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On the Montgomery–Odlyzko method regarding gaps between zeros of the zeta-function

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ABSTRACT

Assuming the Riemann Hypothesis, it is known that there are infinitely many consecutive pairs of zeros of the Riemann zeta-function within 0.515396 times the average spacing. This is obtained using the method of Montgomery and Odlyzko. We prove that this method can never find infinitely many pairs of consecutive zeros within 0.5042 times the average spacing.

1. Introduction

The existence of Landau–Siegel zeros (or the Alternative Hypothesis) implies that there are long ranges where all the zeros of the Riemann zeta-function are always spaced no closer than one half of the average spacing. Numerical evidence, however, strongly agrees with the GUE model that suggests there is a positive proportion of consecutive zeros within any small multiple of the average spacing, a conclusion that is also a consequence of Montgomery's pair correlation conjecture. There are three methods in the literature used to study small spacings between zeros of the zeta-function (see [7] and [2], [6], and [9].) The Montgomery–Odlyzko method [7] produces superior results, albeit for infinitely many consecutive zeros and under assumption of the Riemann Hypothesis (RH). Nevertheless, we are interested in how far one can push this method.

Let us state the problem more precisely. Write the nontrivial zeros of the Riemann zeta-function $\zeta(s)$ as $\rho = \beta + i\gamma$, where $\beta \in (0,1)$ and $\gamma \in \mathbb{R}$. Let $0 < \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \leq \cdots$ denote the ordinates of the nontrivial zeros of $\zeta(s)$ in the upper half-plane. Since

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$$N(T) = \sum_{0<\gamma\le T} 1 \sim \frac{T}{2\pi}\log T,$$

it follows that the gap between consecutive zeros $\gamma_{n+1} - \gamma_n$ is $2\pi/\log\gamma_n$ on average. To examine how far gaps deviate from the average, define

$$\mu = \liminf_{n\to\infty} \frac{\gamma_{n+1}-\gamma_n}{2\pi/\log\gamma_n} \quad \text{and} \quad \lambda = \limsup_{n\to\infty} \frac{\gamma_{n+1}-\gamma_n}{2\pi/\log\gamma_n}.$$

Trivially, we have that $\mu \le 1 \le \lambda$, and it is expected that $\mu = 0$ and $\lambda = \infty$. After much work, the best current result for small gaps under RH is $\mu \le 0.515396$ by Preobrazhenskiĭ [8] and for large gaps under RH is $\lambda \ge 3.18$ by Bui and Milinovich [1]. We refer the reader to [4] for the history of and progress on this problem.

The result of [8] is obtained by an argument based on a method introduced by Montgomery and Odlyzko [7] and simplified by Conrey, Ghosh, and Gonek [2]. Assuming RH, let $n_h(t)$ be the number of zeros $1/2+i\gamma$ of the Riemann zeta-function with γ in the interval $[t-\frac{h}{2}, t+\frac{h}{2}]$ of length h. We assume $t \asymp T$, where T is large, and take $h = 2\pi c/\log T$ so that the interval has length c times the average spacing between zeros in this range. Define, for $T \ge 2$, $c > 0$, $y = T^{1-\delta}$ for some $\delta > 0$, and a_k a sequence of complex numbers,

$$H(c) := \frac{\displaystyle\int_T^{2T} n_h(t) \left|\sum_{k\le y} \frac{a_k}{k^{it}}\right|^2 dt}{\displaystyle\int_T^{2T} \left|\sum_{k\le y} \frac{a_k}{k^{it}}\right|^2 dt}.$$

Clearly if $H(c) > 1$ for all sufficiently large T with some choice of the a_k's, c, and a small positive δ, then there must exist a value of $t \in [T, 2T]$ with $n_h(t) > 1$, which implies $n_h(t) \ge 2$ for this value of t. Hence $\mu \le c$. Similarly, for large gaps, if we have $H(c) < 1$ then we obtain $\lambda \ge c$. By [7] or [2] one obtains $H(c) \sim h(c)$ as $T \to \infty$, with $y = T^{1-\delta}$ for some small $\delta > 0$, and

$$h(c) := c - \frac{\displaystyle\Re \sum_{kn\le y} a_k \overline{a_{kn}} g(n) \frac{\Lambda(n)}{n^{1/2}}}{\displaystyle\sum_{k\le y} |a_k|^2}, \qquad \text{where} \qquad g(n) = \frac{2\sin\left(\frac{\pi c \log n}{\log T}\right)}{\pi \log n}. \tag{1}$$

Notice that $g(n)$ is continuous and differentiable in c for all $c > 0$, and therefore so is $h(c)$.

Conrey, Ghosh, and Gonek [2] showed that, for any choice of a_k

$$h(c) < 1 \qquad \text{if} \qquad c < 1/2, \tag{2}$$

which shows that the Montgomery–Odlyzko method is unable to obtain $\mu < 1/2$. Due to the connection to Landau–Siegel zeros, which is described explicitly in [3], it is a tantalizing hope that we might nevertheless reach this barrier. We prove, however, that the Montgomery–Odlyzko method falls well short of being able to prove $\mu \le 1/2$. Thus a new idea is needed to make further progress on this problem.

Theorem 1. *If $c < 0.5042$, then $h(c) < 1$.*

We note that a much weaker version of this result has been known to the experts for some time via unpublished work of the first-named author. We also mention the following information concerning limitations of the Montgomery–Odlyzko method for large gaps between zeros. Conrey, Ghosh, and Gonek [2, p. 423] showed that $h(c) > 1$ if $c \geq 6.2$, whence, the Montgomery–Odlyzko method cannot prove the existence of gaps at least 6.2 times the average spacing. In a note added in the proof stage of their paper, Conrey, Ghosh, and Gonek remark that 6.2 may be replaced by 3.74. Correcting for a misprint in their paper, their first result is based on the inequality

$$h(c) \geq c - 2\left(\frac{c}{\pi}\int_0^1 \frac{|\sin \pi c v|}{v}\,dv\right)^{1/2}. \tag{3}$$

Using *Mathematica* one finds that $h(c) > 1$ for $c \geq 5.5602\ldots$. Their second improvement result can be obtained from the inequality

$$h(c) \geq c - 2\left(\frac{c}{\pi}\int_0^{\pi c} \left(\frac{\sin v}{v}\right)^2 dv\right)^{1/2} \tag{4}$$

proved by a small change in the proof of the previous bound. One now finds with *Mathematica* that $h(c) > 1$ if $c \geq 3.6747\ldots$.

We note that the work by Bui and Milinovich [1] uses a different method based on the work of Hall [5] and hence is not limited in this way.

2. Proof of Theorem 1

We take $0 < c < 1$. Letting $a_k = b_k k^{-1/2}$, we obtain from (1) that

$$h(c) \leq c + \frac{S}{\sum_{k\leq y}\frac{|b_k|^2}{k}}, \quad \text{where} \quad S = \sum_{kn\leq y} \frac{|b_k||b_{kn}||g(n)|\Lambda(n)}{kn}. \tag{5}$$

For any α, $\beta > 0$ with $4\alpha\beta \geq 1$, we have $|ab| \leq \alpha|a|^2 + \beta|b|^2$, and therefore

$$S \leq \alpha \sum_{kn\leq y} \frac{|b_k|^2}{k}|g(n)|\frac{\Lambda(n)}{n} + \beta \sum_{kn\leq y} \frac{|b_{kn}|^2}{kn}|g(n)|\Lambda(n) =: \alpha S_1 + \beta S_2. \tag{6}$$

Using $|\sin x| \leq |x|$, we have for $1 \leq u \leq y$ and $0 < c < 1$

$$0 < g(u) = \frac{2\sin\left(\frac{\pi c \log u}{\log T}\right)}{\pi \log u} \leq \frac{2c}{\log T}. \tag{7}$$

To evaluate S_1, we have

$$S_1 = \sum_{k\leq y} \frac{|b_k|^2}{k} H(y/k), \qquad \text{where} \qquad H(x) := \sum_{n\leq x} g(n)\frac{\Lambda(n)}{n}. \tag{8}$$

Using partial summation with the elementary asymptotic formula

$$L(x) := \sum_{n\leq x} \frac{\Lambda(n)}{n} = \log x + O(1), \tag{9}$$

we have

$$
\begin{aligned}
H(x) &= \int_1^x g(u)dL(u) = L(u)g(u)\Big|_1^x - \int_1^x L(u)g'(u)\,du \\
&= L(x)g(x) - \int_1^x (\log u + O(1))\,g'(u)\,du \\
&= \Big(g(x)\log x + O(g(x)\Big) - \left(g(x)\log x - \int_1^x \frac{g(u)}{u}\,du + O\left(\int_1^x |g'(u)|\,du\right)\right) \\
&= \int_1^x \frac{g(u)}{u}\,du + O(g(x)) + O\left(\int_1^x |g'(u)|\,du\right).
\end{aligned}
$$

By (7) $g(u) \ll 1/\log T$, and since $x\cos x - \sin x \ll x^3$ for $0 \le x \ll 1$,

$$
g'(u) = \frac{2}{\pi}\left(\frac{\cos\left(\frac{\pi c \log u}{\log T}\right)\frac{\pi c\log u}{\log T} - \sin\left(\frac{\pi c\log u}{\log T}\right)}{u\log^2 u}\right) \ll \frac{\log u}{u\log^3 T},
$$

we have $\int_1^x |g'(u)|\,du \ll \log^2 x/\log^3 T$, and hence

$$
H(x) = \int_1^x \frac{g(u)}{u}\,du + O\left(\frac{\log^2 x}{\log^3 T}\right).
$$

Since $y \le T$, we thus conclude

$$
\begin{aligned}
S_1 &= \sum_{k\le y}\frac{|b_k|^2}{k}\left(\int_1^{y/k}\frac{g(u)}{u}\,du + O\left(\frac{1}{\log T}\right)\right) \\
&= \sum_{k\le y}\frac{|b_k|^2}{k}\left(\frac{2}{\pi}\int_0^{\frac{\pi c\log(y/k)}{\log T}}\frac{\sin v}{v}\,dv + O\left(\frac{1}{\log T}\right)\right),
\end{aligned}
$$

where we made the change of variable $v = \pi c\log u/\log T$ in the last integral.

For S_2 we use (7) and the elementary relation $\sum_{d|n}\Lambda(d) = \log n$, to obtain

$$
S_2 \le \frac{2c}{\log T}\sum_{kn\le y}\frac{|b_{kn}|^2}{kn}\Lambda(n) = \frac{2c}{\log T}\sum_{m\le y}\frac{|b_m|^2}{m}\sum_{n|m}\Lambda(n) = \frac{2c}{\log T}\sum_{k\le y}\frac{|b_k|^2}{k}\log k.
$$

Hence from (6) we obtain

$$
\begin{aligned}
S &\le \sum_{k\le y}\frac{|b_k|^2}{k}\left(\frac{2\beta c\log k}{\log T} + \frac{2\alpha}{\pi}\int_0^{\frac{\pi c\log(y/k)}{\log T}}\frac{\sin v}{v}\,dv + O\left(\frac{1}{\log T}\right)\right) \\
&= \sum_{k\le y}\frac{|b_k|^2}{k}G\left(\frac{\log k}{\log T}\right) + O\left(\frac{1}{\log T}\right),
\end{aligned}
\tag{10}
$$

where, for $0 \le x \le 1-\delta$,

$$G(x) = G(x,\alpha,\beta,c) := 2\beta cx + \frac{2\alpha}{\pi} \int_0^{\pi c(1-\delta-x)} \frac{\sin v}{v}\, dv. \tag{11}$$

We thus conclude, for $0 < c < 1$,

$$\frac{S}{\sum_{k\le y} \frac{|b_k|^2}{k}} \le \max_{0\le x\le 1-\delta} G(x) + O\left(\frac{1}{\log T}\right). \tag{12}$$

Since $G(x)$ is continuous and differentiable on $[0, 1-\delta]$, the maximum above exists and occurs at either a critical point or at an endpoint of the interval. It is also clear from (6) that the smallest maximum occurs when

$$4\alpha\beta = 1, \tag{13}$$

which we henceforth assume. By the fundamental theorem of calculus, for $0 \le x \le 1-\delta$,

$$G'(x) = 2\alpha c\Big(\frac{\beta}{\alpha} - \operatorname{sinc}(\pi c(1-\delta-x))\Big), \quad \text{where} \quad \operatorname{sinc}(u) := \frac{\sin u}{u}. \tag{14}$$

Case 1. Suppose $\beta \ge \alpha$. Since $\operatorname{sinc}(u) \le 1$ and equal to 1 if and only if $u = 0$, from (14) $G(x)$ is increasing on $[0, 1-\delta]$ and $\max G(x) = G(1-\delta) = 2\beta c(1-\delta) \le 2\beta c$. We choose the smallest value of β by taking $\beta = \alpha = 1/2$, which from (5) and (12) recovers (2).[1]

Case 2. Suppose $\beta < \alpha$. Thus by (13) $\beta < 1/2$. Since $\operatorname{sinc}(\pi c(1-\delta-x))$ increases for $x \in [0, 1-\delta]$, we see $G'(x)$ decreases through this interval. By (14) $G'(0) > 0$ when $4\beta^2 > \operatorname{sinc}(\pi c(1-\delta))$, and it is easy to check with *Mathematica* that

$$G'(0) > 0 \qquad \text{provided} \qquad 0.5 \le c \le 0.52,\ 0.42 \le \beta < 0.5,\ \text{and } 0 \le \delta \le 0.1, \tag{15}$$

and we may immediately see that $G'(1-\delta) = 2\alpha c\left(\frac{\beta}{\alpha} - 1\right) < 0$. Thus, with β, c, and δ in the range given in (15), there is a unique value $x = x_0$ with $G'(x_0) = 0$ with $x_0 \in [0, 1-\delta]$, and thus $G(x)$ has a relative maximum at $x = x_0$ which is also the absolute maximum on $[0, 1-\delta]$.

We conclude by (12) that

$$\frac{S}{\sum_{k\le y} \frac{|b_k|^2}{k}} \le G(x_0) + O\left(\frac{1}{\log T}\right), \tag{16}$$

where x_0 satisfies

$$\operatorname{sinc}\left(\pi c(1-\delta-x_0)\right) = \frac{\beta}{\alpha} = 4\beta^2, \tag{17}$$

and by (5)

$$h(c) \le c + G(x_0) + O\left(\frac{1}{\log T}\right). \tag{18}$$

[1] See the last section for comments on how this approach differs from that of [2].

Using *Mathematica* it is easy to compute the largest c obtainable from (11), (17), and (18) for which $h(c) < 1$. In performing computations δ can be taken arbitrarily smaller than the accuracy being used in the calculations, and therefore for computations we can take $\delta = 0$ and thus $y = T$ in (16). Given values for c and β we can use (17) to find x_0 and thus $G(x_0)$, or alternatively, we can directly compute the maximum of $G(x)$. We start with an initial choice of $c = c_1 = 0.5$. Searching with a grid of values of β we determine an interval for β where $h(c_1) < 1$. Next we replace c_1 by a slightly larger value c_2 and repeat the process. This method quickly converges. We can stop this process whenever we attain as many digits of accuracy as we desire, at which point we have found values β_n, c_n, and use (17) to compute x_n where the maximum of $G(x)$ occurs. We now check directly that $h(c_n) \le c_n + G(x_n, 1/4\beta_n, \beta_n, c_n) < 1$. In this way we find $c_n = 0.5042$, $\beta_n = 0.476$, $x_n = 0.51974624430935\ldots$, and $h(c_n) = 0.99999350103135\ldots$.

3. A comment on the approach

In the previous section we recovered the result (2) of [2] in the simple case that $\beta \ge \alpha$. Conrey, Ghosh, and Gonek's proof of (2) in [2, pp. 422-423] is different, which we describe here for the interested reader. There the authors use (7) and (9) to obtain

$$S_1 \le \frac{2c}{\log T} \sum_{k \le y} \frac{|b_k|^2}{k} \sum_{n \le y/k} \frac{\Lambda(n)}{n} = \frac{2c}{\log T} \sum_{k \le y} \frac{|b_k|^2}{k} \Big(\log(y/k) + O(1) \Big). \tag{19}$$

Thus, in place of (10) they obtain

$$S \le \frac{2c}{\log T} \sum_{k \le y} \frac{|b_k|^2}{k} \big(\alpha \log(y/k) + \beta \log k + O(1) \big).$$

Letting $f(u) = \alpha \log(y/u) + \beta \log u$, one finds that $f(1) = \alpha \log y$, $f(y) = \beta \log y$, and $f'(u) = \frac{\beta - \alpha}{u}$, and thus $f(u) \le \max(\alpha, \beta) \log y$ for $1 \le u \le y$. The optimal bound is obtained by taking $\alpha = \beta = 1/2$, and with this choice

$$S \le \frac{c \log y + O(1)}{\log T} \sum_{k \le y} \frac{|b_k|^2}{k} \le c \sum_{k \le y} \frac{|b_k|^2}{k}.$$

Substituting into (5) the authors obtain $h(c) \le 2c$, and thus (2). Actually in [2] the usual choice $\alpha = \beta = 1/2$ was used in the argument, which we now see is also the optimal choice when using (19).

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