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Global regularity and stability analysis of the Patlak–Keller–Segel system with flow advection in a bounded domain: A semigroup approach

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ABSTRACT

We discuss the problem of suppression of singularity via flow advection in chemotaxis modeled by the Patlak–Keller–Segel (PKS) equations. It is well-known that for the system without advection, singularity of the solution may develop at finite time. Specifically, if the initial condition is above certain critical threshold, the solution may blow up in finite time by concentrating positive mass at a single point. In this work, we mainly focus on the global regularity and stability analysis of the PKS system in the presence of flow advection in a bounded domain $\Omega \subset \mathbb{R}^d, d = 2, 3$, by using a semigroup approach. We will show that the global well-posedness can be obtained as long as the semigroup generated by the associated advection–diffusion operator has a rapid decay property. We will also show that for cellular flows in rectangle-like domains, such property can be achieved by rescaling both the cell size and the flow amplitude. This is analogous to the result established by Iyer, Xu and Zlatoš (2021) on the torus $\mathbb{T}^d, d = 2, 3$.

1. Introduction

This work is concerned with the global regularity of the parabolic–elliptic Patlak–Keller–Segel system proposed in [1] and its asymptotic behavior in the presence of flow advection in a bounded domain. The Patlak–Keller–Segel system is one of the classical models that describe the chemotaxis on the movement of cells in response to a chemical stimulus. Specially, the movement has a preference directed by the gradient of the chemo-attractant, which is emitted by the cells. The detailed study of this model and reviews can be found in (e.g. [1–11]).

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$, be an open bounded and connected domain with a smooth boundary Γ (corners may be allowed (see Section 3)). Consider that the motion of the cells is advected by the ambient fluid flow. The flow velocity $\mathbf{v}$ is assumed to be divergence-free and time-independent. Let θ be the density of the cells

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and c be the concentration of a chemoattractant produced by the cells, then the motion of the cells can be described by

$$\frac{\partial \theta}{\partial t} = \Delta\theta - A\mathbf{v}\cdot\nabla\theta - \nabla\cdot(\theta\chi\nabla c) \quad \text{in} \quad \Omega, \tag{1.1}$$

$$-\Delta c + c = \theta \quad \text{in} \quad \Omega, \tag{1.2}$$

$$\nabla\cdot\mathbf{v} = 0 \quad \text{in} \quad \Omega, \tag{1.3}$$

with Neumann boundary conditions for both θ and c, no-penetration condition for $\mathbf{v}$

$$\frac{\partial \theta}{\partial n} = \frac{\partial c}{\partial n} = 0 \quad \text{and} \quad \mathbf{v}\cdot n = 0 \quad \text{on} \quad \Gamma, \tag{1.4}$$

and the initial condition

$$\theta(x,0) = \theta_0(x) \quad \text{in} \quad \Omega, \tag{1.5}$$

where $\chi > 0$ is a sensitivity parameter of the cells to the chemo-attractant c, $A \in \mathbb{R}$ is a parameter regulating the strength of the flow, and n is the outward unit normal vector to the domain boundary Γ. Here we set $A > 0$. The case with $A < 0$ can be treated similarly by letting $\mathbf{v}$ be $-\mathbf{v}$. Without the advection term or the drift, the solution of the PKS equations can exhibit dramatic collapsing behavior. It is shown in (e.g. [4,12–15]) that if the initial density is above certain critical threshold, the solution may blow up in finite time by concentrating positive mass at a single point. However, in the presence of flow advection, it is proven in [16] that for any initial condition there exists an incompressible fluid flow such that the solution to (1.1)–(1.5) remains globally regular. In particular, two classes of flows are shown to be able to prevent the singularity formation in the solutions. One is the relaxation-enhancing flow based on [17] which is time-independent, and the other one is the time-dependent exponentially mixing flow based on the construction in [18]. However, these flows are rather complicated to generate. A recent work by Iyer, Xu and Zlatoš in [19] proved that as long as the flows have small dissipation times, the global well-posedness result can be obtained. They further showed that the flows with arbitrarily small dissipation times can be constructed by rescaling a general class of smooth (time-independent) cellular flows. Other related work, for example, on shear flows can be also found in (e.g. [20]).

The aforementioned work on suppression of singularity via flow advection mainly considered the problem in $\mathbb{R}^d$ or $\mathbb{T}^d, d = 2,3$. The objective of this work is to establish the global regularity and stability results in general bounded domains by employing the tools of analytic semigroup theory, which will pave a way for addressing more practical questions such as how to optimally control the flow advection for preventing finite time blow-up or for steering the chemotaxis towards the desired trajectory. In this work, we first focus on the properties of the semigroup generated by the advection–diffusion operator $\Delta - \mathbf{v}_A\cdot\nabla$, where the flow velocity $\mathbf{v}_A$ continues to be divergence-free and time-independent and depends on the parameter A, but does not necessarily take the form of $\mathbf{v}_A = A\mathbf{v}$ as in (1.1). For example, in the case of cellular flows, we can set $\mathbf{v}_A = A\mathbf{v}(Ax)$ and adjust A to rescale both the cell size and the flow amplitude (see Section 3). We will show that the global well-posedness of the PKS system can be established in an appropriate Hilbert space H if the analytic semigroup generated by $\Delta - \mathbf{v}_A\cdot\nabla$, denoted by $S_A(t), t \geq 0$, has a *rapid decay property* on H. That is, there exist $M_0 > 0$ and $\omega_A > 0$ such that

$$\|S_A(t)\|_{\mathscr{L}(H)} \leq M_0 e^{-\omega_A t}, \quad t \geq 0, \tag{1.6}$$

where ω_A can be made arbitrarily large by choosing a suitable parameter A and M_0 is independent of ω_A. Here $\mathscr{L}^2(H)$ stands for the set of bounded linear operators on H and $\|\cdot\|_{\mathscr{L}^2(H)}$ stands for the operator norm. In fact, the rapid decay property can be achieved for $\mathbf{v}_A = A\mathbf{v}$ by increasing A with the help of the Gearhart-Prüss type theorem established in [21], if the flow $\mathbf{v}$ is *relaxation-enhancing* [17, Def. 1.1]. It is clear that one can enhance the dissipation of the solution in time if the semigroup has such a property. In the last section of this work, we will show that for cellular flows in rectangle-like domains, one can obtain the rapid decay property by rescaling the flows.

1.1. Enhancement of diffusion via flow advection

We first recall some well-established results on enhancement of diffusion via flow advection for the advection–diffusion equation with Neumann boundary condition (e.g. [17,21])

$$\frac{\partial \psi_A}{\partial t} + \mathbf{v}_A \cdot \nabla \psi_A = \Delta \psi_A \quad \text{in} \quad \Omega, \tag{1.7}$$

$$\nabla \cdot \mathbf{v}_A = 0 \quad \text{in} \quad \Omega, \tag{1.8}$$

$$\frac{\partial \psi_A}{\partial n} = 0, \quad \mathbf{v}_A \cdot n = 0 \quad \text{on} \quad \Gamma, \tag{1.9}$$

$$\psi_A(x, 0) = \psi_0 \quad \text{in} \quad \Omega. \tag{1.10}$$

Let $\mathscr{A} = \Delta$ with domain $D(\mathscr{A}) = \{\psi \in H^2(\Omega) : \frac{\partial \psi}{\partial n}|_\Gamma = 0\}$. Define the advection–diffusion operator L_A by

$$L_A = \mathscr{A} - \mathbf{v}_A \cdot \nabla$$

with domain $D(L_A) = D(\mathscr{A})$. For $\psi_0 \in L^2(\Omega)$, using the variation of parameters formula we can express the solution to (1.7)–(1.10) as

$$\psi_A(x, t) = S_A(t)\psi_0(x), \tag{1.11}$$

where $S_A(t) = e^{L_A t}, t \geq 0$, is the analytic semigroup generated by operator L_A on $L^2(\Omega)$. It is well-known that the solution $\psi(x, t)$ will converge to its average

$$\bar{\psi} = \frac{1}{|\Omega|} \int_\Omega \psi_A \, dx = \frac{1}{|\Omega|} \int_\Omega \psi_0 \, dx,$$

as t goes to infinity, which is the constant eigenfunction corresponding to the zero eigenvalue of L_A. In fact, by Stokes formula it is easy to see that

$$\begin{aligned} \frac{\partial \int_\Omega \psi_A \, dx}{\partial t} &= -\int_\Omega \mathbf{v}_A \cdot \nabla \psi_A \, dx + \int_\Omega \Delta \psi_A \, dx \\ &= -\left(\int_\Gamma \mathbf{v}_A \cdot n \psi_A \, dx - \int_\Omega \nabla \cdot \mathbf{v}_A \psi_A \, dx\right) + \int_\Gamma \frac{\partial \psi_A}{\partial n} \, dx = 0, \end{aligned}$$

so $\int_\Omega \psi_A \, dx = \int_\Omega \psi_0 \, dx$ for any $t > 0$.

For $\mathbf{v}_A = A\mathbf{v}$, one of the major questions in the existing literature is to understand how the convergence rate of the solution to its average depends on the properties of the flow, especially how this relates to the parameter A (e.g. [17,22,23]). It is shown in [17] that the solution ψ_A can be arbitrarily close to its average at any given finite time by increasing A if and only if the operator $\mathbf{v} \cdot \nabla$ has no eigenfunctions in $H^1(\Omega)$ other than the constant function. In this case, the incompressible flow $\mathbf{v}$ is relaxation enhancing. The proof of this result utilizes dynamical estimates based on the RAGE theorem (e.g. [24]). A recent work in [21] provided an alternative proof by using a Gearhart-Prüss type theorem and showed how the parameter A is related to the decay property of $e^{L_A t}, t \geq 0$. To be more precise, first note that the advection–diffusion operator L_A in $L^2(\Omega)$ is m-accretive as the left open half-plane is contained in its resolvent set $\varrho(L_A)$ and

$$(L_A + \lambda)^{-1} \in \mathscr{L}^2(L^2(\Omega)), \quad \|(L_A + \lambda)^{-1}\| \leq \Re\lambda^{-1} \quad \text{for } \Re\lambda > 0.$$

As in [21], we let

$$\Psi(L_A) = \inf\{\|(L_A - i\lambda)\phi\|_{L^2} : \phi \in D(L_A), \lambda \in \mathbb{R}, \|\phi\|_{L^2} = 1\}. \tag{1.12}$$

Then it is proven in [21, Theorems 1.3] that for the m-accretive operator L_A in $L^2(\Omega)$,

$$\|e^{L_A t}\|_{\mathscr{L}^2(L^2(\Omega))} \leq M_0 e^{-\Psi(L_A)t}, \quad t \geq 0, \tag{1.13}$$

where $M_0 = e^{\pi/2}$. Moreover, let

$$H = L_0^2(\Omega) = \{\psi \in L^2(\Omega) : \int_\Omega \psi \, dx = 0\} \tag{1.14}$$

be the subspace of mean zero functions and set $\mathcal{L}_A = L_A$ with $D(\mathcal{L}_A) = D(L_A) \cap H$. Then zero is no longer the eigenvalue of $\mathcal{L}_A$ and the proof of [21, Theorem 1.3] yields that for $\mathbf{v}_A = A\mathbf{v}$,

$$\Psi(\mathcal{L}_A) \to +\infty, \quad \text{as} \quad A \to +\infty, \tag{1.15}$$

if and only if

$$\mathbf{v} \cdot \nabla \text{ has no eigenfunctions in } H^1(\Omega) \cap H. \tag{1.16}$$

On the other hand, applying a direct L^2-estimate to (1.7)–(1.10) follows

$$\|\psi_A\|_{L^2} = \|e^{\mathcal{L}_A t}\psi_0\|_{L^2} \le e^{\lambda_1 t}\|\psi_0\|_{L^2}, \quad t \ge 0,$$

for any $\psi_0 \in H$, where $\lambda_1 < 0$ is the first Neumann eigenvalue of the Laplacian operator $\mathscr{A}$ on X. Therefore,

$$\|e^{\mathcal{L}_A t}\|_{\mathscr{L}(H)} \le e^{\lambda_1 t}, \quad t \ge 0. \tag{1.17}$$

Remark 1.1. The relations (1.13) and (1.15)–(1.16) indicate that if the velocity $\mathbf{v}$ satisfies (1.16), i.e., relaxation enhancing, in defining the advection–diffusion operator $\mathcal{L}_A = \mathscr{A} - A\mathbf{v}\cdot\nabla$, then the semigroup $e^{\mathcal{L}_A}, t \ge 0$, generated by such operator possesses the rapid decay property (1.6) on H.

In the current work, we will mainly employ the rapid decay property together with the analytic semigroup theory to establish the global regularity and stability property of the PKS system governed by (1.1)–(1.5). The results are presented in Section 2. However, constructing relaxation enhancing flows is not straightforward and the flow geometries are rather complex (e.g. [17,18,25]). Iyer, Xu and Zlatoš in [19] constructed cellular flows with arbitrarily small dissipation times by rescaling the cell size and the flow amplitude. This operation essentially transfers the energy of the solution from the lower to the higher frequencies, and thus enhances dissipation. The proof was based on the probabilistic method. Alternatively, using a direct estimate in Section 3 we show that in rectangle-like domains, one can make $\Psi(\mathcal{L}_A)$ arbitrarily large by rescaling the cellular flows so that the rapid decay property holds.

2. Well-posedness of the PKS system

To start with, we let $\vartheta = \theta - \bar{\theta}$, then $\bar{\vartheta} = 0$. In the rest of our discussion, we set $\chi = 1$. According to (1.1)–(1.5), ϑ satisfies

$$\frac{\partial \vartheta}{\partial t} = \Delta\vartheta - \mathbf{v}_A \cdot \nabla\vartheta - \nabla \cdot ((\vartheta + \bar{\theta})\nabla c) \quad \text{in} \quad \Omega, \tag{2.1}$$

$$-\Delta c + c = \vartheta + \bar{\theta} \quad \text{in} \quad \Omega, \tag{2.2}$$

$$\nabla \cdot \mathbf{v}_A = 0 \quad \text{in} \quad \Omega, \tag{2.3}$$

$$\frac{\partial \vartheta}{\partial n} = \frac{\partial c}{\partial n} = 0 \quad \text{and} \quad \mathbf{v}_A \cdot n = 0 \quad \text{on} \quad \Gamma, \tag{2.4}$$

$$\vartheta(x, 0) = \theta_0(x) - \bar{\theta} \quad \text{in} \quad \Omega. \tag{2.5}$$

We further introduce the operator $\mathcal{A} = -\Delta + I$ with domain $D(\mathcal{A}) = \{\phi \in H^2(\Omega) : \frac{\partial \phi}{\partial n}|_{\partial\Omega} = 0\}$. Then $\mathcal{A}$ is strictly positive and $\mathcal{A}^{-1}\bar{\theta} = \bar{\theta}$. Thus

$$c = \mathcal{A}^{-1}(\vartheta + \bar{\theta}) = \mathcal{A}^{-1}\vartheta + \bar{\theta}. \tag{2.6}$$

In the sequel, we always consider the state space $H = L_0^2(\Omega)$ defined by (1.14). The symbol C denotes a generic positive constant, which is allowed to depend on the domain as well as on indicated parameters.

2.1. Local in time regularity

In this section, we first show the local in time regularity of the PKS system (2.1)–(2.5) using a direct L^2-estimate.

Proposition 2.1. *Let $\vartheta_0 \in H$ and $\mathbf{v}_A \in L^\infty(\Omega)$ be a divergence-free vector field satisfying $\mathbf{v}_A \cdot n|_\Gamma = 0$. If ϑ is the solution to (2.1)–(2.5), then there exists a $t^* = t^*(\vartheta_0, \bar\theta, d) > 0$ such that*

$$\|\vartheta(t^*)\|_{L^2}^2 \le 2\|\vartheta_0\|_{L^2}^2 + 1 \quad \text{and} \quad \int_0^{t^*} \|\nabla\vartheta\|_{L^2}^2\, dt \le 2\|\vartheta_0\|_{L^2}^2 + 1. \tag{2.7}$$

Proof. Taking the inner product of (2.1) with ϑ yields

$$\begin{aligned}
\frac{1}{2}\frac{d\|\vartheta\|_{L^2}^2}{dt} + \|\nabla\vartheta\|_{L^2}^2 &= -\int_\Omega \nabla\cdot((\vartheta+\bar\theta)\nabla c)\vartheta\, dx \\
&= \int_\Gamma (\vartheta+\bar\theta)\nabla c\cdot n\vartheta\, dx - \int_\Omega (\vartheta+\bar\theta)\nabla c\cdot\nabla\vartheta\, dx \\
&= -\int_\Omega (\vartheta+\bar\theta)\nabla c\cdot\nabla\vartheta\, dx \\
&= -\int_\Omega \vartheta\nabla c\cdot\nabla\vartheta\, dx - \bar\theta\int_\Omega \nabla c\cdot\nabla\vartheta\, dx \\
&= I_1 + I_2,
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
I_1 &= \frac{1}{2}\int_\Omega \nabla c\cdot\nabla\vartheta^2\, dx = \frac{1}{2}\Big(\int_\Gamma (\nabla c\cdot n)\vartheta^2\, dx - \int_\Omega \vartheta^2\Delta c\, dx\Big) \\
&= -\frac{1}{2}\int_\Omega \vartheta^2\Delta c\, dx = \frac{1}{2}\int_\Omega \vartheta^2(\vartheta+\bar\theta-c)\, dx \\
&= \frac{1}{2}\int_\Omega \vartheta^3\, dx + \frac{1}{2}\bar\theta\int_\Omega \vartheta^2\, dx - \frac{1}{2}\int_\Omega \vartheta^2 c\, dx.
\end{aligned}$$

Using the Gagliardo–Nirenberg and Young's inequalities follows

$$\begin{aligned}
\int_\Omega \vartheta^3\, dx &\le \|\vartheta\|_{L^3}^3 \le C\|\vartheta\|_{L^2}^{3-\frac{d}{2}}\|\nabla\vartheta\|_{L^2}^{\frac{d}{2}} \\
&\le \frac{1}{2}\|\nabla\vartheta\|_{L^2}^2 + C\|\vartheta\|_{L^2}^{\frac{12-2d}{4-d}}, \quad d = 2,3.
\end{aligned}$$

Moreover, by Agmon's inequalities we have

$$\begin{aligned}
\int_\Omega \vartheta^2 c\, dx &\le \|\vartheta\|_{L^2}^2\|c\|_{L^\infty} \le C\|\vartheta\|_{L^2}^2\|c\|_{H^{d/2+\epsilon}} \\
&\le C\|\vartheta\|_{L^2}^2\|\vartheta+\bar\theta\|_{H^{d/2-2+\epsilon}} \le C(\|\vartheta\|_{L^2}^3 + |\bar\theta|\|\vartheta\|_{L^2}^2)
\end{aligned}$$

for any $\epsilon > 0$. Therefore,

$$I_1 \le \frac{1}{4}\|\nabla\vartheta\|_{L^2}^2 + C\|\vartheta\|_{L^2}^{\frac{12-2d}{4-d}} + C(\|\vartheta\|_{L^2}^3 + |\bar\theta|\|\vartheta\|_{L^2}^2). \tag{2.9}$$

For I_2, with the help of (2.6) we get

$$\|\nabla c\|_{L^2} = \|\nabla(\mathcal{A}^{-1}\vartheta + \bar\theta)\|_{L^2} \le C\|\vartheta\|_{L^2},$$

so

$$I_2 \le |\bar{\theta}| \|\nabla c\|_{L^2} \|\nabla \vartheta\|_{L^2} \le C\bar{\theta}^2 \|\vartheta\|_{L^2}^2 + \frac{1}{4}\|\nabla \vartheta\|_{L^2}^2. \tag{2.10}$$

Combining (2.8) with (2.9)–(2.10) follows

$$\begin{aligned}\frac{1}{2}\frac{d\|\vartheta\|_{L^2}^2}{dt} + \|\nabla\vartheta\|_{L^2}^2 \le & \frac{1}{4}\|\nabla\vartheta\|_{L^2}^2 + C\|\vartheta\|_{L^2}^{\frac{12-2d}{4-d}} + C(\|\vartheta\|_{L^2}^3 + |\bar{\theta}|\|\vartheta\|_{L^2}^2) \\ & + C\bar{\theta}^2\|\vartheta\|_{L^2}^2 + \frac{1}{4}\|\nabla\vartheta\|_{L^2}^2,\end{aligned}$$

which implies

$$\frac{d\|\vartheta\|_{L^2}^2}{dt} + \|\nabla\vartheta\|_{L^2}^2 \le C\|\vartheta\|_{L^2}^{\frac{12-2d}{4-d}} + C(\bar{\theta})\|\vartheta\|_{L^2}^2 + C\|\vartheta\|_{L^2}^3, \tag{2.11}$$

and hence

$$\frac{d\|\vartheta\|_{L^2}^2}{dt} \le C\|\vartheta\|_{L^2}^{\frac{12-2d}{4-d}} + C(\bar{\theta})\|\vartheta\|_{L^2}^2 + C\|\vartheta\|_{L^2}^3.$$

Let $X = \|\vartheta\|_{L^2}^2 + 1$. Then

$$\frac{dX}{dt} \le C_m(\bar{\theta}) X^{\frac{6-d}{4-d}} \tag{2.12}$$

for some $C_m(\bar{\theta}) > 0$. Integrating this inequality with respect to t yields

$$-\frac{4-d}{2}\left(X(t)^{-\frac{2}{4-d}} - X(0)^{-\frac{2}{4-d}}\right) \le C_m(\bar{\theta})t,$$

and thus

$$X(t) \le \left(\frac{X(0)^{\frac{2}{4-d}}}{1 - \frac{2}{4-d}X(0)^{\frac{2}{4-d}}C_m(\bar{\theta})t}\right)^{\frac{4-d}{2}}.$$

Choose $t_0 > 0$ such that

$$t_0 \le \left(1 - \left(\frac{1}{2}\right)^{\frac{2}{4-d}}\right)\frac{4-d}{2C_m(\bar{\theta})}X(0)^{-\frac{2}{4-d}},$$

then for any $t \in [0, t_0]$,

$$X(t) \le 2X(0) \quad \text{or} \quad \|\vartheta(t)\|_{L^2}^2 \le 2\|\vartheta_0\|_{L^2}^2 + 1. \tag{2.13}$$

By (2.11)–(2.13) we have

$$\int_0^t \|\nabla\vartheta\|_{L^2}^2\,dt \le C_m(\bar{\theta})(2\|\vartheta_0\|_{L^2}^2 + 2)^{\frac{6-d}{4-d}}t + \|\vartheta_0\|_{L^2}^2 + 1,$$

for any $t \in [0, t_0]$. Choose $t^* > 0$ such that

$$t^* \le \min\left\{t_0, \frac{\|\vartheta_0\|_{L^2}^2}{C_m(\bar{\theta})(2\|\vartheta_0\|_{L^2}^2 + 2)^{\frac{6-d}{4-d}}}\right\}, \tag{2.14}$$

then

$$\int_0^{t^*} \|\nabla\vartheta\|_{L^2}^2\,dt \le 2\|\vartheta_0\|_{L^2}^2 + 1, \tag{2.15}$$

which completes the proof. $\square$

2.2. Global regularity

We now show the global regularity of the PKS system using a semigroup approach. First define the nonlinear operator $\mathcal{N}: H^1(\Omega) \to H$ by

$$\mathcal{N}\vartheta = -\nabla \cdot ((\vartheta + \bar{\theta})\nabla c). \tag{2.16}$$

Then system can be rewritten as an abstract Cauchy problem in the state space H

$$\dot{\vartheta} = \mathcal{L}_A\vartheta + \mathcal{N}\vartheta, \tag{2.17}$$

$$\vartheta(0) = \vartheta_0. \tag{2.18}$$

We will investigate the well-posedness and stability of the nonlinear system (2.17)–(2.18) by applying the classic tools of analytic semigroup theory for semilinear equations. To this end, we first define the map

$$(\mathcal{T}\vartheta)(t) = e^{\mathcal{L}_A t}\vartheta_0 + \int_0^t e^{\mathcal{L}_A(t-\tau)}(\mathcal{N}\vartheta)(\tau)\, d\tau \tag{2.19}$$

for any $\vartheta_0 \in H$ and $\vartheta \in C([0,\infty); H)$. Recall that if ϑ is a *mild* solution to (2.17)–(2.18), then by the variation of parameters formula (e.g. [26, Def. 2.3, p. 106])

$$\vartheta(t) = (\mathcal{T}\vartheta)(t). \tag{2.20}$$

Note that the mild solution is a weak solution. The following theorem states the main result of this section. For simplicity, we denote $\Psi(\mathcal{L}_A)$ by Ψ_A.

Theorem 2.2. *Let $\vartheta_0 \in H$ and $\mathbf{v}_A \in L^\infty(\Omega)$ be a divergence-free vector field satisfying $\mathbf{v}_A \cdot n|_\Gamma = 0$. If $\Psi_A = \Psi_A(\vartheta_0, \bar{\theta}) > 0$ is sufficiently large, then there exists a unique mild solution ϑ to (2.17)–(2.18) satisfying*

$$\vartheta \in C([0,\infty); H) \cap L^2_{loc}(0,\infty; H^1(\Omega))$$

and

$$\sup_{t\geq 0} \|\theta\|_{L^2} \leq 2\|\vartheta_0\|_{L^2} + 1. \tag{2.21}$$

Moreover, there exist constants $M_* \geq 1$ and $\omega_0 > 0$ such that

$$\|\vartheta\|_{L^2} \leq M_* e^{-\omega_0 t}\|\vartheta_0\|_{L^2}. \tag{2.22}$$

We will mainly employ the Banach fixed-point theorem to establish the proof. To proceed, we first recall some basic properties of $\mathcal{L}_A$. Since $-\mathcal{L}_A$ is a strictly positive elliptic operator, the fractal powers $(-\mathcal{L}_A)^\sigma$ for $\sigma > 0$ are well-defined with domain $D((-\mathcal{L}_A)^\sigma)$ dense in H (e.g. [26, p. 69]). Moreover, $D((-\mathcal{L}_A)^\sigma)$ equipped with the norm $\|\phi\|_{D((-\mathcal{L}_A)^\sigma)} = \|(-\mathcal{L}_A)^\sigma\phi\|_{L^2}$ for $\phi \in D((-\mathcal{L}_A)^\sigma)$ is the completion of the Hilbert space H under this norm (e.g. [26, p. 195]). By interpolation (e.g. [27–29] and the references cited therein) we know that

$$D((-\mathcal{L}_A)^{-\sigma}) = (D((-\mathcal{L}_A)^\sigma))', \quad \sigma \geq 0, \tag{2.23}$$

where $(D((-\mathcal{L}_A)^\sigma))'$ is the dual space of $D((-\mathcal{L}_A)^\sigma)$. Furthermore, according to (e.g. [28,30]), the Neumann boundary condition allows us to identify the domains of $(-\mathcal{L}_A)^\sigma$ for $0 \leq \sigma \leq 1$ as

$$D((-\mathcal{L}_A)^\sigma) = H^{2\sigma}(\Omega) \cap H, \quad 0 \leq \sigma < \frac{3}{4}, \tag{2.24}$$

$$D((-\mathcal{L}_A)^{3/4}) \subset H^{3/2}(\Omega) \cap H, \tag{2.25}$$

$$D((-\mathcal{L}_A)^\sigma) = \left\{\phi \in H^{2\sigma}(\Omega) \cap H : \frac{\partial\phi}{\partial n}|_\Gamma = 0\right\}, \quad \frac{3}{4} < \sigma \leq 1. \tag{2.26}$$

The concrete characterization of $D((-\mathcal{L}_A)^{3/4})$ can be found in [30, Theorem 2, p. 83].

The following results are concerned with the properties of the analytic semigroup generated by $\mathcal{L}_A$ and the nonlinear operator $\mathcal{N}$ defined by (2.16).

Lemma 2.3. *For $\alpha \geq 0$, there exists a constant $M_\alpha > 0$ only dependent on $M_0 = e^{\pi/2}$ and α such that*

$$\|(-\mathcal{L}_A)^\alpha e^{\mathcal{L}_A t}\|_{\mathscr{L}(H)} \leq M_\alpha t^{-\alpha} e^{-\Psi_A t}, \quad t \geq 0. \tag{2.27}$$

The proof can be easily shown by using (1.13) and [26, Theorem 6.13, p. 74].

Lemma 2.4. *For $\vartheta \in H^1(\Omega)$, then there is a constant $C_1 > 0$ such that*

$$\|\mathcal{N}\vartheta\|_{L^2} \leq C_1(\|\vartheta\|_{H^1}^2 + |\bar{\theta}|\|\vartheta\|_{L^2}). \tag{2.28}$$

Moreover, for $\vartheta \in L^2(\Omega)$, there is a constant $C_2 > 0$ such that

$$\|(-\mathcal{L}_A)^{-\frac{3}{4}}(\mathcal{N}\vartheta)\|_{L^2} \leq C_2(\|\vartheta\|_{L^2}^2 + |\bar{\theta}|\|\vartheta\|_{L^2}). \tag{2.29}$$

Proof. The estimate (2.28) has been established in the proof of [19, Lemma 3.1] on $\mathbb{T}^d, d = 2, 3$. We provide a complete proof for the convenience of the reader. First we have

$$\begin{aligned}
\|\mathcal{N}\vartheta\|_{L^2} &= \|\nabla \cdot ((\vartheta + \bar{\theta})\nabla c)\|_{L^2} \\
&\leq \|\nabla\vartheta \cdot \nabla c\|_{L^2} + \|(\vartheta + \bar{\theta})\Delta c\|_{L^2} \\
&\leq \|\nabla\vartheta\|_{L^2}\|\nabla\mathcal{A}^{-1}\vartheta\|_{L^\infty} + \|\vartheta\Delta\mathcal{A}^{-1}\vartheta\|_{L^2} + |\bar{\theta}|\|\Delta\mathcal{A}^{-1}\vartheta\|_{L^2} \\
&\leq c_1(\|\nabla\vartheta\|_{L^2}\|\vartheta\|_{H^{d/2-1+\epsilon}} + \|\vartheta\|_{H^1}^2 + |\bar{\theta}|\|\vartheta\|_{L^2}) \\
&\leq C_1(\|\vartheta\|_{H^1}^2 + |\bar{\theta}|\|\vartheta\|_{L^2})
\end{aligned}$$

for some constants $c_1, C_1 > 0$ only dependent on Ω, d and ϵ. Moreover, by (2.23) we know that

$$\begin{aligned}
\|(-\mathcal{L}_A)^{-\frac{3}{4}}(\mathcal{N}\vartheta)\|_{L^2} &= \|\nabla \cdot ((\vartheta + \bar{\theta})\nabla c)\|_{(D((-\mathcal{L}_A)^{3/4}))'} \\
&= \sup_{0\neq\phi\in D((-\mathcal{L}_A)^{3/4})} \frac{\int_\Omega \nabla \cdot ((\vartheta + \bar{\theta})\nabla c)\phi\, dx}{\|(-\mathcal{L}_A)^{3/4}\phi\|_{L^2}} \\
&\leq \sup_{0\neq\phi\in D((-\mathcal{L}_A)^{3/4})} \frac{\|\vartheta + \bar{\theta}\|_{L^2}\|\nabla c\|_{L^{2d}}\|\nabla\phi\|_{L^{\frac{2d}{d-1}}}}{\|(-\mathcal{L}_A)^{3/4}\phi\|_{L^2}} \\
&\leq c_2 \sup_{0\neq\phi\in D((-\mathcal{L}_A)^{3/4})} \frac{\|\vartheta + \bar{\theta}\|_{L^2}\ \|_{L^2}\ \|\nabla c\|_{H^{\frac{d-1}{2}}}\|\nabla\phi\|_{H^{1/2}}}{\|(-\mathcal{L}_A)^{3/4}\phi\|_{L^2}} &(2.30)\\
&\leq c_3\|\vartheta + \bar{\theta}\|_{L^2}\|\nabla c\|_{H^1} &(2.31)\\
&\leq C_2(\|\vartheta\|_{L^2}^2 + |\bar{\theta}|\|\vartheta\|_{L^2}), &(2.32)
\end{aligned}$$

for some constants $c_2, c_3, C_2 > 0$ only dependent on Ω and d, where from (2.30) to (2.31) we used $\|\nabla\phi\|_{H^{1/2}} \leq C\|\phi\|_{H^{3/2}} \leq C\|(-\mathcal{L}_A)^{3/4}\phi\|_{L^2}$ based on Poincaré inequality and (2.25). This completes the proof. □

With Lemmas 2.3–2.4 at our disposal, we are in a position to show that the map $\mathcal{T}$ defined by (2.20) has a fixed point in $C([0,\infty); H)$.

Proposition 2.5. *For $\vartheta_0 \in H$, let $\mathcal{O}(0,r) \subset C([0,\infty); H)$ be the ball centered at the origin with any radius $r \geq 2\|\vartheta_0\|_{L^2}$, that is,*

$$\mathcal{O}(0,r) = \{\phi \in C([0,\infty); H) \colon \sup_{t\geq 0}\|\phi\|_{L^2} \leq r\}.$$

Then there exists $\Psi_A = \Psi_A(\vartheta_0, \bar{\theta}) > 0$ large enough such that

(1) $\mathcal{O}(0,r)$ is invariant under $\mathcal{T}$;
(2) $\mathcal{T}$ defined in (2.19) is a contraction mapping on $\mathcal{O}(0,r)$.

Proof (*1*)**.** We first show that $\mathcal{O}(0,r)$ is invariant under $\mathcal{T}$. Using the variation of parameters formula (2.19) together with (1.17) follows

$$\begin{aligned}\sup_{t\geq 0}\|(\mathcal{T}\vartheta)(t)\|_{L^2} &\leq \sup_{t\geq 0}\|e^{\mathcal{L}_A t}\vartheta_0\|_{L^2} + \sup_{t\geq 0}\|\int_0^t e^{\mathcal{L}_A(t-\tau)}(\mathcal{N}\vartheta)(\tau)\,d\tau\|_{L^2}\\ &\leq \|\vartheta_0\|_{L^2} + \sup_{t\geq 0}\int_0^t \|(-\mathcal{L}_A)^{3/4}e^{\mathcal{L}_A(t-\tau)}(-\mathcal{L}_A)^{-3/4}(\mathcal{N}\vartheta)(\tau)\|_{L^2}\,d\tau\\ &\leq \|\vartheta_0\|_{L^2} + M\sup_{t\geq 0}\int_0^t \frac{e^{-\Psi_A(t-\tau)}}{(t-\tau)^{3/4}}\|(-\mathcal{L}_A)^{-3/4}(\mathcal{N}\vartheta)(\tau)\|_{L^2}\,d\tau \qquad (2.33)\\ &\leq \|\vartheta_0\|_{L^2} + M\int_0^\infty \frac{e^{-\Psi_A t}}{t^{3/4}}\,dt\cdot\sup_{t\geq 0}\|(-\mathcal{L}_A)^{-3/4}(\mathcal{N}\vartheta)(t)\|_{L^2} \qquad (2.34)\\ &\leq \|\vartheta_0\|_{L^2} + MC_2(\int_0^{\tilde{t}}\frac{e^{-\Psi_A t}}{t^{3/4}}\,dt + \int_{\tilde{t}}^\infty \frac{e^{-\Psi_A t}}{t^{3/4}}\,dt)\cdot\sup_{t\geq 0}(\|\vartheta\|_{L^2}^2 + |\bar{\theta}|\|\vartheta\|_{L^2})\\ &\leq \|\vartheta_0\|_{L^2} + MC_2(4\tilde{t}^{1/4} + \frac{1}{\tilde{t}^{3/4}\Psi_A})((\sup_{t\geq 0}\|\vartheta\|_{L^2})^2 + |\bar{\theta}|\sup_{t\geq 0}\|\vartheta\|_{L^2}), \qquad (2.35)\end{aligned}$$

for some $\tilde{t} > 0$, where $M = M_{3/4} > 0$ given by (2.27). From (2.33) to (2.34) we used Young's inequality. Letting $\Psi_A \geq \tilde{t}^{-1}$, we have $\frac{1}{\tilde{t}^{3/4}\Psi_A} \leq \tilde{t}^{1/4}$, and therefore

$$4\tilde{t}^{1/4} + \frac{1}{\tilde{t}^{3/4}\Psi_A} \leq 5\tilde{t}^{1/4}. \qquad (2.36)$$

If $\tilde{t}$ is chosen such that

$$5MC_2\tilde{t}^{1/4}(r^2 + |\bar{\theta}|r) \leq \frac{r}{2}, \quad \text{i.e.,} \quad \tilde{t} \leq \frac{1}{\left[10MC_2(2\|\vartheta_0\|_{L^2} + |\bar{\theta}|)\right]^4}, \qquad (2.37)$$

then from (2.35) it follows that

$$\sup_{t\geq 0}\|(\mathcal{T}\vartheta)(t)\|_{L^2} \leq \frac{r}{2} + \frac{r}{2} = r,$$

and hence $\mathcal{T}\vartheta \in \mathcal{O}(0,r)$. In this case, we need

$$\Psi_A \geq \left[10MC_2(2\|\vartheta_0\|_{L^2} + |\bar{\theta}|)\right]^4. \qquad (2.38)$$

Next we show that

$$\mathcal{T}\vartheta \in C([0,\infty);H). \qquad (2.39)$$

Since $e^{\mathcal{L}_A t}, t \geq 0$, is an analytic C_0-semigroup on H,

$$e^{\mathcal{L}_A t}\colon X \to C([0,\infty);H)$$

is continuous. Moreover, recall that

$$\int_0^t e^{\mathcal{L}_A(t-\tau)}(\mathcal{N}\vartheta)(\tau)\,d\tau = \int_0^t (-\mathcal{L}_A)^{\frac{3}{4}}e^{\mathcal{L}_A(t-\tau)}(-\mathcal{L}_A)^{-\frac{3}{4}}(\mathcal{N}\vartheta)(\tau)\,d\tau.$$

By (2.29) in Lemma 2.4 we have $(-\mathcal{L}_A)^{-\frac{3}{4}}\mathcal{N}\vartheta \in C([0,\infty);H)$ for $\vartheta \in C([0,\infty);H)$. Furthermore, by the property of convolution we know that

$$\int_0^t (-\mathcal{L}_A)^{\frac{3}{4}}e^{\mathcal{L}_A(t-\tau)}\cdot\,d\tau\colon C([0,\infty);H) \to C([0,\infty);H)$$

is continuous (e.g. [29, Prop. 01, p. 4]). Thus $\int_0^t e^{\mathcal{L}_A(t-\tau)}(\mathcal{N}\vartheta)(\tau)\,d\tau \in C([0,\infty);H)$ and hence (2.39) holds.

(2) Now we show that $\mathcal{T}$ is a contraction mapping on $\mathcal{O}(0,r)$. For any $\vartheta_1, \vartheta_2 \in C([0,\infty); H)$,

$$\begin{aligned}
\sup_{t\geq 0} \|\mathcal{T}\vartheta_1 - \mathcal{T}\vartheta_2\|_{L^2} &= \sup_{t\geq 0} \| \int_0^t e^{\mathcal{L}_A(t-\tau)} \left[(\mathcal{N}\vartheta_1)(\tau) - (\mathcal{N}\vartheta_2)(\tau)\right] d\tau \|_{L^2} \\
&\leq \sup_{t\geq 0} \int_0^t \|(-\mathcal{L}_A)^{\frac{3}{4}} e^{\mathcal{L}_A(t-\tau)}\|_{\mathscr{L}(X)} \|((-\mathcal{L}_A)^{-\frac{3}{4}}\mathcal{N}\vartheta_1)(\tau) - ((-\mathcal{L}_A)^{-\frac{3}{4}}\mathcal{N}\vartheta_2)(\tau)\|_{L^2}\, d\tau \\
&\leq M \int_0^\infty \frac{e^{-\Psi_A t}}{t^{3/4}}\, dt \cdot \sup_{t\geq 0} \|((-\mathcal{L}_A)^{-\frac{3}{4}}\mathcal{N}\vartheta_1)(\tau) - ((-\mathcal{L}_A)^{-\frac{3}{4}}\mathcal{N}\vartheta_2)(\tau)\|_{L^2},
\end{aligned} \tag{2.40}$$

where by (2.33)–(2.36) we have $\int_0^\infty \frac{e^{-\Psi_A t}}{t^{3/4}}\, dt \leq 5\tilde{t}^{1/4}$ for $\Psi_A \geq \tilde{t}^{-1}$. Using the similar estimates as in (2.32) follows

$$\begin{aligned}
&\|(-\mathcal{L}_A)^{-\frac{3}{4}}\mathcal{N}\vartheta_1 - (-\mathcal{L}_A)^{-\frac{3}{4}}\mathcal{N}\vartheta_2\|_{L^2} \\
&\quad = \sup_{0\neq\phi\in D((-\mathcal{L}_A)^{3/4})} \frac{\int_\Omega \nabla\cdot((\vartheta_1+\bar{\theta})\nabla c_1 - (\vartheta_2+\bar{\theta})\nabla c_2)\phi\, dx}{\|(-\mathcal{L}_A)^{3/4}\phi\|_{L^2}} \\
&\quad \leq \sup_{0\neq\phi\in D((-\mathcal{L}_A)^{3/4})} \frac{\int_\Omega \left|\left[(\vartheta_1+\bar{\theta})\nabla c_1 - (\vartheta_2+\bar{\theta})\nabla c_2\right]\cdot\nabla\phi\right|\, dx}{\|(-\mathcal{L}_A)^{3/4}\phi\|_{L^2}} \\
&\quad = \sup_{0\neq\phi\in D((-\mathcal{L}_A)^{3/4})} \frac{\int_\Omega \left|\left[(\vartheta_1+\bar{\theta} - (\vartheta_2+\bar{\theta}))\nabla c_1 + (\vartheta_2+\bar{\theta})(\nabla c_1 - \nabla c_2)\right]\cdot\nabla\phi\right|\, dx}{\|(-\mathcal{L}_A)^{3/4}\phi\|_{L^2}} \\
&\quad \leq c_3\left(\|\vartheta_1-\vartheta_2\|_{L^2}\|\nabla c_1\|_{H^1} + \|\vartheta_2+\bar{\theta}\|_{L^2}\|\nabla(c_1-c_2)\|_{H^1}\right) \\
&\quad \leq C_2\left(\|\vartheta_1-\vartheta_2\|_{L^2}\|\vartheta_1\|_{L^2} + \|\vartheta_2+\bar{\theta}\|_{L^2}\|\vartheta_1-\vartheta_2\|_{L^2}\right) \\
&\quad \leq C_2\left(\|\vartheta_1\|_{L^2} + \|\vartheta_2\|_{L^2} + |\bar{\theta}|\right)\|\vartheta_1-\vartheta_2\|_{L^2},
\end{aligned}$$

where c_3 and C_2 are the same constants as in (2.31)–(2.32). Therefore, (2.40) becomes

$$\sup_{t\geq 0} \|\mathcal{T}\vartheta_1 - \mathcal{T}\vartheta_2\|_{L^2} \leq 5M\tilde{t}^{1/4}C_2(2r+|\bar{\theta}|)\cdot \sup_{t\geq 0}\|\vartheta_1-\vartheta_2\|_{L^2}. \tag{2.41}$$

With the help of (2.37) we get $5M\tilde{t}^{1/4}C_2(2r+|\bar{\theta}|) < 2\cdot 5M\tilde{t}^{1/4}C_2(r+|\bar{\theta}|) \leq 1$, thus $\mathcal{T}$ is a contraction mapping on $\mathcal{O}(0,r)$. This completes the proof. □

Proof of Theorem 2.2. Based on Proposition 2.1 and Banach fixed-point theorem, there exists a unique solution to system (2.17)–(2.18) satisfying $\vartheta \in C([0,\infty); H)$ and (2.21).

To show that $\theta \in L^2_{loc}(0,\infty; H^1(\Omega))$, we make use of (2.11)–(2.13) and obtain that for any $t_f > t^*$, where t^* is given by (2.14),

$$\int_{t^*}^{t_f} \|\nabla\vartheta\|_{L^2}^2\, dt \leq C_m(\bar{\theta})((\sup_{t\in[t^*,t_f]} \|\vartheta(t)\|)_{L^2}^2 + 1)^{\frac{6-d}{4-d}}(t_f - t^*) + \|\vartheta(t^*)\|_{L^2}^2 \tag{2.42}$$

$$\leq C_m(\bar{\theta})(4\|\vartheta_0\|_{L^2}^2 + 1)^{\frac{6-d}{4-d}}(t_f-t^*) + 4\|\vartheta_0\|_{L^2}^2. \tag{2.43}$$

Combining (2.43) with (2.15) yields the desired result.

To establish the exponential decay of the solution in (2.22), using (2.28) in Lemma 2.4 we have for $0 \leq t \leq \tau^*$,

$$\begin{aligned}
\|\vartheta(\tau^*)\|_{L^2} &\leq \|e^{\mathcal{L}_A\tau^*}\vartheta_0\|_{L^2} + \|\int_0^{\tau^*} e^{\mathcal{L}_A(\tau^*-s)}(\mathcal{N}\vartheta)(s)\, ds\|_{L^2} \\
&\leq M_0 e^{-\Psi_A\tau^*}\|\vartheta_0\|_{L^2} + \int_0^{\tau^*} \|(-\mathcal{L}_A)^{3/4}e^{\mathcal{L}_A(\tau^*-s)}(-\mathcal{L}_A)^{-3/4}(\mathcal{N}\vartheta)(s)\|_{L^2}\, ds
\end{aligned}$$

$$\begin{aligned}
&\leq M_0 e^{-\Psi_A \tau^*}\|\vartheta_0\|_{L^2} + M\int_0^{\tau^*}\frac{e^{-\Psi_A t}}{t^{3/4}}\,dt\cdot\sup_{0\leq t\leq\tau^*}\|\mathcal{L}_A^{-3/4}(\mathcal{N}\vartheta)(t)\|_{L^2}\\
&\leq M_0 e^{-\Psi_A \tau^*}\|\vartheta_0\|_{L^2} + 4MC_2{\tau^*}^{1/4}((\sup_{0\leq t\leq\tau^*}\|\vartheta\|_{L^2})^2 + |\bar\theta|\sup_{0\leq t\leq\tau^*}\|\vartheta\|_{L^2})\\
&\leq M_0 e^{-\Psi_A \tau^*}\|\vartheta_0\|_{L^2} + 4MC_2{\tau^*}^{1/4}(4\|\vartheta_0\|_{L^2}^2 + 2|\bar\theta|\|\vartheta_0\|_{L^2}). \qquad (2.44)
\end{aligned}$$

Let $\tau^* > 0$ satisfy

$$4MC_2{\tau^*}^{1/4}(4\|\vartheta_0\|_{L^2}^2 + 2|\bar\theta|\|\vartheta_0\|_{L^2}) \leq \frac{1}{2}\|\vartheta_0\|_{L^2},$$

$$\text{i.e.,}\quad \tau^* \leq \frac{1}{\left[16MC_2(2\|\vartheta_0\|_{L^2}+|\bar\theta|)\right]^4}, \qquad (2.45)$$

and set Ψ_A large enough such that

$$\frac{1}{2} + M_0 e^{-\Psi_A\tau^*} \leq \eta < 1,$$

for some η satisfying $\frac{1}{2} < \eta < 1$, that is,

$$\Psi_A \geq \frac{\ln M_0 - \ln(\eta-\frac{1}{2})}{\tau^*} \geq \left(\frac{\pi}{2} - \ln\big(\eta - \frac{1}{2}\big)\right)\left[16MC_2(2\|\vartheta_0\|_{L^2}+|\bar\theta|)\right]^4, \qquad (2.46)$$

where $\ln(\eta-\frac{1}{2}) < 0$. Then from (2.44) we have $\|\vartheta(\tau^*)\|_{L^2} \leq \eta\|\vartheta_0\|_{L^2}$. Note that if Ψ_A satisfies (2.46), then it also satisfies (2.38).

Finally, through an iterative process we obtain

$$\|\vartheta(m\tau^*)\|_{L^2} \leq \eta\|\vartheta((m-1)\tau^*)\|_{L^2} \leq \eta^m\|\vartheta_0\|_{L^2}, \quad m = 1, 2, \ldots \qquad (2.47)$$

for a fixed τ^* satisfying (2.45). Once (2.47) is established, the exponential decay of (2.22) holds immediately following the same procedure as in [31, Remark, p. 178]. □

3. Rapid decay via rescaling the cellular flows

Many flows are not necessarily relaxation-enhancing, yet the associated semigroups can still have the rapid decay property via rescaling the flows. To demonstrate the idea, we employ the cellular flows as shown in [19] for generating the velocity fields in rectangle-like domains. In two dimensions, the cellular flows have closed obits and particles away from the boundary are nearly trapped in them, and therefore are not mixing. In this section, we present that in rectangles ($d = 2$) and parallelepipeds ($d = 3$), rescaling the flows can make $\Psi(\mathcal{L}_A)$ arbitrarily large and hence the rapid decay property of the semigroup $e^{\mathcal{L}_A t}$ can be achieved. Note that for such domains the interpolations results (2.24)–(2.26) also hold (e.g. [28, section 6.3]), so do the Gagliardo–Nirenberg, Agmon's and Sobolev inequalities used in our proofs of Proposition 2.1 and Lemma 2.4. Therefore, Proposition 2.5 and Theorem 2.2 still hold.

Without loss of generality, we let $\Omega = (0,1)^d, d = 2, 3$. Here we consider a typical 2D cellular flow given by

$$\mathbf{v}(x_1,x_2) = \nabla^\perp \sin(2\pi x_1)\sin(2\pi x_2) = 2\pi\begin{bmatrix}-\sin(2\pi x_1)\cos(2\pi x_2)\\ \cos(2\pi x_1)\sin(2\pi x_2)\end{bmatrix}. \qquad (3.1)$$

For $d = 3$, we consider the flow with cubic cells given by (e.g. [19,32,33])

$$\mathbf{v}(x_1,x_2,x_3) = (\Phi_{x_1}(x_1,x_2)W'(x_3), \Phi_{x_2}(x_1,x_2)W'(x_3), 8\pi^2\Phi(x_1,x_2)W(x_3)), \qquad (3.2)$$

where $\Phi(x_1,x_2) = \cos(2\pi x_1)\cos(2\pi x_2)$ and $W(x_3) = \sin(2\pi x_3)$. Since the cellular flows and the basis functions are periodic, they can be naturally extended to $N\Omega = (0,N)^d$ for $N \in \mathbb{N}^+$.

Let $\mathcal{L}_1 = \Delta - \mathbf{v}(x)\cdot\nabla$, where $x = (x_1,\ldots,x_d)$ and $\mathbf{v} = (v_1,\ldots,v_d)$, $d = 2,3$, with $D(\mathcal{L}_1) = \{\psi \in H^1(\Omega)\cap H : \frac{\partial\psi}{\partial n}|_\Gamma = 0\}$. Let $v_{i_N}(x) = v_i(Nx)$ for $x\in\Omega$ and $\mathbf{v}_N = (v_{1_N},\ldots,v_{d_N})$. According to (3.1)–(3.2) the rescaled cellular flow velocity $\mathbf{v}_N$ is still sufficiently smooth and periodic, yet with higher frequency compared to $\mathbf{v}$. Now define

$$\mathcal{L}_N = \Delta - N\mathbf{v}_N(x)\cdot\nabla, \quad N\in\mathbb{N}^+,$$

with $D(\mathcal{L}_N) = D(\mathcal{L}_1)$. One can establish the following property.

Proposition 3.1. *Let $\Psi(\mathcal{L}_N)$ be defined as in (1.12) for $N\in\mathbb{N}^+$. We have*

$$\Psi(\mathcal{L}_N) = N^2\,\Psi(\mathcal{L}_1). \tag{3.3}$$

Remark 3.2. As a result of (1.13) and (3.3), we obtain that

$$\|e^{\mathcal{L}_N t}\|_{\mathscr{L}(X)} \le M_0 e^{-\Psi(\mathcal{L}_N)t} = M_0 e^{-N^2\,\Psi(\mathcal{L}_1)t}, \quad t\ge 0.$$

Therefore, the decay rate of the semigroup $e^{\mathcal{L}_N t}$ can be made arbitrarily fast if N is sufficiently large. In other words, the semigroup generated by $\mathcal{L}_N$ associated with the cellular flows possesses the rapid decay property (1.6).

Proof of Proposition 3.1. The proof of the relation (3.3) for cellular flows in rectangle-like domains directly utilizes the definition of $\Psi(\mathcal{L}_N)$ and the properties of the eigenvalues and the corresponding eigenfunctions of the Laplacian operator defined in such domains. To be more precise, recall that the eigenvalues and the corresponding eigenfunctions of the Laplacian operator $\mathscr{A} = \Delta$ with $D(\mathscr{A}) = \{\psi \in H^1(\Omega)\cap H : \frac{\partial\psi}{\partial n}|_\Gamma = 0\}$ are given by

$$\lambda_{n_1,\ldots,n_d} = \lambda^{(1)}_{n_1} + \cdots + \lambda^{(d)}_{n_d} \quad \text{and} \quad \psi_{n_1,\ldots,n_d}(x_1,\ldots,x_d) = \psi^{(1)}_{n_1}(x_1)\ldots\psi^{(d)}_{n_d}(x_d),$$

where

$$\lambda^{(i)}_n = -n^2\pi^2, \quad \psi^{(i)}_n(x_i) = \cos(n\pi x_i), \quad i = 1,\ldots d,$$

for $n = 1,2,\ldots$ (e.g. [34]), and $\{\psi_{n_1,\ldots,n_d}\}^\infty_{n_1,\ldots,n_d=1}$ forms a complete orthogonal basis of H. Since

$$\int_0^1 \cos(n\pi x_i)\cos(m\pi x_i)\,dx_i = \begin{cases} 0 & \text{if } m\ne n,\\ \frac{1}{2} & \text{if } m = n,\end{cases} \tag{3.4}$$

we have $\|\psi_{n_1,\ldots,n_d}\|^2_{L^2} = \frac{1}{2^d}$ for any $n_i\in\mathbb{N}^+, i = 1,\ldots,d$. For any $\phi\in D(\mathcal{L}_1)$, it can be expressed as

$$\phi(x) = \sum_{n_1,\ldots,n_d=1}^{\infty} \phi_{n_1,\ldots,n_d}\psi_{n_1,\ldots,n_d}(x), \tag{3.5}$$

where $\phi_{n_1,\ldots,n_d} = 2^d\int_\Omega \phi(x)\psi_{n_1,\ldots,n_d}(x)\,dx$. Let

$$\phi_N(x) = \sum_{n_1,\ldots,n_d=1}^{\infty} \phi_{n_1,\ldots,n_d}\psi_{n_1,\ldots,n_d}(Nx), \tag{3.6}$$

where $\psi_{n_1,\ldots,n_d}(Nx) = \cos(Nn_1\pi x_1)\ldots\cos(Nn_d\pi x_d) = \psi_{Nn_1,\ldots,Nn_d}(x)$ are also the eigenfunctions of $\mathscr{A}$. Then $\phi_N\in D(\mathcal{L}_N)$ and $\|\phi_N\|_{L^2} = \|\phi\|_{L^2}$. Rescaling the cell size and the flow amplitude essentially transits the energy of ϕ from the lower to the higher eigenmodes.

Next we show that for each basis function $\psi_{n_1,\ldots,n_d}$,

$$\|(\Delta_x - N\mathbf{v}_N(x)\cdot\nabla_x - iN^2\lambda)\psi_{Nn_1,\ldots,Nn_d}\|_{L^2} = N^2\|(\Delta_x - \mathbf{v}(x)\cdot\nabla_x - i\lambda)\psi_{n_1,\ldots,n_d}\|_{L^2}, \tag{3.7}$$

for $\lambda \in \mathbb{R}$. Let $\mathbf{x}_i = Nx_i, i = 1, \dots, d$, and $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_d)$. We have

$$\begin{aligned}
&\|(\Delta_x - N\mathbf{v}_N(x)\cdot\nabla_x - iN^2\lambda)\psi_{Nn_1,\dots,Nn_d}\|_{L^2}^2 \\
&\quad = \int_{(0,1)^d}((\Delta_x - N\mathbf{v}_N(x)\cdot\nabla_x - iN^2\lambda)\psi_{Nn_1,\dots,Nn_d}(x))^2\,dx \\
&\quad = \frac{1}{N}\int_{(0,N)^d}((N^2\lambda_{n_1,\dots,n_d} - iN^2\lambda - N^2\mathbf{v}(\mathbf{x})\cdot\nabla_{\mathbf{x}})\psi_{n_1,\dots,n_d}(\mathbf{x}))^2\,d\mathbf{x} \\
&\quad = N^3(\lambda_{n_1,\dots,n_d} - i\lambda)^2\int_{(0,N)^d}\psi^2_{n_1,\dots,n_d}(\mathbf{x})\,d\mathbf{x} + N^3\int_{(0,N)^d}(\mathbf{v}(\mathbf{x})\cdot\nabla_{\mathbf{x}}\psi_{n_1,\dots,n_d}(\mathbf{x}))^2\,d\mathbf{x},
\end{aligned} \tag{3.8}$$

where

$$\int_{(0,N)^d}\psi^2_{n_1,\dots,n_d}(\mathbf{x})\,d\mathbf{x} = N\int_{(0,1)^d}\psi^2_{Nn_1,\dots,Nn_d}(x)\,dx = N\int_{(0,1)^d}\psi^2_{n_1,\dots,n_d}(x)\,dx \tag{3.9}$$

and

$$\begin{aligned}
&\int_{(0,N)^d}(\mathbf{v}(\mathbf{x})\cdot\nabla_{\mathbf{x}}\psi_{n_1,\dots,n_d}(\mathbf{x}))^2\,d\mathbf{x} = N\int_{(0,1)^d}(\mathbf{v}_N(x)\cdot\nabla_x\psi_{Nn_1,\dots,Nn_d}(x)\frac{1}{N})^2\,dx \\
&\quad = \frac{1}{N}\int_{(0,1)^d}\Big(\sum_{i=1}^d v_{i_N}(x)\partial_{x_i}(\psi^{(1)}_{Nn_1}(x_1)\dots\psi^{(d)}_{Nn_d}(x_d))\Big)^2\,dx \\
&\quad = \frac{1}{N}\int_{(0,1)^d}\Big(\sum_{i=1}^d v_{i_N}(x)(-Nn_i\pi)\sin(Nn_i\pi x_i)\Pi^d_{j=1,j\neq i}\cos(Nn_j\pi x_j)\Big)^2\,dx \qquad (3.10)\\
&\quad = \frac{1}{N}\int_{(0,1)^d}N^2\Big(\sum_{i=1}^d v_i(x)(-n_i\pi)\sin(n_i\pi x_i)\Pi^d_{j=1,j\neq i}\cos(n_j\pi x_j)\Big)^2\,dx \qquad (3.11)\\
&\quad = N\int_{(0,1)^d}(\mathbf{v}(x)\cdot\nabla_x\psi_{n_1,\dots,n_d}(x))^2\,dx. \qquad (3.12)
\end{aligned}$$

From (3.10) to (3.11), we used the following elementary properties of the products of trigonometric functions involving sin and cos in the cellular flow velocity defined by (3.1)–(3.2):

$$\begin{aligned}
&\int_0^1(\sin(2\pi Nx_i)\sin(n_i\pi Nx_i))^2\,dx_i = \frac{1}{4} \\
&\int_0^1(\cos(2\pi Nx_i)\cos(n_i\pi Nx_i))^2\,dx_i = \frac{1}{4}, \\
\text{and}\quad &\int_0^1\sin(2\pi Nx_i)\sin(n_i\pi Nx_i)\cos(2\pi Nx_i)\cos(n_i\pi Nx_i)\,dx_i = 0, \quad i = 1,\dots,d.
\end{aligned}$$

Note that these identities are independent of N. Combining (3.8) with (3.9) and (3.12) follows

$$\begin{aligned}
&\|(\Delta_x - N\mathbf{v}_N(x)\cdot\nabla_x - iN^2\lambda)\psi_{Nn_1,\dots,Nn_d}\|_{L^2}^2 \\
&\quad = N^4(\lambda_{n_1,\dots,n_d} - i\lambda)^2\int_{(0,1)^d}\psi^2_{n_1,\dots,n_d}(x)\,dx + N^4\int_{(0,1)^d}(\mathbf{v}(x)\cdot\nabla_x\psi_{n_1,\dots,n_d}(x))^2\,dx \\
&\quad = N^4\int_{(0,1)^d}(\Delta_x - i\lambda + \mathbf{v}(x)\cdot\nabla_x\psi_{n_1,\dots,n_d}(x))^2\,dx,
\end{aligned} \tag{3.13}$$

which establishes (3.7). Similarly, one can show that (3.7) also holds if replacing $\psi_{n_1,\dots,n_d}$ and $\psi_{Nn_1,\dots,Nn_d}$ by ϕ and ϕ_N defined by (3.5) and (3.6), respectively. That is,

$$\|(\mathcal{L}_N - iN^2\lambda)\phi_N\|_{L^2} = N^2\|(\mathcal{L}_1 - i\lambda)\phi\|_{L^2}, \quad \lambda \in \mathbb{R}, \tag{3.14}$$

for any $\phi \in D(\mathcal{L}_1)$. This implies

$$\Psi(\mathcal{L}_N) \leq N^2\,\Psi(\mathcal{L}_1).$$

If there exist some $\hat{\phi}$ and $\hat{\lambda} \in \mathbb{R}$ such that $\|(\mathcal{L}_N - i\hat{\lambda})\hat{\phi}\|_{L^2} < N^2 \Psi(\mathcal{L}_1)$. Then

$$\|(\mathcal{L}_1 - i\frac{\hat{\lambda}}{N^2})\hat{\phi}(\frac{x}{N})\|_{L^2} = \frac{1}{N^2}\|(\mathcal{L}_N - i\hat{\lambda})\hat{\phi}\|_{L^2} < \Psi(\mathcal{L}_1),$$

which leads to a contradiction. Therefore, $\Psi(\mathcal{L}_N) = N^2 \Psi(\mathcal{L}_1)$. This completes the proof. □

The cellular flows and rectangle-like domains considered in this section are very special (which have certain symmetric structures and the cellular flows are also periodic). A natural question that arises here is what are the characterizations of the flows so that a simple rescaling is able to enhance dissipation. This question is nontrivial and certainly merits our further investigation in our future work.

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