IEEE Control Systems Letters paper presented at
2023 American Control Conference (ACC)
San Diego, CA, USA. May 31 - June 2, 2023

Optimal Placement of PMUs in Power Networks: Modularity Meets A Priori Optimization

Mohamad H. Kazma and Ahmad F. Taha[◇]

***Abstract*— This paper revisits the optimal *p*hasor measurement unit (PMU) *p*lacement *p*roblem ($\mathbf{P}^3$) in transmission networks. We examine $\mathbf{P}^3$ from a control-theoretic and dynamic systems perspectives. Relevant prior literature studied this problem through formulations that are based on empirical observability maximization for nonlinear dynamic power system models. While such studies addressed a plethora of challenges, they mostly adopt a simple representation of system dynamics, ignore basic algebraic equations modeling power flows, forgo including renewables and their uncertainty. This paper offers a fresh perspective on this problem by leveraging the observability matrix's modularity property under a moving horizon estimation theoretic. A nonlinear differential algebraic representation of the system is implicitly discretized while explicitly accounting for uncertainty. To that end, the posed challenges are addressed for the optimal $\mathbf{P}^3$ via a computationally tractable integer program formulation. The validity of the approach is illustrated on an IEEE 39-bus power system.**

I. Introduction

The optimal sensor placement problem exists widely in various dynamic networks such as water distribution networks, electric power systems, and transportation networks. In power systems, PMU placement is critical for accurate fast monitoring and control of the transmission network [1]. The placement refers to the process of selecting the buses or nodes on which PMUs should be installed—naturally an offline design problem. Signal fault detection, communication channel limitations, static power flow considerations, topological network changes, and some socio-economics are several factors that are a basis for $\mathbf{P}^3$ [2]. However, in the scope of this work we focus on the problem from observability- and systems-theoretic perspectives. That being said, a system can have PMUs located at each bus and achieve full observability, however this is not feasible economically [3]. As such, it is necessary to solve for optimal PMU placement that achieves maximum observability given a fixed number of to-be-installed PMUs [4].

From a dynamic system perspective, $\mathbf{P}^3$ can be understood as obtaining the minimal number of PMUs such that the power network model is observable—or at least detectable. For linear dynamic systems modeled as $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$, this translates to the observability matrix being full rank or that the pair $(\boldsymbol{A}, \boldsymbol{C})$ satisfy the Popov-Belevitch-Hautus (PBH) test [5]—a classical linear systems theory result. Matrix $\boldsymbol{C}$ encodes the binary PMU placement variables. This also extends to nonlinear, multi-machine network model through the empirical observability Gramian [6], [7] which is more cumbersome and, as its adjective suggests, is empirical.

Nonetheless, not much research has been conducted on PMU placement jointly with dynamic state estimation (DSE). The literature that addresses optimal PMU placement from a DSE approach in power networks establish that such PMU selection problem is not well understood and is solved via heuristics or greedy approach [8]. Consequently, conventional schemes for observing and estimating power networks are not computationally efficient for larger networks. In [9], the authors formulate $\mathbf{P}^3$ as a maximization of the empirical observability Gramian metrics, however (i) it is performed under typical flow conditions and is then assessed for robustness, (ii) it is computationally expensive, and (iii) it doesn't consider the joint estimation of differential and algebraic states.

Several studies [8]–[12] have extended the $\mathbf{P}^3$ formulation developed in [9], but the studies also neglected the aforementioned drawbacks. Commonly, the differential equations are included in the state system representation of the model, whereas the algebraic equations are neglected due to the computational burden and overall stability implications [13]. A complete representation of a power system includes both differential and algebraic equation forming a system of nonlinear differential algebraic equations (NDAE). The advantages of using an NDAE formulation of the power system are: (i) linking of network dynamics with power flow equations [3], (ii) incorporating renewables and loads, whilst modeling their uncertainty in DSE routines [14], and (iii) rendering the selection of non-generator buses feasible. It is to the best of our knowledge that the observability-based $\mathbf{P}^3$ in power systems represented as a NDAE has not yet been investigated.

Paper's Approach and Contributions: Motivated by the aforementioned limitations within the literature, we revisit $\mathbf{P}^3$ by performing the optimal PMU placement while jointly estimating both dynamic and algebraic states of the NDAE representation of a power system. Compared to [8], [9] we approach formulating $\mathbf{P}^3$—for a NDAE power system—on the basis of exploiting the modularity of the observability matrix. The significance of such optimal PMU placement formulation proposed within the scope of this work are as follows.

- We formulate an approximate implicit discrete-time representation of the NDAE system that retains the structure of NDAEs while attaining mathematical properties of a nonlinear ordinary differential equations (ODE) model.

[◇]Corresponding author. This work is supported by National Science Foundation under Grants 2152450 and 2151571. The authors are with the Civil and Environmental Engineering Department, Vanderbilt University, 2201 West End Ave, Nashville, Tennessee 37235. Emails: mohamad.h.kazma@vanderbilt.edu, ahmad.taha@vanderbilt.edu.

- We perform joint state estimation on dynamic and algebraic states by adopting a moving horizon estimation (MHE) framework. We approach solving the state estimation problem—posed as a nonlinear least-squares problem—using the Gauss-Newton (GN) algorithm.
- We leverage the modularity of the observability matrix to extract *a priori* observability information in order to pose $\mathbf{P}^3$ as an integer program (IP). The use of the *a priori* contribution from each PMU placement extenuates the computational complexity of an optimization instance and therefore results in a computationally tractable approach for PMU placement in lager networks. We demonstrate $\mathbf{P}^3$ on an IEEE 39–bus power system.

The remainder of this paper is structured as follows. In Section II, we introduce the NDAE power system and develop its discrete-time approximate model. The optimal $\mathbf{P}^3$ under the MHE framework is formulated in Section III. The proposed optimization scheme is validated in Section IV and the paper is concluded in Section V.

Paper's Notation: Let $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{p\times q}$ denote the set of real numbers, and real-valued row vectors with size of n, and p-by-q real matrices. The symbol $\otimes$ denotes the Kronecker product. The cardinality of the a set $\mathcal{N}$ is denoted by $|\mathcal{N}|$. The operators $\det(\boldsymbol{A})$ and $\mathrm{trace}(\mathrm{A})$ return the determinant and trace of $\boldsymbol{A}$, and $\mathrm{blkdiag}(\boldsymbol{A})$ constructs a block diagonal matrix.

II. Nonlinear Power System DAE Model

The power system dynamics$(\mathcal{N}, \mathcal{E})$ can be represented as a nonlinear descriptor system, where $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of transmission lines, $\mathcal{N} = \mathcal{G} \cup \mathcal{L}$ is the set of buses within the network such that $|\mathcal{N}| := N$, while $\mathcal{G}$ and $\mathcal{L}$ are the set of generator and load buses where $|\mathcal{G}| := G$, and $|\mathcal{L}| := L$.

In this work, we consider the standard two axis 4^{th} order transient model of a synchronous generator [15], meaning that each generator has four states and two control inputs. The state-space formulation of the NDAE system representing generator dynamics and algebraic constraints can be written as

$$\textit{generator dynamics}: \quad \dot{\boldsymbol{x}}_d = \boldsymbol{f}(\boldsymbol{x}_d, \boldsymbol{x}_a, \boldsymbol{u}) \tag{1a}$$
$$\textit{algebraic constraints}: \quad 0 = \boldsymbol{g}(\boldsymbol{x}_d, \boldsymbol{x}_a), \tag{1b}$$

where $\boldsymbol{x}_d := \boldsymbol{x}_d(t) = [\boldsymbol{\delta}^\top \; \boldsymbol{\omega}^\top \; \boldsymbol{E}'^\top \; \boldsymbol{T}_{\mathrm{M}}{}^\top]^\top \in \mathbb{R}^{4G}$ and $\boldsymbol{x}_a := \boldsymbol{x}_a(t) = [\boldsymbol{P}_{\mathrm{G}}^\top \; \boldsymbol{Q}_{\mathrm{G}}^\top \; \boldsymbol{v}^\top \; \boldsymbol{\theta}^\top]^\top \in \mathbb{R}^{2G+2N}$ represent the differential and algebraic states of the system, and $\boldsymbol{u} := \boldsymbol{u}(t) = [\boldsymbol{E}_{\mathrm{fd}}^\top \; \boldsymbol{T}_{\mathrm{r}}^\top]^\top \in \mathbb{R}^{2G}$ represents the system inputs. $\boldsymbol{f}(\cdot): \mathbb{R}^{4G} \times \mathbb{R}^{2G} \times \mathbb{R}^{2G} \rightarrow \mathbb{R}^{4G}$ and $\boldsymbol{g}(\cdot): \mathbb{R}^{4G} \times \mathbb{R}^{2G} \times \mathbb{R}^{2N} \rightarrow \mathbb{R}^{2G+2N}$ are nonlinear mapping functions. Readers can refer to [15, Ch. 7] for the full description of the power network utilized in this work. The generator states $\boldsymbol{x}_d$ are: $\boldsymbol{\delta}$ the rotor angle, $\boldsymbol{\omega}$ rotor speed, $\boldsymbol{E}'$ transient voltage, and $\boldsymbol{T}_{\mathrm{M}}$ mechanical torque. Generator inputs $\boldsymbol{u}$ are: $\boldsymbol{E}_{\mathrm{fd}}$ generator internal field voltage, $\boldsymbol{T}_{\mathrm{r}}$ governor reference signal. The algebraic states $\boldsymbol{x}_a$ are: $\boldsymbol{P}_{\mathrm{G}}$ and $\boldsymbol{Q}_{\mathrm{G}}$, the real and reactive power, $\boldsymbol{\theta}$ the bus angle, and $\boldsymbol{v}$ the bus voltage.

A. Discrete-Time Modeling of the NDAE

This section describes the discrete-time modeling methodology of the nonlinear descriptor dynamics. Several methods that solve DAEs have been presented and analyzed in the literature. An efficient and stable approach for simulating stiff nonlinear descriptor systems refers to the use of implicit numerical methods. Those of which include: Backward differential formulas (BDF) known as Gear's method [16], Implicit Runge-Kutta (IRK) [17] and Trapezoidal Implicit (TI) methods [18].

In this work, the implicit TI method is used to simulate the discrete-time dynamics of the nonlinear descriptor power system. TI method has been shown to be an efficient method for simulating power systems for transient stability analysis [19]. Further investigation of the other methods in-particular, Gear's method, is important but outside the scope of this work. Accordingly, the discrete-time representation of (1) can be written as (2) for time step k with step size h, such that $\boldsymbol{x}_k := \boldsymbol{x}_{kh}$. We define vectors $\boldsymbol{z}_k := [\boldsymbol{x}_{d,k}, \boldsymbol{x}_{a,k}, \boldsymbol{u}_k]^\top$ and $\boldsymbol{x}_k := [\boldsymbol{x}_{d,k}, \boldsymbol{x}_{a,k}]^\top$ for time step k, and $\tilde{h} := 0.5h$ as the discretization constant.

$$\boldsymbol{x}_{d,k} - \boldsymbol{x}_{d,k-1} = \tilde{h}\big(\boldsymbol{f}(\boldsymbol{z}_k) + \boldsymbol{f}(\boldsymbol{z}_{k-1})\big) \tag{2a}$$
$$0 = \boldsymbol{g}(\boldsymbol{x}_k). \tag{2b}$$

The solvability of the discretized system in (2) entails finding a solution to a set of implicit nonlinear equations, i.e, finding $\boldsymbol{x}_d$ and $\boldsymbol{x}_a$ for each time step k. The Newton-Raphson (NR) method [19] is implemented at each time-step to solve the set of equations under iteration index i. The method is iterated until a convergence criterion—that is, a relatively small error of the $\mathcal{L}_2$ norm of the iteration increment—is attained.

Before showcasing how the NR method is used to simulate the NDAE power system, we introduce a mathematical structural transformation to the NDAE. This transformation entails formulating the system in (2) from an NDAE into a nonlinear ODE representation. A descriptor system of index-n 1 can be represented as an ODE system by differentiating the algebraic equations until a set of differential equations is obtained.

Definition 1: The index-n of descriptor system (2) is the number of times needed to differentiate the DAEs with respect to independent time variable (t) to obtain system of ODEs.

For the power system (1) it has been presented that it is of index-1 [3]. As such, only one differentiation is required to transform system (1) from a DAE to an ODE representation. However, constructing the observability Gramian of the resulting ODE system is rather complex for reasons that are beyond the scope of this work. On such basis, we move forward with transforming the NDAE system by replacing the left hand side in (1) by $\mu\dot{\boldsymbol{x}}_a$ where μ is a relatively small factor that simulates the nonlinear descriptor dynamics while satisfying the algebraic constraint equations. The plausibility of such approximation stems from the low index of the power system model. The validity of this approach is showcased in Section IV of this paper. Given such approximation of the NDAE that we refer to as μ-NDAE system, the discrete-time representation of the power system in (2) can be rewritten in implicit form as

$$0 = \boldsymbol{x}_{d,k} - \boldsymbol{x}_{d,k-1} - \tilde{h}\big(\boldsymbol{f}(\boldsymbol{z}_k) + \boldsymbol{f}(\boldsymbol{z}_{k-1})\big) \tag{3a}$$
$$0 = \mu\boldsymbol{x}_{a,k} - \mu\boldsymbol{x}_{a,k-1} - \tilde{h}\big(\boldsymbol{g}(\boldsymbol{x}_k) + \boldsymbol{g}(\boldsymbol{x}_{k-1})\big). \tag{3b}$$

We solve the system using NR method by first representing (3) under iteration index i denoted by $\boldsymbol{\phi}(\boldsymbol{z}_k^{(i)}, \boldsymbol{x}_{k-1}) :=$

$\boldsymbol{\phi}(\boldsymbol{z}_k^{(i)})$, where $\boldsymbol{z}_k^{(i)}$ retains the same previous definition however under the NR iteration index i. The NR method entails calculating the Jacobian of the nonlinear dynamics. At each time step k the increment $\Delta\boldsymbol{x}_k^{(i)}$, defined as (4), is computed and is then used to update state variable $\boldsymbol{x}_k^{(i+1)} = \boldsymbol{x}_k^{(i)} + \Delta\boldsymbol{x}_k^{(i)}$ for each iteration until the convergence criterion is satisfied.

$$\Delta\boldsymbol{x}_k^{(i)} = \left[\boldsymbol{A}_g(\boldsymbol{z}_k^{(i)})\right]^{-1}\left[\boldsymbol{\phi}(\boldsymbol{z}_k^{(i)})\right], \tag{4}$$

where the Jacobian $\boldsymbol{A}_g(\boldsymbol{z}_k^{(i)}) := \left[\frac{\partial\boldsymbol{\phi}(\boldsymbol{z}_k^{(i)})}{\partial\boldsymbol{x}}\right]$ is given as

$$\boldsymbol{A}_g(\boldsymbol{z}_k^{(i)}) = \begin{bmatrix}\boldsymbol{I}_{n_d} - \tilde{h}\boldsymbol{F}_{\boldsymbol{x}_d}(\boldsymbol{z}_k^{(i)}) & -\tilde{h}\boldsymbol{F}_{\boldsymbol{x}_a}(\boldsymbol{z}_k^{(i)}) \\ -\tilde{h}\boldsymbol{G}_{\boldsymbol{x}_d}(\boldsymbol{x}_k^{(i)}) & \mu\boldsymbol{I}_{n_a} - \tilde{h}\boldsymbol{G}_{\boldsymbol{x}_a}(\boldsymbol{x}_k^{(i)})\end{bmatrix}. \tag{5}$$

We define $n_d := 4G$ and $n_a := 2G + 2N$ as the number of differential and algebraic states, and $n := n_d + n_a$ as the total number of states. Matrices $\boldsymbol{F}_{\boldsymbol{x}_d} \in \mathbb{R}^{n_d \times n_d}$ and $\boldsymbol{F}_{x_a} \in \mathbb{R}^{n_d \times n_a}$ represent the Jacobian of (3a) with respect to $\boldsymbol{x}_d$ and $\boldsymbol{x}_a$. Matrices $\boldsymbol{G}_{\boldsymbol{x}_d} \in \mathbb{R}^{n_a \times n_d}$ and $\boldsymbol{G}_{\boldsymbol{x}_a} \in \mathbb{R}^{n_a \times n_a}$ represent the Jacobian of (3b) with respect to $\boldsymbol{x}_d$ and $\boldsymbol{x}_a$. Matrices $\boldsymbol{I}_{n_d} \in \mathbb{R}^{n_d \times n_d}$ and $\boldsymbol{I}_{n_a} \in \mathbb{R}^{n_a \times n_a}$ are identity matrices.

III. Observability-Based PMU Placement

In this section, we discuss the framework under which we address the $\mathbf{P}^3$ of the descriptor power system (1). Based on the discretized μ-NDAE model developed in Section II-A, the discrete-time power system dynamics with measurements takes the following form

$$\boldsymbol{E}_\mu\boldsymbol{x}_k = \boldsymbol{E}_\mu\boldsymbol{x}_{k-1} + \tilde{h}\boldsymbol{I}_n\begin{bmatrix}\boldsymbol{f}(\boldsymbol{z}_k) + \boldsymbol{f}(\boldsymbol{z}_{k-1}) \\ \boldsymbol{g}(\boldsymbol{x}_k) + \boldsymbol{g}(\boldsymbol{x}_{k-1})\end{bmatrix} \tag{6a}$$

$$\boldsymbol{y}_k = \tilde{\boldsymbol{C}}\boldsymbol{x}_k + \boldsymbol{\Gamma}\boldsymbol{v}_k, \tag{6b}$$

we define $\mathcal{N}_p \subseteq \mathcal{N}$ as the set of buses at which PMUs can be installed, such that $|\mathcal{N}_p| = N_p$. Diagonal matrix $\boldsymbol{E}_\mu \in \mathbb{R}^{n \times n}$ has ones on its diagonal for $\boldsymbol{x}_d$ and μ for $\boldsymbol{x}_a$.

Matrix $\tilde{\boldsymbol{C}} := \boldsymbol{\Gamma}\boldsymbol{C} \in \mathbb{R}^{n_p \times n}$ is the mapping of states variables under the selected sensor configuration, which in this case measures $[\boldsymbol{v}^\top\ \boldsymbol{\theta}^\top]^\top$ such that, $n_p := 2N_p$ represents the number of measured states. Diagonal matrix $\boldsymbol{\Gamma}$ defines the placement of PMUs within the network which is defined as $\boldsymbol{\Gamma} := \mathrm{diag}(\gamma_z)$ with $\gamma_z = [0, 1]^p$, whereby, $\gamma_z = 1$, if a PMU bus is selected and $\gamma_z = 0$, otherwise. Variable $p \leq n_p$ is the number of selected PMUs within the transmission network and $\boldsymbol{v}_k \in \mathbb{R}^{n_p}$ is the measurement noise.

We formulate the observability-based $\mathbf{P}^3$ based on the concept of observability under a MHE approach developed in [20]. The reasons for choosing this approach are two-fold. (i) MHE is robust against measurement noise [21], and (ii) as argued by [20], this framework is most scalable for stiff nonlinear networks amongst the other approaches in literature.

A. Initial State Estimation

MHE is a state estimation approach that uses a series of past measurements that contain noise and inaccuracies to estimate the states of a dynamic system. As such, we begin with denoting the observation horizon as N_o. Then, we define a nonlinear vector function of the initial state $\boldsymbol{g}(\boldsymbol{\Gamma}, \boldsymbol{x}_0) := \boldsymbol{g}(\boldsymbol{x}_0) : \mathbb{R}^{n_p} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n_p}$, such that the objective here is minimize the nonlinear least-square error on $\boldsymbol{g}(\boldsymbol{x}_0)$ posed as

$$(\mathbf{P1}) \quad \underset{\boldsymbol{x}_0}{\text{minimize}} \quad ||\boldsymbol{g}(\boldsymbol{x}_0)||_2^2 \tag{7a}$$

$$\text{subject to} \quad \underline{\boldsymbol{x}}_0 \leq \boldsymbol{x}_0 \leq \overline{\boldsymbol{x}}_0, \tag{7b}$$

where $\underline{\boldsymbol{x}}_0$ and $\overline{\boldsymbol{x}}_0$ are the lower and upper bounds on initial state variables. From a power systems perspective, the upper and lower bounds on algebraic variables are obtained from MATPOWER [22]. The vector function $\boldsymbol{g}(\cdot)$ represented in (8) is defined as $\boldsymbol{g}(\boldsymbol{x}_0) := \boldsymbol{y}(\boldsymbol{x}_0) - \boldsymbol{w}(\boldsymbol{\Gamma}, \boldsymbol{x}_0)$.

$$\mathrm{col}\left\{\boldsymbol{g}(\boldsymbol{x}_i)\right\}_{i=0}^{\mathrm{N}_o-1} = \mathrm{col}\left\{\boldsymbol{y}_i\right\}_{i=0}^{\mathrm{N}_o-1} - \mathrm{col}\left\{\tilde{\boldsymbol{C}}\boldsymbol{x}_i\right\}_{i=0}^{\mathrm{N}_o-1}, \tag{8}$$

where $\boldsymbol{y}(\boldsymbol{x}_i) \in \mathbb{R}^{n_p}$ represents the set of observations over N_o of the discretized system and $\boldsymbol{w}(\boldsymbol{\Gamma}, \boldsymbol{x}_0) := \boldsymbol{w}(\boldsymbol{x}_0) : \mathbb{R}^{n_p} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n_p}$ is the measurement mapping vector function.

Remark 1: Vector $\boldsymbol{g}(\cdot)$ is a function of initial state $\boldsymbol{x}_0$. This is due to the coupling of the k-th state $\boldsymbol{x}_k$ to initial state $\boldsymbol{x}_0$ through the postulated discrete state-space representation.

Accordingly, we can define the observability of system (6) such that for all inputs $\boldsymbol{u}_k$ the initial state $\boldsymbol{x}_0$ can be uniquely determined from a set of measurements over observation horizon N_o. A sufficient condition for $\boldsymbol{g}(\cdot)$ to be injective with respect to $\boldsymbol{x}_0$ is that the Jacobian of $\boldsymbol{g}(\cdot)$ around $\boldsymbol{x}_0$ is of full rank; $\mathrm{rank}\left(\boldsymbol{J}(\boldsymbol{g}(\cdot))\right) = n_d + n_a = n\ \forall\ \boldsymbol{x}_0$ [23].

We move forward with solving the nonlinear least squares objective function (7) by exploiting the discrete nature of the system using a numerical GN algorithm. The reason for choosing a numerical approach rather than using an existing least-square solver are two-fold. (i) GN approach is computationally more efficient, and (ii) for large systems (e.g., IEEE 200-bus case) MATLAB's *lsqminorm* solver could not converge to an initial state estimate.

To that end, in order to solve **P1** using GN, we re-define the objective—posed in **P2**—as minimizing the $\mathcal{L}_2$-norm of the residual function vector formed from (i) measurement equation and (ii) discretized NDAE model.

$$(\mathbf{P2}) \quad \underset{\boldsymbol{q}_0}{\text{minimize}} \quad ||\boldsymbol{r}(\boldsymbol{\Gamma}, \boldsymbol{q})||_2^2, \tag{9}$$

where the vector $\boldsymbol{q} \in \mathbb{R}^{\mathrm{N}_o \cdot n}$ concatenates the differential and algebraic states over horizon N_o. Such that $\boldsymbol{q} := [\boldsymbol{x}_{d,0}^\top\ \boldsymbol{x}_{a,0}^\top\ \ldots\ \boldsymbol{x}_{d,\mathrm{N}_o-1}^\top\ \boldsymbol{x}_{a,\mathrm{N}_o-1}^\top]$. The residual $\boldsymbol{r}(\boldsymbol{\Gamma}, \boldsymbol{q}) := \boldsymbol{r}(\boldsymbol{q}) \in \mathbb{R}^{\mathrm{N}_o \cdot n_p + \mathrm{N}_o \cdot n}$ is defined as $\boldsymbol{r}(\boldsymbol{q}) = [\boldsymbol{r}_{\boldsymbol{y}}\ \boldsymbol{r}_{\boldsymbol{x}}]^T$ where $\boldsymbol{r}_{\boldsymbol{y}} := \boldsymbol{y}(\boldsymbol{x}_0) := [\boldsymbol{r}_{\boldsymbol{y}_0}^\top\ \ldots\ \boldsymbol{r}_{\boldsymbol{y}_{\mathrm{N}_o-1}}^\top] \in \mathbb{R}^{\mathrm{N}_o \cdot n_p}$ and $\boldsymbol{r}_{\boldsymbol{x}} := \boldsymbol{w}(\boldsymbol{x}_0) := [\boldsymbol{r}_{\boldsymbol{x}_0}^\top\ \ldots\ \boldsymbol{r}_{\boldsymbol{x}_{\mathrm{N}_o-1}}^\top] \in \mathbb{R}^{\mathrm{N}_o \cdot n}$. The vector $\boldsymbol{r}_{\boldsymbol{y}} \in \mathbb{R}^{\mathrm{N}_o \cdot n_p}$ is the residual function of the measurement equation for N_o observations that is defined as $\boldsymbol{r}_{\boldsymbol{y}_k} := \boldsymbol{y}_k - \tilde{\boldsymbol{C}}\boldsymbol{x}_{\boldsymbol{k}}$ and $\boldsymbol{r}_{\boldsymbol{x}_k} \in \mathbb{R}^{\mathrm{N}_o \cdot n}$, defined in (10), is the residual of the TI discretized μ-NDAE model represented in (3).

$$\boldsymbol{r}_{\boldsymbol{x}_k} := \begin{bmatrix}\boldsymbol{x}_{d,k} - \boldsymbol{x}_{d,k-1} - \tilde{h}\big(\boldsymbol{f}(\boldsymbol{z}_k) + \boldsymbol{f}(\boldsymbol{z}_{k-1})\big) \\ \boldsymbol{x}_{a,k} - \mu\boldsymbol{x}_{a,k-1} - \tilde{h}\big(\boldsymbol{g}(\boldsymbol{x}_k) + \boldsymbol{g}(\boldsymbol{x}_{k-1})\big)\end{bmatrix}. \tag{10}$$

Now that the residual has been defined, we move forward with solving the minimization problem using GN iterative method by updating state vector $\boldsymbol{q}$ such that (9) is minimized. The GN update for iteration i is given as (11) with a GN step size denoted by h_g.

$$\boldsymbol{q}^{(i+1)} = \boldsymbol{q}^{(i)} - h_g(\boldsymbol{J}_g(\boldsymbol{q}^{(i)})^\top\boldsymbol{J}_g(\boldsymbol{q}^{(i)}))^{-1}\boldsymbol{J}_g(\boldsymbol{q}^{(i)})^\top\boldsymbol{r}(\boldsymbol{q}^{(i)}). \tag{11}$$

The Jacobian matrix in (11) is defined as

$$\boldsymbol{J}_g(\boldsymbol{\Gamma}, \boldsymbol{q}^{(i)}) := \boldsymbol{J}_g(\boldsymbol{q}^{(i)}) = \begin{bmatrix} \boldsymbol{M} \\ \boldsymbol{N} \end{bmatrix}, \tag{12}$$

where $\boldsymbol{M} := \text{blkdiag}(-\tilde{\boldsymbol{C}}) \in \mathbb{R}^{\mathrm{N}_o \cdot n_p \times \mathrm{N}_o \cdot n}$ is the Jacobian matrix of residual function $\boldsymbol{r_y}$, and $\boldsymbol{N} := \text{blkdiag}(\boldsymbol{A}_g) \in \mathbb{R}^{\mathrm{N}_o \cdot n \times \mathrm{N}_o \cdot n}$ is the Jacobian matrix of residual function $\boldsymbol{r_x}$. Here $\boldsymbol{A}_g \in \mathbb{R}^{n \times n}$ is the Jacobian of the discretized NDAE (6a) which is evaluated for observation horizon N_o. The GN method is performed until the $\mathcal{L}_2$–norm of the residual is minimized.

B. Optimal PMU Placement

In this section, the observability-based optimal PMU placement under a MHE framework is formulated. The concept of observability Gramian is used to quantify the NDAE system's observability under a PMU placement. Quantifying observability that is based on the Gramian matrix can be evaluated under several well-known metrics: condition number, rank, trace, etc. A more elaborate study on the different metrics that quantify observability of the Gramian matrix is presented in [9]. For our formulation, we focus on studying the trace of the observability Gramian matrix. The trace quantifies the average observability in all directions of the state-space.

For observability-based sensor selection problem within networks, the optimal $\mathbf{P}^3$ can be posed as a set function optimization problem. This is a common approach widely used in combinatorial optimization that leverages the submodularity of the objective function. In particular for power systems, the work of [8] presented proofs that the observability metrics retain submodularity, defined as follows.

Definition 2: A function $\mathcal{F} : 2^V \to \mathbb{R}$ is *submodular* if for every $A, B \subseteq V$ it holds that

$$\mathcal{F}(A \cap B) + \mathcal{F}(A \cup B) \leq \mathcal{F}(A) + \mathcal{F}(B). \tag{13a}$$

In other words, a submodular function has an incremental additive property. Accordingly, $\mathbf{P}^3$ can be formulated under a set function optimization framework that can be posed as

$$(\mathbf{P3}) \quad \underset{\mathcal{Z}}{\text{minimize}} \quad -\,\text{trace}\left(\boldsymbol{W}_o(\mathcal{Z}, \boldsymbol{x}_0)\right) \tag{14a}$$

$$\text{subject to} \quad |\mathcal{Z}| = p, \; \mathcal{Z} \subseteq \mathcal{N}_p, \tag{14b}$$

where $\mathcal{Z}$ is the set of selected sensors. Such that, $\mathcal{Z} \subseteq \mathcal{N}_p$ is subset of the total number of buses at which PMUs can be placed, and p is the number of PMUs selected.

The mapping of PMUs in set $\mathcal{Z}$ is encoded by the matrix $\tilde{\boldsymbol{C}}$. As such, we define $\boldsymbol{W}_o(\mathcal{Z}, \boldsymbol{x}_0) := \boldsymbol{W}_o(\boldsymbol{\Gamma}, \boldsymbol{x}_0) \in \mathbb{R}^{n \times n}$ in (15) as the observability Gramian of the nonlinear system.

$$\boldsymbol{W}_o(\boldsymbol{\Gamma}, \boldsymbol{x}_0) = \boldsymbol{J}^T(\boldsymbol{\Gamma}, \boldsymbol{x}_0)\boldsymbol{J}(\boldsymbol{\Gamma}, \boldsymbol{x}_0), \tag{15}$$

where $\boldsymbol{J}(\cdot) \in \mathbb{R}^{\mathrm{N}_o \cdot n_p \times n}$ is the Jacobian over observation horizon N_o of function $\boldsymbol{g}(\cdot) = 0$ around $\boldsymbol{x}_0$ and is given by

$$\boldsymbol{J}(\boldsymbol{\Gamma}, \boldsymbol{x}_0) = \left[\boldsymbol{I}_n \otimes \tilde{\boldsymbol{C}}\right] \text{col}\left\{\frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{x}_0}\right\}_{i=0}^{\mathrm{N}_o - 1}. \tag{16}$$

Expressing the Jacobian requires the knowledge of $\boldsymbol{x}_k \; \forall \; i = 1, \ldots, \mathrm{N}_o - 1$, which can be obtained by simulating the system dynamics over N_o. Applying the chain rule, the j-th partial derivative can be evaluated as $\frac{\partial \boldsymbol{x}_j}{\partial \boldsymbol{x}_0} = \frac{\partial \boldsymbol{x}_j}{\partial \boldsymbol{x}_{j-1}} \cdots \frac{\partial \boldsymbol{x}_1}{\partial \boldsymbol{x}_0}$.

However given the implicit nature of the discretized system presented in (3), representing the partial derivative for system is not straightforward and depends on the discretization method followed [20]. We note here that if we use of the NDAE system instead of the approximate μ-NDAE representation—that retains an ODE structure—the process of expressing $\frac{\partial \boldsymbol{x}_{d_j}}{\partial \boldsymbol{x}_{d_{j-1}}}$ in explicit form for the algebraic variable becomes non-trivial and hence the main reason for such approximate transformation. The procedure for explicitly expressing the partial derivative and therefore the Jacobian has been omitted for brevity.

We note that submodular set minimization problem is NP-hard to solve. A greedy heuristics approach is a tractable approach that achieves a sub-optimal solution for maximizing monotone increasing* submodular functions. Although being considered a computationally tractable approach, it yields a sub-optimal solution that is at least $(1 - 1/e) = 63\%$ of the optimal solution [24].

Based on the above considerations, we revisit **P3** and instead solve the submodular set optimization problem using an *a priori set optimization program* that is considered a convex integer program (IP). The framework revolves around computing the observability matrix's singular contribution resulting from each sensor placement and then performing the optimal placement based on the *a priori* contribution of from each sensor. The plausibility of this approach is a result of the fact that the observability Gramian $\boldsymbol{W}_o(\cdot)$ is a modular † set function. Summers *et al.* [8] provided proof pertaining to the structural property—modular set function—of the observability matrix for linear models. In the context of nonlinear systems, we prove that the observability matrix under PMU placement is modular with respect to decision variable $\boldsymbol{\Gamma}$, however for brevity the proof is omitted.

Modular functions form positive linear combinations of the single elements that form the modular set. Intuitively, this means that a modular function is analogous to linear functions and that each element of the set has an independent contribution to the function value. Accordingly, the next proposition formulates the observability matrix $\boldsymbol{W}_o(\cdot)$ as a linear combination of its individual elements—the proof is also omitted for brevity. Readers can check the extended version of this manuscript; Google Scholar is your best friend.

Proposition 1: The observability matrix $\boldsymbol{W}_o(\cdot)$ can be written as a linear combination of the individual contributions on observability from each PMU placement as follows

$$\boldsymbol{W}_o^1(\mathcal{Z}, \boldsymbol{x}_0) = \sum_{i=1}^{N_p} W_{o,i}(\mathcal{Z}_i, \boldsymbol{x}_0).$$

Given this result, the *a priori set optimization program* for optimal PMU placement denoted by **P4** can be posed as

$$(\mathbf{P4}) \quad \underset{\mathcal{Z}}{\text{minimize}} \quad -\,\text{trace}\left(\boldsymbol{W}_o^1(\mathcal{Z}, \boldsymbol{x}_0)\right) \tag{17a}$$

$$\text{subject to} \quad |\mathcal{Z}| = p, \; \mathcal{Z} \subseteq \mathcal{N}_p, \tag{17b}$$

where $\mathcal{Z}_i$ corresponds to the selected i-th sensor that is encoded in matrix $\tilde{\boldsymbol{C}}$, such that $\boldsymbol{\Gamma}$ has a 1 on the diagonal corresponding

*A set function $\mathcal{F} : 2^V \to \mathbb{R}$ is monotone increasing if $\forall \; A, B \subseteq V$ the following holds true; $A \subseteq B \to \mathcal{F}(A) \leq \mathcal{F}(B)$

†A set function is modular if it is both submodular and supermodular, such that $\forall \; A, B \subseteq V$ the following holds true; $\mathcal{F}(A \cap B) + \mathcal{F}(A \cup B) = \mathcal{F}(A) + \mathcal{F}(B)$.

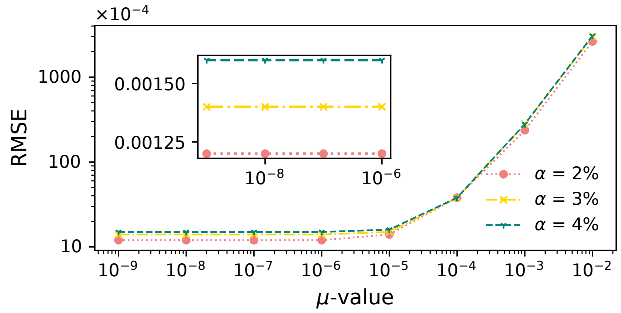

Fig. 1. RMSE for dynamic and algebraic states between the NDAE and μ-NDAE discrete-time models of the power system.

to that sensor location, and zeros elsewhere.

The idea of *a priori* optimization was introduced by [25] as a strategy when solving optimization problems in randomly distributed networks. The concept entails having *a priori* knowledge of instance contributions and then solving the combinatorial problem without exponentially complex computations at each optimization instance. This allows one to perform combinatorial optimization with minimal computing power. Having provided *a priori* information on an particular instance which is possible given the modular nature of the observability matrix, then **P4** which is categorized as a convex integer program (IP) is therefore computational less exhaustive and scalable.

IV. Case Study

In this section, we first validate the discrete-time model of the μ-NDAE system developed in Section II-A and then demonstrate the proposed PMU placement problem **P4**. The PMU placement program is interfaced on MATLAB R2021b through YALMIP [26] and implement using a standard brand and bound method (BNB) with Gurobi [27] as the solver.

We investigate the proposed approach on an IEEE 39-Bus system with 10 generators. The generator parameters are extracted from PST case file $\mathrm{datane.m}$. Regulation and chest time constants for the generators are chosen as $R_{\mathrm{D}i} = 0.2\ \mathrm{Hz/sec}$ and $T_{\mathrm{CH}i} = 0.2\ \mathrm{sec}$, since they are not included in the PST case file. The steady state initial conditions for the power system are generated from the power flow solution obtained from MATPOWER. The synchronous speed is set to $\omega_0 = 120\pi\ \mathrm{rad/sec}$ and a power base to $100\ \mathrm{MVA}$.

A. Simulating the Discrete μ-NDAE Dynamics

We set the discretization step size $h = 0.1$ and simulations time $t = 15$ sec for the discrete time-modeling of the power system. Starting from the initial steady state conditions, we introduce a load disturbance at $t > 0$ on initial load $(\mathrm{P}^0_{\mathrm{L}}, \mathrm{Q}^0_{\mathrm{L}})$ and on initial renewables loads $(\mathrm{P}^0_{\mathrm{R}}, \mathrm{Q}^0_{\mathrm{R}}) = (0.2\mathrm{P}^0_{\mathrm{L}}, 0.2\mathrm{Q}^0_{\mathrm{L}})$ that are modeled as a negative load into the network. The load disturbance magnitude (α_L) is computed as $(\tilde{\mathrm{P}}^0_{\mathrm{L}}, \tilde{\mathrm{Q}}^0_{\mathrm{L}}) = (1 + \frac{\alpha_L}{100})(\mathrm{P}^0_{\mathrm{L}}, \mathrm{Q}^0_{\mathrm{L}})$ and renewables disturbance magnitude (α_R) is computed as $(\tilde{\mathrm{P}}^0_{\mathrm{r}}, \tilde{\mathrm{Q}}^0_{\mathrm{r}}) = (1 + \frac{\alpha_R}{100})(\mathrm{P}^0_{\mathrm{R}}, \mathrm{Q}^0_{\mathrm{R}})$. Under the scope of this paper, we demonstrate simulating the system dynamics with load disturbance magnitude $\alpha = (2, 3, 4)\%$ of the unperturbed initial loads and with renewable disturbance magnitude $\alpha_R = \alpha_L$ of the unperturbed initial renewable loads. We implement the NR method to simulate the discretized dynamics under the TI method. We demonstrate the validity of the approximate μ-NDAE for the three load cases over a range of $\mu = (10^{-2}, 10^{-9})$. It is worth mentioning that for any $\mu < 10^{-2}$ the system does not converge to a solution i.e, the power balance equations are not satisfied.

To further assess the accuracy of this transformation we obtain the root mean square error (RSME) of the TI discretization over time period t that is calculated as $\mathrm{RSME} := \sqrt{\sum_{k=1}^{t} \boldsymbol{e}_k}$ where $\boldsymbol{e}_k := |\hat{\boldsymbol{x}}_k - \boldsymbol{x}_k|$ is the difference between the states of the two system representation with $\hat{\boldsymbol{x}}_k$ corresponding to the NDAE system and $\boldsymbol{x}_k$ to the μ-NDAE system. The RMSE for the different load disturbances and μ values is depicted in Fig. 1. It is obvious that the μ-NDAE system approximates the NDAE with a relatively small error and that this error tends to decrease as μ approaches zero, which is intuitive since with μ reaching zero we go back to having a NDAE. As such, the discrete-time modeling methodology for the power system has been validated. We choose μ to be equal 10^{-6} moving forward; evidently from Fig. 1 one can discern that for each of the different load perturbations the RSME is small when $\mu = 10^{-6}$.

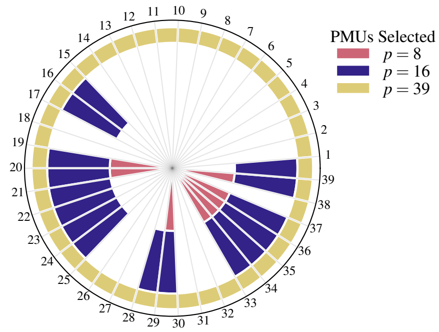

Fig. 2. PMU placements for varying number of PMUs p.

B. Optimal PMU Placement: IEEE 39-Bus system

We now solve the optimal $\mathbf{P}^3$ posed as **P4**. The objective is to obtain an optimal configuration of PMU placement represented by set $\mathcal{Z}^*$ that is constraint by the maximum number of PMUs p to be employed within the network. We initialize the power system under assumed initial conditions $\tilde{x}_0$ and then simulate the discretized measurement model in (6) and under $v = 2\%$ measurement noise over observation horizon N_o. Then, we perform initial state estimation assuming full PMU placement, that is $|\mathcal{Z}| = n_p$, using the GN method to obtain initial state estimate $\hat{\boldsymbol{x}}_0$. The GN algorithm constants are: (i) time step constant $h_g = 0.1$ and (ii) tolerance on residual as 10^{-4}. Based on the initial state estimate, $\mathbf{P}^3$ is solved to obtain optimal set $\mathcal{Z}^*$. Finally the estimation error resulting from the optimal PMU placement is computed as $\varepsilon := \frac{\|\hat{\boldsymbol{x}}_0 - \boldsymbol{x}_0\|_2}{\|\boldsymbol{x}_0\|_2}$. We note here that **P4** is classified as a convex integer program (IP) since the presumed initial state estimate $\hat{\boldsymbol{x}}_0$ is fixed and binary vector $\boldsymbol{\Gamma}$ is the optimization variable. **P4** is successfully solved while being constraint by number of PMUs to be employed within the network. Two cases are taken into consideration: (1) with $p = 8$ PMU $\approx 0.2 \times n_p$ and (2) with $p = 16$ PMUs $\approx 0.4 \times n_p$. The optimal PMU

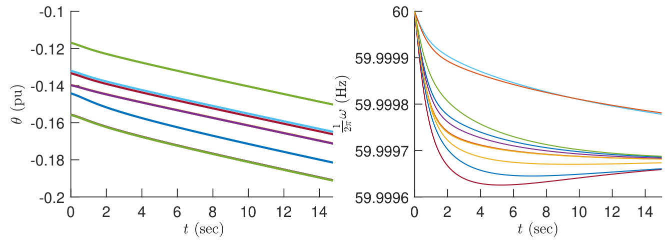

Fig. 3. Dynamic and algebraic state trajectories under load disturbance.

placement over the generator and buses node locations for the two case are depicted in Fig. 2.

Two key aspects can be pointed out from the observability-based PMU placement program solved. The first is through the coupling of dynamics and algebraic states, load buses are selected and thus included in the optimal set $\mathcal{Z}^*$. The second is that it is evident that modularity is retained with the increase of PMUs selected. Whereby, the optimal set for $p = 16$ PMUs included the same buses chosen from the optimal set of the $p = 8$ PMUs optimization case.

Moreover, the initial state estimation error on the PMU placements decreases as the number of PMUs placed on the buses with the power system increases. Bus angles θ_i and generator rotor speeds ω_i for case with $p = 8$ PMUs are depicted in Fig. 3. The estimation error in this case is $\varepsilon = 1.294 \times 10^{-4}$ which then decreased to $\varepsilon = 1.088 \times 10^{-4}$ under $p = 16$ PMUs. This concurs with the observability based estimation framework that the state estimation is based on. As such, the results altogether validate our approach and prospects future investigations with regards to this observability-based framework approach for NDAE power systems. Future work will include investigating $\mathbf{P}^3$ under different discretization techniques, under other observability metrics on the placement, and under the effects of structural system change resulting from fault lines and on that note we end this section.

V. Conclusion

This paper revisits the $\mathbf{P}^3$ for power system. The power system is based on a NDAE representation which allows coupling of the differential and algebraic states within the network. The NDAE system is discretized using TI method and is transformed into a μ-NDAE which retains the mathematical structure of an ODE. We adopt a MHE approach to perform optimal PMU placement with such network by exploiting the modularity of the observability matrix. As such, we posed $\mathbf{P}^3$ as an *a priori set optimization program* which extenuates the computational burden from performing complex computation at each optimization instance of the combinatorial problem.

References

[1] S. Ghosh, Y. Isbeih, Y. Isbeih, S. K. Azman, M. S. El Moursi, and E. F. El-Saadany, "Optimal PMU Allocation Strategy for Completely Observable Networks with Enhanced Transient Stability Characteristics," *IEEE Transactions on Power Delivery*, pp. 1–1, jan 2022.

[2] M. A. Cruz, H. R. Rocha, M. H. Paiva, J. A. Silva, E. Camby, and M. E. Segatto, "PMU placement with multi-objective optimization considering resilient communication infrastructure," *International Journal of Electrical Power and Energy Systems*, vol. 141, no. April, p. 108167, 2022.

[3] S. A. Nugroho, A. Taha, N. Gatsis, and J. Zhao, "Observers for Differential Algebraic Equation Models of Power Networks: Jointly Estimating Dynamic and Algebraic States," *IEEE Transactions on Control of Network Systems*, vol. 5870, no. c, 2022.

[4] W. Yuill, A. Edwards, S. Chowdhury, and S. P. Chowdhury, "Optimal PMU placement: A comprehensive literature review," in *IEEE Power and Energy Society General Meeting*, 2011.

[5] C.-T. Chen, *Linear system theory and design / Chi-Tsong Chen.*, ser. HRW series in electrical and computer engineering. New York: Holt, Rinehart, and Winston, 1984.

[6] S. Lall, J. E. Marsden, and S. Glavaski, "Empirical model reduction of controlled nonlinear systems," *International Federation of Automatic Control*, pp. 2598–2603, 1999.

[7] A. J. Krener and K. Ide, "Measures of unobservability," *Proceedings of the IEEE Conference on Decision and Control*, pp. 6401–6406, 2009.

[8] T. H. Summers, F. L. Cortesi, and J. Lygeros, "On Submodularity and Controllability in Complex Dynamical Networks," *IEEE Transactions on Control of Network Systems*, vol. 3, no. 1, pp. 91–101, mar 2016.

[9] J. Qi, K. Sun, and W. Kang, "Optimal PMU Placement for Power System Dynamic State Estimation by Using Empirical Observability Gramian," *IEEE Transactions on Power Systems*, vol. 30, no. 4, pp. 2041–2054, jul 2015.

[10] J. Qi, K. Sun, and W.Kang, "Adaptive Optimal PMU Placement Based on Empirical Observability Gramian," *IFAC-PapersOnLine*, vol. 49, no. 18, pp. 482–487, 2016.

[11] A. A. Saleh, A. S. Adail, and A. A. Wadoud, "Optimal phasor measurement units placement for full observability of power system using improved particle swarm optimisation," *IET Generation, Transmission and Distribution*, vol. 11, no. 7, pp. 1794–1800, 2017.

[12] Z. Zheng, Y. Xu, L. Mili, Z. Liu, M. Korkali, and Y. Wang, "Observability Analysis of a Power System Stochastic Dynamic Model Using a Derivative-Free Approach," *IEEE Transactions on Power Systems*, vol. 36, no. 6, pp. 5834–5845, 2021.

[13] Y. Liu and K. Sun, "Solving Power System Differential Algebraic Equations Using Differential Transformation," *IEEE Transactions on Power Systems*, vol. 35, no. 3, pp. 2289–2299, may 2020.

[14] T. Groß, S. Trenn, and A. Wirsen, "Solvability and stability of a power system DAE model," *Systems and Control Letters*, vol. 97, pp. 12–17, 2016.

[15] P. W. Sauer, M. A. Pai, and J. H. Chow, *Power System Dynamics and Stability: With Synchrophasor Measurement and Power System Toolbox 2e*, 1st ed. Wiley and Sons Ltd, 2017.

[16] C. W. Gear, "Simultaneous Numerical Solution of Differential-Algebraic Equations," *IEEE Transactions on Circuit Theory*, vol. 18, no. 1, pp. 89–95, 1971.

[17] U. M. Ascher and L. R. Petzold, "Projected Implicit Runge-Kutta methods for differential-algebraic equations," *SIAM Journal on Numerical Analysis*, vol. 28, no. 4, pp. 1097–1120, 1991.

[18] F. A. Potra, M. Anitescu, B. Gavrea, and J. Trinkle, "A linearly implicit trapezoidal method for integrating stiff multibody dynamics with contact, joints, and friction," *International Journal for Numerical Methods in Engineering*, vol. 66, no. 7, pp. 1079–1124, 2006.

[19] F. Milano, "Semi-Implicit Formulation of Differential-Algebraic Equations for Transient Stability Analysis," *IEEE Transactions on Power Systems*, vol. 31, no. 6, pp. 4534–4543, 2016.

[20] A. Haber, F. Molnar, and A. E. Motter, "State Observation and Sensor Selection for Nonlinear Networks," *IEEE Transactions on Control of Network Systems*, vol. 5, no. 2, pp. 694–708, 2018.

[21] A. Alessandri, M. Baglietto, and G. Battistelli, "Moving-horizon state estimation for nonlinear discrete-time systems: New stability results and approximation schemes," *Automatica*, vol. 44, no. 7, pp. 1753–1765, 2008.

[22] R. D. Zimmerman, C. E. Murillo-sánchez, and R. J. Thomas, "MATPOWER : Steady-State Operations , Systems Research and Education," *IEEE Transactions on Power Systems*, vol. 26, no. 1, pp. 12–19, 2011.

[23] S. Hanba, "On the "Uniform" Observability of Discrete-Time Nonlinear Systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 8, pp. 1925–1928, 2009.

[24] G. Calinescu, C. Chekuri, and J. Pàl, Martin; Vondrák, "Maximizing a Monotone Submodular Function Subject to a Matroid Constraint," *SIAM Journal of Computing*, vol. 40, no. 6, pp. 1740–1766, 2011.

[25] H. Maros and S. Juniar, "A Priori Optimization," *Operations Research*, vol. 38, no. 6, pp. 1–23, 1990.

[26] J. Löfberg, "YALMIP : A Toolbox for Modeling and Optimization in MATLAB," in *In Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.

[27] Gurobi Optimization, LLC, "Gurobi Optimizer Reference Manual," 2022.