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ORIGINAL PAPER

Improved Lower Bounds on the Extrema of Eigenvalues of Graphs

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Abstract
In this note, we improve the lower bounds for the maximum size of the kth largest eigenvalue of the adjacency matrix of a graph for several values of k. In particular, we show that closed blowups of the icosahedral graph improve the lower bound for the maximum size of the fourth largest eigenvalue of a graph, answering a question of Nikiforov.

Keywords Eigenvalues · Icosahedral graph · kth largest eigenvalue

Mathematics Subject Classification 05C50 · 05D99

1 Introduction

How large can the kth largest eigenvalue of a graph G on n vertices be? The graphs $kK_{\frac{n}{k}}$ show that the kth largest eigenvalue can be at least $\frac{n}{k} - 1$ (we assume n is a multiple of k here for simplicity). Can this easy lower bound be improved?

To fix notation, for a graph G on n vertices, we denote the eigenvalues of the adjacency matrix of G by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Following Nikiforov [4], we define $\lambda_k(n) = \max_{|V(G)|=n} \lambda_k(G)$ and $c_k = \sup\{\lambda_k(G)/n : |V(G)| = n, n \geq k\}$. In fact, Nikiforov shows $c_k = \lim_{n\to\infty} \lambda_k(n)/n$, by methods introduced in [5].

The question of providing good upper and lower bounds on the kth largest eigenvalue λ_k of a graph was apparently first stated by Hong [3]. Nikiforov was able to prove the following bounds on c_k.

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Table 1 Lower bounds for c_k

k	$c_k \geq$	Graph
4	$\frac{1+\sqrt{5}}{12} \approx 0.26967$	Icosahedral graph
5	$\frac{2}{9} \approx 0.2222$	Paley graph on 9 vertices [4]
6	$\frac{1}{5} = 0.2$	Petersen graph [4], $J(6, 2)$, $J(6, 3)$, Line graph of Petersen graph
7	$\frac{4}{21} \approx 0.190476$	$J(7, 2)$
8	$\frac{5}{28} \approx 0.178571$	$J(8, 2)$, Gosset graph
9	$\frac{1}{6} \approx 0.1666$	$J(9, 2)$
10	$\frac{7}{45} \approx 0.1555$	$J(10, 2)$
11	$\frac{8}{55} \approx 0.14545$	$J(11, 2)$
12	$\frac{3}{22} \approx 0.13636$	$J(12, 2)$
13	$\frac{5}{39} \approx 0.128205$	$J(13, 2)$
14	$\frac{11}{91} \approx 0.1208791$	$J(14, 2)$
15	$\frac{4}{35} \approx 0.1142857$	$J(15, 2)$
16	$\frac{13}{120} \approx 0.108333$	$J(16, 2)$
17	$\frac{2}{19} \approx 0.10526$	srg(57, 24, 11, 9)
18	$\frac{2}{19} \approx 0.10526$	srg(57, 24, 11, 9)
19	$\frac{2}{19} \approx 0.10526$	srg(57, 24, 11, 9)
20	$\frac{13}{125} = 0.104$	srg(125, 72, 45, 36)
21	$\frac{13}{125} = 0.104$	srg(125, 72, 45, 36)
22	$\frac{13}{126} \approx 0.10317$	srg(126, 60, 33, 24)
23	$\frac{25}{243} \approx 0.10288$	srg(243, 132, 81, 60)
24	$\frac{56}{552} \approx 0.101449$	Taylor graph from Conway group Co_3

Theorem 1 (Nikiforov [4]) *Let $k \geq 2$. Then,*

$$c_k \leq \frac{1}{2\sqrt{k-1}}.$$

Furthermore, there exists an integer k_0 such that for any $k > k_0$,

$$c_k \geq \frac{1}{2\sqrt{k-1} + \sqrt[3]{k}}.$$

Nikiforov also showed that $c_k \geq \frac{1}{k-\frac{1}{2}}$ for all $k \geq 5$, improving on the lower bound given by $kK_{\frac{n}{k}}$. On the other hand, $c_k = \frac{1}{k}$ for $k = 1$ and $k = 2$, leaving only the cases $k = 3$ and $k = 4$ open for the question in the beginning paragraph.

Question 1 (Nikiforov [4]) Is $c_3 = \frac{1}{3}$? Is $c_4 = \frac{1}{4}$?

In this note, we answer half of Nikiforov's question, improving the lower bound on c_4.

Theorem 2

$$c_4 \geq \frac{1+\sqrt{5}}{12} \approx 0.26967.$$

We can also improve the best known lower bound on c_k for many other small values of k.

Theorem 3 *For* $6 \leq k \leq 16$,

$$c_k \geq \frac{2(k-3)}{k(k-1)}.$$

The lower bound in Theorem 3 is in fact valid for all $k \geq 4$, but there are better bounds for $4 \leq k \leq 5$ and $k \geq 17$. Furthermore, for sufficiently large values of k the bound is much worse than the bound given by Theorem 1. On the other hand, Theorem 3 also easily shows that $c_k > \frac{1}{k}$ for $k \geq 6$.

2 Proofs of Theorems 2 and 3

Our improved lower bounds are derived from constructions of closed blowups of explicit graphs. Recall that for an integer $t \geq 1$, the closed blowup $G^{[t]}$ of a graph G is the graph obtained by replacing each vertex of G with a t-clique and replacing each edge in G with a complete bipartite graph $K_{t,t}$ on the vertices of the t-cliques. The eigenvalues of the closed blowup $G^{[t]}$ are $t\lambda_1 + t - 1, t\lambda_2 + t - 1, \ldots, t\lambda_n + t - 1$, along with $(t-1)n$ additional -1s, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of G [4, Proposition 5.4].

Proof of Theorem 2 Let G be the icosahedral graph. G is a graph on 12 vertices with spectrum $5^1(\sqrt{5})^3(-1)^5(-\sqrt{5})^3$ [1]. Therefore, the closed blowups of G satisfy $\lambda_4(G^{[t]}) = t\sqrt{5} + t - 1$, so

$$c_4 \geq \sup_t \frac{\lambda_4(G^{[t]})}{12t} = \sup_t \frac{t\sqrt{5}+t-1}{12t} = \frac{1+\sqrt{5}}{12}.$$

□

Proof of Theorem 3 The Johnson graphs $J(k,2)$ for $k \geq 4$ have kth largest eigenvalue $k-4$ (see [2, Theorem 6.3.2], for example, for the complete spectrum of Johnson graphs). Therefore, the closed blowups $J(k,2)^{[t]}$ satisfy $\lambda_k(J(k,2)^{[t]}) = t(k-4) + t - 1$, so

$$c_k \geq \sup_t \frac{t(k-4)+t-1}{t\binom{k}{2}} = \frac{2(k-3)}{k(k-1)}.$$

□

3 Concluding Remarks

Perhaps the most immediate open question stemming from the work presented here is to decide if $c_3 > \frac{1}{3}$. We have been unable to find a construction of a graph G with $\lambda_3 > \frac{n}{3}$. Besides the construction $3K_{\frac{n}{3}}$ mentioned in the beginning of the paper, other examples of graphs with $\lim_{n\to\infty} \frac{\lambda_3(G)}{n} = \frac{1}{3}$ include the closed blowups of the 6-cycle.

One could also attempt to find better constructions which improve the lower bound on c_k for other values of k. As an aid to researchers who might be interested in studying this question further, we conclude with a table of the best lower bound constructions that we know for small values of k. In all cases, the construction is a closed blowup of the graph or graphs listed.

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Declarations

Conflict of interest The author has no relevant financial or non-financial interests to disclose.

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