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THE EXTENDED HAAGERUP FUSION CATEGORIES

BY PINHAS GROSSMAN, SCOTT MORRISON, DAVID PENNEYS,
EMILY PETERS AND NOAH SNYDER

ABSTRACT. – We show there are exactly four fusion categories in the Morita equivalence class of the Extended Haagerup ($\mathcal{EH}$) subfactor, and a unique Morita equivalence between each pair. The $\mathcal{EH}$ subfactor corresponds to the Morita equivalence between $\mathcal{EH}_1$ and $\mathcal{EH}_2$. The new categories $\mathcal{EH}_3$ and $\mathcal{EH}_4$ give new exotic subfactors. The $\mathcal{EH}$ categories are the only known fusion categories unrelated to (quantum) groups or Izumi quadratic categories.

To construct $\mathcal{EH}_3$ and $\mathcal{EH}_4$, we give a general computational recipe to construct fusion categories in the Morita equivalence class of a subfactor. We show that subfactor planar algebra embeddings from $\mathcal{P}_\bullet$ into graph planar algebras are equivalent to pivotal module C* categories over $\mathcal{P}_\bullet$. We construct $\mathcal{EH}_3$ and $\mathcal{EH}_4$ by embedding the $\mathcal{EH}$ planar algebra inside the graph planar algebras of two new graphs. This technique answers a long-standing question of Jones: which graph planar algebras contain a given subfactor planar algebra?

RÉSUMÉ. – Nous montrons qu'il existe exactement quatre catégories de fusion dans la classe d'équivalence au sens de Morita du sous-facteur « Extended Haagerup » ($\mathcal{EH}$), et unicité de l'équivalence entre chaque paire. Le sous-facteur $\mathcal{EH}$ correspond à l'équivalence de Morita entre $\mathcal{EH}_1$ et $\mathcal{EH}_2$. Les nouvelles catégories $\mathcal{EH}_3$ et $\mathcal{EH}_4$ donnent de nouveaux exemples de sous-facteurs exotiques. Les catégories $\mathcal{EH}$ sont les seules catégories de fusion connues qui ne sont pas reliées à un groupe (quantique) ou à une catégorie quadratique d'Izumi.

Pour construire $\mathcal{EH}_3$ et $\mathcal{EH}_4$, nous élaborons une construction générale de catégories de fusion au sein d'une classe d'équivalence de Morita d'un sous-facteur. Nous montrons que les plongements de l'algèbre planaire de sous-facteurs $\mathcal{P}_\bullet$ dans les algèbres planaires de graphe sont en équivalence avec les catégories de modules de pivot C* sur $\mathcal{P}_\bullet$. Nous construisons $\mathcal{EH}_3$ et $\mathcal{EH}_4$ en plongeant l'algèbre planaire $\mathcal{EH}$ dans les algèbres planaires de deux nouveaux graphes. Cette technique répond à une question de Jones de longue date : quelle algèbre planaire de graphe contient une algèbre planaire de sous-facteur donnée?

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1. Introduction

Group theory provides a unifying language for symmetries across classical mathematics, but in many settings related to quantum mechanics, a more general notion of *quantum symmetry* is required. One of the first appearances of this new kind of symmetry was in the theory of subfactors, i.e., inclusions of von Neumann factors, developed by Jones, Ocneanu, Popa, and others [56, 35, 78, 89, 90]. Here, the appropriate notion of 'Galois theory' requires considering structures more general than groups. But such symmetries have since appeared in many other places, most notably the representation theory of groups of Lie type, polynomial link invariants, topological quantum field theory, conformal field theory, and topological phases of matter. Tensor categories [24, 21] provide the modern language to describe these more general quantum symmetries. Roughly speaking, a tensor category is a category that looks like the category of representations of a group—namely, the category has tensor products and duals. But critically, this tensor product can be noncommutative, so that $X \otimes Y$ and $Y \otimes X$ are not identified, and need not even be isomorphic. The simplest and most widely studied tensor categories are fusion categories, which have strong finiteness and semisimplicity properties analogous to the category of representations of a finite group over a field of characteristic prime to the size of the group.

The most well-known examples of tensor categories come from Lie theory. Following Drinfeld and Jimbo [19, 52], one can deform the universal enveloping algebra of a Lie group, and the category of representations of this quantized universal enveloping algebra is a tensor category. These are not fusion categories, because they are too large. For SL_2, Reshetikhin and Turaev constructed fusion categories built from these quantum groups specialized to roots of unity [92], and this construction was generalized to classical groups by Turaev-Wenzl [100] and all semisimple Lie groups by Andersen and Gelfand-Kazhdan [1, 31]. The big question which motivates this article is whether there are examples of 'exotic' fusion categories which do not 'come from' quantum groups at roots of unity [49]. This is an inherently vague question, because there are many constructions (often of a group-theoretical nature) that can be applied to a fusion category to get a new one. It is possible for a fusion category to look exotic at first, but a later construction might provide a connection to quantum groups. A version of this question was posed by Moore and Seiberg [70] in 1990. Even the simplest special case of this question, whether weakly integral fusion categories come from applying known constructions to the trivial fusion category Vec, remains open [23].

The first 'exotic' examples which appeared to be unrelated to quantum groups came from Haagerup's small index subfactor classification program [44], namely the even parts of the Haagerup and Asaeda-Haagerup subfactors constructed in [2] and the even part of the Extended Haagerup subfactor constructed in [6] (after numerical evidence for existence was given by [50]). However, with time, the first two of these three examples were shown to be related to the more general story of Izumi quadratic fusion categories. (1) Izumi generalized the Haagerup subfactor to a possibly infinite family of quadratic 3^G subfactors [51, 26]. Recently in [38], Grossman-Izumi-Snyder found all fusion categories Morita equivalent

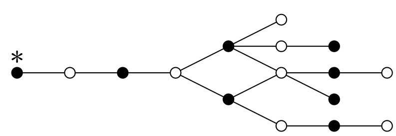
(1) It is a very interesting open question whether these quadratic categories can be constructed from quantum groups via Evans-Gannon's conjectural grafting process [26, 37].

to the Asaeda-Haagerup fusion categories, and discovered that one is an Izumi quadratic category. Thus the only known examples of fusion categories which appear unrelated to quantum groups at roots of unity or Izumi quadratic categories are the Extended Haagerup fusion categories.

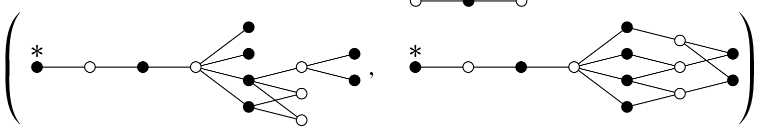
The Extended Haagerup subfactor gives a Morita equivalence between two unitary fusion categories called $\mathcal{EH}_1$ and $\mathcal{EH}_2$. The goal of this paper is to find all fusion categories Morita equivalent to these fusion categories. We find two new fusion categories, seven new subfactors (along with their duals and reduced subfactors—see Section 2), and several interesting new intermediate subfactor lattices. Unlike in the Asaeda-Haagerup case where one of the new categories was a quadratic category, in the Extended Haagerup case neither new category has nontrivial invertible objects and so neither can be a quadratic category. This means the Extended Haagerup fusion categories appear to be more exotic than the Haagerup and Asaeda-Haagerup fusion categories.

Theorem 1.1. – *There are exactly two further fusion categories in the Morita equivalence class of $\mathcal{EH}_1$ and $\mathcal{EH}_2$, which we call $\mathcal{EH}_3$ and $\mathcal{EH}_4$. Between any two of these four fusion categories, there is exactly one Morita equivalence.*

For every choice of simple object in each of these Morita equivalences, we get a subfactor. In addition to the original 7-supertransitive Extended Haagerup subfactor, we get two new 3-supertransitive subfactors: one is self-dual and comes from the Morita auto-equivalence of $\mathcal{EH}_3$ and the other comes from the Morita equivalence between $\mathcal{EH}_3$ and $\mathcal{EH}_4$. Their principal graphs are:



and



The structures of $\mathcal{EH}_3$ and $\mathcal{EH}_4$ are explained in more detail in Section 2. Neither appears to be easily understood using any general techniques, but we encourage the reader to look for a new way to construct them which could give a better understanding of the Extended Haagerup subfactor.

The proof of the main theorem has two parts. On the one hand we need to limit the possible fusion categories and Morita equivalences, and on the other hand we need to construct the remaining possibilities. The former is an application of the techniques introduced in [43], using combinatorial restrictions for compatible fusion rules for the hypothetical fusion categories and bimodule categories.

We construct $\mathcal{EH}_3$ and $\mathcal{EH}_4$ using a general graph planar algebra [57] technique for finding module categories over any fusion category where we have a good skein theoretic description. This technique can be viewed as a generalization of the Ocneanu cell technique for $\mathrm{SU}(n)$ ([79], [27], [87]) to arbitrary tensor categories with good skein theoretic descriptions. From our combinatorial calculation we know that there is at most one module category over each of $\mathcal{EH}_1$ and $\mathcal{EH}_2$ whose dual can be $\mathcal{EH}_3$ (or $\mathcal{EH}_4$). So if we can construct a module category with the correct fusion rules, we will have a construction of $\mathcal{EH}_3$ (or $\mathcal{EH}_4$) as the commutant category.

We can package $\mathcal{EH}_1$ and $\mathcal{EH}_2$ together with their Morita equivalence into a single multifusion category which is the Extended Haagerup planar algebra. The fusion rules for tensoring simple objects in a module with the single strand in Extended Haagerup give a bipartite graph Γ. We prove a graph planar algebra embedding theorem for module categories, which shows such module categories for this planar algebra correspond (up to gauging and graph automorphism) to embeddings of the Extended Haagerup subfactor planar algebra inside the graph planar algebra of Γ. This generalizes the original graph planar algebra embedding theorem [62] which only applied to the principal graphs.

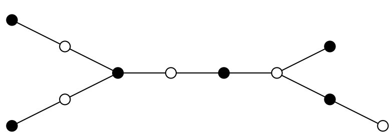
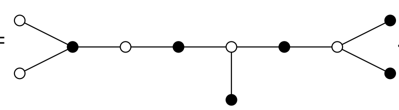
Theorem 1.2. – *Suppose $\mathcal{P}_\bullet$ is a finite depth subfactor planar algebra. Let $\mathscr{C}$ denote the unitary pivotal multifusion category of projections in $\mathcal{P}_\bullet$ with distinguished object $X = \mathrm{id}_{1,+} \in \mathcal{P}_{1,+}$. There is an equivalence between:*

1. *Planar †-algebra embeddings $\mathcal{P}_\bullet \to \mathcal{GPA}(\Gamma)_\bullet$, where Γ is a finite connected bipartite graph, and*
2. *indecomposable finitely semisimple pivotal left $\mathscr{C}$-module C^* categories $\mathcal{M}$ whose fusion graph with respect to X is Γ.*

This theorem answers a long-standing problem of Vaughan Jones: given a finite depth subfactor planar algebra $\mathcal{P}_\bullet$, determine all bipartite graphs Γ for which $\mathcal{P}_\bullet$ embeds in $\mathcal{GPA}(\Gamma)_\bullet$.

By the skein theoretic description of Extended Haagerup in [6], in order to construct a map from Extended Haagerup into a graph planar algebra, we need only to specify a number for each loop of length 16 and check a large number of linear and quadratic equations in these numbers.

Theorem 1.3. – *The Extended Haagerup planar algebra can be embedded into the graph planar algebras of each of the following bipartite graphs:*

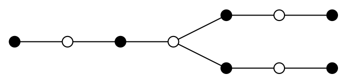
$\Gamma_3 =$  *and* $\Gamma_4 =$  .

Thus, the existence of $\mathcal{EH}_3$ and $\mathcal{EH}_4$ is now a corollary to Theorems 1.2 and 1.3.

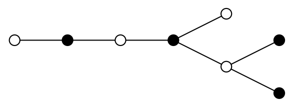
From our complete classification of module categories, we see that there are exactly four graphs whose graph planar algebras take maps from the Extended Haagerup planar algebra: the principal and dual principal graphs for the original subfactor, and the two graphs in the previous theorem. (One may think of these embeddings as giving four independent constructions of the Extended Haagerup subfactor planar algebra.)

Theorem 1.2 also connects the results of [86] and [42] to complete the classification of graph planar algebra embeddings for the Haagerup planar algebra. In the last section of [86], three embeddings of the Haagerup planar algebra into graph planar algebras were found, corresponding to the two principal graphs and the 'broom' graph. However, it was not proven there could not be others. The main result of [42] shows there are exactly three module categories over the Haagerup subfactor planar algebra. Thus we have:

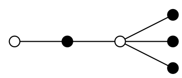
Corollary 1.4. – *The Haagerup subfactor planar algebra embeds into* $\mathcal{GPA}(\Gamma)_\bullet$ *if and only if* Γ *is one of the following:*



(Haagerup principal graph)



(dual principal graph)



('broom' graph)

Here, the two unshaded vertices of the 'broom' graph correspond to the third $\mathcal{H}_2$*-module from* [42, Cor. 3.16].

Section 2 summarizes the combinatorial structure of the four Extended Haagerup fusion categories and the Morita equivalences between them. In particular, we describe the fusion rules for each fusion category, and give the principal and dual principal graphs for all the subfactors coming from small objects in the bimodule categories. This section concludes with a table of all lattices of intermediate subfactors coming from these fusion categories, which can be read off from the fusion rules of the bimodule categories following [106, Cor. 2.4]. There are several particularly interesting examples: a (3,3)-supertransitive non-commuting but cocommuting quadrilateral with indices $(7.0283\ldots, 8.0283\ldots)$, a (2,2)-supertransitive non-commuting but cocommuting quadrilateral with indices $(13.3305\ldots, 14.3305\ldots)$, and a hexagonal intermediate subfactor lattice whose lower and upper inclusions are the 7-supertransitive index $4.3772\ldots$ extended Haagerup subfactor, and whose middle inclusions are 2-supertransitive with index $13.3305\ldots$ The first of these quadrilaterals is the smallest known example of index above 4 in Class II of the Grossman-Izumi classification [36] of highly supertransitive non-commuting quadrilaterals. It is striking that the smallest examples in Class III and Class IV are respectively the Asaeda-Haagerup and Haagerup subfactors.

The main goal of Section 3 is to prove Theorem 1.2. We begin by recalling some key background about module categories, bimodule categories, Brauer-Picard groupoids, and the Maximal Atlas. This background is used throughout the paper. We also recall the relationship between module categories and functors $\mathscr{C} \to \mathrm{End}(M)$. We then relate $\mathrm{End}(M)$ to a version of graph planar algebra. We hope this exposition will make graph planar algebra techniques accessible to readers with a background in tensor categories. In particular, we prove a purely algebraic analogue of Theorem 1.2 directly from the action map $\mathscr{C} \to \mathrm{End}(M)$. In order to adapt this simple algebraic argument to prove Theorem 1.2, we recall some technical background on the definition and classification of unitary pivotal structures from [83], and the correct unitary pivotal analogues of module categories (analogous to [16] in the unitary non-pivotal case and [96] in the non-unitary pivotal case). Note that the unitary pivotal version of this result is essential to our main results, because the characterization of maps out of the Extended Haagerup planar algebra in [6] relies on positive definiteness in order to check more complicated skein relations based on only a few simple skein relations.

In Section 4 we show that there are at most four fusion categories in the Extended Haagerup Morita equivalence class and exactly one Morita equivalence between each pair of categories in this class. Furthermore we determine all the fusion rules between objects in each of these potential fusion categories and bimodule categories. This closely follows the techniques introduced in [43] by computing compatible fusion rules for the hypothetical fusion categories and bimodule categories.

In Section 5 we recall and slightly modify the characterization of maps out of the Extended Haagerup planar algebra proved in [6]. We then prove Theorem 1.3, again following the outline of [6].

The version of this paper on the ArXiv [41] also contains two appendices, not intended for publication. The first online appendix gives an alternate construction of $\mathcal{EH}_3$ by directly constructing a Q-system on $1 \oplus f^{(6)}$ in the Extended Haagerup planar algebra using its explicit embedding into the graph planar algebra of its principal graph. This approach does not work for $\mathcal{EH}_4$ because the smallest Q-system yielding $\mathcal{EH}_4$ lives in too large a box space for computer calculations to be feasible. The second online appendix outlines one promising skein theoretic approach which could give a more natural description of $\mathcal{EH}_3$ and $\mathcal{EH}_4$.

Throughout the paper we will use the notation $\mathscr{C}(X \to Y)$ to denote the morphisms between objects X and Y in the category $\mathscr{C}$.

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2. Facts about the Extended Haagerup fusion categories

In this section, we describe the Extended Haagerup fusion categories from Theorem 1.1. The logic here is somewhat convoluted; the statements of this section logically depend on the later sections (and we're careful not to use the statements here in those sections!). We have decided to put this summary first in order to make the structure of these new fusion categories as accessible as possible.

Recall that by the main results of this paper, there are exactly four unitary fusion categories $\mathcal{EH}_1, \mathcal{EH}_2, \mathcal{EH}_3$, and $\mathcal{EH}_4$ in the Extended Haagerup Morita equivalence class, and between any two of these four, there is exactly one Morita equivalence.

Given any two invertible bimodule categories ${}_{\mathcal{A}}\mathcal{K}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{L}_{\mathscr{C}}$ the composition ${}_{\mathcal{A}}\mathcal{K}_{\mathcal{B}} \boxtimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{L}_{\mathscr{C}}$ is again invertible. But if we fix one invertible bimodule category in each equivalence class, there may be several equivalences ${}_{\mathcal{A}}\mathcal{K}_{\mathcal{B}} \boxtimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{L}_{\mathscr{C}} \to {}_{\mathcal{A}}\mathcal{M}_{\mathscr{C}}$. More specifically, they form a torsor for the group of invertible objects in the center of $\mathcal{A}$. The center of the Extended Haagerup fusion category has no nontrivial invertible objects [73]; indeed, $\mathcal{EH}_1$ has no nontrivial invertible objects, and the induction matrices computed in [73] show that only the identity object forgets to $1_{\mathcal{EH}_1}$. Thus there's a unique composition functor for each composable pair of invertible bimodules between fusion categories in the Extended Haagerup Morita equivalence class.

NOTATION 2.1. – For $1 \leq i, j \leq 4$, we denote by $\mathcal{EH}_{ij}$ the unique invertible $\mathcal{EH}_i - \mathcal{EH}_j$ bimodule category, and notice that $\mathcal{EH}_{ii} = \mathcal{EH}_i$. One may view $\mathcal{EH} = (\mathcal{EH}_{ij})_{i,j=1}^4$ as a single 4×4 unitary multifusion category.

We interpret the fusion ring $EH := K_0(\mathcal{EH})$ for $\mathcal{EH}$ as a single ring whose basis consists of the disjoint union of a set of representatives of simple objects $\mathrm{Irr}(\mathcal{EH}_{ij})$ for each $\mathcal{EH}_{ij}$. We denote by EH_{ij} the span of the set of representatives of simple objects in $\mathrm{Irr}(\mathcal{EH}_{ij})$, and notice that $EH_{ii} = K_0(\mathcal{EH}_i)$. Of course the products of objects which are not composable are declared to be zero; that is, EH is faithfully graded by the standard system of matrix units for $M_4(\mathbb{C})$.

Within each $\mathcal{EH}_{ij}$ we order simple objects by increasing dimension, so O_{ij}^k denotes the kth smallest simple object in $\mathcal{EH}_{ij}$ (or, abusing notation, the corresponding basis element in the Grothendieck ring). When there are duplicate dimensions the ties are broken arbitrarily.

We describe the fusion ring EH in the Mathematica notebook `EHmult.nb`, which is a wrapper for the data file `EHmult.txt`, both of which are bundled with the arXiv sources of this article. Therein, we supply a 6-dimensional tensor T whose (i, j, k, x, y, z)-entry is the coefficient of the z-th basis element of EH_{ik} in the product of the x-th basis element of EH_{ij} and the y-th basis element of EH_{jk}. On the level of categories, these coefficients are the dimensions of Hom spaces between simple objects and tensor products of pairs of simple objects. That is,

$$O_{ij}^x \otimes O_{jk}^y \cong \bigoplus_z T(i, j, k, x, y, z) O_{ik}^z.$$

In this section, we gather information on all the Extended Haagerup fusion categories, including fusion rules, the simplest Q-systems, and intermediate subfactor lattices.

We begin by recalling the well-known dictionary between finite index *overfactors* M of a II_1 factor N and simple Q-systems ($\mathrm{End}_{Q-Q}(Q) = \mathbb{C}$) in $\mathsf{Bim}(N)$. Given such a subfactor $N \subset M$, $L^2(M) \in \mathsf{Bim}(N)$ is a simple Q-system. Conversely, given a Q-system $Q \in \mathsf{Bim}(N)$, one can recover M directly as the bounded vectors in Q. This folklore result is certainly known to experts (see, e.g., [68, 45], [54, 55], or [8, §4] for more details).

REMARK 2.2. – The relationship between unitary fusion categories and finite index, finite depth subfactors has a difficult analytic part, and a well-understood algebraic part. The difficult analytic part is, given a particular factor N, to understand all the ways a given unitary fusion category $\mathscr{C}$ can be realized as a category of N-N bimodules. This analytic part clearly depends on which factor you look at. For the hyperfinite II_1 factor, this question was completely answered by Ocneanu and Popa [78, 89] (see [46] for a complete proof in the language of unitary fusion categories based on the classification results of [98]), while there are many interesting results for other factors [101, 102, 17, 28]. In this paper we only address the algebraic part, i.e., understanding Q-systems in unitary fusion categories. Once you realize $\mathscr{C}$ as a category of bimodules over a factor N, then one can apply the above well-known dictionary to recover the associated subfactors.

Let us quickly recall how one can read off the Q-systems (or, in the algebraic setting, separable algebra objects) from a tensor category and the module categories over it following [67, 69, 75, 80]. Namely, if $\mathscr{C}$ is a unitary fusion category, $\mathcal{M}$ is a module category over it, and m is a simple object in $\mathcal{M}$, then the internal Hom $m \otimes \overline{m}$ gives a Q-system [77, Thm. A.1]. Moreover, all irreducible Q-systems appear this way.(2) See [38, §2] for a more detailed summary. Since a finite index, finite depth subfactor $N \subset M$ gives a unitary fusion category with a Q-system, these are algebraic analogues of subfactors.

There are two key constructions which construct a new subfactor from an old one. First, given a subfactor coming from a simple object m in a module category, one can keep the same module category but change the choice of simple object. Provided m' appears inside a tensor power of m, this is called the *reduced subfactor construction*. Second, $\mathcal{M}$ gives a Morita equivalence between $\mathscr{C}$ and the dual category $\mathrm{End}_{\mathscr{C}}(\mathcal{M})$, so there is a dual subfactor coming from m as an object in $\mathcal{M}$ thought of as module over the dual category. Thus for each bimodule category, choosing a favorite simple (typically the smallest one), all the other subfactors coming from that bimodule arise as reduced subfactors after possibly taking the dual subfactor. In our case there are $4 + \binom{4}{2}$ bimodule categories, three of which only involve $\mathcal{EH}_1$ and $\mathcal{EH}_2$ and so were already known. This is why there are 7 genuinely new subfactors in this article, plus their duals and their many reduced subfactors.

NOTATION 2.3. – Our convention for principal graphs of subfactors and fusion graphs of fusion categories is that we always tensor on the *right*. In particular, the fusion graph for X has $\dim(\mathcal{EH}(A \otimes X \to B))$ oriented edges between simples A and B. Later, in §4.4, we will discuss fusion graphs for left module categories, which use the opposite convention.

Finally, we recall that a subfactor is k-*supertransitive* if there are no branches in the principal graph before depth k; in particular a subfactor is 2-supertransitive if the corresponding algebra object is of the form $1 \oplus X$ with X simple.

2.1. Structure of $\mathcal{EH}_1$

The fusion rules for $\mathcal{EH}_1$ (which is the dual even half of the Extended Haagerup subfactor) are given by

(2) More precisely, the 2-groupoid of algebra objects in $\mathscr{C}$, algebra isomorphisms, and equalities is equivalent to the 2-groupoid whose objects are pairs $(\mathcal{M}, m)$, 1-morphisms are pairs $(\mathcal{F}, f)$ where $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ is a module equivalence and $f : \mathcal{F}(m) \to n$ is an isomorphism, and 2-morphisms natural isomorphisms compatible with the pointing isomorphisms f. In particular, for any invertible object g in the dual category, the internal endomorphisms of the objects m and mg in $\mathcal{M}$ give isomorphic algebras. Since none of our fusion categories have non-trivial invertible objects, this over-counting will not be relevant.

$\otimes$	$f^{(2)}$	$f^{(4)}$	$f^{(6)}$
$f^{(2)}$	$1 \oplus f^{(2)} \oplus f^{(4)}$	$f^{(2)} \oplus f^{(4)} \oplus f^{(6)}$	$f^{(4)} \oplus f^{(6)} \oplus P' \oplus Q'$
$f^{(4)}$	$f^{(2)} \oplus f^{(4)} \oplus f^{(6)}$	$1 \oplus f^{(2)} \oplus f^{(4)}$ $\oplus f^{(6)} \oplus P' \oplus Q'$	$f^{(2)} \oplus f^{(4)} \oplus 2f^{(6)}$ $\oplus 3P' \oplus Q'$
$f^{(6)}$	$f^{(4)} \oplus f^{(6)} \oplus P' \oplus Q'$	$f^{(2)} \oplus f^{(4)} \oplus 2f^{(6)}$ $\oplus 3P' \oplus Q'$	$1 \oplus f^{(2)} \oplus 2f^{(4)}$ $\oplus 4f^{(6)} \oplus 5P' \oplus 3Q'$
P'	$f^{(6)} \oplus 2P' \oplus Q'$	$f^{(4)} \oplus 3f^{(6)}$ $\oplus 3P' \oplus 2Q'$	$f^{(2)} \oplus 3f^{(4)} \oplus 5f^{(6)}$ $\oplus 6P' \oplus 3Q'$
Q'	$f^{(6)} \oplus P'$	$f^{(4)} \oplus f^{(6)}$ $\oplus 2P' \oplus Q'$	$f^{(2)} \oplus f^{(4)} \oplus 3f^{(6)}$ $\oplus 3P' \oplus 2Q'$

$\otimes$	P'	Q'
$f^{(2)}$	$f^{(6)} \oplus 2P' \oplus Q'$	$f^{(6)} \oplus P'$
$f^{(4)}$	$f^{(4)} \oplus 3f^{(6)}$ $\oplus 3P' \oplus 2Q'$	$f^{(4)} \oplus f^{(6)}$ $\oplus 2P' \oplus Q'$
$f^{(6)}$	$f^{(2)} \oplus 3f^{(4)} \oplus 5f^{(6)}$ $\oplus 6P' \oplus 3Q'$	$f^{(2)} \oplus f^{(4)} \oplus 3f^{(6)}$ $\oplus 3P' \oplus 2Q'$
P'	$1 \oplus 2f^{(2)} \oplus 3f^{(4)}$ $\oplus 6f^{(6)} \oplus 7P' \oplus 4Q'$	$f^{(2)} \oplus 2f^{(4)} \oplus 3f^{(6)}$ $\oplus 4P' \oplus 2Q'$
Q'	$f^{(2)} \oplus 2f^{(4)} \oplus 3f^{(6)}$ $\oplus 4P' \oplus 2Q'$	$1 \oplus f^{(4)} \oplus 2f^{(6)}$ $\oplus 2P' \oplus Q'$

Here we have given more informative names to the objects, corresponding to those used in [6] rather than merely naming them O_{11}^x.

The dimensions of the objects $(f^{(2)}, f^{(4)}, f^{(6)}, P', Q')$ are roughly $(3.4, 7.0, 13.3, 16.0, 8.7)$, and this determines the ordering used in the O_{11}^x notation. In particular $O_{11}^1 = \mathbf{1}$, $O_{11}^2 = f^{(2)}$, $O_{11}^3 = f^{(4)}$, $O_{11}^4 = Q'$ (as it has the next smallest dimension), $O_{11}^5 = f^{(6)}$, and $O_{11}^6 = P'$.

A Morita equivalence between $\mathscr{C}$ and $\mathcal{D}$ gives a braided equivalence $Z(\mathscr{C}) \cong Z(\mathcal{D})$. Any such $\mathcal{D}$ is of the form A-mod for A a Lagrangian algebra in $Z(\mathscr{C})$ [95, 22, 13]. In general there might be several Lagrangian algebras yielding a given $\mathcal{D}$, but in our case since the Brauer-Picard group is trivial there is a unique Lagrangian algebra for each $\mathcal{D}$. Using the notation of [30] for the objects in the center $Z(\mathcal{EH})$, the Lagrangian algebra giving $\mathcal{EH}_1$ has underlying object:

$$\omega_0 \oplus \omega_1 \oplus \omega_2 \oplus \alpha_1 \oplus \alpha_2 \oplus \alpha_3.$$

2.2. Structure of $\mathcal{EH}_2$

The fusion rules for $\mathcal{EH}_2$ (which is the principal even half of the Extended Haagerup subfactor) are given by

$\otimes$	$f^{(2)}$	$f^{(4)}$	$f^{(6)}$
$f^{(2)}$	$1 \oplus f^{(2)} \oplus f^{(4)}$	$f^{(2)} \oplus f^{(4)} \oplus f^{(6)}$	$f^{(4)} \oplus W$
$f^{(4)}$	$f^{(2)} \oplus f^{(4)} \oplus f^{(6)}$	$1 \oplus f^{(2)} \oplus f^{(4)} \oplus W$	$f^{(2)} \oplus f^{(4)} \oplus A \oplus B \oplus 2W$
$f^{(6)}$	$f^{(4)} \oplus W$	$f^{(2)} \oplus f^{(4)} \oplus A \oplus B \oplus 2W$	$1 \oplus W \oplus Z$
P	$A \oplus W$	$f^{(4)} \oplus B \oplus 2W$	$f^{(6)} \oplus Q \oplus Z$
Q	$B \oplus W$	$f^{(4)} \oplus A \oplus 2W$	$f^{(6)} \oplus P \oplus Z$
A	P	$f^{(6)} \oplus Q$	$f^{(4)} \oplus B \oplus W$
B	Q	$f^{(6)} \oplus P$	$f^{(4)} \oplus A \oplus W$

$\otimes$	P	Q	A	B
$f^{(2)}$	$B \oplus W$	$A \oplus W$	Q	P
$f^{(4)}$	$f^{(4)} \oplus A \oplus 2W$	$f^{(4)} \oplus B \oplus 2W$	$f^{(6)} \oplus P$	$f^{(6)} \oplus Q$
$f^{(6)}$	$f^{(6)} \oplus Q \oplus Z$	$f^{(6)} \oplus P \oplus Z$	$f^{(4)} \oplus B \oplus W$	$f^{(4)} \oplus A \oplus W$
P	$1 \oplus P \oplus Z$	$f^{(6)} \oplus Z$	$f^{(2)} \oplus A \oplus W$	$f^{(4)} \oplus W$
Q	$f^{(6)} \oplus Z$	$1 \oplus Q \oplus Z$	$f^{(4)} \oplus W$	$f^{(2)} \oplus B \oplus W$
A	$f^{(4)} \oplus W$	$f^{(2)} \oplus A \oplus W$	$f^{(6)}$	$1 \oplus P$
B	$f^{(2)} \oplus B \oplus W$	$f^{(4)} \oplus W$	$1 \oplus Q$	$f^{(6)}$

using the abbreviations $W = f^{(6)} + P + Q$ and $Z = A + B + f^{(2)} + 2f^{(4)} + 3f^{(6)} + 3P + 3Q$.

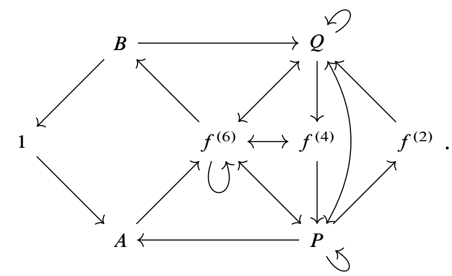
REMARK 2.4. – In the arXiv sources of this article [41, Appendix A], we give an alternate construction of $\mathcal{EH}_3$ by constructing a Q-system on $1 + f^{(6)}$ in $\mathcal{EH}_2$ whose dual category is $\mathcal{EH}_3$. This construction is viable because

$$\dim(\mathrm{Hom}(f^{(6)} \otimes f^{(6)} \to f^{(6)})) = 4$$

is not too large.

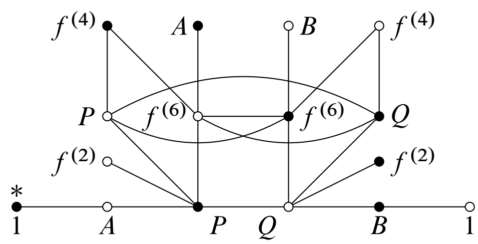
The dimensions of $(f^{(2)}, f^{(4)}, f^{(6)}, P, Q, A, B)$ are roughly (3.4, 7.0, 13.3, 12.3, 12.3, 3.7, 3.7), which again determines the ordering in the O_{22}^{x} naming.

Since A and $B = \overline{A}$ have small dimension we also record the fusion graph for tensoring with A and the principal graph for the corresponding subfactor. The fusion graph is:



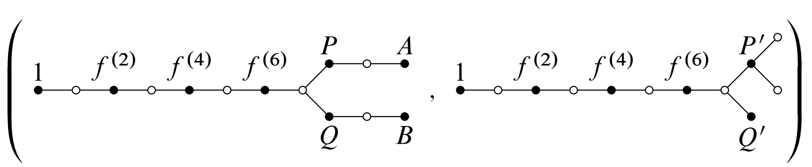
The corresponding subfactor $1 \subset A\overline{A}$ has index roughly 13.3. It has the following principal and dual principal graph. Notice that every simple appears twice—once as an even vertex and once as an odd vertex.

(2.1)



The Extended Haagerup subfactor planar algebra constructed in [6] (see §5) provides a Morita equivalence between $\mathcal{EH}_1$ and $\mathcal{EH}_2$. The generating object X in the Morita equivalence is the smallest object O_{12}^{1} in the unique invertible bimodule category between these two fusion categories. The principal graphs are

(2.2)



Notice all even vertices are self dual except for $\overline{A} \cong B$.

The Lagrangian algebra giving $\mathcal{EH}_2$ has underlying object:

$$\omega_0 \oplus \omega_1 \oplus \omega_2 \oplus 2\alpha_1 \oplus \alpha_2.$$

2.3. Structure of $\mathcal{EH}_3$

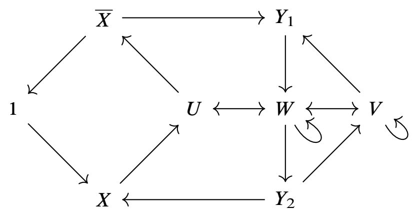
The fusion rules for $\mathcal{EH}_3$ are as follows.

$\otimes$	X	$\overline{X}$	Y_1	Y_2
X	U	$1\oplus Y_2$	$V\oplus X$	W
$\overline{X}$	$1\oplus Y_1$	U	W	$\overline{X}\oplus V$
Y_1	W	$\overline{X}\oplus V$	$1\oplus V\oplus W\oplus Y_1$	$U\oplus V\oplus W$
Y_2	$V\oplus X$	W	$U\oplus V\oplus W$	$1\oplus V\oplus W\oplus Y_2$
U	$\overline{X}\oplus W$	$W\oplus X$	$U\oplus V\oplus W\oplus Y_2$	$U\oplus V\oplus W\oplus Y_1$
V	$V\oplus W\oplus Y_1$	$V\oplus W\oplus Y_2$	$U\oplus 2V\oplus 2W\oplus X\oplus Y_1\oplus Y_2$	$\overline{X}\oplus U\oplus 2V\oplus 2W\oplus Y_1\oplus Y_2$
W	$U\oplus V\oplus W\oplus Y_2$	$U\oplus V\oplus W\oplus Y_1$	$\overline{X}\oplus U\oplus 2V\oplus 3W\oplus Y_1\oplus Y_2$	$U\oplus 2V\oplus 3W\oplus X\oplus Y_1\oplus Y_2$

$\otimes$	U	V	W
X	$\overline{X}\oplus W$	$V\oplus W\oplus Y_2$	$U\oplus V\oplus W\oplus Y_1$
$\overline{X}$	$W\oplus X$	$V\oplus W\oplus Y_1$	$U\oplus V\oplus W\oplus Y_2$
Y_1	$U\oplus V\oplus W\oplus Y_2$	$\overline{X}\oplus U\oplus 2V\oplus 2W\oplus Y_1\oplus Y_2$	$U\oplus 2V\oplus 3W\oplus X\oplus Y_1\oplus Y_2$
Y_2	$U\oplus V\oplus W\oplus Y_1$	$U\oplus 2V\oplus 2W\oplus X\oplus Y_1\oplus Y_2$	$\overline{X}\oplus U\oplus 2V\oplus 3W\oplus Y_1\oplus Y_2$
U	$1\oplus U\oplus V\oplus W\oplus Y_1\oplus Y_2$	$U\oplus 2V\oplus 3W\oplus Y_1\oplus Y_2$	$\overline{X}\oplus U\oplus 3V\oplus 3W\oplus X\oplus Y_1\oplus Y_2$
V	$U\oplus 2V\oplus 3W\oplus Y_1\oplus Y_2$	$1\oplus\overline{X}\oplus 2U\oplus 4V\oplus 5W\oplus X\oplus 2Y_1\oplus 2Y_2$	$\overline{X}\oplus 3U\oplus 5V\oplus 6W\oplus X\oplus 2Y_1\oplus 2Y_2$
W	$\overline{X}\oplus U\oplus 3V\oplus 3W\oplus X\oplus Y_1\oplus Y_2$	$\overline{X}\oplus 3U\oplus 5V\oplus 6W\oplus X\oplus 2Y_1\oplus 2Y_2$	$1\oplus\overline{X}\oplus 3U\oplus 6V\oplus 7W\oplus X\oplus 3Y_1\oplus 3Y_2$

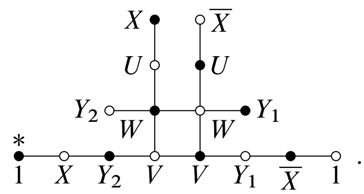
The dimensions of $(X, \overline{X}, Y_1, Y_2, U, V, W)$ are approximately $(2.6, 2.6, 6.0, 6.0, 7.0, 13.3, 15.9)$. They are listed in the order O_{33}^2, O_{33}^3, etc.

The fusion graph for $X \in \mathcal{EH}_3$ is given by



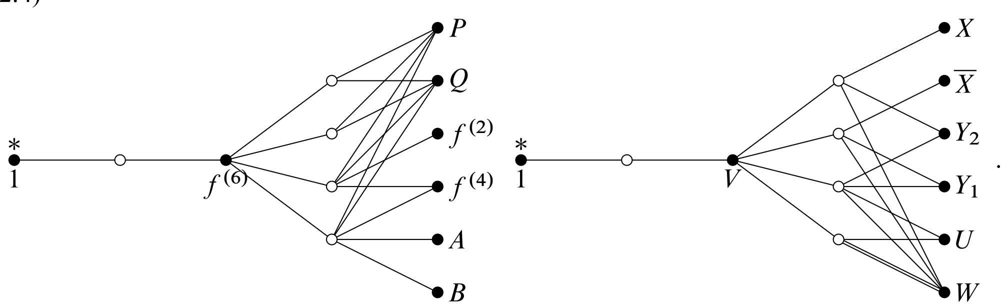
The principal graph of the subfactor corresponding to the algebra object $X \otimes \overline{X} \in \mathcal{EH}_3$ is given below. It has index roughly 7.0. Notice that each simple of $\mathcal{EH}_3$ appears twice—once as an even vertex and once as an odd vertex.

(2.3)



This paper establishes that $\mathcal{EH}_3$ is a categorification of the above ring, but we do not know that this is the only such categorification. Thus in order to construct the Extended Haagerup subfactor from an alternative proposed construction of $\mathcal{EH}_3$ one would need to do additional work. Since the even parts of the Extended Haagerup subfactor are the only categorifications of their fusion rings, it would be enough to construct a categorification of the $\mathcal{EH}_3$ fusion ring plus an algebra structure on $1 \oplus U$, and check that the fusion ring of the commutant category corresponding to the algebra $1 \oplus U$ is the fusion ring of $\mathcal{EH}_2$.

We are able to list all Q-systems in $\mathcal{EH}_3$. The Q-system corresponding to O^1_{23} has relatively small dimension and its underlying object is $1 \oplus U$. The dual algebra in this case is $1 \oplus f^{(6)}$ in $\mathcal{EH}_2$. The index of this subfactor is approximately 14.3, and the principal graphs are:
(2.4)



The Lagrangian algebra in $Z(\mathcal{EH})$ giving $\mathcal{EH}_3$ has underlying object:

$$\omega_0 \oplus \omega_1 \oplus \omega_2 \oplus 2\alpha_1 \oplus \alpha_2.$$

Note that this is the same underlying object as the Lagrangian algebra corresponding to $\mathcal{EH}_2$; the algebra structures must be different.

2.4. Structure of $\mathcal{EH}_4$

We now turn to describing some of the combinatorial structure of $\mathcal{EH}_4$. The fusion rules for $\mathcal{EH}_4$ are as follows.

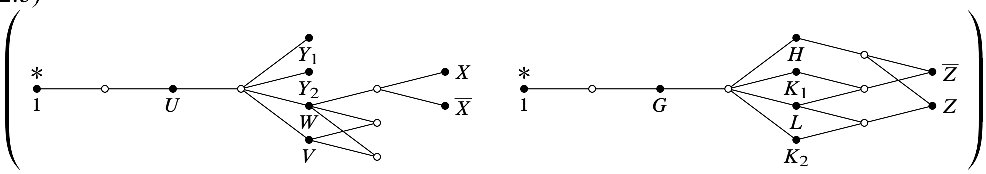
$\otimes$	Z	$\overline{Z}$	G	H
Z	$G\oplus K_1\oplus K_2\oplus L$	$1\oplus G\oplus H\oplus K_2\oplus L$	$H\oplus K_2\oplus L\oplus\overline{Z}\oplus Z$	$G\oplus H\oplus K_1\oplus K_2\oplus L\oplus Z$
$\overline{Z}$	$1\oplus G\oplus H\oplus K_1\oplus L$	$G\oplus K_1\oplus K_2\oplus L$	$H\oplus K_1\oplus L\oplus\overline{Z}\oplus Z$	$G\oplus H\oplus K_1\oplus K_2\oplus L\oplus\overline{Z}$
G	$H\oplus K_1\oplus L\oplus\overline{Z}\oplus Z$	$H\oplus K_2\oplus L\oplus\overline{Z}\oplus Z$	$1\oplus G\oplus H\oplus K_1\oplus K_2\oplus L$	$G\oplus H\oplus K_1\oplus K_2\oplus L\oplus\overline{Z}\oplus Z$
H	$G\oplus H\oplus K_1\oplus K_2\oplus L\oplus Z$	$G\oplus H\oplus K_1\oplus K_2\oplus L\oplus\overline{Z}$	$G\oplus H\oplus K_1\oplus K_2\oplus L\oplus\overline{Z}\oplus Z$	$1\oplus G\oplus H\oplus K_1\oplus K_2\oplus 2L\oplus\overline{Z}\oplus Z$
K_1	$H\oplus K_1\oplus K_2\oplus 2L\oplus\overline{Z}$	$G\oplus H\oplus K_1\oplus K_2\oplus L\oplus\overline{Z}\oplus Z$	$G\oplus H\oplus K_1\oplus K_2\oplus 2L\oplus\overline{Z}$	$G\oplus H\oplus 2K_1\oplus K_2\oplus 2L\oplus\overline{Z}\oplus Z$
K_2	$G\oplus H\oplus K_1\oplus K_2\oplus L\oplus\overline{Z}\oplus Z$	$H\oplus K_1\oplus K_2\oplus 2L\oplus Z$	$G\oplus H\oplus K_1\oplus K_2\oplus 2L\oplus Z$	$G\oplus H\oplus K_1\oplus 2K_2\oplus 2L\oplus\overline{Z}\oplus Z$
L	$G\oplus H\oplus K_1\oplus 2K_2\oplus 2L\oplus\overline{Z}\oplus Z$	$G\oplus H\oplus 2K_1\oplus K_2\oplus 2L\oplus\overline{Z}\oplus Z$	$G\oplus H\oplus 2K_1\oplus 2K_2\oplus 2L\oplus\overline{Z}\oplus Z$	$G\oplus 2H\oplus 2K_1\oplus 2K_2\oplus 3L\oplus\overline{Z}\oplus Z$

$\otimes$	K_1	K_2	L
Z	$G\oplus H\oplus K_1\oplus K_2\oplus L\oplus\overline{Z}\oplus Z$	$H\oplus K_1\oplus K_2\oplus 2L\oplus\overline{Z}$	$G\oplus H\oplus 2K_1\oplus K_2\oplus 2L\oplus\overline{Z}\oplus Z$
$\overline{Z}$	$H\oplus K_1\oplus K_2\oplus 2L\oplus Z$	$G\oplus H\oplus K_1\oplus K_2\oplus L\oplus\overline{Z}\oplus Z$	$G\oplus H\oplus K_1\oplus 2K_2\oplus 2L\oplus\overline{Z}\oplus Z$
G	$G\oplus H\oplus K_1\oplus K_2\oplus 2L\oplus Z$	$G\oplus H\oplus K_1\oplus K_2\oplus 2L\oplus\overline{Z}$	$G\oplus H\oplus 2K_1\oplus 2K_2\oplus 2L\oplus\overline{Z}\oplus Z$
H	$G\oplus H\oplus 2K_1\oplus K_2\oplus 2L\oplus\overline{Z}\oplus Z$	$G\oplus H\oplus K_1\oplus 2K_2\oplus 2L\oplus\overline{Z}\oplus Z$	$G\oplus 2H\oplus 2K_1\oplus 2K_2\oplus 3L\oplus\overline{Z}\oplus Z$
K_1	$1\oplus G\oplus 2H\oplus K_1\oplus 2K_2\oplus 2L\oplus\overline{Z}\oplus Z$	$G\oplus H\oplus 2K_1\oplus 2K_2\oplus 2L\oplus\overline{Z}\oplus Z$	$2G\oplus 2H\oplus 2K_1\oplus 2K_2\oplus 3L\oplus\overline{Z}\oplus 2Z$
K_2	$G\oplus H\oplus 2K_1\oplus 2K_2\oplus 2L\oplus\overline{Z}\oplus Z$	$1\oplus G\oplus 2H\oplus 2K_1\oplus K_2\oplus 2L\oplus\overline{Z}\oplus Z$	$2G\oplus 2H\oplus 2K_1\oplus 2K_2\oplus 3L\oplus 2\overline{Z}\oplus Z$
L	$2G\oplus 2H\oplus 2K_1\oplus 2K_2\oplus 3L\oplus 2\overline{Z}\oplus Z$	$2G\oplus 2H\oplus 2K_1\oplus 2K_2\oplus 3L\oplus\overline{Z}\oplus 2Z$	$1\oplus 2G\oplus 3H\oplus 3K_1\oplus 3K_2\oplus 4L\oplus 2\overline{Z}\oplus 2Z$

The dimensions of the objects $(Z, \overline{Z}, G, H, K_1, K_2, L)$ are approximately (6.3, 6.3, 7.0, 8.6, 9.6, 9.6, 13.3). These are already in increasing order, so the objects O_{44}^x appear in this order.

None of the objects in $\mathcal{EH}_4$ is small enough to have a nice fusion graph. The Q-system corresponding to O_{34}^1 does give a 3-supertransitive subfactor. The underlying object of this Q-system is $1 \oplus G$ and the dual Q-system is $1 \oplus U$ in $\mathcal{EH}_3$, and the index is roughly 8.0. The principal graphs for this subfactor are given by:

(2.5)



As with $\mathcal{EH}_2$ and $\mathcal{EH}_3$, the Lagrangian algebra giving $\mathcal{EH}_4$ has underlying object:

$$\omega_0 \oplus \omega_1 \oplus \omega_2 \oplus 2\alpha_1 \oplus \alpha_2.$$

We thus see that this object must have three distinct algebra structures on it.

2.5. Intermediate subfactors

In this section we describe all the lattices of intermediate subfactors for subfactors coming from the simple objects in the Extended Haagerup bimodule categories.

For a subfactor $N \subseteq M$, an intermediate subfactor is a factor P between N and M, i.e., $N \subseteq P \subseteq M$. One of the original motivations for subfactor theory was an analogy with Galois theory. Nakamura and Takeda showed that for a fixed-point subfactor of an outer action of a finite group on a factor, the intermediate subfactors are precisely the fixed-point algebras of the subgroups [76]. In the 1990s, Watatani proposed studying more generally a lattice of intermediate subfactors as a noncommutative analogue of the subgroup lattice of a group [103]. Of particular interest are quadrilaterals, consisting of a pair of intermediate subfactors P and Q such that $P \wedge Q = N$ and $P \vee Q = M$.

Following [106], we can read the intermediate subfactor lattices directly from the bimodule fusion rules in the Brauer-Picard groupoid (see Section 3.2). For an intermediate subfactor $N \subseteq P \subseteq M$, we have ${}_N L^2(P)_P \otimes_P {}_P L^2(M)_M \cong {}_N L^2(M)_M$. Conversely, if an object ${}_{\mathscr{C}} Z_{\mathcal{D}}$ in a bimodule category over unitary fusion categories factors as ${}_{\mathscr{C}} X_{\mathcal{E}} \otimes_{\mathcal{E}} {}_{\mathcal{E}} Y_{\mathcal{D}}$, then ${}_{\mathscr{C}} X_{\mathcal{E}} \otimes_{\mathcal{E}} {}_{\mathcal{E}} \overline{X}_{\mathscr{C}}$ is a subalgebra of the algebra object ${}_{\mathscr{C}} Z_{\mathcal{D}} \otimes_{\mathcal{D}} {}_{\mathcal{D}} \overline{Z}_{\mathscr{C}}$ in $\mathscr{C}$, and determines an intermediate subfactor of the corresponding subfactor (see Remark 2.2 and the adjacent discussion and references for the relationship between subfactors, algebra objects, and objects in (bi)module categories). So to determine the intermediate subfactors of a subfactor corresponding to a given object in a bimodule category, we only need to know when different factorizations of the object correspond to the same subalgebra/intermediate subfactor. Note that it is rare to have a pair of simple objects X, Y such that the product $X \otimes Y$ is simple, and this explains why irreducible subfactors typically have few intermediates.

In full generality, understanding when two factorizations correspond to the same subalgebra is somewhat delicate, but for Extended Haagerup these complications do not arise. For this reason, we plan to fully address this question in a subsequent paper considering intermediate subfactors related to the Asaeda-Haagerup subfactor where more complicated phenomena occur. Here is a very terse sketch of why there is no over-counting in the case of Extended Haagerup. Recall as in Footnote 2 that a similar phenomenon happened for algebras themselves, where pairs $(\mathcal{M}, m)$ classify algebra objects, but naively they over-count algebra objects, as pairs $(\mathcal{M}, m)$ and $(\mathcal{M}, m')$ can be equivalent as pairs via $(\mathcal{F}, f)$ where $\mathcal{F}$ is a nontrivial autoequivalence of $\mathcal{M}$. Here, we can classify subalgebras of the algebra corresponding to $(\mathcal{M}, m)$ in terms of a tuple $(\mathcal{D}, \mathcal{N}, \mathcal{L}, \mathcal{F}, n, \ell, f)$ where $\mathcal{D}$ is a tensor category, $\mathcal{N}$ is a Morita equivalence between $\mathscr{C}$ and $\mathcal{D}$, $\mathcal{L}$ is a module category over $\mathcal{D}$, n is an object in $\mathcal{N}$, ℓ is an object in $\mathcal{L}$, $\mathcal{F}$ is a module equivalence $\mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{L} \to \mathcal{M}$, and f is an isomorphism $\mathcal{F}(n \otimes \ell) \to m$. A Type III subfactor version of this result was proved by Xu in [106, Cor 2.4]. In particular, we get over-counting from tensor autoequivalences of $\mathcal{D}$, bimodule autoequivalences of $\mathcal{N}$ (which are a torsor for invertible objects in the center of $\mathscr{C}$), and module autoequivalences of $\mathcal{M}$ (which correspond to invertible objects in the dual category). However, if none of the tensor categories Morita equivalent to $\mathscr{C}$ have nontrivial autoequivalences, nor nontrivial invertible objects, nor nontrivial invertible objects in the center (which is the case for the Extended Haagerup fusion categories), then there is no over-counting, and intermediate subalgebras correspond precisely to factorizations of objects in bimodule categories as products of objects in other bimodule categories.

Each row of the following table lists triples of simple objects $X = O_{ij}^x$, $Y = O_{jk}^y$, and their simple product $X \otimes Y = O_{ik}^z$. The columns labeled 'ST' and 'Index' indicate the supertransitivity and index of the corresponding subfactors. We only list one representative of each dual pair, so in the tables we always have $i < j$. We have grouped rows together according to the identity of the large object Z, as these rows are all intermediate subfactors of the same large subfactor.

We recall that:

— O_{21}^1 and O_{12}^1 correspond to the Extended Haagerup subfactor and its dual shown in Equation (2.2),

— O_{33}^2 corresponds to the subfactor with principal graph shown in Equation (2.3),

— O_{34}^1 and O_{43}^1 correspond to the subfactor with principal graph shown in Equation (2.5),

— O_{22}^3 and O_{22}^4 both correspond to the subfactor with principal graph shown in Equation (2.1), and

— O_{23}^1 and O_{32}^1 correspond to the subfactor with principal graph shown in Equation (2.4).

Z	Index(Z)	X	ST(X)	Index(X)	Y	ST(Y)	Index(Y)
O_{11}^6	255.411	O_{12}^4	1	58.3502	O_{21}^1	7	4.3772
		O_{12}^3	1	58.3502	O_{21}^1	7	4.3772
		O_{12}^1	7	4.3772	O_{21}^3	1	58.3502
		O_{12}^1	7	4.3772	O_{21}^4	1	58.3502
O_{12}^3	58.3502	O_{12}^1	7	4.3772	O_{22}^4	2	13.3305
O_{12}^4	58.3502	O_{12}^1	7	4.3772	O_{22}^3	2	13.3305

O_{12}^6	329.743	O_{13}^1	1	23.0099	O_{32}^1	2	14.3305
		O_{13}^2	1	23.0099	O_{32}^1	2	14.3305
		O_{12}^2	1	24.736	O_{22}^3	2	13.3305
		O_{12}^2	1	24.736	O_{22}^4	2	13.3305
		O_{11}^4	1	75.3318	O_{12}^1	7	4.3772
O_{13}^4	62.7274	O_{12}^1	7	4.3772	O_{23}^1	2	14.3305
O_{13}^5	161.72	O_{13}^1	1	23.0099	O_{33}^2	3	7.0283
		O_{13}^2	1	23.0099	O_{33}^3	3	7.0283
O_{13}^6	262.439	O_{14}^1	1	32.6893	O_{43}^1	3	8.0283
		O_{14}^2	1	32.6893	O_{43}^1	3	8.0283
		O_{13}^3	1	37.3404	O_{33}^2	3	7.0283
		O_{13}^3	1	37.3404	O_{33}^3	3	7.0283
		O_{11}^2	1	11.4055	O_{13}^1	1	23.0099
		O_{11}^2	1	11.4055	O_{13}^2	1	23.0099
O_{22}^6	152.041	O_{22}^2	1	11.4055	O_{22}^4	2	13.3305
		O_{22}^3	2	13.3305	O_{22}^2	1	11.4055
O_{22}^7	152.041	O_{22}^2	1	11.4055	O_{22}^3	2	13.3305
		O_{22}^4	2	13.3305	O_{22}^2	1	11.4055
O_{22}^8	177.702	O_{22}^3	2	13.3305	O_{22}^3	2	13.3305
		O_{22}^4	2	13.3305	O_{22}^4	2	13.3305
O_{23}^2	100.719	O_{23}^1	2	14.3305	O_{33}^3	3	7.0283
		O_{21}^1	7	4.3772	O_{13}^2	1	23.0099
O_{23}^3	100.719	O_{23}^1	2	14.3305	O_{33}^2	3	7.0283
		O_{21}^1	7	4.3772	O_{13}^1	1	23.0099
O_{23}^4	163.446	O_{24}^1	1	20.3588	O_{43}^1	3	8.0283
		O_{24}^2	1	20.3588	O_{43}^1	3	8.0283
		O_{22}^2	1	11.4055	O_{23}^1	2	14.3305
		O_{21}^1	7	4.3772	O_{13}^3	1	37.3404
O_{23}^5	191.032	O_{22}^3	2	13.3305	O_{23}^1	2	14.3305
		O_{22}^4	2	13.3305	O_{23}^1	2	14.3305
O_{24}^3	115.049	O_{23}^1	2	14.3305	O_{34}^1	3	8.0283
O_{24}^4	143.088	O_{21}^1	7	4.3772	O_{14}^1	1	32.6893
		O_{21}^1	7	4.3772	O_{14}^2	1	32.6893

O_{24}^{5}	271.392	O_{22}^{4}	2	13.3305	O_{24}^{1}	1	20.3588
		O_{22}^{3}	2	13.3305	O_{24}^{2}	1	20.3588
		O_{21}^{1}	7	4.3772	O_{14}^{3}	1	62.0013
O_{33}^{6}	49.3969	O_{33}^{2}	3	7.0283	O_{33}^{2}	3	7.0283
		O_{33}^{3}	3	7.0283	O_{33}^{3}	3	7.0283
O_{33}^{8}	255.411	O_{33}^{5}	1	36.3404	O_{33}^{3}	3	7.0283
		O_{33}^{4}	1	36.3404	O_{33}^{2}	3	7.0283
		O_{33}^{3}	3	7.0283	O_{33}^{4}	1	36.3404
		O_{33}^{2}	3	7.0283	O_{33}^{5}	1	36.3404
O_{34}^{2}	56.4252	O_{33}^{2}	3	7.0283	O_{34}^{1}	3	8.0283
		O_{33}^{3}	3	7.0283	O_{34}^{1}	3	8.0283
O_{34}^{5}	291.751	O_{33}^{4}	1	36.3404	O_{34}^{1}	3	8.0283
		O_{33}^{5}	1	36.3404	O_{34}^{1}	3	8.0283
		O_{32}^{1}	2	14.3305	O_{24}^{1}	1	20.3588
		O_{32}^{1}	2	14.3305	O_{24}^{2}	1	20.3588

Let us briefly summarize the interesting subfactor lattices encoded in the above table.

There are four lines with $Z = O_{11}^{6}$, so there are four intermediate subfactors of the index 255.411... subfactor corresponding to O_{11}^{6}. Note that O_{11}^{6} denotes the 6th smallest object in $\mathcal{EH}_1$, which is P', so this subfactor is the reduced subfactor corresponding to P'. For two of these intermediates the lower inclusion is Extended Haagerup while the upper inclusions have index 58.3502... and come from the reduced subfactor construction for O_{12}^{3} or O_{12}^{4} (these are the odd vertices near the ends of the Extended Haagerup principal graph). For the other two, the lower and upper parts are switched. We next want to see how these fit together into a lattice. From the next two lines in the table we see that the index 58.3502... subfactors themselves each have a single intermediate, with one inclusion being Extended Haagerup and the other being the 2-supertransitive subfactor of index 13.3305... with principal graph shown in Equation (2.1). It follows that the lattice is a hexagon, where the upper and lower edges are the Extended Haagerup subfactors and the middle edges are the index 13.3305... subfactors.

Note that since none of the other entries in the X or Y columns also occurs in the Z column, other than the hexagon every lattice will just be M_n, the lattice with one maximal element, one minimal element, and n incomparable elements between them. The number of such incomparable entries is simply the number of rows with that Z; for example, O_{13}^{6} has intermediate subfactor lattice M_6.

In addition to the hexagon there are a few notable examples of quadrilaterals where all inclusions are at least 2-supertransitive (such quadrilaterals are called (2, 2)-supertransitive).

Sano and Watatani [94] introduced a notion of angle between two subfactors of the same factor. For a quadrilateral, one can compute two such angles, one for $P, Q \subset M$ and one for the dual subfactors to $N \subset P, Q$. When the first angle is $\pi/2$, we say that the quadrilateral commutes, and when the second angle is $\pi/2$ we say that the quadrilateral cocommutes. This notion of commutativity coincides with the notion of commuting square in the subfactor literature, which means that the two orders of taking conditional expectations around the square commute [89, 90].

In [39, Theorem 3.21] it is shown that an irreducible quadrilateral of finite-index subfactors $N \subseteq P, Q \subseteq M$ commutes iff the $N-N$ bimodule map from $P \otimes_N Q$ to M given by multiplication is injective, and cocommutes iff this map is surjective. In particular, for a commuting and cocommuting quadrilateral, we have $P \otimes_N Q \cong M$ as $N-N$-bimodules, and the indices of all four sides of the quadrilateral are the same. For a non-commuting quadrilateral, the complements of N in the $N-N$ bimodules P and Q must contain a common subobject.

In [36, Theorem 4.3] it is shown that a noncommuting, cocommuting $(3,3)$-supertransitive quadrilateral necessarily satisfies $[M : P] = [M : Q] = [P : N] - 1 = [Q : N] - 1$ (in fact a slightly weaker assumption is sufficient). Moreover, the Galois group of the inclusion $N \subseteq M$ is necessarily a subgroup of the symmetric group $\mathcal{S}_3$ (with equality only realized for the fixed point subfactor of an outer $\mathcal{S}_3$ action). Therefore, such quadrilaterals were divided into three cases (called Classes II, III, and IV, while Class I referred to non-cocommuting) based on whether the Galois group is trivial, cyclic of order 2, or cyclic of order 3.

Here we have:

- A quadrilateral from $O_{33}^6 = U$ which follows from $X \otimes X \cong U \cong \overline{X} \otimes \overline{X}$. This quadrilateral is commuting (since $X \otimes \bar{X} = ... \ncong ... = \bar{X} \otimes X$) and cocommuting (since it is self-dual).
- A quadrilateral from $O_{22}^8 = f^{(6)}$ which follows from $A \otimes A \cong f^{(6)} \cong f^{(6)} \cong B \otimes B$ in $\mathcal{EH}_2$. This quadrilateral is also commuting and cocommuting, for the same reason as above.
- A quadrilateral from O_{23}^5. This is a $(2,2)$-supertransitive quadrilateral where the upper subfactors are the index $14.3305\ldots$ ones from Equation (2.4) and the lower subfactors are the index $13.3305\ldots$ from Equation (2.1). This quadrilateral again commutes, but does not cocommute since the indices are not equal.
- A quadrilateral from O_{34}^2. This is a $(3,3)$-supertransitive commuting but noncocommuting quadrilateral where the upper subfactors are the index $8.0283\ldots$ ones from Equation (2.5) and the lower subfactors are the index $7.0283\ldots$ from Equation (2.3).

The (dual quadrilateral of) fourth example fits into Class II (the Galois group is trivial since there are no nontrivial invertible objects), and the third example is similar, although only $(2,2)$-supertransitive. We expect that the fourth example is the smallest quadrilateral in Class II with indices above 4. Note that the smallest example with indices above 4 in Class III comes from the Asaeda-Haagerup subfactor, while the smallest example with indices above 4 in Class IV comes from the Haagerup subfactor. It is striking that Extended Haagerup appears to be the smallest example in the remaining case.

3. Module categories and graph planar algebra embeddings

The graph planar algebra embedding theorem from [62] states that any subfactor planar algebra embeds in the graph planar algebra [57] of either of its principal graphs. Peters observed in [86] that it is possible for a subfactor planar algebra to embed in the graph planar algebra of other graphs; in particular she found that the Haagerup planar algebra embeds in the graph planar algebra of a third graph, called the "broom." In this section we strengthen the graph planar algebra embedding theorem, to obtain a classification of embeddings in graph planar algebras. In particular, we show that a subfactor planar algebra embeds into the graph planar algebra of a bipartite graph *if and only if* the graph is the fusion graph of a unitary module category with a compatible trace.

We begin with the simple observation that a module category $\mathcal{M}$ for a tensor category $\mathscr{C}$ is exactly the data of a tensor functor $\mathscr{C} \to \mathrm{End}(\mathcal{M})$. As we proceed through this section, we elaborate this observation in various directions, eventually obtaining our theorem. This involves four adjustments:

— describing endofunctors in $\mathrm{End}(\mathcal{M})$ as graphs,

— adapting to the shaded setting required for subfactor planar algebras,

— working in the unitary setting, and

— understanding the additional data corresponding to pivotal structures.

Note that in order to be able to characterize maps out of the Extended Haagerup subfactor planar algebra (and hence characterize modules for these fusion categories), we will rely on the unitary pivotal structure (see Remark 5.6). Thus even if the reader is only interested in the algebraic classification of modules over the Extended Haagerup fusion categories, he/she still needs to understand the unitary pivotal version of the graph planar algebra embedding theorem!

We will assume that the reader is familiar with tensor categories following [21], but we will not assume previous familiarity with graph planar algebras. We take this pedagogical approach for several reasons. First, it was this algebraic perspective that allowed us to see that one should expect a GPA embedding theorem for modules. Thus this approach unifies (unitary) module category classification results like [25, 16] and GPA embedding constructions like [58, 86, 6], which will hopefully make GPA embeddings more accessible to algebraists. Second, an independent purely subfactor theoretic proof of our classification of embeddings recently appeared in [10], using towers of algebras. Subfactor experts may prefer to read that paper as a replacement for this section.

Here is a more detailed breakdown of this section. In §3.1 we discuss Cauchy completeness, especially in the context of C*-categories. In §3.2 we recall some background on module categories, Morita equivalences, and the endofunctor embedding theorem (that giving a module category $\mathcal{M}$ over $\mathscr{C}$ is the same as giving a functor $\mathscr{C} \to \mathrm{End}(\mathcal{M})$). In §3.3 we introduce an "unbiased" definition of monoidal categories which we call monoidal algebras in the spirit of [99, 105] (see Remark 3.25). Monoidal algebras are an analogue of planar algebras without rotational symmetry. In Section 3.4 we introduce the graph monoidal algebra (which is an analogue of the graph planar algebra), explain its relationship to $\mathrm{End}(\mathcal{M})$, and see that the endomorphism embedding theorem yields a graph monoidal algebra embedding theorem for module categories. This is the simplest non-technical version of our main result, and contains the major idea of this section.

The rest of the section is dedicated to adapting the graph monoidal algebra embedding theorem for module categories to the pivotal and unitary pivotal settings where it becomes the graph planar algebra embedding theorem for appropriate pivotal and unitary pivotal analogues of module categories. These analogues of module categories involve both *structure* on $\mathcal{M}$ and *compatibility* of that structure with the module action. In the semisimple pivotal setting Schaumann [96] showed that the appropriate structure is a choice of trace on $\mathcal{M}$. In §3.5, we recall the definitions of planar algebra, unitary dual functors, and unitary pivotal structure, and we explain the relationship between planar algebras and unitary pivotal fusion categories. In §3.6, we recall Schaumann's notion of trace and modify this notion to the unitary setting, we then define (unitary) pivotal modules, prove a (unitary) pivotal analogue of endofunctor embedding, and translate that into the desired graph planar algebra embedding theorem.

3.1. Cauchy complete categories

In this paper we focus on $\mathbb{C}$-linear Cauchy complete categories. Here $\mathbb{C}$-*linear* means the hom spaces are finite dimensional $\mathbb{C}$-vector spaces and composition is bilinear. A $\mathbb{C}$-linear category $\mathscr{C}$ is *Cauchy complete* if it has direct sums of objects and all idempotents split (i.e., if $e : c \to c$ is *idempotent*, i.e., $e^2 = e$, then c has a corresponding direct sum decomposition). Equivalently, $\mathscr{C}$ has all absolute colimits.

Cauchy completeness is a mild condition to impose on a category, because every $\mathbb{C}$-linear category $\mathscr{C}$ has a *Cauchy completion* $\mathsf{Cauchy}(\mathscr{C})$. This completion is built in two stages: first, take the additive completion (where objects are formal direct sums of objects and morphisms are formal matrices of morphisms), and then take the idempotent (also called Karoubi) completion (where objects are pairs of an object and an idempotent, and morphisms make the obvious square commute). A category $\mathscr{C}$ is Cauchy complete if and only if the obvious inclusion $\mathscr{C} \hookrightarrow \mathsf{Cauchy}(\mathscr{C})$ is an equivalence. The Cauchy completion satisfies the universal property that every linear functor $F : \mathscr{C} \to \mathcal{D}$ where $\mathcal{D}$ is Cauchy complete factors uniquely through the Cauchy completion $\mathsf{Cauchy}(\mathscr{C})$. Furthermore, the Cauchy completion of a monoidal category inherits a natural monoidal structure. See [11, §2.5-2.6] for more details.

A Cauchy complete category is *semisimple* if every object's endomorphism algebra is semisimple. This is equivalent to the definition in [75, §2.1]. A semisimple category has a collection of simple objects with the properties that

- the simple objects satisfy Schur's lemma, i.e., for two simple objects $a, b \in \mathscr{C}$, either a and b are isomorphic, or $\mathscr{C}(a \to b) = (0) = \mathscr{C}(b \to a)$, and
- every object is isomorphic to a direct sum of simple objects.

Definition 3.1. – A *multitensor category* over $\mathbb{C}$ is a $\mathbb{C}$-linear Cauchy complete category together with a linear tensor product functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, a unit object $1_{\mathscr{C}}$, and unitors and associators satisfying natural axioms, where every object X has both left and right duals ${}^{\vee}X$ and $X^{\vee}$. We call $\mathscr{C}$ a *tensor category* if $1_{\mathscr{C}}$ is simple. A *multifusion category* is a semisimple multitensor category with finitely many isomorphism classes of simples, and a *fusion category* is a multifusion category with $1_{\mathscr{C}}$ simple.

WARNING 3.2. – Note that our definition of multitensor category does not agree with that of [21], because we work in the Cauchy complete setting rather than the locally-finite abelian setting. Our definition agrees with the notion of multi-pseudo-tensor category in [12]. However, our main results are in the semisimple setting, and semisimple Cauchy complete categories are automatically abelian and thus tensor in the usual sense. Working in the Cauchy complete setting allows for a cleaner exposition without adding extra adjectives, like semisimple, to theorems which do not require extra assumptions in the Cauchy complete setting. One should take care adapting these techniques to the non-semisimple abelian setting.

DEFINITION 3.3. – Suppose $\mathscr{C}$ is a multitensor category. We say a set of objects $\mathscr{S} := \{X_s\}_{s\in S} \subset \mathscr{C}$ *tensor generates* $\mathscr{C}$ if every object of $\mathscr{C}$ is isomorphic to a direct summand of a direct sum of tensor products of objects in $\mathscr{S}$.

In this case, we define $\mathscr{C}_{\mathscr{S}}$ as the full monoidal subcategory of $\mathscr{C}$ whose objects are tensor products of objects in $\mathscr{S}$. Observe that the Cauchy completion of $\mathscr{C}_{\mathscr{S}}$ is equivalent to our original multitensor category $\mathscr{C}$.

3.1.1. *Cauchy complete* C*-*categories*

DEFINITION 3.4. – A *dagger structure* on a $\mathbb{C}$-linear category $\mathscr{C}$ consists of an antilinear map $\dagger : \mathscr{C}(a \to b) \to \mathscr{C}(b \to a)$ for all $a, b \in \mathscr{C}$ satisfying $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$ for all composable morphisms f, g, and $f^{\dagger\dagger} = f$ for all morphisms f. The pair $(\mathscr{C}, \dagger)$ is called a *dagger category*.

Following [74, Prop. 2.1], a dagger category (with finite dimensional hom spaces) is called a C*-*category* if

— (positive definite) for every $f \in \mathscr{C}(a \to b)$, $f^\dagger \circ f = 0$ implies $f = 0$.

By Roberts' 2×2 trick [32, Lem. 2.6], the positive definite condition above is equivalent to

— (2×2 linking C*-algebras) for all $a, b \in \mathscr{C}$, the *linking algebra*

$$\mathcal{L}(a, b) := \begin{pmatrix} \mathscr{C}(a \to a) & \mathscr{C}(b \to a) \\ \mathscr{C}(a \to b) & \mathscr{C}(b \to b) \end{pmatrix}$$

with the obvious matrix multiplication and $\dagger$-transpose operation is a finite dimensional C*-algebra. (3)

REMARK 3.5. – Since finite dimensional C*-algebras are semisimple, Cauchy complete C*-categories (which we assume here have finite dimensional hom spaces) are automatically semisimple.

Starting with a C*-category $\mathscr{C}$, its Cauchy completion $\mathsf{Cauchy}(\mathscr{C})$ is not a C*-category, as not all idempotents are orthogonal projections. However, we may take the *unitary* Cauchy completion $\mathsf{Cauchy}^\dagger(\mathscr{C})$ of $\mathscr{C}$. Again, this completion is built in two stages.

(3) Being a C* algebra is a property of a complex $*$-algebra and not extra structure. Indeed, every C* algebra has a unique C* norm, which can be recovered from the spectral radius, which is defined purely algebraically.

1. First, we take the orthogonal additive completion, which has objects formal orthogonal direct sums of objects and morphisms formal matrices of morphisms. Here, an object $\bigoplus_{i=1}^n c_i$ with isometries $v_j : c_j \to \bigoplus_{i=1}^n c_i$ is called the *orthogonal direct sum* of $c_1, \dots, c_n$ if $\sum_{i=1}^n v_i v_i^\dagger = \mathrm{id}_{\bigoplus_{i=1}^n c_i}$ and $v_j^\dagger v_j = \mathrm{id}_{c_j}$ for all j.
2. Second, we take the orthogonal idempotent completion, where objects are pairs of an object $c \in \mathcal{C}$ and an orthogonal projection $p \in \mathcal{C}(c \to c)$ (satisfying $p^2 = p = p^\dagger$), and morphisms make the obvious square commute.

Observe that $\mathsf{Cauchy}^\dagger(\mathcal{C})$ has the structure of a C*-category where † is given by the †-transpose operation. We say that a C*-category $\mathcal{C}$ is *unitarily Cauchy complete* if the obvious inclusion dagger functor $\mathcal{C} \hookrightarrow \mathsf{Cauchy}^\dagger(\mathcal{C})$ is a dagger equivalence. The unitary Cauchy completion satisfies the universal property that every linear dagger functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is unitarily Cauchy complete factors uniquely throught $\mathsf{Cauchy}^\dagger(\mathcal{C})$.

REMARK 3.6. – It is natural to ask whether given a C*-category $\mathcal{C}$, if we take the Cauchy completion of the underlying linear category $\mathcal{C}^\natural$, is this equivalent to the underlying category of the unitary Karoubi completion $\mathsf{Cauchy}^\dagger(\mathcal{C})^\natural$? This is indeed the case, but we leave the verification to a future article. The interested reader can prove this fact using the polar decomposition for invertible morphisms in $\mathcal{C}$.

3.2. Module categories, Morita equivalences, and endofunctor embedding

Tensor categories can be thought of as categorical analogues of ordinary algebras. Many ordinary algebraic notions have analogues for tensor categories, and in particular the analogues of modules, bimodules, and Morita equivalences play a key role in studying tensor categories, as pioneered by Ocneanu, Müger, Ostrik, and others [79, 75, 80]. For example, a left $\mathcal{C}$-module category $\mathcal{M}$ is a $\mathbb{C}$-linear Cauchy complete category together with a left action functor $\triangleright$ and unitor and associators satisfying natural axioms. Similarly, a right $\mathcal{C}$-module category has a right action functor $\triangleleft$, and a $\mathcal{C} - \mathcal{C}$ bimodule category has two actions and an associator commuting both actions. Again see [21, §7] for further details (changing abelian to Cauchy complete throughout).

In particular, we have the following two important problems about the "representation theory" of fusion categories.

PROBLEM 3.7 (Classification of Modules). – Classify all indecomposable semisimple module categories over a given fusion category $\mathcal{C}$.

PROBLEM 3.8 (Morita Equivalence). – Classify all fusion categories $\mathcal{D}$ (up to tensor equivalence) which are Morita equivalent to $\mathcal{C}$, and all the Morita equivalences between them (up to bimodule equivalence). Furthermore, understand the Brauer-Picard groupoid, which describes the compositions of these Morita equivalences under balanced tensor product ${}_{\mathcal{C}}\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}_{\mathcal{E}}$.

From a higher categorical perspective it is somewhat unnatural to only study equivalence classes, and it is more natural to consider Etingof-Nikshych-Ostrik's Brauer-Picard 3-groupoid [22] which consists of fusion categories Morita equivalent to $\mathcal{C}$, Morita equivalences between them, bimodule equivalences between these Morita equivalences, and bimodule natural isomorphisms. The higher structure of this 3-groupoid is essential for classifying G-graded extensions of fusion categories. The Morita equivalence problem asks for the fundamental 1-groupoid of this 3-groupoid. As it turns out, for the examples considered in this paper, the higher structure of the 3-groupoid is trivial (see Corollary 4.7).

The following theorem shows that the two problems above are closely related. Recall that if X is an invertible object, then conjugation by X is an inner autoequivalence.

THEOREM 3.9 ([22, Prop. 4.2 and §4.3], [21, §7.12]). – *If $\mathcal{C}$ is a fusion category and $\mathcal{M}$ is a semisimple $\mathcal{C}$-module category, then $\mathcal{C}-\mathcal{D}$ bimodule category structures on $\mathcal{M}$ which extend the $\mathcal{C}$-module structure correspond exactly to functors $\mathcal{F}:\mathcal{D}\to \mathrm{End}_{\mathcal{C}}(\mathcal{M})$, and such a bimodule is a Morita equivalence if and only if $\mathcal{F}$ is an equivalence of multitensor categories. Two such bimodule categories are equivalent if and only if the functors differ by an inner autoequivalence. Furthermore, $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$ is a tensor category (with simple unit object) if and only if $\mathcal{M}$ is indecomposable.*

In particular, in order to solve Problem 3.8 about Morita equivalence, it is enough to solve Problem 3.7 about modules, and further solve the following.

PROBLEM 3.10 (Outer Automorphisms). – For each $\mathcal{D}$ in the Morita equivalence class of $\mathcal{C}$, find the outer automorphism group of $\mathcal{D}$.

None of the fusion categories we study in this article have outer automorphisms. Thus classifying modules and Morita equivalences are essentially the same. However, the reader should note that given $\mathcal{C}$ and $\mathcal{M}$, actually calculating the structure of the dual category $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$ may be quite difficult. The dual categories $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$ are essentially the same thing as the dual parts of GHJ subfactors [35]. We refer the reader to [64] for a notable concrete example where understanding the detailed structure of the dual category is difficult.

All of the above problems are "external" problems, relating $\mathcal{C}$ to other tensor categories and module categories. However, they are closely related by a theorem of Ostrik [80] to "internal" problems about algebra objects or Q-systems inside $\mathcal{C}$. Two such algebras are internally Morita equivalent if there is an invertible bimodule object between them.

THEOREM 3.11 ([80]). – *Given $A\in\mathcal{C}$, a connected semisimple algebra, $\mathsf{Mod}_{\mathcal{C}}(A)$ is an indecomposable module category. Moreover every indecomposable $\mathcal{C}$-module category $\mathcal{M}$ is equivalent to one of this form, by taking $A=\underline{\mathrm{End}}_{\mathcal{C}}(m)$ for any simple $m\in\mathcal{M}$.*

The collection of connected semisimple algebras $\{B \mid \mathsf{Mod}_{\mathcal{C}}(B)\cong \mathsf{Mod}_{\mathcal{C}}(A)\}$ is exactly the internal Morita equivalence class of A.

The dual category $\mathrm{End}_{\mathcal{C}}(\mathsf{Mod}_{\mathcal{C}}(A))$ is canonically identified with the category of $A-A$ bimodules in $\mathcal{C}$.

This theorem shows that the above problems are closely related to Ocneanu's "maximal atlas" [79].

DEFINITION 3.12. – Let $\mathscr{C}$ be a fusion category. A *maximal atlas* for $\mathscr{C}$ is a choice of a semisimple connected algebra A in each internal Morita equivalence class. From such a maximal atlas, one gets a collection of fusion categories $\mathsf{Bim}_{\mathscr{C}}(A, A)$ and Morita equivalences $\mathsf{Bim}_{\mathscr{C}}(A, B)$. [(4)]

In general, a maximal atlas will contain less information than the Brauer-Picard groupoid, because it does not remember the tensor equivalences between the fusion categories $\mathsf{Bim}_{\mathscr{C}}(A, A)$

EXAMPLE 3.13. – For $\mathscr{C} = \mathsf{Vec}(\mathbb{Z}/3\mathbb{Z})$, a maximal atlas is given by 1 and the group algebra $A = \mathbb{C}[\mathbb{Z}/3\mathbb{Z}]$ (with each group element in its own grade). The category of bimodules $\mathsf{Bim}_{\mathscr{C}}(A, A)$ is $\mathsf{Rep}(\mathbb{Z}/3\mathbb{Z})$, which is (non-canonically!) equivalent to $\mathsf{Vec}(\mathbb{Z}/3\mathbb{Z})$. The outer automorphism group of $\mathsf{Vec}(\mathbb{Z}/3\mathbb{Z})$ is the group of units $(\mathbb{Z}/3\mathbb{Z})^\times$ acting by permuting simple objects, so we get two distinct equivalences $\mathsf{Rep}(\mathbb{Z}/3\mathbb{Z}) \cong \mathsf{Vec}(\mathbb{Z}/3\mathbb{Z})$. Thus the aforementioned maximal atlas of $\mathscr{C}$ consists of two tensor categories (which happen to be tensor equivalent) and a single bimodule between the two, while the Brauer-Picard groupoid consists of one tensor category and four Morita autoequivalences.

One can then determine the group structure of this set of four autoequivalences. By a result of Etingof-Nikshych-Ostrik [22, Cor. 1.2], this Brauer-Picard group must be the split orthogonal group $O_2(\mathbb{F}_3 \oplus \mathbb{F}_3^*)$, which is the Klein four group. Note that in the maximal atlas formalism one cannot even ask about the structure of this group. In a sense the maximal atlas is a "universal cover" of the Brauer-Picard groupoid, and has lost all the interesting topological information about the latter (while still retaining the combinatorial information). However, for all examples in this article, the Brauer-Picard group is trivial, and so these subtleties between the Brauer-Picard groupoid and the maximal atlas do not play an important role. (In contrast, this distinction was critical in the study of the Asaeda-Haagerup subfactor [43], which has Brauer-Picard group the Klein four group.)

Just as a module M over an algebra A is equivalent to a homomorphism $A \to \mathrm{End}(M)$, module categories $\mathcal{M}$ over $\mathscr{C}$ are equivalent to tensor functors $\mathscr{C} \to \mathrm{End}(\mathcal{M})$ [21, Prop. 7.1.3.]. Thus the module classification problem is equivalent to the following.

PROBLEM 3.14 (Endofunctor embedding). – Classify all semisimple categories $\mathcal{M}$ and all tensor functors $\mathscr{C} \to \mathrm{End}(\mathcal{M})$, up to conjugation by an autoequivalence of $\mathcal{M}$.

The following omnibus theorem summarizes much of the above.

THEOREM 3.15. – *Suppose that $\mathscr{C}$ is a fusion category.*

- *Module category structures on a semisimple category $\mathcal{M}$ correspond exactly to tensor functors $\mathscr{C} \to \mathrm{End}(\mathcal{M})$.*
- *A fusion category $\mathcal{D}$ is Morita equivalent to $\mathscr{C}$ if and only if there is an indecomposable semisimple $\mathscr{C}$-module category $\mathcal{M}$ such that $\mathcal{D}$ is tensor equivalent to $\mathrm{End}_{\mathscr{C}}(\mathcal{M})$. Furthermore the Morita equivalences ${}_{\mathscr{C}}\mathcal{N}_{\mathcal{D}}$ such that ${}_{\mathscr{C}}\mathcal{N}$ is equivalent to ${}_{\mathscr{C}}\mathcal{M}$ are a torsor for the group of outer automorphisms $\mathrm{Out}(\mathcal{D})$.*

[(4)] The distinction between thinking of the maximal atlas as a collection of algebras and bimodules or as a collection of tensor categories and Morita equivalences is often elided.

— *Pairs* $(\mathcal{M}, m)$, *where* $\mathcal{M}$ *is a semisimple indecomposable* $\mathscr{C}$*-module category and* $m \in \mathcal{M}$ *is a simple object, correspond exactly to connected semisimple algebras* A *in* $\mathscr{C}$, *via* $A \mapsto (\mathsf{Mod}_{\mathscr{C}}(A), A)$ *and* $(\mathcal{M}, m) \mapsto \underline{\mathrm{End}}_{\mathscr{C}}(m)$. *The dual category* $\mathrm{End}_{\mathscr{C}}(\mathcal{M})$ *corresponding to* A *is the category of* A*-*A *bimodules in* $\mathscr{C}$.

3.2.1. *Modules for multifusion categories.* – Recall that a multifusion category $\mathscr{C}$ is like a fusion category, except $1_{\mathscr{C}}$ is no longer simple. Since $\mathscr{C}$ is semisimple and $\mathscr{C}(1_{\mathscr{C}} \to 1_{\mathscr{C}})$ is a commutative algebra, $1_{\mathscr{C}}$ breaks up as a sum of r distinct simple objects $1_{\mathscr{C}} = \bigoplus_{i=1}^{r} 1_i$. We call such a multifusion category r-*shaded*. We denote by $\mathscr{C}_{ij}$ the summand $1_i \otimes \mathscr{C} \otimes 1_j$.

PROPOSITION 3.16. – *If* $\mathscr{C}$ *is an* r*-shaded multifusion category, then each* $\mathscr{C}_{ii}$ *is a fusion category. When* $\mathscr{C}$ *is indecomposable as a multifusion category each* $\mathscr{C}_{ij}$ *is a Morita equivalence between* $\mathscr{C}_{ii}$ *and* $\mathscr{C}_{jj}$. *Furthermore, the tensor product map* $\mathscr{C}_{ij} \boxtimes_{\mathscr{C}_{jj}} \mathscr{C}_{jk} \to \mathscr{C}_{ik}$ *is an equivalence.*

Conversely, given fusion categories $\mathcal{D}_{11}, \dots, \mathcal{D}_{rr}$ *and a Morita equivalence* $\mathcal{D}_{1j}$ *between* $\mathcal{D}_{11}$ *and* $\mathcal{D}_{jj}$ *for each* $1 < j \le r$, *we define* $\mathcal{D}_{ik} := \mathcal{D}_{1i}^{-1} \boxtimes_{\mathcal{D}_{11}} \mathcal{D}_{1k}$ *for each* $i, k \in \{1, \dots, r\}$ *to get an indecomposable multifusion category* $\mathcal{D} = \bigoplus_{i,k=1}^{r} (\mathcal{D}_{ik})$. *These constructions are mutually inverse.* (5)

Proof. – The forward direction is [22, Thm. 6.1] where instead of a grading group we have a grading by the groupoid of standard matrix units E_{ij}. The proof for groupoids is parallel to the proof for groups. (See also [21, Prop. 7.17.5] which shows the first two parts.)

(5) One can avoid the relative tensor product to obtain a multifusion category equivalent to $\mathcal{D}$ as follows. First, choose a simple object $d_i \in \mathcal{D}_{1i}$ for all $i = 1, \dots, r$, and consider the connected algebra objects $A_i = \underline{\mathrm{End}}_{\mathcal{D}_{11}}(d_i)$. Then $\mathcal{D}$ is equivalent to the category of $A{-}A$ bimodules internal to $\mathcal{D}_{11}$ where $A = \bigoplus_{i=1}^{r} A_i$.

For the converse direction $\bigoplus_{i,k=1}^r \mathcal{D}_{1i}^{-1} \boxtimes_{\mathcal{D}_{11}} \mathcal{D}_{1k}$ has a monoidal structure given by

$$\begin{aligned}\left(\mathcal{D}_{1i}^{-1} \boxtimes_{\mathcal{D}_{11}} \mathcal{D}_{1k}\right) \boxtimes \left(\mathcal{D}_{1k}^{-1} \boxtimes_{\mathcal{D}_{11}} \mathcal{D}_{1\ell}\right) &\to \mathcal{D}_{1i}^{-1} \boxtimes_{\mathcal{D}_{11}} \mathcal{D}_{1k} \boxtimes_{\mathcal{D}_{kk}} \mathcal{D}_{1k}^{-1} \boxtimes_{\mathcal{D}_{11}} \mathcal{D}_{1\ell}\\ &\to \mathcal{D}_{1i}^{-1} \boxtimes_{\mathcal{D}_{11}} \mathcal{D}_{11} \boxtimes_{\mathcal{D}_{11}} \mathcal{D}_{1\ell} \to \mathcal{D}_{1i}^{-1} \boxtimes_{\mathcal{D}_{11}} \mathcal{D}_{1\ell}\end{aligned}$$

That this category is rigid follows from the proof of [14, Thm. 8.5], namely letting $\mathcal{D}_{jk} = \mathcal{D}_{1j}^{-1} \boxtimes_{\mathcal{D}_{11}} \mathcal{D}_{1k}$ we have that tensoring with an object $x \in \mathcal{D}_{jk}$ thought of as a functor $\mathcal{D}_{jj} \to \mathcal{D}_{jk}$ has both a left and a right adjoint module functor which lives in $\mathrm{Hom}_{\mathcal{D}_{jj}}(\mathcal{D}_{jk}, \mathcal{D}_{jj}) \cong \mathcal{D}_{kj}$ using invertibility. Thus tensoring with x is adjoint to tensoring with some other object $y \in \mathcal{D}_{kj}$ which is thus its dual object. (See [18, Cor. 2.11] for an alternate proof, though some care needs to be taken to adapt the notion of weak rigidity to the multifusion setting.) ☐

REMARK 3.17. – The right way to think about Proposition 3.16 is that the main results of [21] classifying group extensions using obstruction theory also work for "groupoid extensions." Here we're looking at extensions by a trivial groupoid, so the obstructions automatically vanish and the extension must exist. The proofs in that paper go through for groupoids with minimal changes.

REMARK 3.18. – To each multifusion category $\mathscr{C}$ there is a corresponding rigid 2-category whose objects are the indices, whose 1-morphisms are the objects in the $\mathscr{C}_{ij}$, and whose 2-morphisms are the morphisms in $\mathscr{C}_{ij}$. There is not an important difference between this 2-category and the multifusion category, but in this paper, we use the multifusion language to align with the results of [21, 83].

The non-pivotal algebraic analogue of an irreducible finite depth subfactor $N \subset M$ is a pair $(\mathscr{C}, A)$ where $\mathscr{C}$ is a fusion category (which corresponds to the $N-N$ bimodules generated by M) and $A \in \mathscr{C}$ is a semisimple connected algebra object (which corresponds to M). Given such an $A \in \mathscr{C}$, we get a Morita equivalence $\mathsf{Mod}_{\mathscr{C}}(A) = \mathsf{Bim}_{\mathscr{C}}(1_{\mathscr{C}}, A)$ between $\mathscr{C}$ and $\mathsf{Bim}_{\mathscr{C}}(A)$, and we get a 2-shaded indecomposable multifusion category as in Footnote 5 by

$$\mathsf{Bim}_{\mathscr{C}}(1_{\mathscr{C}} \oplus A, 1_{\mathscr{C}} \oplus A) = \begin{pmatrix} \mathsf{Bim}_{\mathscr{C}}(1_{\mathscr{C}}, 1_{\mathscr{C}}) & \mathsf{Bim}_{\mathscr{C}}(1_{\mathscr{C}}, A) \\ \mathsf{Bim}_{\mathscr{C}}(A, 1_{\mathscr{C}}) & \mathsf{Bim}_{\mathscr{C}}(A, A) \end{pmatrix}$$

with tensor product given by $\otimes_{\mathscr{C}}$, $\otimes_A$, or zero as appropriate.

Notice that $\mathsf{Bim}_{\mathscr{C}}(1_{\mathscr{C}}, 1_{\mathscr{C}}) = \mathscr{C}$.

There's an analogue of Prop. 3.16 for module categories.

PROPOSITION 3.19. – *Suppose that $\mathcal{D}$ is an r-shaded multifusion category with components $\mathcal{D}_{ij}$. Suppose that $\mathcal{M}$ is an indecomposable module category over $\mathcal{D}$. Then $\mathcal{M} = \bigoplus_{j=1}^r \mathcal{M}_j$ where $\mathcal{M}_i = 1_i \rhd \mathcal{M}$. Furthermore, the action maps $\mathcal{D}_{ij} \boxtimes_{\mathcal{D}_{jj}} \mathcal{M}_j \to \mathcal{M}_i$ are equivalences.*

Conversely, given an indecomposable module category $\mathcal{M}_1$ over $\mathcal{D}_{11}$, we define

$$\mathcal{F}(\mathcal{M}_1) := \bigoplus_{i=1}^r \mathcal{D}_{i1} \boxtimes_{\mathcal{D}_{11}} \mathcal{M}_1.$$

We can endow $\mathcal{F}(\mathcal{M}_1)$ with the structure of a $\mathcal{D}$-module category via

$$\mathcal{D}_{kj} \boxtimes \mathcal{F}(\mathcal{M}_1) \cong \bigoplus_{i=1}^{r} \mathcal{D}_{kj} \boxtimes \mathcal{D}_{i1} \boxtimes_{\mathcal{D}_{11}} \mathcal{M}_1 \to \mathcal{D}_{kj} \boxtimes_{\mathcal{D}_{jj}} \mathcal{D}_{j1} \boxtimes_{\mathcal{D}_{11}} \mathcal{M}_1 \cong \mathcal{D}_{k1} \boxtimes_{\mathcal{D}_{11}} \mathcal{M}_1 \subseteq \mathcal{F}(\mathcal{M}_1).$$

These constructions are mutually inverse with the isomorphism $\mathcal{F}(\mathcal{M}_1) \to \mathcal{M}$ being the direct sum of the action maps $\mathcal{D}_{i1} \boxtimes_{\mathcal{D}_{11}} \mathcal{M}_1 \cong \mathcal{M}_i$.

Proof. – The only nontrivial step is that $\mathcal{D}_{ij} \boxtimes_{\mathcal{D}_{jj}} \mathcal{M}_j \to \mathcal{M}_i$ is an equivalence. This follows either by the techniques of [22, Thm. 6.1] or of [21, Prop. 7.17.5]. Choose a simple object m_j in $\mathcal{M}_j$, and let $B_j = \underline{\mathrm{End}}_{\mathcal{D}_{jj}}(m_j)$ be the internal endomorphisms of m_j in $\mathcal{D}_{jj}$. Similarly, choose a simple object x_{ij} in $\mathcal{D}_{ij}$ and let $A_{ij} = {}^{\vee}x_{ij} \otimes x_{ij}$ be its internal endomorphism algebra in $\mathcal{D}_{jj}$. We have an equivalence:

$$\mathcal{M} \to \mathsf{Bim}_{\mathcal{D}}(A_{ij}, B_j)$$

via $m \mapsto {}^{\vee}x_{ij} \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(m, m_j)$. The restriction of this functor to $\mathcal{M}_i$ then gives an inverse to the map $\mathcal{D}_{ij} \boxtimes_{\mathcal{D}_{jj}} \mathcal{M}_j \to \mathcal{M}_i$. □

Thus classifying modules for $\mathcal{D}_{11}$ (answering Problem 3.7) is equivalent to classifying modules for $\mathcal{D}$. In particular, given an algebraic analogue of a subfactor $A \in \mathscr{C}$, we can instead solve the module problem over the corresponding indecomposable 2-shaded multifusion category $\mathsf{Bim}_{\mathscr{C}}(1_{\mathscr{C}} \oplus A, 1_{\mathscr{C}} \oplus A)$ which is the purely algebraic, non-pivotal analogue of the subfactor planar algebra. That is, we construct module categories for $\mathcal{EH}_1$ by constructing module categories over the indecomposable 2-shaded multifusion category which combines $\mathcal{EH}_1$ and $\mathcal{EH}_2$. This strategy is successful because the extended Haagerup subfactor planar algebra has a better skein theoretic description than either of the fusion categories $\mathcal{EH}_1$ and $\mathcal{EH}_2$ individually.

3.2.2. *Module* C* *categories for unitary multitensor categories*. – In the nomenclature of [83], a *unitary multitensor category* $\mathscr{C}$ is a Cauchy complete rigid tensor C* category, which is semisimple by [69]. We call $\mathscr{C}$ a *unitary tensor category* if $1_{\mathscr{C}}$ is simple. Similar to the above characterization of module categories, given a (Cauchy complete) C* category $\mathcal{M}$, endowing $\mathcal{M}$ with the structure of a $\mathscr{C}$-module C* category is equivalent to supplying a dagger tensor functor $\mathscr{C} \to \mathrm{End}^{\dagger}(\mathcal{M})$, the C* category of dagger endofunctors of $\mathcal{M}$.(6) We provide a proof for those less familiar with C* categories, which also appears as [15, Lem. A.4.1]. This is the C* version of the first bullet point in Theorem 3.15.

LEMMA 3.20. – *Suppose $\mathscr{C}$ is a unitary multitensor category and $\mathcal{M}$ is a* C* *category. Equipping $\mathcal{M}$ with the structure of a $\mathscr{C}$-module* C* *category is equivalent to supplying a dagger tensor functor* $(\Psi, \mu) : \mathscr{C} \to \mathrm{End}^{\dagger}(\mathcal{M})$.

Proof. – We show how each structure induces the other, and we leave it to the reader to check these two processes are mutually inverse (up to dagger equivalence).

Suppose $\mathcal{M}$ is a $\mathscr{C}$-module dagger category. Note that $c \rhd -$ is a dagger functor in $\mathrm{End}^{\dagger}(\mathcal{M})$ for each $c \in \mathscr{C}$. Moreover, if $f \in \mathscr{C}(a \to b)$, then $(f \rhd -)^{\dagger} = f^{\dagger} \rhd -$.

(6) In order for $\mathrm{End}^{\dagger}(\mathcal{M})$ to be C*, we only work with bounded natural transformations, i.e., those $\theta : F \Rightarrow G$ such that $\sup_{c \in \mathscr{C}} \|\theta_c\| < \infty$. One then defines $\theta^{\dagger}$ component-wise: $(\theta^{\dagger})_c := (\theta_c)^{\dagger}$.

Hence $\Psi : \mathscr{C} \to \mathrm{End}^\dagger(\mathcal{M})$ given by $\Psi(c) = c \triangleright -$ and $\Psi(f) = f \triangleright -$ defines a dagger functor. Now defining

$$\mu^{a,b} : \Psi(a) \circ \Psi(b) = a \triangleright b \triangleright - \Rightarrow a \otimes b \triangleright - = \Psi(a \otimes b)$$

by $\mu_m^{a,b} := \alpha_{a,b,m} : a \triangleright b \triangleright m \to a \otimes b \triangleright m$ defines a unitary natural isomorphism, equipping Ψ with the structure of a dagger tensor functor.

Conversely, suppose $(\Psi, \mu) : \mathscr{C} \to \mathrm{End}^\dagger(\mathcal{M})$ is a dagger tensor functor. For $c \in \mathscr{C}$ and $m \in \mathcal{M}$, define $c \triangleright m := \Psi(c)(m)$. For $c \in \mathscr{C}$ and $g \in \mathcal{M}(m \to n)$, define $\mathrm{id}_c \triangleright g := \Psi(c)(f)$. For $f \in \mathscr{C}(a \to b)$ and $m \in \mathcal{M}$, define $f \triangleright \mathrm{id}_m := \Psi(f)_m$. To show that $\triangleright$: $\mathscr{C} \times \mathcal{M} \to \mathcal{M}$ defines a bifunctor, it suffices to prove the exchange relation, which follows immediately from naturality. That is, for $f \in \mathscr{C}(a \to b)$ and $g \in \mathcal{M}(m \to n)$, the following diagrams commute:

$$\begin{array}{ccc} \Psi(a)(m) & \xrightarrow{\Psi(f)_m} & \Psi(b)(m) \\ \downarrow{\scriptstyle \Psi(a)(g)} & & \downarrow{\scriptstyle \Psi(b)(g)} \\ \Psi(a)(n) & \xrightarrow{\Psi(f)_n} & \Psi(b)(n) \end{array} \quad = \quad \begin{array}{ccc} a \triangleright m & \xrightarrow{f \triangleright \mathrm{id}_m} & b \triangleright m \\ \downarrow{\scriptstyle \mathrm{id}_a \triangleright g} & & \downarrow{\scriptstyle \mathrm{id}_b \triangleright g} \\ a \triangleright m & \xrightarrow{f \triangleright \mathrm{id}_n} & b \triangleright n. \end{array}$$

We define the natural unitary associator isomorphism $\alpha_{a,b,m} \in \mathcal{M}(a \triangleright b \triangleright m \to a \otimes b \triangleright m)$ by $\alpha_{a,b,m} := \mu_m^{a,b} : [\Psi(a) \circ \Psi(b)](m) \to \Psi(a \otimes b)(m)$.

Notice that $\mu^{a,b} : \Psi(a) \circ \Psi(b) \Rightarrow \Psi(a \otimes b)$ is unitary if and only if $\mu_m^{a,b}$ is unitary for all $m \in \mathcal{M}$. Now one calculates $(f \triangleright \mathrm{id}_m)^\dagger = \Psi(f)_m^\dagger := (\Psi(f)^\dagger)_m = \Psi(f^\dagger)_m = f^\dagger \triangleright \mathrm{id}_m$ and $(\mathrm{id}_c \triangleright g)^\dagger = \Psi(c)(g)^\dagger = \Psi(c)(g^\dagger) = \mathrm{id}_c \triangleright g^\dagger$. Thus $\mathcal{M}$ is a $\mathscr{C}$-module dagger category. □

WARNING 3.21. – We do not state a C* version of the other bullet points of Theorem 3.15, which implicitly use rigidity for the statements on Morita equivalence and algebras. When $\mathscr{C}$ is C*, it is natural to impose compatibility conditions between the duality functor (implementing rigidity) and the dagger structure. We will explain this in detail in §3.5.1 below.

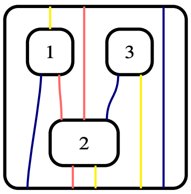
3.3. Monoidal algebras

Most algebraic structures have both a biased definition, like the usual definition of an algebra which emphasizes multiplying exactly two elements together, and an unbiased definition, like the definition of an algebra in which you can multiply arbitrary strings. (7) The usual definition of monoidal category is biased as it emphasizes tensoring two objects and composing two morphisms. In Definition 3.23 below, we give an unbiased definition of monoidal category using the graphical calculus; we will see in §3.6.3 below that planar algebras are the analogous unbiased definition of a pivotal monoidal category.

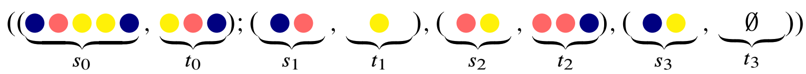
DEFINITION 3.22. – A *monoidal tangle* with label set S is a rectangle, with several smaller rectangles (with edges parallel to those of the big one) removed, and some non-crossing smooth strings labeled by elements of S which are oriented upward, have no local minima nor local maxima, and begin and end on the tops or bottoms of the rectangles. We say a monoidal tangle T has *type* $((s_0, t_0); (s_1, t_1), \ldots, (s_k, t_k))$ where $s_0, \ldots, s_k, t_0, \ldots, t_k$ are finite words on S if the tangle T has k input rectangles, and there are $|s_i|, |t_i|$ strings attached to the

(7) See [88] for a delightful elementary discussion of the unbiased definition of an algebra.

bottom and top respectively of the i-th rectangle (the zeroth rectangle is the output rectangle and $1 \leq i \leq k$ corresponds to the i-th input rectangle), which are labeled by the characters in the words s_i, t_i respectively. Here is an example of a tangle with $S = \{$●, ●, ●$\}$, where we color the strings instead of labeling them:

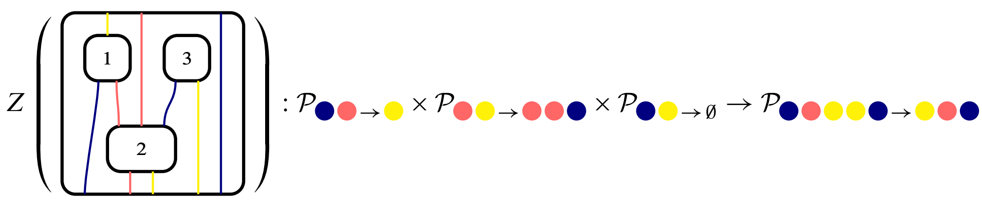


has type



Monoidal tangles are considered up to isotopy (through diagrams that again have no minima or maxima). Monoidal tangles form a colored operad, because you can insert monoidal tangles into the rectangles of a large monoidal tangle to get a new monoidal tangle.

Definition 3.23. – A *monoidal algebra* with label set S is an algebra in finite dimensional vector spaces for the operad of monoidal tangles with label set S. Unpacking this definition, a monoidal algebra $\mathcal{P}_{\bullet\to\bullet}$ consists of a family of finite dimensional vector spaces $\mathcal{P}_{s\to t}$ where s, t are finite words in S, together with an action of monoidal tangles. To each monoidal tangle T of type $((s_0, t_0); (s_1, t_1), \ldots, (s_k, t_k)))$, we associate a multilinear map $Z(T) : \prod_{j=1}^{k} \mathcal{P}_{s_j \to t_j} \to \mathcal{P}_{s_0 \to t_0}$, and composition of monoidal tangles corresponds to composition of multilinear maps. Here is an example:



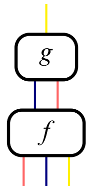
Example 3.24. – Suppose $\mathcal{C}$ is a Cauchy complete linear monoidal category with a set of objects $\mathcal{S} := \{X_s\}_{s\in S}$ which *tensor generates* $\mathcal{C}$, i.e., every object in $\mathcal{C}$ is a direct summand of a direct sum of tensor products of objects in $\mathcal{S}$. We define a monoidal algebra $\mathcal{P}(\mathcal{C}, \mathcal{S})_{\bullet\to\bullet}$ with label set S as follows. For $s_1, \ldots, s_k, t_1, \ldots, t_\ell \in S$, we define

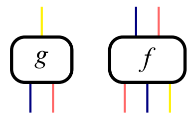
$$\mathcal{P}(\mathcal{C}, \mathcal{S})_{s_1\cdots s_k \to t_1\cdots t_\ell} := \mathcal{C}(X_{s_1} \otimes \cdots \otimes X_{s_k} \to X_{t_1} \otimes \cdots \otimes X_{t_\ell}).$$

We use the convention that if $\emptyset$ is the empty word on S, then the empty tensor product of objects is $1_{\mathcal{C}}$. The action of tangles is just the graphical calculus for tensor categories. See [85, 93, 63] for a summary of many versions of the graphical calculus; additional resources include [97] and [47, §2.1 and 2.3].

Remark 3.25. – The monoidal algebra $\mathcal{P}(\mathcal{C}, X)_{\bullet\to\bullet}$ is similar in spirit to the way the term 'monoidal algebra' is used in the work of Wenzl on constructing and classifying subfactors and fusion categories from quantum groups [99, 105] which is based on the original towers of algebras approach to subfactor theory [56, 104, 35, 91].

Conversely, from a monoidal algebra we can construct a Cauchy complete linear monoidal category following [72, 33]. This takes place in two steps, first we construct a linear monoidal category, and then we take its Cauchy completion as discussed in §3.1. The objects in this category are the words s in the label set S, and the morphism spaces are $\mathcal{P}_{s\to t}$. Composition is given by vertical stacking, and the monoidal structure given by horizontal juxtaposition. Here are examples of composition and monoidal product respectively:

$g \circ f :=$ 

$g \otimes f :=$ 

This construction is inverse to the construction in Example 3.24. We thus have the following theorem.

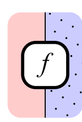
Theorem 3.26. – *There is an equivalence of categories*[8]

$$\{\,\textit{Monoidal algebras } \mathcal{P}_{\bullet\to\bullet} \textit{ with label set } S\,\} \cong \left\{\begin{array}{l}\text{Pairs } (\mathscr{C}, \{X_s\}_{s\in S}) \text{ with } \mathscr{C} \text{ a Cauchy}\\ \text{complete linear monoidal category with}\\ \text{generators } X_s \in \mathscr{C} \text{ for } s \in S\end{array}\right\}.$$

3.3.1. *Shaded monoidal algebras and monoidal categories.* – We next extend the discussion of monoidal algebras to r-shaded multifusion categories. Suppose $\mathscr{C}$ is a Cauchy complete linear monoidal category. A decomposition $1_{\mathscr{C}} = \bigoplus_{i\in R} 1_i$ where each 1_i is non-zero, but not necessarily simple, is called an R-*shading* on $\mathscr{C}$. We write $\mathscr{C}_{ij} = 1_i \otimes \mathscr{C} \otimes 1_j$, and we note that $\mathscr{C} = \bigoplus_{i,j=1}^{r} \mathscr{C}_{ij}$. We also have distinguished idempotents $p_i \in \mathscr{C}(1_{\mathscr{C}} \to 1_{\mathscr{C}})$ corresponding to the summand 1_i for $1 \le i \le r$. In the graphical calculus, we represent these projections, which freely float about in their regions, as a single shading. For example, we could denote

 $= p_i$  $= p_j$

Then for objects $a, b \in \mathscr{C}_{ij}$, we would denote a morphism $f \in \mathscr{C}(a \to b)$ by

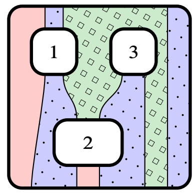


This motivates the following definition.

Definition 3.27. – An R-*shaded monoidal tangle* with label set S is a monoidal tangle with label set S whose regions are shaded by the elements of R such that each element $x \in S$ has a left *source* shading $s_x \in R$ and a right *target* shading $t_y \in R$. For example,

[8] Pairs $(\mathscr{C}, \{X_s\}_{s\in S})$ form a 2-category where between any two 1-morphisms, there is at most one 2-morphism, which is necessarily invertible when it exists [48, Lem. 3.5]. Hence this 2-category is equivalent to its truncation to a 1-category.

for the shading set $R = \{\ ,\ ,\ \}$, and the label set $S = \{\ ,\ ,\ ,\ \}$, we have the following R-shaded monoidal tangle with label set S:



Definition 3.28. – An *R-shaded monoidal algebra* with label set S is an algebra over the operad of R-shaded monoidal tangles with label set S. Notice this means that the spaces $\mathcal{P}_{x \to y}$ are only well-defined when consecutive characters in the words x and y have compatible target and source shadings, and the source and target shadings of the words x and y agree.

We have the following shaded version of Theorem 3.26 which is proved in an analogous way.

Proposition 3.29. – *There is an equivalence of categories (see footnote 8, p. 619)*

$$\left\{\begin{array}{l} R\text{-shaded monoidal algebras} \\ \mathcal{P}_{\bullet\to\bullet} \text{ with label set } S \end{array}\right\} \cong \left\{\begin{array}{l} \text{Pairs } (\mathscr{C}, \{X_y\}_{y \in S}) \text{ with } \mathscr{C} \text{ a Cauchy} \\ \text{complete linear monoidal category with} \\ R\text{-shading } 1 = \bigoplus_{i \in R} 1_i \text{ and tensor genera-} \\ \text{tors } X_y \in \mathscr{C}_{s_y, t_y} \text{ for } y \in S \end{array}\right\}.$$

3.3.2. *Unitary monoidal algebras*

Definition 3.30. – A *dagger monoidal algebra* with label set S is a monoidal algebra $\mathcal{P}_{\bullet\to\bullet}$ with label set S equipped with antilinear maps $\dagger : \mathcal{P}_{s \to t} \to \mathcal{P}_{t \to s}$ for all words s, t on S such that

— $\dagger \circ \dagger = \mathrm{id}$ and
— for every monoidal tangle T, $T^\dagger(x_1^\dagger, \ldots, x_k^\dagger) = T(x_1, \ldots, x_k)^\dagger$ where $T^\dagger$ denotes the vertical reflection of T about the x-axis.

A dagger monoidal algebra is called a C* *monoidal algebra* or a *unitary monoidal algebra* if in addition

— (positive definite) for all $f \in \mathcal{P}_{s \to t}$, $f^\dagger \circ f = 0$ implies $f = 0$.

Here, it is important to note that every $\mathcal{P}_{s \to t}$ was assumed to be finite dimensional. As in Definition 3.4 above, the positive definite condition above is equivalent to

— (2 × 2 linking C*-algebra) for all words s, t on S, the *linking algebra*

$$\mathcal{L}(s,t) := \begin{pmatrix} \mathcal{P}_{s \to s} & \mathcal{P}_{t \to s} \\ \mathcal{P}_{s \to t} & \mathcal{P}_{t \to t} \end{pmatrix}$$

with the obvious matrix multiplication and †-transpose operation is a finite dimensional C*-algebra (see Footnote 3).

When $\mathscr{C}$ is a C* monoidal category, we say a set of objects $\mathscr{S} := \{X_s\}_{s\in S}$ *unitarily tensor generates* $\mathscr{C}$ if every object in $\mathscr{C}$ is unitarily isomorphic to an orthogonal direct summand of an orthogonal direct sum of tensor products of objects in $\mathscr{S}$.

Similar to the previous section, we can define an R-shading as an orthogonal decomposition $1_{\mathscr{C}} = \bigoplus_{i\in R} 1_i$. We have the following unitary version Theorem 3.26 and Proposition 3.29.

PROPOSITION 3.31. – *There is an equivalence of categories (see footnote 8, p. 619)*

$$\left\{\begin{array}{l}R\text{-shaded C}^*\text{ monoidal alge-}\\ \text{bras } \mathcal{P}_{\bullet\to\bullet} \text{ with label set } S\end{array}\right\} \cong \left\{\begin{array}{l}\text{Pairs } (\mathscr{C}, \{X_y\}_{y\in S}) \text{ with } \mathscr{C} \text{ a Cauchy}\\ \text{complete monoidal C}^* \text{ category with}\\ R\text{-shading } 1 = \bigoplus_{i\in R} 1_i \text{ and unitary tensor}\\ \text{generators } X_y \in \mathscr{C}_{s_y,t_y} \text{ for } y\in S\end{array}\right\}.$$

3.4. Graph monoidal algebra embedding

In this section we relate endofunctor embeddings $\mathscr{C} \to \mathrm{End}(\mathcal{M})$ to embeddings of monoidal algebras into graph monoidal algebras, which is the non-pivotal analog of embedding planar algebras into graph planar algebras. We give a 2-shaded multifusion version which applies to an algebraic analog of a finite depth subfactor standard invariant.

DEFINITION 3.32. – Let J be a finite set. The tensor category $\mathsf{Vec}(J\times J)$ of *bi-J-graded vector spaces* has objects finite dimensional vector spaces which decompose as direct sums $V = \bigoplus_{i,j\in J} V_{ij}$, morphisms linear maps which preserve the bi-grading, i.e., $f : V \to W$ is a sum $f = \sum_{ij} f_{ij} : V_{ij} \to W_{ij}$, and composition the composition of linear maps. The tensor product of two bi-graded vector spaces is given by convolution

$$(V\otimes W)_{ik} := \bigoplus_{j\in J} V_{ij}\otimes W_{jk},$$

as is the tensor product of morphisms, i.e., if $f : V^1 \to V^2$ and $g : W^1 \to W^2$, then

$$(f\otimes g)_{ik} := \bigoplus_{j\in J} f_{ij}\otimes g_{jk} : \bigoplus_{j\in J} V^1_{ij}\otimes W^1_{jk} \longrightarrow \bigoplus_{j\in J} V^2_{ij}\otimes W^2_{jk}.$$

It is straightforward to see that $\mathsf{Vec}(J\times J)$ is a finitely semisimple rigid tensor category. A set of representatives of the simple objects is given by $\{E_{ij}\}_{i,j\in J}$, where E_{ij} has a copy of $\mathbb{C}$ in the ij-graded component and the zero vector space everywhere else. The dual of V is given by $(V^\vee)_{ij} := (V_{ji})^\vee$, the space of linear functionals $V_{ji} \to \mathbb{C}$, with obvious evaluation and coevaluation maps. Indeed, it is straightforward to verify that $\mathsf{Vec}(J\times J)$ is monoidally equivalent to the tensor category $\mathsf{Vec}[\mathcal{G}_r]$ of $\mathcal{G}_r$-graded vector spaces, where $\mathcal{G}_r$ is the groupoid with $r := |J|$ objects and a unique isomorphism between any two objects. In turn, $\mathsf{Vec}[\mathcal{G}_r]$ is easily seen to be monoidally equivalent to $\mathrm{End}(\mathcal{M})$ where $\mathcal{M}$ is a finitely semisimple category such that a set of representatives of the simple objects $\mathrm{Irr}(\mathcal{M})$ is in bijection with J.

DEFINITION 3.33. – Given a bi-graded vector space $V \in \mathsf{Vec}(J\times J)$, we may think of it as a Vec-*enriched graph* $\vec{\Gamma} = (J, V)$, whose vertices are the set J, and whose edges are the finite dimensional vector spaces V_{ij}. We call a bi-graded vector space *connected* if given any two vertices $i,k \in J$, there is a sequence of vertices $(i = j_0, j_1, \ldots, j_n = k)$ such that $V_{j_{\ell-1}j_\ell} \neq (0)$ for all $\ell = 1, \dots, n$. Observe that Γ is connected if and only if Γ Cauchy tensor generates $\mathsf{Vec}(J \times J)$.

Given a (connected) Vec-graph $\vec{\Gamma} = (J, V)$, we get an honest (connected) graph Γ with vertex set J and whose edges from i to j are some choice of basis for the space V_{ij}. Clearly picking different bases yields isomorphic graphs.

REMARK 3.34. – This approach is very similar to that in the classification of Temperley-Lieb module categories using weighted graphs from [25]. In [16], the authors classify unitary Temperley-Lieb module categories using bi-graded Hilbert spaces, which we discuss briefly (with a warning) in §3.4.2 below.

DEFINITION 3.35. – Suppose Γ is a connected directed graph with vertex set J. For an edge ε in Γ, we write $s(\varepsilon)$ and $t(\varepsilon)$ for the *source* and *target* of ε.

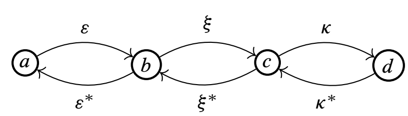
We define the *graph monoidal algebra* $\mathcal{GMA}(\Gamma)_{\bullet\to\bullet}$ as follows. For $m, n \geq 0$, we define $\mathcal{GMA}(\Gamma)_{m\to n}$ to be the $\mathbb{C}$-vector space with distinguished basis the set of pairs (p, q) where p, q are paths on Γ of length m, n respectively whose sources and targets agree.

The action of tangles is given by a state-sum model similar to a graph planar algebra:

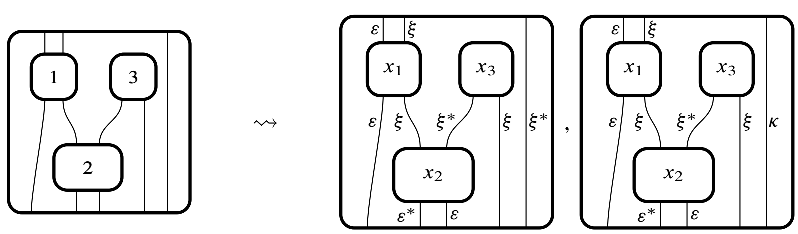
$$T((p_1, q_1), \dots, (p_k, q_k)) := \sum_{\text{states } \sigma \text{ on } T} \prod_{1\leq i\leq k} \delta_{\sigma|_i = (p_i, q_i)} \cdot \sigma|_0 \tag{3.1}$$

Here, T is a monoidal tangle with k input disks, and the (p_i, q_i) are basis elements (pairs of paths) in $\mathcal{GMA}(\Gamma)_{m_i\to n_i}$. A *state* σ on a monoidal tangle T is an assignment of vertices and edges of Γ to the regions and strings of T respectively such that if a string labeled by ε separates the left region R_ℓ from the right region R_r, then R_ℓ is labeled by $s(\varepsilon)$ and R_r is labeled by $t(\varepsilon)$. Now $\sigma|_i$ denotes the pair of paths in $\mathcal{GMA}(\Gamma)_{m_i\to n_i}$ obtained from reading the top and bottom boundaries of the i-th input disk from left to right. In other words, we only sum over states which are 'compatible' with the paths we input.

EXAMPLE 3.36. – Consider the following directed graph:



For the monoidal tangle displayed below on the left, there are exactly two compatible states for the input $(x_1 = (\varepsilon\xi, \varepsilon\xi), x_2 = (\xi\xi^*, \varepsilon^*\varepsilon), x_3 = (\emptyset, \xi^*\xi))$, which are displayed below on the right.



Hence the output of the tangle on the left applied to the input (x_1, x_2, x_3) is

$$(\varepsilon\xi\xi^*, \varepsilon\varepsilon^*\varepsilon\xi\xi^*) + (\varepsilon\xi\kappa, \varepsilon\varepsilon^*\varepsilon\xi\kappa).$$

The graph monoidal algebra of Γ is the non-pivotal analog of the graph planar algebra of Γ. The reader is encouraged to compare the above definition with that of the graph planar algebra of a bipartite graph in Definition 3.75 below.

THEOREM 3.37. – *Given a connected* Vec*-graph* $\vec{\Gamma} = (J, V) \in \mathsf{Vec}(J \times J)$*, the monoidal algebra* $\mathcal{P}(\mathsf{Vec}(J \times J), \vec{\Gamma})_{\bullet\to\bullet}$ *from Example 3.24 is isomorphic to the graph monoidal algebra* $\mathcal{GMA}(\Gamma)_{\bullet\to\bullet}$.

Proof. – Denoting $\vec{\Gamma}^{\otimes n} = (J, V^{\otimes n})$, we have a canonical isomorphism

$$V_{ik}^{\otimes n} \cong \bigoplus_{j_1,\dots,j_{n-1}\in J} V_{ij_1} \otimes V_{j_1,j_2} \otimes \cdots \otimes V_{j_{n-1}k}.$$

Observe that an $f \in \mathrm{Hom}_{\mathsf{Vec}(J\times J)}(\vec{\Gamma}^{\otimes m} \to \vec{\Gamma}^{\otimes n})$ is completely determined by its component maps

$$\left\{ f_{i\ell} : \bigoplus_{j_1,\dots,j_{m-1}\in J} V_{ij_1} \otimes V_{j_1 j_2} \otimes \cdots \otimes V_{j_{m-1}\ell} \to \bigoplus_{k_1,\dots,k_{n-1}\in J} V_{ik_1} \otimes V_{k_1 k_2} \otimes \cdots \otimes V_{k_{n-1}\ell}. \right\}_{i,\ell\in J}.$$

Now fix a basis $\{\varepsilon_{i\ell}^k\}$ for each $V_{i\ell}$, and for each pair of paths on Γ from i to ℓ

$$p = \varepsilon_{ij_1}^{p_1} \otimes \cdots \otimes \varepsilon_{j_{m-1}\ell}^{p_m} \qquad\qquad q = \varepsilon_{ik_1}^{q_1} \otimes \cdots \otimes \varepsilon_{k_{n-1}\ell}^{q_n},$$

of lengths m and n respectively, we let $F_{p\to q}^{i\ell} \in \mathrm{Hom}_{\mathsf{Vec}(J\times J)}(\vec{\Gamma}^{\otimes m} \to \vec{\Gamma}^{\otimes n})$ be the unique $i\ell$-component map sending p to q and all other paths p' from i to ℓ of length m to zero. We see then that

(3.2)
$$\mathcal{P}(\mathsf{Vec}(J \times J), \vec{\Gamma})_{m\to n} := \mathrm{Hom}_{\mathsf{Vec}(J\times J)}(\vec{\Gamma}^{\otimes m} \to \vec{\Gamma}^{\otimes n}) = \bigoplus_{i,\ell\in J} \mathrm{span}_{\mathbb{C}} \left\{ F_{p\to q}^{i\ell} \right\}_{p,q \text{ paths } i \text{ to } \ell}.$$

Now it is straightforward to verify that the linear extension

$$\Phi_{m\to n} : \mathcal{P}(\mathsf{Vec}(J \times J), \vec{\Gamma})_{m\to n} \to \mathcal{GMA}(\Gamma)_{m\to n}$$

of $F_{p\to q}^{i\ell} \mapsto (p, q)$ is a linear isomorphism for all $m, n \geq 0$.

It remains to see that this isomorphism is compatible with the action of monoidal tangles. It suffices to show that Φ intertwines the actions of a single vertical strand with no input disk, vertical stacking tangles, and horizontal concatenation tangles, as these tangles generate the monoidal operad. The vertical strand in $\mathcal{P}(\mathsf{Vec}(J \times J), \vec{\Gamma})_{1\to 1}$ is given by

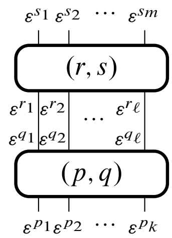
$$\bigoplus_{i,j\in J} \mathrm{id}_{V_{ij}} = \bigoplus_{i,j\in J} \sum_k F_{\varepsilon_{ij}^k \to \varepsilon_{ij}^k}^{ij} \overset{\Phi_{1\to 1}}{\longmapsto} \bigoplus_{i,j\in J} \sum_k (\varepsilon_{ij}^k, \varepsilon_{ij}^k) = \mathrm{id}_{\mathcal{GMA}(\Gamma)_{1\to 1}} = \boxed{\,|\,}.$$

Hence $\Phi_{1\to 1}$ preserves the strand.

To see that $\Phi_{\bullet\to\bullet}$ preserves composition, we check on our basis (3.2). Suppressing subscripts on edges for simplicity, suppose

$$p = \varepsilon^{p_1} \otimes \cdots \otimes \varepsilon^{p_k} \qquad q = \varepsilon^{q_1} \otimes \cdots \otimes \varepsilon^{q_\ell} \qquad r = \varepsilon^{r_1} \otimes \cdots \otimes \varepsilon^{r_\ell} \qquad s = \varepsilon^{s_1} \otimes \cdots \otimes \varepsilon^{s_m}$$

are paths on Γ from i to j. Then

$$\Phi_{k\to\ell}(F^{ij}_{r\to s}) \circ \Phi_{\ell\to m}(F^{ij}_{p\to q}) =$$


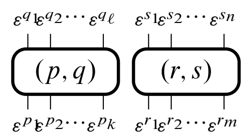
$$= \delta_{q=r}(p, s) = \delta_{q=r}\Phi_{k\to m}(F^{ij}_{p\to s}) = \Phi_{k\to m}(F^{ij}_{r\to s} \circ F^{ij}_{p\to q}).$$

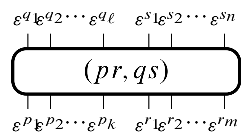
As composition is multi-linear, the general case follows by taking linear combinations.

Finally, to show $\Phi_{\bullet\to\bullet}$ preserves tensor product, we again work with our basis (3.2). Again suppressing subscripts for simplicity, suppose

$$p^{gh} = \varepsilon^{p_1} \otimes \cdots \otimes \varepsilon^{p_k} \qquad q^{gh} = \varepsilon^{q_1} \otimes \cdots \otimes \varepsilon^{q_\ell}$$
$$r^{ij} = \varepsilon^{r_1} \otimes \cdots \otimes \varepsilon^{r_m} \qquad s^{ij} = \varepsilon^{s_1} \otimes \cdots \otimes \varepsilon^{s_n}$$

are paths on Γ, where p^{gh}, q^{gh} go from g to h, and r^{ij}, s^{ij} go from i to j. We calculate

$$\Phi_{k\to\ell}(F^{gh}_{p\to q}) \otimes \Phi_{m\to n}(F^{ij}_{r\to s}) =$$


$$= \delta_{h=i}$$


$$= \delta_{h=i}\Phi_{k+m\to\ell+n}(F^{gj}_{pr\to qs}) = \Phi_{k+m\to\ell+n}(F^{gh}_{p\to q} \otimes F^{ij}_{r\to s}).$$

Again, the general case follows by taking linear combinations.

Since the actions of the generating tangles agree, we are finished. □

Definition 3.38. – Suppose $\mathscr{C}$ is a semisimple monoidal category Cauchy tensor generated by X, and $\mathcal{M}$ is a finitely semisimple module category. Let $\mathrm{Irr}(\mathcal{M}) = \{m_1, \ldots, m_r\}$ be a set of representatives of simple objects of $\mathcal{M}$, and define $J := \{1, \ldots, r\}$. The *fusion* Vec-*graph* $\vec{\Gamma}$ of $\mathcal{M}$ with respect to X is the Vec-graph whose vertices are J and whose edge spaces are given by

$$V_{ij} := \mathcal{M}(X \rhd m_i \to m_j). \tag{3.3}$$

Proposition 3.39 (Graph monoidal algebra embedding). – *Suppose $\mathscr{C}$ is a semisimple monoidal category Cauchy tensor generated by X, $\mathcal{M}$ is a finitely semisimple category with J a set of representatives of the isomorphism classes of simple objects, and $\vec{\Gamma}$ is a connected* Vec-*graph whose vertices are J. Equipping $\mathcal{M}$ with the structure of an indecomposable left $\mathscr{C}$-module category whose connected fusion* Vec-*graph with respect to X is $\vec{\Gamma}$ is equivalent to embedding the monoidal algebra $\mathcal{P}(\mathscr{C}, X)_{\bullet\to\bullet}$ into $\mathcal{GMA}(\Gamma)_{\bullet\to\bullet}$. More precisely, the category of $\mathscr{C}$-module structures on $\mathcal{M}$ and module functor structures on the identity functor is equivalent to the category of monoidal algebra embeddings and gaugings.* (9)

(9) By gauging we mean 'conjugating' the embedding by placing a fixed invertible element (or its inverse) on each strand as in [71]. In all our examples, gauging does not change the embedding, which is easily seen from our calculations. Indeed, we always get a discrete set of embeddings, as all our fusion graphs are trees (cf. [25]). This means gauging is not important in this article.

Proof. – As we discussed previously, $\mathscr{C}$-module structures on $\mathcal{M}$ are equivalent to tensor functors $\mathscr{C} \to \mathrm{End}(\mathcal{M})$. By semisimplicity, $\mathrm{End}(\mathcal{M}) \cong \mathsf{Vec}(J \times J)$. Notice that every linear tensor functor $\mathscr{C}_X \to \mathsf{Vec}(J \times J)$ uniquely extends to its Cauchy completion $\mathscr{C}$, and every linear tensor functor $\mathscr{C}_X \to \mathsf{Vec}(J \times J)$ has essential image in $\mathsf{Vec}(J \times J)_{\vec{\Gamma}}$. The result now follows from Theorem 3.37 together with the equivalence of categories from Theorem 3.26. □

REMARK 3.40. – When $\mathscr{C}$ is fusion, the Frobenius-Perron dimension of X is the norm of the underlying graph Γ.

3.4.1. *Embedding multifusion categories into multishaded graph monoidal algebras.* – We now adapt Proposition 3.39 to more closely approximate subfactor planar algebras, which have two shadings. On the Vec-graph side, we will see this translates into our Vec-graphs $\vec{\Gamma} = (J, V)$ being *bipartite*, i.e., $J = J_+ \amalg J_-$, and $V_{ij} = (0)$ whenever $i \in J_\pm$ and $j \in J_\pm$.

All the results and definitions in the beginning of this section about the graph tensor category and the (graph) monoidal algebra have straightforward 2-shaded/bipartite generalizations to multifusion categories.

DEFINITION 3.41. – Suppose $\mathcal{D}$ is a 2-shaded multifusion category with Cauchy tensor generator X in $\mathcal{D}_{12}$ and $\mathcal{M}$ is a finitely semisimple module category. As in Definition 3.38, we define the *fusion* Vec-*graph* $\vec{\Gamma}$ of $\mathcal{M}$ with respect to X to have vertices corresponding to simple objects in $\mathcal{M}$ and edge spaces

$$V_{ij} := \mathcal{M}(X \rhd m_i \to m_j).$$

Observe that since $\mathcal{D}$ is 2-shaded and $X \in \mathcal{D}_{12}$, $\vec{\Gamma}$ is bipartite.

PROPOSITION 3.42 (2-shaded graph monoidal algebra embedding).

Suppose $\mathcal{D}, \mathcal{M}, \vec{\Gamma}$ are as in Definition 3.41. Indecomposable left $\mathcal{D}$-module category structures on $\mathcal{M}$ whose fusion graph is Γ correspond to embeddings of the 2-shaded monoidal algebra $\mathcal{P}(\mathcal{D}, X)_{\bullet\to\bullet}$ into $\mathcal{GMA}(\Gamma)_{\bullet\to\bullet}$

This is the purely algebraic version of our graph planar algebra embedding theorem.

REMARK 3.43. – In §3.6.3 below, we will define the notion of graph planar algebra by beginning with $\mathcal{GMA}(\Gamma)_{\bullet\to\bullet}$ and extending the action of monoidal tangles to all planar tangles. In particular, it follows just from the results of this section that a graph planar algebra embedding yields a module category. This result alone is enough to show the existence of $\mathcal{EH}_3$ and $\mathcal{EH}_4$ as tensor categories from the GPA embeddings constructed in §5, but not to determine whether these tensor categories are unitary.

3.4.2. *Embedding unitary multifusion categories.* – Recall that a finitely semisimple C* category $\mathcal{M}$ is †-equivalent to $\mathsf{Hilb}^{|\operatorname{Irr}(\mathcal{M})|}$. Similar to the algebraic and multishaded settings, one can adapt to the unitary setting by first considering the unitary multifusion category $\mathsf{Hilb}(J \times J)$ of bi-J-graded Hilbert spaces, which is †-equivalent to $\operatorname{End}^\dagger(\mathcal{M})$ for any C* category $\mathcal{M}$ where $|\operatorname{Irr}(\mathcal{M})| = |J|$. (This is the approach to classifying unitary Temperley-Lieb modules in [16].) Analogous to Definition 3.33, we may identify the objects of $\mathsf{Hilb}(J \times J)$ with Hilb-enriched graphs $\vec{\Gamma} = (J, H)$, and we obtain honest graphs by choosing orthonormal bases for the edge Hilbert spaces.

Now the graph monoidal algebra $\mathcal{GMA}(\Gamma)_{\bullet\to\bullet}$ carries an obvious †-structure by the anti-linear extension of $(p, q) \mapsto (q, p)$ where p, q are paths on Γ whose sources and targets agree. It is straightforward to show that this †-structure is compatible with the vertical reflection of tangles about the x-axis, and that it satisfies the positivity axioms, making $\mathcal{GMA}(\Gamma)_{\bullet\to\bullet}$ a unitary monoidal algebra. Similar to Theorem 3.37, we have a †-isomorphism of unitary monoidal algebras $\mathcal{GMA}(\Gamma)_{\bullet\to\bullet} \cong \mathcal{P}(\mathsf{Hilb}(J \times J), \vec{\Gamma})_{\bullet\to\bullet}$.

We may pass to the unitary 2-shaded setting by working with bipartite Hilb-graphs. There is an "obvious" unitary version of Propositions 3.39 and 3.42.

Warning 3.44. – One should be careful not to use the Formula (3.3) to define the corresponding bi-$\operatorname{Irr}(\mathcal{M})$-graded Hilbert space from a †-functor in $\operatorname{End}^\dagger(\mathcal{M})$, as it would require choosing Hilbert space structures on the hom spaces of $\mathcal{M}$. We will see in Remark 3.61 below that this extra structure corresponds to a unitary trace on $\mathcal{M}$, which gives a distinguished choice of unitary pivotal structure on $\operatorname{End}^\dagger(\mathcal{M})$ by Proposition 3.67 below. Instead, one should simply use the †-equivalence between $\mathsf{Hilb}(\operatorname{Irr}(\mathcal{M}) \times \operatorname{Irr}(\mathcal{M}))$ and $\operatorname{End}^\dagger(\mathcal{M})$.

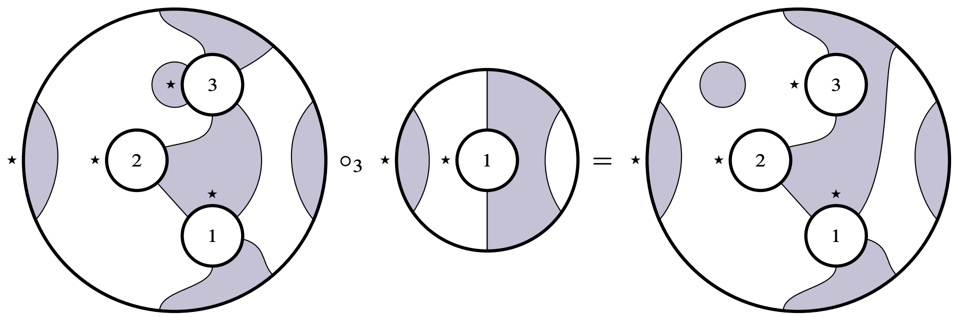
Observe that when $\mathcal{M}$ is C*, $\mathcal{M}(m \to m), \mathcal{M}(n \to n)$ are C* algebras for all $m, n \in \mathcal{M}$, and $\mathcal{M}(m \to n)$ has the canonical structure of a Hilbert C* $\mathcal{M}(m \to m) - \mathcal{M}(n \to n)$ bimodule. We will not discuss this further as it would take us too far afield.

The remaining sections of this chapter are dedicated to adapting Proposition 3.42 to the pivotal and unitary pivotal settings. We will see this adaptation naturally becomes the module embedding theorem for graph planar algebras.

3.5. Planar algebras

In §3.3, we defined the notion of a (shaded) monoidal algebra. As alluded to earlier, the pivotal analog of a monoidal algebra is a planar algebra. To simplify the exposition, we will only define (2-)shaded planar algebras with a single strand type following [60]; we refer the reader to [61, 59] (see also [9]) for a host of other notions of planar algebra.

Definition 3.45. – A (2-)*shaded planar tangle* consists of a disk with smaller internal input disks, together with non-intersecting strings between the disks, a checkerboard shading, and a distinguished interval marked by $\star$ for each disk. We consider shaded planar tangles up to isotopy. We say a shaded planar tangle has *type* $((n_0, \pm_0); (n_1, \pm_1), \dots, (n_k, \pm_k))$, where each $\pm_i \in \{+, -\}$, if the i-th disk has $2n_i$ strings connected to it, and its distinguished interval is in an unshaded/shaded region corresponding to $\pm_i$. The collection of shaded planar tangles forms a colored operad by inserting tangles into the input disks to get a new shaded planar tangle, making sure the distinguished intervals align. We include below an example of a composite of a tangle of type $((4,-);(2,-),(1,+),(3,-))$ with a tangle of type $((3,-);(1,+))$ resulting in a tangle of type $((4,-);(2,-),(1,+),(1,+))$:

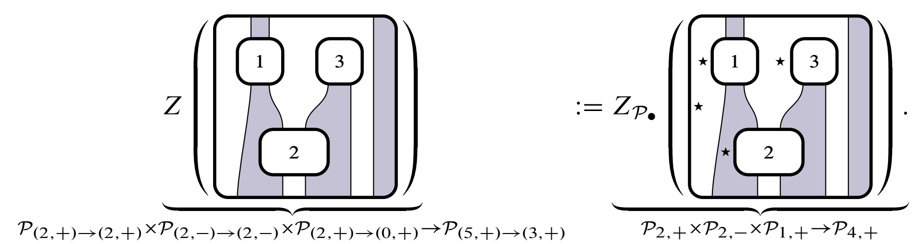


Definition 3.46. – A (2-)*shaded planar algebra* is an algebra in finite dimensional vector spaces for the shaded planar operad. Unpacking this definition, we have a vector space $\mathcal{P}_{n,\pm}$ for each color $(n,\pm)$, and for each tangle T of type $((n_0,\pm_0);(n_1,\pm_1),\dots,(n_k,\pm_k))$, we have a multi-linear map $Z(T)\ :\ \prod_{i=1}^{k}\mathcal{P}_{n_i,\pm_i}\ \to\ \mathcal{P}_{n_0,\pm_0}$. Composition of tangles then corresponds to the composition of multi-linear maps.

Notice that any shaded planar algebra gives us a canonical R-shaded monoidal algebra with region shadings $R=\{\bigcirc,\ \bullet\}$ and label set $S=\{\ |\ ,\ |\ \}$ as follows. Whenever $n_1\equiv n_2 \bmod 2$ and $\pm_1=\pm_2$, we set

$$\mathcal{P}_{(n_1,\pm_1)\to(n_2,\pm_2)} := \mathcal{P}_{(n_1+n_2)/2,\pm_1}, \tag{3.4}$$

and the action of monoidal tangles is given by adding a $\star$ to the left of every input rectangle. (Notice that for this R and S, every monoidal tangle must have a checkerboard shading.) Here is an explicit example:

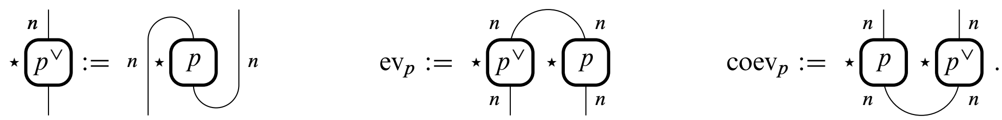


The article [33] provides a dictionary between shaded planar algebras and triples $(\mathscr{C},X,\varphi)$ where $\mathscr{C}$ is a Cauchy complete category, $1_{\mathscr{C}}=1_+\oplus 1_-$ is a decomposition (not necessarily into simples!), $X=1_+\otimes X\otimes 1_-$ Cauchy tensor generates $\mathscr{C}$, and $\varphi\ :\ \mathrm{id}\Rightarrow\vee\circ\vee$ is a trivialization of the double dual functor known as a *pivotal structure* (see Section 3.5.1 below for the precise definition). This dictionary actually gives an equivalence of categories similar to [48].

THEOREM 3.47. – *There is an equivalence of categories* (10)

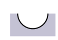
$$\left\{\begin{array}{l}\text{2-shaded planar}\\ \text{algebras } \mathcal{P}_\bullet\end{array}\right\} \cong \left\{\begin{array}{l}\text{Triples } (\mathcal{C},\varphi,X) \text{ with } (\mathcal{C},\varphi) \text{ a pivotal multitensor cate-}\\ \text{gory, and } X\in\mathcal{C} \text{ a Cauchy tensor generator with a decom-}\\ \text{position } 1_{\mathcal{C}} = 1_+ \oplus 1_- \text{ such that } X = 1_+\otimes X\otimes 1_-\end{array}\right\}.$$

This theorem is exactly the pivotal analog of Proposition 3.31, which provides the equivalence of the underlying 2-shaded monoidal algebras and linear Cauchy complete monoidal categories. Indeed, given such a planar algebra, we see that its category of idempotents $\mathcal{C}$ is rigid. Indeed, given an idempotent $p\in\mathcal{C}$, its dual is given by its π-rotation, and the evaluation and coevaluation are given by the ordinary cap and cup in the planar algebra cut down by p and $p^\vee$:



Choosing $\varphi_p := p \in \operatorname{Hom}(p\to p)$ endows $\mathcal{C}$ with a pivotal structure. Conversely, one passes from triples $(\mathcal{C},\varphi,X)$ to planar algebras via the graphical calculus to produce a monoidal algebra, and one then gets cups and caps by defining

 $:= \mathrm{ev}_X$  $:= \mathrm{coev}_X$

 $:= \mathrm{ev}_{X^\vee}\circ(\varphi_X\otimes \mathrm{id}_{X^\vee})$  $:= (\mathrm{id}_{X^\vee}\otimes\varphi_X^{-1})\circ \mathrm{coev}_{X^\vee}$.

3.5.1. *Unitary dual functors for unitary multifusion categories*. – In order to discuss the unitary version of Theorem 3.47, we rapidly recall the relevant notions for unitary dual functors and unitary pivotal structures from [97, 83]. We do so only for *unitary multifusion categories*, which are finitely semisimple multitensor C* categories, which substantially simplifies the presentation. For $\mathcal{C}$ a unitary multifusion category, we have that $1_{\mathcal{C}}$ decomposes into an orthogonal direct sum of simples as $\bigoplus_{i=1}^r 1_i$, and we let $p_i\in\mathcal{C}(1_{\mathcal{C}}\to 1_{\mathcal{C}})$ be the minimal projection corresponding to the summand 1_i for $i=1,\ldots,r$. In the exposition below, we assume $\mathcal{C}$ is *indecomposable*, i.e., $\mathcal{C}$ is not equivalent to the direct sum of two non-zero unitary multifusion categories. We write $\mathcal{C}_{ij} = 1_i\otimes\mathcal{C}\otimes 1_j$, and we note $\mathcal{C} = \bigoplus_{i,j}\mathcal{C}_{i,j}$ is a faithful grading of $\mathcal{C}$ by the groupoid $\mathcal{G}_r$ with r objects and a unique isomorphism between any two objects, which can also be viewed as the standard system of matrix units $\{E_{ij}\}$ for $M_r(\mathbb{C})$.

A dual functor $\vee:\mathcal{C}\to\mathcal{C}$ consists of a choice of dual object $c^\vee$ for each $c\in\mathcal{C}$ together with morphisms $\mathrm{ev}_c, \mathrm{coev}_c$ which satisfy the zig-zag axioms. On morphisms $f\in\mathcal{C}(a\to b)$, we define $\vee$ by

$$f^\vee = (\mathrm{ev}_b\otimes\mathrm{id}_{a^\vee})\circ(\mathrm{id}_{b^\vee}\otimes f\otimes\mathrm{id}_{a^\vee})\circ(\mathrm{id}_{b^\vee}\otimes\mathrm{coev}_a).$$

A dual functor has a canonical anti-tensorator $\nu_{a,b} : a^\vee\otimes b^\vee \to (b\otimes a)^\vee$ built from evaluations and coevaluations. Any two dual functors are uniquely monoidally naturally isomorphic.

(10) Similar to footnote 8, triples $(\mathcal{C},\varphi,X)$ form a 2-category where between any two 1-morphisms, there is at most one 2-morphism, which is necessarily invertible when it exists [48, Lem. 3.5]. Hence this 2-category is equivalent to its truncation to a 1-category.

A *pivotal structure* is a pair $(\vee, \varphi)$ consisting of a dual functor $\vee$ and a monoidal natural isomorphism $\varphi : \mathrm{id}_{\mathscr{C}} \Rightarrow \vee \circ \vee$. If a pivotal structure exists for a multitensor category, the equivalence classes of pivotal structures form a torsor over the group $\mathrm{Hom}(\mathcal{U} \to \mathbb{C}^\times)$ with group law given by pointwise multiplication, where $\mathcal{U}$ is the universal grading groupoid of $\mathscr{C}$ (see [21, §4.14] and [83, §3.3] for more details).

A dual functor is called *unitary* if it is a dagger tensor functor, i.e., $\nu_{a,b}$ is unitary for all $a, b \in \mathscr{C}$, and $f^{\vee\dagger} = f^{\dagger\vee}$ for all $f \in \mathscr{C}(a \to b)$. Each unitary dual functor induces a canonical *unitary pivotal structure* by $\varphi_c := (\mathrm{coev}_c^\dagger \otimes \mathrm{id}_{c^{\vee\vee}}) \circ (\mathrm{id}_c \otimes \mathrm{coev}_{c^\vee})$, which is unitary. As in [97, §7.3], the term 'unitary pivotal structure' should be viewed as a synonym for 'the canonical unitary pivotal structure induced from a unitary dual functor.'

REMARK 3.48. – Equivalently, we can say that a pivotal structure $\varphi_c : c \to c^{\vee\vee}$ is compatible with the dagger structure if $\mathrm{coev}_c^\dagger = \mathrm{ev}_{c^\vee} \circ (\varphi_c \otimes \mathrm{id}_{c^\vee}) : c \otimes c^\vee \to 1_{\mathscr{C}}$, and define a unitary pivotal structure as a pivotal structure which is compatible with the dagger structure. It is easy to see that the only compatible pivotal structure is the canonical one. Note that this compatibility condition is needed in order for unitary pivotal categories to have the correct diagram calculus where dagger corresponds to reflection of diagrams, since $\mathrm{coev}_c^\dagger$ and $\mathrm{ev}_{c^\vee} \circ (\varphi_c \otimes \mathrm{id}_{c^\vee})$ both are represented graphically by the same oriented cap.

REMARK 3.49. – We found the relationship between pivotal structures and unitary pivotal structures very confusing, and so we'd like to pause to explain why it's so confusing. In both the algebraic and unitary settings a pivotal structure consists of two parts: a choice of dual functor and a choice of trivialization of the double dual functor subject to a compatibility condition. In the algebraic setting, the dual functor is essentially unique (i.e., any two choices are canonically naturally isomorphic) and the compatibility condition is vacuous, so the only interesting part is the trivialization of the double dual functor. By contrast, in the unitary setting, once you've chosen a unitary dual functor, the compatibility condition guarantees that there's a unique compatible trivialization of the double dual, so the only interesting part is the choice of unitary dual functor. This means even though the two definitions can be made parallel, the *interesting parts* of the two definitions are disjoint!

Note that a unitary pivotal structure φ is *pseudounitary*, i.e., all dimensions of simple objects are strictly positive. Here, the left and right dimensions of a non-simple object $c \in \mathscr{C}$ are the matrices in $M_r(\mathbb{C})$ determined by

$$\mathrm{Dim}_L^\varphi(c)_{ij}\, \mathrm{id}_{1_j} = \mathrm{tr}_L^\varphi(p_i \otimes \mathrm{id}_c \otimes p_j) \qquad \mathrm{Dim}_R^\varphi(c)_{ij}\, \mathrm{id}_{1_i} = \mathrm{tr}_R^\varphi(p_i \otimes \mathrm{id}_c \otimes p_j).$$

When $c \in \mathscr{C}$ is simple, $\mathrm{Dim}_L^\varphi(c), \mathrm{Dim}_R^\varphi(c)$ have exactly one non-zero entry, which we call $\dim_L^\varphi(c), \dim_R^\varphi(c)$ respectively.

For our indecomposable unitary multifusion category $\mathscr{C}$, there exists a canonical spherical structure [69, 107, 5, 83] which satisfies for all simples $c \in \mathscr{C}$, $\dim_L^\varphi(c) = \dim_R^\varphi(c)$. By picking this basepoint, we identify the torsor of pivotal structures with the group $\mathrm{Hom}(\mathcal{U} \to \mathbb{C}^\times)$. Polar decomposition gives us a group isomorphism $\mathbb{C}^\times \cong U(1) \times \mathbb{R}_{>0}$, which gives us a group isomorphism

$$\mathrm{Hom}(\mathcal{U} \to \mathbb{C}^\times) \cong \mathrm{Hom}(\mathcal{U} \to U(1)) \times \mathrm{Hom}(\mathcal{U} \to \mathbb{R}_{>0}).$$

It follows that the unitary pivotal structures correspond to the subgroup $1 \times \mathrm{Hom}(\mathcal{U} \to \mathbb{R}_{>0})$ as all dimensions must be strictly positive.

Now in the case of a unitary multifusion category, the universal grading groupoid $\mathcal{U}$ is finite. If $\mathcal{G} \subseteq \mathcal{U}$ is a subgroup (with only one object), then given a $\pi \in \mathrm{Hom}(\mathcal{U} \to \mathbb{R}_{>0})$, we must have $\pi(\mathcal{G}) = \{1\}$. Hence for our indecomposable unitary multifusion category $\mathscr{C}$ such that $1_{\mathscr{C}} = \bigoplus_{i=1}^{r} 1_i$ is a decomposition into simples, the relevant grading groupoid to see all unitary pivotal structures is exactly $\mathcal{G}_r$.

Summarizing, we have:

THEOREM 3.50. – *Let $\mathscr{C}$ be a unitary multifusion category. There is a bijective correspondence between*

1. *unitary equivalence classes of unitary dual functors and their induced unitary pivotal structures*
2. $\mathrm{Hom}(\mathcal{G}_r \to \mathbb{R}_{>0})$.

See [83] for more details.

REMARK 3.51. – Notice that a homomorphism $\pi \in \mathrm{Hom}(\mathcal{G}_r \to \mathbb{R}_{>0})$ is uniquely determined by its image on $E_{i+1,i}$ for $1 \le i \le r-1$.

Explicitly, starting with a unitary dual functor $\vee$ with its induced unitary pivotal structure φ, we get our $\pi \in \mathrm{Hom}(\mathcal{G}_r \to \mathbb{R}_{>0})$ by taking the ratio of left to right quantum dimensions of simple objects:

$$\pi(E_{ij}) := \frac{\dim_L^{\varphi}(c)}{\dim_R^{\varphi}(c)} \qquad \text{for all simple } c \in \mathscr{C}_{ij}.$$

Conversely, we can choose for each $c \in \mathscr{C}$ a unique balanced dual $(\overline{c}, \mathrm{ev}_c, \mathrm{coev}_c)$ up to unique isomorphism. One then obtains all other unitary dual functors from homomorphisms $\pi \in \mathrm{Hom}(\mathcal{G}_r \to \mathbb{R}_{>0})$ by rescaling the evaluations and coevaluations on simple objects $c \in \mathscr{C}_{ij}$ by

$$\mathrm{ev}_c^{\pi} := \pi(E_{ij})^{1/4}\, \mathrm{ev}_c . \qquad \mathrm{coev}_c^{\pi} := \pi(E_{ij})^{-1/4}\, \mathrm{coev}_c .$$

3.5.2. *Unitary planar algebras*. –

DEFINITION 3.52. – A *planar †-algebra* is a planar algebra equipped with antilinear maps $\dagger : \mathcal{P}_{n,\pm} \to \mathcal{P}_{n,\pm}$ such that

- $\dagger \circ \dagger = \mathrm{id}$, and
- for every planar tangle T, $T^{\dagger}(x_1^{\dagger}, \ldots, x_k^{\dagger}) = T(x_1, \ldots, x_k)^{\dagger}$ where $T^{\dagger}$ denotes the reflection of T about *any* axis.

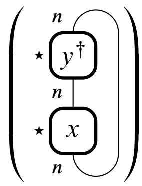
A planar †-algebra is called a C* *planar algebra* or a *unitary planar algebra* if its underlying dagger monoidal algebra from (3.4) is C*. (11)

REMARK 3.53. – Observe that the planar algebra of a bipartite graph from [57] (see also Definition 3.75 below) is unitary.

(11) Our definition of unitary planar †-algebra from [84, Def. 2.3] was vague about what positive definite means for $\mathcal{P}_{0,\pm}$. The above definition clarifies this omission.

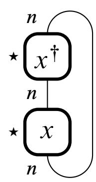
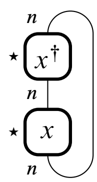
We now show the above definition of C* planar algebra is equivalent to [61, Def. 1.37].

Lemma 3.54. – *A planar* †*-algebra* $\mathcal{P}_\bullet$ *is unitary if and only if there exists a faithful tracial state* $\psi_\pm$ *on* $\mathcal{P}_{0,\pm}$ *such that for every* $n \geq 0$*, the sesquilinear form*

(3.5) $\langle x, y\rangle^{\psi}_{n,\pm} := \psi_\pm$ 

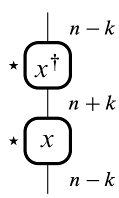
is a positive definite inner product.

Proof. – Suppose $\mathcal{P}_\bullet$ is unitary, so $\mathcal{P}_{0,\pm}$ is a finite dimensional C* algebra as a corner of a 2×2 linking C*-algebra. Choose a faithful tracial state $\psi_\pm$ on $\mathcal{P}_{0,\pm}$. Suppose $x \in \mathcal{P}_{n,\pm}$. Since the 2×2 linking algebra $\mathcal{L}(0, 2n)$ is C*, the $\mathcal{P}_{0,\pm}$-valued pairing inside $\psi_\pm$ in (3.5) is positive definite, i.e.,

 ≥ 0 and $\left(\right.$  $= 0 \implies x = 0 \left.\right)$.

Hence the sesquilinear form (3.5) above is positive definite by positivity and faithfulness of ψ.

Conversely, suppose we have $\psi_\pm$ on $\mathcal{P}_{0,\pm}$ such that (3.5) is positive definite for all $n \geq 0$. Suppose $x \in \mathcal{P}_{n,\pm}$ and $0 \leq k \leq n$ such that

 $= 0.$

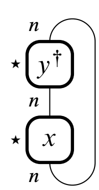
Then capping off the remaining strings in the diagram on the left hand side and applying $\psi_\pm$, we see that $\langle x, x\rangle^{\psi}_{n,\pm} = 0$, and thus $x = 0$. Hence the underlying dagger monoidal algebra is C*, and we are finished. □

Definition 3.55. – A *subfactor planar algebra* is a 2-shaded planar †-algebra satisfying the following axioms:

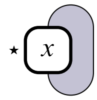
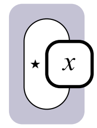
— (connected) $\mathcal{P}_{0,\pm} \cong \mathbb{C}$ via the map which sends the empty diagram to $1_{\mathbb{C}}$,

— (finite dimensional) $\dim(\mathcal{P}_{n,\pm}) < \infty$ for all $n \geq 0$.

— (positive) For every $n \geq 0$, the sesquilinear form on $\mathcal{P}_{n,\pm}$ given by

$\langle x, y\rangle_n :=$ 

is a positive definite inner product, and

— (spherical) For every $x \in \mathcal{P}_{1,\pm}$,  $=$ .

By Lemma 3.54, a subfactor planar algebra is a 2-shaded unitary planar algebra.

The following result, which appears in [83, §4], is the unitary analog of Theorem 3.47, which uses unitary dual functors instead of a pivotal structure.

THEOREM 3.56 ([83, §4]). – *There is an equivalence of categories (see Footnote 10)*

$$\left\{\begin{array}{l}\text{2-shaded unitary}\\ \text{planar algebras } \mathcal{P}_\bullet\end{array}\right\} \cong \left\{\begin{array}{l}\text{Triples } (\mathscr{C}, \vee, X) \text{ with } \mathscr{C} \text{ a unitary multitensor}\\ \text{category, } \vee \text{ a unitary dual functor, and a gener-}\\ \text{ator } X \in \mathscr{C} \text{ with an orthogonal decomposition}\\ 1_{\mathscr{C}} = 1_+ \oplus 1_- \text{ such that } X = 1_+ \otimes X \otimes 1_-\end{array}\right\}.$$

Moreover, under this equivalence,

- *finite depth planar algebras correspond to triples* $(\mathscr{C}, \vee, X)$ *where* $\mathscr{C}$ *is unitary multifusion, and*
- *subfactor planar algebras correspond to triples* $(\mathscr{C}, \vee, X)$ *where* $1_\pm$ *are simple and* $\vee$ *is the canonical spherical dual functor.*

REMARK 3.57. – In the unitary setting, Proposition 3.31 gives us an equivalence between the underlying 2-shaded unitary monoidal algebras and unitary multitensor categories. Starting with a 2-shaded unitary planar algebra $\mathcal{P}_\bullet$, we get a unitary dual functor on the projection category $\mathscr{C}$ by taking the π-rotation in $\mathcal{P}_\bullet$. Conversely, given a tuple $(\mathscr{C}, \vee, X)$, by Remark 3.48, $\mathrm{coev}_c^\dagger = \mathrm{ev}_{X^\vee} \circ (\varphi_X \otimes \mathrm{id}_{X^\vee})$ and $\mathrm{ev}_c^\dagger = (\mathrm{id}_{X^\vee} \otimes \varphi_X^{-1}) \circ \mathrm{coev}_{X^\vee}$. This means the cups and caps are alternately described by

 $:= \mathrm{ev}_X$  $:= \mathrm{coev}_X$  $:= \mathrm{coev}_X^\dagger$  $:= \mathrm{ev}_X^\dagger$.

Now in order for a 2-shaded unitary planar algebra $\mathcal{P}_\bullet$ to have *scalar* loop modulus, we choose the *standard* unitary dual functor $\vee_{\text{standard}}$ on $\mathscr{C}$ *with respect to* X following [34], which is clarified in [83]. First, define $n_\pm := \dim(\mathrm{End}_{\mathscr{C}}(1_\pm))$, and denote the summands of 1_+ and 1_- by V_+ and V_- respectively. Let D_X be the $n_+ \times n_-$ matrix whose uv-th entry is $\dim(u \otimes X \otimes v)$, using the canonical spherical structure. Let $d_X > 0$ such that $d_X^2 = \|D_X D_X^T\| = \|D_X^T D_X\|$, and let μ and ν be the Frobenius-Perron eigenvectors of $D_X D_X^T$ and $D_X^T D_X$ respectively normalized so that

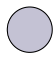
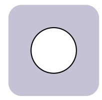
$$\sum_{u \in V_+} \mu(u)^2 = 1 = \sum_{v \in V_-} \nu(v)^2.$$

We denote by λ the vector in $\mathbb{R}_{>0}^{n_+ + n_-}$ obtained by concatenating μ and ν.

DEFINITION 3.58. – The *standard* unitary dual functor with respect to X corresponds to the standard groupoid homormorphism $\mathcal{G}_r \to \mathbb{R}_{>0}$ given by

$$\pi^{\text{standard}}(E_{u,v}) := \left(\frac{\lambda(u)}{\lambda(v)}\right)^2 \tag{3.6}$$

under Theorem 3.50. It is straightforward to verify that the shaded planar algebra corresponding to $(\mathscr{C}, \vee_{\text{standard}}, X)$ under Theorem 3.56 has scalar loop moduli given by

(3.7)  $= d_X \, \mathrm{id}_{\mathcal{P}_{0,+}}$  $= d_X \, \mathrm{id}_{\mathcal{P}_{0,-}}$.

3.6. The graph planar algebra module embedding theorem

In this section, we finally prove the unitary pivotal module embedding theorem. We begin by defining the notion of a trace on a semisimple category in §3.6.1, and then discussing Schaumann's notion of a pivotal module for a pivotal category from [96] in §3.6.2. As both of these concepts have unitary versions, we treat both the algebraic and unitary setting in parallel; the reader should include the parenthetical statements for the unitary setting, and may omit these statements in the non-unitary setting. Finally, in §3.6.3, we see how our Main Theorem 3.80 in this section is a natural generalization of the embedding theorems from §3.4.

3.6.1. *Traces on semisimple categories*. – In this section we now discuss (unitary) traces on finitely semisimple (C*) categories. Throughout we denote the semisimple category with trace by $\mathcal{M}$ because in our applications we will be looking at traces on module categories, but nothing in this section uses a module structure.

Definition 3.59. – A *trace* on a semisimple category $\mathcal{M}$ is a family of linear functionals $\mathrm{Tr}_m : \mathrm{End}_{\mathcal{M}}(m) \to \mathbb{C}$ for $m \in \mathcal{M}$ such that $\mathrm{Tr}_m(g \circ f) = \mathrm{Tr}_n(f \circ g)$ for all $f \in \mathcal{M}(m \to n)$ and $g \in \mathcal{M}(n \to m)$. We call a trace *nondegenerate* if the bilinear forms $\mathrm{Hom}_{\mathcal{M}}(m,n) \times \mathrm{Hom}_{\mathcal{M}}(n,m) \to \mathbb{C}$ via $(f,g) \mapsto \mathrm{Tr}_m(g \circ f)$ are non-degenerate ($\mathrm{Tr}_m(g \circ f) = 0$ for all $g \in \mathrm{Hom}(n,m)$ implies $f = 0$). For convenience, all traces that follow are assumed to be nondegenerate unless stated otherwise.

When $\mathcal{M}$ is a semisimple C* category, we call a trace *unitary* if in addition for every $m, n \in \mathcal{M}$, the sesquilinear form $\langle f, g \rangle := \mathrm{Tr}_m(g^\dagger \circ f)$ on $\mathrm{Hom}_{\mathcal{M}}(m,n)$ is a positive definite inner product.

Remark 3.60. – We do not require $\mathrm{Tr}_m^{\mathcal{M}}$ to be normalized; that is $\mathrm{Tr}_m^{\mathcal{M}}(\mathrm{id}_m)$ is typically not 1. Instead we think of $\mathrm{Tr}_m^{\mathcal{M}}(\mathrm{id}_m)$ as specifiying a notion of the dimension of $m \in \mathcal{M}$.

For example, any trace on the $n \times n$ matrices $M_n(\mathbb{C})$ is a scalar multiple of the standard matrix trace. A trace on Vec is a collection of traces on $M_n(\mathbb{C})$ for all n; however the condition $\mathrm{Tr}^{\mathcal{M}}(f \circ g) = \mathrm{Tr}^{\mathcal{M}}(g \circ f)$ applied to maps between vector spaces of different dimensions restricts the normalizations of the different traces. In particular, the standard trace ($\mathrm{Tr}_V(\mathrm{id}_V) = \dim V$) on each $\mathrm{End}(V)$ gives a trace on Vec, but the normalized trace ($\mathrm{tr}_V(\mathrm{id}_V) = 1$) on each $\mathrm{End}(V)$ does not give a trace on Vec.

Remark 3.61. – Similar to [96], in the non-unitary setting, traces on $\mathcal{M}$ are in bijection with families of natural isomorphisms $\mathcal{M}(m \to n) \cong \mathcal{M}(n \to m)^*$ for all $m, n \in \mathcal{M}$.

Unitary traces on $\mathcal{M}$ are in bijection with *2-Hilbert space* structures on $\mathcal{M}$ [3] (see also [4, §3 and 5.6]), i.e., for every $m, n \in \mathcal{M}$, a Hilbert space structure on $\mathcal{M}(m \to n)$ such that for all $f \in \mathcal{M}(m \to n)$, $g \in \mathcal{M}(n \to p)$, and $h \in \mathcal{M}(m \to p)$,

$$\langle g \circ f, h \rangle_{\mathcal{M}(m\to p)} = \langle f, g^\dagger \circ h \rangle_{\mathcal{M}(m\to n)} = \langle g, h \circ f^\dagger \rangle_{\mathcal{M}(n\to p)}. \quad (3.8)$$ [12]

For the remainder of this section, we simultaneously develop the theory of traces on semisimple categories and on C* categories; the extra adjectives and conditions required in the latter case appear parenthetically. We denote by G the multiplicative group $\mathbb{C}^\times$ in the algebraic setting or $\mathbb{R}_{>0}$ in the unitary setting.

NOTATION 3.62. – Suppose $\mathcal{M}$ is finitely semisimple and $\mathrm{Irr}(\mathcal{M}) := \{x_1, \ldots, x_r\}$ is a choice of representatives of the isomorphism classes of simple objects in $\mathcal{M}$. Let $\mathcal{E}(\mathcal{M})$ denote $\mathrm{End}(\mathcal{M})$ in the algebraic setting, and $\mathrm{End}^\dagger(\mathcal{M})$ in the unitary setting. If $\mathcal{N}$ is a (C*) category and $y_1, \ldots, y_r$ are objects in $\mathcal{N}$, then there is a (dagger) functor $\mathcal{F}_{y_1,\ldots,y_r} : \mathcal{M} \to \mathcal{N}$ that is unique up to unique (unitary) isomorphism such that $\mathcal{F}(x_i) = y_i$. Furthermore, any (dagger) functor out of $\mathcal{M}$ is of this form. In particular, we let $E_{ij} \in \mathcal{E}(\mathcal{M})$ denote the (dagger) functor which sends x_i to x_j and sends x_k to the zero object for all $k \neq i$. Then $\{E_{ij} \mid 1 \le i, j \le r\}$ is a choice of representatives of isomorphism classes of simple objects in $\mathcal{E}(\mathcal{M})$. Thus in the algebraic setting, $\mathcal{E}(\mathcal{M})$ is equivalent to the category of $\mathcal{G}_r$-graded vector spaces $\mathsf{Vec}[\mathcal{G}_r]$, and in the unitary setting, $\mathcal{E}(\mathcal{M})$ is dagger equivalent to $\mathsf{Hilb}[\mathcal{G}_r]$. In either case, the universal grading groupoid of $\mathcal{E}(\mathcal{M})$ is $\mathcal{G}_r$.

LEMMA 3.63. – *Let $\mathcal{V}$ be* Vec *(respectively* Hilb*). The function from (unitary) traces on $\mathcal{V}$ to G given by $\mathrm{Tr}^{\mathcal{V}} \longmapsto \mathrm{Tr}^{\mathcal{V}}_{\mathbb{C}}(\mathrm{id}_{\mathbb{C}})$ is a bijection.*

Proof. – For surjectivity, we note that if $\lambda \in G$, then $(\lambda\,\mathrm{Tr})(V) := \lambda \dim(V)$ is a trace on $\mathcal{V}$ which satisfies $(\lambda\,\mathrm{Tr})_{\mathbb{C}}(\mathrm{id}_{\mathbb{C}}) = \lambda$.

For injectivity, we prove that $\mathrm{Tr}^{\mathcal{V}}$ is determined by $\mathrm{Tr}^{\mathcal{V}}_{\mathbb{C}}(\mathrm{id}_{\mathbb{C}})$. Let $V \in \mathcal{V}$ and choose a(n) (orthonormal) basis $v_1, \ldots v_n$ for V. Let $\pi_j : V \to \mathbb{C}$ be the projection $\sum_i a_i v_i \mapsto a_j$, and let $\iota_k : \mathbb{C} \to V$ be the inclusion $\lambda \mapsto \lambda v_j$. The composites $\iota_k \pi_j$ span $\mathrm{End}(V)$, and

$$\mathrm{Tr}^{\mathcal{V}}_V(\iota_k \pi_j) = \mathrm{Tr}^{\mathcal{V}}_{\mathbb{C}}(\pi_j \iota_k) = \delta_{j=k}\,\mathrm{Tr}^{\mathcal{V}}_{\mathbb{C}}(\mathrm{id}_{\mathbb{C}}).$$

Hence $\mathrm{Tr}^{\mathcal{V}}$ is completely determined by $\mathrm{Tr}^{\mathcal{V}}_{\mathbb{C}}(\mathrm{id}_{\mathbb{C}})$, which proves injectivity. □

PROPOSITION 3.64. – *The function from (unitary) traces on $\mathcal{M}$ to G^r, where r is the rank of $\mathcal{M}$, given by*

$$\mathrm{Tr}^{\mathcal{M}} \longmapsto (\mathrm{Tr}^{\mathcal{M}}_{x_1}(\mathrm{id}_{x_1}), \ldots, \mathrm{Tr}^{\mathcal{M}}_{x_r}(\mathrm{id}_{x_r}))$$

is a bijection.

Proof. – If $\mathcal{M}$ has r distinct isomorphism classes of simple objects, $\mathcal{M}$ is (dagger) equivalent to a(n orthogonal) direct sum $\bigoplus_{i=1}^r \mathsf{Vec}$ (respectively $\bigoplus_{i=1}^r \mathsf{Hilb}$). Since there are no maps between objects in the different summands, a (unitary) trace on $\bigoplus_{i=1}^r \mathsf{Vec}$ (respectively $\bigoplus_{i=1}^r \mathsf{Hilb}$) is equivalent to independently giving a (unitary) trace on each of the r copies of Vec (respectively Hilb). The result now follows from Lemma 3.63. □

[12] We note that the second equality in (3.8) holds if and only if for each $m \in \mathcal{M}$, the linear functor $\mathcal{M}(- \to m) : \mathcal{M}^{\mathrm{op}} \to \mathsf{Hilb}$ is a dagger functor. In this case, the first equality in (3.8) holds if and only if the Yoneda embedding $m \mapsto \mathcal{M}(- \to m)$ is a (fully faithful) dagger functor $\mathcal{M} \hookrightarrow \mathsf{Fun}^\dagger(\mathcal{M}^{\mathrm{op}} \to \mathsf{Hilb})$.

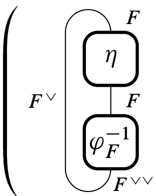
PROPOSITION 3.65. – *The function from (unitary) pivotal structures on $\mathcal{E}(\mathcal{M})$ to G^{r-1} given by*

$$\varphi \longmapsto \left(\dim_L^{\varphi}(E_{i+1,i})\right)_{i=1}^{r-1}$$

is a bijection.

Proof. – There exists a canonical (unitary) spherical structure on $\mathcal{E}(\mathcal{M})$ where all objects have left and right dimension 1. Thus the (unitary) pivotal structures on $\mathcal{E}(\mathcal{M})$ form a torsor over $\operatorname{Hom}(\mathcal{G}_r \to G)$. Such a homomorphism is uniquely determined by its image on $E_{i+1,i}$ for $1 \leq i \leq r-1$ as in Remark 3.51. □

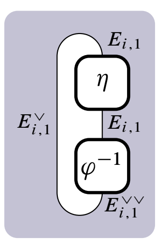
Given a pivotal structure φ on $\mathcal{E}(\mathcal{M})$, the left pivotal trace $\operatorname{tr}_L^{\varphi}$ takes values in $\mathcal{E}(\mathcal{M})(\mathrm{id} \Rightarrow \mathrm{id}) \cong \mathbb{C}^r$. Choosing a simple object x_i induces a $\mathbb{C}$-valued trace on $\mathcal{E}(\mathcal{M})$ by projecting to the x_i-component of $1_{\mathcal{E}(\mathcal{M})} := \mathrm{id}_{\mathcal{M}}$. That is, if $F \in \mathcal{E}(\mathcal{M})$ and $\eta : F \Rightarrow F$ is a natural transformation, we define $\operatorname{Tr}_F^{\mathcal{E}(\mathcal{M}),x_i}(\eta)$ by the formula

(3.9) $\operatorname{Tr}_F^{\mathcal{E}(\mathcal{M}),x_i}(\eta) \cdot \mathrm{id}_{x_i} := \Bigg($  $\Bigg)_{x_i} = \operatorname{tr}_L^{\varphi}(\eta)_{x_i}.$

We define the j-th column functor $F_j : \mathcal{M} \to \mathcal{E}(\mathcal{M})$ by letting $F_j(m)$ be the (dagger) functor which sends x_j to m and all other simples to the zero object. We denote by $\mathcal{E}(\mathcal{M})_j$ the essential image of $\mathcal{M}$ under F_j, which consists of (orthogonal) direct sums of the objects E_{ij} for $i = 1, \ldots, r$. Notice that F_i is a (dagger) equivalence $\mathcal{M} \cong \mathcal{E}(\mathcal{M})_j$. This is the categorical analogue of identifying a vector space with matrices supported on the j-th column.

We now choose the simple object $x_1 \in \mathcal{M}$ giving us our scalar-valued trace $\operatorname{Tr}^{\mathcal{E}(\mathcal{M}),x_1}$ on $\mathcal{E}(\mathcal{M})$. By restriction, we get a (unitary) trace on $\mathcal{E}(\mathcal{M})_1 \cong \mathcal{M}$, which we denote by $\operatorname{Tr}^{\mathcal{E}(\mathcal{M})_1}$. Notice that taking the x_1-component of $\operatorname{tr}_L^{\varphi}$ can be viewed as cutting down $\mathcal{E}(\mathcal{M})(\mathrm{id} \Rightarrow \mathrm{id})$ by the (orthogonal) projection onto the summand $E_{11} \subset \mathrm{id}_{\mathcal{E}(\mathcal{M})}$. Denoting this projection by a shading, we get the following diagrammatic formula for $\operatorname{Tr}_{E_{i1}}^{\mathcal{E}(\mathcal{M})_1}(\eta)$ for $\eta : E_{i1} \Rightarrow E_{i1}$:

(3.10)

$\operatorname{Tr}_{E_{i1}}^{\mathcal{E}(\mathcal{M})_1}(\eta) :=$  $\in \mathcal{E}(E_{1,1} \to E_{1,1}) \cong \mathbb{C}$  $:= \operatorname{proj}_{E_{1,1}}.$

We have thus proved:

PROPOSITION 3.66. – *The function $\varphi \mapsto \operatorname{tr}_L^{\varphi}|_{\mathcal{E}(\mathcal{M})_1}$ together with the (dagger) equivalence $\mathcal{E}(\mathcal{M})_1 \cong \mathcal{M}$, induces a function Δ from the set of (unitary) equivalence classes of pivotal structures on $\mathcal{E}(\mathcal{M})$ to the set of (unitary) equivalence classes of (unitary) traces on $\mathcal{M}$.*

We now construct a left inverse to the function Δ.

PROPOSITION 3.67. – *The function Λ defined by*

$$\left\{\text{Traces } \mathrm{Tr}^{\mathcal{M}}\right\} \underset{\text{Prop. 3.64}}{\cong} G^r \ni (a_1, \dots, a_r) \mapsto \left(\frac{a_2}{a_1}, \dots, \frac{a_r}{a_{r-1}}\right) \in G^{r-1}$$
$$\underset{\text{Prop. 3.65}}{\cong} \left\{\text{Pivotal structures } \varphi^{\mathcal{E}}\right\}$$

is surjective and provides a left inverse to Δ from Proposition 3.66. Moreover, under Λ, two (unitary) traces map to the same (unitary) pivotal structure if and only if they are proportional.

Proof. – Surjectivity of Λ is obvious. Notice that $a_{i+1}/a_i = b_{i+1}/b_i$ for all $1 \le i \le r-1$ if and only if $a_i/b_i = a_{i+1}/b_{i+1}$ for $1 \le i \le r-1$ if and only if $(a_1, \dots, a_r)$ is proportional to $(b_1, \dots, b_r)$.

Finally, we show $\Lambda \circ \Delta = \mathrm{id}$. Let $\mathrm{Tr}^{\mathcal{M}}$ be the (unitary) trace on $\mathcal{M}$ corresponding to $(a_1, \dots, a_r) \in G^r$ under Proposition 3.64, and let φ be the corresponding (unitary) pivotal structure on $\mathcal{E}$ corresponding to $(a_2/a_1, \dots, a_r/a_{r-1})$. It suffices to prove that $\mathrm{tr}^{\varphi}|_{\mathcal{N}}$ is proportional to $\mathrm{Tr}^{\mathcal{M}}$ under the equivalence $\mathcal{N} \cong \mathcal{M}$, since proportional traces give rise to (unitarily) equivalent pivotal structures under Λ. Indeed, for a fixed $1 \le j \le r$, by monoidality of φ, we have

$$\mathrm{tr}_L^{\varphi}(\mathrm{id}_{E_{j,1}}) = \dim_L^{\mathcal{E}}(E_{j,1}) = \prod_{i=1}^{j-1} \dim_L^{\varphi}(\mathrm{id}_{E_{i+1,i}}) = \prod_{i=1}^{j-1} \frac{a_{i+1}}{a_i} = \frac{a_j}{a_1} = \frac{1}{a_1} \mathrm{Tr}_{x_j}^{\mathcal{M}}(\mathrm{id}_{x_j})$$

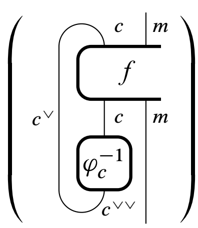
as in the proof of Proposition 3.64. Hence $\mathrm{tr}_L^{\varphi} = a_1^{-1} \mathrm{Tr}^{\mathcal{M}}$ under the (dagger) equivalence $\mathcal{N} \cong \mathcal{M}$. □

In particular, if we change our choice of simple object x_1 this only rescales the trace on $\mathcal{M}$.

3.6.2. *Pivotal module categories for pivotal categories.* – We now expand on the previous section to the scenario where $\mathcal{M}$ is equipped with the structure of a $\mathscr{C}$-module (C*) category, where $\mathscr{C}$ is a (unitary) multitensor category. Some other interesting results related to the non-unitary multifusion case were recently obtained in [20, §2.6].

DEFINITION 3.68 ([96]). – If $(\mathscr{C}, \varphi)$ is a semisimple (unitary) pivotal multifusion category and $\mathcal{M}$ is a semisimple left $\mathscr{C}$-module (C*) category with a (unitary) trace $\mathrm{Tr}^{\mathcal{M}}$, then $(\mathcal{M}, \mathrm{Tr}^{\mathcal{M}})$ is called a *pivotal* $\mathscr{C}$-module (C*) category if we have the following compatibility of $\mathrm{Tr}^{\mathcal{M}}$ with the left partial trace in $\mathscr{C}$: for all $c \in \mathscr{C}$, $m \in \mathcal{M}$, and $f \in \mathcal{M}(c \triangleright m \to c \triangleright m)$,

(3.11)
$$\mathrm{Tr}_{c \triangleright m}^{\mathcal{M}}(f) = \mathrm{Tr}_m^{\mathcal{M}}[(\mathrm{ev}_c \triangleright \mathrm{id}_m) \circ (\mathrm{id}_{c^\vee} \otimes f) \circ (\mathrm{id}_{c^\vee} \otimes (\varphi_c)^{-1} \triangleright \mathrm{id}_m) \circ (\mathrm{coev}_{c^\vee} \triangleright \mathrm{id}_m)]$$

$= \mathrm{Tr}_m^{\mathcal{M}}$  .

Here, we use the diagrammatic convention of [5] for left $\mathscr{C}$-module categories, where the coupons in $\mathcal{M}$ are drawn cut open on the right hand side to indicate the absence of any right action.

REMARK 3.69. – In [96, §4.1], it is shown that when $(\mathscr{C}, \varphi)$ is pivotal fusion and $\mathcal{M}$ is indecomposable, traces on $\mathcal{M}$ which satisfy (3.11) are unique up to scaling. Moreover, by [96, §5], when $\mathscr{C}$ is pseudo-unitary equipped with its canonical spherical structure, every indecomposable module category $\mathcal{M}$ admits a trace $\operatorname{Tr}^{\mathcal{M}}$ which satisfies (3.11).

When $\mathscr{C}$ is unitary, every indecomposable unitary module category $\mathcal{M}$ is of the form $\mathsf{Mod}_{\mathscr{C}}(A)$ for A an irreducible Q-system (normalized C* Frobenius algebra) in $\mathscr{C}$ by [77, Thm. A.1]. This is enough to get a unique unitary trace $\operatorname{Tr}^{\mathcal{M}}$ which satisfies (3.11).

REMARK 3.70. – In fact, pivotal structures on the 2-shaded multifusion category built from $\mathscr{C}$, $\mathcal{M}$, and its dual category correspond exactly to module traces on $\mathcal{M}$ *not up to rescaling*. That is rescaling the choice of trace changes the pivotal structure on the odd part of the 2-shaded multifusion category, but in the even parts this rescaling cancels out.

DEFINITION 3.71. – Given a tensor functor between pivotal categories $(\Psi, \mu) : (\mathscr{C}, \varphi^{\mathscr{C}}) \to (\mathcal{D}, \varphi^{\mathcal{D}})$, where our convention for the tensorator natural isomorphism is $\mu_{a,b} : \Psi(a) \otimes \Psi(b) \to \Psi(a \otimes b)$, we get a canonical anti-monoidal natural isomorphism $\delta_c : \Psi(c^\vee) \to \Psi(c)^\vee$ given by

$$\delta_c := ([\Psi(\mathrm{ev}_c) \circ \mu_{c^\vee,c}] \otimes \mathrm{id}_{\Psi(c)^\vee}) \circ (\mathrm{id}_{\Phi(c^\vee)} \otimes \mathrm{coev}_{\Psi(c)}). \tag{3.12}$$

We call (Ψ, μ) *pivotal* if $\delta_c^\vee \circ \varphi_{\Psi(c)} = \delta_{c^\vee} \circ \Psi(\varphi_c)$ for all $c \in \mathscr{C}$.

THEOREM 3.72. – *Suppose $\mathcal{M}$ is a finitely semisimple left $\mathscr{C}$-module (C*) category, and let $(\Psi, \mu) : \mathscr{C} \to \mathcal{E}(\mathcal{M})$ be the corresponding (dagger) tensor functor from Lemma 3.20. The following are equivalent for a (unitary) trace $\operatorname{Tr}^{\mathcal{M}}$ on $\mathcal{M}$ and its induced (unitary) pivotal structure φ on $\mathcal{E}(\mathcal{M})$ from Proposition 3.67.*

1. *Compatibility condition* (3.11) *holds.*
2. *The corresponding (dagger) tensor functor* (Ψ, μ) *is pivotal.*

(Note that (2) implies (1) is relatively straightforward since one can use the graphical calculi for pivotal categories and module categories with trace. But for (1) implies (2) since we do not know that the functor is pivotal we cannot use a standard graphical calculus and need to keep track of all of the structure maps. This explains why the formulas in the following proof have a lot of explicit structure maps .)

Proof. – As in the discussion right before Proposition 3.66, there is a (dagger) equivalence

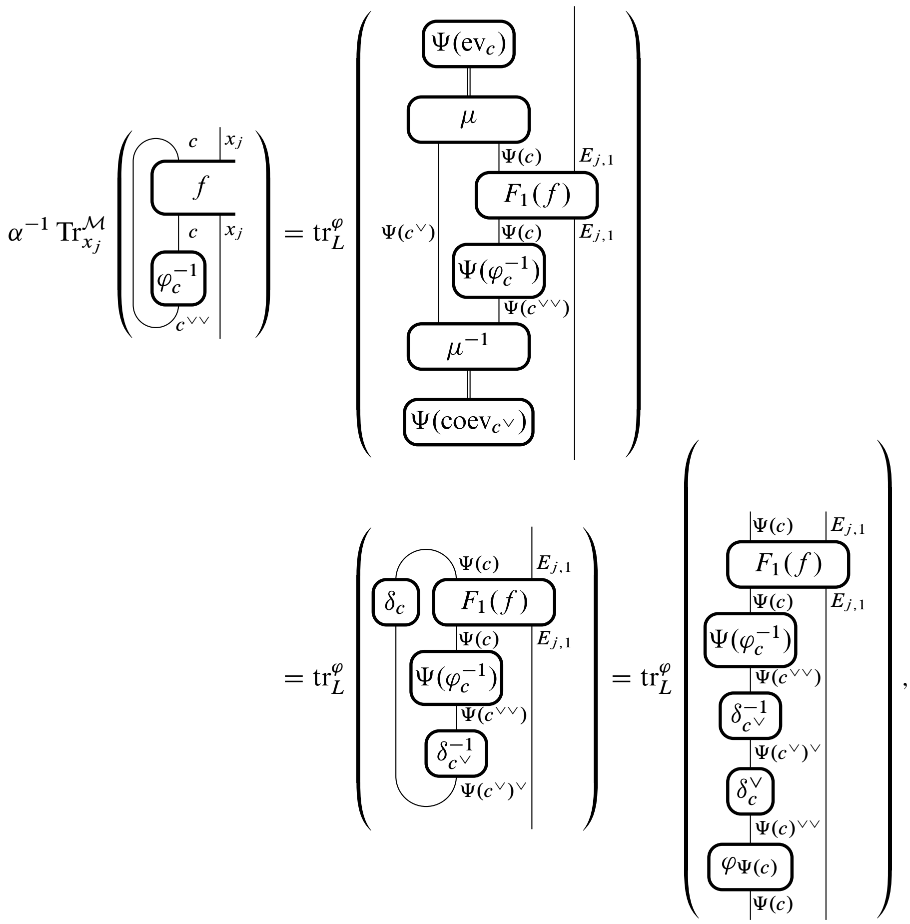
$$F_1 : \mathcal{M} \xrightarrow{\sim} \mathcal{E}(\mathcal{M})_1 := \operatorname{span}\{E_{i,1} \mid 1 \le i \le r\} \subset \mathcal{E}(\mathcal{M}) \qquad x_j \mapsto E_{j,1}.$$

In fact, using the tensorator of Ψ, we can equip the (dagger) equivalence F_1 with a (unitary) *modulator*

$$\begin{aligned}\nu_{a,b,m} : \Psi(a) \otimes F_1(b \rhd x_j) &= \Psi(a) \otimes (\Psi(b) \otimes E_{j,1}) = (\Psi(a) \otimes \Psi(b)) \otimes E_{j,1} \\ &\cong \Psi(a \otimes b) \otimes E_{j,1} \cong F_1(a \otimes b \rhd x_j),\end{aligned}$$

extending it to a (dagger) equivalence of $\mathscr{C}$-module (C*) categories. As in the proof of Proposition 3.67, there is a non-zero scalar $\alpha \in G$ such that for every $f \in \mathcal{M}(m \to m)$, $\mathrm{Tr}^{\mathcal{M}}(f) = \alpha\, \mathrm{tr}^{\varphi}_L(F(f))$. Thus for $1 \leq j \leq r$ and $f \in \mathcal{M}(c \rhd x_j \to c \rhd x_j) \cong \mathcal{E}(\mathcal{M})(\Psi(c) \otimes E_{j,1} \to \Psi(c) \otimes E_{j,1})$, we always have

(3.13)



where $\delta_c \in \mathcal{E}(\mathcal{M})(\Psi(c^\vee) \to \Psi(c)^\vee)$ is the canonical isomorphism from (3.12).

(1) $\Rightarrow$ (2): Suppose (3.11) holds. Then for all $1 \leq j \leq r$ and

$$f \in \mathcal{M}(c \rhd x_j \to c \rhd x_j) \cong \mathcal{E}(\Psi(c)\otimes E_{j,1} \to \Psi(c)\otimes E_{j,1}), \quad \mathrm{tr}^{\varphi}_L(F_1(f)) = \alpha^{-1}\,\mathrm{Tr}^{\mathcal{M}}_{c\rhd x_j}(f),$$

which is equal to the right hand side of (3.13). Hence

$$\mathrm{tr}^{\varphi}_L(f \circ [(\mathrm{id}_{\Psi(c)} - \Psi(\varphi_c^{-1}) \circ \delta_{c^\vee}^{-1} \circ \delta_c^\vee \circ \varphi_{\Psi(c)}) \otimes \mathrm{id}_{E_{j,1}}]) = 0.$$

Since tr^{φ}_L is nondegenerate (e.g., see [83, Lem. 2.6]), we must have

$$(\mathrm{id}_{\Psi(c)} - \Psi(\varphi_c^{-1}) \circ \delta_{c^\vee}^{-1} \circ \delta_c^\vee \circ \varphi_{\Psi(c)}) \otimes \mathrm{id}_{E_{j,1}} = 0$$

for all $1 \leq j \leq r$. Now taking right partial traces in $\mathcal{E}(\mathcal{M})$, we must have $\mathrm{id}_{\Psi(c)} = \Psi(\varphi_c^{-1}) \circ \delta_{c^\vee}^{-1} \circ \delta_c^\vee \circ \varphi_{\Psi(c)}$, so (Ψ, μ) is pivotal.

(2) $\Rightarrow$ (1): Suppose that (Ψ, μ) is pivotal, so that $\delta_c^\vee \circ \varphi_{\Psi(c)} = \delta_{c^\vee} \circ \Psi(\varphi_c)$. Then for any $f \in \mathcal{M}(c \triangleright m \to c \triangleright m)$, the right hand side of (3.13) is equal to $\mathrm{tr}_L^\varphi(F_1(f)) = \alpha^{-1} \operatorname{Tr}^{\mathcal{M}}_{c \triangleright x_j}(f)$, and thus (3.11) holds. □

We have an analogous omnibus theorem in the pivotal and unitary pivotal settings.

THEOREM 3.73. – *Suppose that $\mathscr{C}$ is a (unitary) pivotal fusion category then*

— *Module category structures with a (unitary) trace on a* (C^*) *category $\mathcal{M}$ correspond exactly to (unitary) pivotal tensor functors $\mathscr{C} \to \mathcal{E}(\mathcal{M})$.*
— *A (unitary) pivotal fusion category $\mathcal{D}$ is (unitary) pivotal Morita equivalent to $\mathscr{C}$ if and only if there is an indecomposable semisimple pivotal* (C^*) *module category* $(\mathcal{M}, \operatorname{Tr}^{\mathcal{M}})$ *such that $\mathcal{D}$ is (unitary) pivotal tensor equivalent to* $\operatorname{End}_{\mathscr{C}}(\mathcal{M})$*, the $\mathscr{C}$-linear (dagger) endofunctors of $\mathcal{M}$. Furthermore, the pivotal left $\mathscr{C}$-module* (C^*) *categories* $(\mathcal{M}, \operatorname{Tr}^{\mathcal{M}})$ *which realize a (unitary) pivotal Morita equivalence between $\mathscr{C}$ and $\mathcal{D}$ are a torsor for the group of (unitary) pivotal outer automorphisms* $\operatorname{Out}(\mathcal{D})$.
— *Tuples* $(\mathcal{M}, \operatorname{Tr}^{\mathcal{M}}, m)$ *where* $(\mathcal{M}, \operatorname{Tr}^{\mathcal{M}})$ *is an indecomposable semisimple pivotal $\mathscr{C}$-module* (C^*) *category and $m \in \mathcal{M}$ is a chosen simple object correspond exactly to connected normalized Frobenius algebras (irreducible Q-systems* [7]*) A in $\mathscr{C}$. The dual category corresponding to A is the category of A-A bimodules in $\mathscr{C}$* [75].

REMARK 3.74. – This theorem is analogous to the purely algebraic Theorem 3.15. We warn the reader that if $\mathscr{C}$ is a (unitary) pivotal fusion category, the answers to our main problems might in principle be *different* in the algebraic and pivotal (and unitary pivotal) settings.

For example, there might be several pivotal $\mathscr{C}$-module (C^*) categories which are equivalent just as algebraic $\mathscr{C}$-module categories, or there may be an algebraic module category which cannot be endowed with a (unitary) compatible trace (or even a dagger structure!). These phenomena do not happen for Extended Haagerup, but it is interesting to ask whether they ever occur.

3.6.3. *The embedding theorem for pivotal module categories.* – In this section, we finally prove the embedding theorem for pivotal module categories. We begin with a discussion of the planar algebra of a bipartite graph [57]. Our definition will simply use a Frobenius-Perron vertex weighting on our finite graph to extend the action of 2-shaded monoidal tangles for a bipartite graph monoidal algebra to an action of shaded planar tangles. We then show how to recover the usual definition of the graph planar algebra from [57].

DEFINITION 3.75. – Let $\Gamma = (V_+, V_-, E)$ be a finite connected bipartite graph with even/$+$ vertices V_+, odd/$-$ vertices V_-, and edges E. We consider an edge $\varepsilon \in E$ as directed from $+$ to $-$ with source $s(\varepsilon) \in V_+$ and target $t(\varepsilon) \in V_-$. We write ε^* for the same edge with the opposite direction. Let λ denote any Frobenius-Perron eigenvector of the adjacency matrix of Γ. (13)

(13) The definition of the graph planar algebra $\mathcal{G}_\bullet$ does not depend on the normalization of the Frobenius-Perron eigenvector λ. In Remark 3.76 below, we will define a spherical faithful state on $\mathcal{G}_\bullet$ using a particular normalization of λ.

The 2-shaded graph monoidal algebra $\mathcal{G}_{\bullet\to\bullet} = \mathcal{GMA}(\Gamma)_{\bullet\to\bullet}$ is defined analogously to the unshaded version in Definition 3.35. The $\mathbb{C}$-vector spaces $\mathcal{GMA}(\Gamma)_{(m\to n),\pm}$ are spanned by pairs of paths (p, q) of length m, n respectively which start at the same $\pm$ vertex and end at the same vertex. Note that $\mathcal{GMA}(\Gamma)_{(m\to n),\pm}$ is defined only when $m \equiv n \mod 2$. The action of shaded monoidal tangles is given by the state-sum Formula (3.1). Note that $\mathcal{G}_{\bullet\to\bullet}$ is unitary with †-structure given by the conjugate-linear extension of $(p, q)^\dagger = (q, p)$.

Now given a shaded *planar* tangle T of type $((t_0, \pm_0); (t_1, \pm_1), \ldots, (t_k, \pm_k))$ whose input and output disks are rectangles with the star on the left, where the i-th disk has n_i strings emanating from the top and m_i from the bottom with $m_i + n_i = 2t_i$, we describe its action on tuples of basis elements $(p_i, q_i) \in \mathcal{G}_{m_i\to n_i,\pm}$ by the weighted state-sum formula

$$T((p_1, q_1), \ldots, (p_k, q_k)) := \sum_{\text{states } \sigma \text{ on } T} c(T;\sigma) \left(\prod_{1\le i\le k} \delta_{\sigma|_i,(p_i,q_i)} \right) \sigma|_0. \tag{3.14}$$

A *state* σ on the tangle T is an assignment of even vertices to unshaded regions, odd vertices to shaded regions, and edges to strings such that if a string labeled by ϵ separates two regions, then $s(\epsilon)$ is assigned to that unshaded region, and $t(\epsilon)$ is assigned to that shaded region. Now $\sigma|_i$ denotes the pair of paths in $\mathcal{GMA}(\Gamma)_{m_i\to n_i}$ obtained from reading the bottom and top boundaries of the i-th input disk from left to right. In other words, we sum only over states that are 'compatible' with the loops we start with. To define the constant $c(T;\sigma)$, we first isotope T so that strings are sufficiently smooth. Now consider the set $E(T)$ of all local maxima and minima of strings of T. Then

$$c(T;\sigma) = \prod_{e\in E(T)} \left(\frac{\lambda(\sigma(e_{\text{convex}}))}{\lambda(\sigma(e_{\text{concave}}))} \right)^{1/2},$$

where e_{convex} is the convex region of the extremum e, and e_{concave} is the concave region of e.

This definition appears to be highly dependent on the choice of numbers of strings m_i, n_i emanating from the bottom and top of each input and output disk. However, every space $\mathcal{G}_{m\to n,\pm}$ is canonically isomorphic to $\mathcal{G}_{m+n,\pm} := \mathcal{G}_{m+n\to 0,\pm}$ by

$$(p, q) \mapsto \left(\frac{\lambda(t(p))}{\lambda(s(p))} \right)^{1/2} pq^*. \tag{3.15}$$

Here, instead of writing the pair of paths $(pq^*, \emptyset)$ where the second has length zero, we only write the first path pq^*, which is actually a loop of length $2t = m + n$. Indeed, by post-composing with instances of the above isomorphism and precomposing with instances of its inverse as appropriate, we see that the action of planar tangles does not depend on the decomposition $2t_i = m_i + n_i$. As in [57, Th. 3.1], changing a tangle by a Morse cancelation or rotating a single input out output disk by 2π does not change the action of the tangle. Hence the isomorphisms (3.15) endow the spaces $\mathcal{G}_{n,\pm}$ with the structure of a shaded planar algebra called the *graph planar algebra*, denoted $\mathcal{G}_\bullet$. We recover the definition from [57] by always choosing $m_i = n_i = t_i$ for every input and output disk of T.

The †-structure of $\mathcal{G}_\bullet$ is inherited from the graph monoidal algebra $\mathcal{G}_{\bullet\to\bullet}$. Since $\dagger : \mathcal{G}_{m\to n,\pm} \to \mathcal{G}_{n\to m,\pm}$, the identification of both spaces with $\mathcal{G}_{m+n,\pm}$ means that $(pq^*)^\dagger = qp^*$, i.e., † is the conjugate-linear extension of reversing a loop. It is straightforward to see that the † structure on $\mathcal{G}_\bullet$ is compatible with (3.15), and is thus compatible with the reflection of planar tangles. As the underlying monoidal algebra is unitary, so is the graph planar algebra.

REMARK 3.76. – The graph planar algebra, and hence its projection category, is in general not spherical. For example, taking any edge ε which connects two vertices of distinct weights, the projection $\varepsilon\varepsilon^* \in \mathcal{G}_{1,+}$ has distinct left and right traces. However, if we normalize the Frobenius-Perron eigenvector λ so that $\sum_{u\in V_+} \lambda_u^2 = 1 = \sum_{v\in V_-} \lambda_v^2$, then $\psi(p_v) := \lambda_v^2$ defines a spherical faithful state on $\mathcal{G}_\bullet$ [57, Prop. 3.4].

NOTATION 3.77. – To state the main theorems of this section, we fix the following notation.

— $\Gamma = (V_+, V_-, E)$ is a connected bipartite graph
— λ is any Frobenius-Perron eigenvector of Γ.
— $\mathcal{G}_\bullet$ is the bipartite graph planar algebra of Γ
— $\mathcal{M} = \mathsf{Hilb}^{V_+} \oplus \mathsf{Hilb}^{V_-}$ is one copy of Hilb for each vertex of Γ.
— $\mathrm{Tr}^{\mathcal{M}}$ is the unitary trace on $\mathcal{M}$ corresponding to $\lambda \in G^{V_+ \amalg V_-}$ under Proposition 3.64.
— $\mathrm{End}^\dagger(\mathcal{M})$ is the unitary multifusion category of dagger endofunctors of $\mathcal{M}$.
— $\mathcal{U}$ is the universal grading groupoid of $\mathrm{End}^\dagger(\mathcal{M})$, which is the groupoid with $n_+ + n_-$ objects, and a unique isomorphism between any two objects.
— $F = \bigoplus_{\varepsilon\in E} E_{t(\varepsilon),s(\varepsilon)} \in \mathrm{End}^\dagger(\mathcal{M})$.
— $\vee_{\text{standard}}$ is the standard unitary dual functor with respect to F from [83, 34], which is induced by the standard groupoid homomorphism defined from λ as in (3.6).
— $\mathcal{H}_\bullet$ is the planar algebra corresponding to $(\mathrm{End}^\dagger(\mathcal{M}), \vee_{\text{standard}}, F)$ under Theorem 3.56.

THEOREM 3.78. – *With the above notation, the †-isomorphism of the underlying monoidal algebras $\mathcal{H}_\bullet \cong \mathcal{G}_\bullet$ from Theorem 3.37 gives a †-isomorphism of unitary planar algebras.*

Proof. – As the isomorphism from Theorem 3.37 identifies the underlying unitary monoidal algebras, we only need to check that the actions of cup and cap agree. Since cup is always the † of cap in a unitary planar algebra as discussed in Remark 3.48, we only need to check each shading of cap agrees.

First, the standard evaluation and coevaluation with respect to F are given by

$$\mathrm{ev}^{\text{standard}}_{E_{u,v}} := \left(\frac{\lambda(u)}{\lambda(v)}\right)^{1/2} \qquad \mathrm{coev}^{\text{standard}}_{E_{u,v}} := \left(\frac{\lambda(v)}{\lambda(u)}\right)^{1/2}.$$

Indeed, it is straightforward to check that the ratio $\pi^{\text{standard}}(E_{u,v})$ of the left to right standard pivotal dimension of $E_{u,v}$ is given exactly by (3.6). Thus we see from the graphical calculus for $\mathrm{End}^\dagger(\mathcal{M})$ that under the isomorphism of underlying monoidal algebras from Theorem 3.37, the formula for each shading of cap is given by [14]

 $$= \sum_{\varepsilon\in E} \left(\frac{\lambda(t(\varepsilon))}{\lambda(s(\varepsilon))}\right)^{1/2} \varepsilon\varepsilon^*$$  $$= \sum_{\varepsilon\in E} \left(\frac{\lambda(s(\varepsilon))}{\lambda(t(\varepsilon))}\right)^{1/2} \varepsilon^*\varepsilon.$$

[14] Turning all strings down via (3.15) turns (p, q) into $(pq^*, \emptyset)$, and we suppress this second empty loop.

These are exactly the same formulas for each shading of cap given by the state-sum Formula (3.14). □

The following corollary follows immediately from Theorem 3.56.

COROLLARY 3.79. – *Let* $\Gamma = (V_+, V_-, E)$ *be a finite connected bipartite graph, and let* $\mathcal{G}_\bullet$ *be its graph planar algebra. Let* $\mathcal{M} = \mathsf{Hilb}^{n_+} \oplus \mathsf{Hilb}^{n_-}$ *where* $n_\pm = |V_\pm|$.

— *The idempotent category of* $\mathcal{G}_\bullet$ *is equivalent to* $\mathrm{End}(\mathcal{M})$*, as multifusion categories.*
— *The projection* C^* *category of* $\mathcal{G}_\bullet$ *is dagger equivalent to* $\mathrm{End}^\dagger(\mathcal{M})$*, as unitary multifusion categories.*

We now prove a version of the graph planar algebra embedding theorem [62] for module categories. Below, we fix a finite depth subfactor planar algebra $\mathcal{P}_\bullet$, and we denote by $(\mathscr{C}, X)$ the unitary multifusion category of projections of $\mathcal{P}_\bullet$ with distinguished object X corresponding to the unshaded-shaded strand of $\mathcal{P}_\bullet$. We endow $\mathscr{C}$ with the canonical spherical structure from [69, 107, 5, 83].

THEOREM 3.80. – *The following are equivalent:*

1. *An embedding of shaded planar* †*-algebras* $\mathcal{P}_\bullet \hookrightarrow \mathcal{G}_\bullet$
2. *A pivotal dagger tensor functor* $(\Psi, \mu) : (\mathscr{C}, \vee_{\text{spherical}}) \to (\mathrm{End}^\dagger(\mathcal{M}), \vee_{\text{standard}})$ *such that* $\Psi(X) = F$ *and* $\Psi(\mathrm{id}_X) = \mathrm{id}_F$*, and*
3. *an indecomposable left* $\mathscr{C}$*-module structure on* $\mathcal{M}$*, compatible with the dagger structures of* $\mathscr{C}$ *and* $\mathcal{M}$*, together with a unitary trace* $\mathrm{Tr}^{\mathcal{M}}$ *defined up to scalar satisfying the compatibility condition* (3.11)*, whose fusion graph with respect to* X *is* Γ*. More precisely, the category of* $\mathscr{C}$*-module structures on* $\mathcal{M}$ *and module-functor structures on the identity is equivalent to the category of embeddings and guagings.*

Proof. – The equivalence of (1) and (2) follows from Theorem 3.56 together with Corollary 3.79. The equivalence of (2) and (3) follows from Lemma 3.20 together with Proposition 3.67 and Theorem 3.72. □

WARNING 3.81. – Since Theorem 3.80 is about module structures on a fixed $\mathcal{M}$, it "overcounts" module categories in the following sense. If the graph Γ has a graph automorphism, then two different module structures on $\mathcal{M}$ will be equivalent to each other via a non-identity functor built from the graph automorphism.

4. Combinatorics of potential (bi)module categories for Extended Haagerup

4.1. Summary of the combinatorial techniques for classifying module and bimodule categories

In this section we prove a partial classification of all fusion categories Morita equivalent to the Extended Haagerup fusion categories and all Morita equivalences between them. Specifically, we show that there are at most four fusion categories in the Morita equivalence class, that there is exactly one Morita equivalence between any two that actually exist, and determine the fusion rules for all possible fusion categories and bimodule categories. This argument closely follows the outline of [42, 43], so we begin by briefly summarizing the techniques of these articles.

Given a fusion category $\mathscr{C}$, one gets a fusion ring $C := K_0(\mathscr{C})$ with basis consisting of the isomorphism classes of simple objects in $\mathscr{C}$ and non-negative structure constants N_{ij}^k for multiplication coming from the fusion rules $X_i \otimes X_j \cong \bigoplus N_{ij}^k X_k$. This fusion ring has an involution corresponding to taking duals. It is natural to wonder: given a candidate fusion ring, is it categorifiable into a fusion category, and if so, how many fusion categories categorify our fusion ring? Typically, each of these questions is quite difficult [82, 81, 66]; combinatorics alone tells you very little about a single fusion category.

Given several fusion categories $\mathscr{C}_i$ and some Morita equivalences $\mathcal{M}_{ij}^k$ between $\mathscr{C}_i$ and $\mathscr{C}_j$, one gets several fusion rings $C_i = K_0(\mathscr{C}_i)$, several fusion bimodules $M_{ij}^k = K_0(\mathcal{M}_{ij}^k)$, and many "composition rules" $M_{ij}^k \otimes_{C_j} M_{j\ell}^{k'} \to M_{i\ell}^{k''}$. This collection of data satisfies many combinatorial constraints. It is again natural to wonder: given a collection of fusion rings, fusion bimodules, and composition rules, are they categorifiable, and if so, in how many different ways? In general, this question is again quite difficult. However, in a small handful of examples coming from the small index subfactor classification program, we have seen that candidates which satisfy the many combinatorial constraints have been uniquely categorified. In contrast to the situation for a single fusion category, combinatorics often tells you quite a lot about the full Morita equivalence class of a known fusion category with a few known Morita equivalences.

Here is the outline in more detail. We start with some fusion categories $\mathscr{C}_i$ with fusion algebras C_i, and some Morita equivalences between them which we understand well. We first use a computer to list the fusion modules over the fusion rings C_i. (By 'fusion modules' we mean based modules satisfying some additional properties—see [43]. Sometimes the term 'NIMrep' is used in the literature; this is an abbreviation for non-negative integer matrix representations [29].) We identify a few of these fusion modules as coming from the known Morita equivalences, and we use some additional arguments to see that the known categorification is the only possible realization of these modules. (In our case, this step uses the uniqueness of the Extended Haagerup subfactor [6], which is much easier than existence.)

Second, we try to determine the possible fusion rings of the dual categories for each (real or hypothetical) module category. Using a computer, we can sometimes uniquely determine the dual fusion ring from a fusion module combinatorially, or at least produce a relatively small list. We then compute the fusion modules over each of these new fusion rings, as well as the fusion bimodules between each pair of rings in our collection.

At this point, we have a collection of rings, bimodules between them, and some information about categorification of some of the bimodules (coming from known algebra objects). We now use the following key fact: given a triple of fusion categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$, invertible bimodule categories ${}_{\mathcal{A}}\mathcal{K}_{\mathcal{B}}, {}_{\mathcal{B}}\mathcal{L}_{\mathcal{C}}, {}_{\mathcal{A}}\mathcal{M}_{\mathcal{C}}$, and a tensor equivalence

$${}_{\mathcal{A}}\mathcal{K}_{\mathcal{B}} \boxtimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{L}_{\mathcal{C}} \cong {}_{\mathcal{A}}\mathcal{M}_{\mathcal{C}},$$

we get an induced map on the decategorified bimodules over the fusion rings:

$${}_{A}K_{B} \otimes_{B} {}_{B}L_{C} \to {}_{A}M_{C}.$$

This induced map preserves positivity of coefficients and Frobenius-Perron dimensions. Moreover, the existence of such a map can be checked with a computer. If such a map does not exist, we say that the triple of fusion bimodules is *not multiplicatively compatible*. Thus categorification of many fusion modules or bimodules can be ruled out due to not being multiplicatively compatible with those fusion bimodules which have known categorifications. A similar argument can be used to compare the number of categorifications of different bimodules. This stage of the argument is a bit similar to playing Sudoku, since each time you rule out one possible bimodule, then the composites which were only compatible with the eliminated one are now themselves incompatible.

Following this outline, we can often deduce a lot of information about the Brauer-Picard groupoid from a relatively small amount of input data (such as existence of a few small objects which are known to have unique algebra structures). In particular, for the Extended Haagerup fusion categories, we start with our two fusion categories $\mathcal{EH}_1$ and $\mathcal{EH}_2$, the existence and uniqueness of the Extended Haagerup Morita equivalence between them, and the lack of automorphisms of the Extended Haagerup planar algebra. This data is sufficient to successfully run the above procedure to obtain the entire Morita equivalence class, as evidenced by Theorem 4.13 below.

4.2. The Brauer-Picard groupoid of Extended Haagerup

The Extended Haagerup subfactor gives a Morita equivalence between two fusion categories which are not tensor equivalent. The fusion rules for these two categories are given in §2.1 and 2.2; one of the categories has commuting fusion rules and the other one does not. We will call the category with commuting fusion rules $\mathcal{EH}_1$ and the other category $\mathcal{EH}_2$.

We refer the reader to [43] for precise definitions of fusion modules and bimodules and multiplicative compatibility of triples of modules/bimodules. Detailed descriptions of the computer algorithms used to search for fusion (bi)modules and to check for multiplicative compatibility are also described there.

LEMMA 4.1. – *There are exactly* 7 *fusion modules over* EH_1 *and exactly* 5 *fusion modules over* EH_2.

Proof. – Checked with computer. □

The data of the (right) fusion modules are presented in accompanying text files `EH1modules.txt` and `EH2modules.txt`. Each fusion module of rank r over the fusion ring of rank s is described by a list of r non-negative integer matrices of size $s \times r$. The (i, j)-th entry of the k-th matrix is the coefficient of the module basis element m_j in the product $x_i m_k$ (where x_i is a ring basis element). The bases for the fusion rings and modules are ordered with increasing Frobenius-Perron dimension.

From the list of matrices for a given fusion module, one can read off a corresponding list of objects in the fusion category which are of the form $\underline{\mathrm{End}}_{\mathcal{EH}_i}(m)$, the internal endomorphism object associated to a simple object m in a module category categorification. Such an internal endomorphism object necessarily admits an algebra structure if the module can be categorified. The multiplicity vector of the simple objects in $\mathcal{EH}_i$ in each such (hypothetical) algebra object is given by the j-th column of the j-th matrix. In particular, the first column of the first matrix in the data of the fusion module gives the multiplicity vector of the internal endomorphism algebra object with the smallest Frobenius-Perron dimension for any module category realization. Therefore if we can classify algebra objects with the given multiplicity vector, we can classify module category realizations of the fusion module.

We refer to the five fusion modules over EH_2 as EH_2-Modules 1-5, using the same order as in the text file. In the notation of Section §2.2, the corresponding (hypothetical) smallest algebra objects are given by $1 + 2f_2 + 2f_4 + f_6$, $1 + f_6$, $1 + f_4 + P$ (or $1 + f_4 + Q$), $1 + f_2$, and 1.

LEMMA 4.2. – *EH_2-Modules 4 and 5 are each realized by a unique right $\mathcal{EH}_2$-module category.*

Proof. – In any fusion category, the object 1 has a unique algebra structure. The object $1 + f_2$ has a (necessarily unique by 3-supertransitivity [42, Lemma 3.13]) algebra structure by the existence of the Extended Haagerup subfactor [6]. □

REMARK 4.3. – The argument in [42, Lemma 3.13] shows that 3-supertransitivity implies uniqueness (but not existence) of an algebra/Q-system structure on $1 + f_2$ in the pivotal and unitary pivotal settings as well.

To go further, we consider fusion bimodules, which we again enumerate with a computer. The full data is in the accompanying text file `EHBimodules.txt`. There are two EH_1-EH_1 fusion bimodules. There are three EH_1-EH_2 fusion bimodules, exactly one of which corresponds to the algebra $1 + f_2$ in $\mathcal{EH}_2$ (i.e., the Extended Haagerup subfactor). There are three EH_2-EH_2 fusion bimodules, one of which has rank 3, and the other two of which each contain basis elements with Frobenius-Perron dimension 1.

LEMMA 4.4. – *The rank* 3 *EH_2-EH_2 fusion bimodule is not realized by an $\mathcal{EH}_2$-$\mathcal{EH}_2$ bimodule category.*

Proof. – Looking at the (computer-generated) lists of multiplicatively compatible modules and bimodules in the accompanying text file `EHbimodulecomposition.txt`, we find that there is no possible way to tensor a realization of the rank 3 EH_2-EH_2 fusion bimodule (which is the first one on the list of EH_2-EH_2 bimodules) with any invertible $\mathcal{EH}_1$-$\mathcal{EH}_2$ bimodule category. □

LEMMA 4.5. – *The automorphism group of $\mathcal{EH}_2$ is trivial.*

Proof. – This argument is the same as the corresponding ones for Haagerup and Asaeda-Haagerup [42, 43]. There is a unique algebra object in $\mathcal{EH}_2$ giving the Extended Haagerup planar algebra and this algebra tensor generates $\mathcal{EH}_2$. Thus automorphisms of $\mathcal{EH}_2$ correspond to automorphisms of the Extended Haagerup planar algebra (see [48, Thm A] for details). Any automorphism of the Extended Haagerup planar algebra must send the generator to a multiple of itself (because it is uncappable) and the quadratic relation says that this scalar must be one. Thus the Extended Haagerup planar algebra does not admit non-trivial automorphisms. □

THEOREM 4.6. – *The Brauer-Picard group of the Extended Haagerup fusion categories is trivial.*

Proof. – Since by Lemma 4.4 the only realizable EH_2-EH_2-bimodules each contain a basis element of Frobenius-Perron dimension 1, any bimodule category realization of one of these bimodules is equivalent to the trivial module category as either a left or right module category. Since $\mathcal{EH}_2$ has no outer automorphisms, any such bimodule category is in fact the trivial bimodule category. Thus $\mathcal{EH}_2$ does not admit any non-trivial invertible bimodule categories, and the Brauer-Picard group is trivial. □

COROLLARY 4.7. – *The Brauer-Picard* 3*-groupoid has the homotopy type of* $K(\mathbb{C}^\times, 3)$. *Any* G*-graded extension of an Extended Haagerup fusion category is of the form* $\mathscr{C} \boxtimes \mathsf{Vec}(G, \omega)$ *for* $\omega \in H^3(G, \mathbb{C}^\times)$.

Proof. – The Brauer-Picard 3-groupoid is connected, has trivial π_1 (since the Brauer-Picard group is trivial), has trivial π_2 (by [40, Cor. 3.7] since $\mathcal{EH}_1$ has no invertible objects and no non-trivial gradings), and has $\pi_3 = \mathbb{C}^\times$ (by [22, Prop. 7.1]). Hence it is a $K(\mathbb{C}^\times, 3)$.

The classification of obstructions follows from the main result of [22]. Since the Brauer-Picard group is trivial, the obstructions O_3 and O_4 vanish. Since π_2 is trivial, extensions are classified by $H^3(G, \mathbb{C}^\times)$ and it is easy to see that $\mathscr{C} \boxtimes \mathsf{Vec}(G, \omega)$ realizes these extensions. □

COROLLARY 4.8. – *Exactly one of the three* EH_1*-*EH_2 *fusion bimodules is realized by a bimodule category (the one corresponding to the Extended Haagerup subfactor).*

LEMMA 4.9. – EH_2*-Module 1 is not realized by any module category.*

Proof. – Again looking at the lists of multiplicatively compatible modules and bimodules in the file `EHbimodulecomposition.txt`, we find that there is no possible way to tensor a right $\mathcal{EH}_2$-module category realizing EH_2-Module 1 with the known existing $\mathcal{EH}_2$-$\mathcal{EH}_1$ bimodule category (which corresponds to the third $EH_2 - EH_1$ bimodule on the list in the text files). This implies that EH_2-Module 1 is not realized by a module category. □

We are now left to classify categorifications of EH_2-Modules 2 and 3. For each of EH_2-Module 2/3, we can use multiplicative compatibility with the realized EH_1-EH_2-bimodule to uniquely identify a corresponding fusion module over EH_1 which would have to be realized as well for any realization of EH_2-Module 2/3.

From the lists in `EHbimodulecomposition.txt`, we see that EH_2-Module 2 corresponds to EH_1-Module 6 and EH_2-Module 3 corresponds to EH_1-Module 7.

We now introduce fusion rings EH_3 and EH_4 (whose multiplication tables were described in the preceding section). We compute the lists of fusion modules over EH_3 and EH_4; fusion bimodules over EH_i-EH_j, $1 \leq i, j, \leq 4$; and multiplicative compatibility between all of these modules and bimodules. This data is all included in the accompanying text files.

The reason for introducing these rings is the following:

LEMMA 4.10. – *If* EH_2*-Module 2 is realized by a right* $\mathcal{EH}_2$*-module category, then the fusion ring of the dual category is* EH_3. *If* EH_2*-Module 3 is realized by a right* $\mathcal{EH}_2$*-module category, then the fusion ring of the dual category is* EH_4.

Proof. – We use a computer to find the fusion rings of the dual categories of realizations of these fusion modules. It turns out that it is easier to compute the dual rings for the corresponding EH_1-Modules 6 and 7. Since any module category $\mathcal{K}_{\mathcal{EH}_2}$ realizing EH_2-Module 2 can be tensored with the $\mathcal{EH}_2$-$\mathcal{EH}_1$ Morita equivalence to give a module category $\mathcal{L}_{\mathcal{EH}_1}$ realizing EH_1-Module 6 and having the same dual category as $\mathcal{K}$ (and similarly for EH_2-Module 3), this is sufficient. □

LEMMA 4.11. – *EH_2-Module 2 and EH_2-Module 3 are each realized by at most one module category.*

Proof. – Let $\mathcal{K}_{\mathcal{EH}_2}$ and $\mathcal{L}_{\mathcal{EH}_2}$ be realizations of EH_2-Module 2 with dual categories $\mathcal{C}$ and $\mathcal{D}$. Then by the previous lemma $\mathcal{C}$ and $\mathcal{D}$ each have fusion ring EH_3. Then

$$\mathcal{M} = \mathcal{K}_{\mathcal{EH}_2} \boxtimes_{\mathcal{EH}_2} {}_{\mathcal{EH}_2}\mathcal{L}^{\mathrm{op}}$$

is an invertible $\mathcal{C}$-$\mathcal{D}$ bimodule category with realizes some EH_3-EH_3 fusion bimodule. Looking at the list of EH_3-EH_3-fusion bimodules, we see that every such bimodule has a basis element with Frobenius-Perron dimension 1. Therefore $\mathcal{M}$ is trivial as a left $\mathcal{C}$ module category. This means that $\mathcal{C} \cong \mathcal{D}$. Since the Brauer-Picard group is trivial, this implies that $\mathcal{K}_{\mathcal{EH}_2} \cong \mathcal{L}_{\mathcal{EH}_2}$.

The proof for EH_2-Module 3 is similar. □

Since ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is a Morita equivalence if and only if $\mathcal{B}$ is isomorphic to the dual category $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, we have the following corollary.

COROLLARY 4.12. – *There is at most one fusion category Morita equivalent to $\mathcal{EH}_2$ with fusion ring EH_3 and at most one fusion category Morita equivalent to $\mathcal{EH}_2$ with fusion ring EH_4.*

Putting this all together, we obtain the following result.

THEOREM 4.13. – *In addition to $\mathcal{EH}_1$ and $\mathcal{EH}_2$, the Morita equivalence class of the Extended Haagerup fusion categories contains:*

- *at most one fusion category with fusion ring EH_3;*
- *at most one fusion category with fusion ring EH_4;*
- *and no other fusion categories.*

The main result of this paper, Theorem 1.1 asserts the existence of fusion categories $\mathcal{EH}_3$ and $\mathcal{EH}_4$ in the Extended Haagerup Morita equivalence class with fusion rings EH_3 and EH_4, respectively.

REMARK 4.14. – There are analogous versions of Theorem 4.13 for the pivotal and unitary pivotal settings with the analogous conclusion as Theorem 4.13.

- In the pivotal setting, the pivotal Morita equivalence class of the Extended Haagerup pivotal fusion categories contains at most one pivotal fusion category with each of the fusion rings EH_3 and EH_4 and no other pivotal fusion categories.

— In the unitary pivotal setting, the unitary pivotal Morita equivalence class of the Extended Haagerup unitary fusion categories contains at most one unitary fusion category with each of the fusion rings EH_3 and EH_4 and no other unitary fusion categories.

The proofs of these theorems are completely analogous to the above argument inserting adjectives as necessary. It is important to note that there is no obvious way to derive these theorems from each other; rather we must use the same argument separately in each setting. The key point is that we already know that the Extended Haagerup subfactor is unique in all contexts (algebraically, pivotally, and unitary pivotally) by Remark 4.3. That is, we need to know that not only is there a unique algebra structure on $1 + f^{(2)}$ in $\mathcal{EH}_3$, but we also have a unique C* algebra structure [53, 55], a unique normalized Frobenius structure [75, Defn. 3.13], and a unique Q-system structure.

In principle, we might still have that $\mathcal{EH}_3$ or $\mathcal{EH}_4$ exists as say a fusion category, but not as a unitary pivotal fusion category. However, note that existence of $\mathcal{EH}_3$ and $\mathcal{EH}_4$ in the unitary pivotal setting (which is what we actually prove!) implies existence in all settings.

4.3. The fusion ring of the groupoid

Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are fusion categories and ${}_{\mathcal{A}}\mathcal{K}_{\mathcal{B}}$, ${}_{\mathcal{B}}\mathcal{L}_{\mathcal{C}}$, and ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{C}}$ are Morita equivalences such that there is a bimodule equivalence

$$\Phi : {}_{\mathcal{A}}\mathcal{K}_{\mathcal{B}} \boxtimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{L}_{\mathcal{C}} \cong {}_{\mathcal{A}}\mathcal{M}_{\mathcal{C}}.$$

In general there may be multiple such equivalences Φ, which are parametrized by invertible objects in the (common) Drinfeld center $Z(\mathcal{A})$. If the Drinfeld center has no non-trivial invertible objects then the equivalence Φ is uniquely determined by $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{M}$. There are no invertible central objects for the Extended Haagerup categories, as can be read off from the complete description of $Z(\mathcal{EH})$ in [73] or can be seen from [40, Corollary 4.2]. Therefore it makes sense to define the tensor product of simple objects in $\mathcal{K}$ and $\mathcal{L}$ as a direct sum of simple objects of $\mathcal{M}$. Thus for Extended Haagerup, we can define the fusion ring of the Brauer-Picard groupoid, with basis consisting of isomorphism classes of simple objects in each invertible bimodule category.

Notation 4.15. – For $1 \leq i, j \leq 4$, we denote by C_{ij} the unique $EH_i - EH_j$ fusion bimodule which was calculated using a computer and discussed in the last section. The rank of C_{ij} is the ij-th entry of the following matrix:

$$R := \begin{pmatrix} 6 & 6 & 6 & 6 \\ 6 & 8 & 5 & 5 \\ 6 & 5 & 8 & 5 \\ 6 & 5 & 5 & 8 \end{pmatrix}. \tag{4.1}$$

We may view $(C_{ij})_{i,j=1}^4$ as one fusion ring whose basis consists of the union of the distinguished bases of each C_{ij}. Multiplication of basis elements is determined by the relative tensor product of the ambient bimodules (and defined to be zero when the ambient bimodules don't compose).

We describe the fusion ring in the Mathematica notebook `EHmult.nb`, which is a wrapper for the data file `EHmult.txt`, both of which are bundled with the arXiv sources of this article. Therein, we supply a 6-dimensional tensor T whose (i, j, k, x, y, z)-entry is the coefficient of z-th basis element of $C_{i,k}$ in the product of the x-th basis element of C_{ij} and the y-th basis element of C_{jk}. That is,

$$ {}_iX_j \otimes {}_jY_k = \sum_Z T(i, j, k, x, y, z) {}_iZ_k \qquad \langle {}_iX_j \otimes {}_jY_k, {}_iZ_k \rangle := T(i, j, k, x, y, z) $$

where ${}_iX_j$ is the x-th basis of EH_{ij}, and similarly for ${}_jY_k$ and ${}_iZ_k$.

Notation 4.16. – We denote by $(\mathcal{EH}_{ij})_{i,j=1}^2$ and $(EH_{ij})_{i,j=1}^2$ the projection unitary multifusion category of the Extended Haagerup subfactor planar algebra and its fusion ring, where the 2 corresponds to an unshaded region and a 1 corresponds to a shaded region.

Remark 4.17. – By Theorem 4.13, there is at most one way to extend the unitary multifusion category $(\mathcal{EH}_{ij})_{i,j=1}^2$ to a unitary multifusion category $(\mathcal{EH}_{ij})_{i,j=1}^4$ such that $\mathcal{EH}_{ij}$ categorifies EH_{ij} for all $1 \le i, j \le 4$.

4.4. Fusion graphs from EH_2-Modules

We continue using Notations 4.15 and 4.16 from the previous section. Notice that for $1 \le k \le 4$, we get a left C-module M_k given by

$$ M_k := \begin{pmatrix} EH_{1k} \\ EH_{2k} \end{pmatrix}. $$

Definition 4.18. – For $1 \le k \le 4$, the *fusion graph* Γ_k for M_k with respect to X is the bipartite graph consisting of

— odd, shaded vertices given by the basis elements of EH_{1k},

— even, unshaded vertices given by the basis elements of EH_{2k}, and

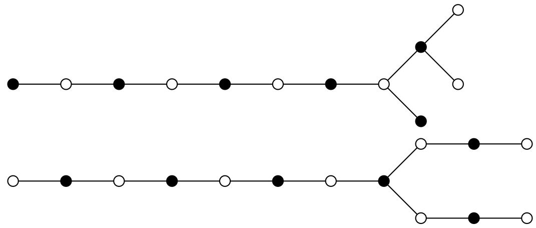
— $\langle {}_2X_1 \otimes {}_1Y_k, {}_2Z_k \rangle = T(2, 1, k, 1, y, z)$ edges between the y-th basis element ${}_1Y_k \in EH_{1k}$ and the z-th basis element ${}_2Z_k \in EH_{2k}$.

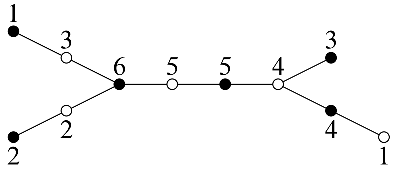
Note this convention is opposite to the one used for principal graphs of subfactors and fusion graphs for fusion categories in §2 (see Notation 2.3).

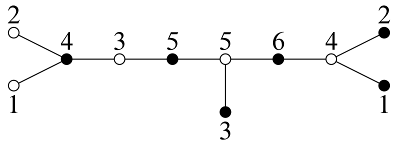
Proposition 4.19. – *The fusion graphs Γ_k for $1 \le k \le 4$ are given by*

$\Gamma_1 =$

$\Gamma_2 =$



$\Gamma_3 =$ 

$\Gamma_4 =$ 

REMARK 4.20. – The labelings on Γ_3 and Γ_4 match the indexing of objects in `EHmult.nb`. As we only need labelings on Γ_3 and Γ_4 in the following section, we have not labeled Γ_1 and Γ_2.

Our convention for shading the above vertices is that all vertices in $\mathcal{EH}_{1k}$ are shaded, whereas all vertices in $\mathcal{EH}_{2k}$ are unshaded. This corresponds to the fact that the unshaded region of the Extended Haagerup planar algebra $\mathcal{EH}_\bullet$ corresponds to $\mathcal{EH}_2$, and the shaded region corresponds to $\mathcal{EH}_1$.

Proof of Proposition 4.19. – The first two are exactly the definition of the dual principal graph and the principal graph of the extended Haagerup subfactor. The second two are obtained via computer in the Mathematica notebook `EHmult.nb` included with the arXiv sources of this article. □

REMARK 4.21. – By the complete classification of possible module categories for $\mathcal{EH}_1$ and $\mathcal{EH}_2$ in Theorem 4.13 together with Corollary 1.2, the graphs in Proposition 4.19 are the only bipartite graphs which could accept a planar algebra embedding map from the Extended Haagerup subfactor planar algebra.

COROLLARY 4.22. – *If the extended Haagerup subfactor planar algebra embeds into the graph planar algebra of Γ_k for $k = 3, 4$, then M_k is categorifiable as a $(\mathcal{EH}_{ij})_{i,j=1}^2$-module* C^**-category, and $\mathcal{EH}_k$ exists.*

Proof. – Fix $3 \leq k \leq 4$. By Corollary 1.2, the embedding of shaded planar algebras gives us a $(\mathcal{EH}_{ij})_{i,j=1}^2$-module C^*-category $\mathcal{M}_k$ which categorifies M_k and whose fusion graph with respect to the unshaded-shaded strand is given by Γ_k. We see that $\mathcal{M}_k$ is equivalent to a direct sum $\mathcal{EH}_{1k} \oplus \mathcal{EH}_{2k}$ where $\mathcal{EH}_{jk}$ is a left module category over $\mathcal{EH}_{jj} := \mathcal{EH}_j$ for $j = 1, 2$. By analyzing the fusion rules with X, by Theorem 4.13, we may conclude that $\mathcal{EH}_{jk}$ categorifies the fusion bimodule EH_{jk} for $j = 1, 2$. Specializing to $j = 2$, since EH_{2k} is a $EH_{22} - EH_{kk}$ bimodule, by Theorem 4.13, the dual category $\mathcal{EH}_{kk}$ of the $\mathcal{EH}_{22}$-module $\mathcal{EH}_{2k}$ must categorify EH_{kk}. Again by Theorem 4.13, $\mathcal{EH}_{kk}$ is equivalent to $\mathcal{EH}_k$. □

REMARK 4.23. – We can perform a similar (simpler) calculation for the Haagerup fusion categories. It was shown in [42] that there are exactly three fusion categories in the Morita equivalence class of the Haagerup subfactor, which we will denote by $\mathcal{H}_k$, $k = 1, 2, 3$; and a unique Morita equivalence between each pair. The category $\mathcal{H}_2$ has six simple objects, labeled 1, g, g^2, X, gX, and g^2X, which satisfy the fusion rules

$$g^3 = 1, \quad X^2 = 1 + X + gX + g^2X, \quad gX = Xg^2.$$

(Here we have used decategorified notation, and suppressed tensor product, direct sum, and isomorphism symbols).

The category $\mathcal{H}_3$ is the category of bimodules in $\mathcal{H}_2$ over the algebra $1 + g + g^2$; it has the same fusion ring as $\mathcal{H}_2$, and we will label its simple objects by 1, h, h^2, Y, gY, and g^2Y. The category $\mathcal{H}_1$ can described as the category of bimodules over the algebra $1 + X$ in $\mathcal{H}_2$, or as the category of bimodules over $1 + Y + hY$ in $\mathcal{H}_3$. The Haagerup planar algebra is the planar algebra corresponding to the generator K of the $\mathcal{H}_1$-$\mathcal{H}_2$ Morita equivalence whose right internal end $\overline{K}K$ is $1 + X$. Let L be the object in the $\mathcal{H}_2$-$\mathcal{H}_3$ Morita equivalence whose left internal end $L\overline{L}$ is $1 + g + g^2$ (and whose right internal end $\overline{L}L$ is $1 + h + h^2$). Let M be the object in the $\mathcal{H}_1$-$\mathcal{H}_3$ Morita equivalence whose right internal end $\overline{M}M$ is $1 + Y + hY$.

The $\mathcal{H}_2$-$\mathcal{H}_3$ Morita equivalence has rank two, with simple objects $L = gL = Lh$ and $XL = LY$. The $\mathcal{H}_1$-$\mathcal{H}_3$ Morita equivalence has rank four, with simple objects KL, M, Mh, and Mh^2. The fusion graph for the module corresponding to $\mathcal{E}\mathcal{H}_3$ is then determined by tensoring each of the two simple 1-2 objects on the left by K and decomposing into simple 2-3 objects. Clearly there is a single edge from L to KL and no other edges out of L. We now want to find the vertices adjacent to XL, i.e., the summands of KXL. By Frobenius reciprocity, using $(\cdot,\cdot)$ to denote the dimension of the hom space,

$$\begin{aligned}(KXL, KXL) &= (\overline{K}K, XL\overline{L}\overline{X}) = (1 + X, X(1 + g + g^2)\overline{X}) \\ &= (1 + X, 1 + g + g^2 + 3X + 3gX + 3g^2X) = 4.\end{aligned}$$

So KXL has either four distinct simple summands or a single simple summand with multiplicity two. But

$$(KXL, KL) = (\overline{K}KX, L\overline{L}) = ((1+X)X, 1+g+g^2) = (1+2X+gX+g^2X, 1+g+g^2) = 1,$$

so KL appears with multiplicity one in KXL. Thus KXL has four distinct summands and there is a single edge from XL to each of the four simple 2-3 objects. This gives the broom graph of Corollary 1.4.

5. Graph planar algebra embeddings for Extended Haagerup

To specify a map out of a planar algebra presented by generators and relations, we need only to assign values to the generators and check the relations. In particular, once we have a nice presentation of a planar algebra, we can easily calculate all pivotal (C*) module categories over it. For example, if we want to calculate all pivotal (C*) module categories over the Termperley-Lieb-Jones planar algebra, we have no generators, and the only relation is the loop modulus, so we get a unique module category for every planar graph with the correct Frobenius-Perron eigenvector [25, 16]. The $SU(3)_q$ planar algebra is presented by two trivalent vertices satisfying certain relations using Kuperberg's spider description [65], and finding elements in a graph planar algebra corresponding to these two trivalent vertices is exactly solving Ocneanu's cell conditions [79, 27, 87].

One of the main results of [6] is to give a similar characterization of maps out of the Extended Haagerup planar algebra denoted $\mathcal{EH}_\bullet$, which we recall in Proposition 5.7. Using this result, we give the embeddings of the extended Haagerup subfactor planar into the graph planar algebras of Γ_3 and Γ_4, by solving the equations specified in Proposition 5.7 in the appropriate graph planar algebras. This is closely analogous to the original construction of $\mathcal{EH}_\bullet$ by embedding it in the graph planar algebra of its principal graph, and we are able to reuse the same code. There are associated Mathematica notebooks (`module-GPAs-EH3.nb` and `module-GPAs-EH4.nb`) which demonstrate the messy process of solving these equations. Here we simply exhibit particular solutions. Thus by Corollary 4.22, M_3 and M_4 are categorifiable as $(\mathcal{EH}_{ij})_{i,j=1}^2$-module C*-categories, and $\mathcal{EH}_3$ and $\mathcal{EH}_4$ exist.

5.1. The lopsided graph planar algebra convention

Suppose $\mathcal{P}_\bullet$ is a semisimple shaded planar algebra with pivotal projection multitensor category $(\mathscr{C}, X, \varphi)$ where $X \in \mathcal{P}_{1,+}$ is the shaded-unshaded strand. By just rescaling cups and caps in $\mathscr{C}$ for X as in [71, §1.1],

(5.1) 

we obtain another semisimple shaded planar algebra $\mathcal{P}_\bullet^{\cap x,y}$ with the same underlying projection multitensor category. To describe the action of tangles, we first write the tangles in standard form, where each box has the same number of strings emanating from the top and bottom. The action of tangles is obtained from the action of tangles for $\mathcal{P}_\bullet$, where in addition, we multiply by factors of x, y, x^{-1}, y^{-1} corresponding to appearances of cups and caps as in (5.1) in the standard form for the tangle.

It is straightforward to verify that this is a well-defined action of planar tangles which is independent of the choice of standard form of a tangle. One first verifies that the zig-zag relations hold and 2π-rotation is still the identity. One then appeals to the folklore theorem ([61, Proof of Thm. 4.2.1], similar to [48, Prop. 4.5]) that any two standard forms of a tangle are related by a finite number of moves including Morse cancelation, 2π-rotation, and exchanging the heights of two input boxes. Thus $\mathcal{P}_\bullet^{\cap x,y}$ is a shaded planar algebra.

While the underlying projection multitensor category $\mathscr{C}$ has not changed, the pivotal structure $\varphi^{\cap x,y}$ on $\mathscr{C}$ corresponding to $\mathcal{P}_\bullet^{\cap x,y}$ has changed! Indeed, pivotal structures on a semisimple multitensor category are completely determined by the left and right pivotal dimensions [83, Lem. 2.12]. The left and right $\varphi^{\cap x,y}$ pivotal dimensions on $\mathscr{C}$, denoted $\dim_{L/R}^{\cap x,y}$, are related to the left and right φ pivotal dimensions, denoted $\dim_{L/R}^{\varphi}$, as follows:

$$(5.2)\qquad (\dim_L^{\cap x,y}(c), \dim_R^{\cap x,y}(c)) = \begin{cases} (\dim_L^{\varphi}(c), \dim_R^{\varphi}(c)) & \text{if } c \in \mathscr{C}_{00} \\ (xy^{-1}\dim_L^{\varphi}(c), yx^{-1}\dim_R^{\varphi}(c)) & \text{if } c \in \mathscr{C}_{01} \\ (yx^{-1}\dim_L^{\varphi}(c), xy^{-1}\dim_R^{\varphi}(c)) & \text{if } c \in \mathscr{C}_{10} \\ (\dim_L^{\varphi}(c), \dim_R^{\varphi}(c)) & \text{if } c \in \mathscr{C}_{11} \end{cases}$$

Notice that we may write (5.2) as simply one equation:

$$(\dim_L^{\cap x,y}(c), \dim_R^{\cap x,y}(c)) = (x^j x^{-i} y^i y^{-j} \dim_L^{\varphi}(c), x^i x^{-j} y^j y^{-i} \dim_R^{\varphi}(c)) \qquad \forall c \in \mathscr{C}_{ij}.$$

DEFINITION 5.1. – Suppose $\mathcal{P}_\bullet$ is a semisimple shaded planar algebra in which the shaded/unshaded closed loops are multiplicative scalars $\delta_\pm \in \mathcal{P}_{0,\pm}$ respectively. We call $\mathcal{P}_\bullet$ *lopsided* if $\delta_+ = 1$.

Given a semisimple shaded planar algebra $\mathcal{P}_\bullet$ with scalar loop moduli $\delta_\pm$ as in Definition 5.1, we can always obtain a lopsided planar algebra $\mathcal{P}_\bullet^{\text{lopsided}} := \mathcal{P}_\bullet^{\cap\delta_+,1}$. Notice that the shaded/unshaded loop moduli in $\mathcal{P}_\bullet^{\text{lopsided}}$ are now 1 and $\delta_+\delta_-$ respectively.

EXAMPLE 5.2 ([71, §1.1]). – Let $\mathcal{G}_\bullet$ be the graph planar algebra of a finite bipartite graph $\Gamma = (V_+, V_-, E)$, whose shaded and unshaded loop moduli are both $\delta = \|\Gamma\|$. The *lopsided* graph planar algebra is $\mathcal{G}_\bullet^{\text{lopsided}} := \mathcal{G}_\bullet^{\cap\delta,1}$. Notice that the lopsided pivotal structure is obtained from the standard pivotal structure by *only* rescaling cups and caps which are shaded above by a multiplicative factor of $\delta^{\pm1}$, where the sign is the sign of the critical point ($+1$ for caps and -1 for cups).

WARNING 5.3. – The corresponding projection unitary multifusion category of $\mathcal{G}_\bullet$ is $\mathrm{End}^\dagger(\mathsf{Hilb}^{|V_+|+|V_-|})$, which is equipped with the standard unitary dual functor $\vee_{\text{standard}}$ with respect to the object X representing Γ.

The lopsided pivotal structure on $\mathrm{End}^\dagger(\mathsf{Hilb}^{|V_+|+|V_-|})$ induced by $\mathcal{G}_\bullet^{\text{lopsided}}$ is not unitary as noted in the first paragraph of [71, §1.1], as $y^{-1} = 1 \neq \delta = \overline{x}$. However, it is computationally easier to work with the non-unitary lopsided pivotal structure as introducing square roots increases the degree of the number fields involved. Moreover, by [71], one can pass back and forth between the non-unitary lopsided convention and the unitary standard convention, so we do not lose any examples.

5.2. The Extended Haagerup subfactor planar algebra

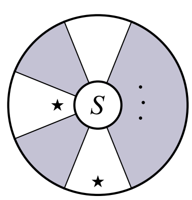
The Extended Haagerup subfactor planar algebra $\mathcal{EH}_\bullet$ is a shaded planar †-algebra, generated by an 8-box called S which satisfies the relations given below.

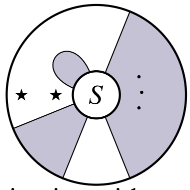
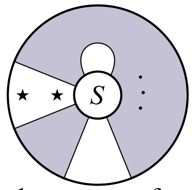
The presentation given in [6] uses the spherical pivotal structure, and here we also give a presentation with the lopsided pivotal structure, as this is necessary for computations later. The translation follows the discussion on p. 3 of [71].

— *Modulus:* With [2] the largest root of $x^6 - 8x^4 + 17x^2 - 5 = 0$, approximately 2.09218, in the lopsided pivotal structure we have the shaded loop equal to 1 and the unshaded loop equal to $[2]^2$, while in the spherical pivotal structure both loops are equal to [2].

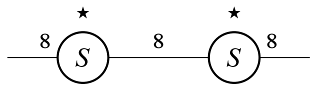
 (In the remainder of these formulas, coefficients are given using quantum numbers defined in the usual way, $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$.)

— *Self-adjoint:* $S = S^*$.

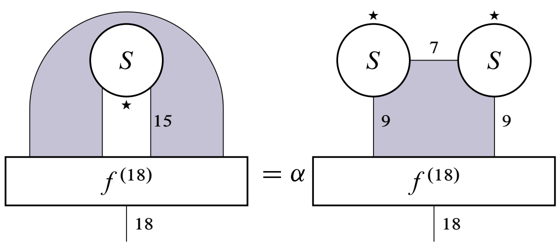
— *Rotational eigenvector:*  $= -S$.

— *Uncappable:*  $= 0$ and  $= 0$

(and in combination with rotation, all placements of a cap on a generator S are zero).

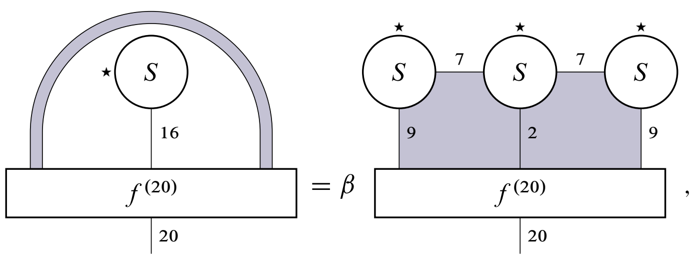
— *Multiplication relation:* $S^2 =$  $= f^{(8)}$.

— *One strand jellyfish relation:*



with $\alpha = i\dfrac{\sqrt{[8][10]}}{[2]^4[9]}$ in the lopsided case, or $\alpha = i\dfrac{\sqrt{[8][10]}}{[9]}$ in the spherical case.

— *Two strand jellyfish relation:*



with $\beta = \frac{[20]}{[2]^6[9][10]}$ in the lopsided case, or $\beta = \frac{[2][20]}{[9][10]}$ in the spherical case.

These relations are sufficient to evaluate all closed diagrams in S, via the 'jellyfish algorithm' which pulls copies of S to the exterior and then cancels them in pairs. Note that in addition to the above relations, to give a complete description of the Extended Haagerup subfactor planar algebra we also quotient by the negligible elements. Moreover, there is a non-zero representation of this abstract planar †-algebra in the graph planar algebra of the principal graph, which proves the existence of the Extended Haagerup subfactor planar algebra. We refer the reader to [6] for more details.

Below we use the constant λ for the largest purely imaginary root of $\lambda^6+2\lambda^4-3\lambda^2-5=0$, approximately $1.54i$.

LEMMA 5.4 (Variation of [6, Prop. 3.12]). – *Let Γ be a finite bipartite graph with norm* [2] *as above. Suppose $S \in \mathcal{GPA}(\Gamma)_{8,+}$ is a self-adjoint, uncappable, rotational eigenvector with eigenvalue -1, and has the Extended Haagerup moments*

$$\text{tr}(S^2) = [9] \qquad \text{tr}(S^3) = 0 \qquad \text{tr}(S^4) = [9] \qquad \text{tr}(\rho^{1/2}(S)^3) = i\frac{[18]}{\sqrt{[8][10]}}. \tag{5.3}$$

Let $\mathcal{PA}(S)_\bullet$ *be the planar* †*-subalgebra of* $\mathcal{GPA}(\Gamma)_\bullet$ *generated by* S*. Then* $\mathcal{PA}(S)_\bullet \cong \mathcal{EH}_\bullet$.

Proof. – The proof that $\mathcal{PA}(S)_\bullet$ is an irreducible subfactor planar algebra with principal graph Γ_2 from Proposition 4.19 is identical to the proof of [6, Prop. 3.12], which never used that $\Gamma = \Gamma_2$. The final claim that $\mathcal{PA}(S)_\bullet \cong \mathcal{EH}_\bullet$ follows by uniqueness of the Extended Haagerup subfactor planar algebra [44]. □

REMARK 5.5. – In fact, Lemma 5.4 holds if we replace $\mathcal{GPA}(\Gamma)_\bullet$ with any unitary shaded planar algebra $\mathcal{P}_\bullet$ with a spherical faithful tracial state $\psi_\pm$ on $\mathcal{P}_{0,\pm}$ (see Remark 3.76 or [83, §5]) whose shaded and unshaded loop values are both [2] as above.

REMARK 5.6. – We want to emphasize that the proof [6, Prop. 3.12] uses unitarity in an essential way. The key step, following [86], is that using only the moments you can prove the Jellyfish relations by checking that the inner product of each relation with itself is 0.

PROPOSITION 5.7. – *Suppose* $\mathcal{P}_\bullet$ *is any unitary shaded planar algebra with a spherical faithful tracial state* $\psi_\pm$ *on* $\mathcal{P}_{0,\pm}$ *whose shaded and unshaded loop values are both* [2] *as above. Planar* †*-algebra morphisms* $\mathcal{EH}_\bullet \to \mathcal{P}_\bullet$ *are in bijection with choices of self-adjoint uncappable elements* $S' \in \mathcal{P}_{8,+}$ *with rotational eigenvalue* -1*, satisfying*

$$S'^2 = f^{(8)} \tag{5.4}$$

$$\begin{aligned}\rho^{-1/2}(S')^2 &= \tfrac{2}{5}\left(-\lambda^5 - 2\lambda^3 + 3\lambda\right)[2]^{-1}\rho^{-1/2}(S') + \left(\lambda^2 - 2\right)[2]^{-2} f^{(8)} \\ &= i(\check{r}^{1/2} - \check{r}^{-1/2})\rho^{-1/2}(S) - f^{(8)},\end{aligned} \tag{5.5}$$

where $\check{r} = \frac{[10]}{[8]}$.

Proof. – By Lemma 5.4 and Remark 5.5, we only need to show that S' satisfies the Extended Haagerup moments (5.3) if and only if (5.4) and (5.5) hold. Clearly if (5.4) and (5.5) hold, then S' satisfies the Extended Haagerup moments (5.3). Conversely, suppose S' satisfies the Extended Haagerup moments (5.3). By [6, Prop. 3.7], S' so (5.4) holds, together with the one and two strand jellyfish relations. As the principal graphs must be those of Extended Haagerup, again by Lemma 5.4, we can apply [6, Eq. (3.3)] (essentially from [60]), which gives (5.5) above for S'. □

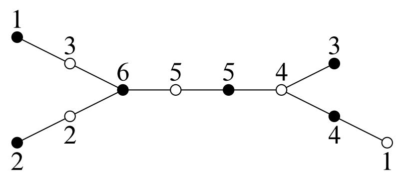
COROLLARY 5.8. – *Planar algebra homomorphisms* $\mathcal{EH}_\bullet^{\cap\delta,1} \to \mathcal{P}_\bullet^{\cap\delta,1}$ *between the lopsided planar algebras are in bijection with choices of uncappable elements* $S' \in \mathcal{P}_{8,+}$ *with rotational eigenvalue* -1 *satisfying* (5.4) *and*

$$\rho^{-1/2}(S)^2 = \frac{2}{5}\left(-\lambda^5 - 2\lambda^3 + 3\lambda\right)\rho^{-1/2}(S) + \left(\lambda^2 - 2\right) f^{(8)}, \tag{5.6}$$

rather than (5.5).

5.3. $\mathcal{EH}_3$

In this section, we use Corollary 5.8 to find $\mathcal{EH}_\bullet$ in the graph planar algebra of the bipartite graph

$\Gamma_3 =$ 

The eigenspace of uncappable elements with 16 boundary points, and rotational eigenvalue -1, is 18-dimensional. An element in this eigenspace is determined by its values c_i on the following loops ℓ_i based at unshaded/even vertices:

$$\begin{array}{lll} \ell_1 = 5655434556554345 & \ell_2 = 5543455622263626 & \ell_3 = 4556265626365543 \\ \ell_4 = 5636265626554345 & \ell_5 = 2636265626362636 & \ell_6 = 4556263626265543 \\ \ell_7 = 4556222655554345 & \ell_8 = 2636263626263622 & \ell_9 = 4345562626313655 \\ \ell_{10} = 4345563626363655 & \ell_{11} = 4556313655454345 & \ell_{12} = 5631365636554345 \\ \ell_{13} = 2636265626263136 & \ell_{14} = 2226362631362636 & \ell_{15} = 2631362636362636 \\ \ell_{16} = 5631362626554345 & \ell_{17} = 2631362626362636 & \ell_{18} = 2226313622263136 \end{array}$$

There are exactly two solutions to the equations, and these are related by $S' = -\overline{S}$, or by applying the unique graph automorphism (and hence corresponds to an equivalent module category as in Warning 3.81). The element S has coefficients in $\mathbb{Q}(\mu)$, where μ is the root of $\mu^{12} + 718\mu^{10} + 679145\mu^8 + 43340550\mu^6 + 43588750\mu^4 - 625000\mu^2 + 390625 = 0$ which is approximately $-0.229025 - 0.202916i$. The values of c_i written as polynomials in μ are quite horrific (coefficients rational numbers with numerators and denominators having up to 30 digits), so we instead express them directly in terms of their minimal polynomials. (The associated Mathematica notebook contains their values in the number field.) We use the notation $\lambda^x_{a_0,\ldots,a_k}$ to denote the root of $a_0 + a_1\lambda + \cdots + a_k\lambda^k = 0$ which is closest to the approximate number x (and we're careful to write x with enough precision that this is unambiguous).

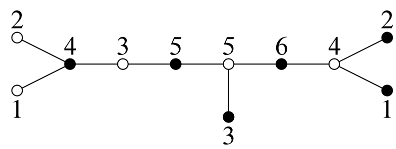
$$\begin{aligned}
c_1 &= \lambda^{(0.0080256i)}_{1,0,112942,0,-1940695,0,-125} \\
c_2 &= \lambda^{(0.1672-0.0995i)}_{625,0,58550,0,1877265,0,24363782,0,119192086,0,-4303080,0,172225} \\
c_3 &= \lambda^{(0.03538+0.16258i)}_{9765625,0,822187500,0,5692096250,0,704926450,0,34457185,0,774362,0,6889} \\
c_4 &= \lambda^{(0.03272-0.15038i)}_{15625,0,47736250,0,11814953125,0,1219921150,0,49538050,0,927928,0,6889} \\
c_5 &= \lambda^{(0.0061335)}_{25,0,4235,0,26582,0,-1} \\
c_6 &= \lambda^{(0.10306+0.06133i)}_{9765625,0,100312500,0,287121250,0,166019450,0,31036785,0,-421822,0,6889} \\
c_7 &= \lambda^{(-0.048654i)}_{5,0,183,0,-422,0,-1} \\
c_8 &= \lambda^{(-0.1672)}_{125,0,1490,0,137,0,-5} \\
c_9 &= \lambda^{(0.24287-0.03754i)}_{625,0,30300,0,164710,0,6266122,0,18530421,0,-2194130,0,70225}
\end{aligned}$$

$$c_{10} = \lambda^{(0.049520-0.029468i)}_{15625,0,1045000,0,25515750,0,222706550,0,624079625,0,-1976682,0,6889}$$
$$c_{11} = \lambda^{(0.05260i)}_{125,0,-205,0,-362,0,-1}$$
$$c_{12} = \lambda^{(-0.09532-0.05673i)}_{15625,0,5448750,0,470120625,0,259808550,0,42457870,0,-493928,0,6889}$$
$$c_{13} = \lambda^{(-0.045805+0.040583i)}_{625,0,17950,0,679145,0,1733622,0,69742,0,-40,0,1}$$
$$c_{14} = \lambda^{(0.09647i)}_{25,0,-622,0,-543,0,-5}$$
$$c_{15} = \lambda^{(-0.049520)}_{625,0,17450,0,365,0,-1}$$
$$c_{16} = \lambda^{(-0.138433-0.021397i)}_{15625,0,3842500,0,55831750,0,-4013550,0,7389525,0,-273698,0,2809}$$
$$c_{17} = \lambda^{(0.013563+0.012017i)}_{390625,0,24156250,0,2220203125,0,1165172950,0,9182770,0,-608,0,1}$$
$$c_{18} = \lambda^{(-0.3117i)}_{5,0,-222,0,-279,0,-25}.$$

It is then a simple matter to directly verify the equations (this takes less than a minute on a modern CPU); this verification can be found in `module-GPAs-EH3.nb`.

5.4. $\mathcal{EH}_4$

In this section, we use Corollary 5.8 to find $\mathcal{EH}_\bullet$ in the graph planar algebra of the bipartite graph

$\Gamma_4 =$ 

The eigenspace of uncappable elements with 16 boundary points, and rotational eigenvalue -1, is 20 dimensional. An element in this eigenspace is determined by its values c_i on the following loops ℓ_i based at unshaded/even vertices:

$\ell_1 = 3553565355553424$ $\ell_2 = 5653555356535653$ $\ell_3 = 3555535646553414$
$\ell_4 = 5646535653564653$ $\ell_5 = 4146535534243556$ $\ell_6 = 5553564146535653$
$\ell_7 = 4146535641465356$ $\ell_8 = 4246553424355356$ $\ell_9 = 4246535534243556$
$\ell_{10} = 5553564246535653$ $\ell_{11} = 5646535646424653$ $\ell_{12} = 4146535642465356$
$\ell_{13} = 4246535642465356$ $\ell_{14} = 3556424146553424$ $\ell_{15} = 5646535653564146$
$\ell_{16} = 5356424146535646$ $\ell_{17} = 5642414646535653$ $\ell_{18} = 5641424646414653$
$\ell_{19} = 5642414246424653$ $\ell_{20} = 4142414641424142$

There are four solutions to these equations, and the graph automorphism group acts freely and transitively on them (and hence they all give equivalent module categories as in Warning 3.81). The solutions have coefficients in $\mathbb{Q}(\mu)$, where μ is the root of

$$\mu^{12} - 74510\mu^{10} + 1753550625\mu^8 - 8889717968750\mu^6 + 23050129394531250\mu^4 + 42850952148437500\mu^2 + 95367431640625 = 0,$$

which is approximately $-0.0472042i$. One of the four solutions has coefficients:

$$c_1 = \lambda^{(0.04828+0.07374i)}_{3125,49250,56580,53520,1597,-200,53}$$

$$
\begin{aligned}
c_2 &= \lambda^{(-0.038842i)}_{125,0,-1285982,0,-1789244179,0,-2699449}\\
c_3 &= \lambda^{(-0.02632-0.06233i)}_{3125,-18750,31575,-20540,4443,186,25}\\
c_4 &= \lambda^{(0.05003i)}_{5,0,2882,0,-249683,0,-625}\\
c_5 &= \lambda^{(-0.063152-0.039778i)}_{48828125,-195312500,386718750,-344687500,126334375,-3725000,-6388300,43560,201947,23420,1230,36,1}\\
c_6 &= \lambda^{(0.0086287i)}_{125,0,150048,0,92084512,0,8056764288,0,285286080768,0,296306688,0,20480}\\
c_7 &= \lambda^{(0.40535i)}_{125,0,197208,0,81755664,0,-661557632,0,3025487360,0,515469312,0,20480}\\
c_8 &= \lambda^{(-0.30264+0.07970i)}_{125,3750,27250,-64700,141035,-2100,103848,105108,29242,2034,-122,-10,1}\\
c_9 &= \lambda^{(0.38771+0.10211i)}_{48828125,-195312500,386718750,-344687500,126334375,-3725000,-6388300,43560,201947,23420,1230,36,1}\\
c_{10} &= \lambda^{(-0.031052i)}_{125,0,150048,0,92084512,0,8056764288,0,285286080768,0,296306688,0,20480}\\
c_{11} &= \lambda^{(0.032477i)}_{5,0,1282,0,-4739,0,-5}\\
c_{12} &= \lambda^{(-0.2194i)}_{1,0,-293,0,-118,0,-5}\\
c_{13} &= \lambda^{(0.0063040i)}_{125,0,197208,0,81755664,0,-661557632,0,3025487360,0,515469312,0,20480}\\
c_{14} &= \lambda^{(-0.1850-0.1190i)}_{15625,-37500,2375,-850,-832,-156,13}\\
c_{15} &= \lambda^{(-0.069636i)}_{125,0,-9582,0,821981,0,-28226758,0,2200643514,0,16166708,0,26645}\\
c_{16} &= \lambda^{(0.0099201i)}_{125,0,-2047,0,-50809,0,-5}\\
c_{17} &= \lambda^{(-0.026344i)}_{125,0,455738,0,13472487051,0,-26481195508,0,28428109059,0,58134938,0,26645}\\
c_{18} &= \lambda^{(-0.00188817i)}_{625,0,-74510,0,2805681,0,-22757678,0,94413330,0,280828,0,1}\\
c_{19} &= \lambda^{(0.0544863i)}_{625,0,-74510,0,2805681,0,-22757678,0,94413330,0,280828,0,1}\\
c_{20} &= \lambda^{(0.1063i)}_{1,0,4982,0,-2155,0,-25}.
\end{aligned}
$$

Again, it is easy to verify this gives a solution, shown in `module-GPAs-EH4.nb`.

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