1) with initial data (6.2).

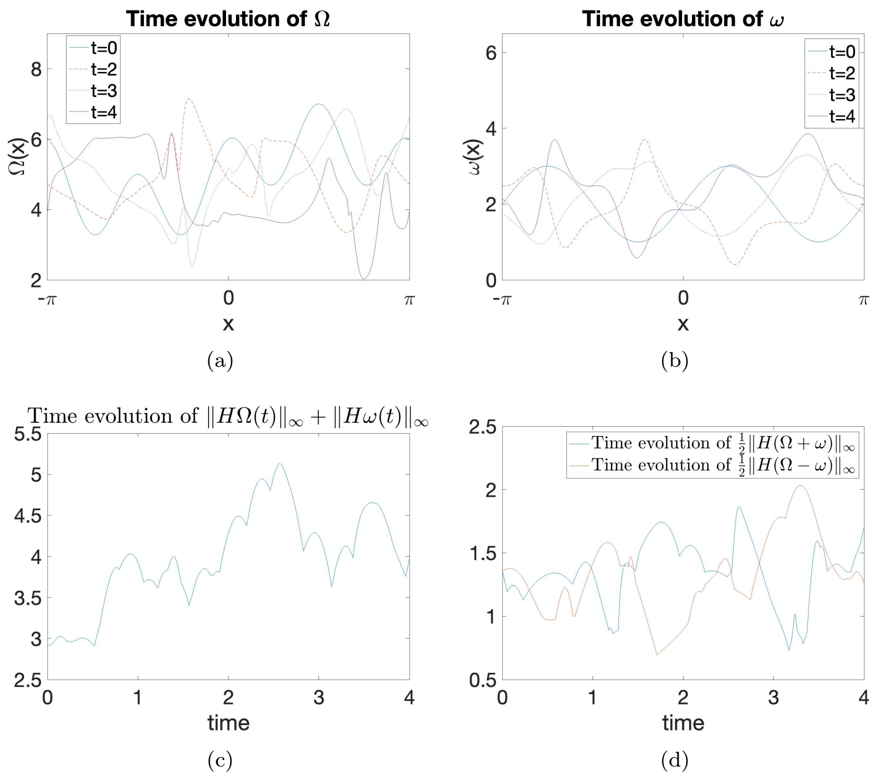



FIGURE 4. $a = -2$ in (6.1) with initial data (6.2).

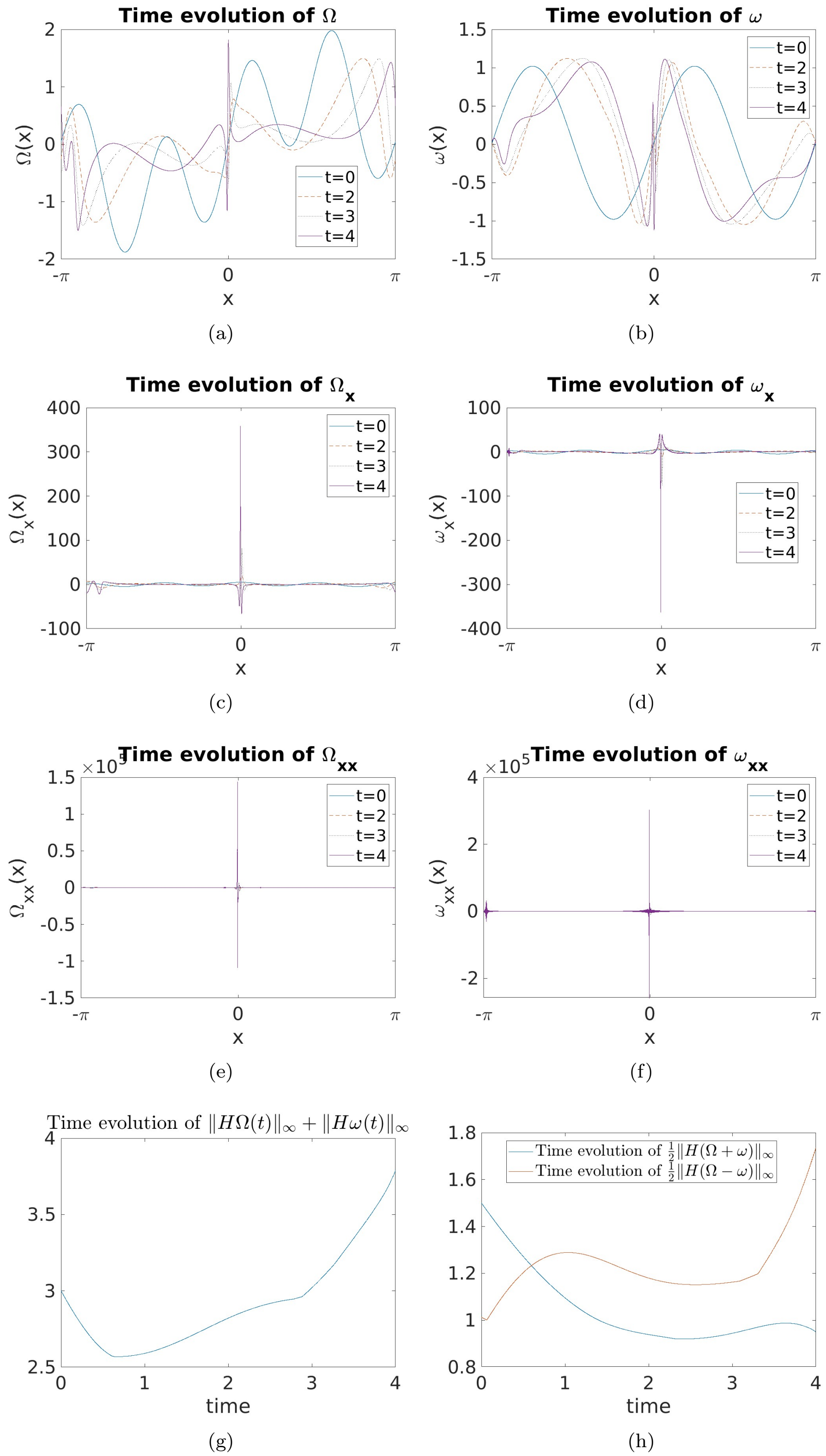



FIGURE 5. $a = 1$ in (6.1) with initial data (6.3).

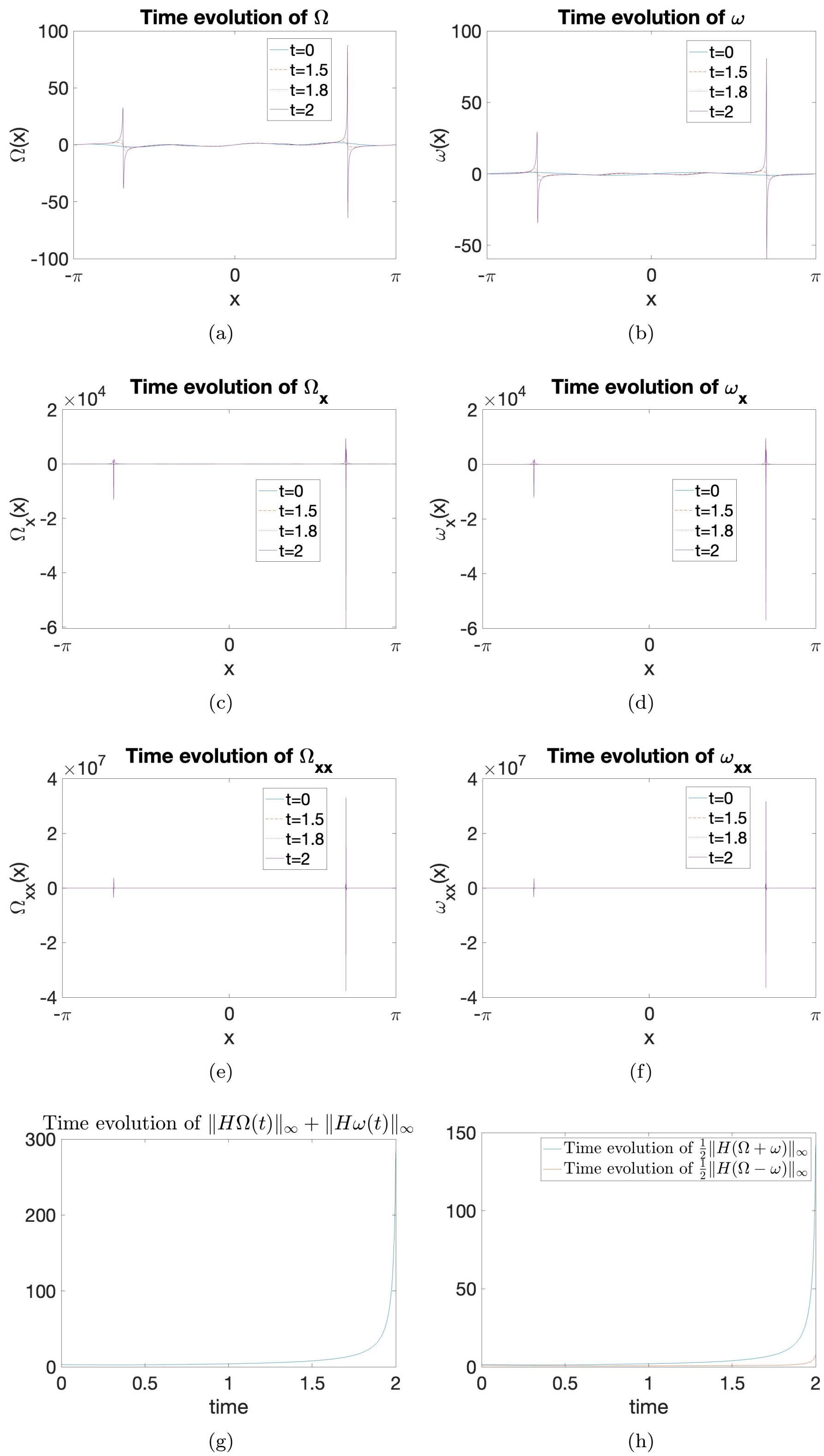
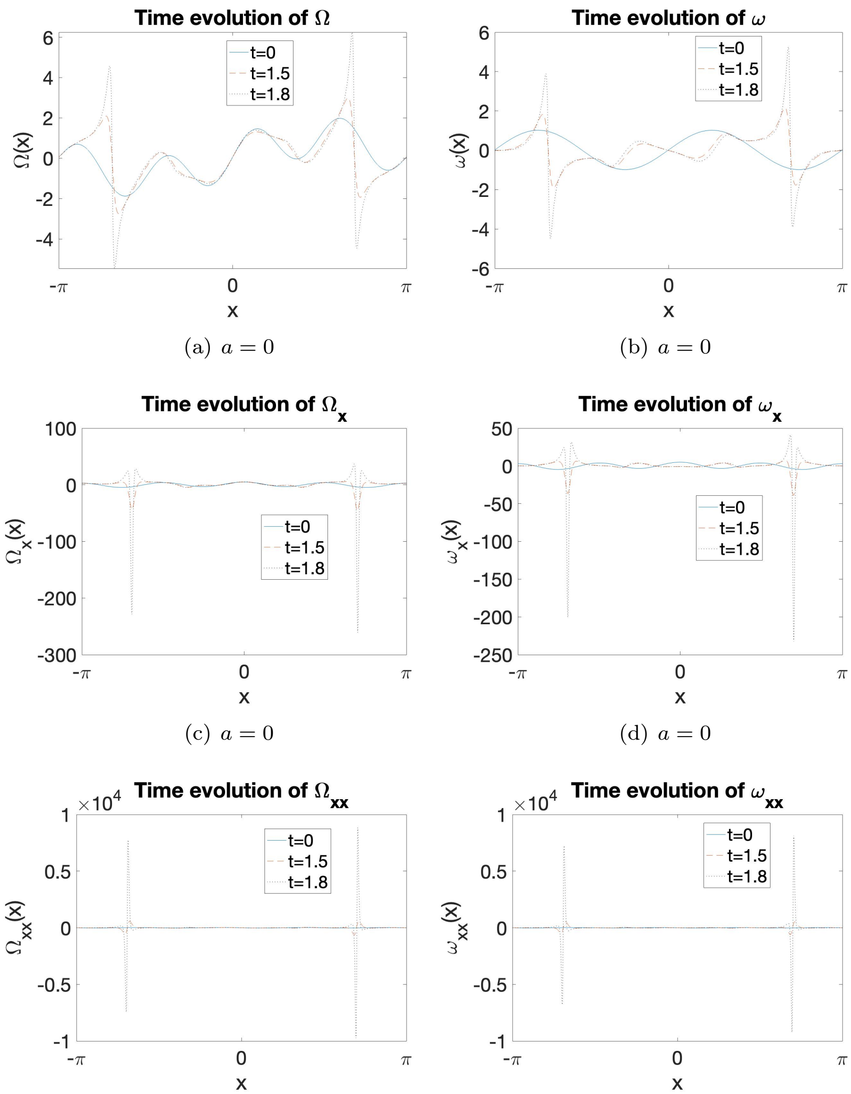



FIGURE 6. $a = 0$ in (6.1) with initial data (6.3).




(a) $a = 0$ (b) $a = 0$

(c) $a = 0$ (d) $a = 0$

(e) $a = 0$ (f) $a = 0$

FIGURE 7. The smooth solutions in early time in (6.1) using $a = 0$ with initial data (6.3).

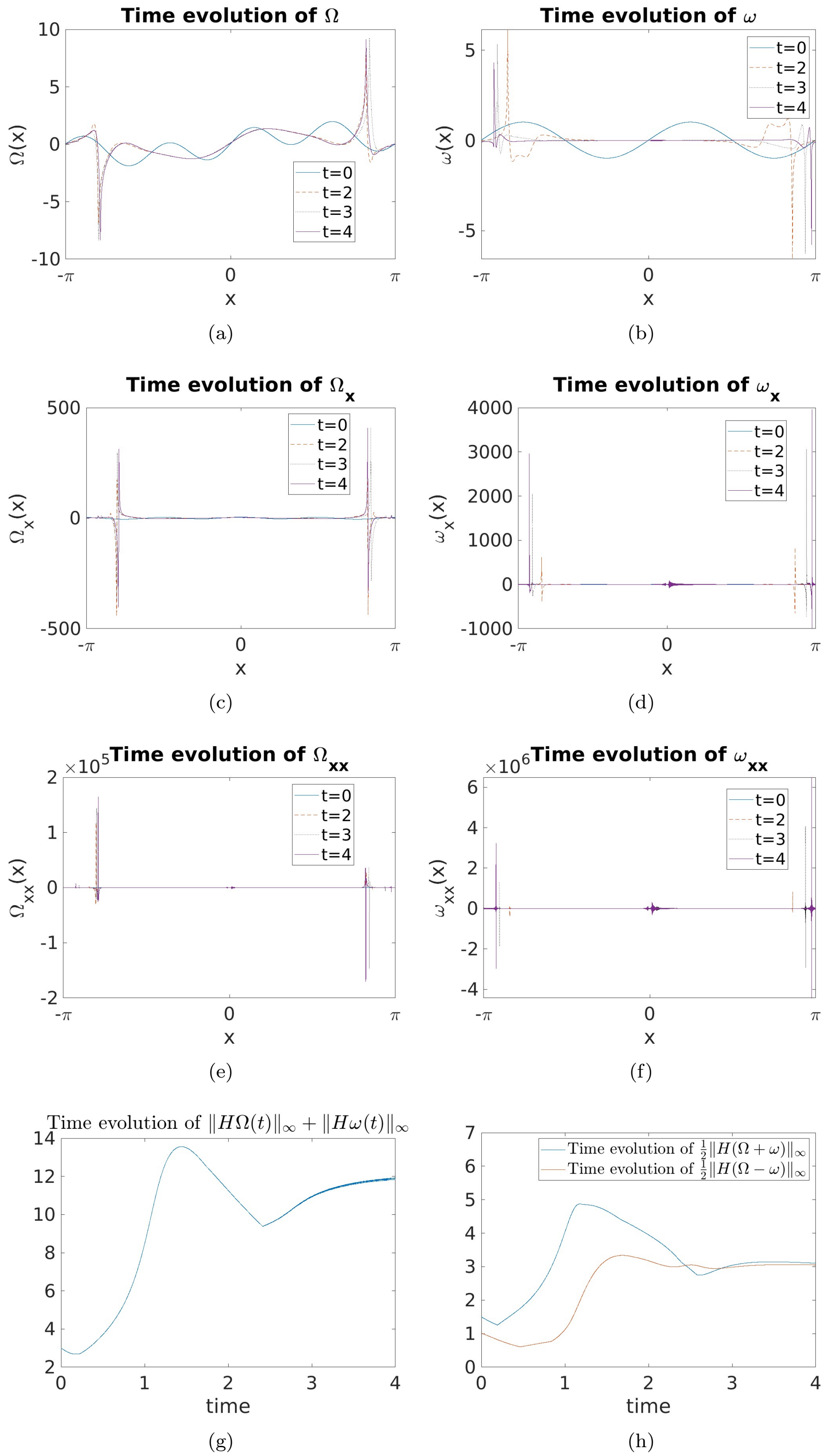



FIGURE 8. $a = -1$ in (6.1) with initial data (6.3).

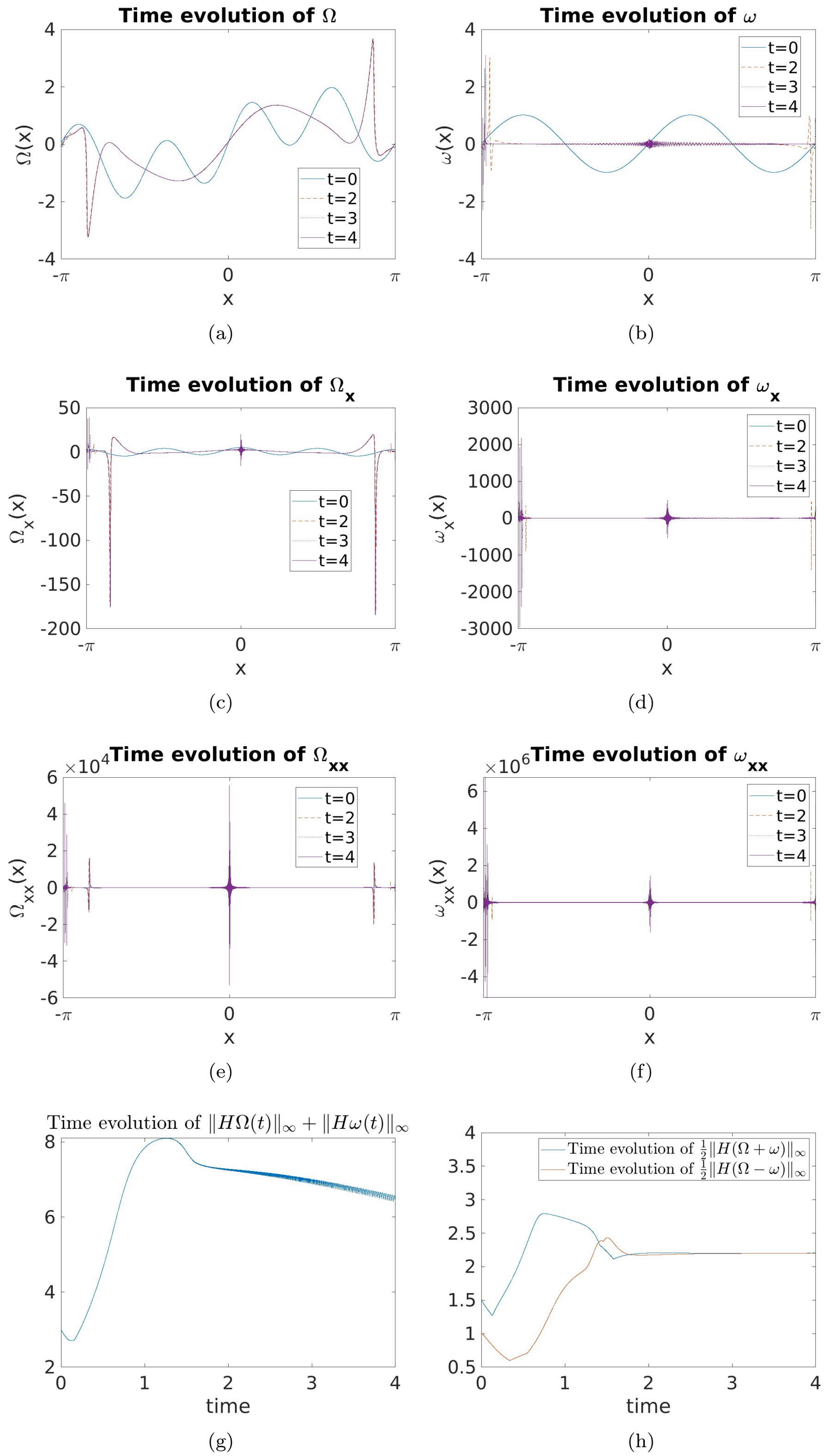



FIGURE 9. $a = -2$ in (6.1) with initial data (6.3).

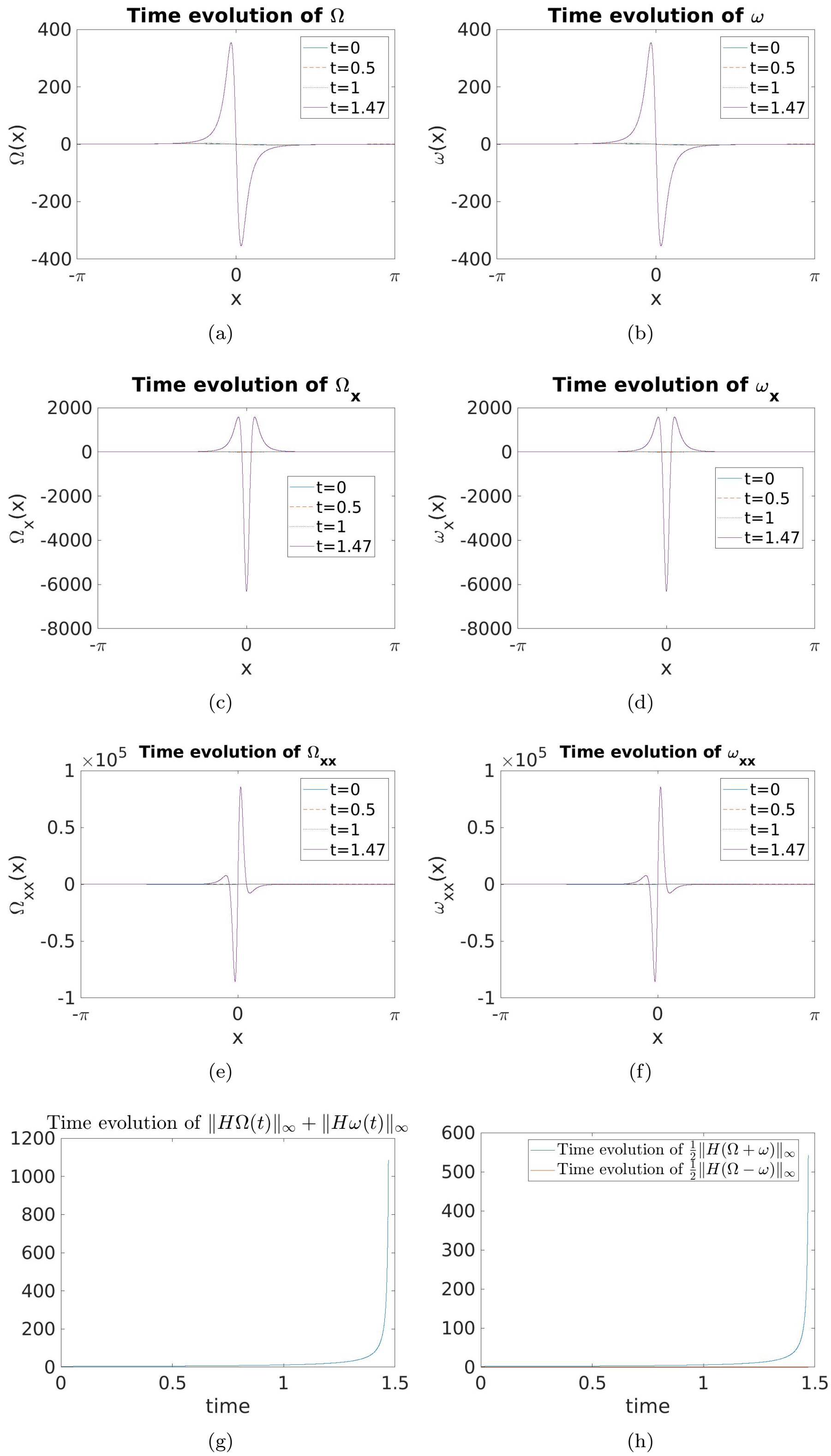



FIGURE 10. $a = 0.5$ in (6.1) with initial data (6.4).

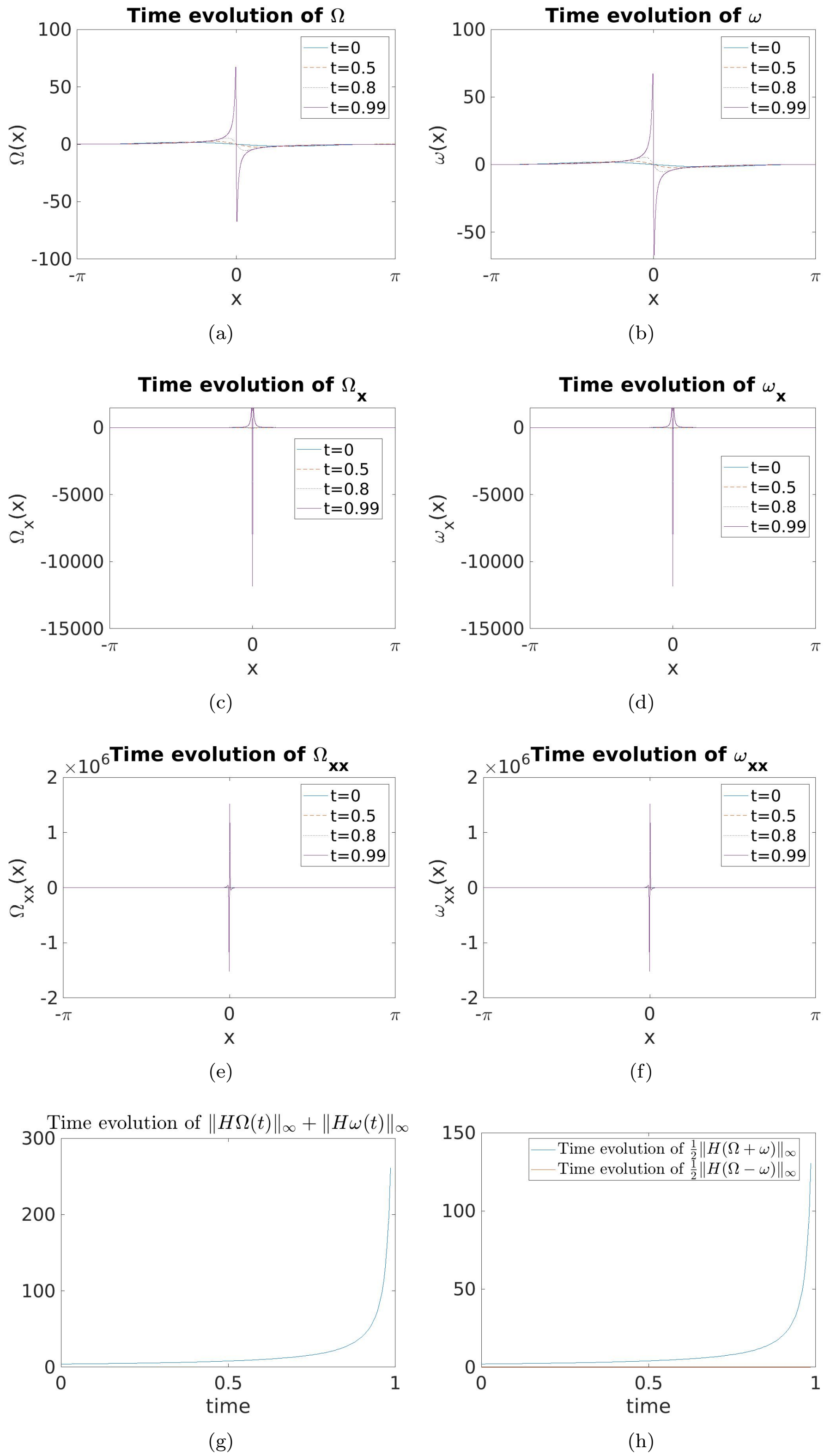
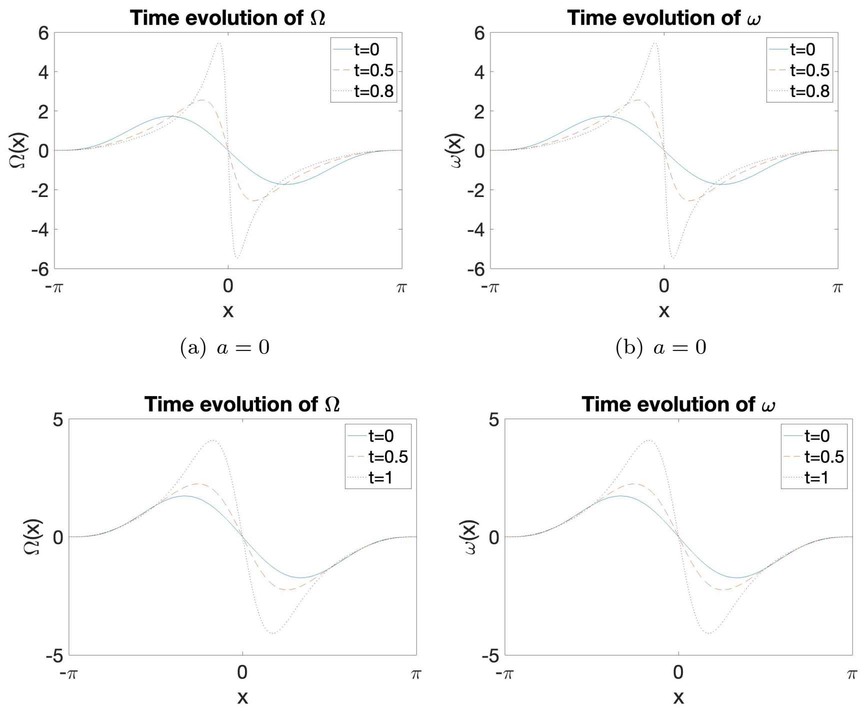


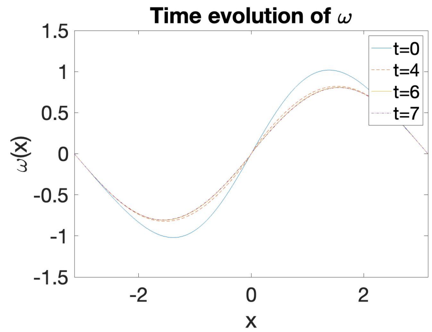

FIGURE 11. $a = 0$ in (6.1) with initial data (6.4).



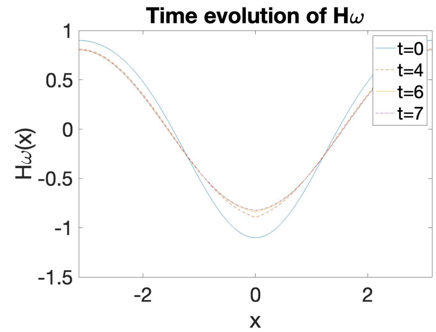

(a) $a = 0$ (b) $a = 0$

(c) $a = 0.5$ (d) $a = 0.5$

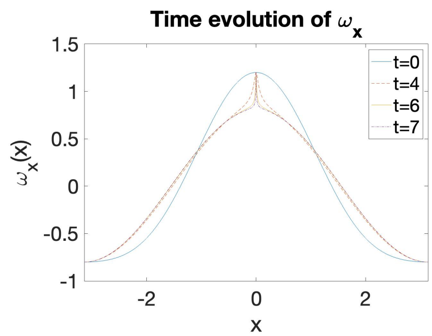
Figure 12. The smooth solutions in early time in (6.1) with initial data (6.4).



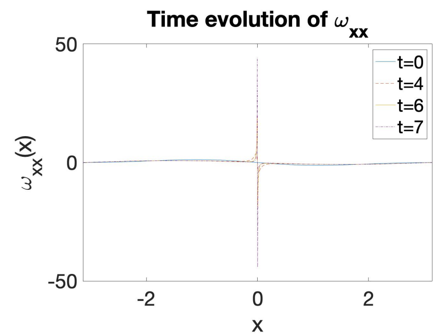

(a)



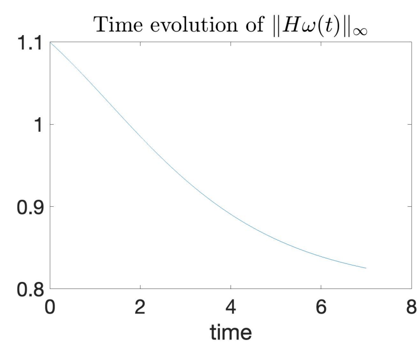

(b)




(c)




(d)




(e)

FIGURE 13. The De Gregorio model.