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The permuto-associahedron revisited

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a b s t r a c t

A classic problem connecting algebraic and geometric combinatorics is the realization problem: given a poset, determine whether there exists a polytope whose face lattice is the poset. In 1990s, Kapranov defined a poset as a hybrid between the face poset of a permutohedron and that of an associahedron, and he asked whether this poset is realizable. Shortly after his question was posed, Reiner and Ziegler provided a realization. Based on our previous work on the nested braid fan, we provide in this paper a different realization of Kapranov's poset by constructing the vertex set and the normal fan of a permuto-associahedron simultaneously.

1. Introduction

A *polytope* is the convex hull of finitely many points in an Euclidean space. Equivalently, a polytope can also be defined as a bounded solution set of a finite system of linear inequalities. Either definition provides a direct geometric embedding of a given polytope. However, sometimes people are more interested in combinatorial properties of a polytope, which is captured by its face poset. The *face poset* of a polytope P, denote $\mathcal{F}(P)$, is the poset of all faces of P ordered by inclusion. This leads to a classic question: given a poset $\mathcal{F}$, determine whether there exists a polytope P such that $\mathcal{F} = \mathcal{F}(P)$. If the answer is yes, we say $\mathcal{F}$ is *realizable*, and P is a *realization* of $\mathcal{F}$. We say a polytope arises or is defined *abstractly* if it was initially constructed as an answer to a realization problem.

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Two of the most studied polytopes in geometric combinatorics are the *permutohedron* and the *associahedron*, where the former was constructed by providing a geometric embedding, and the latter arised abstractly. Several generalizations have been explored and developed; we mention a few: Chapoton and Fomin [9], Fomin and Reading [10], Pilaud and Pons [24], and Lange and Pilaud [18]. Particularly relevant for the present paper is the work of Gelfand, Kapranov, and Zelevinsky [12], and Postnikov [25]. We also mention the work of Reading [27], Stella [34], and subsequent generalizations by Hohlweg, Lange, and Thomas [14], and by Hohlweg, Pilaud, and Stella [15].

1.1. Motivation: Realizing Kapranov's poset

The main purpose of this paper is to give a realization of a poset defined by Kapranov [17] as a hybrid between the face poset of a permutohedron and that of an associahedron. (See Definition 5.1.) Before giving more details about Kapranov's poset, we give a brief introduction to permutohedra and associahedra.

Permutohedron*.* A d-dimensional *permutohedron* or a *d-permutohedron* is the convex hull of all coordinate permutations of a fixed (generic) point in $\mathbb{R}^{d+1}$. After being first introduced and studied by Schoute in 1911 [30], the family of permutohedra naturally appeared in many different fields of mathematics. They can be described as the set of diagonal vectors of hermitian matrices with a fixed spectrum [2, Chapter II.6], as simple zonotopes [35, Section 7] or as the Newton polytopes of the Schur polynomials [3]. It is known that the face poset of a permutohedron is the poset of ordered set partitions [2, Chapter VI, Proposition 2.2].

Associahedron*.* A d-dimensional *associahedron* or a *d-associahedron* is a polytope defined by the following property: its vertices v_B are in bijection with full bracketings B on $(\ell_1 \square \ell_2 \square \dots \square \ell_{d+2})$, and two vertices v_B and $v_{B'}$ form an edge if and only if the bracketings B and B' are related by a single application of the associative law. This description of the associahedron can be translated into an equivalent description in terms of complete binary threes, using a connection between full bracketings and complete binary trees. (See Remark 3.6 for this connection.) We want to also mention another well-known equivalent but different way of defining a d-associahedron abstractly: its vertices v_T are in bijection with triangulations T of a $(d+3)$-gon and two vertices v_T and $v_{T'}$ form an edge of if and only if the triangulations T and T' differ by a flip. All three descriptions above only define vertices and edges of associahedra. A complete description of the face poset of the associahedron will be given in Definition 3.3 using the language of trees. We remark that it is not a coincidence that full bracketings, complete binary trees, and triangulations of a polygon all belong to Catalan families.

The associahedron was initially defined abstractly by Stasheff [32] and for a while whether there exists a geometric realization was an open problem. See [8] for an historical account. After decades of insights by many mathematicians, several realizations of the associahedron have been found. Below we mention two that are relevant to this paper:

(a) Gelfand, Kapranov and Zelevinsky [12, Chapter 7] provided a realization by considering regular triangulations of polytopes of arbitrary dimension. This realization was further generalized by Billera and Sturmfels [4] in their construction of fiber polytopes.
(b) Loday gave a realization by providing explicit coordinates for vertices of associahedra [19] and showed that this realization is a deformation of the regular permutohedron. His construction recovers the realization of Stasheff and Shnider [33, Appendix B] of associahedra that proceeds by truncating faces of a standard simplex. Later Postnikov [25] showed that Loday's associahedron can be expressed as a Minkowski sum of simplices.

Permuto-associahedron*.* Motivated by providing a geometric proof for MacLane's coherence theorem for associativities and commutativities in monoidal categories [21], Kapranov constructed a poset whose elements are ordered set partitions with bracketings and ordered by either removing bracketings or merging blocks. (See Definition 5.1 for a precise definition for this poset.) He then showed that his poset can be realized as a CW-ball. Using this, he provided a short proof for MacLane's coherence theorem. In the introduction of [17], Kapranov asked a natural question: does his poset have a geometric realization as a polytope? Shortly after Kapranov's question was posed, Reiner and Ziegler [29] gave an affirmative answer with an explicit construction, using Gelfand, Kapranov, and Zelevinsky's realization of the associahedron mentioned above. A second realization was obtained by Gaiffi [11].

Reiner–Ziegler's [29] and Gaiffi's [11] work are both related to the construction that will be given in this paper, but in different ways. Our construction is connected to Reiner and Ziegler's topological proof for the result that Kapranov's poset is a CW-ball, but is very different from their geometric realization. Meanwhile, the polytopes we construct have the same normal fan as Gaiffi's. However, the constructed families of polytopes are different and more importantly our approaches are different: Whereas his starting point is the work of Stasheff and Shnider [33], ours is Loday's [19]. In Section 7, we will compare our construction with both Reiner–Ziegler's and Gaiffi's, discussing both similarities and differences.

1.2. Our construction

In our previous work [7], we have defined and studied a family of polytopes called *nested permutohedra*, which interpolate the structures of two permutohedra of consecutive dimensions. In the present paper we use the tools and ideas developed in [7] to construct a permuto-associahedron as a *deformation* of a nested permutohedron. In other words, we obtain a permuto-associahedron by altering the inequality description of a nested permutohedron without overrunning any vertex. We call our realization the *nested permuto-associahedron*.

Our main strategy for constructions in both this article and [7] involves a primal/dual argument which we lay out below: Suppose we want to construct a polytope whose vertices are in bijection with a certain set S (e.g., if we try to realize a poset $\mathcal{F}$, then S is the set of rank-1 elements of $\mathcal{F}$). Then we do the following steps:

(1) Define a point v_s for each $s \in S$.
(2) Define a cone σ_s for each $s \in S$, and let $\mathcal{N}$ be the fan induced by the set $\{\sigma_s\}$. (If we want to realize a poset $\mathcal{F}$, we need to make sure that the face poset of $\mathcal{N}$ is dual to the poset $\mathcal{F}$.)
(3) Verify that the set $\{v_s\}$ and the set $\{\sigma_s\}$ ''match''. (See the hypothesis in Lemma 2.4 for a precise statement.)

After the above procedure, particularly after the last verification step, we can immediately conclude that the set $\{v_s\}$ constructed in step (1) and the fan $\mathcal{N}$ constructed in step (2) are the vertex set and the normal fan respectively of the desired polytope. Moreover, we can immediately apply Lemma 2.5 to obtain an inequality description of the constructed polytope, which is another benefit of our construction strategy.

In this paper, we first describe a realization of the associahedron using the 3-step method outlined above. Explicit coordinates for Step (1) are given as a generalization of Loday's construction, and the cones in Step (2) are given by a union of *braid cones* [26]. We then combine this realization with our previous results on nested permutohedra [7] to give a realization of Kapranov's poset which we call *nested permuto-associahedra*.

We finish this part by mentioning one more related construction of polytopes. Recently Baralic, Ivanovic, and Petric [1] constructed a *simple* permuto-associahedron. Since Kapranov's permuto-associahedron which is not simple, these two families of permuto-associahedra clearly have different combinatorial structures.

Organization of the paper

After reviewing basic preliminary material in Section 2 we proceed, in Sections 3 and 4, to describe a realization of the associahedron. Note that Section 3 serves as a preliminary section for Section 4. In particular, Section 3.1, Section 3.2 and Section 3.3 introduce basic combinatorial objects that have been used in various other realizations of the associahedron, experts on associahedra may

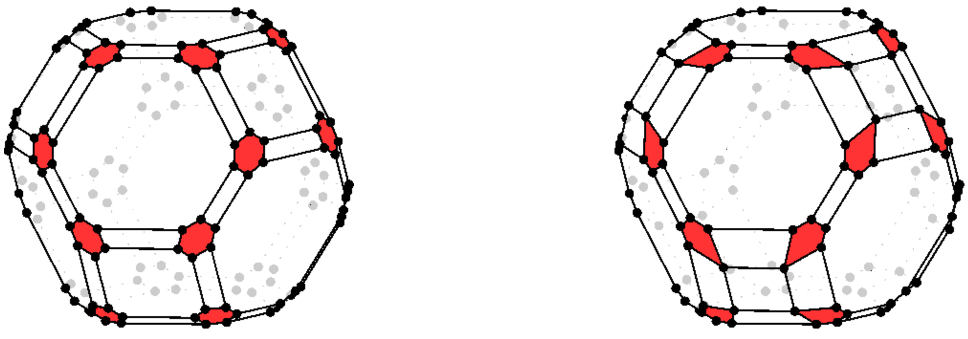

Fig. 1. A side by side comparison of a nested permutohedron (on the left) and a nested permuto-associahedron (on the right).

skip these parts. In Sections 5 and 6, we use the results of previous sections together with results on nested permutohedra [7] to realize Kapranov's poset.

Finally, in Section 7, we compare our construction with Reiner–Ziegler's and Gaiffi's realizations.

2. Preliminaries

Recall that $[n]$ denotes the set $\{1, 2, \ldots, n\}$ for any positive integer n and $[a, b]$ denotes the *integer interval* $\{z \in \mathbb{Z} : a \leq z \leq b\}$ for any integers a and b satisfying $a - 1 \leq b$. By convention $[a, a - 1]$ denotes the *empty integer interval*.

2.1. Preorder and preposet

Let A be a finite set. A *preorder* $\preceq$ on A is a binary relation that is both reflexive and transitive. If $i \preceq j$ and $j \preceq i$, we write $i \equiv j$. The relation $\equiv$ is an equivalence relation on A, and thus it decomposes A into equivalence classes. We denote by $\bar{i}$ the equivalence class of i and $A/_{\equiv}$ the set of equivalence classes. *preposet* is an ordered pair $(A, \preceq)$ where $\preceq$ is a preorder on A. A *poset* is a preposet such that $i \equiv j$ if and only if $i = j$. Note that if a preorder $\preceq$ is antisymmetric, which implies that $i \equiv j$ if and only if $i = j$, then $\preceq$ is a partial order on A, and the preposet $(A, \preceq)$ is a poset. (See [31, Chapter 3] for concepts related to partial orders and posets.)

One sees that a preorder $\preceq$ on A induces a partial order on $A/_{\equiv}$ in which $\bar{i} \preceq \bar{j}$ if $i \preceq j$ in A. The poset $(A/_{\equiv}, \preceq)$ and the preposet $(A, \preceq)$ are closely related. Hence, we can conveniently extend many concepts for the former to the latter: A *covering relation* $i \lessdot j$ in the preposet $(A, \preceq)$ is a pair of elements (i, j) such that $\bar{i}$ is covered by $\bar{j}$ in the poset $(A/_{\equiv}, \preceq)$. The *Haase diagram* of a preposet $(A, \preceq)$ is the Haase diagram of the poset $(A/_{\equiv}, \preceq)$ except that for convenience when we mark vertices with equivalence classes $\bar{i}$, we omit the parentheses around sets, see Fig. 6 for an example.

Suppose $\preceq_1$ and $\preceq_2$ are two preorders on A. We say the preorder $\preceq_1$ is a *contraction* of the preposet $\preceq_2$ if the Haase diagram of the former is obtained by contracting some edges of the Haase diagram of the latter and merging the corresponding equivalence classes.

An *order-preserving* map from a preposet $(A_1, \preceq_1)$ to another preposet $(A_2, \preceq_2)$ is a bijection $f : A_1 \to A_2$ such that $f(i) \preceq_2 f(j)$ whenever $i \preceq_1 j$ for any $i, j \in A_1$. An order-preserving map is an *isomorphism* if it is invertible and its inversion is order-preserving as well. Two preposets are *isomorphic* if there exists an isomorphism between them.

Note that any subset C of $\mathbb{Z}$ (or $\mathbb{R}$) as is totally ordered with respect to $\leq$ and thus can be considered as a preposet (or a poset); we use the letter C alone to indicate the preposet $(C, \leq)$ for simplicity.

Suppose $(A, \preceq)$ is a poset. The *dual* of $(A, \preceq)$ is the poset $(A, \preceq^*)$ where $i \preceq^* j$ if and only if $j \preceq i$. An order-preserving map from the poset $(A, \preceq)$ to the set $\{1, 2, \dots, |A|\}$ is called a *linear extension* of $(A, \preceq)$. We denote by $L[A, \preceq]$ the set of linear extensions of $(A, \preceq)$.

A poset $(A, \preceq)$ is *graded* if there exists a function $\rho : A \to \mathbb{Z}_{\geq 0}$ such that $\rho(i) = 0$ for every minimal element of the poset and $\rho(j) = \rho(i) + 1$ whenever $i \lessdot j$ is a covering relation. We call ρ (which is uniquely defined) the *rank function* of $(A, \preceq)$, and $\rho(i)$ the *rank* of i for each element i. The *rank* of a graded poset $(A, \preceq)$ is defined as $r(A, \preceq) := \max_{i \in A} \rho(i)$.

We denote by $\hat{0}$ and $\hat{1}$ the minimum and maximum of a poset (if they exist).

Our setup:

In our paper, we will mostly fix $A = [n]$ where n is either d or $d+1$. Hence, when n is fixed, the preorder $\preceq$ on $[n]$ is the only variable that changes. Whenever it is clear that $\preceq$ is a partial order on $A = [n]$, we will omit A and just write $L[\preceq]$ for the set of linear extensions of $([n], \preceq)$. Note that $L[\preceq]$ is a subset of the symmetric group S_n.

2.2. Polytopes and fans

Let $V \subseteq \mathbb{R}^D$ be a d-dimensional vector space in the D-dimensional Euclidean space and W is the dual space of V which consists of all linear functionals on V. Thus, we may consider W is a quotient space of $\mathbb{R}^D$, and the perfect pairing between V and W : $\langle \cdot, \cdot \rangle : W \times V \to \mathbb{R}$ is just the dot product on $\mathbb{R}^D$.

Let $U \subseteq \mathbb{R}^D$ be an affine space that is a translation of V. A *polyhedron* $P \subseteq U$ is the solution set of a finite set of linear inequalities:

$$P = \{\mathbf{x} \in U \ : \ \langle \boldsymbol{a}_i, \mathbf{x} \rangle \leq b_i, \ i \in I\}, \tag{2.1}$$

A *face* of a polyhedron is a subset $F \subseteq P$ such that there exists $\mathrm{w} \in W$ such that

$$F = \{\mathbf{x} \in P \ : \ \langle \mathrm{w}, \mathbf{x} \rangle \geq \langle \mathrm{w}, \boldsymbol{y} \rangle, \quad \forall \boldsymbol{y} \in P\}.$$

An inequality $\langle \boldsymbol{a}, \mathbf{x} \rangle \leq b$ is *facet-defining* on P if the corresponding equality defines a facet of P, i.e., $\{\mathbf{x} \in P : \langle \boldsymbol{a}, \mathbf{x} \rangle = b\}$ is face of P of dimension $\dim(P) - 1$.

Suppose a polyhedron P is defined by (2.1). We say (2.1) is a *facet-defining inequality description* for P if each inequality in (2.1) is facet-defining. (However, it is possible multiple inequalities determine a same facet.) We say (2.1) is a *minimal inequality description* if P has exactly $|I|$ facets. Thus, when (2.1) is minimal, the equality obtained for each $i \in I$ determines exactly one facet of P. A *polytope* is a bounded polyhedron. A k-dimensional polytope is *simple* if each vertex is incident to exactly k edges. A *(polyehdral) cone* is a polyhedron defined by homogeneous linear inequalities. A cone is *pointed* if it does not contain a line. A k-dimensional cone is *simplicial* if it is spanned by exactly k (linearly independent) rays. Note that any simplicial cone is pointed.

By convention we always consider $\emptyset$ to be a face of a polyhedron P. The set of all faces of P partially ordered by inclusion forms the *face poset* $\mathcal{F}(P)$ of P. A *fan* in W is a collection Σ of cones that is a simplicial complex. The collection together with the partial order given by inclusion forms a poset $\mathcal{F}(\Sigma)$ called its *face poset*. A fan Σ is *simplicial* if every cone in it is simplicial. A fan Σ in W is *complete* if the union of its cones is W. The following definition gives a standard example of complete fans that arises from polytopes.

Definition 2.1. Suppose V, W and U are given as above, and $P \subseteq U$ is a polytope. Given a nonempty face F of P, the *normal cone* of P at F is defined to be

$$\mathrm{ncone}(F, P) := \left\{ \mathrm{w} \in W \ : \ \langle \mathrm{w}, \boldsymbol{y} \rangle \geq \langle \mathrm{w}, \boldsymbol{y}' \rangle, \quad \forall \boldsymbol{y} \in F, \quad \forall \boldsymbol{y}' \in P \right\}.$$

Therefore, $\mathrm{ncone}(F, P)$ is the collection of linear functionals w in W such that w attains maximum value at F over all points in P. The *normal fan* of P, denoted by $\Sigma(P)$, is the collection of all normal cones of P as we range over all nonempty faces of P.

Lemma 2.2. *The map $F \mapsto \operatorname{ncone}(F, P)$ for nonempty faces F induces a poset isomorphism from $\mathcal{F}(P) \setminus \{\emptyset\}$ to the dual poset of $\mathcal{F}(\Sigma(P))$.*

If Q is a polytope such that $\Sigma(Q)$ is a coarsening of $\Sigma(P)$, i.e., if every cone in the former is a union of cones in the latter, we say that Q is a *deformation* of P.

As we mentioned above, $\Sigma(P)$ is always a complete fan in W. Moreover, any fan that is a normal fan of a polytope is called a *projective* fan. Once we know that a projective fan Σ is the normal fan of a polytope, one can check that the polytope is full-dimensional if and only if $0 \in \Sigma$, i.e., all cones in Σ are pointed.

Given a fan Σ in W, the set $\mathcal{M} \subseteq \Sigma$ of maximal cones (in terms of dimension) determines Σ. More precisely, the set of cones in $\mathcal{M}$, together with all their faces, forms the fan Σ. In this case, we say Σ is *induced* by $\mathcal{M}$. Therefore, we often focus on the description of the maximal cones of a fan, which has the property of being the conic dissection of W.

Definition 2.3. A *conic dissection* of W is a set $\mathcal{M}$ of full-dimensional cones such that the union of the cones in $\mathcal{M}$ is equal to W, and for any distinct $\sigma_1, \sigma_2 \in \mathcal{M}$, their relative interiors σ_1° and σ_2° have no intersection. We say a conic dissection $\mathcal{M}$ is *pointed* (and *simplicial* resp.) if all the cones in $\mathcal{M}$ are *pointed* (and *simplicial* resp.)

We remark that a conic dissection does not necessarily induce a fan, since cones in the dissection may not intersect in proper faces.

The primal/dual argument in the following lemma was used in the proof of Proposition 3.5 of our previous work [7]. We summarize it here since it will be our main tool in verifying the constructions of associahedra and permuto-associahedra.

Lemma 2.4. *Let $\mathcal{M} = \{\sigma_1, \dots, \sigma_k\}$ be a conic dissection of W and $\{v_1, \dots, v_k\} \subset U$ a set of points such that for each $i = 1, \dots, k$ we have*

$$\langle w, v_i \rangle > \langle w, v_j \rangle \qquad \forall w \in \sigma_i^\circ \text{ and } j \neq i. \tag{2.2}$$

Let P be the polytope defined by $P := \operatorname{ConvexHull}\{v_1, \dots, v_k\}$. Then the followings are true:

(1) The set $\{v_1, \dots, v_k\}$ is the vertex set of P.
(2) For each $i = 1, 2, \dots, k$, we have $\sigma_i = \operatorname{ncone}(v_i, P)$.

As a consequence, the conic dissection $\mathcal{M}$ induces the normal fan $\Sigma(P)$ of P, which is a complete projective fan in W. Moreover, if $\mathcal{M}$ is pointed, then $0 \in \Sigma(P)$ and thus P is full-dimensional in U.

Proof. For each $i = 1, 2, \dots, k$, it follows from condition (2.2) that v_i does not lie in $\operatorname{ConvexHull}(v_j : j \neq i)$, and thus v_i must be a vertex of P. Hence, (1) follows. Next, condition (2.2) also implies that for each i we have $\sigma_i \subseteq \operatorname{ncone}(v_i, P)$. However, since both $\{\sigma_i\}$ and $\{\operatorname{ncone}(v_i, P)\}$ are conic dissections of W, we must have $\sigma_i = \operatorname{ncone}(v_i, P)$. □

Lemma 2.5. *Suppose P is a full-dimensional polytope in U with vertex set $\{v_1, \dots, v_k\}$, and $\sigma_i = \operatorname{ncone}(v_i, P)$ for each $1 \le i \le k$. Let $\{\rho_1, \rho_2, \dots, \rho_m\}$ be the set of one dimensional cones in $\Sigma(P)$ and for each $1 \le j \le m$, let $\boldsymbol{n}_j$ be a nonzero vector in the cone ρ_j (or equivalently a generator for ρ_j). Then the polytope P has the following minimal inequality description:*

$$P = \left\{ \boldsymbol{x} \in U \ : \ \langle \boldsymbol{n}_j, \boldsymbol{x} \rangle \le \max_{1 \le i \le k} \langle \boldsymbol{n}_j, v_i \rangle, \quad 1 \le j \le m \right\}.$$

Moreover, for each $1 \le j \le m$, if we choose i_j such that $\rho_j \subseteq \sigma_{i_j}$, then

$$\max_{1 \le i \le k} \langle \boldsymbol{n}_j, v_i \rangle = \langle \boldsymbol{n}_j, v_{i_j} \rangle.$$

2.3. Permutohedra and braid cones

In this paper, we always have $D = d+1$ where D is the dimension of the ambient space $\mathbb{R}^D$ and d is the dimension of the polytopes we consider. The d-dimensional vector space we use is $V_d = \{\mathbf{x} \in \mathbb{R}^{d+1} : \langle \mathbf{1}, \mathbf{x}\rangle = 0\} \subseteq \mathbb{R}^{d+1}$ and its dual space is $W_d = \mathbb{R}^{d+1}/\mathbf{1}$, where $\mathbf{1} = (1, 1, \dots, 1)$ denotes the all-one vector in $\mathbb{R}^{d+1}$. For $\alpha \in \mathbb{R}^{d+1}$, let

$$U_d^{\alpha} := \left\{ \mathbf{x} \in \mathbb{R}^{d+1} : \langle \mathbf{1}, \mathbf{x}\rangle = \sum_{i=1}^{d+1} \alpha_i \right\} \tag{2.3}$$

be a translation of V_d. Our polytopes will be defined in these affine spaces.

Given a strictly increasing sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \mathbb{R}^{d+1}$, for any $\pi \in \mathfrak{S}_{d+1}$, we use the notation below, following [7]:

$$\mathbf{v}_{\pi}^{\alpha} := \left(\alpha_{\pi(1)}, \alpha_{\pi(2)}, \dots, \alpha_{\pi(d+1)} \right) = \sum_{i=1}^{d+1} \alpha_i \boldsymbol{e}_{\pi^{-1}(i)}. \tag{2.4}$$

Then we define the *usual permutohedron*

$$\mathrm{Perm}(\alpha) = \mathrm{ConvexHull}\left(\mathbf{v}_{\pi}^{\alpha} : \pi \in \mathfrak{S}_{d+1} \right) \subseteq U_d^{\alpha}.$$

It is well-known that $\mathrm{Perm}(\alpha)$ is a full-dimensional polytope in U_d^{α}, and it has the following minimal inequality description, writing $\boldsymbol{e}_S := \sum_{i \in S} \boldsymbol{e}_i$:

$$\mathrm{Perm}(\alpha) = \left\{ \mathbf{x} \in U_d^{\alpha} : \langle \boldsymbol{e}_S, \mathbf{x} \rangle \leq \sum_{i=d+2-|S|}^{d+1} \alpha_i, \quad \forall\, \emptyset \neq S \subsetneq [d+1] \right\}. \tag{2.5}$$

A *generalized permtuhohedron* is a deformation of a usual permutohedron $\mathrm{Perm}(\alpha)$ for some α.

Definition 2.6. For each $\pi \in \mathfrak{S}_{d+1}$, we define the *braid cone associated to* π to be:

$$\sigma(\pi) := \{ \mathbf{w} \in W_d : w_{\pi^{-1}(1)} \leq w_{\pi^{-1}(2)} \leq \cdots \leq w_{\pi^{-1}(d+1)} \}. \tag{2.6}$$

Let $\mathcal{B}_d := \{\sigma(\pi) : \pi \in \mathfrak{S}_{d+1}\}$ be the collection of braid cones in W_d.

One checks that the relative interior of $\sigma(\pi)$ is

$$\sigma^{\circ}(\pi) := \{ \mathbf{w} \in W_d : w_{\pi^{-1}(1)} < w_{\pi^{-1}(2)} < \cdots < w_{\pi^{-1}(d+1)} \}. \tag{2.7}$$

Thus, we clearly have the following:

Lemma 2.7. *The collection of braid cones* $\mathcal{B}_d = \{\sigma(\pi) : \pi \in \mathfrak{S}_{d+1}\}$ *forms a simplicial conic dissection of* W_d.

As an example of the usefulness of Lemma 2.4, we verified in [7] that the collection of braid cones $\mathcal{B}_d$ and the set of points $\{\mathbf{v}_{\pi}^{\alpha} : \pi \in \mathfrak{S}_{d+1}\}$ satisfy the hypotheses of the lemma. Consequently, we proved that $\mathcal{B}_d$ induces the well-known *braid fan* Br_d, and that the braid fan Br_d is the normal fan of the usual permutohedron $\mathrm{Perm}(\alpha)$. Thus a polytope is a generalized permutohedron if and only if its normal fan coarsens the braid fan Br_d for some d.

Finally, the face poset of a permutohedron has a nice combinatorial description.

Definition 2.8. We say the ordered tuple $\mathcal{S} = (S_1, \dots, S_k)$ is an *ordered (set) partition* of $[d+1]$ with k *blocks* if $S_1, \dots, S_k$ are k disjoint sets whose union is $[d+1]$. We denote by $\mathcal{O}_{d+1}$ the set of all ordered partitions of $[d+1]$ and by $\mathcal{O}_{d+1,k}$ the set of all ordered partitions of $[d+1]$ with k parts.

We define a partial order $\preceq$ on the set $\mathcal{O}_{d+1} \cup \{\hat{0}\}$ by declaring $\mathcal{S}_1 \preceq \mathcal{S}_2$ if $\mathcal{S}_1 \in \mathcal{O}_{d+1}$ refines $\mathcal{S}_2 \in \mathcal{O}_{d+1}$ and $\hat{0} \preceq \mathcal{S}$ for all $\mathcal{S} \in \mathcal{O}_{d+1}$. We denote the poset $(\mathcal{O}_{d+1} \cup \{\hat{0}\}, \preceq)$ by $\mathcal{O}_{d+1}$.

The set $\mathcal{O}_{d+1,d+1}$ of rank 1 elements of $\mathcal{O}_{d+1}$ is in bijection with $\mathfrak{S}_{d+1}$. More precisely, for each permutation $\pi \in \mathfrak{S}_{d+1}$, we let

$$\mathcal{S}(\pi) := (\{\pi^{-1}(1)\}, \{\pi^{-1}(2)\}, \dots, \{\pi^{-1}(d+1)\}). \tag{2.8}$$

to be the ordered set partition that corresponds to π. Clearly, $\pi \mapsto \mathcal{S}(\pi)$ is a bijection from $\mathfrak{S}_{d+1}$ to $\mathcal{O}_{d+1,1}$.

Notation 2.9. When we write examples of ordered partitions we often omit commas and brackets for convenience. For example, $(\{1, 2, 3\}, \{4\}, \{5, 6\})$ will be written as $(123, 4, 56)$.

Theorem 2.10. *[2, Section VI.2] Suppose $\alpha \in \mathbb{R}^{d+1}$ is a strictly increasing sequence. Then the face poset of the usual permutohedron* $\mathrm{Perm}(\alpha)$ *is isomorphic to* $\mathcal{O}_{d+1}$.

Note that by Lemma 2.2, there is a poset isomorphism from the poset $\mathcal{O}_{d+1} \setminus \{\hat{0}\}$ to the dual of the face poset $\mathcal{F}(\mathrm{Br}_d)$ of the braid fan Br_d. See Remark 2.12 below.

2.4. Preorder cones

In [26, Section 3.4], the authors defined a "braid cone" as a polyhedral cone in W_{n-1} whose defining inequalities are of the form $\mathbf{w}_i \le \mathbf{w}_j$ for some $i, j \in [n]$. To avoid confusion with Definition 2.6, we will refer to these cones as *preorder cones*.

Definition 2.11. For each preorder $\preceq$ on the set $[n]$, we define the *preorder cone* associated to $\preceq$ to be

$$\sigma_{\preceq} := \{\mathbf{w} \in W_{n-1} \ : \ \mathbf{w}_i \le \mathbf{w}_j \text{ if } i \preceq j\}.$$

It is clear from the definition that every face of a preorder cone is itself a preorder cone.

Remark 2.12. For any $\mathcal{S} = (S_1, \dots, S_k) \in \mathcal{O}_{d+1}$, it determines a unique preorder $\preceq_{\mathcal{S}}$ on $[d+1]$ by letting

$$i \preceq_{\mathcal{S}} j \text{ if } i \in S_a, j \in S_b \text{ with } a \le b. \tag{2.9}$$

Then we define the cone $\sigma(\mathcal{S}) := \sigma_{\preceq_{\mathcal{S}}}$. The set $\{\sigma(\mathcal{S}) \ : \ \mathcal{S} \in \mathcal{O}_{d+1}\}$ consists of all cones in Br_d. In particular, for $\pi \in \mathfrak{S}_d$ the cone $\sigma(\mathcal{S}(\pi))$ is the braid cone $\sigma(\pi)$.

The map $\mathcal{S} \to \sigma(\mathcal{S})$ induces the poset isomorphism from $\mathcal{O}_{d+1} \setminus \{\hat{0}\}$ to $\mathcal{F}(\mathrm{Br}_d)$ that is asserted by Lemma 2.2 with P being the usual permutohedron $\mathrm{Perm}(\alpha)$.

We state the following facts from [26] that will be useful in our constructions.

Lemma 2.13 (*Proposition 3.5 in [26]*). *Let $\preceq$ be a (fixed) preorder on the set $[n]$. Then the following statements hold.*

(1) The preorder cone has the following facet-defining inequality description:

$$\sigma_{\preceq} = \{\mathbf{w} \in \bar{W}_n \ : \ \mathbf{w}_i \le \mathbf{w}_j \text{ if } i \prec\!\!\cdot\, j\}.$$

Thus, the relative interior of $\sigma_{\preceq}$ is

$$\sigma^{\circ}_{\preceq} = \{\mathbf{w} \in \bar{W}_n \ : \ \mathbf{w}_i < \mathbf{w}_j \text{ if } i \prec\!\!\cdot\, j\}.$$

(2) The preorder cone $\sigma_{\preceq}$ is simplicial if and only if the Haase diagram of $\preceq$ is a tree. (Recall that a tree is a connected acyclic graph.)

(3) The preorder cone $\sigma_{\preceq'}$ is a face of $\sigma_{\preceq}$ if and only if $\preceq'$ is a contraction of $\preceq$.

(4) Suppose $\preceq$ is a partial order on $[n]$. Then its associated cone $\sigma_{\preceq}$ is a union of braid cones. More precisely, $\sigma_{\preceq} = \bigcup_{\pi \in \mathcal{L}[\preceq]} \sigma(\pi)$.

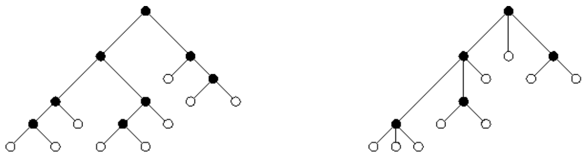

Fig. 2. Example of unlabeled plane rooted trees that are strictly branching.

3. Trees and the associahedron

In this section, we will review the combinatorics of the associahedron and develop results that will be needed in our constructions of associahedra and permuto-associahedra. As mentioned in the introduction, instead of using bracketings, we will define the face poset of the associahedron in terms of trees.

3.1. Strictly branching trees

We assume the readers are familiar with terminologies on graphs as presented in [31, Appendix], and review ones that are relevant to our paper.

A *rooted tree* is tree with a special vertex which is called the *root* of the tree. For any edge $\{i, j\}$ of a rooted tree T, if i is closer to the root of T than j, then we say i is the *parent* of j and j is a child of i. We call a vertex of a rooted tree a *leaf* if it has no children, and an *internal vertex* otherwise. An edge of a rooted tree is *internal* if it connects two internal vertices. A rooted tree is *strictly branching* if each of its internal vertex has at least two children. An *unlabeled plane rooted tree* T is a rooted tree whose vertices are considered to be indistinguishable, but the subtrees at any vertex are linearly ordered.

For $n \in \mathbb{Z}_{\geq 0}$, let T_n be the set of unlabeled plane rooted trees that are strictly branching and have $n+1$ leaves. For $0 \leq k \leq n$, let $\mathrm{T}_{n,k}$ be the set of trees in T_n that has k internal vertices. Note that $\mathrm{T}_{n,0} = \emptyset$ unless $n = 0$, and $\mathrm{T}_0 = \mathrm{T}_{0,0}$ consists of the only rooted tree on one vertex. It is easy to see that for any positive integer n the followings are true:

(1) $\mathrm{T}_n = \bigsqcup_{k=1}^{n} \mathrm{T}_{n,k}$.
(2) $\mathrm{T}_{n,1}$ consists of only one tree in which the root is the only internal vertex.
(3) $\mathrm{T}_{n,n}$ consists of all the complete binary trees with $n+1$ leaves. Recall that a *complete binary tree* is an unlabeled plane rooted tree whose internal vertices all have exactly two children. For each internal vertex of a complete binary tree, we call its first child its *Left* child, and second child its *Right* child. As a consequence, we call the two corresponding subtrees its *Left* subtree and *Right* subtree, and the two connecting edges a *Left* internal edge and a *Right* internal edge.

See Fig. 2 for examples: the tree on the left is a complete binary tree in $\mathrm{T}_{8,8}$ and the tree on the right is a plane rooted tree in $\mathrm{T}_{8,5}$. The leaves of both trees were enumerated by left-to-right order.

Definition 3.1. Let $\mathrm{T}, \mathrm{T}' \in \mathrm{T}_{n,n}$. We say T is obtained from T′ by a *flip (of an internal edge)* and T′ is obtained from T by a *flip (of an internal edge)* if there exist a Left internal edge e of T and a Right internal edge e' of T′ such that after contracting e in T and contracting e' in T′, we obtain exactly the same tree.

See Fig. 3 for a demonstration of how a flip of an internal edge works.

3.2. The associahedron

We can now define associahedron abstractly. Here we replace n with $d+1$ in T_n or $\mathrm{T}_{n,k}$.

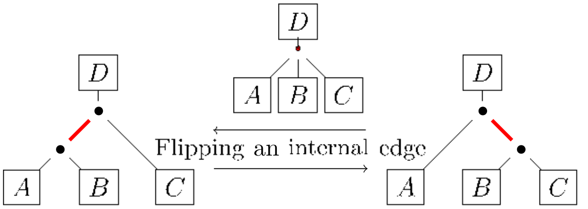

Fig. 3. Flips on complete binary trees.

Definition 3.2. A *d-associahedron* is a *d*-dimensional polytope such that its vertices are in bijection with complete binary trees in $T_{d+1,d+1}$, and two vertices form an edge if and only if their corresponding complete binary trees are obtained from one another by a flip.

Every complete binary tree $T \in T_{d+1,d+1}$ has exactly d internal edges, and thus is connected with exactly d complete binary trees in $T_{d+1,d+1}$ via flips. This means that an associahedron is a simple polytope. A classical result of Blind and Mani [5] (with a very short alternative proof given by G.Kalai in [16]) states that the graph[3] of a simple polytope determines its face poset.[4] As a consequence, all realizations of d-associahedra share the same face poset which we describe below.

Definition 3.3. We define a partial order $\leq_K$ on the set $T_{d+1} \cup \{\hat{0}\}$ by declaring $T_1 \leq_K T_2$ whenever $T_2 \in T_{d+1}$ is obtained from $T_1 \in T_{d+1}$ by contracting *internal edges*, and $\hat{0} \leq_K T$ for all $T \in T_{d+1}$. We denote the poset $(T_{d+1} \cup \{\hat{0}\}, \leq_K)$ by K_d.

It is easy to verify that the poset K_d is graded of rank $d+1$ with a unique minimal element $\hat{0}$. For each $1 \leq k \leq d+1$, the set $T_{d+1,k}$ consists of all elements of K_d of rank $d+2-k$. In particular, the only tree in $T_{d+1,1}$ is the unique maximal element of K_d.

The following lemma is a well-known result, and often is taken as the definition of associahedra. See [6, Proof of Lemma 3.3] for a proof.

Lemma 3.4. *A polytope is a d-associahedron if and only if its face poset is* K_d.

3.3. Enumerating leaves and labeling internal vertices

For each tree in T_n, we canonically enumerate its leaves by left-to-right order, denoted by $\ell_1, \ell_2, \ell_3, \ldots, \ell_{n+1}$. For each tree $T \in T_n$, we also assign labels $1, 2, \ldots, n$ to its internal vertices: If an internal vertex v is the closest common ancestor of ℓ_i and ℓ_{i+1} we label v with i.

Suppose T' is a subtree of T. We denote by $I_T(T')$ the collection of labels on the internal vertices of T' (as a subtree of T).

Example 3.5. In Fig. 4 we depict the internal vertices of the trees of Fig. 2 together with the labelings with the set [8]. Let T be the tree on the left of Fig. 4, and T' the right subtree of T. Then $I_T(T') = \{7, 8\}$.

The name associahedron historically comes from the interpretation of vertices as bracketings and flips as applications of associativity, see following remark.

[3] The graph $G(P) = (V, E)$ of a polytope P is defined by taking V be the vertex set of P and two vertices are joined by an edge if they form a one dimensional face of P.

[4] By duality we have that the facet-ridge graph of a simplicial polytope determines the rest of the face poset. In [5, Question 1] it is asked whether the same property hold for simplicial spheres. To our best knowledge this generalization remains open to this day.

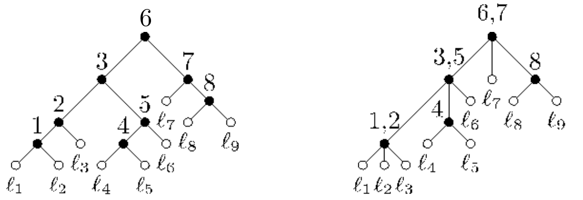

Fig. 4. Internal labelings associated to Fig. 2.

Remark 3.6. The labeling on the leaves gives a canonical bijection between T_n and the set of all possible "bracketings" on $(\ell_1 \ast \ell_2 \ast \dots \ast \ell_{n+1})$, which induces a bijection between complete binary trees in $\mathrm{T}_{n,n}$ and "full bracketings" on $(\ell_1 \ast \ell_2 \ast \dots \ast \ell_{n+1})$. Furthermore, flipping internal edges on complete binary trees correspond to applying "associative law" on full bracketings on $(\ell_1 \ast \ell_2 \ast \dots \ast \ell_{n+1})$.

The following lemma states properties of the labels on the internal vertices of trees in T_n. The proof of it is straightforward, and thus is omitted.

Lemma 3.7. *Let $T \in \mathrm{T}_n$.*

(1) Each internal vertex of T is labeled by a nonempty set of numbers, and each number in $[n]$ appears exactly once. Thus, if $T \in \mathrm{T}_{n,n}$ each internal vertex has a unique label.

(2) Let T' be a subtree of T. Then $I(T')$ is an integer interval, and it is empty if and only if T' is a leave of T. Furthermore, suppose T' is rooted at an internal vertex of T, thus $I(T') = [a, b]$ for some integers $a \le b$. Then the labels of internal vertices of T' when we treat it as a rooted tree by itself are obtained from subtracting $(a - 1)$ from those when we treat it as a subtree of T.

(3) Suppose T_1 and T_2 are two subtrees below an internal vertex of T. If T_1 is to the left of T_2, then any label appearing on an internal vertex of T_1 is smaller than any label appearing on an internal vertex of T_2.

(4) Suppose T' is obtained from T by contracting one internal edge $\{x, y\}$, and we call the new vertex z. Then the internal vertex labeling of T' can be obtained from that of T by labeling z with the union of the labels for x and y and keeping labels for all the other internal vertices.

We want to note that given a tree in T_n or $\mathrm{T}_{n,n}$, the labels we put on its leaves (and internal vertices) are uniquely determined. Hence, those labels do not carry extra information.

3.4. From trees to cones

We finish this section by defining a cone for each of our trees.

Let $T \in \mathrm{T}_n$. We define G(T) to be the induced subtree of T on its internal vertices, together with the labeling for internal vertices we described above. One sees that G(T) is the Hasse diagram of a preorder on $[n]$, which we denote by $\preceq_T$. Recall that we have defined linear extensions and preorder cones in Section 2.1. For convenience and brevity, we denote

$$\mathrm{L}[T] := \mathrm{L}[[n], \preceq_T] \quad \text{and} \quad \sigma(T) := \sigma_{\preceq_T}. \tag{3.1}$$

The following lemma follows immediately from definitions and Lemma 2.13 (3).

Lemma 3.8. *Let $T, T' \in \mathrm{T}_n$. Then the following statements are equivalent:*

(1) T' is a contraction of T (i.e., $T \le_K T'$).
(2) The preorder $\preceq_{T'}$ is a contraction of $\preceq_T$.
(3) The cone $\sigma(T')$ is a face of $\sigma(T)$.

The next lemma gives a connection between permutations and complete binary trees.

Lemma 3.9. *Let $n \in \mathbb{Z}_{\geq 0}$. For every $\pi \in \mathfrak{S}_n$, there exists a* unique *complete binary tree $T \in \mathsf{T}_{n,n}$ such that $\pi \in L[T]$.*

The key idea of the proof is to construct a tree T by inserting the permutation π in a binary search tree. Such a construction was given as the map ψ by Loday [19, Page 2] and a proof can be found in Loday and Ronco's paper [20, Section 2].

4. Realization of the associahedron

In this section, we follow the method outlined in the introduction to construct Loday's associahedron as in [19]. In particular we explicitly describe its normal fan, a fan we called Loday fan. This fan can be seen as an example of a *Cambrian* fan, studied by Reading and Speyer [28] and it is also an example of a *permutreehedral* fan studied by Pilaud and Pons [24]. We redo the construction here because particular details of it are relevant for our realization of the permuto-associahedron in Section 6.

4.1. Vertices of Loday associahedra

We begin by defining a set of points that are the candidates for vertices of the polytope.

Definition 4.1. Suppose $\alpha \in \mathbb{R}^n$ and $\mathsf{T} \in \mathsf{T}_{m,m}$ where $1 \leq m \leq n$. Define

$$\mathrm{val}(\alpha, \mathsf{T}) := \sum_{k=1}^{t} \alpha_k - \sum_{k=1}^{a} \alpha_k - \sum_{k=1}^{b} \alpha_k, \tag{4.1}$$

where t, a, and b are the number of total internal vertices in T, the Left subtree of T, and the Right subtree of T, respectively. (Note that we have $t = a + b + 1$.)

Given a (strictly increasing) sequence $\alpha \in \mathbb{R}^n$ and $\mathsf{T} \in \mathsf{T}_{n,n}$, we define

$$\mathrm{v}_{\mathsf{T}}^{\alpha} := \sum_{i=1}^{n} \mathrm{val}\left(\alpha, \mathsf{T}_{(i)}\right) \boldsymbol{e}_i = \left(\mathrm{val}\left(\alpha, \mathsf{T}_{(1)}\right), \mathrm{val}\left(\alpha, \mathsf{T}_{(2)}\right), \ldots, \mathrm{val}\left(\alpha, \mathsf{T}_{(n)}\right)\right),$$

where $\mathsf{T}_{(i)}$ denotes the subtree of T that is rooted at the internal vertex labeled by i.

The definition given above is a generalization of Loday: Note that if we choose $\alpha = (1, 2, \ldots, n)$, then the right hand side of (4.1) becomes $(a + 1)(b + 1)$. This recovers the vertex coordinates of the associahedron constructed by Loday [19]. Another generalization of Loday's coordinates was given by Masuda, Thomas, Tonks, and Vallette [22] using weights on the leaves (rather than on internal vertices as we do). The reason we consider our generalization is twofold: On the one hand we want a polytope related to Perm(α), see Corollary 4.15 below. On the other hand, we will need some flexibility on α later when we construct the permuto-associahedron.

We remark that some of the definitions and results in this section will be stated using the variable n in which case the readers can assume $n = d + 1$ (where d is the dimension of the constructed associahedron). The reason to do this is that in later sections we used these ideas with $n = d$.

Example 4.2. Let $\alpha = (2, 5, 6, 14, 17, 21, 22, 24)$ and T be the complete binary tree on the left of Fig. 4. Then $\mathsf{T}_{(3)}$ is the Left subtree of the root of T. It has 5 internal vertices, and both of its Left subtree and Right subtree have 2 internal vertices. Hence,

$$\mathrm{val}\left(\alpha, \mathsf{T}_{(3)}\right) = \sum_{k=1}^{5} \alpha_k - \sum_{k=1}^{2} \alpha_k - \sum_{k=1}^{2} \alpha_k = 44 - 7 - 7 = 30,$$

which gives the 3rd entry of $\mathrm{v}_{\mathsf{T}}^{\alpha}$. We can compute the other entries similarly, and get $\mathrm{v}_{\mathsf{T}}^{\alpha} = (2, 5, 30, 2, 5, 60, 5, 2)$.

We can check that the sum of coordinates of $\mathrm{v}_{\mathsf{T}}^{\alpha}$ is 111 which is the same as that of α. Hence, $\mathrm{v}_{\mathsf{T}}^{\alpha} \in U_7^{\alpha}$. This is true in general, as we state in the lemma below.

Lemma 4.3. *Suppose* $\alpha \in \mathbb{R}^n$ *and* $T \in \mathsf{T}_{n,n}$. *Let* T' *be a subtree of* T *with* t *internal vertices, and suppose (by Lemma 3.7 (2)) that* $I_T(T') = [c, c+t-1]$ *for some integer* c. *Then*

$$\sum_{i=c}^{c+t-1} \mathrm{val}(\alpha, T_{(i)}) = \sum_{k=1}^{t} \alpha_k. \tag{4.2}$$

In particular, if $T' = T$, *we obtain that sum of the coordinates of* v_T^{α} *is* $\sum_{k=1}^{n} \alpha_k$. *Hence,* v_T^{α} *is a point in* U_{n-1}^{α}. *(Recall the affine space* U_d^{α} *is defined in (2.3).)*

Remark 4.4. If we let $I = \mathrm{I}_{\mathrm{T}}(\mathrm{T}')$, then (4.2) can be rewritten as $\langle \boldsymbol{e}_I, \mathrm{v}_{\mathrm{T}}^{\alpha} \rangle = \sum_{k=1}^{|I|} \alpha_k$.

Proof of Lemma 4.3. We prove by induction on t, the number of internal vertices in T'. If $t = 0$, we have that $\mathrm{I}_{\mathrm{T}}(\mathrm{T}') = \emptyset$ or $[c, c-1]$. So (4.2) clearly holds.

Suppose $t \geq 1$ and (4.2) holds for any T' with less than t internal vertices. Now we consider T' has t internal vertices. Let T_L and T_R be the Left and Right subtrees of T' respectively, and suppose a and b are the number of internal vertices of T_L and T_R. (Note we have $t = a + b + 1$.) Then by Lemma 3.7 (specifically numerals (1),(2), and(3)), we must have that $\mathrm{T}' = \mathrm{T}_{(c+a)}$ is the subtree of T rooted at the internal vertex with label $c + a$, and

$$\mathrm{I}_{\mathrm{T}}(\mathrm{T}_L) = [c, c+a-1] \quad \text{and} \quad \mathrm{I}_{\mathrm{T}}(\mathrm{T}_R) = [c+a+1, c+a+b] = [c+a+1, c+t-1].$$

Since both T_L and T_R have less than t internal vertices, by induction hypothesis,

$$\sum_{i=c}^{c+a-1} \mathrm{val}(\alpha, \mathrm{T}_{(i)}) = \sum_{k=1}^{a} \alpha_k \quad \text{and} \quad \sum_{i=c+a+1}^{c+t-1} \mathrm{val}(\alpha, \mathrm{T}_{(i)}) = \sum_{k=1}^{b} \alpha_k.$$

Summing these two equations and that $\mathrm{val}(\alpha, \mathrm{T}_{(c+a)}) = \mathrm{val}(\alpha, \mathrm{T}') = \sum_{k=1}^{t} \alpha_k - \sum_{k=1}^{a} \alpha_k - \sum_{k=1}^{b} \alpha_k$, we obtain (4.2). □

Definition 4.5. Let $\alpha \in \mathbb{R}^{d+1}$ be a strictly increasing sequence. We define the following polytope

$$\mathrm{LodAsso}(\alpha) := \mathrm{ConvexHull}\{\mathrm{v}_{\mathrm{T}}^{\alpha} : \mathrm{T} \in \mathsf{T}_{d+1,d+1}\} \subset U_d^{\alpha}. \tag{4.3}$$

The polytope LodAsso(α) is called the *Loday associahedron* in reference of Jean-Louis Loday who first study it in the case $\alpha = (1, 2, \ldots, d+1)$. The following is the main theorem of this section.

Theorem 4.6. *Let* $\alpha \in \mathbb{R}^{d+1}$ *be a strictly increasing sequence. Then the face poset of the Loday associahedron LodAsso(α) is* K_d. *(Recall that* K_d *is defined in Definition 3.3.) Moreover, LodAsso(α) is a d-associahedron, and is a generalized permutohedron as well.*

4.2. Normal fan of Loday associahedra

In this part we construct a conic dissection of W_d and then apply Lemma 2.4 to show that the conic dissection induces the normal fan of LodAsso(α).

Recall that in Section 3.4 we define a cone $\sigma(\mathrm{T}) \subseteq W_{n-1}$ for each tree $\mathrm{T} \in \mathsf{T}_n$. The following result gives us the conic dissection we need.

Lemma 4.7. *Each cone* $\sigma(T)$ *for* $T \in \mathsf{T}_{n,n}$ *is a union of braid cones. Furthermore the collection of cones* $\mathsf{M}_{n-1} := \{\sigma(T) : T \in \mathsf{T}_{n,n}\}$ *is a simplicial conic dissection of* W_{n-1}.

Proof. Note that for $\mathrm{T} \in \mathsf{T}_{n,n}$, the preorder $\preceq_{\mathrm{T}}$ on $[n]$ is a partial order. Hence, the first statement follows from Lemma 2.13 (4). Then the second statement follows from part (2) of Lemma 2.13, and Lemmas 2.7 and 3.9. □

We need one preliminary lemma before establishing the condition we need on M_{n-1} and $\{\mathrm{v}_{\mathrm{T}}^{\alpha}\}$ in order to apply Lemma 2.4.

Lemma 4.8. *Let $T \in \mathcal{T}_{n,n}$ and $w \in \sigma^\circ_T$. For every $T' \in \mathcal{T}_{n,n}$ that is different from T, there exists T'' obtained from T' by a flip of an internal edge such that $\langle w, v^\alpha_{T'}\rangle < \langle w, v^\alpha_{T''}\rangle$.*

Proof. Let $w \in \sigma^\circ(T)$. Lemma 4.7 states that $\mathcal{M}_{n-1}$ is a conic dissection of W_{n-1}. Hence, given that $T' \neq T$, we must have that $w \notin \sigma(T')$. Using the description of a preorder cone given in Lemma 2.13 (1), we conclude that there exists a covering relation $j \lessdot_{T'} i$ in $\preceq_{T'}$ such that $x_i < x_j$. Let e be the internal edge of T' corresponding to this covering relation, and let T'' be the tree obtained from T' by flipping e. It can be checked (see [6, Lemma 4.8]) that $v^\alpha_{T'} - v^\alpha_{T''} = \lambda(\boldsymbol{e}_i - \boldsymbol{e}_j)$ for some $\lambda > 0$. Thus,

$$\langle w, v^\alpha_{T'} - v^\alpha_{T''}\rangle = \langle w, \lambda(\boldsymbol{e}_i - \boldsymbol{e}_j)\rangle = \lambda(x_i - x_j) < 0. \quad \square \quad \square$$

Corollary 4.9. *Let $T, T' \in \mathcal{T}_{n,n}$ be two complete binary trees. Then for every $w \in \sigma^\circ(T)$, we have $\langle w, v^\alpha_T\rangle \geq \langle w, v^\alpha_{T'}\rangle$, where the equality holds if and only if $T = T'$.*

Proof. It follows from Lemma 4.8 that we can construct a sequence of complete binary trees in $\mathcal{T}_{n,n}$: $T_0 = T', T_1, T_2, \ldots,$ such that for each i, $\langle w, v^\alpha_{T_i}\rangle < \langle w, v^\alpha_{T_{i+1}}\rangle$. The construction can continue as long as $T_i \neq T$. Since $\mathcal{T}_{n,n}$ is finite, this procedure must ends with a tree $T_k = T$. Then the conclusion follows. □

The following proposition is the key result of this section, characterizing the vertex set and the normal fan of the Loday associahedron. It also provides the main ingredients we need for proving Theorem 4.6.

Proposition 4.10. *Let $\alpha \in \mathbb{R}^{d+1}$ be a strictly increasing sequence.*

(1) The Loday associahedron $LodAsso(\alpha)$ is full-dimensional in U^α_d and its vertex set is $\{v^\alpha_T : T \in \mathcal{T}_{d+1,d+1}\}$.
(2) For each $T \in \mathcal{T}_{d+1,d+1}$, we have $\sigma(T) = \mathrm{ncone}(v^\alpha_T, LodAsso(\alpha))$.
(3) The normal fan of $LodAsso(\alpha)$ is $\Lambda_d := \{\sigma(T) : T \in \mathcal{T}_{d+1}\}$. Hence, Λ_d is a complete projective fan in W_d.

Proof. It follows from Lemma 4.7 and Corollary 4.9 that the set of cones $\mathcal{M}_d = \{\sigma(T) : T \in \mathcal{T}_{d+1,d+1}\}$ in W_d and the set of points $\{v^\alpha_T\ T \in \mathcal{T}_{d+1,d+1}\}$ in U^α_d satisfy the hypothesis of Lemma 2.4. Hence, we conclude that (1) and (2) are true, and that the $\mathcal{M}_d$ induces the normal fan of LodAsso(α). Therefore, it is left to show that Λ_d is induced by $\mathcal{M}_d$. However, this easily follows from Lemma 2.13 (3), Lemma 3.8, and the fact that $\mathcal{T}_{d+1}$ consists of all contractions of $\mathcal{T}_{d+1,d+1}$. □

Because of the results given above, we call

$$\Lambda_d := \{\sigma(T) : T \in \mathcal{T}_{d+1}\}. \tag{4.4}$$

the *Loday fan*. The following lemma gives a characterization of the Loday fan.

Lemma 4.11. *The Loday fan Λ_d has the following properties:*

(1) The set $\{\sigma(T) : T \in \mathcal{T}_{d+1,k}\}$ consists of all the $(k-1)$-dimensional cones in Λ_d.
(2) For any $T, T' \in \mathcal{T}_{d+1}$, we have that $\sigma_{T'}$ is a face of σ_T if and only if T' is obtained from T by contracting internal edges of T.
(3) The face poset of Λ_d is isomorphic to the poset dual to $\mathcal{K}_d \setminus \hat{0}$.
(4) The Loday fan Λ_d is a simplicial fan.
(5) The Loday fan Λ_d is a coarsening of the braid fan Br_d.

Proof. One sees that (1), (2), and (4) follow from Lemma 2.13 and the definition of $\sigma(T)$. Item (3) follows from (2) and the definition of $\mathcal{K}_d$ in Definition 3.3. By Lemma 4.7, each cone in $\mathcal{M}_d$ is a union of braid cones. Since Λ_d is induced by $\mathcal{M}_d$, it must coarsen Br_d. □

We are now ready to prove the main theorem of this section.

Proof of Theorem 4.6. By Proposition 4.10 (3) and Lemmas 4.11 (3) and 2.2, we have that the face poset of LodAsso(α) is K_d. Moreover, by Lemma 3.4, we conclude that LodAsso(α) is a d-associahedron. Finally, it follows from Proposition 4.10 (3) and Lemma 4.11 (5) that LodAsso(α) is a generalized permutohedron. □

4.3. Inequality description for Loday associahedra

It follows from Proposition 4.10 that we can apply Lemma 2.5 to find an inequality description for the Loday associahedron LodAsso(α). Note that by Lemma 4.11 (1), the set $\{\sigma(T) : T \in T_{d+1,2}\}$ consists of all one dimensional cones of Λ_d. Thus, our first step is to choose a generator for each of the cones in this set.

Definition 4.12. Let $T \in T_{d+1,2}$. Since T has two internal vertices, it has a unique non-root internal vertex, say v. Then by Lemma 3.7 (2), the labels of v is a nonempty integer interval, denoted by Itv(T), and called the *defining interval* of T.

The following proposition is straightforward.

Proposition 4.13. *The map $T \mapsto \mathrm{Itv}(T)$ gives a one-to-one correspondence between the set $T_{d+1,2}$ and the set*

$$I_d := \{I : I \text{ is an integer interval and } \emptyset \neq I \subsetneq [d+1]\} \tag{4.5}$$

Moreover, for each $T \in T_{d+1,2}$, if let $I = \mathrm{Itv}(T)$, then the one-dimensional cone $\sigma_T \in \Lambda_d$ is generated by $e_{\bar{I}}$, where $\bar{I} := [d+1] \setminus I$ is the complement of I.

Theorem 4.14. *Let $\alpha \in \mathbb{R}^{d+1}$ be a strictly increasing sequence. Then we have the following minimal inequality description for LodAsso(α) :*

$$LodAsso(\alpha) = \left\{ x \in U_d^{\alpha} \ : \ \langle e_{\bar{I}}, x\rangle \geq \sum_{i=|I|+1}^{d+1} \alpha_i, \quad \forall I \in I_d \right\}. \tag{4.6}$$

We remark that the inequality $\langle e_{\bar{I}}, x\rangle \geq \sum_{i=|I|+1}^{d+1} \alpha_i$ is equivalent to $\langle e_I, x\rangle \leq \sum_{i=1}^{|I|} \alpha_i$ since the sum of all coordinates is fixed. We present this way so that it is consistent with using outer normal vectors for the normal fan.

Proof. Applying the first part of Lemma 2.5 together with Proposition 4.13, one sees that it is left to show for any $T' \in T_{d+1,2}$, if we let $I = \mathrm{Itv}(T')$, then $\max_{T \in T_{d+1,d+1}} \langle e_{\bar{I}}, v_T^{\alpha}\rangle = \sum_{i=|I|+1}^{d+1} \alpha_i$. We choose $T \in T_{d+1,d+1}$ such that $T \leq_K T'$ in K_d, equivalently, T' can be obtained from T by contracting internal edges. Thus, by Lemma 4.11 (2), we have $\sigma_{T'}$ is a face of σ_T, where the latter is the normal cone of LodAsso(α) at v_T. Therefore, by the second part of Lemma 2.5, we just need to show $\langle e_{\bar{I}}, v_T^{\alpha}\rangle = \sum_{i=|I|+1}^{d+1} \alpha_i$.

By the definition of I and how the internal vertices of trees in T_n are labeled, one sees that there exists a subtree T_0 of T such that $I = I_T(T_0)$. Therefore, by Lemma 4.3 and Remark 4.4, we have $\langle e_I, v_T^{\alpha}\rangle = \sum_{k=1}^{|I|} \alpha_k$, and $\langle \mathbf{1}, v_T^{\alpha}\rangle = \sum_{k=1}^{d+1} \alpha_k$. Thus, the conclusion follows. □

Following terminology from V.Pilaud [23], if a polytope is defined by a subset of a system of linear inequalities that defines a permutohedron Perm(α), then we call it an α-*removohedron*.

Corollary 4.15. *Let $\alpha \in \mathbb{R}^{d+1}$ be a strictly increasing sequence. Then the associahedron LodAsso(α) is an α-removohedron.*

Proof. Comparing the inequality description obtained in Theorem 4.14 with the inequality description for Perm(α) given in Eq. (2.5), we see the conclusion follows. □

Corollary 4.16. *Let* $\alpha = (\alpha_1, \ldots, \alpha_{d+1}) \in \mathbb{R}^{d+1}$ *be a strictly increasing sequence and* $\delta = (\delta_1, \ldots, \delta_d)$, *where* $\delta_k = \alpha_k - \alpha_{k-1}$ *for* $k = 1, 2, \ldots, d+1$, *the vector of first differences (and setting* $\alpha_0 = 0$*). We have the following description using Mikowski sums*

$$LodAsso(\alpha) = \delta_{d+1}\Delta_{[d+1]} + \sum_{I \in \mathrm{I}_d} \delta_{|I|}\Delta_I, \tag{4.7}$$

where $\Delta_I := \mathrm{ConvexHull}\{\boldsymbol{e}_i : i \in I\} \subset \mathbb{R}^{d+1}$.

The corollary can be proved by applying [25, Proposition 6.3] to give an inequality description for the Minkowski sum of simplices in the right hand side of Eq. (4.7) and verifying that it is the same as (4.6).

Remark 4.17. If one takes Eq. (4.7) as the definition for LodAsso(α), most of the constructions and results can be derived using results of [25, Section 7]. In particular, [25, Theorem 7.9] provides explicit coordinates for vertices of nestohedra and [25, Theorem 7.10] describes the normal cone at each vertex. We write this section in the order as presented because this is how we come up with our construction and we want to use it as an example to demonstrate our methods of using Lemmas 2.4 and 2.5.

In [8] the authors called any polytope of the form $\sum_I a_I \Delta_I$, where $a_I > 0$ summed over all integer intervals, a *Postnikov associahedron*. This family of polytopes contains Loday associahedra defined in this paper, but is strictly larger, since in Eq. (4.7) the scalar factors are the same for integer intervals of the same size. Furthermore, Corollary 4.15 is not valid for Postnikov associahedra.

5. Kapranov poset, partition labeled trees and their associated cones

In this section we define the Kapranov poset in terms of trees labeled by partitions. Then we associate cones to these trees by using our ideas from [7]. Finally, we prove some basic properties of these cones which are fundamental for our construction of the permuto-associahedron in the next section.

5.1. The Kapranov poset

In this part, we will introduce the poset defined by Kapranov as a hybrid of the face poset of a permutohedron and the face poset of an associahedron, and define the permuto-associahedron abstractly.

Recall the face poset of a usual d-permutohedron is the poset O_{d+1} on ordered set partitions on $[d+1]$ (defined in Definition 2.8) and the face poset of a d-associahedron is the poset K_d on strictly branching trees in T_{d+1} (defined in Definition 3.3). Below we construct Kapranov's poset as a poset on pairs of ordered set partitions and these trees.

Definition 5.1. Let $\mathrm{T}_{\le d} := \bigcup_{i=0}^{d} \mathrm{T}_i$ be the set of strictly branching trees with at most $d+1$ leaves. A *([d+1]-)partition labeled tree* consists of a pair $(\mathrm{S}, \mathrm{T}) \in \mathrm{O}_{d+1} \times \mathrm{T}_{\le d}$, where the tree T has $k \le d+1$ leaves and the partition S has k blocks $S_1, \ldots, S_k$ that we use to label the leaves of T from left to right. The set of all $[d+1]$-partition labeled trees is denoted P_d.

For any $(\mathrm{S}_1, \mathrm{T}_1), (\mathrm{S}_2, \mathrm{T}_2)$ in P_d, we define $(\mathrm{S}_1, \mathrm{T}_1) \lessdot_{KP} (\mathrm{S}_2, \mathrm{T}_2)$ if one of the following two conditions hold:

i. The tree T_2 is obtained from T_1 by contracting a single internal edge of T_1, and $\mathrm{S}_1 = \mathrm{S}_2$.
ii. There exists a minimal [5] internal vertex p of T_1 such that T_2 is obtained from T_1 by contracting all edges between p and its children in T_1 while labeling this new leaf with the union of the labels of the children of p in T_1.

5 An internal vertex is a *minimal* internal vertex if all of its children are leaves.

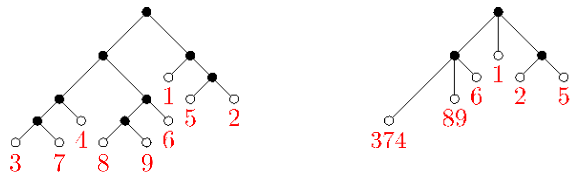

Fig. 5. Two [9]-partition labeled trees.

This relation $\lessdot_{KP}$ extends to a partial order $\leq_{KP}$ on P_d where $\lessdot_{KP}$ is the covering relation. We also add a minimum element $\{\hat{0}\}$ by declaring $\hat{0} \leq_{KP} (S, T)$ for any $(S, T) \in P_d$. Finally, we denote the poset $(P_d \sqcup \{\hat{0}\}, \hat{\leq}_{KP})$ by $K\Pi_d$ and call it the *Kapranov poset*.

Example 5.2.
In Fig. 5 we present two elements of P_8 with partition labelings
(3, 7, 4, 8, 9, 6, 1, 5, 2) and (374, 89, 6, 1, 5, 2) respectively. The one on the left is smaller than the one on the right in the Kapranov poset.

The next lemma collects useful facts about the Kapranov poset. Some are proven in [17] in a different language and some are straightforward, so we omit proofs.

Lemma 5.3. *The Kapranov poset $K\Pi_d$ is graded of rank $d+1$. Furthermore, the following facts hold for its elements:*

(1) The rank of a $[d+1]$-partition labeled tree (S, T) is $d - i(T) + 1$, where $i(T)$ is the number of internal vertices of the tree.
(2) It has a unique maximum element (S_0, T_0) where S_0 is the trivial partition $([d+1])$, and T_0 is the only rooted tree on a single vertex which is the unique element in T_0.
(3) The elements of rank 1 are in bijection with $S_{d+1} \times T_{d,d}$. More precisely, each $(\pi, T) \in S_{d+1} \times T_{d,d}$ defines a rank-1 element $(S(\pi), T)$ of $K\Pi_d$, and every rank-1 element arises this way. (Recall that $S(\pi)$ is defined in (2.8).)
(4) The elements of rank d are in bijection with $\overline{O_{d+1}} := \bigsqcup_{k=2}^{d+1} O_{d+1,k}$, the set of all non-trivial ordered partitions of $[d+1]$. More precisely, each $S \in O_{d+1,k}$ (for some $2 \leq k \leq d+1$) defines a rank-d element (S, T_k) of $K\Pi_d$, where T_k is the unique tree with one internal vertex and k leaves, and every rank-d element of $K\Pi_d$ arises this way (for some k).

Definition 5.4. A *d-permuto-associahedron* is a *d*-dimensional polytope whose face poset is isomorphic to $K\Pi_d$.

5.2. Nested combinatorics

In this part, we will review results on nested permutohedra and nested braid fans obtained in [7]. Recall that $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{d+1}\}$ is the standard basis for $\mathbb{R}^{d+1}$. For any permutation $\pi \in S_{d+1}$ and integer $i \in [d]$, we define the vectors

$$\mathbf{f}_i^{\pi} := \boldsymbol{e}_{\pi^{-1}(i+1)} - \boldsymbol{e}_{\pi^{-1}(i)},$$

and the linear transformations $D^{\pi} : \mathbb{R}^{d+1} \to \mathbb{R}^d$ as

$$D^{\pi}\mathbf{w} := (D_1^{\pi}\mathbf{w}, D_2^{\pi}\mathbf{w}, \ldots, D_d^{\pi}\mathbf{w}),$$

where the *i*th coordinate is

$$D_i^{\pi}\mathbf{w} := \langle \mathbf{w}, \mathbf{f}_i^{\pi} \rangle = w_{\pi^{-1}(i+1)} - w_{\pi^{-1}(i)}.$$

If π is the identity permutation, we may omit π and write $D\mathbf{w}$ instead.

It is easy to verify that $D^\pi(w) = D^\pi(w + k\mathbf{1})$ for any $k \in \mathbb{R}$. Hence, we may consider the domain of the map D^π is W_d, and D^π is a map from W_d to $\mathbb{R}^d$.

We note that with the above definition, the braid cone can be expressed as

$$\sigma(\pi) = \{w \in W_d \ : \ D^\pi w \geq 0\}.$$

Example 5.5. Let $\pi = 791386245$. Then $\pi^{-1} = 374896152$. Thus $w \in \sigma(\pi)$ means that

$$D^\pi_1 w = w_7 - w_3 \geq 0, \quad D^\pi_3 w = w_8 - w_4 \geq 0, \quad D^\pi_5 w = w_6 - w_9 \geq 0, \quad D^\pi_7 w = w_2 - w_1 \geq 0,$$
$$D^\pi_2 w = w_4 - x_7 \geq 0, \quad D^\pi_4 w = w_9 - w_8 \geq 0, \quad D^\pi_6 w = w_1 - w_6 \geq 0, \quad D^\pi_8 w = w_5 - w_2 \geq 0,$$

or equivalently,

$$w_3 \leq w_7 \leq w_4 \leq w_8 \leq w_9 \leq w_6 \leq w_1 \leq w_5 \leq w_2.$$

For convenience, for $w \in \sigma(\pi)$, we often let $u_i = w_{\pi^{-1}(i)}$ for each i, which allows us to express w as

$$w = \sum_{i=1}^{d+1} u_i \boldsymbol{e}_{\pi^{-1}(i)}. \tag{5.1}$$

Then one sees that $D^\pi w = D\boldsymbol{u}$ (where $\boldsymbol{u} = (u_1, \dots, u_{d+1})$).

Given strictly increasing sequences $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \mathbb{R}^{d+1}$ and $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{R}^d$, for any $(\pi, \tau) \in S_{d+1} \times S_d$, we define

$$v^{(\alpha,\beta)}_{\pi,\tau} := \sum_{i=1}^{d+1} \alpha_i \boldsymbol{e}_{\pi^{-1}(i)} + \sum_{i=1}^{d} \beta_i \mathbf{f}^\pi_{\tau^{-1}(i)} \in U^\alpha_d. \tag{5.2}$$

We call (α, β) an *appropriate pair* (of strictly increasing sequences) if for any (π, τ), when we write $v^{(\alpha,\beta)}_{\pi,\tau} = \sum \gamma_i \boldsymbol{e}_{\pi^{-1}(i)}$ we have $\gamma_1 < \gamma_2 < \cdots < \gamma_{d+1}$. It is not hard to see, by scaling the vector α if necessary, that appropriate pairs exist. Then for any appropriate pair (α, β), we define the *usual nested permutohedron* (associated with the pair (α, β)) to be

$$\mathrm{Perm}(\alpha, \beta) := \mathrm{ConvexHull}\left(v^{(\alpha,\beta)}_{\pi,\tau} \ : \ (\pi, \tau) \in S_{d+1} \times S_d\right). \tag{5.3}$$

A *generalized nested permtuhohedron* is a deformation of a usual nested permutohedron. We need the following definition before defining *nested braid cones*.

Definition 5.6. Let $\tau \in S_d$. We define $\check{\sigma}(\tau) := \{\mathbf{x} \in \mathbb{R}^d \ : \ x_{\tau^{-1}(1)} \leq x_{\tau^{-1}(2)} \leq \cdots \leq x_{\tau^{-1}(d)}\}$.

Clearly $\check{\sigma}(\tau)$ maps to $\sigma(\tau)$ under the quotient map $\mathbb{R}^d \to W_{d-1}$. Notice that $\check{\sigma}(\tau)$ is not pointed as it contains the line spanned by $\mathbf{1}$. Similar to Lemma 2.7, the collection of cones $\{\check{\sigma}(\tau) : \pi \in S_d\}$ forms a conic dissection of $\mathbb{R}^d$.

Definition 5.7. For each $(\pi, \tau) \in S_{d+1} \times S_d$, we define the *nested braid cone* $\sigma(\pi, \tau)$ to be

$$\sigma(\pi, \tau) := \left\{ w \in W_d \ : \ \begin{array}{c} w \in \sigma(\pi), \\ D^\pi w \in \check{\sigma}(\tau). \end{array} \right\} \tag{5.4}$$

Note here $\sigma(\pi)$ is in W_d and $\check{\sigma}(\tau)$ is in $\mathbb{R}^d$.

One can check that $\sigma(\pi, \tau)$ is a well-defined d-dimensional cone in W_d, and has a minimal inequality description:

$$\sigma(\pi, \tau) = \left\{ w \in W_d \ : \ 0 \leq D^\pi_{\tau^{-1}(1)} w \leq D^\pi_{\tau^{-1}(2)} w \leq \cdots \leq D^\pi_{\tau^{-1}(d)} w \right\}.$$

Therefore, the relative interior of $\sigma(\pi, \tau)$ is given by

$$\sigma^\circ(\pi, \tau) = \left\{ w \in W_d \ : \ 0 < D^\pi_{\tau^{-1}(1)} w < D^\pi_{\tau^{-1}(2)} w < \cdots < D^\pi_{\tau^{-1}(d)} w \right\}.$$

Let $\mathcal{B}_d^2 := \{\sigma(\pi,\tau) : (\pi,\tau) \in \mathfrak{S}_{d+1} \times \mathfrak{S}_d\}$ be the collection of nested braid cones in W_d. In [7] we proved that $\mathcal{B}_d^2$ induces a projective fan Br_d^2 which we call the *nested braid fan*. More precisely, in [7, Proposition 4.6] we prove that for any appropriate pair (α, β), the normal fan of $\mathrm{Perm}(\alpha, \beta)$ is equal to Br_d^2. Thus a polytope P is a generalized nested permutohedron if and only if its normal fan coarsens the nested braid fan Br_d^2 for some d.

Analogous to Lemma 2.7 we have the following.

Lemma 5.8. *The collection of nested braid cones $\mathcal{B}_d^2$ forms a conic dissection of W_d.*

Recall we define $\boldsymbol{e}_S = \sum_{i\in S} \boldsymbol{e}_i$ for each $S \subseteq [d+1]$. For each element $\mathcal{S} = (S_1, \dots, S_k) \in \mathcal{O}_{d+1}$, we define

$$\boldsymbol{e}_{\mathcal{S}} := \sum_i i\boldsymbol{e}_{S_i}. \tag{5.5}$$

We also define the *type* of $\mathcal{S}$, denoted by $\mathrm{Type}(\mathcal{S})$, to be the sequence $(t_1, t_2, \dots, t_{k-1})$, where

$$t_i = \sum_{j=1}^{i} |S_j|, \quad \text{for } 0 \le i \le k.$$

The following theorem gives inequality descriptions for usual nested permutohedra, recalling that $\overline{\mathcal{O}_{d+1}} = \bigsqcup_{i=2}^{d+1} \mathcal{O}_{d+1,i}$ is the set of all non-trivial ordered partitions of $[d+1]$:

Theorem 5.9 (*Theorem 4.20 in [7]*). *Suppose $(\alpha, \beta) \in \mathbb{R}^{d+1} \times \mathbb{R}^d$ is a pair of strictly increasing sequences $(\alpha, \beta) \in \mathbb{R}^{d+1} \times \mathbb{R}^d$ that is appropriate. Suppose $\boldsymbol{b} \in \mathbb{R}^{\overline{\mathcal{O}_{d+1}}}$ is defined as follows: for each $\mathcal{S} \in \overline{\mathcal{O}_{d+1}}$, if* $\mathrm{Type}(\mathcal{S}) = (t_1, t_2, \dots, t_{k-1})$, *let*

$$b_{\mathcal{S}} = \left(\sum_{i=1}^{k} i \left(\sum_{j=t_{i-1}+1}^{t_i} \alpha_j\right) + \sum_{j=d+2-k}^{d} \beta_j\right), \tag{5.6}$$

where by convention we set $t_0 = 0$ and $t_k = d+1$. Then we have the following minimal inequality description for $\mathrm{Perm}(\alpha, \beta)$*:*

$$\mathrm{Perm}(\alpha, \beta) = \left\{ \boldsymbol{x} \in U_d^{\alpha} : \langle \boldsymbol{e}_{\mathcal{S}}, \boldsymbol{x}\rangle \le b_{\mathcal{S}}, \quad \forall \mathcal{S} \in \overline{\mathcal{O}_{d+1}} \right\}. \tag{5.7}$$

5.3. Cones associated to partition labeled trees

Let $(\mathcal{S}, \mathcal{T}) \in \mathcal{P}_d$. Suppose $\mathcal{S} = (S_1, S_2, \dots, S_k)$ and $\mathrm{Type}(\mathcal{S}) = (t_1, \dots, t_{k-1})$. Hence, $\mathcal{T}$ has k leaves. Recall in Section 3.3, we describe a way to label internal vertices of $\mathcal{T}$ with the set $[k-1]$. We apply the same procedure on $\mathcal{T}$ first, and then replace each label i with t_i. Hence, we obtain a labeling on internal vertices of $\mathcal{T}$ with the set $\{t_1, t_2, \dots, t_{k-1}\}$. Let $G(\mathcal{S}, \mathcal{T})$ be the induced subtree of $\mathcal{T}$ on its internal vertices together with the new labeling. One sees that $G(\mathcal{S}, \mathcal{T})$ is the Hasse diagram of a preorder on $\{t_1, t_2, \dots, t_{k-1}\}$, which we denote by $\preceq_{(\mathcal{S},\mathcal{T})}$. For convenience, we also treat $\mathrm{Type}(\mathcal{S})$ as a set, and thus we can say that $\preceq_{(\mathcal{S},\mathcal{T})}$ is a preorder on $\mathrm{Type}(\mathcal{S})$.

Example 5.10. Suppose $(\mathcal{S}, \mathcal{T})$ is the $[9]$-partition labeled tree on the right of Fig. 5. Then $\mathcal{S} = (374, 89, 6, 1, 2, 5)$ and $\mathrm{Type}(\mathcal{S}) = (3, 5, 6, 7, 8)$. In Fig. 6, on the left we show $\mathcal{T}$ together with its old internal vertex labeling considered in Section 3.3, and in the middle we show $\mathcal{T}$ with its internal vertices labeled by $\mathrm{Type}(\mathcal{S})$, and finally the right side is $G(\mathcal{S}, \mathcal{T})$, which defines the preorder $\preceq_{(\mathcal{S},\mathcal{T})}$.

Remark 5.11. Given the way we construct the labeling of $\mathcal{T}$ using the set $\mathrm{Type}(\mathcal{S})$, one sees that a lot of properties we discussed in Section 3.3 on internal vertex labelings of $\mathcal{T}$, e.g., Lemma 3.7, can be modified to a version that works for the current version of labeling.

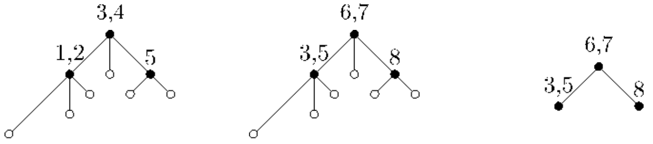

Fig. 6. An example of the construction of $G(S, T)$.

For each $(S, T) \in P_d$, we will define a preorder cone associated to it using the preorder $\preceq_{(S,T)}$. We need a preliminary lemma before giving such a definition.

Recall that for any $S \in O_{d+1}$, we associate a preorder $\preceq_S$ on $[d+1]$ to it (see (2.9)). Let

$$W_S := \{w \in W_d : w_i = w_j, \text{ if } i \equiv_S j\}. \tag{5.8}$$

Recall $S(\pi)$ is defined in (2.8).

Lemma 5.12. *Suppose* $S \in O_{d+1}$ *and* $\pi, \pi' \in S_{d+1}$. *If* $S(\pi)$ *and* $S(\pi')$ *both refine* S, *then for any* $w \in W_S$, *we have*

$$D_i^{\pi} w = D_i^{\pi'} w, \text{ for all } i.$$

We will not give a proof for the above lemma, which is straightforward by checking the definition. Instead, we will give an example to demonstrate why it is true.

Example 5.13. Let $S = (374, 89, 6, 1, 2, 5)$. Then $w \in W_S$ if and only if

$$w_3 = w_7 = w_4 \quad \text{and} \quad w_8 = w_9. \tag{5.9}$$

Let $\pi = 791386245$ and $\pi' = 793286145$. Then $S(\pi) = (3, 7, 4, 8, 9, 6, 1, 5, 2)$ and $S(\pi') = (7, 4, 3, 8, 9, 6, 1, 5, 2)$, both of which refine S. Clearly, for each $4 \le i \le 8$, we have $D_i^{\pi} w = D_i^{\pi'} w$ for any $w \in W_d$. Now if $w \in W_S$, we have $D_i^{\pi} w = 0 = D_i^{\pi'} w$ for $i = 1, 2$, and

$$D_3^{\pi} w = w_8 - w_4 = w_8 - w_3 = D_3^{\pi'} w.$$

Lemma 5.12 allows us to give the following definition.

Definition 5.14. Let $(S, T) \in P_d$. Choose $\pi \in S_{d+1}$ such that $S(\pi)$ refines S. Then we define

$$\sigma(S, T) := \left\{ w \in W_S : \begin{array}{l} D_i^{\pi} w \le D_j^{\pi} w, \text{ if } i \preceq_{(S,T)} j \\ D_\ell^{\pi} w \ge 0, \text{ if } \ell \text{ is a minimal element of } \preceq_{(S,T)} \end{array} \right\}. \tag{5.10}$$

Note that by Lemma 5.12, the definition of $\sigma(S, T)$ does not depend on the choice of π as long as $S(\pi)$ refines S.

Example 5.15. Let (S, T) be the $[9]$-partition labeled tree on the right of Fig. 5. Then $S = (374, 89, 6, 1, 2, 5)$ and its preorder $\preceq_{(S,T)}$ has been discussed in Example 5.10. We choose $\pi = 791386245$ which we have shown that $S(\pi)$ refines S. Thus, we have that $w \in \sigma(S, T)$ if and only if both condition in (5.9) and the condition below hold:

$$0 \le D_3^{\pi} w = D_5^{\pi} w \le D_6^{\pi} w = D_7^{\pi} w \text{ and } 0 \le D_8^{\pi} w \le D_6^{\pi} w. \tag{5.11}$$

5.4. Face structure of $\sigma(S, T)$ *our cones*

In Section 6, we will show (in Proposition 6.6) that the collection of cones

$$\Xi_d := \{\sigma(S, T) : (S, T) \in P_d\} \tag{5.12}$$

is the normal fan of the permuto-associahedron that we construct. Therefore, we call Ξ_d the *nested Loday fan*. In the remaining of this section, we will explore properties of $\sigma(S, T)$ and Ξ_d. The main goal of this subsection is to prove the following proposition, which establishes the connection between Ξ_d and the Kapranov poset.

Proposition 5.16. *The map* $(\mathcal{S}, T) \mapsto \sigma(\mathcal{S}, T)$ *gives a bijection from* $\mathcal{P}_d$ *to* Ξ_d. *Furthermore, for any* $(\mathcal{S}_1, T_1), (\mathcal{S}_2, T_2) \in \mathcal{P}_d$, *we have that* $(\mathcal{S}_1, T_1) \leq_{KP} (\mathcal{S}_2, T_2)$ *in the Kapranov poset* $K\Pi_d$ *if and only if the cone* $\sigma(\mathcal{S}_2, T_2)$ *is a face of* $\sigma(\mathcal{S}_1, T_1)$.

First we prove the following lemma with various technical but basic facts about the cones $\sigma(\mathcal{S}, T)$.

Lemma 5.17. *Let* $(\mathcal{S}, T) \in \mathcal{P}_d$ *and choose* $\pi \in \mathfrak{S}_{d+1}$ *such that* $\mathcal{S}(\pi)$ *refines* $\mathcal{S}$. *Recall* $W_{\mathcal{S}}$ *is defined in* (5.8). *We define the affine space*

$$W_{(\mathcal{S},T)} := \left\{ \mathbf{w} \in W_{\mathcal{S}} : D_i^\pi \mathbf{w} = D_j^\pi \mathbf{w}, \text{ if } i \equiv_{(\mathcal{S},T)} j \right\} = \left\{ \mathbf{w} \in W_d : \begin{array}{l} w_i = w_j, \text{ if } i \equiv_{\mathcal{S}} j \\ D_i^\pi \mathbf{w} = D_j^\pi \mathbf{w} \text{ if } i \equiv_{(\mathcal{S},T)} j \end{array} \right\}. \tag{5.13}$$

Then the cone $\sigma(\mathcal{S}, T)$ *has the following inequality description:*

$$\sigma(\mathcal{S}, T) = \left\{ \mathbf{w} \in W_{(\mathcal{S},T)} : \begin{array}{l} D_i^\pi \mathbf{w} \leq D_j^\pi \mathbf{w}, \text{ if } i \preceq_{(\mathcal{S},T)} j \\ D_\ell^\pi \mathbf{w} \geq 0, \text{ if } \ell \text{ is a minimal element of } \preceq_{(\mathcal{S},T)} \end{array} \right\}. \tag{5.14}$$

Moreover, the following statements hold:

1. *(1) The cone* $\sigma(\mathcal{S}, T)$ *is full-dimensional in* $W_{(\mathcal{S},T)}$, *whose dimension is equal to the number of internal vertices of* T.
2. *(2) The inequality description* (5.14) *for* $\sigma(\mathcal{S}, T)$ *is facet-defining.*
3. *(3) For every facet* F *of* $\sigma(\mathcal{S}, T)$, *there exists* $(\mathcal{S}', T')$ *covering* $(\mathcal{S}, T)$ *in* $K\Pi_d$ *such that* $F = \sigma(\mathcal{S}', T')$. *Conversely, for every* $(\mathcal{S}', T')$ *in* $K\Pi_d$ *that covers* $(\mathcal{S}, T)$, *its associated cone* $\sigma(\mathcal{S}', T')$ *is a facet of* $\sigma(\mathcal{S}, T)$.

Proof. First, we observe that given the definition of $W_{(\mathcal{S},T)}$, the inequality description (5.14) clearly defines $\sigma(\mathcal{S}, T)$. Hence, it is left to verify statements (1)–(3).

In order to make notation easy, we let $V(\mathcal{S}, T)$ and $K(\mathcal{S}, T)$ be the images of $W_{(\mathcal{S},T)}$ and $\sigma(\mathcal{S}, T)$ under the map $D^\pi : W_d \to \mathbb{R}^d$. Then

$$V(\mathcal{S}, T) = \left\{ \mathbf{u} \in \mathbb{R}^d : \begin{array}{l} u_\ell = 0, \text{ if } \ell \notin \text{Type}(\mathcal{S}) \\ u_i = u_j, \text{ if } i \equiv_{(\mathcal{S},T)} j \end{array} \right\},$$

and

$$K(\mathcal{S}, T) = \left\{ \mathbf{u} \in V(\mathcal{S}, T) : \begin{array}{l} u_i \leq u_j, \text{ if } i \preceq_{(\mathcal{S},T)} j \\ u_\ell \geq 0, \text{ if } \ell \text{ is a minimal element of } \preceq_{(\mathcal{S},T)} \end{array} \right\}. \tag{5.15}$$

One important observation is that the linear map D^π is an isomorphism from W_d to $\mathbb{R}^d$. Hence, it suffices to show the following three corresponding statements:

- (C1) The cone $K(\mathcal{S}, T)$ is full-dimensional in $V(\mathcal{S}, T)$, whose dimension is equal to the number of internal vertices of T.
- (C2) The inequality description (5.15) for $K(\mathcal{S}, T)$ is facet-defining.
- (C3) For every facet K of $K(\mathcal{S}, T)$, there exists $(\mathcal{S}', T')$ covering $(\mathcal{S}, T)$ in $K\Pi_d$ such that $K = K(\mathcal{S}', T')$. Conversely, for every $(\mathcal{S}', T')$ in $K\Pi_d$ that covers $(\mathcal{S}, T)$, the cone $K(\mathcal{S}', T')$ is a facet of $K(\mathcal{S}, T)$.

We prove (C1) first. Recall that $\preceq_{(\mathcal{S},T)}$ is a preorder on Type($\mathcal{S}$). One sees that $V(\mathcal{S}, T) \cong \{\mathbf{u} \in \mathbb{R}^{\text{Type}(\mathcal{S})} : u_i = u_j, \text{ if } i \equiv_{(\mathcal{S},T)} j\}$. Clearly, the dimension of the latter is the number of equivalence classes in Type$(\mathcal{S})/\equiv_{(\mathcal{S},T)}$, which is exactly the number of internal vertices of T.

We now show that $K(\mathcal{S}, T)$ is full-dimensional in $V(\mathcal{S}, T)$. Given that $K(\mathcal{S}, T)$ is described by (5.15), it is enough to show there exists $\mathbf{u} \in V(\mathcal{S}, T)$ such that

(a) $u_i < u_j$, if $i \preceq_{(\mathcal{S},T)} j$ and (b) $u_k > 0$ if k is a minimal element of $\preceq_{(\mathcal{S},T)}$.

We can construct such a $\boldsymbol{u}$ easily: First set $u_k = 0$ for each $k \in [d] \setminus \mathrm{Type}(S)$. Next, noticing that the Haase diagram $G(S, T)$ for the preorder $\preceq_{(S,T)}$ without labels is just T which is a rooted tree, we set $u_i = d - k$ for each $i \in \mathrm{Type}(S)$ that is a label for an internal vertex that is k-distance away from the root of T. It is easy to see a vector $\boldsymbol{u}$ constructed this way satisfies the desired conditions. Thus, $K(S, T)$ is full-dimensional in $V(S, T)$ (whose dimension is the number of internal vertices of T.) Hence, statement (C1) holds.

Next we prove (C2). There are two kinds of inequalities in (5.15). We treat them separately.

i. Suppose $i \preceq_{(S,T)} j$, and assume that i and j are labels of the internal vertices x and y of T. Let T' be the tree obtained from T by contracting the internal edge $\{x, y\}$ and let $S' := S$. Clearly, we have $\mathrm{Type}(S') = \mathrm{Type}(S)$, and thus $\preceq_{(S,T)}$ and $\preceq_{(S',T')}$ are two preorders on the same set. By Lemma 3.7 (4) and Remark 5.11, we conclude that $\preceq_{(S',T')}$ is obtained from $\preceq_{(S,T)}$ by merging the equivalence classes $\bar{i}$ and $\bar{j}$. Hence, one sees that

$$K(S, T) \cap \{\boldsymbol{u} : u_i = u_j\} = K(S', T').$$

Since T' has one less internal vertex than T, it follows from property (C1) that $K(S', T')$ is a facet of $K(S, T)$. Therefore, the inequality $u_i \le u_j$ if facet-defining.

ii. Suppose ℓ is a minimal element of $\preceq_{(S,T)}$, and assume ℓ is a label of the internal vertex p of (S, T). Clearly, p is a minimal internal vertex of T.
Let (S', T') be the partition labeled tree obtained from (S, T) by contracting all edges between p and its children in T while labeling this new leaf with the union of the labels of the children of p in T. More precisely, suppose $S = (S_1, S_2, \ldots, S_k)$, and assume the children of p are labeled by $S_a, S_{a+1}, \ldots, S_b$, then

$$S' = \left(S_1, S_2, \ldots, S_{a-1}, \bigcup_{i=a}^{b} S_i, S_{b+1}, S_{b+2}, \ldots, S_k\right).$$

Hence, if $\mathrm{Type}(S) = (t_1, t_2, \ldots, t_{k-1})$ then $\mathrm{Type}(S') = (t_1, t_2, \ldots, t_{a-1}, t_b, t_{b+1}, \ldots, t_{k-1})$. One checks that $G(S', T')$ is obtained from $G(S, T)$ by removing p together with its adjacent edge and its labeling (which is $\{t_a, t_{a+1}, \ldots, t_{b-1}\}$). Finally, using all these, we can verify that

$$K(S, T) \cap \{\boldsymbol{u} : u_\ell = 0\} = K(S', T').$$

Similarly, since T' has one less internal vertex than T, we have that $K(S', T')$ is a facet of $K(S, T)$, and thus the inequality $u_\ell \ge 0$ if facet-defining.

Finally, we prove (C3). Note that in the proof of (C2), the partition labeled tree (S', T') we constructed in each case covers (S, T) in $K\Pi_d$, and every (S', T') covers (S, T) arises from one of the inequality discussion. Therefore, (C3) follows. □

Example 5.18. Let (S, T) the $[9]$-partition labeled tree on the left of Fig. 5. Since T has 8 internal vertices, by Lemma 5.17 (1), we have that $\dim \sigma(S, T) = 8$. Recall there are two types, type i and type ii, of covering relations in $K\Pi_d$ defined in Definition 5.1. In the poset $K\Pi_8$, our partition labeled tree (S, T) are covered by 10 elements, 7 of which are obtained from (S, T) by contracting a single internal edge of T and the remaining 3 are obtained by contracting the pairs of leaves $\{3, 7\}, \{8, 9\}, \{5, 2\}$ in T, respectively. By Lemma 5.17 (3), the cone $\sigma(S, T)$ has 10 facets, hence it is not simplicial.

Lemma 5.19. *If $(S, T), (S', T') \in P_d$ are distinct, then $\sigma(S, T) \neq \sigma(S', T')$ as cones in W_d.*

In order to prove the above lemma, we need a modified version of preorder cones that are also in bijection with preorders. For every preorder $\preceq$ on $[n]$, we define

$$\tilde{\sigma}_{\preceq} := \{\mathbf{x} \in \mathbb{R}^n_{\ge 0} : x_i \le x_j \text{ if } i \preceq j\}. \quad (5.16)$$

The following result is straightforward to check.

Lemma 5.20. *The map $\preceq \mapsto \tilde{\sigma}_{\preceq}$ is an injection from the set of all preorders on $[n]$ to the set of cones in $\mathbb{R}_{\geq 0}$.*

Proof of Lemma 5.19. Suppose $\sigma(S, T) = \sigma(S', T')$. We need to show $(S, T) = (S', T')$.

We first prove that $S = S'$. Suppose $S = (S_1, \dots, S_k)$. It follows from Lemma 5.17 that for any w in the interior of $\sigma(S, T)$, we have $w_i = w_j$ if $i, j \in S_a$ for some a, and $w_i < w_j$ if $i \in S_a$ and $j \in S_b$ for some $a < b$. Since $\sigma(S, T) = \sigma(S', T')$, they have exactly the same interior. Thus, we must have that $S = S'$.

Next, we show $T = T'$. We choose π such that $S(\pi)$ refines $S = S'$, and let $K(S, T) = D^{\pi}(\sigma(S, T))$ and $K(S', T') = D^{\pi}(\sigma(S', T'))$ as defined in the proof of Lemma 5.17. Since $\sigma(S, T) = \sigma(S', T')$, we must have that $K(S, T) = K(S', T')$. Recall that $\preceq_1 := \preceq_{(S,T)}$ and $\preceq_2 := \preceq_{(S',T')}$ are preorders on $\mathrm{Type}(S) = \mathrm{Type}(S')$. We consider the modified version of preorder cones

$$\tilde{\sigma}_{\preceq_\ell} = \{\boldsymbol{u} \in \mathbb{R}^{\mathrm{Type}(S)}_{\geq 0} \ : \ u_i \leq u_j, \ \text{if } i \preceq_\ell j\}, \quad \ell = 1, 2,$$

defined in (5.16). By description of $K(S, T)$ given in the proof of Lemma 5.17, one sees that $K(S, T) \cong \tilde{\sigma}_{\preceq^1}$ and $K(S', T') \cong \tilde{\sigma}_{\preceq^2}$. Thus, $\tilde{\sigma}_{\preceq^1} = \tilde{\sigma}_{\preceq^2}$. Then it follows from Lemma 5.20 that $\preceq_1 = \preceq_{(S,T)}$ and $\preceq_2 = \preceq_{(S',T')}$ are the same preorder on $\mathrm{Type}(S)$. Since we can recover the tree from the Hasse diagram of the preorder, we must have that $T = T'$, completing the proof.

Proof of Proposition 5.16. By the definition of Ξ_d, the map $(S, T) \mapsto \sigma(S, T)$ clearly is a surjection from P_d to Ξ_d. Moreover, by Lemma 5.19, the map is injective. Therefore, the first conclusion of the proposition follows.

The second conclusion follows from Lemma 5.17 (3) and the transitivity of the poset. □

5.5. Nested Loday fan

Recall the nested Loday fan Ξ_d is defined by Eq. (5.12). (As we mentioned before, the proof for that Ξ_d is a fan will be completed in the next section. However, for convenience we still refer to Ξ_d as the nested Loday fan.) In this subsection, we explore further properties of Ξ_d, and summarize useful results that will be used in Section 6.

Let $K_d \subseteq \Xi_d$ be the set of maximal cones (in terms of dimension). It then follows from Proposition 5.16 that the set Ξ_d is induced by K_d. By Lemma 5.17 (1) and Lemma 5.3 (3), we see that

$$K_d = \{\sigma(S, T) : (S, T) \text{ is a rank-1 element of } K\Pi_d\} = \{\sigma(S(\pi), T) : (\pi, T) \in S_{d+1} \times T_{d,d}\},$$

and each cone has dimension d (and thus is full-dimensional in W_d). For ease of notation we define for any pair $(\pi, T) \in S_{d+1} \times T_{d,d}$,

$$\sigma(\pi, T) := \sigma(S(\pi), T). \tag{5.17}$$

Then for these maximal dimensional cones in K_d, their descriptions in Definition 5.14 and expressions for their interiors can be simplified.

Recall $\check{\sigma}(\tau)$ is defined in Definition 5.6. Similar to how braid cones $\sigma(\pi)$ are generalized to preorder cones $\sigma_{\preceq}$, for any preorder $\preceq$ on $[n]$, we define

$$\check{\sigma}_{\preceq} := \{\mathbf{x} \in \mathbb{R}^n \ : \ x_i \leq x_j \text{ if } i \preceq j\}, \tag{5.18}$$

and for any $T \in T_n$, we set $\check{\sigma}(T) := \check{\sigma}_{\preceq_T}$.

With these notations, we have

$$\sigma(\pi, T) = \left\{ w \in W_d \ : \ \begin{array}{c} w \in \sigma(\pi) \\ D^{\pi} w \in \check{\sigma}(T) \end{array} \right\}, \quad \text{and } \sigma^{\circ}(\pi, T) = \left\{ w \in W_d \ : \ \begin{array}{c} w \in \sigma^{\circ}(\pi) \\ D^{\pi} w \in \check{\sigma}^{\circ}(T) \end{array} \right\}, \tag{5.19}$$

where

$$\check{\sigma}^{\circ}(T) = \{\mathbf{x} \in \mathbb{R}^d \ : \ x_i < x_j \text{ if } i \prec_T j\}. \tag{5.20}$$

Lemma 5.21. *Each cone $\sigma \in \mathrm{K}_d$ is a union of nested braid cones. Furthermore, the collection of cones K_d is a pointed conic dissection of W_d.*

Proof. Let $\sigma \in \mathrm{K}_d$. Then $\sigma = \sigma(\pi, \mathrm{T})$ for some $(\pi, \mathrm{T}) \in \mathrm{S}_{d+1} \times \mathrm{T}_{d,d}$. By an analogous proof of Lemma 2.13 (3), we have that $\check{\sigma}(\mathrm{T}) = \bigcup_{\tau \in \mathrm{L}[\mathrm{T}]} \check{\sigma}(\tau)$, so combining Eq. (5.19) together with the definition of nested braid cone as in Eq. (5.4) we obtain

$$\sigma(\pi, \mathrm{T}) = \bigcup_{\tau \in \mathrm{L}[\mathrm{T}]} \sigma(\pi, \tau). \tag{5.21}$$

Furthermore, for each fixed $\pi \in \mathrm{S}_{d+1}$, it follows from Lemma 3.9 that each permutation $\tau \in \mathrm{S}_d$ appears in the above expression for exactly one $\mathrm{T} \in \mathrm{T}_{d,d}$. By Lemma 5.8 the collection $\{\sigma(\pi, \tau) : (\pi, \tau) \in \mathrm{S}_{d+1} \times \mathrm{S}_d\}$ is a conic dissection of W_d, so we conclude that so is K_d. Finally, by (5.19), we see that for any $(\pi, \mathrm{T}) \in \mathrm{S}_{d+1} \times \mathrm{T}_{d,d}$, the cone $\sigma(\pi, \mathrm{T}) \subseteq \sigma(\pi)$. Since the latter is pointed, so is the former. □

Lemma 5.23 below summarizes properties of Ξ_d, recalling $\overline{\mathrm{O}_{d+1}} = \bigsqcup_{k=2}^{d+1} \mathrm{O}_{d+1,k}$ is the set of all non-trivial ordered partitions of $[d+1]$.

Notation 5.22. For any $\mathrm{S} \in \overline{\mathrm{O}_{d+1}}$, if S has k blocks, we define T_S to be the unique tree with one internal vertex and k leaves.

Lemma 5.23. *The nested Loday fan Ξ_d has the following properties:*

(1) The set $\mathrm{K}_d = \{\sigma(\pi, T) : (\pi, T) \in \mathrm{S}_{d+1} \times \mathrm{T}_{d,d}\}$ consists of all the d-dimensional cones in Ξ_d.
(2) The set $\{\sigma(\mathrm{S}, T_\mathrm{S}) : \mathrm{S} \in \overline{\mathrm{O}_{d+1}}\}$ consists of all the 1-dimensional cones of Ξ_d. Moreover, for each $\mathrm{S} \in \overline{\mathrm{O}_{d+1}}$, the vector $\boldsymbol{e}_\mathrm{S} \in W_d$ is a generator for the 1-dimensional cone $\sigma(\mathrm{S}, T_\mathrm{S})$.
(3) The face poset of Ξ_d is isomorphic to the poset dual to $\mathrm{K}\Pi_d \setminus \hat{0}$.
(4) The nested Loday fan Ξ_d is not simplicial for $d \geq 3$.
(5) The nested Loday fan Ξ_d is a coarsening of the nested braid fan Br_d^2.

Proof. Condition (1) follows from the our discussion on maximal dimensional cones, condition (3) follows from Proposition 5.16, and condition (5) follows from Lemma 5.21.

The first assertion in (2) follows from Lemma 5.17 (1) and Lemma 5.3 (4). Suppose $\mathrm{S} = (S_1, S_2, \ldots, S_k) \in \mathrm{O}_{d+1,k}$. Then it is easy to check that

$$\sigma(\mathrm{S}, \mathrm{T}_\mathrm{S}) = \left\{ \mathrm{w} \in W_d : \begin{array}{c} \mathrm{w}_i = \mathrm{w}_j \text{ if } i, j \in S_a \text{ for some } a \\ \mathrm{w}_i \leq \mathrm{w}_j \text{ if } i \in S_a, j \in S_{a+1} \text{ for some } a \\ \mathrm{w}_j - \mathrm{w}_i = \mathrm{w}_\ell - \mathrm{w}_k \text{ if } i \in S_a, j \in S_{a+1}, k \in S_b, \ell \in S_{b+1} \text{ for some } a, b \end{array} \right\}.$$

One can verify that the vector $\boldsymbol{e}_\mathrm{S}$ is a nonzero vector in the above cone. Therefore, the second assertion in (2) follows.

Example 5.18 shows a particular example which is not simplicial. In general it follows from Lemma 5.17 (3) that the number of facets of a d-dimensional cone $\sigma(\pi, \mathrm{T}) \in \mathrm{K}_d$ is $a + b$ where a is the number of internal edges of T and b is the number of vertices adjacent to exactly two leaves. One checks that any tree $\mathrm{T} \in \mathrm{T}_{d,d}$ has $a = 2d - (d+1) = d - 1$, and if $d \geq 3$, there exists $\mathrm{T} \in \mathrm{T}_{d,d}$ such that $b \geq 2$. Hence, there exist non simplicial cones for every $d \geq 3$. □

6. Realization of the permuto-associahedron

Recall that a d-permuto-associahedron is a d-dimensional polytope whose face poset is isomorphic to $\mathrm{K}\Pi_d$. Similar to Section 4, we follow the method outlined in the introduction to construct a realization for the permuto-associahedron, which is the majority of the content of this section. In the last part of this section, we compare our realization with the one given by Reiner and Ziegler in [29].

6.1. Vertices of the nested permuto-associahedron

In this part, we will give a set of points that are vertex candidates of our realization. We start by adapting notation from Definition 4.1.

Notation 6.1. For any $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$ and $T \in \mathcal{T}_{d,d}$, we let

$$\beta_{T,i} := \mathrm{val}(\beta, T_{(i)}).$$

Therefore, $v_T^{\beta} = \sum_{i=1}^{d} \beta_{T,i}\, e_i$.

Given strictly increasing sequences $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \mathbb{R}^{d+1}$ and $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{R}^d$, for any $(\pi, T) \in S_{d+1} \times \mathcal{T}_{d,d}$, we define

$$v_{\pi,T}^{(\alpha,\beta)} := \sum_{i=1}^{d+1} \alpha_i e_{\pi^{-1}(i)} + \sum_{i=1}^{d} \beta_{T,i} f_i^{\pi}. \tag{6.1}$$

It is easy to see that $v_{\pi,T}^{(\alpha,\beta)}$ lies in U_d^{α} since the sum of its coordinates is $\sum_{i=1}^{d+1} \alpha_i$.

After rearranging coordinates in (6.1), we get the following expression:

$$v_{\pi,T}^{(\alpha,\beta)} = \sum_{i=1}^{d+1} \left(\alpha_i + \left(\beta_{T,i-1} - \beta_{T,i} \right) \right) e_{\pi^{-1}(i)}, \tag{6.2}$$

where by convention we let $\beta_{T,0} = \beta_{T,d+1} = 0$.

Parallel to Section 5.2, we say that (α, β) is a $\mathcal{T}$ *-appropriate pair* (of strictly increasing sequences), if for any complete binary tree $T \in \mathcal{T}_{d,d}$, the coefficients of $e_{\pi^{-1}(i)}$ in the above expansion increase strictly as i increases.

Remark 6.2. We remark that being an ''appropriate pair'' and being a ''$\mathcal{T}$-appropriate pair'' are not equivalent. We can show for $d = 3$ that any $\mathcal{T}$-appropriate pair (α, β) is an appropriate pair. We suspect that this implication is true in general; however, we do not have a proof. Since this is not relevant to the discussion of this paper, we leave it to interested readers. In any case, it is not hard to see that by scaling α with a sufficiently large factor, we can make (α, β) both ''appropriate'' and ''$\mathcal{T}$-appropriate''.

Definition 6.3. Suppose $(\alpha, \beta) \in \mathbb{R}^{d+1} \times \mathbb{R}^d$ is a $\mathcal{T}$-appropriate pair of strictly increasing sequences. We define the *nested permuto-associahedron*

$$\mathrm{PermAsso}(\alpha, \beta) := \mathrm{ConvexHull}\left(v_{\pi,T}^{(\alpha,\beta)} : (\pi, T) \in S_{d+1} \times \mathcal{T}_{d,d} \right). \tag{6.3}$$

The next result is the main theorem of this paper.

Theorem 6.4. *Let* $(\alpha, \beta) \in \mathbb{R}^{d+1} \times \mathbb{R}^d$ *be a* $\mathcal{T}$ *-appropriate pair of strictly increasing sequences. Then the face poset of the nested permuto-associahedron* $\mathrm{PermAsso}(\alpha, \beta)$ *is the Kapranov poset* $K\Pi_d$. *Moreover,* $\mathrm{PermAsso}(\alpha, \beta)$ *is a d-dimensional permuto-associahedron, and is a generalized nested permutohedron as well.*

6.2. Normal fan of nested permuto-associahedra

Recall that in Section 5.5 we have defined $\mathcal{K}_d$ to be the set of maximal cones in Ξ_d, and have shown that $\mathcal{K}_d$ is a conic dissection of W_d (see Lemma 5.21). The goal of this part is to use Lemma 2.4 to show that $\mathcal{K}_d$ induces the normal fan of $\mathrm{PermAsso}(\alpha, \beta)$, as well as confirm the set of points $\left\{ v_{\pi,T}^{(\alpha,\beta)} \right\}$ defined above is indeed the vertex set of $\mathrm{PermAsso}(\alpha, \beta)$.

Lemma 6.5. *Suppose* $(\alpha, \beta) \in \mathbb{R}^{d+1} \times \mathbb{R}^d$ *is a* $\mathcal{T}$ *-appropriate pair of strictly increasing sequences. Let* $(\pi, T), (\pi', T') \in \mathfrak{S}_{d+1} \times \mathcal{T}_{d,d}$*. Then for every* $\mathbf{w} \in \sigma^\circ(\pi, T)$*, we have*

$$\left\langle \mathbf{w}, \mathbf{v}_{\pi,T}^{(\alpha,\beta)} \right\rangle \geq \left\langle \mathbf{w}, \mathbf{v}_{\pi',T'}^{(\alpha,\beta)} \right\rangle \tag{6.4}$$

where the equality holds if and only if $(\pi, T) = (\pi', T')$*.*

Proof. We will prove the inequality by introducing an intermediate product and showing

$$\left\langle \mathbf{w}, \mathbf{v}_{\pi,T}^{(\alpha,\beta)} \right\rangle \geq \left\langle \mathbf{w}, \mathbf{v}_{\pi,T'}^{(\alpha,\beta)} \right\rangle \geq \left\langle \mathbf{w}, \mathbf{v}_{\pi',T'}^{(\alpha,\beta)} \right\rangle, \tag{6.5}$$

where the first equality holds if and only if $T = T'$ and the second equality holds if and only $\pi = \pi'$.

We let $u_i = w_{\pi^{-1}(i)}$ for each i, which allows us to express $\mathbf{w}$ as in (5.1) and have $D^\pi \mathbf{w} = D\boldsymbol{u}$. Then because $\mathbf{w} \in \sigma^\circ(\pi, T)$, we have the following conditions from Eq. (5.19):

(1) $D\boldsymbol{u} > 0$, which is equivalent to $u_1 < u_2 < \cdots < u_{d+1}$, and
(2) $D_i\boldsymbol{u} < D_j\boldsymbol{u}$ if $i \triangleleft_T j$.

Expression (5.1), together with (6.2), allows us to compute products in (6.5) easily. Since the pair (α, β) is $\mathcal{T}$ -appropriate, we have that $\left(\alpha_i + \left(\beta_{T',i-1} - \beta_{T',i}\right)\right)$ strictly increases as i increases. This, together with condition (1) above and the Rearrangement Inequality [13, Theorem 368] gives us the second inequality in (6.5) and that the equality holds if and only if $\pi = \pi'$.

Next we see that the first inequality in (6.5) is equivalent to

$$\sum_{i=1}^{d+1} \left(\beta_{T,i-1} - \beta_{T,i}\right) u_i \geq \sum_{i=1}^{d+1} \left(\beta_{T',i-1} - \beta_{T',i}\right) u_i.$$

After rearranging summations, the above inequality becomes

$$\left\langle D\boldsymbol{u}, \mathbf{v}_T^\beta \right\rangle = \sum_{i=1}^{d+1} (u_{i+1} - u_i)\beta_{T,i} \geq \sum_{i=1}^{d+1} (u_{i+1} - u_i)\beta_{T',i} = \left\langle D\boldsymbol{u}, \mathbf{v}_{T'}^\beta \right\rangle. \tag{6.6}$$

Then because $D\boldsymbol{u} \in \sigma^\circ(T)$, it follows from Corollary 4.9 that the inequality (6.6) holds and its equality holds if and only if $T = T'$. □

The following proposition is the key result of this subsection, characterizing the vertex set and the normal fan of the nested permuto-associahedron. It also provides the main ingredients we need for proving Theorem 6.4.

Proposition 6.6. *Let* $(\alpha, \beta) \in \mathbb{R}^{d+1} \times \mathbb{R}^d$ *be a* $\mathcal{T}$ *-appropriate pair of strictly increasing sequences.*

(1) The nested permuto-associahedron PermAsso(α, β) *is full-dimensional in* U_d^α *and its vertex set is* $\left\{ \mathbf{v}_{\pi,T}^{(\alpha,\beta)} : (\pi, T) \in \mathfrak{S}_{d+1} \times \mathcal{T}_{d,d} \right\}$.

(2) For each $(\pi, T) \in \mathfrak{S}_{d+1} \times \mathcal{T}_{d,d}$*, we have* $\sigma(\pi, T) = \text{ncone}\left(\mathbf{v}_{\pi,T}^{(\alpha,\beta)}, \text{PermAsso}(\alpha, \beta)\right)$.

(3) The normal fan of PermAsso(α, β) *is* $\Xi_d = \{\sigma(S, T) : (S, T) \in \mathcal{P}_d\}$*. Hence,* Ξ_d *is a complete projective fan in* W_d*.*

Proof. It follows from Lemmas 5.21 and 6.5 that the set of cones $\mathcal{K}_d = \{\sigma(\pi, T) : (\pi, T) \in \mathfrak{S}_{d+1} \times \mathcal{T}_{d,d}\}$ in W_d and the set of points $\left\{ \mathbf{v}_{\pi,T}^{(\alpha,\beta)} : (\pi, T) \in \mathfrak{S}_{d+1} \times \mathcal{T}_{d,d} \right\}$ in U_d^α satisfy the hypothesis of Lemma 2.4. Hence, we conclude that the first two statements are true, and that the $\mathcal{K}_d$ induces the normal fan of PermAsso(α, β). However, since $\mathcal{K}_d$ contains all the maximal cones in Ξ_d, by Proposition 5.16, we have that Ξ_d is induced by $\mathcal{K}_d$. Therefore, (3) follows. □

We can now prove our main theorem.

Proof of Theorem 6.4. By Proposition 6.6 (3) and Lemmas 5.23 (3) and 2.2, we have that the face poset of PermAsso(α, β) is the Kapranov poset $K\Pi_d$. Hence, we conclude that PermAsso(α, β) is a d-permuto-associahedron. Finally, it follows from Proposition 6.6 (3) and Lemma 5.23 (5) that PermAsso(α, β) is a generalized nested permutohedron.

6.3. Inequality description for nested permuto-associahedra

It follows from Proposition 6.6 that we can apply Lemma 2.5 to find an inequality description for the nested permuto-associahedron PermAsso(α, β).

Theorem 6.7. *Let* $(\alpha,\beta) \in \mathbb{R}^{d+1} \times \mathbb{R}^d$ *be a* $\mathcal{T}$*-appropriate pair of strictly increasing sequences* $(\alpha,\beta) \in \mathbb{R}^{d+1} \times \mathbb{R}^d$*. Suppose* $\boldsymbol{b} \in \mathbb{R}^{\mathcal{O}_{d+1}}$ *is defined as follows: for each* $S = (S_1, S_2, \dots, S_k) \in \overline{\mathcal{O}_{d+1}}$*, if* $\mathrm{Type}(S) = (t_0, t_1, t_2 \dots, t_k)$*, let*

$$b_S = \left(\sum_{i=1}^{k} i \sum_{j=t_{i-1}+1}^{t_i} \alpha_j \right) + \left(\sum_{j=1}^{d} \beta_j - \sum_{i=1}^{k} \sum_{j=1}^{|S_i|} \beta_j \right). \tag{6.7}$$

Then we have the following facet-defining inequality description for PermAsso(α, β)*:*

$$\mathrm{PermAsso}(\alpha,\beta) = \left\{ \boldsymbol{x} \in U_d^{\alpha} \ : \ \langle \boldsymbol{e}_S, \boldsymbol{x}\rangle \le b_S, \quad \forall S \in \overline{\mathcal{O}_{d+1}} \right\}. \tag{6.8}$$

Proof. Recall $S(\pi)$ and T_S is defined in (2.8) and Notation 5.22, respectively. Applying Lemma 2.5 together with Proposition 5.16, Lemma 5.23 (1)(2), and Proposition 6.6 (2) one sees it is left to show that for any $S \in \overline{\mathcal{O}_{d+1}}$, if we choose $(\pi, T) \in \mathfrak{S}_{d+1} \times \mathcal{T}_{d,d}$ such that $(S(\pi), T) \le_{KP} (S, T_S)$ in the Kapranov poset, then

$$\left\langle \boldsymbol{e}_S, v^{(\alpha,\beta)}_{\pi,T} \right\rangle = b_S, \tag{6.9}$$

where b_S is given by Eq. (6.7).

We compute from the definitions in Eqs. (5.5) and (6.1):

$$\begin{aligned} \left\langle \boldsymbol{e}_S, v^{(\alpha,\beta)}_{\pi,T} \right\rangle &= \left\langle \sum_{i=1}^{k} i\boldsymbol{e}_{S_i}, \sum_{j=1}^{d+1} \alpha_j \boldsymbol{e}_{\pi^{-1}(j)} + \sum_{j=1}^{d} \beta_{T,j} \mathbf{f}^{\pi}_j \right\rangle \\ &= \left\langle \sum_{i=1}^{k} i\boldsymbol{e}_{S_i}, \sum_{j=1}^{d+1} \alpha_j \boldsymbol{e}_{\pi^{-1}(j)} \right\rangle + \left\langle \sum_{i=1}^{k} i\boldsymbol{e}_{S_i}, \sum_{j=1}^{d} \beta_{T,j} \mathbf{f}^{\pi}_j \right\rangle. \end{aligned} \tag{6.10}$$

We will show that the two terms in (6.10) are equal to the two terms in (6.7).

Note that the leaves of the partition labeled tree $(S(\pi), T)$ are labeled by $S(\pi) = (\{\pi^{-1}(1)\}, \{\pi^{-1}(2)\}, \dots, \{\pi^{-1}(d+1)\})$ from left to right. Since $(S(\pi), T) \le_{KP} (S, T_S)$, by the definition of the covering relation of the Kapranov poset, the followings are true:

(i) $S(\pi)$ refines S. Hence,

$$\pi^{-1}(j) \in S_i \text{ if and only if } t_{i-1} + 1 \le j \le t_i. \tag{6.11}$$

(ii) For each $1 \le i \le k$, there exists a subtree T_i of T such that the leaves of T_i are labeled by $\{\pi^{-1}(j) \ : \ t_{i-1} + 1 \le j \le t_i\}$. Thus, $I_T(T_i) = \{j \ : \ t_{i-1} + 1 \le j \le t_i - 1\}$.

Clearly, by (6.11), the first term in (6.10) is equal to the first term in (6.7).

Next, applying (6.11) again, we obtain that $\left\langle \sum_i i\boldsymbol{e}_{S_i}, \mathbf{f}^{\pi}_j \right\rangle$ is 1 if $j \in \mathrm{Type}(S)$ and is 0 otherwise. Therefore, the second term in (6.10) is equal to $\sum_{j \in \mathrm{Type}(S)} \beta_{T,j}$. By condition (ii) above, the set of labels for internal vertices of T that do not appear in any of $T_1, \dots, T_k$ is exactly Type(S). Therefore,

$$\sum_{j \in \mathrm{Type}(S)} \beta_{T,j} = \sum_{j=1}^{d} \beta_{T,j} - \sum_{i=1}^{k} \sum_{j \in I_T(T_i)} \beta_{T,j}.$$

By Lemma 4.3, we have that

$$\sum_{j=1}^{d} \beta_{T,j} = \sum_{j=1}^{d} \beta_j \quad \text{and} \quad \sum_{j \in I_T(T_i)} \beta_{T,j} = \sum_{j=1}^{|S_i|} \beta_j.$$

Therefore, we conclude that the second term in (6.10) is equal to the second term in (6.7), completing the proof. □

Notice in particular that the set of facets of PermAsso(α, β) and Perm(α, β) are in bijection; in contrast to Corollary 4.15, the nested permuto-associahedron PermAsso(α, β) cannot be obtained by removing facets from a nested permutohedron. We also remark that we do not have a Minkowski sum decomposition as in Corollary 4.16 for PermAsso(α, β).

7. Comparison to previous work

In this section, we highlight some differences and similarities between our realization, Reiner–Ziegler's and Gaiffi's. We start by noting a common similarity among all three realizations: All constructions can be obtained by symmetrizing an embedding of an $(d-1)$-associahedron in $\mathbb{R}^{d+1}$. The constructions and proofs are different insomuch as they use different associahedra.

7.1. Comparison to Reiner–Ziegler's realization

Reiner and Ziegler's paper [29] has two distinct parts. In the first part, they prove that the *dual* of Kapranov's poset can be realized as the face poset of a CW-ball [29, Theorem 1], which they called the *sphericity* theorem. Contrary to Kapranov's realization as a CW-ball, Reiner and Ziegler's approach is purely combinatorial. In the second part of [29], they provide the first polytopal realization of Kapranov's poset [29, Theorem 2] using methods that independent from the first part.

Interestingly, our approach turns out to be more related to the proof of Reiner and Ziegler's sphericity theorem. The approach in [29] to prove sphericity is to glue together cells of the CW-ball arising from the second barycentric subdivision of a simplex. See the first row of [29, Figure 5]. The cells of the second barycentric subdivision of a simplex are in natural bijection with the cones of the nested braid fan [7, Section 6]. The fan Ξ_d is a coarsening of the nested braid fan and it groups nested braid cones in the same way that Reiner and Ziegler glue the cells. In this sense, the present work completes the discussion started in [29, Section 2] by showing that the proposed gluing results in a polytope, not just a topological ball.

Below we highlight the differences between our realization and Reiner–Ziegler's:

(1) Whereas we use Loday's associahedra, they use a secondary polytope of a specific cyclic polygon.
(2) The resulting realizations in [29] and Section 6 have different normal fans. In fact, the sets of rays of these two normal fans are different, although for both cases a ray is constructed for each ordered set partition: For each $S = (S_1, \ldots, S_k) \in O_{d+1}$, we associate to S a ray spanned by $\mathbf{e}_S = \sum_{i=1}^{k} i\mathbf{e}_{S_i} \in W_d$, and Reiner and Ziegler associate to S a ray spanned by $\sum_{i=1}^{k} (t_i + t_{i-1})\mathbf{e}_{S_i} \in W_d$, where $(t_1, \ldots, t_{k-1})$ is the type of S and by convention $t_0 = 0$ and $t_k = d+1$. It follows from above descriptions of rays that the two normal fans are not even linearly equivalent.
(3) Reiner and Ziegler's construction is surprisingly *inscribable* (all vertices lie on a sphere), whereas ours never is. Indeed any nested permuto-associahedron PermAsso(α, β) of dimension greater than one will have a Loday pentagon as a face and these pentagons are never inscribable.

We remark that Reiner and Ziegler also extended their construction to both type-*B* and type-*D*, and showed that the permuto-associahedron (which correspond to type-*A*) arises as a facet of their type-*B* version.

7.2. Comparison to Gaiffi

The second realization of the permuto-associahedron was given by Gaiffi [11]. His construction is quite general: he constructed permutonestohedra for any nestohedron [25, Section 7] of which the associahedron is an example. Furthermore, he does it for general root systems. When the root system is of type-A and the nestohedron is the Stasheff–Shnider associahedron, his construction becomes a realization of the permuto-associahedron. We remark that when the root system is of type-B, Gaiffi's version is different from Reiner and Ziegler's.

Even though Gaiffi's approach and ours are manifestly different, a closer inspection reveals that the rays for the normal fan of his construction are generated by the vectors $\mathbf{e}_S$ we defined in Section 5.2 for ordered set partitions S, and the rays form the maximal cones in the fan in exactly the same we as in our realization. It follows that Gaiffi's and ours permuto-associahedra have the same normal fan. This is not surprising, as Gaiffi's starting point is Stasheff–Shnider's construction for associahedra and we start with Loday's construction, and Stasheff–Shnider's and Loday's associahedra have the same normal fan.

Below we list a few more differences between the approaches in our realization in this article and Gaiffi's [11]:

(1) Gaiffi starts by choosing a *suitable* list $\varepsilon_1 < \cdots < \varepsilon_d$ of real numbers. We use an *appropriate* pair of $(\alpha, \beta) \in \mathbb{R}^{d+1} \times \mathbb{R}^d$. Gaiffi's definition of suitable requires that each one is sufficiently larger than the previous one; see [11, Definition 3.1]. Our definition of appropriateness is a bit more flexible: By Remark 6.2, as long as $\alpha \in \mathbb{R}^{d+1}$ is fixed, any increasing sequence $\beta \in \mathbb{R}^d$ is appropriate if every entry is smaller than a global constant. So it turns out that our flexibility in the choice of α allows us to relax the conditions on the choice of β.

(2) Gaiffi describes his realization by providing inequality description, and then describing vertices as intersections of d facets (even though they will eventually be contained in more facets). However, he did not provide explicit description for vertex coordinates of his permuto-associahedra. We start by explicitly constructing vertices and the normal fan of our permuto-associahedra, and then provide explicitly inequality description as a consequence of our method.

(3) For each ordered set partition $S = (S_1, \dots, S_k)$ with type $(t_1, t_2, \dots, t_{k-1})$, as mentioned above Gaiffi and we both associate to a same normal vector $\mathbf{e}_S$. The corresponding inequality in the inequality description of Gaiffi's permuto-associahedron is

$$\langle \mathbf{e}_S, \mathbf{x} \rangle \leq \varepsilon_d - (\varepsilon_{|S_1|} + \cdots + \varepsilon_{|S_k|}), \tag{7.1}$$

and ours is (by Theorem 6.7)

$$\langle \mathbf{e}_S, \mathbf{x} \rangle \leq \left(\sum_{i=1}^{k} i \sum_{j=t_{i-1}+1}^{t_i} \alpha_j \right) + \left(\sum_{j=1}^{d} \beta_j - \sum_{i=1}^{k} \sum_{j=1}^{|S_i|} \beta_j \right). \tag{7.2}$$

The right hand sides of both inequalities depend on sizes of blocks in S. However, (7.2) depends on the *orders* of the blocks in S but (7.1) does not.

Question 7.1. What is the set of all three-dimensional permuto-associahedra that arise both in Gaiffi's and in our construction? It seems that the areas of the pentagonal faces relative to the areas of the square faces behave differently in both constructions. Gaiffi's realization in dimension 3 is depicted in the left hand side of [11, Figure 5] where the pentagons are large in comparison to the little square faces, whereas in Fig. 1 our pentagons are small with respect to the permutohedron, and indeed in our constructions they can be arbitrarily small compare to the other faces.

There are more relations to be explored. Because Gaiffi's permuto-associahedron has the same normal fan as ours, which is a generalized nested permutohedron, maybe all of Gaiffi's permutonestohedra are generalized nested permutohedra as well, in other words, that their normal fans are all coarsening of the nested braid fan [7]. Coincidentally, what we call nested permutohedron is what Gaiffi calls *permutopermutohedron*.

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